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We construct families of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ with Scherk-type ends and arbitrary finite genus in the quotient. The construction follows by gluing small perturbations of pieces of already known minimal surfaces: Scherk minimal surfaces, Costa–Hoffman–Meeks surfaces and KMR examples.

1. Introduction

Besides the plane and the helicoid, the first singly periodic minimal surface in $\mathbb{R}^3$ was discovered by Scherk [1835]. This surface, known as *Scherk's second surface*, is a properly embedded minimal surface in $\mathbb{R}^3$ that is invariant by one translation T we can assume to be along the x_2 axis, and can be seen as the desingularization of two perpendicular planes P_1 and P_2 containing the x_2 axis. We assume P_1 and P_2 are symmetric with respect to the planes $\{x_1 = 0\}$ and $\{x_3 = 0\}$. By changing the angle between P_1 and P_2, we obtain a 1-parameter family of properly embedded singly periodic minimal surfaces, which we will refer to as *Scherk surfaces*. In the quotient $\mathbb{R}^3/T$ by its shortest period T, each Scherk surface has genus zero and four ends asymptotic to flat annuli contained in P_1/T and P_2/T. Such ends are called Scherk-type ends. From now on, T will denote a translation in the x_2 direction.

In 1982, C. Costa [1982; 1984] discovered a genus one minimal surface with three embedded ends: one top catenoidal end, one middle planar end and one bottom catenoidal end. D. Hoffman and W. H. Meeks [1985; 1989; 1990] proved the global embeddedness for this Costa example, and generalized it for higher genus. For each $k \geqslant 1$, the Costa–Hoffman–Meeks surface M_k is a properly embedded minimal surface of genus k and three ends: two catenoidal ends and one middle planar end.

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F. Martín and V. Ramos Batista [Martín and Ramos Batista 2006] have recently constructed a 1-parameter family of properly embedded singly periodic minimal surfaces that have genus one and six Scherk-type ends in the quotient $\mathbb{R}^3/T$. These are called *Scherk–Costa surfaces* and are based on the Costa surface. Roughly speaking, they remove each end of the Costa surface (asymptotic to a catenoid or a plane) and replace it by two Scherk-type ends. In this paper, we obtain surfaces in the same spirit as Martin and Ramos Batista's one, but with a completely different method. We construct properly embedded singly periodic minimal surfaces with genus $k \geqslant 1$ and six Scherk-type ends in the quotient $\mathbb{R}^3/T$ by gluing (in an analytic way) a compact piece of M_k to two halves of a Scherk surface at the top and bottom catenoidal ends, and one flat horizontal annulus P/T with a disk removed at the middle planar end.

Theorem 1.1. *Let T denote a translation in the x_2 direction. For each $k \geqslant 1$, there exists a 1-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ invariant by T whose quotient in $\mathbb{R}^3/T$ has genus k and six Scherk-type ends.*

V. Ramos Batista [2005] constructed a singly periodic Costa minimal surface with two catenoidal ends and two Scherk-type middle ends; this surface has genus one in the quotient $\mathbb{R}^3/T$. This example is not embedded outside a slab in $\mathbb{R}^3/T$ that contains the topology of the surface. We observe that the surface we obtain by gluing a compact piece of M_1 (Costa surface) at its middle planar end to a flat horizontal annulus with a disk removed has the same properties as Ramos Batista's.

In 1988, H. Karcher [1988; 1989] defined a family of properly embedded doubly periodic minimal surfaces, called *toroidal halfplane layers*, which have genus one and four horizontal Scherk-type ends in the quotient. In 1989, W. H. Meeks and H. Rosenberg [1989] developed a general theory for doubly periodic minimal surfaces having finite topology in the quotient, and used a minimax approach to obtain the existence of a family of properly embedded doubly periodic minimal surfaces, also with genus one and four horizontal Scherk-type ends in the quotient. Karcher's and Meeks and Rosenberg's surfaces have been generalized by M. M. Rodríguez [2007], who constructed a 3-parameter family $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ of surfaces, called *KMR examples* (sometimes they are also called toroidal halfplane layers). Such examples have been classified by J. Pérez, M. M. Rodríguez and M. Traizet [Pérez et al. 2005] as the only properly embedded doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient. Each $M_{\sigma,\alpha,\beta}$ is invariant by a horizontal translation T (by the period vector at its ends) and a nonhorizontal one $\widetilde{T}$. We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{R}^3/T$, which has genus zero, infinitely many horizontal Scherk-type ends, and two limit ends.

In [1992], F. S. Wei added a handle to a KMR example $M_{\sigma,0,0}$ in a periodic way, obtaining a properly embedded doubly periodic minimal surface invariant under reflection in three orthogonal planes, which has genus two and four horizontal Scherk-type ends in the quotient. Some years later, W. Rossman, E. C. Thayer and M. Wolgemuth [Rossman et al. 2000] added a different type of handle to a KMR example $M_{\sigma,0,0}$, also in a periodic way, obtaining a different minimal surface with the same properties as Wei's one. They also added two handles to a KMR example, getting doubly periodic examples of genus three in the quotient. L. Mazet and M. Traizet [2008] have added $N \geqslant 1$ handles to a KMR example $M_{\sigma,0,0}$, obtaining a genus N properly embedded minimal surface in $\mathbb{R}^3/T$ with an infinite number of horizontal Scherk-type ends and two limit ends. The idea of the construction is to connect N periods of the doubly periodic example of Wei with two halves KMR example. However they only control the asymptotic behavior in their construction. They have also constructed a properly embedded minimal surface in $\mathbb{R}^3/T$ with infinite genus, adding handles in a quasiperiodic way to a KMR example.

L. Hauswirth and F. Pacard [2007] have constructed higher genus Riemann minimal examples in $\mathbb{R}^3$, by gluing two halves of a Riemann minimal example with the intersection of a conveniently chosen Costa–Hoffman–Meeks surface M_k with a slab. We follow their ideas to generalize Mazet and Traizet's examples by constructing higher genus KMR examples: We construct two 1-parameter families of properly embedded singly periodic minimal examples whose quotient in $\mathbb{R}^3/T$ has arbitrary finite genus, infinitely many horizontal Scherk-type ends and two limit ends. More precisely, we glue a compact piece of a slightly deformed example M_k with tilted catenoidal ends, to two halves of a KMR example $M_{\sigma,\alpha,0}$ or $M_{\sigma,0,\beta}$ (see Figure 1) and a periodic horizontal flat annulus with a disk removed.

Theorem 1.2. *Let T denote a translation in the x_2 direction. For each $k \geqslant 1$, there exist two 1-parameter families $\mathcal{K}_1$ and $\mathcal{K}_2$ of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^3/T$ has genus k, infinitely many horizontal Scherk-type ends and two limit ends. The surfaces in $\mathcal{K}_1$ have a plane of symmetry orthogonal to the x_1 axis, and the surfaces in $\mathcal{K}_2$ have a plane of symmetry orthogonal to the x_2 axis.*

L. Mazet, M. Rodríguez and M. Traizet [2007] have constructed saddle towers with infinitely many ends: They are nonperiodic properly embedded minimal surfaces in $\mathbb{R}^3/T$ with infinitely many ends and one limit end. In this paper, we construct (nonperiodic) properly embedded minimal surfaces in $\mathbb{R}^3/T$ with arbitrary finite genus $k \geqslant 0$, infinitely many ends and one limit end. With this aim, we glue half of a Scherk example with half of a KMR example in the case $k = 0$; when $k \geqslant 1$, we glue a compact piece of M_k to half of a Scherk surface (at the top catenoidal end of M_k), a periodic horizontal flat annulus with a disk removed (at

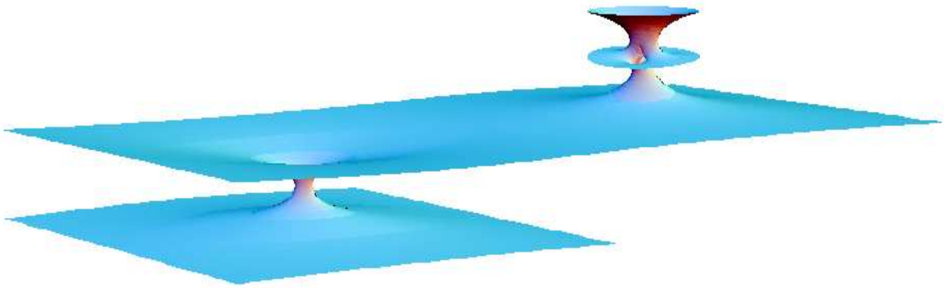

Figure 1. A sketch of half of a KMR example $M_{\sigma,0,0}$ glued to a compact piece of Costa surface.

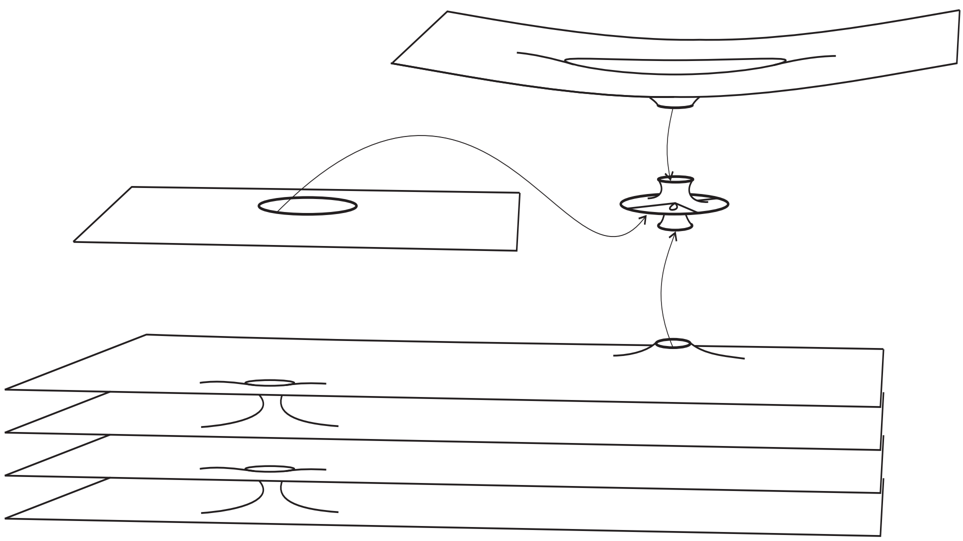

Figure 2. A sketch of a surface in the family of Theorem 1.3.

the middle planar end) and half of a KMR example (at the bottom catenoidal end); see Figure 2.

Theorem 1.3. *Let T denote a translation in the x_2 direction. For each $k \geqslant 0$, there exists a 1-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^3/T$ has genus k, infinitely many horizontal Scherk-type ends and one limit end.*

The family of KMR examples is a three parameter family that contains two subfamilies whose surfaces have a plane of symmetry. In the construction of examples satisfying Theorems 1.2 and 1.3, we need to have at least one plane of symmetry in order to control the kernel of the Jacobi operator on each glued piece.

F. Morabito [2008a] has recently proved there is a bounded Jacobi field that does not come from isometries of $\mathbb{R}^3$ on M_k with tilted ends. For this reason, we are not able to produce a 3-parameter family of KMR examples with higher genus in Theorem 1.2.

The paper is organized as follows. In Section 2 we briefly describe the Costa–Hoffman–Meeks examples M_k and obtain, for each genus k, a 1-parameter family of surfaces $M_k(\xi)$ by bending the catenoidal ends of $M_k = M_k(0)$ while keeping a plane of symmetry. This is used to prescribe the flux of the deformed surface M_k, which has to be the same as the corresponding KMR example we want to glue (to prove Theorem 1.2). To simplify the construction of examples satisfying Theorems 1.1 and 1.3, we consider a "not bent" example M_k. In Section 3 we perturb $M_k(\xi)$ using the implicit function theorem. We get an infinite dimensional family of minimal surfaces that have three boundaries.

In Section 4, we apply an implicit function theorem to solve the Dirichlet problem for the minimal graph equation on a horizontal flat periodic annulus with a disk B removed, prescribing the boundary data on ∂B and the asymptotic direction of the Scherk-type ends. We construct the flat annulus with a disk removed that we will glue to the example M_k at its middle planar end. Varying the asymptotic direction of the ends and the flux of the surface, we obtain the pieces of Scherk surface that we will glue at the top and bottom catenoidal ends of M_k (proving Theorem 1.1) and to half of a KMR example (to prove Theorem 1.3).

In Section 5, we study the KMR examples $M_{\sigma,\alpha,\beta}$ and describe a conformal parameterization of these examples on a cylinder. We also obtain an expansion of pieces of the KMR examples as the flux of $M_{\sigma,\alpha,\beta}$ becomes horizontal (that is, near the catenoidal limit). Section 6 is devoted to the study of the mapping properties of the Jacobi operator about such $M_{\sigma,\alpha,\beta}$ near the catenoidal limit. And we apply in Section 7 the implicit function theorem to perturb half of a KMR example $M_{\sigma,\alpha,0}$ (respectively $M_{\sigma,0,\beta}$), obtaining a family of minimal surfaces asymptotic to half of a $M_{\sigma,\alpha,0}$ (respectively $M_{\sigma,0,\beta}$) and whose boundary is a Jordan curve. We prescribe the boundary data of such a surface. Sections 5, 6, 7 are of independent interest: They are devoted to the global analysis on KMR examples.

Finally, we do the end-to-end construction in Section 8: We explain how the boundary data of the corresponding minimal surfaces constructed in Sections 3, 4 and 7 can be chosen so that their union forms smooth minimal surfaces satisfying Theorems 1.1, 1.2 and 1.3.

2. A Costa–Hoffman–Meeks type surface with bent catenoidal ends

In this section we recall the result shown in [Hauswirth and Pacard 2007] about the existence of a family of minimal surfaces $M_k(\xi)$ close to the Costa–Hoffman–Meeks surface $M_k(0) = M_k$ of genus $k \geqslant 1$, with one planar end and two catenoidal ends slightly bent by an angle ξ.

2.1. ***Costa–Hoffman–Meeks surfaces.*** We briefly present here the family of the surfaces M_k studied in [Costa 1982; 1984; Hoffman and Meeks 1985; 1989; 1990]. For each natural $k \geqslant 1$, M_k is a properly embedded minimal surface of genus k and three ends. After suitable rotations and translations, we can assume its ends are horizontal (in particular, they can be ordered by heights). The surface M_k enjoys the following properties:

(1) M_k has one middle planar end E_m asymptotic to the $\{x_3 = 0\}$ plane, and two catenoidal ends: one top E_t and one bottom E_b asymptotic, respectively, to the upper and lower end of a catenoid having as axis of revolution the x_3 axis.

(2) M_k intersects the $\{x_3 = 0\}$ plane in $k+1$ straight lines, which intersect at equal angles $\pi/(k+1)$ at the origin. The intersection of M_k with any one of the remaining horizontal planes is a single Jordan curve. Thus the intersection of M_k with the upper half-space $\{x_3 > 0\}$ (respectively the lower half-space $\{x_3 < 0\}$) is topologically an open annulus.

(3) The isometry group of M_k is generated by rotations by π about the $k+1$ lines contained in the surface at height zero, together with reflections in planes that bisect those lines. Assume one such plane of symmetry is the $\{x_2 = 0\}$ plane. In particular, M_k is invariant by the rotation by $2\pi/(k+1)$ about the x_3 axis and by the composition of a rotation by $\pi/(k+1)$ about the x_3 axis with a reflection across the $\{x_3 = 0\}$ plane.

Now we give describe locally the surfaces M_k near its ends, and we introduce coordinates that we will use.

The planar end. See [Hauswirth and Pacard 2007]. The planar end E_m of M_k can be parameterized by

$$X_m(x) = \Big(\frac{x}{|x|^2}, u_m(x)\Big) \in \mathbb{R}^3 \quad \text{for } x \in \bar{B}^*_{\rho_0}(0),$$

where $\bar{B}^*_{\rho_0}(0)$ is the punctured closed disk in $\mathbb{R}^2$ of radius $\rho_0 > 0$ small centered at the origin, and $u_m = \mathbb{O}_{C^{2,\alpha}_b}(|x|^{k+1})$ is a solution of

$$|x|^4 \operatorname{div}\Big(\frac{\nabla u}{(1+|x|^4|\nabla u|^2)^{1/2}}\Big) = 0. \tag{1}$$

Moreover, u_m can be extended continuously to the puncture, using Weierstrass representation (in fact, it can be extended as a $C^{2,\alpha}$ function). Here $\mathcal{O}_{C_b^{n,\alpha}}(g)$ denotes a function that, together with its partial derivatives of order no greater than $n+\alpha$, is bounded by a constant times g. In the sequel, where necessary, we will consider on $B_{\rho_0}(0)$ also the polar coordinates $(\rho,\theta)\in[0,\rho_0]\times\mathbb{S}^1$.

If we linearize in $u=0$ the nonlinear Equation (1), we obtain the expression of an operator that is the Jacobi operator about the plane; that is, $\mathscr{L}_{\mathbb{R}^2}=|x|^4\Delta_0$. To be more precise, the linearization of (1) gives

$$L_u v=|x|^4\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|x|^4|\nabla u|^2}}-|x|^4\nabla u\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}}\right). \tag{2}$$

Equation (1) means that the surface Σ_u parameterized by $x\mapsto(x/|x|^2,u(x))$ is minimal. We will express the mean curvature H_{u+v} of Σ_{u+v} in terms of the mean curvature H_u of Σ_u.

Lemma 2.1. *There exists a function Q_u satisfying $Q_u(0,0)=0$ and $\nabla Q_u(0,0)=0$ such that*

$$2H_{u+v}=2H_u+L_uv+|x|^4Q_u(|x|^2\nabla v,|x|^2\nabla^2 v).$$

Proof. Define $f(t)=1/\sqrt{1+|x|^4|\nabla(u+tv)|^2}$ and apply Taylor expansion. □

Since u satisfies (1), $H_u=0$. Then, if we put

$$\mathcal{Q}_u(\,\cdot\,):=\sqrt{1+|x|^4|\nabla u|^2}\,Q_u(|x|^2\nabla\,\cdot\,,|x|^2\nabla^2\,\cdot\,)$$

to simplify the notation, the minimal surface equation satisfied by the function v defined on the graph of the function u is

$$|x|^4\big(\Delta_0 v+\sqrt{1+|x|^4|\nabla u|^2}\,\bar{L}_u v+\mathcal{Q}_u(v)\big)=0, \tag{3}$$

where $\bar{L}_u$ is a second order linear operator whose coefficients are in $\mathcal{O}_{C^{2,\alpha}}(|x|^{k+1})$.

The catenoidal ends. We will denote by X_c the parameterization of the standard catenoid C whose axis of revolution is the x_3 axis. Its expression is

$$X_c(s,\theta):=(\cosh s\,\cos\theta,\cosh s\,\sin\theta,s)\in\mathbb{R}^3,$$

where $(s,\theta)\in\mathbb{R}\times\mathbb{S}^1$. The unit normal vector field about C is

$$n_c(s,\theta):=\frac{1}{\cosh s}(\cos\theta,\sin\theta,-\sinh s)\quad\text{for } (s,\theta)\in\mathbb{R}\times\mathbb{S}^1.$$

Up to a dilation, we can assume that the two ends E_t and E_b of M_k are asymptotic to some translated copy in the vertical direction of the catenoid parameterized by X_c.

Therefore, E_t and E_b can be parameterized, respectively, by

$$X_t := X_c + w_t n_c + \sigma_t e_3 \quad \text{in } (s_0, \infty) \times \mathbb{S}^1,$$
$$X_b := X_c - w_b n_c - \sigma_b e_3 \quad \text{in } (-\infty, -s_0) \times \mathbb{S}^1,$$

where $\sigma_t, \sigma_b \in \mathbb{R}$, and w_t (respectively w_b) is a function defined in $(s_0, \infty) \times \mathbb{S}^1$ (respectively $(-\infty, -s_0) \times \mathbb{S}^1$) that tends exponentially fast to 0 as s goes to $+\infty$ (respectively $-\infty$), reflecting that the ends are asymptotic to a catenoidal end.

We recall that the surface parameterized by $X := X_c + w\, n_c$ is minimal if and only if the function w satisfies the minimal surface equation, which for normal graphs over a catenoid has the form

$$\mathbb{L}_C w + \frac{1}{\cosh^2 s}\left(Q_2\left(\frac{w}{\cosh s}\right) + \cosh s\, Q_3\left(\frac{w}{\cosh s}\right)\right) = 0, \tag{4}$$

where $\mathbb{L}_C$ is the Jacobi operator about the catenoid, that is,

$$\mathbb{L}_C w = \frac{1}{\cosh^2 s}\left(\partial^2_{ss} w + \partial^2_{\theta\theta} w + \frac{2w}{\cosh^2 s}\right),$$

and Q_2 and Q_3 are nonlinear second order differential operators that are bounded in $\mathscr{C}^k(\mathbb{R} \times \mathbb{S}^1)$ for every k and satisfy $Q_2(0) = Q_3(0) = 0$, $\nabla Q_2(0) = \nabla Q_3(0) = 0$, and $\nabla^2 Q_3(0) = 0$ together with

$$\begin{aligned} &\|Q_j(v_2) - Q_j(v_1)\|_{\mathscr{C}^{0,\alpha}([s,s+1]\times\mathbb{S}^1)} \\ &\qquad \leqslant c\Big(\sup_{i=1,2} \|v_i\|_{\mathscr{C}^{2,\alpha}([s,s+1]\times\mathbb{S}^1)}\Big)^{j-1} \|v_2 - v_1\|_{\mathscr{C}^{2,\alpha}([s,s+1]\times\mathbb{S}^1)} \end{aligned} \tag{5}$$

for all $s \in \mathbb{R}$ and all v_1, v_2 such that $\|v_i\|_{\mathscr{C}^{2,\alpha}([s,s+1]\times\mathbb{S}^1)} \leqslant 1$. The constant $c > 0$ does not depend on s.

The family of Costa–Hoffman–Meeks surfaces with bent catenoidal ends. We denote by R_ξ the rotation by ξ about the x_2 axis oriented by e_2. The following result may be proved using an elaborate version of the implicit function theorem and by following [Jleli 2004] and [Kusner et al. 1996].

Theorem 2.2 [Hauswirth and Pacard 2007]. *There exists $\xi_0 > 0$ and a smooth 1-parameter family of minimal surfaces $\{M_k(\xi) \mid \xi \in (-\xi_0, \xi_0)\}$ with the properties that $M_k(0) = M_k$ and each $M_k(\xi)$ is invariant by reflection across the $\{x_2 = 0\}$ plane, has one horizontal planar end E_m and has two catenoidal ends $E_t(\xi)$ and $E_b(\xi)$ asymptotic respectively, up to a translation, to the upper and lower end of the catenoid $R_\xi C$ (that is, the standard catenoid whose axis of revolution is directed by $R_\xi e_3$). Moreover, $E_t(\xi)$ and $E_b(\xi)$ can be parameterized respectively* *by*

$$X_{t,\xi} = R_\xi(X_c + w_{t,\xi} n_c) + \sigma_{t,\xi} e_3 + \varsigma_{t,\xi} e_1, \tag{6}$$

$$X_{b,\xi} = R_\xi(X_c - w_{b,\xi} n_c) - \sigma_{b,\xi} e_3 - \varsigma_{b,\xi} e_1, \tag{7}$$

where the functions $w_{t,\xi}, w_{b,\xi}$ *and the numbers* $\sigma_{t,\xi}, \varsigma_{t,\xi}, \sigma_{b,\xi}, \varsigma_{b,\xi} \in \mathbb{R}$ *depend smoothly on* ξ *and satisfy*

$$|\sigma_{t,\xi} - \sigma_t| + |\sigma_{b,\xi} - \sigma_b| + |\varsigma_{t,\xi}| + |\varsigma_{b,\xi}| + \|w_{t,\xi} - w_t\|_{C^{2,\alpha}_{-2}([s_0,+\infty)\times\mathbb{S}^1)} + \|w_{b,\xi} - w_b\|_{C^{2,\alpha}_{-2}((-\infty,-s_0]\times\mathbb{S}^1)} \leqslant c|\xi|,$$

where

$$\|w\|_{\mathscr{C}^{\ell,\alpha}_\delta([s_0,+\infty)\times\mathbb{S}^1)} = \sup_{s\geqslant s_0}\left(e^{-\delta s}\,\|w\|_{\mathscr{C}^{\ell,\alpha}([s,s+1]\times\mathbb{S}^1)}\right),$$

$$\|w\|_{\mathscr{C}^{\ell,\alpha}_\delta((-\infty,-s_0])\times\mathbb{S}^1)} = \sup_{s\leqslant -s_0}\left(e^{\delta s}\,\|w\|_{\mathscr{C}^{\ell,\alpha}([s-1,s]\times\mathbb{S}^1)}\right).$$

For all $s > s_0$ and $\rho < \rho_0$, we define

$$M_k(\xi, s, \rho) := M_k(\xi) - \left(X_{t,\xi}([s,+\infty)\times\mathbb{S}^1) \cup X_m(B_\rho(0)) \cup X_{b,\xi}((-\infty,-s]\times\mathbb{S}^1)\right). \tag{8}$$

The parameterizations of the three ends of $M_k(\xi)$ induce a decomposition of $M_k(\xi)$ into slightly overlapping components: a compact piece $M_k(\xi, s_0+1, \rho_0/2)$ and three noncompact pieces

$$X_{t,\xi}((s_0,+\infty)\times\mathbb{S}^1), \quad X_{b,\xi}((-\infty,-s_0)\times\mathbb{S}^1), \quad X_m(\bar{B}_{\rho_0}(0)).$$

We define the weighted space of functions on $M_k(\xi)$.

Definition 2.3. Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, we define $\mathscr{C}^{\ell,\alpha}_\delta(M_k(\xi))$ as the space of functions in $\mathscr{C}^{\ell,\alpha}_{\mathrm{loc}}(M_k(\xi))$ invariant by reflections across the $\{x_2 = 0\}$ plane (that is, $w(p) = w(\bar{p})$ for all $p = (p_1, p_2, p_3) \in M_k(\xi)$, where $\bar{p} := (p_1, -p_2, p_3)$) and for which the following norm is finite:

$$\|w\|_{\mathscr{C}^{\ell,\alpha}_\delta(M_k(\xi))} := \|w\|_{\mathscr{C}^{\ell,\alpha}(M_k(\xi,s_0+1,\rho_0/2))} + \|w\circ X_m\|_{\mathscr{C}^{\ell,\alpha}(B_{\rho_0}(0))} + \|w\circ X_{t,\xi}\|_{\mathscr{C}^{\ell,\alpha}_\delta([s_0,+\infty)\times\mathbb{S}^1)} + \|w\circ X_{b,\xi}\|_{\mathscr{C}^{\ell,\alpha}_\delta((-\infty,-s_0]\times\mathbb{S}^1)}.$$

We remark that there is no weight on the planar end E_m of $M_k(\xi)$. In fact, we can compactify this end and consider a weighted space of functions defined on a two-ended surface. In the next section we will consider normal perturbations of $M_k(\xi)$ by functions $u \in \mathscr{C}^{2,\alpha}_\delta(M_k(\xi))$, and the planar end E_m will be just vertically translated.

