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**MULTIPLICITY OF SOLUTIONS FOR A CLASS OF  
RESONANT  $p$ -LAPLACIAN DIRICHLET PROBLEMS**

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## MULTIPLICITY OF SOLUTIONS FOR A CLASS OF RESONANT $p$ -LAPLACIAN DIRICHLET PROBLEMS

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**We consider nonlinear Dirichlet problems driven by the  $p$ -Laplacian, which are resonant at  $+\infty$  with respect to the principal eigenvalue. Using a variational approach based on the critical point theory, we show that the problem has three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative). In the semilinear case, assuming stronger regularity on the nonlinear perturbation  $f(z, \cdot)$  and using Morse theory, we show that the problem has at least four nontrivial smooth solutions, two of constant sign.**

### 1. Introduction

Let  $Z \subseteq \mathbb{R}^N$  be a bounded domain with a  $C^2$ -boundary  $\partial Z$ . In this paper, we study the following nonlinear Dirichlet problem driven by the  $p$ -Laplacian differential operator:

$$(1-1) \quad \begin{aligned} -\Delta_p x(z) &= f(z, x(z)) \quad \text{a.e. on } Z, \\ x|_{\partial Z} &= 0. \end{aligned}$$

Here,  $\Delta_p$  denotes the  $p$ -Laplace differential operator, namely,

$$\Delta_p u = \operatorname{div}(\|Du\|^{p-2} Du), \quad 1 < p < \infty.$$

The aim of this work is to prove the existence of three nontrivial smooth solutions when resonance occurs at infinity and the Euler functional of the problem need not be coercive. Recently, three-solution theorems for the Dirichlet  $p$ -Laplacian were proved in [Carl and Perera 2002; Liu 2006; Liu and Liu 2005; Papageorgiou and Papageorgiou 2007; Zhang et al. 2004]. In all these works, either nonresonance is assumed or the Euler functional is coercive or both. In this work, in addition to the resonance condition at  $+\infty$  with respect to the principal eigenvalue  $\lambda_1 > 0$  of  $(-\Delta_p, W_0^{1,p}(Z))$  near the origin, we require that the quotient (*slope*)  $\frac{F(z,x)}{|x|^p}$  (here  $F(z, x) = \int_0^x f(z, s) ds$  the primitive of  $f(z, \cdot)$ ) stays strictly above  $\lambda_2 > 0$ ,

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the second eigenvalue of  $(-\Delta_p, W_0^{1,p}(Z))$ . This way, we can take advantage of an alternative variational characterization of  $\lambda_2 > 0$  by Cuesta, Figueiredo and Gossez [Cuesta et al. 1999]. Our approach is variational and based on the critical point theory. In the special case  $p = 2$  (semilinear problem), using in addition Morse theory and stronger regularity conditions on  $f(z, \cdot)$ , we are able to produce four nontrivial smooth solutions.

## 2. Mathematical background

We start by recalling some elements from critical point theory and from Morse theory, which we will need in the sequel. So, let  $X$  be a Banach space and let  $X^*$  be its topological dual. By  $\langle \cdot, \cdot \rangle$  we denote the duality brackets for the pair  $(X^*, X)$ . Given  $\varphi \in C^1(X)$ , we say that  $\varphi$  satisfies the *Palais–Smale condition* (the *PS-condition* for short) if every  $\{x_n\}_{n \geq 1} \subseteq X$  such that  $\{\varphi(x_n)\}_{n \geq 1}$  is bounded and  $\varphi'(x_n) \rightarrow 0$  in  $X^*$  as  $n \rightarrow \infty$  admits a strongly convergent subsequence.

The following minimax principle is known in the literature as the *mountain pass theorem*.

**Theorem 2.1.** *If  $\varphi \in C^1(X)$ ,  $\varphi$  satisfies the PS-condition,  $x_0, x_1 \in X$ ,*

$$\|x_1 - x_0\| > r > 0, \quad \max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = c_r$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t))$$

where  $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$ , then  $c \geq c_r$  and  $c$  is a critical value of  $\varphi$ , that is, there exists  $x \in X$  such that  $\varphi(x) = c$  and  $\varphi'(x) = 0$ .

