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We prove a Wiener-type Tauberian theorem for L^1 spherical functions on a semisimple Lie group of arbitrary real rank.

1. Introduction

Let $f \in L^1(\mathbb{R})$ and let $\tilde{f}$ be its Fourier transform. The celebrated Wiener Tauberian theorem says that the ideal generated by f is dense in $L^1(\mathbb{R})$ if and only if $\tilde{f}$ is a nowhere vanishing function on the real line. Ehrenpreis and Mautner [1955] observed that the corresponding result is not true for the commutative algebra of K-biinvariant functions on the semisimple Lie group $\mathrm{SL}(2, \mathbb{R})$. Here K is the maximal compact subgroup $\mathrm{SO}(2)$. However, in the same paper it was also proved that an additional condition of not-too-rapid decay on the spherical Fourier transform of a function suffices to prove an analogue of the Wiener Tauberian theorem. That is, if f is a K-biinvariant integrable function on $G = \mathrm{SL}(2, \mathbb{R})$ and its spherical Fourier transform $\hat{f}$ does not vanish anywhere on the maximal ideal space (which can be identified with a certain strip on the complex plane) then the function f generates a dense subalgebra of $L^1(K\backslash G/K)$ provided $\hat{f}$ does not vanish too fast at ∞. See [Ehrenpreis and Mautner 1955] for precise statements.

There have been a number of attempts to generalize these results to $L^1(K\backslash G/K)$ or $L^1(G/K)$ where G is a noncompact connected semisimple Lie group with finite center. Almost complete results have been obtained when G is a group of real rank one. We refer the reader to [Benyamini and Weit 1992; Ben Natan et al. 1996; Sarkar 1998; Sitaram 1988] for results on the rank-one case. See also [Sarkar 1997] for a result on the whole group $\mathrm{SL}(2, \mathbb{R})$.

Sitaram [1980] proved that under suitable conditions on the spherical Fourier transform of a single function f, an analogue of the Wiener Tauberian theorem holds for $L^1(K\backslash G/K)$, with no assumptions on the rank of G. The purpose of this paper is to prove such a theorem for an arbitrary family of functions with suitable conditions on the spherical Fourier transforms.

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Notation and preliminaries. For convenience, we follow the notation in [Sitaram 1980], so we essentially reproduce its introduction. For unexplained terminology, refer to [Helgason 1994]. Let G be a connected noncompact semisimple Lie group with finite center and K a fixed maximal compact subgroup of G. Fix an Iwasawa decomposition $G = KAN$ and let $\mathbf{a}$ be the Lie algebra of A. Let $\mathbf{a}^*$ be the real dual of $\mathbf{a}$ and $\mathbf{a}^*_{\mathbb{C}}$ its complexification. Let ρ be the half sum of positive roots for the adjoint action of $\mathbf{a}$ on $\mathbf{g}$, the Lie algebra of G. The Killing form induces a positive definite form $\langle\, .\, ,\, .\,\rangle$ on $\mathbf{a}^* \times \mathbf{a}^*$. Extend this form to a bilinear form on $\mathbf{a}^*_{\mathbb{C}}$. We will use the same notation for the extension as well. Let W be the Weyl group of the symmetric space G/K. Then there is a natural action of W on $\mathbf{a}$, $\mathbf{a}^*$ and $\mathbf{a}^*_{\mathbb{C}}$, and $\langle\, .\, ,\, .\,\rangle$ is invariant under this action.

For each $\lambda \in \mathbf{a}^*_{\mathbb{C}}$, let φ_λ be the elementary spherical function associated with λ. Recall that φ_λ is given by the formula

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}\, dk, \qquad x \in G.$$

It is known that $\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda' = s\lambda$ for some $s \in W$. Let l be the dimension of $\mathbf{a}$ and let $F \subset \mathbb{C}^l$ denote the set

$$F = \mathbf{a}^* + iC_\rho, \quad \text{where } C_\rho \text{ is the convex hull of } \{s\rho : s \in W\}.$$

A well-known theorem of Helgason and Johnson states that φ_λ is bounded if and only if $\lambda \in F$.