The Jacobi operator. The Jacobi operator about $M_k(\xi)$ is

$$\mathbb{L}_{M_k(\xi)} := \Delta_{M_k(\xi)} + |A_{M_k(\xi)}|^2,$$

where $|A_{M_k(\xi)}|$ is the norm of the second fundamental form on $M_k(\xi)$.

In the parameterization of the ends of $M_k(\xi)$ introduced above, the volume form $d\mathrm{vol}_{M_k(\xi)}$ can be written as $\gamma_t\, ds d\theta$ (respectively $\gamma_b\, ds d\theta$, $\gamma_m dx_1 dx_2$) near $E_t(\xi)$ (respectively $E_b(\xi)$, E_m). We define globally on $M_k(\xi)$ a smooth function

$$\gamma : M_k(\xi) \to [0, +\infty)$$

that equals 1 on $M_k(\xi, s_0 - 1, 2\rho_0)$ and equals γ_t (respectively γ_b, γ_m) on the end $E_t(\xi)$ (respectively $E_b(\xi)$, E_m). Observe that

$$\begin{aligned} (\gamma \circ X_{t,\xi})(s,\theta) &\sim \cosh^2 s \quad \text{on } (s_0, +\infty) \times \mathbb{S}^1, \\ (\gamma \circ X_{b,\xi})(s,\theta) &\sim \cosh^2 s \quad \text{on } (-\infty, -s_0) \times \mathbb{S}^1, \\ (\gamma \circ X_m)(x) &\sim |x|^{-4} \qquad \text{on } B_{\rho_0}. \end{aligned}$$

Given the defined spaces above, one can check that

$$\mathscr{L}_{\xi,\delta} : \mathscr{C}^{2,\alpha}_\delta(M_k(\xi)) \to \mathscr{C}^{0,\alpha}_\delta(M_k(\xi)), \quad w \mapsto \gamma\, \mathbb{L}_{M_k(\xi)}(w)$$

is a bounded linear operator. The subscript δ is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that the function γ is here to counterbalance the effect of the conformal factor $1/\sqrt{|g_{M_k(\xi)}|}$ in the expression of the Laplacian in the coordinates we use to parameterize the ends of the surface $M_k(\xi)$. This is precisely what is needed to have the operator defined from the space $\mathscr{C}^{2,\alpha}_\delta(M_k(\xi))$ into the target space $\mathscr{C}^{0,\alpha}_\delta(M_k(\xi))$.

To better grasp what is going on, let us linearize the nonlinear Equation (4) at $w = 0$. We get the expression of the Jacobi operator about the standard catenoid

$$\mathbb{L}_C := \frac{1}{\cosh^2 s}\Big(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\Big).$$

The operator $\cosh^2 s\mathbb{L}_C$ maps the space $(\cosh s)^\delta \mathscr{C}^{2,\alpha}((s_0, +\infty) \times \mathbb{S}^1)$ into the space $(\cosh s)^\delta \mathscr{C}^{0,\alpha}((s_0, +\infty) \times \mathbb{S}^1)$.

Similarly, if we linearize the nonlinear Equation (1) at $u = 0$, we obtain (see (2) with $u = 0$) the expression of the Jacobi operator about the plane $\mathbb{L}_{\mathbb{R}^2} := |x|^4 \Delta_0$. Again, the operator $|x|^{-4}\mathbb{L}_{\mathbb{R}^2} = \Delta_0$ clearly maps the space $\mathscr{C}^{2,\alpha}(\bar{B}_{\rho_0})$ into the space $\mathscr{C}^{0,\alpha}(\bar{B}_{\rho_0})$. Now, the function γ plays for the ends of the surface $M_k(\xi)$ the role the function $\cosh^2 s$ plays for the ends of the standard catenoid and the role the function $|x|^{-4}$ plays for the plane. Since the Jacobi operator about $M_k(\xi)$ is asymptotic to $\mathbb{L}_{\mathbb{R}^2}$ at E_m and is asymptotic to $\mathbb{L}_C$ at $E_t(\xi)$ and $E_b(\xi)$, we conclude that the operator $\mathscr{L}_{\xi,\delta}$ maps $\mathscr{C}^{2,\alpha}_\delta(M_k(\xi))$ into $\mathscr{C}^{0,\alpha}_\delta(M_k(\xi))$.

Definition 2.4 [Hauswirth and Pacard 2007]. A surface $M_k(\xi)$ is said to be nondegenerate if $\mathcal{L}_{\xi,\delta}$ is injective for all $\delta < -1$.

It is useful to observe that a duality argument in the weighted Lebesgue spaces implies $\mathcal{L}_{\xi,\delta}$ is injective if and only if $\mathcal{L}_{\xi,-\delta}$ is surjective, provided $\delta \notin \mathbb{Z}$. For details, see [Jleli 2004; Melrose 1993].

The nondegeneracy of $M_k(\xi)$ follows from the study of the kernel of $\mathcal{L}_{\xi,\delta}$.

The Jacobi fields. It is known that a smooth 1-parameter group of isometries containing the identity generates a Jacobi field, that is, a solution of $\mathbb{L}_{M_k(\xi)} u = 0$. The solutions that are invariant under reflection across the $\{x_2 = 0\}$ plane are generated by dilations, vertical translations and horizontal translations along the x_1 axis (see [Hauswirth and Pacard 2007]):

- The vertical translations generated by the Killing vector field $\Xi(p) = e_3$ give rise to the Jacobi field $\Phi^{0,+}(p) := n(p) \cdot e_3$.
- The vector field $\Xi(p) = p$ associated to the 1-parameter group of dilations generates the Jacobi field $\Phi^{0,-}(p) := n(p) \cdot p$.
- The Killing vector field $\Xi(p) = e_1$ that generates the group of translations along the x_1 axis is associated to a Jacobi field $\Phi^{1,+}(p) := n(p) \cdot e_1$.
- Finally, we denote by $\Phi^{1,-}(p) := n(p) \cdot (e_2 \times p)$ the Jacobi field associated to the Killing vector field $\Xi(p) = e_2 \times p$ that generates the group of rotations about the x_2 axis.

There are other Jacobi fields we do not take into account because they are not invariant by reflection across the $\{x_2 = 0\}$ plane.

With these notations, we define the deficiency space

$$\mathcal{D} := \operatorname{Span}\{\chi_t \Phi^{j,\pm}, \chi_b \Phi^{j,\pm} : j = 0, 1\}$$

where χ_t is a cutoff function that equals 1 on $X_{t,\xi}((s_0+1, +\infty) \times \mathbb{S}^1)$, equals 0 on $M_k(\xi) - X_{t,\xi}((s_0, +\infty) \times \mathbb{S}^1)$, is invariant under reflection across the $\{x_2 = 0\}$ plane, and satisfies $\chi_b(\cdot) := \chi_t(-\,\cdot)$. Clearly

$$\widetilde{\mathcal{L}}_{\xi,\delta} : \mathcal{C}^{2,\alpha}_{\delta}(M_k(\xi)) \oplus \mathcal{D} \to \mathcal{C}^{0,\alpha}_{\delta}(M_k(\xi)), \quad w \mapsto \gamma\, \mathbb{L}_{M_k(\xi)}(w)$$

is a bounded linear operator for $\delta < 0$.

A result of S. Nayatani [1992; 1993], which the second author extended in [Morabito 2008b], states that any bounded Jacobi field invariant by reflection across the $\{x_2 = 0\}$ plane is a linear combination of $\Phi^{0,+}$ and $\Phi^{1,+}$.

From that we get the following result about the operator $\mathcal{L}_{\xi,\delta}$.

Proposition 2.5. *We fix $\delta \in (1, 2)$. Then (reducing ξ_0 if this is necessary) the operator $\mathcal{L}_{\xi,\delta}$ is surjective and has a kernel of dimension* 4. *Moreover, there exists*

$G_{\xi,\delta}$, a right inverse for $\mathscr{L}_{\xi,\delta}$ that depends smoothly on ξ and in particular whose norm is bounded uniformly as $|\xi| < \xi_0$.

This fact together with an adaptation to our setting of the linear decomposition lemma proved in [Kusner et al. 1996] for constant mean curvature surfaces (see also [Jleli 2004] for minimal hypersurfaces), allows us to prove the following result.

Proposition 2.6. *We fix $\delta \in (-2, -1)$. Then (reducing ξ_0 if this is necessary) the operator $\widetilde{\mathscr{L}}_{\xi,\delta}$ for $|\xi| < \xi_0$ is surjective and has a kernel of dimension* 4.

3. Infinite dimensional family of minimal surfaces close to $M_k(\xi)$

In this section we consider a truncature of $M_k(\xi)$. First we write portions of the ends of $M_k(\xi)$ as vertical graphs over the $\{x_3 = 0\}$ plane.

We set $r_\varepsilon = 1/(2\sqrt{\varepsilon})$.

Lemma 3.1 [Hauswirth and Pacard 2007]. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $|\xi| \leqslant \varepsilon$, an annular part of the ends $E_t(\xi)$, $E_b(\xi)$ and E_m of $M_k(\xi)$ can be written, respectively, as vertical graphs over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ for the functions*

$$U_t(r,\theta) = \sigma_{t,\xi} + \ln(2r) - \xi r\cos\theta + \mathbb{O}_{\mathscr{C}_b^\infty}(\varepsilon),$$
$$U_b(r,\theta) = -\sigma_{b,\xi} - \ln(2r) - \xi r\cos\theta + \mathbb{O}_{\mathscr{C}_b^\infty}(\varepsilon),$$
$$U_m(r,\theta) = \mathbb{O}_{\mathscr{C}_b^\infty}(r^{-(k+1)}).$$

Here (r,θ) are the polar coordinates in the $\{x_3 = 0\}$ plane. The functions $\mathbb{O}_{\mathscr{C}_b^\infty}(\varepsilon)$ are defined in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ and are bounded in the $\mathscr{C}_b^\infty$ topology by a constant (independent on ε) multiplied by ε, where the partial derivatives are computed with respect to the vector fields $r\partial_r$ and ∂_θ.

In particular, a portion of the two catenoidal ends $E_t(\varepsilon/2)$ and $E_b(\varepsilon/2)$ of $M_k(\varepsilon/2)$ are graphs over the annulus $B_{2\,r_\varepsilon} - B_{r_\varepsilon/2} \subset \{x_3 = 0\}$ for functions U_t and U_b. We set $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$, $\rho_\varepsilon = 2\varepsilon^{1/2}$ and

$$M_k^T(\varepsilon/2) = M_k(\varepsilon/2) - \big(X_{t,\varepsilon/2}((s_\varepsilon, +\infty)\times\mathbb{S}^1) \cup X_{b,\varepsilon/2}((-\infty, -s_\varepsilon)\times\mathbb{S}^1) \cup X_m(B_{\rho_\varepsilon}(0))\big).$$

We prove, following [Hauswirth and Pacard 2007, Section 6], the existence of a family of surfaces close to $M_k^T(\xi)$. In a first step, we modify the parameterization of the ends $E_t(\varepsilon/2)$, $E_b(\varepsilon/2)$, E_m, for appropriates values of s, so that, when $r \in [3r_\varepsilon/4, 3r_\varepsilon/2]$, the curves given by

$$\theta \to (r\ \cos\theta, r\ \sin\theta, U_t(r,\theta)),$$
$$\theta \to (r\ \cos\theta, r\ \sin\theta, U_b(r,\theta)),$$
$$\theta \to (r\ \cos\theta, r\ \sin\theta, U_m(r,\theta))$$

correspond respectively to the curves $\{s=\ln(2r)\}$, $\{s=-\ln(2r)\}$, $\{\rho=1/r\}$.

The second step is the modification of the unit normal vector field on $M_k(\varepsilon/2)$ to produce a transverse unit vector field $\tilde{n}_{\varepsilon/2}$ that coincides with the normal vector field $n_{\varepsilon/2}$ on $M_k(\varepsilon/2)$, is equal to e_3 on the graph over $B_{3r_\varepsilon/2}-B_{3r_\varepsilon/4}$ of the functions U_t and U_b, and interpolates smoothly between the different definitions of $\tilde{n}_{\varepsilon/2}$ in different subsets of $M_k^T(\varepsilon/2)$.

Finally we observe that close to $E_t(\varepsilon/2)$, we can give the estimate

$$\left|\cosh^2 s\left(\mathbb{L}_{M_k(\varepsilon/2)}v-\cosh^{-2}s(\partial^2_{ss}v+\partial^2_{\theta\theta}v)\right)\right|\leqslant c|\cosh^{-2}s\,v|. \tag{9}$$

This follows easily from (4) together with the fact that $w_{t,\xi}$ (see (6)) decays at least like $\cosh^{-2}s$ on $E_t(\varepsilon/2)$. Similar considerations hold close to the bottom end $E_b(\varepsilon/2)$. Near the middle planar end E_m, we have the estimate

$$\left||x|^{-4}\left(\mathbb{L}_{M_k(\varepsilon/2)}v-|x|^4\Delta_0 v\right)\right|\leqslant c\left||x|^{2k+3}\nabla v\right|. \tag{10}$$

This follows easily from (2) and the fact that u_m decays at least like $|x|^{k+1}$ on E_m.

The graph of a function u, using the vector field $\tilde{n}_{\varepsilon/2}$, will be a minimal surface if and only if u is a solution of a second order nonlinear elliptic equation of the form

$$\mathbb{L}_{M_k^T(\varepsilon/2)}u=\tilde{L}_{\varepsilon/2}u+Q_\varepsilon(u),$$

where $\mathbb{L}_{M_k^T(\varepsilon/2)}$ is the Jacobi operator about $M_k^T(\varepsilon/2)$, Q_ε is a nonlinear second order differential operator, and $\tilde{L}_{\varepsilon/2}$ is a linear operator that takes into account the change of the normal vector field (only for the top and bottom ends) and the change of the parameterization. This operator is supported in neighborhoods of $\{\pm s_\varepsilon\}\times\mathbb{S}^1$, where its coefficients are uniformly bounded by a constant times ε^2, and a neighborhood of $\{\rho_\varepsilon\}\times\mathbb{S}^1$, where its coefficients are uniformly bounded by a constant times ε^3.

Now, we consider three even functions $\varphi_t,\varphi_b,\varphi_m\in\mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ such that φ_t and φ_b are L^2-orthogonal to 1 and $\theta\mapsto\cos\theta$, while φ_m is L^2-orthogonal to 1. Assume that they satisfy

$$\|\varphi_t\|_{\mathscr{C}^{2,\alpha}}+\|\varphi_b\|_{\mathscr{C}^{2,\alpha}}+\|\varphi_m\|_{\mathscr{C}^{2,\alpha}}\leqslant\kappa\varepsilon. \tag{11}$$

We set $\Phi:=(\varphi_t,\varphi_b,\varphi_m)$ and we define w_Φ to be the function equal to

(1) $\chi_+(s)H_{\varphi_t}(s_\varepsilon-s,\cdot)$ on the image of $X_{t,\varepsilon/2}$, where χ_+ is a cutoff function that equals 0 for $s\leqslant s_0+1$ and equals 1 for $s\in[s_0+2,s_\varepsilon]$;

(2) $\chi_-(s)H_{\varphi_b}(s+s_\varepsilon,\cdot)$ on the image of $X_{b,\varepsilon/2}$, where χ_- is a cutoff function that equals 0 for $s\geqslant -s_0-1$ and equals 1 for $s\in[-s_\varepsilon,-s_0-2]$;

(3) $\chi_m(\rho)\tilde{H}_{\rho_\varepsilon,\varphi_m}(\cdot,\cdot)$ on the image of X_m, where χ_m is a cutoff function that equals 0 for $\rho\geqslant\rho_0$ and equals 1 for $\rho\in[\rho_\varepsilon,\rho_0/2]$;

(4) 0 on the remaining part of the surface $M_k^T(\varepsilon/2)$,

where $\tilde{H}$ and H are, respectively, harmonic extensions of the operators introduced in Propositions A.2 and A.4.

We would like to prove that, under appropriate hypotheses, the graph over $M_k^T(\varepsilon/2)$ of the function $u = w_\Phi + v$ is a minimal surface. This is equivalent to solving the equation

$$\mathbb{L}_{M_k^T(\varepsilon/2)}(w_\Phi + v) = \tilde{L}_{\varepsilon/2}(w_\Phi + v) + Q_\varepsilon(w_\Phi + v).$$

The solution of this equation is obtained thanks to the fixed point problem

$$v = T(\Phi, v) := G_{\varepsilon/2,\delta} \circ \mathscr{E}_\varepsilon\big(\gamma\left(\tilde{L}_{\varepsilon/2}(w_\Phi + v) - \mathbb{L}_{M_k^T(\varepsilon/2)}\, w_\Phi + Q_\varepsilon(w_\Phi + v)\right)\big), \tag{12}$$

where $\delta \in (1, 2)$, the operator $G_{\varepsilon/2,\delta}$ is the right inverse provided in Proposition 2.5, and $\mathscr{E}_\varepsilon$ is a linear extension operator

$$\mathscr{E}_\varepsilon : \mathscr{C}^{0,\alpha}_\delta(M_k^T(\varepsilon/2)) \to \mathscr{C}^{0,\alpha}_\delta(M_k(\varepsilon/2)).$$

Here $\mathscr{C}^{0,\alpha}_\delta(M_k^T(\varepsilon/2))$ denotes the space of functions of $\mathscr{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))$ restricted to $M_k^T(\varepsilon/2)$, and $\mathscr{E}_\varepsilon$ is defined so that $\mathscr{E}_\varepsilon v$ equals v in $M_k^T(\varepsilon/2)$, vanishes in the image of $[s_\varepsilon + 1, +\infty) \times \mathbb{S}^1$ by $X_{t,\varepsilon/2}$, in the image of $(-\infty, -s_\varepsilon - 1) \times \mathbb{S}^1$ by $X_{b,\varepsilon/2}$ and in the image of $B_{\rho_\varepsilon/2}$ by X_m, and is an interpolation of these values in the remaining part of $M_k(\varepsilon/2)$:

$$\begin{aligned}
(\mathscr{E}_\varepsilon v) \circ X_{t,\varepsilon/2}(s, \theta) &= (1 + s_\varepsilon - s)(v \circ X_{t,\varepsilon/2}(s_\varepsilon, \theta)) \\
&\qquad\qquad \text{for } (s, \theta) \in [s_\varepsilon, s_\varepsilon + 1] \times \mathbb{S}^1, \\
(\mathscr{E}_\varepsilon v) \circ X_{b,\varepsilon/2}(s, \theta) &= (1 + s_\varepsilon + s)(v \circ X_{b,\varepsilon/2}(s_\varepsilon, \theta)) \\
&\qquad\qquad \text{for } (s, \theta) \in [-s_\varepsilon - 1, -s_\varepsilon] \times \mathbb{S}^1, \\
(\mathscr{E}_\varepsilon v) \circ X_m(\rho, \theta) &= \Big(\frac{2\rho}{\rho_\varepsilon} - 1\Big)(v \circ X_m(\rho_\varepsilon, \theta)) \quad \text{for } (\rho, \theta) \in [\rho_\varepsilon/2, \rho_\varepsilon] \times \mathbb{S}^1.
\end{aligned}$$

Remark 3.2. As consequence of the properties of $\mathscr{E}_\varepsilon$, if $\operatorname{supp} v \cap \big(B_{\rho_\varepsilon} - B_{\rho_\varepsilon/2}\big) \neq \varnothing$ then

$$\|(\mathscr{E}_\varepsilon v) \circ X_m\|_{\mathscr{C}^{0,\alpha}(\bar{B}_{\rho_0})} \leqslant c\rho_\varepsilon^{-\alpha} \|v \circ X_m\|_{\mathscr{C}^{0,\alpha}(B_{\rho_0} - B_{\rho_\varepsilon})}.$$

This explosion of the norm does not occur near the catenoidal type ends:

$$\|(\mathscr{E}_\varepsilon v) \circ X_{t,\varepsilon/2}\|_{\mathscr{C}^{0,\alpha}([s_0,+\infty)\times\mathbb{S}^1)} \leqslant c\|v \circ X_{t,\varepsilon/2}\|_{\mathscr{C}^{0,\alpha}([s_0,s_\varepsilon]\times\mathbb{S}^1)}.$$

A similar inequality holds for the bottom end.

In the sequel we will assume $\alpha > 0$ and close to zero.

The existence of a solution $v \in \mathscr{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))$ for Equation (12) is a consequence of the following result, which proves that $T(\Phi, \cdot)$ is a contracting mapping.

Proposition 3.3. *Choose* $\delta \in (1, 2)$, $\alpha \in (0, 1/2)$, $\Phi = (\varphi_t, \varphi_b, \varphi_m) \in [\mathscr{C}^{2,\alpha}(\mathbb{S}^1)]^3$ *satisfying* (11) *and enjoying the properties described above. There exist constants* $c_\kappa > 0$ *and* $\varepsilon_\kappa > 0$ *such that*

$$\|T(\Phi, 0)\|_{\mathscr{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))} \leqslant c_\kappa \varepsilon^{3/2}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant \tfrac{1}{2}\|v_2 - v_1\|_{\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))},$$

$$\|T(\Phi_2, v) - T(\Phi_1, v)\|_{\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant c\varepsilon\|\Phi_2 - \Phi_1\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}, \tag{13}$$

where

$$\|\Phi_2 - \Phi_1\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} = \|\varphi_{t,2} - \varphi_{t,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} + \|\varphi_{b,2} - \varphi_{b,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} + \|\varphi_{m,2} - \varphi_{m,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)},$$

for all $v, v_1, v_2 \in \mathscr{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))$ *such that* $\|v\|_{\mathscr{C}^{2,\alpha}_\delta} \leqslant 2c_\kappa\varepsilon^{3/2}$ *and for all boundary data* $\Phi_1, \Phi_2 \in [\mathscr{C}^{2,\alpha}(\mathbb{S}^1)]^3$ *enjoying the same properties as* Φ.

Proof. We recall that the Jacobi operator associated to $M_k(\varepsilon/2)$ is asymptotic to the operator of the catenoid near the catenoidal ends, and it is asymptotic to the Laplacian near of the planar end. The function w_Φ is identically zero far from the ends where the explicit expression of $\mathbb{L}_{M_k(\varepsilon/2)}$ is not known: This is the reason of our particular choice in the definition of w_Φ. Then from the definition of w_Φ, thanks to Proposition 2.5 and to (9) and (10), we obtain the estimate

$$\begin{aligned}\|\mathscr{E}_\varepsilon(\gamma \mathbb{L}_{M_k^T(\varepsilon/2)} w_\Phi)\|_{\mathscr{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} &\leqslant c\|\cosh^{-2} s(w_\Phi \circ X_{t,\varepsilon/2})\|_{\mathscr{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times\mathbb{S}^1)} \\ &\quad + c\|\cosh^{-2} s(w_\Phi \circ X_{b,\varepsilon/2})\|_{\mathscr{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times\mathbb{S}^1)} \\ &\quad + c\varepsilon^{-\alpha/2}\|\rho^{2k+3}\nabla(w_\Phi \circ X_m)\|_{\mathscr{C}^{0,\alpha}([\rho_\varepsilon,\rho_0]\times\mathbb{S}^1)} \leqslant c_\kappa\varepsilon^{3/2}.\end{aligned}$$

Using the properties of $\tilde{L}_{\varepsilon/2}$, we obtain

$$\begin{aligned}\|\mathscr{E}_\varepsilon(\gamma \tilde{L}_{\varepsilon/2} w_\Phi)\|_{\mathscr{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} &\leqslant c\varepsilon\|w_\Phi \circ X_{t,\varepsilon/2}\|_{\mathscr{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times\mathbb{S}^1)} \\ &\quad + c\varepsilon\|w_\Phi \circ X_{b,\varepsilon/2}\|_{\mathscr{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times\mathbb{S}^1)} \\ &\quad + c\varepsilon^{1-\alpha/2}\|w_\Phi \circ X_m\|_{\mathscr{C}^{0,\alpha}([\rho_\varepsilon,\rho_0]\times\mathbb{S}^1)} \leqslant c_\kappa\varepsilon^{3/2}.\end{aligned}$$

As for the last term, we recall that the operator Q_ε has two different expressions if we consider the catenoidal type end and the planar end (see (4) and (3)). We leave it to the reader to check that

$$\|\mathscr{E}_\varepsilon(\gamma Q_\varepsilon(w_\Phi))\|_{\mathscr{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant c_\kappa\varepsilon^{3/2}.$$

□

Theorem 3.4. *Let*

$$B := \{w \in \mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2)) \mid \|w\|_{\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant 2c_\kappa \varepsilon^{3/2}\} \quad \text{and} \quad \Phi \in [\mathscr{C}^{2,\alpha}(\mathbb{S}^1)]^3$$

be as above. Then the nonlinear mapping $T(\Phi, \cdot)$ defined above has a unique fixed point v in B.

Proof. The previous proposition shows that, if ε is chosen small enough, the nonlinear mapping $T(\Phi, \cdot)$ is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^{3/2}$ in $\mathscr{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))$ into itself. This value follows from the estimate of the norm of $T(\Phi, 0)$. Consequently by the Schauder fixed point theorem, $T(\Phi, \cdot)$ has a unique fixed point w in this ball. □

This argument provides a minimal surface $M_k^T(\varepsilon/2, \Phi)$ that is close to $M_k^T(\varepsilon/2)$ and has three boundaries. This surface is, close to its upper and lower boundary, a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$ whose parameterizations are respectively given by

$$U_t(r,\theta) = \quad \sigma_{t,\varepsilon/2} + \ln(2r) - \tfrac{1}{2}\varepsilon r\cos\theta + H_{\varphi_t}(s_\varepsilon - \ln(2r), \theta) + V_t(r,\theta), \tag{14}$$

$$U_b(r,\theta) = -\sigma_{b,\varepsilon/2} - \ln(2r) - \tfrac{1}{2}\varepsilon r\cos\theta + H_{\varphi_b}(s_\varepsilon - \ln(2r), \theta) + V_b(r,\theta), \tag{15}$$

where $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$. The boundaries of the surface correspond to $r_\varepsilon = \frac{1}{2}\varepsilon^{-1/2}$. Near the middle boundary the surface is a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$, Its parameterization is

$$U_m(r,\theta) = \tilde{H}_{\rho_\varepsilon,\varphi_m}(1/r, \theta) + V_m(r,\theta), \tag{16}$$

where $\rho_\varepsilon = 2\varepsilon^{1/2}$. All the functions V_i for $i = t, b, m$ depend nonlinearly on ε, φ.