Another result from critical point theory, which we will use in the analysis of problem (1-1), is the so-called *second deformation theorem* [Chang 1993, p. 23]. Let  $\varphi \in C^1(X)$  and  $c \in \mathbb{R}$ . We define

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(x) \leq c\} && \text{as the sublevel set of } \varphi \text{ at } c, \\ K &= \{x \in X : \varphi'(x) = 0\} && \text{as the critical set of } \varphi \text{ and} \\ K_c &= \{x \in K : \varphi(x) = c\} && \text{as the critical set of } \varphi \text{ at the level } c. \end{aligned}$$

In the next theorem (the second deformation theorem) we allow  $b = +\infty$  in which case  $\varphi^b \setminus K_b = X$ .

**Theorem 2.2.** *If  $\varphi \in C^1(X)$ ,  $\varphi$  satisfies the PS-condition,  $a \in \mathbb{R}$ ,  $a < b \leq +\infty$ ,  $\varphi$  has no critical values in  $(a, b)$  and  $\varphi^{-1}(a)$  contains at most a finite number of critical points of  $\varphi$ , then there exists a deformation*

$$h : [0, 1] \times (\varphi^b \setminus K_b) \rightarrow \varphi^b$$

such that

- (a)  $h(1, \varphi^b \setminus K_b) \subseteq \varphi^a$ ,
- (b)  $h(t, x) = x$  for all  $(t, x) \in [0, 1] \times \varphi^a$  and
- (c)  $\varphi(h(t, x)) \leq \varphi(h(s, x))$  for all  $s, t \in [0, 1], s \leq t$  and all  $x \in \varphi^b \setminus K_b$ .

Let  $(Y_1, Y_2)$  be a topological pair with  $Y_2 \subseteq Y_1 \subseteq X$ . For every integer  $k \geq 0$ , by  $H_k(Y_2, Y_1)$  we denote the  $k$ -th relative singular homology group of  $(Y_1, Y_2)$  with coefficients in  $\mathbb{Z}$ . The critical groups of  $\varphi \in C^1(X)$  at an isolated critical point  $x_0 \in X$  with  $\varphi(x_0) = c$  are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}) \quad \text{for all } k \geq 0,$$

where  $U$  is a neighborhood of  $x_0$  such that

$$K \cap \varphi^c \cap U = \{x_0\}$$

(see [Chang 1993; Mawhin and Willem 1989]). The excision property of singular homology implies that this definition of critical groups is independent of the particular neighborhood  $U$  we use.

Suppose that  $\varphi \in C^1(X)$  satisfies the PS-condition and  $-\infty < \inf \varphi(K)$ . Choosing  $c < \inf \varphi(K)$ , we define the *critical groups of  $\varphi$  at infinity* by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0$$

(see [Bartsch and Li 1997]). If  $K$  is finite, we set

$$M(t, x) = \sum_{k \geq 0} \text{rank} C_k(\varphi, x) t^k \quad \text{for all } x \in K,$$

$$P(t, \infty) = \sum_{k \geq 0} \text{rank} C_k(\varphi, \infty) t^k.$$

Then the Morse relation holds

$$(2-1) \quad \sum_{x \in K} M(t, x) = P(t, \infty) + (1 + t)Q(t)$$

where

$$Q(t) = \sum_{k \geq 0} \beta_k t^k$$

is a formal series with nonnegative integer coefficients (see [Chang 1993, p. 36] and [Mawhin and Willem 1989, p. 184]).

Let  $X = H$  be a Hilbert space,  $x \in H$  is a critical point of  $\varphi \in C^1(H)$  and for  $U$  a neighborhood of  $x$ ,  $\varphi \in C^2(U)$ . Then the *Morse index* of  $\varphi$  is defined as the supremum of the dimensions of the vector subspaces of  $H$  on which  $\varphi''(x)$  is negative definite.

**Definition 2.3.** A map  $A : X \rightarrow X^*$  is said to be of *type*  $(S)_+$  if for any sequence  $\{x_n\}_{n \geq 1} \subseteq X$  for which  $x_n \xrightarrow{w} x$  in  $X$  and

$$\limsup_{n \rightarrow \infty} \langle A(x_n), x_n - x \rangle \leq 0$$

one has  $x_n \rightarrow x$  in  $X$ .