Let $I(G)$ be the set of all complex valued spherical functions on G:

$$I(G) = \{f : f(k_1 x k_2) = f(x), k_1, k_2 \in K, x \in G\}.$$

Fix a Haar measure dx on G and let $I_1(G) = I(G) \cap L^1(G)$. Then it is well known that $I_1(G)$ is a commutative Banach algebra under convolution and that the maximal ideal space of $I_1(G)$ can be identified with F/W.

For $f \in I_1(G)$, define its spherical Fourier transform, $\hat{f}$ on F by

$$\hat{f}(\lambda) = \int_G f(x)\, \varphi_{-\lambda}(x)\, dx.$$

Then $\hat{f}$ is a W-invariant bounded function on F which is holomorphic in the interior F^0 of F, and continuous on F. Also $\widehat{f * g} = \hat{f}\hat{g}$ where the convolution of f and g is defined by

$$f * g(x) = \int_G f(xy^{-1}) g(y)\, dy.$$

Next, we define the L^1-Schwartz space of K-biinvariant functions on G, which will be denoted by $S(G)$. Let $x \in G$. Then $x = k \exp X,\ k \in K,\ X \in \mathbf{p}$, where $\mathbf{g} = \mathbf{k} + \mathbf{p}$ is the Cartan decomposition of the Lie algebra $\mathbf{g}$ of G. Put $\sigma(x) = \|X\|$, where $\|\, .\, \|$

is the norm on $\mathbf{p}$ induced by the Killing form. For any left-invariant differential operator D on G and any integer $r \geq 0$, we define for a smooth K-biinvariant function f

$$p_{D,r}(f) = \sup_{x \in G} (1+\sigma(x))^r \, |\varphi_0(x)|^{-2} \, |Df(x)|$$

where φ_0 is the elementary spherical function corresponding to $\lambda = 0$. Define

$$S(G) = \{f : p_{D,r}(f) < \infty \text{ for all } D, r\}.$$

Then $S(G)$ becomes a Frechet space when equipped with the topology induced by the family of seminorms $p_{D,r}$.

Let $P = P(\mathbf{a}^*_{\mathbb{C}})$ be the symmetric algebra over $\mathbf{a}^*_{\mathbb{C}}$. Then each $u \in P$ gives rise to a differential operator $\partial(u)$ on $\mathbf{a}^*_{\mathbb{C}}$. Let $Z(F)$ be the space of functions f on F satisfying the following conditions:

(i) f is holomorphic in F^0 (interior of F) and continuous on F.

(ii) If $u \in P$ and $m \geq 0$ is any integer, then

$$q_{u,m}(f) = \sup_{\lambda \in F^0} (1+\|\lambda\|^2)^m \, |\partial(u) f(\lambda)| < \infty.$$

(iii) f is W-invariant.

Then $Z(F)$ is an algebra under pointwise multiplication and a Frechet space when equipped with the topology induced by the seminorms $q_{u,m}$.

If $a \in Z(F)$ we define the "wave packet" ψ_a on G by

$$\psi_a(x) = \frac{1}{|W|} \int_{\mathbf{a}^*} a(\lambda)\, \varphi_\lambda(x)\, |c(\lambda)|^{-2} \, d\lambda,$$

where $c(\lambda)$ is the well known Harish-Chandra c-function. By the Plancherel theorem due to Harish-Chandra we also know that the map $f \to \hat{f}$ extends to a unitary map from $L^2(K\backslash G/K)$ onto $L^2(\mathbf{a}^*, |c(\lambda)|^{-2} d\lambda)$.

Theorem 1.1 [Trombi and Varadarajan 1971]. (i) *If $f \in S(G)$ then $\hat{f} \in Z(F)$.*

(ii) *If $a \in Z(F)$ then the integral defining the "wave packet" ψ_a converges absolutely and $\psi_a \in S(G)$. Moreover, $\hat{\psi_a} = a$.*

(iii) *The map $f \to \hat{f}$ is a topological linear isomorphism of $S(g)$ onto $Z(F)$.*

If G is of real rank one, the maximal ideal space of $I_1(G)$ is given by a certain strip domain in the complex plane that is biholomorphically equivalent to the unit disc in $\mathbb{C}$. Hence function theory in the unit disc (in particular the Beurling–Rudin theorem) can be used to study the ideals in $I_1(G)$. See [Benyamini and Weit 1992] for more details of this method. However, when the real rank of G exceeds one we need different methods. Using the Trombi–Varadarajan theorem just quoted, Sitaram [1980] proved that under certain conditions a single function will generate all of $I_1(G)$. We extend this result to an arbitrary family. Our method is as follows:

(a) From an arbitrary family of functions whose spherical Fourier transforms have no common zero, we manufacture a finite family with the same property. Here we need to use the fact that a complex analytic set admits a stratification.