Lemma 3.5. *The functions $V_i(\varepsilon, \varphi_i)$ for $i = t, b$ satisfy*

$$\begin{aligned} &\|V_i(\varepsilon,\varphi_i)(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \leqslant c\varepsilon, \\ &\|V_i(\varepsilon,\varphi_{i,2})(r_\varepsilon\,\cdot\,,\cdot) - V_i(\varepsilon,\varphi_{i,1})(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \\ &\qquad\qquad \leqslant c\varepsilon^{1-\delta/2}\|\varphi_{i,2} - \varphi_{i,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}. \end{aligned} \tag{17}$$

The function $V_m(\varepsilon, \varphi)$ satisfies

$$\begin{aligned} &\|V_m(\varepsilon,\varphi)(\rho_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \leqslant c\varepsilon, \\ &\|V_m(\varepsilon,\varphi_{m,2})(\rho_\varepsilon\,\cdot\,,\cdot) - V_m(\varepsilon,\varphi_{m,1})(\rho_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \\ &\qquad\qquad \leqslant c\varepsilon\|\varphi_{m,2} - \varphi_{m,1}\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}. \end{aligned} \tag{18}$$

Proof. The first estimate follows from

$$\begin{aligned} &\|V_i(\varepsilon,\varphi_2)(\,\cdot\,,\cdot) - V_i(\varepsilon,\varphi_1)(\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_{2r_\varepsilon} - B_{r_\varepsilon/2})} \\ &\qquad\qquad \leqslant ce^{\delta s_\varepsilon}\|(T(\Phi_2, V_i) - T(\Phi_1, V_i)) \circ X_{i,\varepsilon/2}\|_{\mathscr{C}^{2,\alpha}_\delta(\Omega_i\times\mathbb{S}^1)}, \end{aligned}$$

for $i = t, b$, with $\Omega_t = [s_0, s_\varepsilon]$ and $\Omega_b = [-s_\varepsilon, -s_0]$. The second one follows from

$$\|V_m(\varepsilon, \varphi_2)(\,\cdot\,,\cdot\,) - V_m(\varepsilon, \varphi_1)(\,\cdot\,,\cdot\,)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_{2\rho_\varepsilon} - B_{\rho_\varepsilon/2})} \leqslant c\|(T(\Phi_2, V_m) - T(\Phi_1, V_m)) \circ X_m\|_{\mathscr{C}^{2,\alpha}([\rho_\varepsilon, \rho_0]\times\mathbb{S}^1)}$$

and the estimate (13) of Proposition 3.3. □

4. An infinite family of Scherk-type minimal surfaces close to a horizontal periodic flat annulus

This section has two purposes. The first is to find an infinite family of minimal surfaces close to a horizontal periodic flat annulus Σ with a disk D_s removed. The surfaces of this family have two horizontal Scherk-type ends E_1 and E_2 and will be glued on the middle planar end of a Costa–Hoffman–Meeks surface M_k. We will prescribe the boundary data φ on ∂D_s. Assume the period T of Σ points in the x_2 direction. Then the asymptotic direction of E_1 and E_2 is along x_1 axis.

The second and more general purpose of this section is to show the existence of an infinite family of minimal graphs over $\Sigma - D_s$, whose ends have slightly modified asymptotic directions. When the asymptotic directions are not horizontal, these surfaces are close to half of a Scherk surface, seen as a graph over $\Sigma - D_s$ (see Figure 2). A piece of such a surface will be glued to the catenoidal ends of the surface M_k and to an end of a KMR example $M_{\sigma,0,0}$ introduced in Section 5. We will prescribe the boundary data on ∂D_s. Since we need to prescribe the flux along ∂D_s, we will modify the asymptotic direction of the ends, and we will choose $|T|$ large.

4.1. ***Scherk-type ends.*** Conformally parameterize the annulus $\Sigma \subset \mathbb{R}^3/T$ on $\mathbb{C}^*$, with the notation $(x_1, x_2, x_3) = (x_1 + i x_2, x_3)$, by the mapping

$$A(w) = \Big(-\frac{|T|}{2\pi}\ln(w), 0\Big) \quad \text{for } w \in \mathbb{C}^*.$$

The horizontal Scherk-type end E_1 described above can be written as the graph of a function $h_1 \in \mathscr{C}^{2,\alpha}(B_r^*(0))$, where $B_r^*(0)$ is the punctured disk $B_r(0) - \{0\}$ of radius $r \in (0, 1)$ centered at the origin. The function $h_1(w)$ is bounded and extends to the puncture; see [Hauswirth and Traizet 2002]. The end E_1 can be parameterized by

$$X_1(w) = A(w) + h_1(w)e_3 = \Big(-\frac{|T|}{2\pi}\ln(w), h_1(w)\Big) \in \mathbb{R}^3/T \quad \text{for } w \in B_r^*(0)$$

in the orthonormal frame $\mathscr{F} = (e_1, e_2, e_3)$. The end has asymptotic direction e_1.

The horizontal Scherk-type end E_2 can be parameterized in $\mathbb{C} - B_{r^{-1}}(0)$ similarly. Via an inversion, we can parameterize E_2 by

$$X_2(w) = \Big(-\frac{|T|}{2\pi}\ln(w), h_2(w)\Big) \in \mathbb{R}^3/T \quad \text{for } w \in B_r^*(0)$$

in the frame $\mathscr{F}^- = (-e_1, -e_2, e_3)$, where $h_2 \in \mathscr{C}^{2,\alpha}(B_r^*(0))$ is a bounded function that can be extended to the puncture. Now the end has asymptotic direction $-e_1$.

Let us now parameterize a general Scherk-type end, not necessarily horizontal. Let R_θ denote a rotation in $\mathbb{R}^3/T$ by θ about the x_2 axis (oriented by e_2). We can parameterize a not necessarily horizontal Scherk-type end $\widetilde{E}_1$ with asymptotic direction $\cos\theta_1 e_1 + \sin\theta_1 e_3$ and limit normal vector $R_{\theta_1}(e_3)$, with $\theta_1 \in [0, \pi/2)$, by

$$\widetilde{X}_1(z) = \Big(-\frac{|T|}{2\pi}\ln(z), \widetilde{h}_1(z)\Big) \quad \text{for } z \in B_r^*(0)$$

in the frame $\mathscr{F}(\theta_1) = R_{\theta_1}\mathscr{F}$, where $\widetilde{h}_1 \in \mathscr{C}^{2,\alpha}(B_r^*(0))$ is a bounded function that can be extended to the origin.

Finally, a Scherk-type end $\widetilde{E}_2$ with asymptotic direction $-\cos\theta_2 e_1 + \sin\theta_2 e_3$ and limit normal vector $R_{-\theta_2}(e_3)$, with $\theta_2 \in [0, \pi/2)$, can be parameterized by

$$\widetilde{X}_2(z) = \Big(-\frac{|T|}{2\pi}\ln(z), \widetilde{h}_2(z)\Big) \quad \text{for } z \in B_r^*(0)$$

in the frame $\mathscr{F}^-(\theta_2) = R_{-\theta_2}\mathscr{F}^-$, where $\widetilde{h}_2 \in \mathscr{C}^{2,\alpha}(B_r^*(0))$ is a bounded function that can be extended to the origin.

4.2. ***Construction of the infinite families.*** Given an $r \in (0, 1)$ and a $\Theta = (\theta_1, \theta_2)$ in $[0, \theta_0]^2$, with $\theta_0 > 0$ small, we denote by $A_\Theta : \mathbb{C}^* \to \mathbb{R}^3/T$ the immersion obtained as the smooth interpolation of

$$\begin{array}{ll} (R_{\theta_1} \circ A)(z) & \text{if } |z| < r/2, \\ A(z) & \text{if } r < |z| < r^{-1}, \\ (R_{-\theta_2} \circ A)(z) & \text{if } |z| > 2r^{-1}. \end{array}$$

Let N_Θ be the vector field obtained as the smooth interpolation of $R_{\theta_1}(e_3)$ on $\{|z| < r/2\}$, of e_3 on $\{r < |z| < r^{-1}\}$ and of $R_{-\theta_2}(e_3)$ on $\{|z| > 2r^{-1}\}$. For any $h \in C^{2,\alpha}(\overline{\mathbb{C}})$, we define the immersion

$$X_{\Theta,h}(z) = A_\Theta(z) + h(z)N_\Theta(z) \quad \text{for } z \in \mathbb{C}^*.$$

The immersion $X_{\Theta,h}$ has two Scherk-type ends E_1 and E_2 with asymptotic directions $\cos\theta_1 e_1 + \sin\theta_1 e_3$ and $-\cos\theta_2 e_1 + \sin\theta_2 e_3$, respectively.

At the end E_1 (respectively E_2), $X_{\Theta,h}(z) = A(z) + h_1(z)e_3$ in the orthogonal frame $\mathscr{F}(\theta_1)$ (respectively $X_{\Theta,h}(z) = A(z^{-1}) + h_2(z)e_3$ in the frame $\mathscr{F}^-(\theta_2)$), with $z \in B_r^*(0)$, where $h_1(z) = h(z)$ and $h_2(z) = h(z^{-1})$. L. Hauswirth and M. Traizet [2002] proved that, in terms of the z coordinate, the mean curvature of $X_{\Theta,h}$ at E_i is

$$H = \frac{2\pi^2|z|^2}{|T|^2}\,\mathrm{div}_0(P^{-1/2}\nabla_0 h_i),$$

where $P = 1 + (4\pi^2|z|^2/|T|^2)\|\nabla_0 h_i\|_0^2$ and the subscript 0 means that the corresponding object is computed with respect to the flat metric of the z plane. We denote by λ the smooth function without zeros defined by $\lambda(z) = |T|^2/(4\pi^2|z|^2)$ for $z \in B_r^*(0)$. Then at E_i we have

$$2\lambda H = P^{-1/2}\Delta_0 h_i - \tfrac{1}{2}P^{-3/2}\langle\nabla_0 P, \nabla_0 h_i\rangle_0.$$

So the mean curvature at the end E_i vanishes if h_i satisfies the equation

$$\Delta_0 h - \frac{1}{2P}\langle\nabla_0 P, \nabla_0 h\rangle_0 = 0. \tag{19}$$

Definition 4.1. Given $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ we define $C^{k,\alpha}(\overline{\mathbb{C}})$ as the space of functions $u \in C^{k,\alpha}_{\text{loc}}(\overline{\mathbb{C}})$ such that

$$\|u\|_{C^{k,\alpha}(\overline{\mathbb{C}})} := [u]_{k,\alpha,\overline{\mathbb{C}}} < +\infty,$$

where $[u]_{k,\alpha,\overline{\mathbb{C}}}$ denotes the usual $C^{k,\alpha}$ Hölder norm on $\overline{\mathbb{C}}$.

Let B_s be a disk in $\mathbb{C}^*$ such that

$$D_s = A(B_s) \subset \Sigma = \{z \in \mathbb{C} \mid -|T| < 2y \leqslant |T|\}$$

is a geodesic disk centered at the origin of $\mathbb{R}^3/T$. Denote by $C^{k,\alpha}(\overline{\mathbb{C}} - B_s)$ the space of functions in $C^{k,\alpha}(\overline{\mathbb{C}})$ restricted to $\overline{\mathbb{C}} - B_s$. We denote by $H(\Theta, h)$ the mean curvature of $X_{\Theta,h}$, and $\overline{H}(\Theta, h) = \lambda H(\Theta, h)$, where λ is the smooth function defined in a neighborhood of each puncture by $\lambda(z) = |T|^2/(4\pi^2|z|^2)$. [Hauswirth and Traizet 2002, Lemma 4.1] shows that

$$\overline{H} : \mathbb{R}^2 \times \mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_s) \to \mathscr{C}^{0,\alpha}(\overline{\mathbb{C}} - B_s)$$

is an analytical operator. Denote by $\mathscr{L}_\Theta$ the Jacobi operator about A_Θ. We set $\overline{\mathscr{L}}_\Theta = \lambda\mathscr{L}_\Theta$.

Remark 4.2. The operators H and $\mathscr{L}_\Theta$ are the mean curvature operator and the Jacobi operator with respect to the metric $|dz|^2$ of $\overline{\mathbb{C}}$. Defining operators $\overline{H} = \lambda H$ and $\overline{\mathscr{L}}_\Theta = \lambda\mathscr{L}_\Theta$ means considering a different metric on $\overline{\mathbb{C}}$. Actually, $\overline{H}$ and $\overline{\mathscr{L}}_\Theta$ are the mean curvature operator and Jacobi operator with respect to the metric $g_\lambda = |dz|^2/\lambda$. From the definition of λ, it follows that the volume of $\overline{\mathbb{C}}$ with respect this metric is finite.

The Jacobi operator $\overline{\mathscr{L}}_\Theta$ is a second order linear elliptic operator satisfying $|\overline{\mathscr{L}}_\Theta u - \Delta u| \leqslant c(|\theta_1| + |\theta_2|)|u|$, and the coefficients of $F_\Theta = \Delta - \overline{\mathscr{L}}_\Theta$ have compact support.

Now we fix $s_0 > 0$. Given $\varepsilon > 0$ and $|T| \in [4/\sqrt{\varepsilon}, +\infty)$ large enough, we choose $s \in (0, s_0)$ so that $D_s = A(B_s)$ is the geodesic disk of radius $1/2\sqrt{\varepsilon}$ centered at the origin.

Proposition 4.3. *There exists $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and every $|T| \in (\eta_0, +\infty)$, there exists an operator*

$$G_{\varepsilon,|T|} : \mathscr{C}^{0,\alpha}(\overline{\mathbb{C}} - B_s) \to \mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_s)$$

such that, given $f \in \mathscr{C}^{0,\alpha}(\overline{\mathbb{C}} - B_s)$, $w = G_{\varepsilon,|T|}(f)$ satisfies

$$\begin{cases} \Delta w = f & \text{on } \overline{\mathbb{C}} - B_s, \\ w \in \operatorname{Span}\{1\} & \text{on } \partial B_s, \end{cases}$$

and $\|w\|_{\mathscr{C}^{2,\alpha}} \leq c\|f\|_{\mathscr{C}^{0,\alpha}}$ for some constant $c > 0$ that does not depend on ε or $|T|$.

Proof. Let be u a solution of $\Delta u = f$ on $\overline{\mathbb{C}} - B_s$ with $u = 0$ on ∂B_s. We recall that the metric in use on $\overline{\mathbb{C}}$ is given by $g_\lambda = |dz|^2/\lambda$. With respect to this metric

$$R := \operatorname{vol}(\overline{\mathbb{C}} - B_s) < +\infty \quad \text{and} \quad \int_{\overline{\mathbb{C}} - B_s} u \, d\mathrm{vol}_{g_\lambda} < \infty.$$

We set $w = u - (1/R) \int_{\overline{\mathbb{C}} - B_s} u \, d\mathrm{vol}_{g_\lambda}$. The function w is well defined and satisfies $\int_{\overline{\mathbb{C}} - B_s} w \, d\mathrm{vol}_{g_\lambda} = 0$; also $w \in \operatorname{Span}\{1\}$ on ∂B_s. If the theorem is false, there is a sequence of functions f_n, of solutions w_n, and of real numbers s_n such that

$$\sup_{\overline{\mathbb{C}} - B_{s_n}} |f_n| = 1 \quad \text{and} \quad A_n := \sup_{\overline{\mathbb{C}} - B_{s_n}} |w_n| \to +\infty \quad \text{as } n \to +\infty,$$

where $s_n \in [0, s_0]$. Now we set $\tilde{w}_n := w_n / A_n$. Elliptic estimates imply that s_n and $\tilde{w}_n$ converge up to a subsequence, respectively, to $s_\infty \in [0, s_0]$ and to $\tilde{w}_\infty$ on $\overline{\mathbb{C}} - B_{s_\infty}$. This function satisfies $\Delta \tilde{w}_\infty = 0$. Then $\tilde{w}_\infty$ is constant on $\overline{\mathbb{C}} - B_{s_\infty}$ and $\int_{\overline{\mathbb{C}} - B_{s_\infty}} \tilde{w}_\infty \, d\mathrm{vol}_{g_\lambda} = 0$, which contradicts that $\sup|\tilde{w}_\infty| = 1$. □

Now we fix $|T| \geqslant 4/\sqrt{\varepsilon}$, $\Theta \in (0, \varepsilon)^2$, $s_\varepsilon = 1/(2\sqrt{\varepsilon})$, and let $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ be even (or odd) L^2-orthogonal to 1, with $\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon$ for some $\kappa > 0$. Let w_φ be the unique bounded harmonic extension of φ. We would like to solve the minimal surface equation $H(\Theta, v + w_\varphi) = 0$ with fixed boundary data φ, prescribed asymptotic direction Θ and period $|T|$. Then we have to solve the equation

$$\Delta v = F_\Theta(v + w_\varphi) + Q_\Theta(v + w_\varphi),$$

with Q_Θ a quadratic term such that $|Q_\Theta(v_1) - Q_\Theta(v_2)| \leqslant c|v_1 - v_2|^2$. The resolution of the previous equation is obtained by showing the existence of a fixed point

$$v = S(\Theta, \varphi, v) := G_{\varepsilon,|T|}(F_\Theta(v + w_\varphi) + Q_\Theta(v + w_\varphi)).$$

Proposition 4.4. *Let $\varphi \in \mathbb{S}^1$ satisfy $\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon$ and enjoy the properties described above. There exist $c_\kappa > 0$ and $\varepsilon_\kappa > 0$ such that*

$$\|S(\Theta, \varphi, 0)\|_{\mathscr{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^2 \quad \textit{for all } |T| \geqslant 4/\sqrt{\varepsilon}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\|S(\Theta, \varphi, v_1) - S(\Theta, \varphi, v_2)\|_{\mathscr{C}^{2,\alpha}} \leqslant \tfrac{1}{2}\|v_2 - v_1\|_{\mathscr{C}^{2,\alpha}},$$
$$\|S(\Theta, \varphi_1, v) - S(\Theta, \varphi_2, v)\|_{\mathscr{C}^{2,\alpha}} \leqslant c\varepsilon\|\varphi_2 - \varphi_1\|_{\mathscr{C}^{2,\alpha}}$$

for all $v, v_1, v_2 \in \mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_{s_\epsilon})$ *whose* $\mathscr{C}^{2,\alpha}$ *norm is bounded by* $2c_\kappa\varepsilon^2$, *for all boundary data* $\varphi_1, \varphi_2 \in \mathbb{S}^1$ *with the same properties as* φ *and for all* $\Theta = (\theta_1, \theta_2)$ *such that* $|\theta_1| + |\theta_2| \leqslant \varepsilon$.

Proof. Using Proposition 4.3, the inequality $|\mathscr{L}u - \Delta u| \leqslant c(|\theta_1| + |\theta_2|)|u|$, and the quadratic behavior of Q_Θ, we derive the stated estimate. The details of the proof are left to the reader. □

Theorem 4.5. *Let* $B := \{w \in \mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_{s_\epsilon}) \mid \|w\|_{\mathscr{C}^{2,\alpha}} \leqslant 2c_\kappa\varepsilon^2\}$. *Let* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *as above, and let* $\Theta = (\theta_1, \theta_2)$ *with* $|\theta_1| + |\theta_2| \leqslant \varepsilon$. *Then the nonlinear mapping* $S(\Theta, \varphi, \cdot)$ *defined above has a unique fixed point* v *in* B.

Proof. The previous proposition shows that, if ε is chosen small enough, the nonlinear mapping S is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^2$ in $\mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_{s_\epsilon})$ into itself. This value follows from the estimate of the norm of $S(\Theta, \varphi, 0)$. Consequently by the Schauder fixed point theorem, $S(\Theta, \varphi, \cdot)$ has a unique fixed point v in this ball. □

On the set $B_{2s_\varepsilon} - B_{s_\varepsilon}$, the function $U = v + w_\varphi$ is the solution of Equation (19). Using the vertical translation c_0e_3, we can fix the value $c_0 + \varphi$ at the boundary, obtaining $U = c_0 + w_\varphi + v$.

The function v depends nonlinearly on φ. Using the Schauder estimate for the equation on a fixed bounded domain, we find

$$\|v(\varphi_1) - v(\varphi_2)\|_{\mathscr{C}^{2,\alpha}(\overline{\mathbb{C}} - B_{s_\epsilon})} \leqslant c_\kappa\varepsilon\|\varphi_1 - \varphi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}.$$

This can be done uniformly in (θ_1, θ_2). Now we want to obtain the parametrization of the surface close to the annulus with linear growth ends (from which we have removed D_{s_ϵ}) in a neighbourhood of ∂D_{s_ϵ}. We recall that D_{s_ϵ} corresponds to B_{s_ϵ} by a conformal mapping. From now on, φ will be considered as the boundary data for ∂D_{s_ϵ}. We will denote its harmonic extension by $w_\varphi = \tilde{H}_{s_\epsilon,\varphi}$. We observe that near $\partial D_{s_\varepsilon}$ the function U grows logarithmically. The hypothesis that φ is orthogonal to 1 implies that the function w_φ is also and is bounded. This is not the case for v, which can be seen as the sum of a bounded function that is orthogonal to 1 and of a function of the form $c\ln(r/s_\varepsilon)$, where $c = c(|T|, \theta_1, \theta_2)$, defined in a neighborhood of $\partial D_{s_\varepsilon}$. We can determine c using a flux formula.

Let γ_1 and γ_2 be two closed curves in $\overline{\Sigma}/T$ chosen to correspond by conformal mapping to the boundaries of two circular neighborhoods N_1 and N_2 of the punctures corresponding to the ends with linear growth. Let $\mathscr{S} = \overline{\mathbb{C}} - (B_{s_\varepsilon} \cup N_1 \cup N_2)$.

Now $\int_{\mathcal{S}} \Delta X = 0$ since X is the parameterization of a minimal surface. By the divergence theorem, if $\Gamma = \partial\mathcal{S}$, then

$$0 = \int_{\mathcal{S}} \Delta X = \int_{\Gamma} \frac{\partial X}{\partial \eta} ds = \int_{\gamma_1} \frac{\partial X}{\partial \eta} ds + \int_{\gamma_2} \frac{\partial X}{\partial \eta} ds + \int_{\partial D_{s_\varepsilon}} \frac{\partial X}{\partial \eta} ds,$$

where η denotes the conormal along Γ. This equality implies

$$\int_{\partial D_{s_\varepsilon}} \frac{\partial U}{\partial \eta} ds = \sin\theta_1 |T| + \sin\theta_2 |T|.$$

By integration we can conclude that

$$U = \frac{|T|}{2\pi}(\sin\theta_1 + \sin\theta_2)\ln(r/s_\varepsilon) + c_0 + w_\varphi + v^\perp \quad \text{on } D_{2s_\varepsilon} - D_{s_\varepsilon}, \text{ with } v^\perp \perp 1.$$

We observe that if $\theta_2 = \theta_1 = 0$, there exists an infinite family of minimal surfaces that are close to the surface $\Sigma - D_{s_\varepsilon}$. Let $S_m(\varphi)$ be one such surface. It can be seen as the graph about $D_{2s_\varepsilon} - D_{s_\varepsilon}$ of the function

$$\overline{U}_m(r,\theta) = c_0 + \tilde{H}_{s_\varepsilon,\varphi}(r,\theta) + \overline{V}_m(r,\theta),$$

where $V_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon)$, and it satisfies

$$\|\overline{V}_m(\varphi_1) - \overline{V}_m(\varphi_2)\|_{\mathcal{C}^{2,\alpha}(D_{2s_\varepsilon} - D_{s_\varepsilon})} \leqslant c_\kappa \varepsilon \|\varphi_1 - \varphi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)} \tag{20}$$

for $\varphi_2, \varphi_1 \in C^{2,\alpha}(\mathbb{S}^1)$.

If $(\theta_2,\theta_1) \neq 0$, we choose $|T|$ so that $(|T|/2\pi)(\sin\theta_1 + \sin\theta_2) = 1$. There exists an infinite family of minimal surfaces that are close to the periodic Scherk-type example. After a vertical translation, any such surface can be seen as the graph about $D_{2s_\varepsilon} - D_{s_\varepsilon}$ of the function

$$\overline{U}_t(r,\theta) = \ln(2r) + c_0 + \tilde{H}_{s_\varepsilon,\varphi}(r,\theta) + \overline{V}_t(r,\theta) \tag{21}$$

where $\overline{V}_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon)$, and it satisfies

$$\|\overline{V}_t(\varphi_1) - \overline{V}_t(\varphi_2)\|_{\mathcal{C}^{2,\alpha}(D_{2s_\varepsilon} - D_{s_\varepsilon})} \leqslant c_\kappa \varepsilon \|\varphi_1 - \varphi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)}, \tag{22}$$

for $\varphi_2, \varphi_1 \in C^{2,\alpha}(\mathbb{S}^1)$.

Remark 4.6. If the boundary data φ is an even function, it is clear the surfaces we have just described are symmetric across the vertical plane $\{x_2 = 0\}$. However, if the boundary data φ is an odd function and $\theta_1 = \theta_2$, the surfaces are symmetric across the plane $\{x_1 = 0\}$.

5. KMR examples

Here we briefly present the *KMR examples* $M_{\sigma,\alpha,\beta}$ studied in [Karcher 1988; 1989; Meeks and Rosenberg 1989; Rodríguez 2007] — these are also called *toroidal half-plane layers* — which are the only properly embedded, doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient; see [Pérez et al. 2005].

For each $\sigma \in (0, \pi/2)$, $\alpha \in [0, \pi/2]$ and $\beta \in [0, \pi/2]$ with $(\alpha, \beta) \neq (0, \sigma)$, consider the rectangular torus $\Sigma_\sigma = \{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z^2 + \lambda^2)(z^2 + \lambda^{-2})\}$, where $\lambda = \lambda(\sigma) = \cot(\sigma/2) > 1$. By means of the Weierstrass representation, the KMR example $M_{\sigma,\alpha,\beta}$ is determined by its Gauss map g and the differential of its height function h, which are defined on Σ_σ and given by

$$g(z, w) = \frac{az+b}{i(\bar{a}-\bar{b}z)} \quad \text{and} \quad dh = \mu \frac{dz}{w},$$

with

$$a = a(\alpha, \beta) = \cos\tfrac{1}{2}(\alpha+\beta) + i\cos\tfrac{1}{2}(\alpha-\beta),$$
$$b = b(\alpha, \beta) = \sin\tfrac{1}{2}(\alpha-\beta) + i\sin\tfrac{1}{2}(\alpha+\beta), \qquad \mu = \mu(\sigma) = \frac{\pi \csc\sigma}{\mathcal{K}(\sin^2\sigma)},$$

where $\mathcal{K}(m) = \int_0^{\pi/2} 1/(1 - m\sin^2 u)^{1/2} du$ for $0 < m < 1$ is the complete elliptic integral of first kind. Such μ has been chosen so that the vertical part of the flux of $M_{\sigma,\alpha,\beta}$ along any horizontal level section equals 2π.

Remark 5.1. These statements give us a better understanding of the geometrical meaning of a and b:

(i) $b \to 0$ if and only if $\alpha \to 0$ and $\beta \to 0$, in which case $a \to 1 + i$.

(ii) $|b|^2 + |a|^2 = 2$.