Let  $A : W_0^{1,p}(Z) \rightarrow W^{-1,p'}(Z) = W_0^{1,p}(Z)^*$  ( $\frac{1}{p} + \frac{1}{p'} = 1$ ) be the nonlinear map corresponding to  $-\Delta_p$ , namely,

$$(2-2) \quad \langle A(x), y \rangle = \int_Z \|Dx\|^{p-2} (Dx, Dy)_{\mathbb{R}^N} dz \quad \text{for all } x, y \in W_0^{1,p}(Z).$$

Hereafter by  $\langle \cdot, \cdot \rangle$  we denote the duality brackets for the pair

$$(W^{-1,p'}(Z), W_0^{1,p}(Z)).$$

For the map  $A$ , we have the following result (see [Gasiński and Papageorgiou 2006] for example).

**Proposition 2.4.** *The map  $A : W_0^{1,p}(Z) \rightarrow W^{-1,p'}(Z)$  defined by (2-2) is maximal monotone, strictly monotone and of type  $(S)_+$ .*

Finally let us recall some basic facts about the spectrum of negative Dirichlet  $p$ -Laplacian denoted by  $(-\Delta_p, W_0^{1,p}(Z))$ . For details and additional references, see [Cuesta 2001; Lê 2006]. Of course, if  $p = 2$  then  $\Delta_p = \Delta$ , the usual Laplace differential operator defined on  $H_0^1(Z)$ . We consider the following nonlinear eigenvalue problem:

$$(2-3) \quad \begin{aligned} -\Delta_p u(z) &= \lambda |u(z)|^{p-2} u(z) \quad \text{a.e. on } Z, \\ u|_{\partial Z} &= 0. \end{aligned}$$

Every  $\lambda \in \mathbb{R}$  for which problem (2-3) has a nontrivial solution is said to be an eigenvalue of  $(-\Delta_p, W_0^{1,p}(Z))$ . The smallest eigenvalue  $\lambda_1$  of  $(-\Delta_p, W_0^{1,p}(Z))$  is positive, simple and admits the following variational characterization

$$(2-4) \quad \lambda_1 = \inf \left\{ \frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(Z), u \neq 0 \right\}.$$

Let  $u_1$  be the  $L^p$ -normalized eigenfunction corresponding to  $\lambda_1 > 0$ . We know that  $u_1 \in C_0^1(\bar{Z})$  (see [Lieberman 1988]). The Banach space  $C_0^1(\bar{Z})$  is an ordered Banach space with the order cone

$$C_+ = \{u \in C_0^1(\bar{Z}) : u(z) \geq 0 \text{ for all } z \in \bar{Z}\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_0^1(\bar{Z}) : u(z) > 0 \text{ for all } z \in Z, \frac{\partial u}{\partial n}(z) < 0 \text{ on } \partial Z \right\},$$

where  $n$  denotes the outward unit normal on  $\partial Z$ . The nonlinear strong maximum principle of Vázquez [1984] implies that  $u_1 \in \text{int} C_+$ .

The Ljusternik–Schnirelmann theory in addition to  $\lambda_1 > 0$  provides a whole strictly increasing sequence  $\{\lambda_k\}_{k \geq 1} \subseteq \mathbb{R}_+$  of eigenvalues for problem (2-3), known as the LS-eigenvalues of  $(-\Delta_p, W_0^{1,p}(Z))$ . If  $p = 2$  (linear eigenvalue problem) then the LS-eigenvalues are all the eigenvalues of  $(-\Delta, H_0^1(Z))$ . If  $p \neq 2$  then we do not know if this is true. Nevertheless we know that  $\lambda_2$  is the second eigenvalue of  $(-\Delta_p, W_0^{1,p}(Z))$ . So the Ljusternik–Schnirelmann theory provides a variational characterization of  $\lambda_2 > 0$ . However, for our purposes, that characterization is not convenient. Instead, we will use an alternative one produced by Cuesta, Figueiredo and Gossez [Cuesta et al. 1999]. More precisely, let

$$\begin{aligned} \partial B_1^{L^p} &= \{u \in L^p(Z) : \|u\|_p = 1\}, \\ S &= W_0^{1,p}(Z) \cap \partial B_1^{L^p} \end{aligned}$$

endowed with the relative  $W_0^{1,p}(Z)$ -topology and

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1, 1], S) : \hat{\gamma}(-1) = -u_1, \hat{\gamma}(1) = u_1\}.$$