(b) Next we use a result of Hörmander to generate an appropriate ring of holomorphic functions. Here the not-too-rapid decay of the Fourier transform is crucial.

(c) Finally, as in [Sitaram 1980], the Trombi–Varadarajan isomorphism result (Theorem 1.1) can be applied.

We end this section with a lemma and a proposition that will be needed later.

Lemma 1.2 [Sitaram 1980, Lemma 3.2]. *Let k be a fixed nonnegative integer and let $\varphi_k(z) = e^{\langle z,z\rangle^k}$, $z \in F$. Let X be defined by $X = \{h : h, h\varphi_k \in Z(F)\}$. Then X is a linear dense subspace of $Z(F)$.*

Proposition 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a connected domain and $\{f_\alpha\}_{\alpha\in I}$ be an arbitrary family of bounded holomorphic functions defined on Ω. Suppose that there is no $z \in \Omega$ such that $f_\alpha(z) = 0$ for all $\alpha \in I$. Then there exists functions $g_0, g_1, \dots, g_n$ such that*:

(a) *Each g_i, $i = 0, 1, \dots, n$, is an infinite linear combination of f_α's. More precisely, $g_i = \sum_{k=1}^{\infty} c_k(i) f^i_{\alpha_k}$ with $\sum_k |c_k(i)| < \infty$.*

(b) *There exists no $z \in \Omega$ such that $g_i(z) = 0$ for all $i = 0, 1, \dots, n$.*

Proof. We modify the proof of Proposition 5.7 in [Chirka 1989, page 63]. After multiplying by suitable constants we may assume that each f_α is bounded by 1. Choose a function arbitrarily from the given collection and name it g_n. The zero set Z_{g_n} of g_n is an analytic subset of Ω and so admits a stratification (page 60 of the same reference). Let M^{n-1} denote the $(n-1)$-dimensional stratum of Z_{g_n} (since Ω is connected there is no n-dimensional stratum). Using [Chirka 1989, Theorem 5.4, page 57], write M^{n-1} as a union of its irreducible components: $M^{n-1} = \bigcup_{j=1}^{k} M_j^{n-1}$, where k can be infinite. Choose $a_j \in M_j^{n-1}$ arbitrarily. By the hypothesis there exists f_1 in the given family such that $f_1(a_1) \neq 0$. We define f_j, for $j \geq 2$, as follows: If $f_{j-1}(a_j) \neq 0$ then $f_j = f_{j-1}$. Otherwise, by the hypothesis there exists a function f in the given family such that $f(a_j) \neq 0$. Define f_j to be this function f. Then $f_j(a_j) \neq 0$ for any j. Next, define constants c_j by $c_1 = \frac{1}{4}$,

$$c_j = 4^{-j} |f_1(a_1)| \cdots |f_{j-1}(a_{j-1})|, \quad j \geq 2.$$

Then $0 < c_j \le 4^{-j}$. Now define

$$g_{n-1}(z) = \sum_{k=1}^{\infty} c_k f_k(z).$$

Since $|f_k| \le 1$ the series converges uniformly, so g_{n-1} is holomorphic in Ω. Also,

$$g_{n-1}(a_j) = \left(\sum_{k=l}^{j} c_k\right) f_j(a_j) + c_{j+1} f_{j+1}(a_j) + \cdots$$

for some $1 \le l \le j$. Therefore

$$|g_{n-1}(a_j)| \ge 4^{-j} |f_1(a_1)||f_2(a_2)| \cdots |f_j(a_j)| - \sum_{m=j+1}^{\infty} c_m.$$

But

$$\sum_{m=j+1}^{\infty} c_m \le \frac{4^{-j}}{3} |f_1(a_1)| \cdots |f_j(a_j)|,$$

so $|g_{n-1}(a_j)| > 0$. It follows that $Z_{g_n} \cap Z_{g_{n-1}}$ is an analytic subset of Ω whose dimension is at most $n-2$, as in the proof of Proposition 5.7 on [Chirka 1989, page 63]. We repeat this procedure and finish the proof. □

2. A Wiener Tauberian theorem for $L^1(K\backslash G/K)$

In this section we prove, after some preliminaries, a Wiener Tauberian theorem for K-biinvariant integrable functions on G (Theorem 2.2).