(iii) If $\beta = 0$, then $a = (1 + i)\cos(\alpha/2)$ and $b = (1 + i)\sin(\alpha/2)$, and $b = \mathcal{O}(\alpha)$.

(iv) If $\alpha = 0$, then $a = (1 + i)\cos(\beta/2)$ and $b = (-1 + i)\sin(\beta/2)$, and $b = \mathcal{O}(\beta)$.

(v) In general, $|b/a| = \tan(\varphi/2)$, where φ is the angle between the north pole $(0, 0, 1) \in \mathbb{S}^2$ and the pole of g seen in $\mathbb{S}^2$ via the inverse of the stereographic projection.

The ends of $M_{\sigma,\alpha,\beta}$ correspond to the punctures $\{A, A', A'', A'''\} = g^{-1}(\{0, \infty\})$, and the branch values of g are those with $w = 0$, that is,

$$D = (-i\lambda, 0), \quad D' = (i\lambda, 0), \quad D'' = (i/\lambda, 0), \quad D''' = (-i/\lambda, 0). \tag{23}$$

Seen in $\mathbb{S}^2$, these points form two pairs $D'' = -D$ and $D''' = -D'$ of antipodal points, and each KMR example can be given in terms of the branch values of its Gauss map; see [Rodríguez 2007].

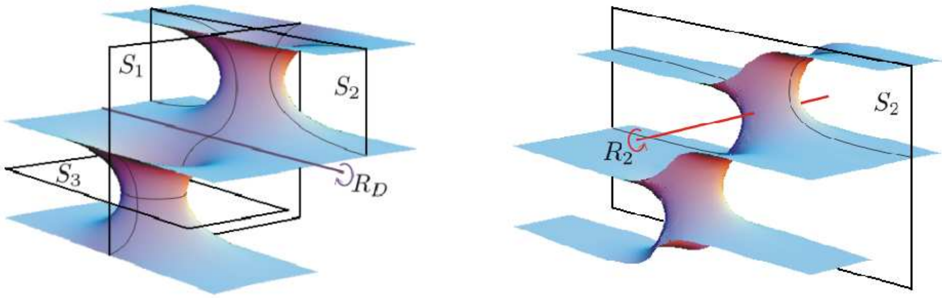

Figure 3. Left: $M_{\sigma,0,0}$, with $\sigma = \pi/4$. Right: $M_{\sigma,\alpha,0}$, with $\sigma = \alpha = \pi/4$.

Denote by T the period of $M_{\sigma,\alpha,\beta}$ at its ends. We focus on two more symmetric subfamilies of KMR examples:

$$\{M_{\sigma,\alpha,0} \mid 0 < \sigma < \tfrac{1}{2}\pi,\ 0 \leqslant \alpha \leqslant \tfrac{1}{2}\pi\} \quad \text{and} \quad \{M_{\sigma,0,\beta} \mid 0 < \sigma < \tfrac{1}{2}\pi,\ 0 \leqslant \beta < \sigma\}.$$

(1) When $\alpha = \beta = 0$, $M_{\sigma,0,0}$ contains four straight lines parallel to the x_1 axis. The isometry group of $M_{\sigma,0,0}$ is generated by the π-rotation R_D around one of the four straight lines contained in the surface, and by three reflection symmetries S_1, S_2, S_3, where each S_i is across the $\{x_i = 0\}$ plane; see Figure 3 left. In this case, $T = (0, \pi\mu, 0)$.

(2) When $0 < \alpha < \pi/2$, the isometry group of $M_{\sigma,\alpha,0}$ is generated by $\mathfrak{D}$ (corresponding to the deck transformation $(z, w) \mapsto (z, -w)$), which represents in $\mathbb{R}^3$ a central symmetry about any of the four branch points of the Gauss map of $M_{\sigma,\alpha,0}$; the reflection S_2 across the $\{x_2 = 0\}$ plane; and the π-rotation R_2 around a line parallel to the x_2 axis that cuts $M_{\sigma,\alpha,0}$ orthogonally; see Figure 3 right. Now $T = (0, \pi\mu t_\alpha, 0)$, with $t_\alpha = \sin\sigma/(\sin^2\sigma\cos^2\alpha + \sin^2\alpha)^{1/2}$.

(3) Suppose that $0 < \beta < \sigma$. Then $M_{\sigma,0,\beta}$ contains four straight lines parallel to the x_1 axis, and the isometry group of $M_{\sigma,0,\beta}$ is generated by the reflection S_1 across the $\{x_1 = 0\}$ plane; the π-rotation R_1 around a line parallel to the x_1 axis that cuts the surface orthogonally; and the π-rotation R_D around any one of the straight lines contained in the surface; see Figure 4. Moreover, $T = (0, \pi\mu t^\beta, 0)$, where $t^\beta = \sin\sigma/(\sin^2\sigma - \sin^2\beta)^{1/2}$.

Finally, it will be useful to see Σ_σ as a branched 2-covering of $\overline{\mathbb{C}}$ through the map $(z, w) \mapsto z$. Thus Σ_σ can be seen as two copies $\overline{\mathbb{C}}_1$ and $\overline{\mathbb{C}}_2$ of $\overline{\mathbb{C}}$ glued along two common cuts γ_1 and γ_2, which can be taken along the imaginary axis: γ_1 from D to D', and γ_2 from D'' to D'''.

5.1. ***$M_{\sigma,\alpha,\beta}$ as a graph over $\{x_3 = 0\}/T$.*** The KMR examples $M_{\sigma,\alpha,\beta}$ converge as $(\sigma, \alpha, \beta) \to (0, 0, 0)$ to a vertical catenoid, since Σ_σ converges to two pinched

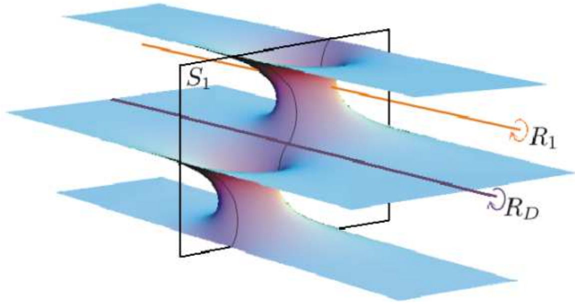

Figure 4. $M_{\sigma,0,\beta}$, where $\sigma = \pi/4$ and $\beta = \pi/8$.

spheres, $g(z) \to z$ and $dh \to \pm dz/z$ as $(\sigma, \alpha, \beta) \to (0, 0, 0)$. In fact, we can obtain two catenoids in the limit, depending on the choice of branch for w (for each copy of $\overline{\mathbb{C}}$ in Σ_σ, we obtain one catenoid in the limit). One of our aims for this paper is to take KMR examples $M_{\sigma,\alpha,0}$ or $M_{\sigma,0,\beta}$ near this catenoidal limit and glue them to a convenient compact piece of the surface $M_k(\varepsilon/2)$. In this subsection, we express part of $M_{\sigma,\alpha,\beta}$ as a vertical graph over the $\{x_3 = 0\}$ plane when σ, α, β are small.

Consider $M_{\sigma,\alpha,\beta}$ near the catenoidal limit, that is, σ, α, β close to zero. Without lost of generality, we can assume $dh \sim -dz/z$ in $\overline{\mathbb{C}}_1$. We are studying the surface in an annulus about one of its ends, say a zero of its Gauss map.

Lemma 5.2. *Consider $\alpha + \beta + \sigma \leqslant \varepsilon$ small. Up to translations, $M_{\sigma,\alpha,\beta}$ can be parameterized in the annulus $\{(z, w) \in \Sigma_\sigma \mid z \in \overline{\mathbb{C}}_1,\ |b/a| < |z| < \nu\}$, for $\nu \in (|b/a|, 1)$ small, by*

$$X_1 + iX_2 = \tfrac{1}{2}(z + 1/\bar{z}) + \frac{(1+i)\bar{b}}{4\bar{z}^2} + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}),$$
$$X_3 = -\ln|z| + \mathcal{O}(\varepsilon^2 z^{-2}),$$

Proof. Recall we have assumed $dh \sim -dz/z$ in the annulus we are working on. More precisely, we have

$$dh = -\frac{\mu\, dz}{\sqrt{(z^2+\lambda^2)(z^2+\lambda^{-2})}} = -\frac{\mu}{\lambda\sqrt{1+\lambda^{-2}z^2+\lambda^{-2}z^{-2}+\lambda^{-4}}}\,\frac{dz}{z}.$$

Since $\mu/\lambda = \pi/((1+\cos(\sigma))\mathcal{K}(\sin^2\sigma)) = 1 + \mathcal{O}(\sigma^4)$, and $\lambda^{-1} = \tan(\sigma/2) = \mathcal{O}(\varepsilon)$, we get

$$dh = -\frac{dz}{z}(1 + \mathcal{O}(\varepsilon^4))(1 + \mathcal{O}(\varepsilon^2 z^2 + \varepsilon^2 z^{-2} + \varepsilon^4)).$$

Since $|z| < \nu < 1$, we have $dh = -(dz/z)(1 + \mathcal{O}(\varepsilon^2 z^{-2}))$. Fix any point $z_0 \in \mathbb{C}_1$, with $z_0 \notin \{-b/a, \bar{a}/\bar{b}\}$ (which correspond to two ends of the KMR example), and recall that $g = -i(az + b)/(\bar{a} - \bar{b}z)$. Straightforward computations give, for

$|b/a| < |z| < 1$,

$$\int_{z_0}^{z} \frac{dh}{g} = \frac{i\bar{b}}{a}\ln z + \frac{2i}{a^2 z} - \frac{2ib}{a^3 z^2} - C_1 + \mathcal{O}(\varepsilon^2 z^{-3}),$$
$$\int_{z_0}^{z} g\,dh = \frac{ib}{\bar{a}}\ln z + \frac{2i}{\bar{a}^2} z - C_2 + \mathcal{O}(\varepsilon^2 z^{-1}),$$

where $C_1, C_2 \in \mathbb{C}$ satisfy $\frac{1}{2}(\overline{C_1} - C_2) = \frac{1}{2}(z_0 + 1/\overline{z_0}) + \mathcal{O}(\varepsilon)$. Taking into account that $a = (1+i) + \mathcal{O}(\varepsilon)$, we obtain

$$\begin{aligned} X_1 + iX_2 &= \frac{1}{2}\Big(\overline{\int_{z_0}^{z}\frac{dh}{g}} - \int_{z_0}^{z} g\,dh\Big) \\ &= -\frac{i}{\bar{a}^2}\Big(z + \frac{1}{\bar{z}}\Big) - \frac{ib}{\bar{a}}\ln|z| + \frac{i\bar{b}}{\bar{a}^3\bar{z}^2} - \frac{1}{2}\Big(z_0 + \frac{1}{\overline{z_0}}\Big) + \mathcal{O}(\varepsilon^2 z^{-3}) \\ &= \frac{1}{2}\Big(z + \frac{1}{\bar{z}}\Big) + \frac{(1+i)\bar{b}}{4\bar{z}^2} - \frac{1}{2}\Big(z_0 + \frac{1}{\overline{z_0}}\Big) + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}). \end{aligned}$$

Similarly, $\int_{z_0}^{z} dh = -\ln z + \ln z_0 + \mathcal{O}(\varepsilon^2 z^{-2})$; hence

$$X_3 = \operatorname{Re}\int_{z_0}^{z} dh = -\ln|z| + \ln|z_0| + \mathcal{O}(\varepsilon^2 z^{-2}). \qquad \square$$

By suitably translating $M_{\sigma,\alpha,\beta}$, we can assume its coordinate functions are as in Lemma 5.2.

Lemma 5.3. *Let (r,θ) denote the polar coordinates in the $\{x_3 = 0\}$ plane, and let $r_\varepsilon = 1/(2\sqrt{\varepsilon})$. If $\alpha + \beta + \sigma \leqslant \varepsilon$ small, then an annular piece of $M_{\sigma,\alpha,\beta}$ can be written as a vertical graph of the function*

$$\widetilde{U}(r,\theta) = \ln(2r) + r(-\kappa_1\cos\theta + \kappa_2\sin\theta) + \mathcal{O}(\varepsilon),$$

for $(r,\theta) \in (r_\varepsilon/2, 2r_\varepsilon) \times [0, 2\pi)$, where $\kappa_1 = \operatorname{Re}(b) + \operatorname{Im}(b)$ and $\kappa_2 = \operatorname{Re}(b) - \operatorname{Im}(b)$.

We denote by $M_{\sigma,\alpha,\beta}(\gamma,\xi)$ the KMR example $M_{\sigma,\alpha,\beta}$ dilated by $1+\gamma$ for some small $\gamma \leqslant 0$, and translated by a vector $\xi = (\xi_1, \xi_2, \xi_3)$. Then an annular piece of $M_{\sigma,\alpha,\beta}(\gamma,\xi)$ can be written as a vertical graph of

$$\widetilde{U}_{\gamma,\xi}(r,\theta) =$$
$$(1+\gamma)\ln(2r) + r\,(-\kappa_1\cos\theta + \kappa_2\sin\theta) - \frac{1+\gamma}{r}(\xi_1\cos\theta + \xi_2\sin\theta) + d + \mathcal{O}(\varepsilon),$$

for $(r,\theta) \in (r_\varepsilon/2, 2r_\varepsilon) \times [0, 2\pi)$, where $d = \xi_3 - (1+\gamma)\ln(1+\gamma)$.

Remark 5.4. Recall that $b = \sin\frac{1}{2}(\alpha - \beta) + i\,\sin\frac{1}{2}(\alpha + \beta)$. Here are some special cases:

- When $\beta = 0$, we have $\kappa_1 = 2\sin\frac{1}{2}\alpha$ and $\kappa_2 = 0$, so

$$\widetilde{U}_{\gamma,\xi}(r,\theta) = (1+\gamma)\ln(2r) - 2r\sin\tfrac{1}{2}(\alpha)\cos\theta - \frac{1+\gamma}{r}(\xi_1\cos\theta + \xi_2\sin\theta) + d + \mathcal{O}(\varepsilon).$$

- When $\alpha = 0$, we have $\kappa_1 = 0$ and $\kappa_2 = 2\sin\frac{1}{2}(\beta)$, so

$$\widetilde{U}_{\gamma,\xi}(r,\theta) = (1+\gamma)\ln(2r) + 2r\sin\tfrac{1}{2}\beta\sin\theta - \frac{1+\gamma}{r}(\xi_1\cos\theta + \xi_2\sin\theta) + d + \mathcal{O}(\varepsilon).$$

In Section 7, we will consider $\xi_1 = 0$ when $\alpha = 0$, and $\xi_2 = 0$ when $\beta = 0$.

Proof. Suppose $|b/a| < |z| < \nu$, with $\nu > |b/a|$ small. From Lemma 5.2, we know the coordinate functions (X_1, X_2, X_3) of the perturbed KMR example $M_{\sigma,\alpha,\beta}(\gamma,\xi)$ are given by

$$\begin{aligned} X_1 + iX_2 &= \tfrac{1}{2}(1+\gamma)(z + 1/\bar{z}) + A(z), \\ X_3 &= -(1+\gamma)\ln|z| + \xi_3 + \mathcal{O}(\varepsilon^2 z^{-2}), \end{aligned} \tag{24}$$

where

$$\begin{aligned} A(z) &= \frac{(1+\gamma)(1+i)\bar{b}}{4\bar{z}^2} + (\xi_1 + i\xi_2) + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}) \\ &= \frac{(1+\gamma)(\kappa_1 + i\kappa_2)}{4\bar{z}^2} + (\xi_1 + i\xi_2) + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}). \end{aligned}$$

If we set $z = |z|e^{i\psi}$ and $X_1 + iX_2 = re^{i\theta}$, then $z + 1/\bar{z} = (|z| + 1/|z|)\, e^{i\psi}$ and

$$\begin{aligned} r\cos\theta &= \tfrac{1}{2}(1+\gamma)\,(|z| + 1/|z|)\cos\psi + A_1, \\ r\sin\theta &= \tfrac{1}{2}(1+\gamma)\,(|z| + 1/|z|)\sin\psi + A_2, \end{aligned}$$

where $A_1 = \mathrm{Re}(A)$ and $A_2 = \mathrm{Im}(A)$. Therefore,

$$r^2 = \tfrac{1}{4}(1+\gamma)^2\Big(|z| + \frac{1}{|z|}\Big)^2\Bigg(1 + \frac{4|z|}{(1+\gamma)(|z|^2+1)}(A_1\cos\psi + A_2\sin\psi) + \frac{4|z|^2}{(1+\gamma)^2(|z|^2+1)^2}(A_1^2 + A_2^2)\Bigg). \tag{25}$$

When $\sqrt{\varepsilon}/R \leqslant |z| \leqslant R\sqrt{\varepsilon}$ for some $R > 0$, the functions A_i are bounded, and we get

$$r = \tfrac{1}{2}(1+\gamma)\Big(|z| + \frac{1}{|z|}\Big)(1 + \mathcal{O}(\sqrt{\varepsilon})) = \frac{1+\gamma}{2|z|} + \mathcal{O}(\sqrt{\varepsilon}). \tag{26}$$

In particular, $r = \mathcal{O}(1/\sqrt{\varepsilon})$. We consider $R > 0$ large enough so that

$$\{r_\varepsilon/2 \leqslant r \leqslant 2r_\varepsilon\} \subset \{\sqrt{\varepsilon}/R \leqslant |z| \leqslant R\sqrt{\varepsilon}\}.$$

From (26), we get $r\big/\big(\frac{1}{2}(1+\gamma)\,(|z| + 1/|z|)\big) = 1 + \mathcal{O}(\sqrt{\varepsilon})$, which gives

$$\frac{X_1 + iX_2}{\frac{1}{2}(1+\gamma)\,(|z| + 1/|z|)} = e^{i\theta}(1 + \mathcal{O}(\sqrt{\varepsilon})).$$

On the other hand,

$$\frac{X_1 + iX_2}{\frac{1}{2}(1+\gamma)\,(|z| + 1/|z|)} = e^{i\psi} + \frac{2|z|A}{(1+\gamma)(1+|z|^2)} = e^{i\psi} + \mathcal{O}(\sqrt{\varepsilon}).$$

Thus $e^{i\psi} = e^{i\theta}(1 + \mathcal{O}(\sqrt{\varepsilon}))$.

From (25) and (26) we can deduce

$$\frac{(1+\gamma)^2(1+|z|^2)^2}{4|z|^2} = r^2\big(1 - (2/r)\,(A_1\cos\psi + A_2\sin\psi) + \mathcal{O}(\varepsilon)\big),$$

from which we obtain

$$\begin{aligned}\frac{1}{|z|^2} &= \Big(\frac{2r}{1+\gamma}\Big)^2\big(1 - (2/r)\,(A_1\cos\psi + A_2\sin\psi) + \mathcal{O}(\varepsilon)\big)(1 + \mathcal{O}(\varepsilon))\\ &= \Big(\frac{2r}{1+\gamma}\Big)^2\big(1 - (2/r)\,(A_1\cos\psi + A_2\sin\psi) + \mathcal{O}(\varepsilon)\big),\end{aligned}$$

and then

$$-\ln|z| = \ln\frac{2r}{1+\gamma} - \frac{1}{r}(A_1\cos\psi + A_2\sin\psi) + \mathcal{O}(\varepsilon). \tag{27}$$

Finally, it is not difficult to prove that

$$\begin{aligned}A_1 &= \frac{1+\gamma}{4|z|^2}(\kappa_1\cos(2\psi) - \kappa_2\sin(2\psi)) + \xi_1 + \mathcal{O}(\sqrt{\varepsilon}),\\ A_2 &= \frac{1+\gamma}{4|z|^2}(\kappa_1\sin(2\psi) + \kappa_2\cos(2\psi)) + \xi_2 + \mathcal{O}(\sqrt{\varepsilon}).\end{aligned}$$

Therefore,

$$\begin{aligned}A_1\cos\psi + A_2\sin\psi &= \frac{1+\gamma}{4|z|^2}\,(\kappa_1\cos\psi - \kappa_2\sin\psi) + \xi_1\cos\psi + \xi_2\sin\psi + \mathcal{O}(\sqrt{\varepsilon})\\ &= \frac{r^2}{1+\gamma}\,(\kappa_1\cos\theta - \kappa_2\sin\theta) + \xi_1\cos\theta + \xi_2\sin\theta + \mathcal{O}(\sqrt{\varepsilon}).\end{aligned}$$

From here, (27) and (24), Lemma 5.3 follows. □

5.2. ***Parameterization of the KMR example on the cylinder.*** In this subsection we want to parameterize the KMR example $M_{\sigma,\alpha,\beta}$ on a cylinder. Recall its conformal compactification Σ_σ only depends on σ. The parameter $\sigma \in (0, \pi/2)$ will remain fixed along this subsection, and we will omit the dependence on σ of the functions we are introducing. Also recall that Σ_σ can be seen as a branched 2-covering of $\overline{\mathbb{C}}$ by gluing $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2$ along two common cuts γ_1 and γ_2 along the imaginary axis joining the branch points D, D' and D'', D''', respectively; see (23).

We introduce the spheroconal coordinates (x, y) on the annulus $\mathbb{S}^2 - (\gamma_1 \cup \gamma_2)$ as in [Jansen 1977]: For any $(x, y) \in \mathbb{S}^1 \times (0, \pi) \equiv [0, 2\pi) \times (0, \pi)$, we define

$$F(x, y) = (\cos x \sin y, \sin x\, m(y), l(x)\, \cos y) \in \mathbb{S}^2 - (\gamma_1 \cup \gamma_2),$$

where

$$m(y) = (1-\cos^2\sigma \cos^2 y)^{1/2} \quad \text{and} \quad l(x) = (1-\sin^2\sigma \sin^2 x)^{1/2}.$$

Geometrically, $\{x = \text{const}\}$ and $\{y = \text{const}\}$ correspond to two closed curves on $\mathbb{S}^2$ that are the intersection of the sphere with two elliptic cones (one with horizontal axis, the other one with vertical axis) having as vertex the center of the sphere.

If we compose $F(x, y)$ with the stereographic projection and enlarge the domain of definition of the function, we obtain a differentiable map $\mathbf{z}$ defined on the torus $\mathbb{S}^1 \times \mathbb{S}^1 \equiv [0, 2\pi) \times [0, 2\pi) \to \overline{\mathbb{C}}$ and given by

$$\mathbf{z}(x, y) = \frac{\cos x \sin y + i \sin x\, m(y)}{1-l(x)\cos y}, \tag{28}$$

which is a branch 2-covering of $\overline{\mathbb{C}}$ with branch values in the four points whose spheroconal coordinates are $(x, y) \in \{\pm\pi/2\} \times \{0, \pi\}$; these correspond to D, D', D'' and D'''. Moreover, $\mathbf{z}$ maps $\mathbb{S}^1 \times (0, \pi)$ onto $\overline{\mathbb{C}} - (\gamma_1 \cup \gamma_2)$. Hence we can parameterize the KMR example by $\mathbf{z}$, via its Weierstrass data

$$g(\mathbf{z}) = \frac{a\mathbf{z}+b}{i(\bar{a}-\bar{b}\mathbf{z})}, \qquad dh = \mu \frac{d\mathbf{z}}{\sqrt{(\mathbf{z}^2+\lambda^2)(\mathbf{z}^2+\lambda^{-2})}},$$

We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{R}^3/T$ by forgetting its nonhorizontal period (that is, its period in homology, $\widetilde{T}$). We can then parameterize $\widetilde{M}_{\sigma,\alpha,\beta}$ on $\mathbb{S}^1 \times \mathbb{R}$ by extending $\mathbf{z}$ to $[0, 2\pi) \times \mathbb{R}$ periodically. But such a parameterization is not conformal, since the spheroconal coordinates $(x, y) \mapsto F(x, y)$ of the sphere are not conformal. As the stereographic projection is a conformal map, it suffices to find new conformal coordinates (u, v) of the sphere defined on the cylinder. In particular, we look for a change of variables $(x, y) \mapsto (u, v)$ for which $|\widetilde{F}_u| = |\widetilde{F}_v|$ and $\langle \widetilde{F}_u, \widetilde{F}_v \rangle = 0$, where $\widetilde{F}(u, v) = F(x(u, v), y(u, v))$.

We observe that

$$|F_x| = \sqrt{k(x, y)}/l(x) \quad \text{and} \quad |F_y| = \sqrt{k(x, y)}/m(y),$$

with $k(x, y) = \sin^2\sigma \cos^2 x + \cos^2\sigma \sin^2 y$. Then it is natural to consider the change of variables $(x, y) \in [0, 2\pi) \times \mathbb{R} \mapsto (u, v) \in [0, U_\sigma) \times \mathbb{R}$ defined by

$$u(x) = \int_0^x \frac{dt}{l(t)} \quad \text{and} \quad v(y) = \int_{\pi/2}^y \frac{dt}{m(t)}, \tag{29}$$

where

$$U_\sigma = u(2\pi) = \int_0^{2\pi} \frac{dt}{\sqrt{1-\sin^2\sigma \sin^2 t}}.$$

Note that U_σ is a function on σ that goes to 2π as σ approaches to zero, and that the change of variables above is well defined because $\sigma \in (0, \pi/2)$.

In these variables (u, v), $\mathbf{z}$ is v-periodic with period

$$V_\sigma = v(2\pi) - v(0) = \int_0^{2\pi} \frac{dt}{\sqrt{1-\cos^2\sigma\cos^2 t}}.$$

The period V_σ goes to $+\infty$ as σ goes to zero (see the proof of Lemma 5.5), which is made clear by taking into account the limits of $M_{\sigma,\alpha,\beta}$ as σ tends to zero.

From all this, we can deduce that $\widetilde{M}_{\sigma,\alpha,\beta}$ (respectively $M_{\sigma,\alpha,\beta}$) is conformally parameterized on $(u, v) \in I_\sigma \times \mathbb{R}$ (respectively $(u, v) \in I_\sigma \times J_\sigma$), where $I_\sigma = [0, U_\sigma]$ and $J_\sigma = [v(0), v(2\pi)]$. In Section 6, which is devoted to the study of the mapping properties of the Jacobi operator of $\widetilde{M}_{\sigma,\alpha,\beta}$, we will use the (u, v) variables.

In Lemma 5.3, an appropriate piece of $\widetilde{M}_{\sigma,\alpha,\beta}$ has been written as a vertical graph over the annulus $\{r_\varepsilon/2 \leqslant r \leqslant 2r_\varepsilon\} \subset \{x_3 = 0\}$. The boundary curve of $\widetilde{M}_{\sigma,\alpha,\beta}$ along which we will glue a piece of the Costa–Hoffman–Meeks surface corresponds to $\{r = r_\varepsilon\}$. Equation (26) says that if r is near r_ε, then z is in a neighborhood of $\{|\mathbf{z}| = \sqrt{\varepsilon}\}$. Next lemma gives us the values of v corresponding to such a neighborhood.