Then

$$(2-5) \quad \lambda_2 = \inf_{\hat{\gamma} \in \hat{\Gamma}} \max_{-1 \leq t \leq 1} \|D\hat{\gamma}(t)\|_p^p.$$

Finally, if  $p = 2$  and  $m \in L^\infty(Z)$ ,  $m^+ \neq 0$  (weight function), we consider the linear eigenvalue problem

$$-\Delta u(z) = \lambda m(z)u(z) \quad \text{a.e. on } Z, \quad u|_{\partial Z} = 0.$$

This problem has a sequence  $\{\hat{\lambda}_k(m)\}_{k \geq 1}$  of positive eigenvalues

$$\hat{\lambda}_1(m) < \hat{\lambda}_2(m) < \dots < \hat{\lambda}_k(m) \rightarrow +\infty$$

and a sequence  $\{\hat{\lambda}_{-k}(m)\}_{k \geq 1}$  of negative eigenvalues

$$0 > \hat{\lambda}_{-1}(m) > \hat{\lambda}_{-2}(m) > \dots > \hat{\lambda}_{-k}(m) \rightarrow -\infty.$$

We know that  $\hat{\lambda}_1(m)$  is simple, isolated and

$$\hat{\lambda}_1(m) = \inf \left\{ \frac{\|Du\|_2^2}{\int_Z mu^2 dz} : u \in H_0^1(Z), u \neq 0 \right\}.$$

Similarly for  $\hat{\lambda}_{-1}(m)$ . If  $m \equiv 1$ , then  $\hat{\lambda}_k(m) = \lambda_k$  for  $k \in \mathbb{Z} \setminus \{0\}$ .

### 3. The nonlinear problem

In this section, using a variational approach, we produce three nontrivial smooth solutions, two of which have constant sign.

The nonlinearity hypotheses on  $f(z, x)$  are the following:

$H_1$ :  $f : Z \times \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $f(z, 0) = 0$  a.e. on  $Z$ .

- (i) For all  $x \in \mathbb{R}$ ,  $z \rightarrow f(z, x)$  is measurable.
- (ii) For almost all  $z \in Z$ ,  $x \rightarrow f(z, x)$  is continuous.
- (iii) For almost all  $z \in Z$  and all  $x \in \mathbb{R}$ ,

$$|f(z, x)| \leq a(z) + c|x|^{r-1}$$

with  $a \in L^\infty(Z)_+$ ,  $c > 0$  and

$$p < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p, \\ +\infty & \text{if } p \geq N. \end{cases}$$

- (iv) For almost all  $z \in Z$  and all  $x \geq 0$ ,  $f(z, x) \geq 0$  and if  $F(z, x) = \int_0^x f(z, s) ds$  then

$$\lim_{x \rightarrow +\infty} \frac{pF(z, x)}{|x|^p} = \lambda_1 \quad \text{and} \quad \lim_{x \rightarrow +\infty} (pF(z, x) - \lambda_1 x^p) = -\infty,$$

both uniformly for almost all  $z \in Z$ .

- (v) There exist  $a < 0$  and  $\xi > 0$  such that for almost all  $z \in Z$ ,  $f(z, a) = 0$ ,  $f(z, x) \leq 0$  for all  $x \in [a, 0]$  and

$$x \rightarrow \xi|x|^{p-2}x + f(z, x)$$

is nondecreasing on  $[a, 0]$ .

- (vi) There exist  $\delta_0 > 0$  and  $\xi_0 > \lambda_2$  such that

$$F(z, x) \geq \frac{\xi_0}{p}|x|^p \quad \text{for a.a. } z \in Z, \text{ all } |x| \leq \delta_0.$$

**Remark 3.1.** Hypothesis  $H_1$ (iv) implies that at  $+\infty$  we have resonance with respect to  $\lambda_1 > 0$  from the left. The hypotheses on  $f(z, \cdot)$  on the negative semiaxis are minimal and allow the Euler functional to be strongly indefinite in the negative direction. Specifically, we only assume  $H_1$ (v). Nevertheless, this condition with suitable truncation techniques and with the use of the nonlinear maximum principle of Vázquez [1984] leads to a negative solution of (1-1). We point out that the conditions on the two semiaxes are asymmetric.