Let p be a plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$. Let $A_p(\Omega)$ denote the ring of holomorphic functions f on Ω such that

$$|f(z)| \le C_1 \exp(C_2 p(z)), \quad z \in \Omega,$$

for some constants C_1 and C_2 possibly depending on f.

Theorem 2.1 [Hörmander 1967]. *Let p be a plurisubharmonic function in the open set $\Omega \subset \mathbb{C}^n$ such that*

(i) *all polynomials belong to $A_p(\Omega)$;*

(ii) *there exist constants $K_1, \ldots, K_4$ such that $z \in \Omega$ and the inequality $|z - \xi| \le \exp(-K_1 p(z) - K_2)$ implies $\xi \in \Omega$ and $p(\xi) \le K_3 p(z) + K_4$.*

Then $f_1, \ldots, f_N \in A_p(\Omega)$ generate $A_p(\Omega)$ if and only if there are positive constants c_1 and c_2 such that

$$|f_1(z)| + \cdots + |f_N(z)| \ge c_1 \exp(-c_2 p(z)), \quad z \in \Omega.$$

Let $l = \dim \mathbf{a} = \dim \mathbf{a}^*$. Write $\rho = (\rho_1, \dots \rho_l)$. We may assume that each $\rho_i \geq 0$. If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_l)$ is an l vector with $\varepsilon_i > 0$ we denote by F_ε the set

$$F_\varepsilon = \mathbf{a}^* + iC_{\rho+\varepsilon}, \quad \text{where } C_{\rho+\varepsilon} \text{ is the convex hull of } \{s(\rho+\varepsilon) : s \in W\}.$$

Theorem 2.2. *Let $\{f_\alpha : \alpha \in I\}$ be a family of functions in $I_1(G)$ such that the spherical Fourier transform $\hat{f}_\alpha$ extend to F_ε^0 as bounded holomorphic functions. Suppose that the collection $\{\hat{f}_\alpha : \alpha \in I\}$ does not have a common zero in F_ε^0. Assume further that there is an $\alpha_0 \in I$ such that*

$$|\hat{f}_{\alpha_0}(z)| \geq \exp\left(-c\sum_{j=1}^{l} |z_j|^m\right)$$

for some $c > 0$, $0 < m \in \mathbb{N}$ and for all large $z \in F_\varepsilon^0$. Then the family $\{f_\alpha : \alpha \in I\}$ generates a dense subset of $I_1(G)$.

Proof. Let Λ denote the given family of functions in $I_1(G)$ and $\hat{\Lambda}$ denote the collection of its spherical Fourier transforms. We may assume that $\|f_\alpha\|_1 \leq 1$ and $\|\hat{f}_\alpha\|_\infty \leq 1$ for all $\alpha \in I$ where $\|g\|_\infty = \sup_{z \in F_\varepsilon} |g(z)|$. Applying Proposition 1.3 we obtain finitely many functions $f_1, \dots, f_N$ in the ideal generated by Λ (in $I_1(G)$) such that $\hat{f}_1, \dots, \hat{f}_N$ have no common zero in F_ε^0. Let $\delta = (\delta_1, \dots, \delta_l)$ be an l vector such that $0 < \delta_i < \varepsilon_i$ for $i = 1, \dots, l$. Consider the domain F_δ^0. Then, by the hypothesis we have

$$|\hat{f}_{\alpha_0}(z)| + |\hat{f}_1(z)| + \cdots + |\hat{f}_N(z)| \geq c_1 \exp\left(-c_2\sum_{j=1}^{l} |z_j|^m\right), \quad z \in F_\delta \tag{2-1}$$

for some $c_1, c_2 > 0$ and $l = \dim A$.