Lemma 5.5. *Consider $\sigma \leqslant \varepsilon$. If $\sqrt{\varepsilon}/R \leqslant |\mathbf{z}| \leqslant R\sqrt{\varepsilon}$, for $R > 0$, then*

$$-\tfrac{1}{2}\ln\varepsilon + c_1 \leqslant v \leqslant -\tfrac{1}{2}\ln\varepsilon + c_2,$$

where c_1 and c_2 are constant. Under the same assumptions, $V_\sigma = -4\ln\varepsilon + \mathcal{O}(1)$.

Proof. Using Equation (28), we can show that, if $\sqrt{\varepsilon}/R \leqslant |\mathbf{z}(x, y)| \leqslant R\sqrt{\varepsilon}$, then $\pi - d_1\sqrt{\varepsilon} \leqslant y \leqslant \pi - d_2\sqrt{\varepsilon}$, where $d_1 > d_2 > 0$ are constant. This means, since v is increasing function of y, that $v(\pi - d_1\sqrt{\varepsilon}) \leqslant v(y) \leqslant v(\pi - d_2\sqrt{\varepsilon})$. Let us compute $v(\pi - d_i\sqrt{\varepsilon})$ for $i = 1, 2$. We have

$$\begin{aligned} v(y) - v(0) &= \int_0^y \frac{ds}{\sqrt{1-\cos^2\sigma\cos^2 s}} = \int_0^y \frac{ds}{\sqrt{1-\cos^2\sigma+\cos^2\sigma\sin^2 s}} \\ &= \frac{1}{\sin\sigma}\int_0^y \frac{ds}{\sqrt{1+\cot^2\sigma\ \sin^2 s}} = \frac{1}{\sin\sigma}\mathcal{F}(y, -\cot^2\sigma), \end{aligned}$$

where $\mathcal{F}(y, m) = \int_0^y (1 - m\sin^2 s)^{-1/2} ds$ is the incomplete elliptic integral of first kind. $\mathcal{F}(y, m)$ is an odd function in y and, if $k \in \mathbb{Z}$,

$$\mathcal{F}(y + k\pi, m) = \mathcal{F}(y, m) + 2k\,\mathcal{K}(m),$$

where $\mathcal{K}(m) = \mathcal{F}(\pi/2, m)$ is the complete elliptic integral of first kind. Since $\sigma = \mathcal{O}(\varepsilon)$, we have

$$\frac{1}{\sin\sigma}\mathcal{F}(d\sqrt{\varepsilon}, -\cot^2\sigma) = -\tfrac{1}{2}\ln\varepsilon + \mathcal{O}(1).$$

On the other hand, if $|m|$ is sufficiently big, then

$$\mathcal{K}(m) = \frac{1}{\sqrt{-m}}\big(\ln 4 + \tfrac{1}{2}\ln(-m)\big)\,(1 + \mathcal{O}(1/m))\,.$$

It follows that

$$\frac{1}{\sin\sigma}\mathcal{K}(-\cot^2\sigma) = -\ln\sigma + \ln 4 + \mathcal{O}(\sigma^2) = -\ln\varepsilon + \mathcal{O}(1).$$

Then, for $i = 1, 2$,

$$\begin{aligned} v(\pi - d_i\sqrt{\varepsilon}) &= \frac{1}{\sin\sigma}\,\big(\mathcal{F}(\pi - d_i\sqrt{\varepsilon}, -\cot^2\sigma) - \mathcal{K}(-\cot^2\sigma)\big) \\ &= \frac{1}{\sin\sigma}\,\big(\mathcal{F}(-d_i\sqrt{\varepsilon}, -\cot^2\sigma) + 2\mathcal{K}(-\cot^2\sigma) - \mathcal{K}(-\cot^2\sigma)\big) \\ &= \frac{1}{\sin\sigma}\,\big(-\mathcal{F}(d_i\sqrt{\varepsilon}, -\cot^2\sigma) + \mathcal{K}(-\cot^2\sigma)\big) = -\tfrac{1}{2}\ln\varepsilon + \mathcal{O}(1). \end{aligned}$$

Hence there exist constants c_1 and c_2 such that $v(\pi - d_1\sqrt{\varepsilon}) \geqslant -\frac{1}{2}\ln\varepsilon + c_1$ and $v(\pi - d_2\sqrt{\varepsilon}) \leqslant -\frac{1}{2}\ln\varepsilon + c_2$.

The result concerning $V_\sigma = v(2\pi) - v(0)$ follows once it is observed that $v(2\pi) = (3/\sin\sigma)\mathcal{K}(-\cot^2\sigma)$ and $v(0) = -(1/\sin\sigma)\mathcal{K}(-\cot^2\sigma)$. □

From Lemma 5.5 it follows that the value of the v corresponding to $|\mathbf{z}| = \sqrt{\varepsilon}$ is $v_\varepsilon = -\frac{1}{2}\ln\varepsilon + \mathcal{O}(1)$.

6. The Jacobi operator about KMR examples

The Jacobi operator for $M_{\sigma,\alpha,\beta}$ is given by $\mathcal{J} = \Delta_{ds^2} + |A|^2$, where $|A|^2$ is the squared norm of the second fundamental form on $M_{\sigma,\alpha,\beta}$ and Δ_{ds^2} is the Laplace–Beltrami operator with respect to the metric $ds^2 = \frac{1}{4}(|g| + |g|^{-1})^2|dh|^2$ on the surface. We consider the metric on the torus Σ_σ obtained as pull-back of the standard metric ds_0^2 on the sphere $\mathbb{S}^2$ by the Gauss map $N : M_{\sigma,\alpha,\beta} \to \mathbb{S}^2$; that is, $dN^*(ds_0^2) = -K\,ds^2$, where $K = -\frac{1}{2}|A|^2$ denotes the Gauss curvature of $M_{\sigma,\alpha,\beta}$. Hence $\Delta_{ds^2} = -K\,\Delta_{ds_0^2}$, and so $\mathcal{J} = -K(\Delta_{ds_0^2} + 2)$. From [Jansen 1977] and taking into account the parameterization of $M_{\sigma,\alpha,\beta}$ on the cylinder given in Section 5.2, we can deduce that, in the (x, y) variables,

$$\Delta_{ds_0^2} := \frac{l(x)m(y)}{k(x, y)}\Big(\partial_x\Big(\frac{l(x)}{m(y)}\partial_x\Big) + \partial_y\Big(\frac{m(y)}{l(x)}\partial_y\Big)\Big).$$

Recall $k(x, y) = \sin^2\sigma\cos^2 x + \cos^2\sigma\sin^2 y$. In the (u, v) variables defined by (29), we have $\mathcal{J} = -(K/k(u, v))\mathcal{L}_\sigma$, where $k(u, v) = k(x(u), y(v))$ and

$$\mathcal{L}_\sigma := \partial^2_{uu} + \partial^2_{vv} + 2k(u, v) \tag{30}$$

is the Lamé operator [Jansen 1977].

Remark 6.1. In Proposition 6.5, we will take $\sigma \to 0$. For such a limit, the torus Σ_σ degenerates into a Riemann surface with nodes consisting of two spheres joined at two common points p_0 and p_1, and the corresponding Jacobi operator equals the Legendre operator on $\mathbb{S}^2 - \{p_0, p_1\}$ given by $\mathcal{L}_0 = \partial^2_{xx} + \sin y\, \partial_y(\sin y\, \partial_y) + 2\sin^2 y$ in the (x, y) variables. When $\sigma = 0$, the change of variables $(x, y) \mapsto (u, v)$ given in (29) is not well defined.

The mapping properties of the Jacobi operator. From now on, we consider the conformal parameterization of $\widetilde{M}_{\sigma,\alpha,\beta}$ on the cylinder $\mathbb{S}^1 \times \mathbb{R} \equiv I_\sigma \times \mathbb{R}$ described in Section 5.2. In this subsection, we study the mapping properties of the operator $\mathcal{J}$. It is clear that it suffices to study the simpler operator $\mathcal{L}_\sigma$ defined by (30), so we will study the problem

$$\begin{cases} \mathcal{L}_\sigma w = f & \text{in } I_\sigma \times [v_0, +\infty[, \\ \quad w = \varphi & \text{on } I_\sigma \times \{v_0\} \end{cases}$$

with $v_0 \in \mathbb{R}$ and consider convenient normed functional spaces for w, f, φ so that the norm of w is bounded by that of f.

We will work in two different functional spaces to solve the Dirichlet problem above. To explain the reason, we recall that the isometry group of $\widetilde{M}_{\sigma,\alpha,\beta}$ depends on the values of the three parameters σ, α, β. When $\beta = 0$, $\widetilde{M}_{\sigma,\alpha,\beta}$ is invariant by reflection about the $\{x_2 = 0\}$ plane; when $\alpha = 0$, it is invariant about the $\{x_1 = 0\}$ plane. We want show there exist families of minimal surfaces close to $\widetilde{M}_{\sigma,\alpha,0}$ and $\widetilde{M}_{\sigma,0,\beta}$ and having the same symmetry properties. Thus the surfaces in the family about $\widetilde{M}_{\sigma,\alpha,0}$ (respectively $\widetilde{M}_{\sigma,0,\beta}$) will be defined as normal graphs of functions defined in $I_\sigma \times \mathbb{R}$ that are even (respectively odd) in the first variable. We will solve the Dirichlet problem above in the first case. The second one follows similarly.

Definition 6.2. Given $\sigma \in (0, \pi/2)$, $\ell \in \mathbb{N}$, $\alpha \in (0, 1)$, $\mu \in \mathbb{R}$, and an interval I, we define $\mathscr{C}^{\ell,\alpha}_\mu(I_\sigma \times I)$ as the space of functions $w = w(u, v)$ in $\mathscr{C}^{\ell,\alpha}_{\text{loc}}(I_\sigma \times I)$ that are even and U_σ-periodic in the variable u and for which the following norm is finite:

$$\|w\|_{\mathscr{C}^{\ell,\alpha}_\mu(I_\sigma\times I)} := \sup_{v\in I} \left(e^{-\mu v}\|w\|_{\mathscr{C}^{\ell,\alpha}(I_\sigma\times[v,v+1])}\right).$$

We observe that the Jacobi operator $\mathcal{L}_\sigma$ becomes a Fredholm operator when restricted to $\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times I)$. Moreover, $\mathcal{L}_\sigma$ has separated variables. Then we consider the operator $L_\sigma = \partial^2_{uu} + 2\sin^2\sigma\cos^2(x(u))$ defined on the space of U_σ-periodic and even functions in I_σ. This operator L_σ is uniformly elliptic and selfadjoint. In particular, L_σ has discrete spectrum $(\lambda_{\sigma,i})_{i\geqslant 0}$, which we assume is arranged so that $\lambda_{\sigma,i} < \lambda_{\sigma,i+1}$ for every i. Each eigenvalue $\lambda_{\sigma,i}$ is simple because we only consider even functions. We denote by $e_{\sigma,i}$ the even eigenfunction associated to $\lambda_{\sigma,i}$ and normalized so that

$$\int_0^{U_\sigma} (e_{\sigma,i}(u))^2\, du = 1.$$

Lemma 6.3. *For every $i \geqslant 0$, the eigenvalue $\lambda_{\sigma,i}$ of the operator L_σ and its associated eigenfunctions $e_{\sigma,i}$ satisfy*

$$-2\sin^2\sigma \leqslant \lambda_{\sigma,i} - i^2 \leqslant 0 \quad \textit{and} \quad \|e_{\sigma,i} - e_{0,i}\|_{\mathscr{C}^2(I_\sigma)} \leqslant c_i \sin^2\sigma,$$

where $e_{0,i}(u) := \cos(i\,x(u))$ for every $u \in I_\sigma$, and the constant $c_i > 0$ depends only on i (it does not depend on σ).

Proof. The bound for $\lambda_{\sigma,i} - i^2$ comes from the variational characterization of the eigenvalue $\lambda_{\sigma,i}$ as

$$\lambda_{\sigma,i} = \sup_{\operatorname{codim} E = i} \left(\inf_{e\in E,\, \|e\|_{L^2}=1} \int_0^{U_\sigma} \left((\partial_u e)^2 - 2\sin^2\sigma\, \cos^2(x(u))e^2\right) du \right),$$

where E is a subset of the space of U_σ-periodic and even functions in $L^2(I_\sigma)$, since it always holds $0 \leqslant 2\sin^2\sigma\,\cos^2(x(u)) \leqslant 2\sin^2\sigma$. The bound for the eigenfunctions follows from standard perturbation theory [Kato 1980]. □

The Hilbert basis $\{e_{\sigma,i}\}_{i\in\mathbb{N}}$ of the space of U_σ-periodic and even functions in $L^2(I_\sigma)$ introduced above induces the Fourier decomposition

$$g(u,v) = \sum_{i\geqslant 0} g_i(v)\, e_{\sigma,i}(u)$$

of functions $g = g(u,v)$ in $L^2(I_\sigma \times \mathbb{R})$ that are U_σ-periodic and even in the variable u. From this, we deduce that the operator $\mathscr{L}_\sigma$ can be decomposed as $\mathscr{L}_\sigma = \sum_{i\geqslant 0} L_{\sigma,i}$, where

$$L_{\sigma,i} = \partial^2_{vv} + 2\cos^2\sigma\,\sin^2(y(v)) - \lambda_{\sigma,i} \quad \text{for every } i \geqslant 0.$$

Since $0 \leqslant 2\cos^2\sigma\,\sin^2(y(v)) \leqslant 2\cos^2\sigma = 2 - 2\sin^2\sigma$, Lemma 6.3 gives us

$$P_{\sigma,i} := 2\cos^2\sigma\,\sin^2(y(v)) - \lambda_{\sigma,i} \leqslant 2 - i^2. \tag{31}$$

This fact allows us to prove the following lemma, which ensures that $\mathscr{L}_\sigma$ is injective when restricted to the set of functions that in the variable u are even and L^2-orthogonal to $e_{\sigma,0}$ and $e_{\sigma,1}$.

Lemma 6.4. *Given $v_0 < v_1$, let w be a solution of $\mathscr{L}_\sigma w = 0$ on $I_\sigma \times [v_0, v_1]$ that is U_σ-periodic and even in the variable u and satisfies*

(i) $w(\,\cdot\,, v_0) = w(\,\cdot\,, v_1) = 0$;

(ii) $\int_0^{U_\sigma} w(u,v)e_{\sigma,i}(u)\, du = 0$ *for every $v \in [v_0, v_1]$ and every $i \in \{0, 1\}$.*

Then $w = 0$.

Proof. By (ii), $w = \sum_{i \geqslant 2} w_i(v) e_{\sigma,i}(u)$. Since the potential $P_{\sigma,i}$ of the operator $L_{\sigma,i}$ is negative for every $i \geqslant 2$ (see (31)) and the operator $L_{\sigma,i}$ is elliptic, the maximum principle holds. We can then conclude that $w = 0$ from (i). □

To study the mapping properties of the Jacobi operator $\mathscr{L}_\sigma$, we need to give a description of the Jacobi fields associated to $M_{\sigma,\alpha,0}$, which are defined as the solutions of $\mathscr{L}_\sigma v = 0$. Since $M_{\sigma,\alpha,0}$ is invariant by reflection across the $\{x_2 = 0\}$ plane, there are only four independent Jacobi fields:

- Two Jacobi fields induced by vertical translations and by horizontal translations in the x_1 direction. These Jacobi fields are clearly periodic and hence bounded.
- A third Jacobi field generated by the 1-parameter group of dilations, which is not bounded (it grows linearly).
- A last Jacobi field obtained by considering the 1-parameter family of minimal surfaces induced by changing the parameter σ. This Jacobi field is not periodic and grows linearly.

The Jacobi fields induced by translation along the x_3 axis and by dilatation are solutions of $\mathscr{L}_\sigma u = 0$ that are collinear to the eigenfunction $e_{\sigma,0}$. The Jacobi fields induced by the horizontal translation and by the variation of the parameter σ are collinear to $e_{\sigma,1}$.

The Jacobi fields of $M_{\sigma,0,\beta}$, which is invariant by reflection across the plane $\{x_1 = 0\}$, are the same as those of $M_{\sigma,\alpha,0}$, with the exception that the one induced by horizontal translations in the x_1 direction is to be replaced by the field induced by horizontal translations in the x_2 direction.

The next proposition states that for an appropriately chosen parameter μ and interval I, there exists a right inverse for $\mathscr{L}_\sigma : \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times I) \to \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times I)$ whose norm is uniformly bounded.

Proposition 6.5. *Given $\mu \in (-2,-1)$, there exists a $\sigma_0 \in (0,\pi/2)$ such that, for every $\sigma \in (0,\sigma_0)$ and $v_0 \in \mathbb{R}$, there exists an operator*

$$G_{\sigma,v_0} : \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0,+\infty)) \to \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_0,+\infty))$$

such that for every $f \in \mathscr{C}^{0,\alpha}_\mu(I_\sigma \times [v_0,+\infty))$, the function $w := G_{\sigma,v_0}(f)$ solves

$$\begin{cases} \mathscr{L}_\sigma w = f & \text{in } I_\sigma \times [v_0,+\infty), \\ \quad w \in \operatorname{Span}\{e_{\sigma,0}, e_{\sigma,1}\} & \text{on } I_\sigma \times \{v_0\}. \end{cases}$$

Moreover $\|w\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant c\|f\|_{\mathscr{C}^{0,\alpha}_\mu}$ for some constant $c > 0$ that depends neither on $\sigma \in (0,\sigma_0)$ nor on $v_0 \in \mathbb{R}$.

Proof. Every $f \in \mathscr{C}^{0,\alpha}_{\mu}(I_\sigma \times [v_0, +\infty))$ can be decomposed as

$$f = f_0 e_{\sigma,0} + f_1 e_{\sigma,1} + \bar{f},$$

where $\bar{f}(\,\cdot\,, v)$ is L^2-orthogonal to $e_{\sigma,0}$ and to $e_{\sigma,1}$ for each $v \in [v_0, +\infty)$.

Step 1. First, let's prove Proposition 6.5 for functions $\bar{f} \in \mathscr{C}^{0,\alpha}_{\mu}(I_\sigma \times [v_0, +\infty))$ that are L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$. By Lemma 6.4, $\mathscr{L}_\sigma$ acts injectively on such a function space. Hence, the Fredholm alternative ensures that there exists for each $v_1 > v_0 + 1$ a unique $\bar{w} \in \mathscr{C}^{2,\alpha}_{\mu}(I_\sigma \times [v_0, v_1])$ in which $\bar{w}(\,\cdot\,, v)$ is L^2-orthogonal to $e_{\sigma,0}, e_{\sigma,1}$ and satisfies

$$\begin{cases} \mathscr{L}_\sigma \bar{w} = \bar{f} & \text{on } I_\sigma \times [v_0, v_1], \\ \bar{w}(\,\cdot\,, v_0) = \bar{w}(\,\cdot\,, v_1) = 0. \end{cases} \tag{32}$$

Claim 6.6. *There exist $c \in \mathbb{R}$ and $\sigma_0 \in (0, \pi/2)$ such that, for every $\sigma \in (0, \sigma_0)$, $v_0 \in \mathbb{R}$, $v_1 > v_0 + 1$ and $\bar{f} \in \mathscr{C}^{0,\alpha}_{\mu}(I_\sigma \times [v_0, v_1])$, there exists $\bar{w} \in \mathscr{C}^{2,\alpha}_{\mu}(I_\sigma \times [v_0, v_1])$ that is L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$ and satisfies* (32) *and*

$$\sup_{I_\sigma \times [v_0, v_1]} (e^{-\mu v} |\bar{w}|) \leqslant c \sup_{I_\sigma \times [v_0, v_1]} (e^{-\mu v} |\bar{f}|). \tag{33}$$

Proof. Suppose by contradiction that Claim 6.6 is false. Then, for every $n \in \mathbb{N}$ there exists $\sigma_n \in (0, 1/n)$, $v_{1,n} > v_{0,n} + 1$ and $\bar{f}_n, \bar{w}_n$ satisfying (32) (but with $\sigma_n, v_{0,n}, v_{1,n}$ instead of σ, v_0, v_1) such that

$$\sup_{I_{\sigma_n} \times [v_{0,n}, v_{1,n}]} (e^{-\mu v} |\bar{f}_n|) = 1 \qquad \text{and}$$

$$A_n := \sup_{I_{\sigma_n} \times [v_{0,n}, v_{1,n}]} (e^{-\mu v} |\bar{w}_n|) \to +\infty \quad \text{as } n \to \infty.$$

Since $I_{\sigma_n} \times [v_{0,n}, v_{1,n}]$ is a compact set, A_n is achieved at a point (u_n, v_n) in it.

After passing to a subsequence, the intervals $I_n = [v_{0,n} - v_n, v_{1,n} - v_n]$ converge to a set I_∞. Elliptic estimates imply that

$$\sup_{I_{\sigma_n} \times [v_{0,n}, v_{0,n}+1/2]} (e^{-\mu v} |\nabla \bar{w}_n|) \leqslant c \Bigg(\sup_{I_{\sigma_n} \times [v_{0,n}, v_{0,n}+1]} (e^{-\mu v} |\bar{f}_n|) + \sup_{I_{\sigma_n} \times [v_{0,n}, v_{0,n}+1]} (e^{-\mu v} |\bar{w}_n|) \Bigg).$$

Hence the supremum of $(e^{-\mu v} |\nabla \bar{w}_n|)$ over $I_{\sigma_n} \times [v_{0,n}, v_{0,n} + 1/2]$ is $\leqslant c(1 + A_n)$. From this estimate for the gradient of $\bar{w}_n$ near $v = v_{0,n}$, it follows that v_n cannot be too close to $v_{0,n}$, where $\bar{w}_n$ vanishes. More precisely, $v_{0,n} - v_n$ remains bounded away from 0, and then it converges to some $\bar{v}_0 \in [-\infty, 0)$. By similar arguments, it is possible to show that $\nabla \bar{w}_n$ is bounded near $v_{1,n}$, and consequently $v_{1,n} - v_n$ converges to some $\bar{v}_1 \in (0, +\infty]$. Then we can conclude that $I_\infty = [\bar{v}_0, \bar{v}_1]$.

We define

$$\tilde{w}_n(u,v) := \frac{e^{-\mu v_n}}{A_n}\overline{w}_n(u, v+v_n) \quad \text{for } (u,v)\in I_{\sigma_n}\times I_n.$$

We observe that

$$|\tilde{w}_n(u,v)| \leqslant e^{\mu v}\frac{e^{-\mu(v+v_n)}|\overline{w}_n(u,v+v_n)|}{A_n} \leqslant e^{\mu v},$$

$$\sup_{I_{\sigma_n}\times I_n}(e^{-\mu v}|\tilde{w}_n|) = 1.$$

Using the above estimate for $e^{-\mu v}|\nabla\overline{w}_n|$, we obtain

$$|\nabla\tilde{w}_n| \leqslant c\frac{1+A_n}{A_n}e^{\mu v} < 2c\, e^{\mu v}.$$

Since the sequences $\{\tilde{w}_n\}_n$ and $\{\nabla\tilde{w}_n\}_n$ are uniformly bounded, the Ascoli–Arzelà theorem ensures that, if $n\to+\infty$, a subsequence of $\{\tilde{w}_n\}_n$ converges on compact sets of $I_0\times I_\infty$ to a function $\tilde{w}_\infty$ that vanishes on $I_0\times\partial I_\infty$ when $\partial I_\infty\neq\varnothing$, and such that $\tilde{w}_\infty(\,\cdot\,,v)$ is L^2-orthogonal to $\{e_{0,0}, e_{0,1}\}$ for each $v\in I_\infty$. Moreover,

$$\sup_{I_0\times I_\infty}(e^{-\mu v}|\tilde{w}_\infty|) = 1. \tag{34}$$

Since $\sigma_n\to 0$ as $n\to\infty$, we can conclude that $\tilde{w}_\infty$ satisfies

$$\begin{cases}\mathscr{L}_0\tilde{w}_\infty = 0 & \text{in } I_0\times I_\infty,\\ \quad\tilde{w}_\infty = 0 & \text{on } I_0\times\partial I_\infty \text{ (if } \partial I_\infty\neq\varnothing).\end{cases}$$

If I_∞ is bounded, the maximum principle allows us to conclude that $\tilde{w}_\infty = 0$ on $I_0\times I_\infty$, which contradicts (34). Hence I_∞ is an unbounded interval.

Recall $\mathscr{L}_0$ is given in terms of the (x,y) variables. The equation $\mathscr{L}_0\tilde{w}_\infty = 0$ becomes

$$\partial^2_{xx}\tilde{w}_\infty + \sin y\,\partial_y(\sin y\,\partial_y\tilde{w}_\infty) + 2\sin^2 y\,\tilde{w}_\infty = 0.$$

Now we consider $\tilde{w}_\infty$ decomposed into eigenfunctions as

$$\tilde{w}_\infty(x,y) = \sum_{j\geqslant 2} a_j(y)\cos(j\,x).$$

Each coefficient a_j with $j\geqslant 2$ must satisfy the associated Legendre differential equation (see Appendix C)

$$\sin y\,\partial_y(\sin y\,\partial_y a_j) - j^2 a_j + 2\sin^2 y\, a_j = 0.$$

We obtain that $a_j(y)$ is the associated Legendre functions of second kind, that is, $a_j(y) = Q_1^j(\cos y)$ for $j\geqslant 2$.

We obtain from (29) that

$$u(x)\to x \quad\text{and}\quad v(y)\to \tfrac{1}{2}\ln|\tan(y/2)| \quad \text{as } \sigma\to 0.$$

In particular, define $y(v) = 2\arctan(e^{2v})$ for $\sigma = 0$. Then

$$\cos(y(v)) = \frac{1-e^{4v}}{1+e^{4v}},$$
$$\tilde{w}_\infty(u,v) = \sum_{j\geqslant 2} Q_1^j\Big(\frac{1-e^{4v}}{1+e^{4v}}\Big)\cos(ju).$$

One can show that $|a_j|$ tends to $+\infty$ as the function $e^{2j|v|}$ does. Since the interval I_∞ is unbounded, we reach a contradiction with (34), proving Claim 6.6. □

Let $c \in \mathbb{R}$ and σ_0 satisfy Claim 6.6. Choose $\sigma \in (0,\sigma_0)$, $v_0 \in \mathbb{R}$ and then an $\bar{f} \in \mathscr{C}_\mu^{0,\alpha}(I_\sigma \times [v_0,+\infty))$. Then, for every $v_1 > v_0+1$, there exists a function $\overline{w}$ that is L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$ and satisfies (32) and (33). Let's take the limit as $v_1 \to \infty$. Clearly

$$e^{-\mu v}|\overline{w}| \leqslant \|\overline{w}\|_{\mathscr{C}_\mu^{0,\alpha}(I_\sigma\times[v_0,v_1])} \leqslant c\|\bar{f}\|_{\mathscr{C}_\mu^{0,\alpha}(I_\sigma\times[v_0,v_1])}.$$

And using Schauder estimates, we get

$$\begin{aligned} e^{-\mu v}|\nabla\overline{w}| &\leqslant \|\overline{w}\|_{\mathscr{C}_\mu^{2,\alpha}(I_\sigma\times[v_0,v_1])} \\ &\leqslant c_1\big(\|\bar{f}\|_{\mathscr{C}_\mu^{0,\alpha}(I_\sigma\times[v_0,v_1])} + \|\overline{w}\|_{\mathscr{C}_\mu^{0,\alpha}(I_\sigma\times[v_0,v_1])}\big) \leqslant c_2\|\bar{f}\|_{\mathscr{C}_\mu^{0,\alpha}(I_\sigma\times[v_0,v_1])}. \end{aligned}$$

Hence the Ascoli–Arzelà theorem ensures that a subsequence of $\{\overline{w}_{v_1}\}_{v_1>v_0+1}$ converges to a function $\overline{w}$ defined on $I_\sigma \times [v_0,+\infty)$, which satisfies

$$\sup_{I_\sigma\times[v_0,+\infty)} e^{-\mu v}|\overline{w}| \leqslant c \sup_{I_\sigma\times[v_0,+\infty)} e^{-\mu v}|\bar{f}|.$$

Using again elliptic estimates we can conclude that $\overline{w}$ satisfies the statement of Proposition 6.5. The uniqueness of the solution follows from Lemma 6.4.