**Example 3.2.** The following function  $f(x)$  satisfies hypotheses  $H_1$  (for the sake of simplicity we drop the  $z$ -dependence):

$$f(x) = \begin{cases} \hat{\xi}|x|^{p-2}x - \hat{\xi}|x|^{q-2}x & \text{if } x \leq 0, \\ \hat{\xi}|x|^{p-2}x & \text{if } x \in [0, 1], \\ \lambda_1|x|^{p-2}x + c|x|^{\tau-2}x & \text{if } x \geq 1, \end{cases}$$

with  $\tau < p < q < p^*$ ,  $\hat{\xi} > \lambda_2$  and  $c = \hat{\xi} - \lambda_1$ .

First using truncation and variational techniques, we will produce two nontrivial smooth solutions of constant sign (one positive and the other negative). To this end, we introduce the following truncation of the nonlinearity  $f(z, \cdot)$ :

$$f_+(z, x) = \begin{cases} 0 & \text{if } x \leq 0, \\ f(z, x) & \text{if } x \geq 0. \end{cases}$$

We set

$$F_+(z, x) = \int_0^x f_+(z, s) ds$$

and introduce the functional  $\varphi_+ : W_0^{1,p}(Z) \rightarrow \mathbb{R}$  defined by

$$\varphi_+(x) = \frac{1}{p} \|Dx\|_p^p - \int_Z F_+(z, x(z)) dz \quad \text{for all } x \in W_0^{1,p}(Z).$$

Evidently  $\varphi_+ \in C^1(W_0^{1,p}(Z))$ .

**Proposition 3.3.** *If hypotheses  $H_1$  hold then  $\varphi_+$  is coercive.*

*Proof.* We proceed by contradiction. So suppose that  $\varphi_+$  is not coercive. Then we can find  $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$  such that

$$(3-1) \quad \|x_n\| \rightarrow \infty \text{ as } n \rightarrow \infty \quad \text{and} \quad \varphi_+(x_n) \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1.$$

Let  $y_n = x_n / \|x_n\|$ , for  $n \geq 1$ . Then  $\|y_n\| = 1$  for all  $n \geq 1$  and we may assume that

$$(3-2) \quad \begin{aligned} y_n &\overset{w}{\rightharpoonup} y \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad y_n \rightarrow y \text{ in } L^p(Z) \text{ as } n \rightarrow \infty \\ &\Rightarrow y_n^\pm \overset{w}{\rightharpoonup} y^\pm \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad y_n^\pm \rightarrow y^\pm \text{ in } L^p(Z) \text{ as } n \rightarrow \infty. \end{aligned}$$

By virtue of hypotheses  $H_1$ (iii)–(iv), we have

$$(3-3) \quad F_+(z, x) \leq \frac{\lambda_1}{p} (x^+)^p + c_1 \quad \text{for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and some } c_1 > 0.$$

From (3-1) we have

$$(3-4) \quad \begin{aligned} M_1 &\geq \frac{1}{p} \|Dx_n^+\|_p^p + \frac{1}{p} \|Dx_n^-\|_p^p - \int_Z F_+(z, x_n(z)) dz \\ &\geq \frac{1}{p} \|Dx_n^+\|_p^p + \frac{1}{p} \|Dx_n^-\|_p^p - \frac{\lambda_1}{p} \|x_n^+\|_p^p - c_2 \end{aligned}$$

for some  $c_2 > 0$ , all  $n \geq 1$  (see (3-3))

$$\begin{aligned} &\Rightarrow M_1 \geq \frac{1}{p} \|Dx_n^-\|_p^p - c_2 \quad \text{for all } n \geq 1 \text{ (see (2-4))} \\ (3-5) \quad &\Rightarrow \{x_n^-\}_{n \geq 1} \subseteq W_0^{1,p}(Z) \text{ is bounded.} \end{aligned}$$

Therefore  $\|x_n^+\| \rightarrow \infty$ ,  $y^- = 0$ ,

$$(3-6) \quad y_n^+ \xrightarrow{w} y \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad y_n^+ \rightarrow y \text{ in } L^p(Z), \quad y \geq 0 \text{ (see (3-2)).}$$