Next, we will apply Theorem 2.1 to these $N+1$ functions. For this, consider

$$p(z) = \log(1/d(z)) + \sum_{j=1}^{l} |z_j|^m, \quad z \in F_\delta^0,$$

where $d(z)$ is the distance of z to the boundary of F_δ^0. Adding a constant to p if necessary, we may assume that p is nonnegative. Since F_δ^0 is a convex domain the function $\log(1/d(z))$ is plurisubharmonic [Hörmander 1990, Theorem 2.6.5]. Hence the function p is plurisubharmonic. It is known that $\log(1/d(z))$ satisfies condition (ii) in Theorem 2.1; see the remarks on [Hörmander 1967, page 944]. Now, it is easy to check that the same holds for our function p defined above and that $A_p(F_\delta^0)$ contains polynomials. Since each of the functions $\hat{f}_1, \dots, \hat{f}_N$ and $\hat{f}_{\alpha_0}$ is bounded (see the construction in Proposition 1.3) they too belong to $A_p(F_\delta^0)$.

The left-hand side of (2-1) is at least $c_3 \exp(-c_4 p(z))$ in the interior of the domain F_δ, for some positive constants c_3 and c_4. Applying Theorem 2.1 we obtain holomorphic functions $g_1, \dots, g_N$ and g_0 in $A_p(F_\delta^0)$ such that

$$g_1(z)\hat{f}_1(z) + \cdots + g_N(z)\hat{f}_N(z) + g_0(z)\hat{f}_{\alpha_0}(z) = 1, \quad z \in F_\delta^0. \tag{2-2}$$

We may assume that the functions g_j are W-invariant. Let $\eta = (\eta_1, \dots, \eta_l)$ be another l vector such that $0 < \eta_j < \delta_j$ for $j = 1, \dots, l$. By the l-dimensional version of Cauchy's formula, all the derivatives of g_j, $j = 0, 1, \dots, N$, satisfy the same growth conditions as the g_j in the domain F_η^0. Hence, if k is a large enough positive integer, $g_j(z)\phi_k(z)$ will belong to $Z(F)$, where

$$\phi_k(z) = e^{-\langle z,z\rangle^k}.$$

That is (by Theorem 1.1), there are functions $\psi_j \in S(G)$ for $j = 0, 1, \dots, N$ such that $\hat{\psi}_j = \phi_k g_j$. Hence if f is any function in the L^1 Schwartz space $S(G)$, from (2-2) we have

$$\hat{f}\phi_k = (\hat{f}\psi_1)\hat{f}_1 + \cdots + (\hat{f}\psi_N)\hat{f}_N + (\hat{f}\psi_0)\hat{f}_{\alpha_0}.$$

Now the proof can be completed as in [Sitaram 1980] using Theorem 1.1 and Lemma 1.2. □

Remarks. (1) The generating family is assumed to have spherical Fourier transforms defined on a larger domain than the maximal ideal space. Even for the case of real rank one this assumption was crucial. See [Benyamini and Weit 1992; Sarkar 1998]. The condition of not-too-rapid decay is assumed on the whole domain F_ε^0; it is a stronger condition than in the rank-one case.

(2) Results similar to Theorem 2.2 can be proved for $L^p(K\backslash G/K)$; see [Sitaram 1980, Theorem 4.1].

3. Rank-one symmetric spaces revisited

In this section we assume that the real rank of G is one. Let G/K be the associated Riemannian symmetric space of noncompact type. Our aim is to derive a Wiener Tauberian theorem for the space $L^1(G/K)$ with the aid of a similar theorem for biinvariant functions and the simplicity criterion (for λ's), under certain decay conditions on the generating functions (instead of the condition of not-too-rapid decay on the Fourier transform). Although a similar result appears in [Sarkar 1998], our proof is simple and different from the one given there, which requires constructing Schwartz class functions on the whole group G with prescribed properties on the Fourier transform. We use the simplicity criterion and averaging over K instead. Moreover, our method is valid for higher-rank cases too, and a strengthening of Theorem 2.2 will readily imply a Wiener Tauberian theorem for $L^1(G/K)$. We shall state our result in terms of the Helgason Fourier transform.