Step 2. Let's now consider $f \in \mathscr{C}_\mu^{0,\alpha}(I_\sigma \times [v_0,+\infty))$ in Span$\{e_{\sigma,0}, e_{\sigma,1}\}$, that is,

$$f(u,v) = f_0(v)e_{\sigma,0}(u) + f_1(v)e_{\sigma,1}(u).$$

We extend the functions $f_0(v)$ and $f_1(v)$ for $v \leqslant v_0$ by $f_0(v_0)$ and $f_1(v_0)$, respectively. Given $v_1 > v_0+1$, consider the problem

$$\begin{cases} L_{\sigma,j}w_j = f_j & \text{in } (-\infty, v_1], \\ w_j(v_1) = \partial_v w_j(v_1) = 0. \end{cases} \tag{35}$$

The Cauchy–Lipschitz theorem and the linearity of the equation ensure the existence and the uniqueness of the solution w_j. We aim to prove the following result.

Claim 6.7. *For some constant c that does not depend on v_1,*

$$\sup_{(-\infty,v_1]} (e^{-\mu v}|w_j|) \leqslant c \sup_{(-\infty,v_1]} (e^{-\mu v}|f_j|).$$

Proof. Suppose to the contrary that for every $n \in \mathbb{N}$ there exists $\sigma_n \in (0, 1/n)$, $v_{1,n} > v_{0,n} + 1$ and $f_{j,n}$, $w_{j,n}$ satisfying (35) such that

$$\sup_{(-\infty, v_{1,n}]} (e^{-\mu v} |\bar{f}_{j,n}|) = 1,$$

$$A_n := \sup_{(-\infty, v_{1,n}]} (e^{-\mu v} |w_{j,n}|) \to +\infty \quad \text{as } n \to \infty.$$

The solution $w_{j,n}$ of the previous equation is a linear combination of the two solutions of the homogeneous problem $L_{\sigma_n, j} w = 0$. They grow at most linearly at ∞ (recall that the Jacobi fields have this rate of growth). Hence the supremum A_n is achieved at a point $v_n \in (-\infty, v_{1,n}]$. We define on $I_n := (-\infty, v_{1,n} - v_n]$ the function

$$\tilde{w}_{j,n}(v) := \frac{e^{-\mu v_n}}{A_n} w_{j,n}(v_n + v).$$

As in Step 1, one shows that the sequence $\{v_{1,n} - v_n\}_n$ remains bounded away from 0 and, after passing to a subsequence, it converges to $\bar{v}_1 \in (0, +\infty]$, and $\{\tilde{w}_{j,n}\}_n$ converges on compact subsets of $I_\infty = (-\infty, \bar{v}_1]$ to a nontrivial function $\tilde{w}_j$ such that

$$\sup_{I_\infty} (e^{-\mu v} |\tilde{w}_j|) = 1 \tag{36}$$

and $\tilde{w}_j(\bar{v}_1) = \partial_v w_j(\bar{v}_1) = 0$ if $\bar{v}_1 < +\infty$. The function $\tilde{w}_j$ solves a second order ordinary differential equation given, in terms of the (x, y) variables, by

$$\sin y\, \partial_y (\sin y\, \partial_y \tilde{w}_j) - j^2\, \tilde{w}_j + 2 \sin^2 y\, \tilde{w}_j = 0. \tag{37}$$

If $\bar{v}_1 < +\infty$, then $\tilde{w}_j = 0$, and this contradicts (36). In the case $\bar{v}_1 = +\infty$ we will try to reach a contradiction by determining the solution of (37). This is again the associated Legendre differential equation; see Appendix C. The solutions of (37) are linear combinations of the associated Legendre functions of first and second kind: $P_1^j(\cos y)$ and $Q_1^j(\cos y)$ for $j = 0, 1$. Specifically, $P_1^0(\cos y) = \cos y$ and $P_1^1(\cos y) = -\sin y$. We change coordinates to express $\tilde{w}_j$ in terms of the (u, v) variables. As $v \to \pm\infty$, one can show that $|Q_1^1(\cos y(v))|$ and $|Q_1^0(\cos y(v))|$ tend to ∞ as $e^{2|v|}$ and $|v|$, respectively. We conclude that the functions $\tilde{w}_1$ and $\tilde{w}_0$ do not satisfy (36) with $\mu \in (-2, -1)$, a contradiction. □

Therefore, $\sup_{(-\infty, v_1]} (e^{-\mu v} |w_j|) \leqslant c \sup_{(-\infty, v_1]} (e^{-\mu v} |f_j|)$. Taking $v_1 \to +\infty$, we get a solution of $L_{\sigma, j}\, w_j = f_j$ defined in $[v_0, +\infty)$ that satisfies

$$\sup_{[v_0, +\infty)} (e^{-\mu v} |w_j|) \leqslant c \sup_{[v_0, +\infty)} (e^{-\mu v} |f_j|).$$

Elliptic estimates allow us to obtain the desired estimates for the derivatives. To prove the uniqueness of solution, it suffices to observe that no solution of $\mathscr{L}_\sigma v = 0$

that is collinear with $e_{\sigma,0}$ and $e_{\sigma,1}$ decays exponentially at ∞. This fact follows from the behavior of the Jacobi fields. □

Remark 6.8. The results proved in this section also follow from considering not $\widetilde{M}_{\sigma,\alpha,0}$ but $\widetilde{M}_{\sigma,0,\beta}$: It is invariant by reflection about the $\{x_1 = 0\}$ plane. To keep such a symmetry, we work with functions that are odd (and not even) in the variable u. Hence $\mathscr{C}^{\ell,\alpha}_{\mu}(I_\sigma \times I)$ will be, in this case, the space of functions $w = w(u, v)$ in $\mathscr{C}^{\ell,\alpha}_{\text{loc}}(I_\sigma \times I)$ that are odd and U_σ-periodic in the variable u, and for which the norm $\|w\|_{\mathscr{C}^{\ell,\alpha}_{\mu}(I_\sigma\times I)}$ is finite. Also, we replace in the above results $e_{0,i}(u) = \cos(ix(u))$ by $\widetilde{e}_{0,i}(u) = \sin(ix(u))$, and $e_{\sigma,i}$ by the normalized odd eigenfunction $\widetilde{e}_{\sigma,i}$ associated to the eigenvector $\lambda_{\sigma,i}$ of the operator L_σ.

7. A family of minimal surfaces close to $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,\alpha,0}$

The aim of this section is to find a family of minimal surfaces near conveniently translated and dilated pieces of $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,\alpha,0}$, with given Dirichlet data on the boundary.

We denote by Z the immersion of the surface $\widetilde{M}_{\sigma,\alpha,\beta}$. The following proposition, proved in Appendix B, states that the linearization of the mean curvature operator about $\widetilde{M}_{\sigma,\alpha,\beta}$ is the Lamé operator $\mathscr{L}_\sigma$ introduced in Section 5.2; see (30).

Proposition 7.1. *The surface parameterized by $Z_f := Z + fN$ is minimal if and only if the function f is a solution of*

$$\mathscr{L}_\sigma f = Q_\sigma(f)$$

where Q_σ is a nonlinear operator that satisfies

$$(38)\quad \|Q_\sigma(f_2) - Q_\sigma(f_1)\|_{\mathscr{C}^{0,\alpha}(I_\sigma\times[v,v+1])} \leqslant c\Big(\sup_{i=1,2}\|f_i\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\Big)\|f_2 - f_1\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}$$

for all functions f_1, f_2 such that $\|f_i\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])} \leqslant 1$. Here the constant $c > 0$ depends neither on $v \in \mathbb{R}$ nor on $\sigma \in (0, \pi/2)$.

In Section 5.1 (see Lemma 5.3) we have written annular pieces of $M_{\sigma,\alpha,0}(\gamma, \xi)$ and $M_{\sigma,0,\beta}(\gamma, \xi)$ as vertical graphs over an annular neighborhood of $\{r = r_\varepsilon\}$ in $\{x_3 = 0\}$ of the functions

$$(39)\quad \widetilde{U}^{\alpha}_{\gamma,\xi_1}(r,\theta) = (1+\gamma)\ln(2r) - 2r\sin\tfrac{1}{2}\alpha\cos\theta - \frac{1+\gamma}{r}\xi_1\cos\theta + d + \mathbb{O}(\varepsilon),$$

$$(40)\quad \widetilde{U}^{\beta}_{\gamma,\xi_2}(r,\theta) = (1+\gamma)\ln(2r) + 2r\sin\tfrac{1}{2}\beta\sin\theta - \frac{1+\gamma}{r}\xi_2\sin\theta + d + \mathbb{O}(\varepsilon),$$

respectively, where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\gamma, \xi_1, \xi_2, \xi_3$ are small. We now truncate the surfaces $\widetilde{M}_{\sigma,\alpha,0}(\gamma, \xi)$ and $\widetilde{M}_{\sigma,0,\beta}(\gamma, \xi)$ at their respective graph curves over

$\{r = r_\varepsilon\}$. We only consider the upper half of these surfaces, which we call M_1 and M_2, respectively. We are interested those minimal normal graphs over M_1 and M_2 that are asymptotic to them, and whose boundary is prescribed.

As a consequence of the dilation of the surfaces by the factor $1+\gamma$, the minimal surface equation becomes

$$\mathscr{L}_\sigma w = \frac{1}{1+\gamma} Q_\sigma\left((1+\gamma)w\right). \tag{41}$$

That is, the normal graph of a function w over the dilated $\widetilde{M}_{\sigma,\alpha,\beta}$ is minimal if and only if w is a solution of (41).

Two more modifications are required: In Lemma 5.5 we showed that the value of the variable v corresponding to $r = r_\varepsilon$ is $v_\varepsilon = -\frac{1}{2}\ln\varepsilon + \mathbb{O}(1)$. Since we are working in the (u, v) variables, we would like to parameterize M_i in $I_\sigma \times [v_\varepsilon, +\infty]$ for $i = 1, 2$. But the boundary of M_i does not correspond to the curve $\{v = v_\varepsilon\}$. We therefore modify the parameterization so that it remains fixed for $v \geqslant v_\varepsilon + \ln 4$, while requiring, in a small annular neighborhood of $\{v = v_\varepsilon\}$, that the curves $\{v = \text{const}\}$ correspond to the vertical graphs of curves $\{r = \text{const}\}$ by the corresponding function (39) or (40). We also want the normal vector field relative to M_i to be vertical near its boundary. This can be achieved by modifying the normal vector field into a transverse vector field $\tilde{N}$ that agrees with N when $v \geqslant v_\varepsilon + \ln 4$, and with e_3 when $v \in [v_\varepsilon, v_\varepsilon + \ln 2]$.

We consider a graph of some function w over M_i, using the modified vector field $\tilde{N}$. This graph will be minimal if and only if the function w is a solution of a nonlinear elliptic equation related to (41). To get the new equation, we take into account the effects of the change of parameterization and the change of the vector field N into $\tilde{N}$. The new minimal surface equation is

$$\mathscr{L}_\sigma w = \tilde{L}_\varepsilon w + \tilde{Q}_\sigma\left(w\right). \tag{42}$$

Here $\tilde{Q}_\sigma$ enjoys the same properties as Q_σ, since it is obtained by a slight perturbation from it. The operator $\tilde{L}_\varepsilon$ is a linear second order operator whose coefficients are supported in $I_\sigma \times [v_\varepsilon, v_\varepsilon + \ln 4]$ and are bounded in the $\mathscr{C}^\infty$ topology by a constant multiplied by $\sqrt{\varepsilon}$, where partial derivatives are computed with respect to the vector fields ∂_u and ∂_v. In fact, if we take into account the effect of the change of the normal vector field, we would obtain by applying the result of [Hauswirth and Pacard 2007, Appendix B] a similar formula in which the coefficients of the corresponding operator $\tilde{L}_\varepsilon$ are bounded by a constant multiplied by ε, since

$$\tilde{N}_\varepsilon \cdot N_\varepsilon = 1 + \mathbb{O}_{\mathscr{C}_b^{2,\alpha}}(\varepsilon) \quad \text{when } v \in [v_\varepsilon, v_\varepsilon + \ln 2].$$

If we take into account the effect of the change in the parameterization, we would obtain a similar formula in which the coefficients of the corresponding operator $\tilde{L}_\varepsilon$

are bounded by a constant multiplied by $\sqrt{\varepsilon}$. The estimate of the coefficients of $\tilde{L}_\varepsilon$ follows from these considerations.

Now we will give a detailed proof of the existence of a family of minimal graphs about M_1 and asymptotic to it. Recall that M_1 is invariant by reflection across the $\{x_2 = 0\}$ plane. The normal graph of the function $w = w(u, v)$ over M_1 inherits the same symmetry property if w is even in the u variable. The corresponding results for M_2 are obtained similarly, considering odd functions instead of even ones.

We consider a function $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ that is even and L^2-orthogonal to $e_{0,0}, e_{0,1}$ and that satisfies

$$\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon, \tag{43}$$

where $\kappa > 0$ is a constant. We define $w_\varphi(u', v) := \overline{\mathscr{H}}_{v_\varepsilon,\varphi}$, where $\overline{\mathscr{H}}_{v_\varepsilon,\varphi}$ is the harmonic extension introduced in Proposition A.5. If $u = (2\pi/U_\sigma)u'$, then $w_\varphi(u, v)$ belongs to $\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$, and $w_\varphi(\,\cdot\,, v_\varepsilon) \in \mathscr{C}^{2,\alpha}(I_\sigma)$ is even and L^2-orthogonal to $e_{\sigma,0}, e_{\sigma,1}$. To solve Equation (42), we choose $\mu \in (-2, -1)$ and look for $w \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$ of the form $w = w_\varphi + g$ for some $g \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$. Using Proposition 6.5, we can rephrase this problem as a fixed point problem

$$g = S(\varphi, g) := G_{\varepsilon,v_\varepsilon}\big(\tilde{L}_\varepsilon(w_\varphi + g) - \mathscr{L}_\sigma w_\varphi + \tilde{Q}_\sigma(w_\varphi + g)\big). \tag{44}$$

where the nonlinear mapping S depends on σ, ε, γ, and operator $G_{\varepsilon,v_\varepsilon}$ is as defined in Proposition 6.5. To prove the existence of a fixed point for (44), we need the next lemma. We will abbreviate by writing $\|\cdot\|_{\mathscr{C}^{2,\alpha}_\mu}$ instead of $\|\cdot\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))}$.

Proposition 7.2. *Let* $0 < \sigma \leqslant \varepsilon$, $\mu \in (-2, -1)$. *Suppose* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *satisfies* (43) *and enjoys the properties given above. Then there exist some constants* $c_\kappa > 0$ *and* $\varepsilon_\kappa > 0$ *such that*

$$\|S(\varphi, 0)\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))} \leqslant c_\kappa \varepsilon^{(3+\mu)/2} \tag{45}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$\begin{aligned} \|S(\varphi, g_2) - S(\varphi, g_1)\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))} &\leqslant \tfrac{1}{2}\|g_2 - g_1\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))}, \\ \|S(\varphi_2, g) - S(\varphi_1, g)\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))} &\leqslant c\varepsilon^{\frac{1}{2}+\mu/2}\|\varphi_2 - \varphi_1\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \end{aligned} \tag{46}$$

for all $g, g_1, g_2 \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$ *such that* $\|g_i\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant 2c_\kappa\,\varepsilon^{(3+\mu)/2}$, *and all* $\varphi_1, \varphi_2 \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *enjoying the same properties as* φ.

Proof. We know from Proposition 6.5 that $\|G_{\varepsilon,v_\varepsilon}(f)\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant c\|f\|_{\mathscr{C}^{0,\alpha}_\mu}$ for some $c > 0$ (throughout the proof, c will denote an arbitrary positive constant). Then

$$\begin{aligned} \|S(\varphi, 0)\|_{\mathscr{C}^{2,\alpha}_\mu} &\leqslant c\|\tilde{L}_\varepsilon w_\varphi - \mathscr{L}_\sigma\, w_\varphi + \tilde{Q}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} \\ &\leqslant c\big(\|\tilde{L}_\varepsilon w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} + \|\mathscr{L}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} + \|\tilde{Q}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu}\big). \end{aligned}$$

So we need to estimate the three terms above.

In the proof of Proposition A.5 we obtain that, for every $v \in [v_\varepsilon, +\infty)$,

$$\|w_\varphi\|_{\mathscr{C}^{2,\alpha}(I_\sigma \times [v,v+1])} \leqslant c e^{-2(v-v_\varepsilon)} \|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}.$$

Therefore,

$$\begin{aligned}\|w_\varphi\|_{\mathscr{C}^{2,\alpha}_\mu} &= \sup_{v\in[v_\varepsilon,+\infty)} \left(e^{-\mu v}\|w_\varphi\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\right)\\ &\leqslant c \sup_{v\in[v_\varepsilon,+\infty)} (e^{-\mu v-2(v-v_\varepsilon)})\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant c e^{-\mu v_\varepsilon}\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa c \varepsilon^{1+\mu/2}.\end{aligned}$$

From this inequality and the estimates of the coefficients of $\tilde{L}_\varepsilon$, it follows that

$$\|\tilde{L}_\varepsilon(w_\varphi)\|_{\mathscr{C}^{0,\alpha}_\mu} \leqslant c\varepsilon^{1/2}\|w_\varphi\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant \kappa c \varepsilon^{(3+\mu)/2}.$$

Since w_φ is an harmonic function, the definition of $\mathscr{L}_\sigma$ in (30) gives the equality

$$\mathscr{L}_\sigma w_\varphi = 2k(u,v) w_\varphi.$$

Recall (see Lemma 5.5) that if $v \geqslant v_\varepsilon$, then $y(v) \geqslant \pi - a_\varepsilon$, where $a_\varepsilon = \mathbb{O}(\sqrt{\varepsilon})$. From the facts that if $|y(v) - \pi| \leqslant a_\varepsilon$, then

$$k(u,v) = \sin^2\sigma \cos^2(x(u)) + \cos^2\sigma \sin^2(y(v)) \leqslant \sin^2\sigma + \sin^2(a_\varepsilon) \leqslant c\varepsilon$$

and that w_φ is an exponentially decaying function, we conclude that

$$\|\mathscr{L}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} \leqslant c\varepsilon\|w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} \leqslant \kappa c\varepsilon^{2+\mu/2}.$$

Finally, $\|\tilde{Q}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}(I_\sigma\times[v,v+1])} \leqslant c\|w_\varphi\|^2_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}$, so

$$\|\tilde{Q}_\sigma w_\varphi\|_{\mathscr{C}^{0,\alpha}_\mu} \leqslant c \sup_{v\in[v_\varepsilon,+\infty)} \left(e^{-\mu v}\|w_\varphi\|^2_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\right) \leqslant c\,\|w_\varphi\|^2_{\mathscr{C}^{2,\alpha}_{\mu/2}} \leqslant \kappa^2 c\,\varepsilon^{2+\mu/2}.$$

Putting together these estimates, we get (45). The details of other the estimates are left to the reader. □

Theorem 7.3. *Consider* $0 < \sigma \leqslant \varepsilon$, $\mu \in (-2,-1)$ *and* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *as above. We define* $B := \{g \in \mathscr{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty)) : \|g\|_{\mathscr{C}^{2,\alpha}_\mu} \leqslant 2c_k\varepsilon^{(3+\mu)/2}\}$. *Then the nonlinear mapping* $S(\varphi,\cdot)$ *has a unique fixed point* g *in* B.

Proof. The previous proposition shows that if ε is chosen small enough, the nonlinear mapping $S(\varphi,\cdot)$ is a contraction mapping from B into itself. Hence Schauder's theorem ensures that $S(\varphi,\cdot)$ has a fixed point g in B. □

Theorem 7.3 provides, for each even function $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ L^2-orthogonal to $e_{0,0}, e_{0,1}$ with $\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon$, a minimal surface $S_{t,\alpha,\gamma,\xi,d}(\varphi)$ close to M_1 (the subindex t reflects the fact we are considering the upper half of $\widetilde{M}_{\sigma,\alpha,0}(\gamma,\xi)$). In a neighborhood of its boundary, this surface can be written as a vertical graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon} \subset \{x_3 = 0\}$ of the function

$$\overline{U}_{t,1}(r,\theta) = (1+\gamma)\ln(2r) - 2r\sin\tfrac{1}{2}(\alpha)\cos\theta - \frac{1+\gamma}{r}\xi\cos\theta + d + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \overline{V}_{t,1}(r,\theta). \tag{47}$$

The function $\overline{V}_{t,1} = \overline{V}_{t,1}(\gamma,\varphi)$ depends nonlinearly on γ and φ, and there exists a $c > 0$ such that

$$\begin{aligned} &\|\overline{V}_{t,1}(\gamma,\varphi)(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\varepsilon,\\ &\|\overline{V}_{t,1}(\gamma,\varphi_1)(r_\varepsilon\,\cdot\,,\cdot) - \overline{V}_{t,1}(\gamma,\varphi_2)(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_1)}\\ &\qquad\qquad \leqslant c\varepsilon^{1/2}\|\varphi_1 - \varphi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)}, \end{aligned} \tag{48}$$

for all even functions $\varphi, \varphi_1, \varphi_2 \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ that are L^2-orthogonal to $e_{0,0}, e_{0,1}$ and whose $\mathscr{C}^{2,\alpha}$-norms are bounded above by $\kappa\varepsilon$. The latter estimate follows from estimate (46) and

$$\begin{aligned} &\|\overline{V}_{t,1}(\gamma,\varphi_1)(r_\varepsilon\,\cdot\,,\cdot) - \overline{V}_{t,1}(\gamma,\varphi_2)(r_\varepsilon\,\cdot\,,\cdot)\|_{\mathscr{C}^{2,\alpha}(\bar{B}_2 - B_1)}\\ &\qquad\qquad \leqslant e^{\mu v_\varepsilon}\|S(\varphi_1,\overline{V}_{t,1}) - S(\varphi_2,\overline{V}_{t,1})\|_{\mathscr{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}. \end{aligned}$$

The boundary of $S_{t,\alpha,\gamma,\xi,d}(\varphi)$ corresponds to the image by $\overline{U}_{t,1}$ of $\{r = r_\varepsilon\}$.

Similar arguments can be followed for the lower half of $\widetilde{M}_{\sigma,\alpha,0}(\gamma,\xi)$, and we obtain a minimal surface $S_{b,\alpha,\gamma,\xi,d}(\varphi)$ close to such a half of $\widetilde{M}_{\sigma,\alpha,0}(\gamma,\xi)$, which can be written in a neighborhood of its boundary as a vertical graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the function

$$\overline{U}_{b,1}(r,\theta) = -(1+\gamma)\ln(2r) - 2r\sin\tfrac{1}{2}\alpha\cos\theta - \frac{1+\gamma}{r}\xi\cos\theta + d + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \overline{V}_{b,1}(r,\theta), \tag{49}$$

where the function $\overline{V}_{b,1} = \overline{V}_{b,1}(\gamma,\varphi)$ enjoys the same properties as $\overline{V}_{t,1}$. The boundary of $S_{b,\alpha,\gamma,\xi,d}(\varphi)$ corresponds to the image by $\overline{U}_{b,1}$ of $\{r = r_\varepsilon\}$.

Analogously, for an odd function $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ that is L^2-orthogonal to $\widetilde{e}_{0,0}, \widetilde{e}_{0,1}$ (see Remark 6.8) and that satisfies $\|\varphi\|_{\mathscr{C}^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon$, we obtain minimal surfaces $\widetilde{S}_{t,\beta,\gamma,\xi,d}(\varphi)$ and $\widetilde{S}_{b,\beta,\gamma,\xi,d}(\varphi)$ near the upper and lower half of $\widetilde{M}_{\sigma,0,\beta}(\gamma,\xi)$ that can be written in a neighborhood of their boundary as vertical graphs over the annulus

$B_{2r_\varepsilon} - B_{r_\varepsilon}$ respectively of the functions

$$\overline{U}_{t,2}(r,\theta) = (1+\gamma)\ln(2r) + 2r\sin\tfrac{1}{2}\beta\sin\theta - \frac{1+\gamma}{r}\xi\sin\theta + d + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \overline{V}_{t,2}(r,\theta),$$

$$\overline{U}_{b,2}(r,\theta) = -(1+\gamma)\ln(2r) + 2r\sin\tfrac{1}{2}\beta\sin\theta - \frac{1+\gamma}{r}\xi\sin\theta + d + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r,\theta) + \overline{V}_{b,2}(r,\theta),$$

where the functions $\overline{V}_{t,2} = \overline{V}_{t,2}(\gamma,\varphi)$ and $\overline{V}_{b,2} = \overline{V}_{b,2}(\gamma,\varphi)$ enjoy the same properties as $\overline{V}_{t,1}$. Their respective boundaries correspond to the image by $\overline{U}_{t,2}$ and $\overline{U}_{b,2}$ of $\{r = r_\varepsilon\}$.

8. The matching of Cauchy data

In this section we shall complete the proof of Theorems 1.1, 1.2 and 1.3.

8.1. ***Proof of Theorem 1.2.*** The proof is articulated in two distinct parts: the proof of the existence of the family $\mathcal{K}_1$ and of the existence of the family $\mathcal{K}_2$.

We start with the second. Its proof is based on an analytical gluing procedure. The surfaces in the family $\mathcal{K}_2$ are symmetric about the plane $\{x_2 = 0\}$, so all the surfaces involved in the proof must have the same property. We will show how to glue a compact piece of a Costa–Hoffman–Meeks-type surface with bent catenoidal end to two halves of the KMR example $\widetilde{M}_{\sigma,\alpha,0}$ along the upper and lower boundaries and to a horizontal periodic flat annulus with a disk removed along the middle boundary. All the surfaces just mentioned have the desired symmetry, as do the surfaces obtained from them by slight perturbation. We recall below the necessary results proved in previous sections.