From (3-4) and (3-5) we have

$$\frac{1}{p} \|Dx_n^+\|_p^p - \frac{\lambda_1}{p} \|x_n^+\|_p^p \leq M_2$$

for some  $M_2 > 0$ , all  $n \geq 1$

$$(3-7) \quad \Rightarrow \frac{1}{p} \|Dy_n^+\|_p^p - \frac{\lambda_1}{p} \|y_n^+\|_p^p \leq \frac{M_2}{\|x_n^+\|_p^p} \quad \text{for all } n \geq 1.$$

Passing to the limit as  $n \rightarrow \infty$  and using (3-6), we obtain

$$\|Dy\|_p^p \leq \lambda_1 \|y\|_p^p \Rightarrow y = 0 \text{ or } y = \eta u_1 \text{ for some } \eta > 0 \text{ (recall } y \geq 0).$$

If  $y = 0$  then from (3-6) and (3-7) it is clear that

$$\|Dy_n^+\|_p \rightarrow 0 \Rightarrow y_n^+ \rightarrow 0 \text{ in } W_0^{1,p}(Z) \Rightarrow y_n \rightarrow 0 \text{ in } W_0^{1,p}(Z) \quad \text{(see (3-5)),}$$

a contradiction to the fact that  $\|y_n\| = 1$  for all  $n \geq 1$ .

If  $y = \eta u_1$  then recalling that  $u_1(z) > 0$  for all  $z \in Z$  we have  $x_n^+(z) \rightarrow +\infty$  for almost all  $z \in Z$  and so, by virtue of  $H_1(\text{iv})$ , we have

$$(3-8) \quad F_+(z, x_n^+(z)) - \frac{1}{p} \lambda_1 x_n^+(z)^p \rightarrow -\infty \quad \text{for a.a. } z \in Z, \text{ as } n \rightarrow \infty.$$

From (3-4) and (3-5), we have

$$\frac{1}{p} \|Dx_n^+\|_p^p - \frac{\lambda_1}{p} \|x_n^+\|_p^p - \int_Z \left( F_+(z, x_n^+(z)) - \frac{1}{p} \lambda_1 x_n^+(z)^p \right) dz \leq M_3$$

for some  $M_3 > 0$ , all  $n \geq 1$

$$(3-9) \quad \Rightarrow - \int_Z \left( F_+(z, x_n^+(z)) - \frac{\lambda_1}{p} x_n^+(z)^p \right) dz \leq M_3 \quad \text{(see (2-4)).}$$

If in (3-9) we pass to the limit as  $n \rightarrow \infty$  and use Fatou's lemma and (3-8), we reach a contradiction. This proves that  $\varphi_+$  is coercive.  $\square$

Also we consider the following truncation of  $f(z, \cdot)$ :

$$\hat{f}(z, x) = \begin{cases} 0 & \text{if } x < a, \\ f(z, x) & \text{if } x \geq a. \end{cases}$$

Set  $\hat{F}(z, x) = \int_0^x \hat{f}(z, s) ds$  and let  $\hat{\varphi} : W_0^{1,p}(Z) \rightarrow \mathbb{R}$  be the functional defined by

$$\hat{\varphi}(x) = \frac{1}{p} \|Dx\|_p^p - \int_Z \hat{F}(z, x(z)) dz \quad \text{for all } x \in W_0^{1,p}(Z).$$

Clearly  $\hat{\varphi} \in C^1(W_0^{1,p}(Z))$ .

**Proposition 3.4.** *If hypotheses  $H_1$  hold then problem (1-1) has a solution*

$$x_0 \in \text{int}C_+$$

which is a local minimizer of the functional  $\hat{\varphi}$ .