The Helgason Fourier transform of a suitable function f on G/K is the function on $\mathbf{a}^* \times K/M$ defined by

$$\tilde{f}(\lambda, k) = \tilde{f}(\lambda, kM) = \int_{G/K} f(x) e^{(i\lambda-\rho)H(x^{-1}k)}\, dx$$

where $\lambda \in \mathbf{a}^*$ and $k \in K$. Here dx denotes the essentially unique left G-invariant measure on G/K. We have a Plancherel theorem, which reads

$$\int_X |f(x)|^2\, dx = \frac{1}{|W|} \int_{\mathbf{a}^* \times K/M} |\tilde{f}(\lambda, kM)|^2\, |c(\lambda)|^{-2}\, d\lambda\, dk.$$

For other properties of this transform we refer to [Helgason 1994].

The domain F^0 defined in the previous sections becomes a horizontal strip in the complex plane (since G is of real rank one) and F^0_ε is an enlarged strip. We shall use the following result for K-biinvariant functions:

Theorem 3.1 [Benyamini and Weit 1992; Sarkar 1998]. *Let $\{f_\alpha : \alpha \in I\}$ be a family of functions in $I_1(G)$ such that the spherical Fourier transform $\hat{f}_\alpha$ extends holomorphically to the strip F^0_ε for some $\varepsilon > 0$. Suppose the collection $\{\hat{f}_\alpha : \alpha \in I\}$ does not vanish simultaneously on any point in F^0_ε. Assume further that there is an $\alpha_0 \in I$ such that $\hat{f}_{\alpha_0}$ satisfies the decay condition $\limsup_{|\lambda|\to\infty} |\hat{f}_{\alpha_0}(\lambda)| \cdot |\exp(ke^{|\lambda|})| > 0$ for all $k > 0$. Then the given collection generates a dense subset of $I_1(G)$.*

Our result is as follows:

Theorem 3.2. *Let f be a function on G/K that satisfies the decay assumption $|f(x)| \le Ce^{-\beta\sigma(x)^2}$, for some $\beta > 0$. Assume further that there is no $\lambda \in F^0_\varepsilon$ such that $\tilde{f}(\lambda, k)$ is identically zero as a function on K/M. Then the left G-translates of f span a dense subset of $L^1(G/K)$.*

Proof. Since f has exponential decay, $\tilde{f}(\lambda, k)$ extends as a holomorphic function to all of $\mathbf{a}^*_{\mathbb{C}}$ and is a C^∞ function in the k variable. Let V_f denote the closed span of left G-translates of the given function f. It suffices to show that $L^1(K\backslash G/K) \subset V_f$.

For each $g \in G$, define a K-biinvariant function $f_g(x) = \int_K f(gkx)\, dx$. Then $f_g \in V_f$ for all $g \in G$ and each f_g satisfies a decay estimate similar to that of f (with a smaller β). Also, the spherical Fourier transform of f_g is

$$\hat{f}_g(\lambda) = f * \varphi_\lambda(g) = \int_K \tilde{f}(\lambda, k)\, e^{(i\lambda-\rho)(H(g^{-1}k))}\, dk,$$

which is the Poisson transform of the function $k \to \tilde{f}(\lambda, k)$ [Helgason 1994]. The Poisson transform is injective if and only if λ is simple, which is the case when $\mathrm{Re}(i\lambda) \ge 0$ [Helgason 1994].

Now consider the collection of K-biinvariant functions $\{f_g : g \in G\}$. For any $\lambda \in F^0_\varepsilon$ with $\mathrm{Re}(i\lambda) \ge 0$ it is not possible to have $\hat{f}_g(\lambda) = 0$ for all $g \in G$, as this will contradict the simplicity of λ. Since $\hat{f}_g$ are even functions, it follows that there is no $\lambda \in F_\varepsilon^0$ such that $\hat{f}_g(\lambda) = 0$ for all $g \in G$. The decay condition for the spherical Fourier transform will be satisfied because of the Hardy uncertainty principles [Sitaram and Sundari 1997; Sarkar 1998, page 356]. By Theorem 3.1 it follows that $I_1(G) \subset V_f$, which finishes the proof. □

Remarks. (1) Using Proposition 4.1 in [Helgason 1994] it is easy to see that this method works well for the higher-rank case too, so long as an analogue of Theorem 3.1 is true. This amounts to weakening the decay condition in Theorem 2.2.

(2) Theorem 3.2 can also be formulated for a family of functions.

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