As we have seen in Section 3, we can construct a minimal surface $\overline{M}^T_{k,\varepsilon}(\varepsilon/2,\Psi)$, with $\Psi = (\psi_t,\psi_b,\psi_m)$, that is close to a truncated genus k Costa–Hoffman–Meeks surface M_k and has three boundaries. The functions $\psi_t,\psi_b,\psi_m \in C^{2,\alpha}(\mathbb{S}^1)$ are even. Also, ψ_m is L^2-orthogonal to 1, and ψ_t and ψ_b are L^2-orthogonal to 1 and to $\cos\theta$. Close to its upper, lower and middle boundaries, the surface $\overline{M}^T_{k,\varepsilon}(\varepsilon/2,\Psi)$ is a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$, respectively, of the functions

$$U_t(r,\theta) = \sigma_t + \ln(2r) - \tfrac{1}{2}\varepsilon r\cos\theta + H_{\psi_t}(s_\varepsilon - \ln(2r),\theta) + \mathbb{O}_{C^{2,\alpha}_b}(\varepsilon),$$
$$U_b(r,\theta) = -\sigma_b - \ln(2r) - \tfrac{1}{2}\varepsilon r\cos\theta + H_{\psi_b}(s_\varepsilon - \ln(2r),\theta) + \mathbb{O}_{C^{2,\alpha}_b}(\varepsilon),$$
$$U_m(r,\theta) = \tilde{H}_{\rho_\varepsilon,\psi_m}(1/r,\theta) + \mathbb{O}_{C^{2,\alpha}_b}(\varepsilon),$$

where $s_\varepsilon = -\ln\sqrt{\varepsilon}$ and $\rho_\varepsilon = 2\sqrt{\varepsilon}$; see Equations (14), (15) and (16).

Using the results of Section 7 we can show the existence of a minimal surface $S_{t,\alpha_t,\gamma_t,\xi_t,d_t}(\varphi_t)$ near the upper half of the KMR example $\widetilde{M}_{\sigma_t,\alpha_t,0}$, and asymptotic to it. Near its boundary, this surface can be parameterized over $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of (see (47))

$$\overline{U}_t(r,\theta) = (1+\gamma_t)\ln(2r) - 2r\sin\tfrac{1}{2}\alpha_t\cos\theta - \frac{1+\gamma_t}{r}\xi_t\cos\theta \\ + d_t + \overline{\mathcal{H}}_{v_\varepsilon,\varphi_t}(\ln 2r,\theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon).$$

We recall that $\varphi_t \in C^{2,\alpha}(\mathbb{S}^1)$ is an even function L^2-orthogonal to 1 and to $\cos\theta$. The surface $S_{t,\alpha_t,\gamma_t,\xi_t,d_t}(\varphi_t)$ will be glued to the upper boundary of $\overline{M}^T_{k,\varepsilon}(\varepsilon/2,\Psi)$.

Near its boundary, the surface $S_{b,\alpha_b,\gamma_b,\xi_b,d_b}(\varphi_b)$ that will be glued along the lower boundary of $\overline{M}^T_{k,\varepsilon}(\varepsilon/2,\Psi)$ can be parameterized in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\overline{U}_b(r,\theta) = -(1+\gamma_b)\ln(2r) - 2r\sin\tfrac{1}{2}\alpha_b\cos\theta - \frac{1+\gamma_b}{r}\xi_b\cos\theta \\ + d_b + \overline{\mathcal{H}}_{v_\varepsilon,\varphi_b}(\ln 2r,\theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon);$$

see (49). Recall that we assumed $\varphi_b \in C^{2,\alpha}(\mathbb{S}^1)$ to be an even function that is L^2-orthogonal to 1 and to $\cos\theta$.

Using the results of Section 4, we can construct a minimal graph $S_m(\varphi_m)$ close to a horizontal periodic flat annulus with a disk removed. Here $\varphi_m \in C^{2,\alpha}(\mathbb{S}^1)$ is an even function L^2-orthogonal to 1. In a neighborhood of its boundary, it can be parameterized (see (21)) as the vertical graph over $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of $\overline{U}_m(r,\theta) = \tilde{H}_{r_\varepsilon,\varphi_m}(r,\theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$.

The functions $\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$ in the formulas above replace the functions V_t, V_b, V_m, $\overline{V}_t$, $\overline{V}_b$ and $\overline{V}_m$ that appear in Equations (14), (15), (16), (47), (49) and (21). They depend nonlinearly on the different parameters and boundary data, but they are bounded by a constant times ε in the $C_b^{2,\alpha}$ topology, where partial derivatives are taken with respect to the vector fields $r\partial_r$ and ∂_θ.

We assume that the parameters and the boundary functions are chosen so that

$$(50)\quad |\gamma_t| + |\gamma_b| + |-\gamma_t\ln\sqrt{\varepsilon} + \eta_t| + |\gamma_b\ln\sqrt{\varepsilon} + \eta_b| \\ + (4\sqrt{\varepsilon})^{-1}\big|-4\sin(\alpha_t/2) + \varepsilon\big| + (4\sqrt{\varepsilon})^{-1}\big|-4\sin(\alpha_b/2) + \varepsilon\big| \\ + 2\sqrt{\varepsilon}(|(1+\gamma_t)\xi_t| + |(1+\gamma_b)\xi_b|) \\ + \|\varphi_t\|_{C^{2,\alpha}(\mathbb{S}^1)} + \|\varphi_b\|_{C^{2,\alpha}(\mathbb{S}^1)} + \|\varphi_m\|_{C^{2,\alpha}(\mathbb{S}^1)} \\ + \|\psi_t\|_{C^{2,\alpha}(\mathbb{S}^1)} + \|\psi_b\|_{C^{2,\alpha}(\mathbb{S}^1)} + \|\psi_m\|_{C^{2,\alpha}(\mathbb{S}^1)} \leqslant \kappa\varepsilon,$$

where $\eta_t = d_t - \sigma_t$ and $\eta_b = d_b + \sigma_b$ for some fixed constant $\kappa > 0$ large enough.

It remains to show that, for all ε small enough, it is possible to choose the parameters and boundary functions so that the surface

$$M^T_k(\varepsilon/2,\Psi) \cup S_{t,\alpha_t,\gamma_t,\xi_t,d_t}(\varphi_t) \cup S_{b,\alpha_b,\gamma_b,\xi_b,d_b}(\varphi_b) \cup S_m(\varphi_m)$$

is a C^1 surface across the boundaries of the different summands. Regularity theory will then ensure that this surface is in fact smooth, and then by construction it has the desired properties. This will therefore complete the proof of the existence of the family of examples $\mathcal{K}_2$.

It is necessary to fulfill the following system of equations on $\mathbb{S}^1$:

$$\begin{cases} U_t(r_\varepsilon,\cdot)=\overline{U}_t(r_\varepsilon,\cdot), & \partial_r U_t(r_\varepsilon,\cdot)=\partial_r\overline{U}_t(r_\varepsilon,\cdot),\\ U_b(r_\varepsilon,\cdot)=\overline{U}_b(r_\varepsilon,\cdot), & \partial_r U_b(r_\varepsilon,\cdot)=\partial_r\overline{U}_b(r_\varepsilon,\cdot),\\ U_m(r_\varepsilon,\cdot)=\overline{U}_m(r_\varepsilon,\cdot), & \partial_r U_m(r_\varepsilon,\cdot)=\partial_r\overline{U}_m(r_\varepsilon,\cdot).\end{cases}$$

The left three equations lead to the system

$$\begin{cases} \gamma_t\ln(2r_\varepsilon)+\eta_t-(1+\gamma_t)(\xi_t/r_\varepsilon)\cos\theta \\ \qquad\qquad +r_\varepsilon(-2\sin\tfrac12\alpha_t+\tfrac12\varepsilon)\cos\theta+\varphi_t-\psi_t=\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon),\\ -\gamma_b\ln(2r_\varepsilon)+\eta_b-(1+\gamma_b)(\xi_b/r_\varepsilon)\cos\theta \\ \qquad\qquad +r_\varepsilon(-2\sin\tfrac12\alpha_b+\tfrac12\varepsilon)\cos\theta+\varphi_b-\psi_b=\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)\\ \qquad\qquad\qquad\qquad \varphi_m-\psi_m=\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon).\end{cases} \tag{51}$$

The right three equations give the system

$$\begin{cases} \gamma_t+(1+\gamma_t)(\xi_t/r_\varepsilon)\cos\theta \\ \qquad\qquad +r_\varepsilon(-2\sin\tfrac12\alpha_t+\tfrac12\varepsilon)\cos\theta+\partial_\theta^*(\varphi_t+\psi_t)=\mathbb{O}_{C_b^{1,\alpha}}(\varepsilon),\\ -\gamma_b+(1+\gamma_b)(\xi_b/r_\varepsilon)\cos\theta \\ \qquad\qquad +r_\varepsilon(-2\sin\tfrac12\alpha_b+\tfrac12\varepsilon)\cos\theta+\partial_\theta^*(\varphi_b+\psi_b)=\mathbb{O}_{C_b^{1,\alpha}}(\varepsilon)\\ \qquad\qquad\qquad\qquad \partial_\theta^*(\varphi_m+\psi_m)=\mathbb{O}_{C_b^{1,\alpha}}(\varepsilon).\end{cases} \tag{52}$$

Here ∂_θ^* denotes the operator that associates to $\phi=\sum_{i\geqslant 1}\phi_i\cos(i\theta)$ the function $\partial_\theta^*\phi=\sum_{i\geqslant 1}i\phi_i\cos(i\theta)$. To obtain this system, we applied the results of Lemmas A.6 and A.7. The functions $\mathbb{O}_{C^{l,\alpha}}(\varepsilon)$ in the above expansions depend nonlinearly on the different parameters and boundary data functions, but they are bounded in the $C^{l,\alpha}$ topology by a constant times ε. The projection of the first two equations of each system over the L^2-orthogonal complement of $\mathrm{Span}\{1,\cos\theta\}$, together with the remaining two equations, gives the system

$$\begin{cases} \varphi_t-\psi_t=\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*\varphi_t+\partial_\theta^*\psi_t=\mathbb{O}_{C_b^{1,\alpha}}(\varepsilon),\\ \varphi_b-\psi_b=\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*\varphi_b+\partial_\theta^*\psi_b=\mathbb{O}_{C_b^{1,\alpha}}(\varepsilon),\\ \varphi_m-\psi_m=\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*\varphi_m+\partial_\theta^*\psi_m=\mathbb{O}_{C_b^{1,\alpha}}(\varepsilon).\end{cases} \tag{53}$$

Lemma 8.1 [Fakhi and Pacard 2000]. *The operator*

$$h:C^{2,\alpha}(\mathbb{S}^1)\to C^{1,\alpha}(\mathbb{S}^1),\quad \varphi\mapsto\partial_\theta^*\varphi$$

is invertible when acting on functions that are even and L^2-orthogonal to 1.

Proof. If we decompose $\varphi = \sum_{j\geqslant 1} \varphi_j \cos(j\theta)$, then

$$h(\varphi) = \sum_{j\geqslant 1} j\varphi_j \cos(j\theta),$$

is clearly invertible from $H^1(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$. Elliptic regularity theory implies that this is still true when this operator is defined between Hölder spaces. □

Using this result, the system (53) can be rewritten as

$$(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m) = \mathbb{O}_{C^{2,\alpha}}(\varepsilon). \tag{54}$$

Recall that the right hand side depends nonlinearly on $\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m$ and also on the parameters $\gamma_t, \gamma_b, \eta_t, \eta_b, \xi_t, \xi_b, \alpha_t, \alpha_b$. We look at this equation as a fixed point problem and fix κ large enough. Thanks to estimates (48), (20),(22), (17) and (18), we can use a fixed point theorem for contracting mappings in the ball of radius $\kappa\varepsilon$ in $(C^{2,\alpha}(\mathbb{S}^1))^6$ to obtain, for all ε small enough, a solution $(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m)$ of (54). Since this solution is a fixed point for a contraction mapping and since the right hand side of (54) is continuous with respect to all data, we see that this fixed point $(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m)$ depends continuously (and in fact smoothly) on the parameters $\gamma_t, \gamma_b, \eta_t, \eta_b, \xi_t, \xi_b, \alpha_t, \alpha_b$. Inserting this solution into (51) and (52), we see that it only remains to solve a system of the form

$$\begin{cases} \gamma_t \ln(2r_\varepsilon) + \eta_t + \Big(-(1+\gamma_t)\dfrac{\xi_t}{r_\varepsilon} + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_t + \tfrac{1}{2}\varepsilon)\Big)\cos\theta = \mathbb{O}(\varepsilon), \\ -\gamma_b \ln(2r_\varepsilon) + \eta_b + \Big(-(1+\gamma_b)\dfrac{\xi_b}{r_\varepsilon} + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_b + \tfrac{1}{2}\varepsilon)\Big)\cos\theta = \mathbb{O}(\varepsilon), \\ \gamma_t + \Big((1+\gamma_t)\dfrac{\xi_t}{r_\varepsilon} + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_t + \tfrac{1}{2}\varepsilon)\Big)\cos\theta = \mathbb{O}(\varepsilon), \\ -\gamma_b + \Big((1+\gamma_b)\dfrac{\xi_b}{r_\varepsilon} + r_\varepsilon(-2\sin\tfrac{1}{2}\alpha_b + \tfrac{1}{2}\varepsilon)\Big)\cos\theta = \mathbb{O}(\varepsilon), \end{cases}$$

where this time the right hand sides only depend nonlinearly on γ_t, γ_b, η_t, η_b, ξ_t, ξ_b, α_t, α_b. There are eight equations that are obtained by projecting this system over 1 and $\cos\theta$. If we set

$$(\overline{\eta}_t, \overline{\eta}_b) = (\gamma_t \ln(2r_\varepsilon) + \eta_t, -\gamma_b \ln(2r_\varepsilon) + \eta_b),$$
$$(\overline{\xi}_t, \overline{\xi}_b) = r_\varepsilon^{-1}((1+\gamma_t)\xi_t, (1+\gamma_b)\xi_b), \qquad (\overline{\alpha}_t, \overline{\alpha}_b) = r_\varepsilon(2\sin\tfrac{1}{2}\alpha_t, 2\sin\tfrac{1}{2}\alpha_t),$$

the previous system can be rewritten as

$$(\gamma_t, \gamma_b, \overline{\xi}_t, \overline{\xi}_b, \overline{\eta}_t, \overline{\eta}_b, \overline{\alpha}_t, \overline{\alpha}_b) = \mathbb{O}(\varepsilon). \tag{55}$$

This time, provided κ has been fixed large enough, we can use the Leray–Schauder fixed point theorem in the ball of radius $\kappa\varepsilon$ in $\mathbb{R}^8$ to solve (55), for all ε small enough. This provides a set of parameters and a set of boundary data such that (51) and (52) hold. Equivalently, we have proved the existence of a solution of systems (51) and (52). So the proof of the first part of Theorem 1.2 is complete.

The proof of the second part uses the same arguments as above, so we will omit most of the details. We wish to show the existence of the family of surfaces $\mathcal{K}_1$, which are symmetric about the plane $\{x_1 = 0\}$. It is important to observe in this proof that the KMR example is obtained by slight perturbation of $\widetilde{M}_{\sigma,0,\beta}$. The symmetry properties of this surface differ from those of the surface close to $\widetilde{M}_{\sigma,\alpha,0}$ involved in the previous gluing procedure. In particular $\widetilde{M}_{\sigma,0,\beta}$ is symmetric about the plane $\{x_1 = 0\}$, whereas the Costa–Hoffman–Meeks-type surface from before is symmetric about the plane $\{x_2 = 0\}$. Thus $\widetilde{M}_{\sigma,0,\beta}$ is not appropriate for gluing with a KMR example of the type described above. To obtain a surface with the desired symmetry about $\{x_1 = 0\}$, we rotate the Costa–Hoffman–Meeks surface with bent catenoidal ends described in Section 3 counterclockwise by $\pi/2$ about the x_3 axis. In the parameterizations of the top and bottom ends, the cosine function is replaced by the sine function, that is,

$$\begin{aligned} U_t(r,\theta) &= \sigma_t + \ln(2r) - \tfrac{1}{2}\varepsilon r \sin\theta + H_{\psi_t}(s_\varepsilon - \ln(2r), \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \\ U_b(r,\theta) &= -\sigma_b - \ln(2r) - \tfrac{1}{2}\varepsilon r \sin\theta + H_{\psi_b}(s_\varepsilon - \ln(2r), \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \end{aligned}$$

where $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$ and $(r,\theta) \in B_{r_\varepsilon} - B_{r_\varepsilon/2}$. As for the planar middle end, the form of its parameterization remains unchanged; see the first part of the proof. Another important remark concerns the Dirichlet boundary data ψ_t, ψ_b, ψ_m. Before, to preserve the symmetry about the plane $\{x_2 = 0\}$, it was required that these were even functions and that ψ_t and ψ_b were orthogonal to 1 and to $\cos\theta$. Now these must be odd functions and ψ_t and ψ_b must be orthogonal to 1 and to $\sin\theta$. Then all results shown in Section 3 continue to hold (see Remark 6.8).

Now we parameterize the surface $\widetilde{S}_{t,\beta_t,\gamma_t,\xi_t,d_t}(\varphi_t)$, the minimal surface obtained by perturbation from the KMR example $\widetilde{M}_{\sigma,0,\beta}$ and asymptotic to it. This surface can be parameterized in the neighborhood $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\begin{aligned} \overline{U}_t(r,\theta) = (1+\gamma_t)\ln(2r) + 2r\sin\tfrac{1}{2}\beta_t \sin\theta - (1+\gamma_t)/r\xi_t \sin\theta \\ + d_t + \overline{\mathcal{H}}_{v_\varepsilon,\varphi_t}(\ln 2r, \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon). \end{aligned}$$

The parameterization of $\widetilde{S}_{b,\beta_b,\gamma_b,\xi_b,d_b}(\varphi_b)$, the surface that we will glue to the Costa–Hoffman–Meeks-type surface along its lower boundary, is given by

$$\begin{aligned} \overline{U}_b(r,\theta) = -(1+\gamma_b)\ln(2r) + 2r\sin\tfrac{1}{2}\beta_t \sin\theta - (1+\gamma_b)/r\xi_b \sin\theta \\ + d_b + \overline{\mathcal{H}}_{v_\varepsilon,\varphi_b}(\ln 2r, \theta) + \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \end{aligned}$$

where $(r,\theta)\in B_{2r_\varepsilon}-B_{r_\varepsilon}$.

To prove the theorem it is necessary to show there is solution to the system

$$\begin{cases} U_t(r_\varepsilon,\cdot)=\overline{U}_t(r_\varepsilon,\cdot), & \partial_r U_b(r_\varepsilon,\cdot)=\partial_r\overline{U}_b(r_\varepsilon,\cdot),\\ U_b(r_\varepsilon,\cdot)=\overline{U}_b(r_\varepsilon,\cdot), & \partial_r U_t(r_\varepsilon,\cdot)=\partial_r\overline{U}_t(r_\varepsilon,\cdot),\\ U_m(r_\varepsilon,\cdot)=\overline{U}_m(r_\varepsilon,\cdot), & \partial_r U_m(r_\varepsilon,\cdot)=\partial_r\overline{U}_m(r_\varepsilon,\cdot)\end{cases}$$

on $\mathbb{S}^1$, under the assumption (50) for the parameters and the boundary functions. It is clear that the existence proof for this system is based on the same arguments seen before. Note that the role played before by the functions $\cos(i\theta)$ is now played by the functions $\sin(i\theta)$. This completes the proof of Theorem 1.2. □

8.2. ***The proof of Theorem 1.1.*** We will glue a compact piece of the surface $M_k^T(\xi)$ with $\xi=0$ described in Section 3 to two halves of a Scherk-type surface along the upper and lower boundary and to a horizontal periodic flat annulus along the middle boundary. The construction of these surfaces was shown in Section 4. In particular, we showed the existence of a minimal graph close to half of a Scherk-type example whose ends have asymptotic directions given by $\cos\theta_1 e_1+\sin\theta_1 e_3$ and $-\cos\theta_2 e_1+\sin\theta_2 e_3$. These surfaces, in the neighborhood $B_{2r_\varepsilon}-B_{r_\varepsilon}$ of the boundary, admit the parameterization

$$\overline{U}_t=d_t+\ln(2r)+\tilde{H}_{r_\varepsilon,\varphi_t}(r,\theta)+\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon),$$
$$\overline{U}_b=d_b-\ln(2r)+\tilde{H}_{r_\varepsilon,\varphi_b}(r,\theta)+\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon),$$

where the Dirichlet boundary data $\tilde{H}_{r_\varepsilon,\varphi_i}\in C^{2,\alpha}(\mathbb{S}^1)$ for $i=t,b$ is required to be even and orthogonal to 1, and $\tilde{H}_{r_\varepsilon,\varphi_i}$ denotes their harmonic extensions. The other surfaces in the gluing procedure have been described in Section 8.1.

The proof is similar to the one given for Theorem 1.2, so we will give only the essentials. We must show there is a solution to the system

$$\begin{cases} U_t(r_\varepsilon,\cdot)=\overline{U}_t(r_\varepsilon,\cdot), & \partial_r U_b(r_\varepsilon,\cdot)=\partial_r\overline{U}_b(r_\varepsilon,\cdot),\\ U_b(r_\varepsilon,\cdot)=\overline{U}_b(r_\varepsilon,\cdot), & \partial_r U_t(r_\varepsilon,\cdot)=\partial_r\overline{U}_t(r_\varepsilon,\cdot),\\ U_m(r_\varepsilon,\cdot)=\overline{U}_m(r_\varepsilon,\cdot), & \partial_r U_m(r_\varepsilon,\cdot)=\partial_r\overline{U}_m(r_\varepsilon,\cdot)\end{cases}$$

on $\mathbb{S}^1$, under an assumption similar to (50). See Section 8.1 for the expressions of $U_t, U_b, U_m, \overline{U}_m$. We point out that here we consider the more symmetric example (with $\xi=0$) in the family $(M_k^T(\xi))_\xi$, so we must replace $\varepsilon/2$ by 0 in the expressions of the functions U_t and U_b of the top and bottom ends.

The boundary data for the surfaces we will glue together do not all share the same orthogonality properties. All are orthogonal to the constant function, but only ψ_t and ψ_b are orthogonal to $\cos\theta$. The functions denoted by $\mathbb{O}_{\mathscr{C}_b^{2,\alpha}}(\varepsilon)$, appearing in the expressions of $\overline{U}_i$ and U_i with $i = t, b, m$, have a Fourier series decomposition containing a term collinear to $\cos\theta$ only if the corresponding boundary data is assumed to be orthogonal only to the constant function. Furthermore the fact that $\xi = 0$ (which reflects that the catenoidal ends are not bent) implies that the functions parameterizing the top and bottom end of $M_k^T(0)$ are orthogonal to $\cos\theta$. In other words, in contrast to the Scherk-type surfaces, we are not able this time to prescribe the coefficients of the eigenfunction $\cos\theta$ for the catenoidal ends of $M_k^T(0)$, because they are required to vanish in this more symmetric setting.

The left three equations lead to the system

$$\begin{cases} \eta_t + \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), \\ \eta_b + \varphi_b - \psi_b = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), \\ \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), \end{cases}$$

where $\eta_t = d_t - \sigma_t$, $\eta_b = d_b + \sigma_b$. The right three equations give the system

$$\begin{cases} \partial_\theta^*(\varphi_t + \psi_t) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \partial_\theta^*(\varphi_b + \psi_b) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \partial_\theta^*(\varphi_m + \psi_m) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon). \end{cases}$$

The proof is completed by the arguments of Section 8.1. □

8.3. *The proof of Theorem 1.3.* To prove this theorem, we treat separately the cases $k = 0$ and $k \geqslant 1$.

The case $k = 0$. We will glue half of a Scherk example with half of a KMR example with $\alpha = \beta = 0$. We observe that this surface is symmetric about the $\{x_1 = 0\}$ and $\{x_2 = 0\}$ planes. The Scherk example is symmetric about the $\{x_2 = 0\}$ plane. To preserve this property of symmetry in the surface obtained by the gluing procedure, we will consider the perturbation of $\tilde{M}_{\sigma,0,0}$ that has the same mirror symmetry. This is the surface denoted by $S_{t,0,\gamma_t,\xi_t,d_t}(\varphi_t)$ with $\gamma_t = \xi_t = 0$ and $d_t = d$. It can be parameterized in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\overline{U}(r,\theta) = \ln(2r) + \bar{d} + \overline{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

The Scherk example is parameterized as the vertical graph of

$$U_t(r,\theta) = \ln(2r) + d + \tilde{H}_{r_\varepsilon,\psi}(r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

As for the Dirichlet boundary data, we assume φ to be an even function orthogonal to the constant function and to $\cos\theta$, and we assume ψ to be even and orthogonal to 1.

To prove the theorem in the case $k=0$, we must show there is a solution to the system

$$\begin{cases} U(r_\varepsilon,\cdot\,) = \overline{U}_t(r_\varepsilon,\cdot\,), \\ \partial_r U(r_\varepsilon,\cdot\,) = \partial_r \overline{U}_t(r_\varepsilon,\cdot\,) \end{cases}$$

on $\mathbb{S}^1$, under appropriate assumptions on the norms of the Dirichlet boundary data and the parameters ξ, d, $\bar{d}$.

These equations lead to the system

$$\begin{cases} \eta+\varphi-\psi = \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \\ \partial_\theta^*(\varphi+\psi) = \mathbb{O}_{C_b^{1,\alpha}}(\varepsilon). \end{cases}$$

where $\eta = \bar{d}-d$. The proof is completed by the arguments of Section 8.1.