*Proof.* Exploiting the compact embedding of  $W_0^{1,p}(Z)$  into  $L^r(Z)$ , we can easily verify that the functional  $\varphi_+$  is sequentially weakly lower semicontinuous. Since  $\varphi_+$  is coercive (see Proposition 3.3), we can apply the Weierstrass theorem and obtain  $x_0 \in W_0^{1,p}(Z)$  such that

$$(3-10) \quad \varphi_+(x_0) = m_+ = \inf \varphi_+.$$

We claim that  $m_+ < 0$ . Indeed, since  $u_1 \in \text{int}C_+$ , we can find  $t > 0$  small such that

$$(3-11) \quad \begin{aligned} 0 \leq tu_1(z) \leq \delta_0 \quad & \text{for all } z \in \bar{Z} \\ \Rightarrow F_+(z, tu_1(z)) \geq \frac{\xi_0}{p} t^p u_1(z)^p \quad & \text{a.e. on } Z \text{ (see } H_1(\text{vi})) \\ \Rightarrow \varphi_+(tu_1) \leq \frac{t^p}{p} (\lambda_1 - \xi_0) < 0 \quad & \text{(since } \|Du_1\|_p^p = \lambda_1 \|u_1\|_p^p) \Rightarrow m_+ < 0. \end{aligned}$$

Hence

$$\varphi_+(x_0) = m_+ < 0 = \varphi_+(0) \quad \text{(see (3-10) and (3-11))} \Rightarrow x_0 \neq 0.$$

From (3-10) we have

$$(3-12) \quad \varphi'_+(x_0) = 0 \Rightarrow A(x_0) = N_+(x_0)$$

where  $N_+(u)(\cdot) = f_+(\cdot, u(\cdot))$  for all  $u \in W_0^{1,p}(Z)$ .

On (3-12) we act with  $-x_0^- \in W_0^{1,p}(Z)$  and obtain

$$\|Dx_0^-\|_p^p = 0 \quad \text{(recall } f_+(z, x) = 0 \text{ for a.a. } z \in Z, \text{ all } x \leq 0) \Rightarrow x_0 \geq 0, x_0 \neq 0.$$

From (3-9) we have

$$(3-13) \quad -\Delta_p x_0(z) = f_+(z, x_0(z)) \geq 0 \quad \text{a.e. on } Z \text{ (see } H_1(\text{iv})).$$

Nonlinear regularity theory (see [Lieberman 1988]) implies that  $x_0 \in C_+ \setminus \{0\}$ . Moreover, from (3-13) we have

$$\Delta_p x_0(z) \leq 0 \quad \text{a.e. on } Z.$$

Invoking the nonlinear strong maximum principle of Vázquez, we conclude that

$$x_0 \in \text{int} C_+.$$

Note that  $\varphi_+|_{C_+} = \hat{\varphi}|_{C_+}$ . So it follows that  $x_0$  is a local  $C_0^1(\bar{Z})$ -minimizer of  $\hat{\varphi}$ . Then, by [García Azorero et al. 2000, Theorem 1],  $x_0$  is also a local  $W_0^{1,p}(Z)$ -minimizer of  $\hat{\varphi}$ .  $\square$

Next we produce a negative solution. For this purpose, we introduce the following truncation of the nonlinearity  $f(z, \cdot)$ :

$$\hat{f}_-(z, x) = \begin{cases} 0 & \text{if } x \leq a, \\ f(z, x) & \text{if } a \leq x \leq 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

We set  $\hat{F}_-(z, x) = \int_0^x \hat{f}_-(z, s) ds$  and then define the functional

$$\begin{aligned} \hat{\varphi}_- : W_0^{1,p}(Z) &\rightarrow \mathbb{R}, \\ \hat{\varphi}_-(x) &= \frac{1}{p} \|Dx\|_p^p - \int_Z \hat{F}_-(z, x(z)) dz \quad \text{for all } x \in W_0^{1,p}(Z). \end{aligned}$$

Again we have  $\hat{\varphi}_- \in C^1(W_0^{1,p}(Z))$ .

**Proposition 3.5.** *If hypotheses  $H_1$  hold then problem (1-1) has a solution*

$$v_0 \in -\text{int} C_+$$

which is a local minimizer of the functional  $\hat{\varphi}_-$ .

*Proof.* Clearly  $\hat{\varphi}_-$  is coercive and it is also sequentially weakly lower semicontinuous. Therefore, we can find  $v_0 \in W_0^{1,p}(Z)$  such that

$$(3-14) \quad \hat{\varphi}_-(v_0) = \hat{m}_- = \inf \hat{\varphi}_-$$

by the Weierstrass theorem.