The case $k \geqslant 1$. The proof in this case is similar the proof of Theorem 1.1. In fact three of the surfaces we are going to glue are ones we used there: a compact piece of the Costa–Hoffman–Meeks example M_k, half of a Scherk-type example, and a horizontal periodic flat annulus. The fourth surface is half of a KMR example, of the type we used in the $k=0$ case. The surfaces are parameterized as vertical graphs over $B_{2r_\varepsilon}-B_{r_\varepsilon}$ of the following functions:

$$\overline{U}_b(r,\theta) = -\ln(2r)+d_b+\tilde{H}_{r_\varepsilon,\varphi_b}(r,\theta)+\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$$

for the Scherk-type example;

$$\overline{U}_m(r,\theta) = \tilde{H}_{r_\varepsilon,\varphi_m}(r,\theta)+\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$$

for the horizontal periodic flat annulus;

$$\overline{U}_t(r,\theta) = \ln(2r)+d_t+\overline{\mathcal{H}}_{v_\varepsilon,\varphi_t}(\ln 2r,\theta)+\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon)$$

for the KMR example; and

$$\begin{aligned} U_t(r,\theta) &= \quad \sigma_t+\ln(2r)+H_{\psi_t}(s_\varepsilon-\ln(2r),\theta)+\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \\ U_b(r,\theta) &= -\sigma_b-\ln(2r)+H_{\psi_b}(s_\varepsilon-\ln(2r),\theta)+\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), \\ U_m(r,\theta) &= \tilde{H}_{\rho_\varepsilon,\varphi_m}(1/r,\theta)+\mathbb{O}_{C_b^{2,\alpha}}(\varepsilon) \end{aligned}$$

for the compact piece of the Costa–Hoffman–Meeks example. We require the Dirichlet boundary data to consist of even functions. The functions ψ_t and ψ_b are orthogonal to 1 and to $\cos\theta$, but ψ_m, φ_t, φ_b and φ_m are orthogonal only to 1. In this case the system of equations to solve is

$$\begin{cases} \eta_t+\varphi_t-\psi_t = \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*(\varphi_t+\psi_t) = \mathbb{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \eta_b+\varphi_b-\psi_b = \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*(\varphi_b+\psi_b) = \mathbb{O}_{C_b^{1,\alpha}}(\varepsilon), \\ \varphi_m-\psi_m = \mathbb{O}_{C_b^{2,\alpha}}(\varepsilon), & \partial_\theta^*(\varphi_m+\psi_m) = \mathbb{O}_{C_b^{1,\alpha}}(\varepsilon), \end{cases}$$

where $\eta_t = d_t - \sigma_t$ and $\eta_b = d_b + \sigma_b$. The details are left to the reader.

Appendix A

Definition A.1. For $\ell \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\nu \in \mathbb{R}$, the space $\mathscr{C}^{\ell,\alpha}_{\nu}(B_{\rho_0}(0))$ is defined to be the space of functions in $\mathscr{C}^{\ell,\alpha}_{\mathrm{loc}}(B_{\rho_0}(0))$ for which the norm $\|\rho^{-\nu}\, w\|_{\mathscr{C}^{\ell,\alpha}(B_{\rho_0}(0))}$ is finite.

Proposition A.2. *There exists an operator* $\tilde{H} : C^{2,\alpha}(\mathbb{S}^1) \to C^{2,\alpha}_{-1}([\bar{\rho}, +\infty) \times \mathbb{S}^1)$, *such that for each even function* $\varphi(\theta) \in C^{2,\alpha}(\mathbb{S}^1)$ *that is* L^2*-orthogonal to* 1, *the function* $w_\varphi = \tilde{H}_{\bar{\rho},\varphi}$ *solves*

$$\begin{cases} \Delta w_\varphi = 0 & \text{on } [\bar{\rho}, +\infty) \times \mathbb{S}^1, \\ w_\varphi = \varphi & \text{on } \{\bar{\rho}\} \times \mathbb{S}^1. \end{cases}$$

Moreover, $\|\tilde{H}_{\bar{\rho},\varphi}\|_{C^{2,\alpha}_{-1}([\bar{\rho},+\infty)\times\mathbb{S}^1)} \leqslant c\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^1)}$ *for some constant* $c > 0$.

Remark A.3. Following the arguments of the proof below, it is possible to state a similar proposition but with the hypothesis that φ is odd.

Proof. We decompose the function φ in the basis $\{\cos(i\theta)\}$ as $\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(i\theta)$. Then the solution w_φ is given by

$$w_\varphi(\rho, \theta) = \sum_{i=1}^{\infty} \left(\frac{\bar{\rho}}{\rho}\right)^i \varphi_i \cos(i\theta).$$

Because $\bar{\rho}/\rho \leqslant 1$, we have $(\bar{\rho}/\rho)^i \leqslant (\bar{\rho}/\rho)$. Thus $|w(r, \theta)| \leqslant c\rho^{-1}|\varphi(\theta)|$ and then $\|w_\varphi\|_{C^{2,\alpha}_{-1}} \leqslant c\|\varphi\|_{C^{2,\alpha}}$. □

Now we state a useful result; for a proof see [Fakhi and Pacard 2000].

Proposition A.4. *There exists an operator* $H : \mathscr{C}^{2,\alpha}(\mathbb{S}^1) \to \mathscr{C}^{2,\alpha}_{-2}([0, +\infty) \times \mathbb{S}^1)$, *such that, for all* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$ *that are even and* L^2*-orthogonal to* 1 *and* $\cos\theta$, *the function* $w = H_\varphi$ *solves*

$$\begin{cases} (\partial_s^2 + \partial_\theta^2) w = 0 & \text{in } [0, +\infty) \times \mathbb{S}^1, \\ w = \varphi & \text{on } \{0\} \times \mathbb{S}^1. \end{cases}$$

Moreover $\|H_\varphi\|_{\mathscr{C}^{2,\alpha}_{-2}} \leqslant c\|\varphi\|_{\mathscr{C}^{2,\alpha}}$ *for some constant* $c > 0$.

Proposition A.5. *There exists an operator*

$$\overline{\mathscr{H}}_{v_0} : C^{2,\alpha}(\mathbb{S}^1) \to C^{2,\alpha}_{\mu}([v_0, +\infty) \times \mathbb{S}^1)$$

for $\mu \in (-2,-1)$ *such that, for every function* $\varphi(u) \in C^{2,\alpha}(\mathbb{S}^1)$ *that is even and* L^2*-orthogonal to* $e_{0,i}(u)$ *with* $i = 0, 1$*, the function* $w_\varphi = \overline{\mathcal{H}}_{v_0,\varphi}$ *solves*

$$\begin{cases} \partial^2_{uu} w_\varphi + \partial^2_{vv} w_\varphi = 0 & \text{on } [v_0,+\infty) \times \mathbb{S}^1, \\ w_\varphi = \varphi & \text{on } \{v_0\} \times \mathbb{S}^1. \end{cases}$$

Moreover, $\|\overline{\mathcal{H}}_{v_0,\varphi}\|_{C^{2,\alpha}_\mu([v_0,+\infty)\times\mathbb{S}^1)} \leqslant c\|\varphi\|_{C^{2,\alpha}(\mathbb{S}^1)}$ *for some constant* $c > 0$.

Proof. We decompose of the function φ in the basis $\{e_{0,i}(u)\}$ as $\varphi = \sum_{i=2}^{\infty} \varphi_i e_{0,i}(u)$. Then the solution w_φ is given by

$$w_\varphi(u,v) = \sum_{i=2}^{\infty} e^{-i(v-v_0)} \varphi_i e_{0,i}(u).$$

We recall that $\mu \in (-2,-1)$, so we have $-i \leqslant \mu$, from which it follows that

$$\begin{aligned} &\|w_\varphi\|_{\mathscr{C}^{2,\alpha}([v,v+1]\times\mathbb{S}^1)} \leqslant e^{\mu(v-v_0)}\|\varphi\|_{C^{2,\alpha}}, \\ &\qquad \|w_\varphi\|_{\mathscr{C}^{2,\alpha}_\mu} = \sup_{v\in[v_0,\infty]} e^{-\mu v}\|w_\varphi\|_{\mathscr{C}^{2,\alpha}([v,v+1]\times\mathbb{S}^1)} \\ &\qquad\qquad \leqslant \sup_{v\in[v_0,\infty]} e^{-\mu v} e^{\mu(v-v_0)}\|\varphi\|_{C^{2,\alpha}} \leqslant e^{-\mu v_0}\|\varphi\|_{C^{2,\alpha}}. \end{aligned}$$ □

Lemma A.6. *Let* $u(r,\theta)$ *be the harmonic extension defined on* $[r_0,+\infty) \times \mathbb{S}^1$ *of the even function* $\varphi = \sum_{i\geqslant 0} \varphi_i \cos(i\theta) \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$*, and suppose* $u(r_0,\theta) = \varphi(\theta)$*. Then* $\partial^*_\theta \varphi(\theta) = r_0 \partial_r u(r,\theta)|_{r=r_0}$*.*

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0} \varphi_i \cos(i\theta)$, then the function u is given by

$$u(r,\theta) = \sum_{i\geqslant 0} \varphi_i \left(\frac{r}{r_0}\right)^i \cos(i\theta).$$

Then $\partial_r u(r,\theta) = \sum_{i\geqslant 1} \varphi_i\, (r/r_0)^i\, i \cos(i\theta)/r$, and $\partial^*_\theta \varphi(\theta) = r_0 \partial_r u(r,\theta)|_{r=r_0}$. □

Lemma A.7. *Let* $u(r,\theta)$ *be the harmonic extension defined on* $[0,r_0] \times \mathbb{S}^1$ *of the even function* $\varphi \in \mathscr{C}^{2,\alpha}(\mathbb{S}^1)$*, with* $u(r_0,\theta) = \varphi(\theta)$*. Then* $\partial^*_\theta \varphi(\theta) = -r_0 \partial_r u(r,\theta)_{|r=r_0}$*.*

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0} \varphi_i \cos(i\theta)$, then

$$u(r,\theta) = \sum_{i\geqslant 0} \varphi_i (r_0/r)^i \cos(i\theta).$$

Then $\partial_r u(r,\theta) = -\sum_{i\geqslant 1} \varphi_i\, (r_0/r)^i\, i \cos(i\theta)/r$, and the result follows. □

Appendix B

Proof of Proposition 7.1. Let Z be the immersion of the surface $\widetilde{M}_{\sigma,\alpha,\beta}$ and N its normal vector. We want to find the differential equation a function f must satisfy so that the surface parameterized by $Z_f = Z + fN$ is minimal. In Section 5.2 we parameterized the surface $\widetilde{M}_{\sigma,\alpha,\beta}$ on the cylinder $\mathbb{S}^1 \times \mathbb{R}$. We introduced the map $\mathbf{z}(x, y) : \mathbb{S}^1 \times [0, \pi[\to \overline{\mathbb{C}}$ where x, y denote the spheroconal coordinates. We start with the conformal variables p and q, defined to be as the real and the imaginary part of $\mathbf{z}$. We have

$$\begin{aligned} |Z_p|^2 = |Z_q|^2 = \Lambda, &\qquad |N_p|^2 = |N_q|^2 = -K\Lambda, \\ \langle N_p, N\rangle = \langle N_q, N\rangle = 0, &\qquad \langle Z_p, Z_q\rangle = 0, \quad \langle N_p, N_q\rangle = 0, \\ \langle N_q, Z_q\rangle = -\langle N_p, Z_p\rangle, &\qquad \langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle, \end{aligned}$$

so

$$\begin{aligned} \langle N_p, Z_p\rangle &= |N_p|\,|Z_p| \cos\gamma_1 = \sqrt{-K}\,\Lambda \cos\gamma_1, \\ \langle N_p, Z_q\rangle &= |N_p|\,|Z_q| \cos\gamma_2 = \sqrt{-K}\,\Lambda \cos\gamma_2. \end{aligned}$$

Here K denotes the Gauss curvature, Z_p, Z_q and N_p, N_q denote the partial derivatives of the vectors Z and N, γ_1 is the angle between the vectors N_p and Z_p, and γ_2 is the angle between the vectors N_p and Z_q.

The proof of Proposition 7.1 is articulated through some lemmas. We denote by E_f, F_f, G_f the coefficients of the second fundamental form for the surface parameterized by Z_f. The first lemma expresses the area energy functional.

Lemma B.1. $A(f) := \int (E_f G_f - F_f^2)^{1/2}\, dp\, dq$, *with*

$$\begin{aligned} E_f G_f - F_f^2 = \Lambda^2 + \Lambda(f_p^2 + f_q^2) + 2K\Lambda^2 f^2 + 2f(f_q^2 - f_p^2)\sqrt{-K}\,\Lambda\cos\gamma_1 \\ - 4 f f_p f_q \sqrt{-K}\,\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4 K^2 \Lambda^2. \end{aligned}$$

Proof. The coefficients of the second fundamental form are

$$\begin{aligned} E_f &= |\partial_p Z_f|^2 = |Z_p|^2 + f_p^2 + f^2|N_p|^2 + 2f\langle N_p, Z_p\rangle, \\ G_f &= |\partial_q Z_f|^2 = |Z_q|^2 + f_q^2 + f^2|N_q|^2 + 2f\langle N_q, Z_q\rangle, \\ F_f &= |\partial_p Z_f \cdot \partial_q Z_f| = f_p f_q + f(\langle Z_p, N_q\rangle + \langle Z_q, N_p\rangle). \end{aligned}$$

Then

$$\begin{aligned} E_f G_f = |Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2) \\ + f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2 + 4f^2(\langle N_p Z_p\rangle)(\langle N_q Z_q\rangle) \\ + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle) + f_p^2 f_q^2 \\ + 2f(\langle N_q, Z_q\rangle|Z_p|^2 + \langle N_p, Z_p\rangle|Z_q|^2) + 2f^3(\langle N_q, Z_q\rangle|Z_p|^2 + \langle N_p, Z_p\rangle|Z_q|^2). \end{aligned}$$

Since $\langle N_q, Z_q\rangle + \langle N_p, Z_p\rangle = 0$ and $|Z_p|^2 = |Z_q|^2$, we can conclude that the last two terms of the previous expression are zero. Since $\langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle$, we have $F_f = f_p f_q + 2f\langle N_p, Z_q\rangle$. Then

$$F_f^2 = f_p^2 f_q^2 + 4f^2(\langle N_p, Z_q\rangle)^2 + 4ff_p f_q\langle N_p, Z_q\rangle.$$

So the expression for $E_f G_f - F_f^2$ is

$$\begin{aligned}|Z_p|^2|Z_q|^2 &+ f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2)\\ &+ f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2 + 4f^2\langle N_p, Z_p\rangle\langle N_q, Z_q\rangle\\ &+ 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle) - 4f^2(\langle N_p, Z_q\rangle)^2 - 4ff_p f_q\langle N_p, Z_q\rangle.\end{aligned}$$

Ordering the terms, we get

$$\begin{aligned}|Z_p|^2|Z_q|^2 &+ f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2)\\ &- 4f^2\langle N_p, Z_q\rangle^2 + 4f^2\langle N_p, Z_p\rangle\langle N_q, Z_q\rangle + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle)\\ &- 4ff_p f_q\langle N_p, Z_q\rangle + f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2.\end{aligned}$$

The expression for $E_f G_f - F_f^2$ becomes

$$\begin{aligned}\Lambda^2 &+ \Lambda(f_p^2 + f_q^2) - 2K\Lambda^2 f^2 + 4f^2K\Lambda^2(\cos^2\gamma_1 + \cos^2\gamma_2)\\ &+ 2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1 - 4ff_p f_q\sqrt{-K}\Lambda\cos\gamma_2\\ &- K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2.\end{aligned}$$

Using the relations $\langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle$ and $\langle N_q, Z_q\rangle = -\langle N_p, Z_p\rangle$, one can see that vectors are pointed so that $\gamma_2 = \pi/2 \pm \gamma_1$. So $\cos^2\gamma_2 = \cos^2(\pi/2 \pm \gamma_1) = \sin^2\gamma_1$ and $\cos^2\gamma_1 + \cos^2\gamma_2 = 1$. Then we can write

$$\begin{aligned}\Lambda^2 &+ \Lambda(f_p^2 + f_q^2) + 2K\Lambda^2 f^2 + 2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1\\ &- 4ff_p f_q\sqrt{-K}\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2. \quad \square\end{aligned}$$

The next lemma completes the proof of Proposition 7.1.

Lemma B.2. *The surface whose immersion is given by $Z + fN$, is minimal if and only if f satisfies*

$$\mathscr{L}_\sigma f + Q_\sigma(f) = 0,$$

where $\mathscr{L}_\sigma$ is the Lamé operator and Q_σ is a second order differential operator that satisfies

$$\|Q_\sigma(f_2) - Q_\sigma(f_1)\|_{\mathscr{C}^{0,\alpha}(I_\sigma\times[v,v+1])} \leqslant c\sup_{i=1,2}\|f_i\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\|f_2 - f_1\|_{\mathscr{C}^{2,\alpha}(I_\sigma\times[v,v+1])}.$$

Proof. The surface parameterized by $Z_f = Z + fN$ is minimal if and only the first variation of $A(f)$ is 0. That is,

$$2DA(g) = \int \frac{1}{(E_f G_f - F_f^2)^{1/2}}\Big|_{f=0} D_f(E_f G_f - F_f^2)(g)\, dp\, dq = 0.$$

By the previous lemma, the integrand above is equal to

$$\begin{aligned}\frac{1}{\Lambda}\Big(2\Lambda(f_p g_p + f_q g_q) + 4K\Lambda^2 fg \\ &+ 2\sqrt{-K}\Lambda\cos\gamma_1\big(2ff_q g_q + gf_q^2 - 2ff_p g_p - gf_p^2\big) \\ &- 4\sqrt{-K}\Lambda\cos\gamma_2\big(ff_q g_p + fg_q f_p + gf_p f_q\big) \\ &\quad - 2K\Lambda\big(fgf_p^2 + f_p g_p f^2 + fgf_q^2 + f_q g_q f^2\big) + 4K^2\Lambda^2 f^3 g\Big),\end{aligned}$$

which, by reordering the summands, becomes

$$\begin{aligned}2\Big(f_p g_p + f_q g_q + 2K\Lambda fg \\ &+ \sqrt{-K}\cos\gamma_1\big(2f(f_q g_q - f_p g_p) + g(f_q^2 - f_p^2)\big) \\ &- 2\sqrt{-K}\cos\gamma_2\big(f(f_q g_p + g_q f_p) + gf_p f_q\big) \\ &\quad - K\big(fg(f_p^2 + f_q^2) + f^2(f_p g_p + f_q g_q)\big) + 2K^2\Lambda f^3 g\Big).\end{aligned}$$

In the next computation we skip the overall factor of 2 in this expression. We find

$$f_p g_p + f_q g_q + 2K\Lambda fg + Q_1(f, f_p, f_q)g - Q_2(f, f_p, f_q)g_p - Q_3(f, f_p, f_q)g_q = 0,$$

where

$$\begin{aligned}Q_1(f, f_p, f_q) &= -(f_p^2 - f_q^2)\sqrt{-K}\cos\gamma_1 - 2f_p f_q\sqrt{-K}\cos\gamma_2 \\ &\qquad - Kf(f_p^2 + f_q^2) + 2K^2\Lambda f^3, \\ Q_2(f, f_p, f_q) &= 2ff_p\sqrt{-K}\cos\gamma_1 + 2ff_q\sqrt{-K}\cos\gamma_2 + Kf^2 f_p, \\ Q_3(f, f_p, f_q) &= -2ff_q\sqrt{-K}\cos\gamma_1 + 2ff_p\sqrt{-K}\cos\gamma_2 + Kf^2 f_q.\end{aligned}$$

An integration by parts and a change of sign give us the equation

$$\begin{aligned}\big(f_{pp} + f_{qq} - 2K\Lambda f - Q_1(f, f_p, f_q) \\ &+ P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) + P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq})\big)g = 0,\end{aligned}$$

where

$$\begin{aligned}P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) &= \partial_p Q_2(f, f_p, f_q), \\ P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) &= \partial_q Q_3(f, f_p, f_q).\end{aligned}$$

That is,

$$
\begin{aligned}
&P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = \\
&\quad 2(f_p^2 + f f_{pp})\sqrt{-K}\cos\gamma_1 + 2(f_p f_q + f f_{pq})\sqrt{-K}\cos\gamma_2 + K(2 f f_p^2 + f^2 f_{pp}) \\
&\qquad + 2f\big(f_p(\sqrt{-K}\cos\gamma_1)_p + f_q(\sqrt{-K}\cos\gamma_2)_p\big) + f^2 f_p K_p, \\
&P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = \\
&\quad -2(f_q^2 + f f_{qq})\sqrt{-K}\cos\gamma_1 + 2(f_p f_q + f f_{pq})\sqrt{-K}\cos\gamma_2 + K(2 f f_q^2 + f^2 f_{qq}) \\
&\qquad + 2f\big(-f_q(\sqrt{-K}\cos\gamma_1)_q + f_p(\sqrt{-K}\cos\gamma_2)_q\big) + f^2 f_q K_q.
\end{aligned}
$$

Now we want to understand how differential equation above changes when passing from the (p, q) to the (u, v) variables. We recall that p and q are the real and imaginary part of the variable $\mathbf{z}$ that is expressed in terms of the spheroconal coordinates x, y in (28). The metric $\bar{g}$ induced on a surface whose immersion Z is given by the Weierstrass representation on a domain of the complex $\mathbf{z}$-plane can be expressed in terms of the metric $d\bar{s}^2 = dp^2 + dq^2$ by $\bar{g} = \Lambda(dp^2 + dq^2)$, where $\Lambda = |Z_p|^2 = |Z_q|^2$. The Laplace–Beltrami operators written with respect to the metrics $d\bar{s}^2$ and $\bar{g}$ are related by $\Delta_{d\bar{s}^2} = (1/\Lambda)\Delta_{\bar{g}}$, that is, they differ by the conformal factor $1/\Lambda$. In Proposition 7.1, we observed that the conformal factor related to the change of coordinates $(x, y) \to (u, v)$ is $-K/k$. So the conformal factor induced by the change $(p, q) \to (u, v)$ is the product of the conformal factors described above. Summarizing, we have

$$f_{pp} + f_{qq} = \frac{-K\Lambda}{k}(f_{uu} + f_{vv}).$$

So we can write

$$\frac{-K\Lambda}{k}(f_{uu} + f_{vv}) + 2(-K\Lambda) f + R_1 + R_2 + R_3 = 0,$$

where

$$
\begin{aligned}
&R_1(f, f_u, f_v) = \\
&\quad -\frac{-K\Lambda}{k}\big(-(f_u^2 - f_v^2)\sqrt{-K}\cos\gamma_1 - 2 f_u f_v\sqrt{-K}\cos\gamma_2 - K f(f_u^2 + f_v^2)\big) - 2K^2\Lambda f^3 \\
&\quad = \frac{-K\Lambda}{k}\big((f_u^2 - f_v^2)\sqrt{-K}\cos\gamma_1 + 2 f_u f_v\sqrt{-K}\cos\gamma_2 + K f(f_u^2 + f_v^2) - 2Kk f^3\big) \\
&\quad = \frac{-K\Lambda}{k}\bar{P}_1(f, f_u, f_v),
\end{aligned}
$$

and

$$
\begin{aligned}
R_2(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) &= \frac{-K\Lambda}{k} P_2(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}), \\
R_3(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) &= \frac{-K\Lambda}{k} P_3(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}).
\end{aligned}
$$

Simplifying the notation, we can write

$$\frac{-K\Lambda}{k}\big(f_{uu}+f_{vv}+2k(u,v)f+\bar{P}_1(f)+P_2(f)+P_3(f)\big)=0.$$

We can recognize the Lamé operator in

$$\mathcal{L}_\sigma f = f_{uu}+f_{vv}+2(\sin^2\sigma\,\cos^2 x(u)+\cos^2\sigma\,\sin^2 y(v))f;$$

then, if we set $Q_\sigma=\bar{P}_1(f)+P_2(f)+P_3(f)$, the equation can be written

$$\mathcal{L}_\sigma f+Q_\sigma(f)=0.$$

To show the estimate of Q_σ, it suffices to show that all its coefficients are bounded. In particular we will show that the Gauss curvature K and its derivatives K_u and K_v are bounded. We start observing that $-K/k(x(u),y(v))$ is bounded. It is well known that the Gauss curvature can be expressed in terms of the Weierstrass data g, dh as

$$K=-16\Big(|g|+\frac{1}{|g|}\Big)^{-4}\Big|\frac{dg}{g}\Big|^2|dh|^{-2}$$

We recall that $dh=\mu\,dz/\sqrt{(z^2+\lambda^2)(z^2+\lambda^{-2})}$. Now $|z^2+\lambda^2||z^2+\lambda^{-2}|$ and $k(x,y)=\sin^2\sigma\,\cos^2 x(u)+\cos^2\sigma\,\sin^2 y(v)$ have the same zeros, that is, the points D, D', D'', D''' are given by (23), so $-K/k$ is bounded, as are its derivatives.

We estimate the derivatives of K and $\sqrt{-K}$. We can write $\sqrt{-K}=\sqrt{k}\sqrt{-K/k}$. From the observations made above, it follows that to show that the derivatives of $\sqrt{-K}$ are bounded, it suffices to study the derivatives of $\sqrt{k}$.

We recall that

$$l(x)=\sqrt{1-\sin^2\sigma\,\sin^2 x}\quad\text{and}\quad m(y)=\sqrt{1-\cos^2\sigma\,\cos^2 y}.$$

From the expression of k, it is easy to get from (29) that

$$\frac{\partial}{\partial u}\sqrt{k}=-\frac{\sin^2\sigma\,\sin 2x(u)}{2\sqrt{k}}l(x(u))\quad\text{and}\quad\frac{\partial}{\partial v}\sqrt{k}=\frac{\cos^2\sigma\,\sin 2y(v)}{2\sqrt{k}}m(y(v)).$$

Then

$$\Big|\frac{\partial}{\partial u}\sqrt{k}\Big|=\frac{\sin^2\sigma\,|\sin 2x(u)|l(x(u))}{2\sqrt{\sin^2\sigma\,\cos^2 x(u)+\cos^2\sigma\,\sin^2 y(v)}}\leqslant\frac{\sin^2\sigma\,|\sin 2x(u)|}{2\sin\sigma\,|\cos x(u)|}\leqslant\sin\sigma,$$

$$\Big|\frac{\partial}{\partial v}\sqrt{k}\Big|=\frac{\cos^2\sigma\,|\sin 2y(v)|m(y(v))}{2\sqrt{\sin^2\sigma\,\cos^2 x(u)+\cos^2\sigma\,\sin^2 y(v)}}\leqslant\frac{\cos^2\sigma\,|\sin 2y(v)|}{2\cos\sigma\,|\sin y(v)|}\leqslant\cos\sigma.$$

Thus the derivatives of $\sqrt{k}$ (and consequently those of $\sqrt{-K}$) are bounded. □

Appendix C

The differential equation

$$\sin y\, \partial_y(\sin y\, \partial_y f) - j^2 f + 2\sin^2 y\, f = 0 \tag{56}$$

is the $l = 1$ case of the associated Legendre differential equation

$$\sin y\, \partial_y(\sin y\, \partial_y f) - j^2 f + l(l+1)\sin^2 y\, f = 0,$$

where $l, j \in \mathbb{N}$. The family of the solutions of (56) (see [Abramowitz and Stegun 1964]) is $c_1 P_l^j(\cos y) + c_2 Q_l^j(\cos y)$ for $l = 1$, where $P_l^j(t)$ and $Q_l^j(t)$ are respectively the associated Legendre functions of first and second kind. If $l = 1$, these functions are defined as follows:

$$P_1^j(t) = \begin{cases} t & \text{if } j = 0, \\ -\sqrt{1-t^2} & \text{if } j = 1, \\ 0 & \text{if } j \geqslant 2, \end{cases}$$

$$Q_1^j(t) = (-1)^j \sqrt{(1-t^2)^j}\, \frac{d^j Q_1^0(t)}{dt^j}$$

$$Q_1^0(t) = \tfrac{1}{2} t \ln\left(\frac{1+t}{1-t}\right) - 1.$$

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