As we did for  $\varphi_+$  (see the proof of Proposition 3.4), by choosing  $t > 0$  small such that

$$\max\{-\delta_0, a\} \leq -tu_1(z) \leq 0$$

for all  $z \in \bar{Z}$ , we can show using hypothesis  $H_1(\text{vi})$  that

$$\hat{\varphi}_-(v_0) = \hat{m}_- < 0 = \hat{\varphi}_-(0) \Rightarrow v_0 \neq 0.$$

From (3-14), we have

$$(3-15) \quad A(v_0) = \hat{N}_-(v_0)$$

where  $\hat{N}_-(u)(\cdot) = \hat{f}_-(\cdot, u(\cdot))$  for all  $u \in W_0^{1,p}(Z)$ .

On (3-15) we act with  $v_0^+ \in W_0^{1,p}(Z)$  and obtain

$$\|Dv_0^+\|_p^p = 0 \quad (\text{since } \hat{f}_-(z, x) = 0 \text{ for a.a. } z \in Z, \text{ all } x \geq 0) \Rightarrow v_0 \leq 0, v_0 \neq 0.$$

Also from (3-15) we have

$$\begin{aligned} \Delta_p(-v_0(z)) &= -\Delta_p v_0(z) = \hat{f}_-(z, v_0(z)) \leq 0 \quad \text{a.e. on } Z \text{ (see } H_1(v)) \\ &\Rightarrow v_0 \in -\text{int}C_+ \quad (\text{see [Vázquez 1984]}). \end{aligned}$$

If we act with  $(a - v_0)^+ \in W_0^{1,p}(Z)$  on (3-15), we obtain

$$\begin{aligned} (3-16) \quad \langle A(v_0), (a - v_0)^+ \rangle &= \int_{\{a > v_0\}} \hat{f}_-(z, v_0)(a - v_0) dz = 0 \\ &\Rightarrow \|D(a - v_0)^+\|_p^p = 0 \Rightarrow a \leq v_0. \end{aligned}$$

By virtue of hypothesis  $H_1(v)$ , we have

$$\begin{aligned} -\Delta_p v_0(z) + \zeta |v_0(z)|^{p-2} v_0(z) &= f(z, v_0(z)) + \zeta |v_0(z)|^{p-2} v_0(z) \geq \zeta |a|^{p-2} a \quad \text{a.e. on } Z \text{ (see (3-16))} \\ &\Rightarrow \Delta_p v_0(z) + \zeta (|a|^{p-2} a - |v_0(z)|^{p-2} v_0(z)) \leq 0 \quad \text{a.e. on } Z. \end{aligned}$$

Involving the tangency principle of Serrin [1970] (see also [Pucci and Serrin 2007, p. 35]), we obtain

$$v_0(z) > a \quad \text{for all } z \in \bar{Z}.$$

Then it follows from the definition of  $\hat{\phi}_-$  that we can find  $\varrho > 0$  such that if

$$\bar{B}_\varrho^{C_0^1(\bar{Z})}(v_0) = \{u \in C_0^1(\bar{Z}) : \|u - v_0\|_{C_0^1(\bar{Z})} \leq \varrho\},$$

then

$$\hat{\phi}_-|_{\bar{B}_\varrho^{C_0^1(\bar{Z})}} = \hat{\phi}|_{\bar{B}_\varrho^{C_0^1(\bar{Z})}}.$$

Hence  $v_0 \in -\text{int}C_+$  is a local  $C_0^1(\bar{Z})$ -minimizer of  $\hat{\phi}$ . Once again, Theorem 1 of [García Azorero et al. 2000], implies that

$$v_0 \in -\text{int}C_+$$

is a local  $W_0^{1,p}(Z)$ -minimizer of  $\hat{\phi}$ . □

Now using minimax techniques from critical point theory, we can produce a third nontrivial smooth solution of (1-1) and have the full multiplicity result (three solutions theorem) for problem (1-1).

**Theorem 3.6.** *If hypotheses  $H_1$  hold then problem (1-1) has at least three nontrivial smooth solutions*

$$x_0 \in \text{int}C_+, \quad v_0 \in -\text{int}C_+ \quad \text{and} \quad y_0 \in C_0^1(\bar{Z}).$$

*Proof.* From Propositions 3.4 and 3.5 we already have two constant sign solutions

$$x_0 \in \text{int} C_+, \quad v_0 \in -\text{int} C_+.$$

Without any loss of generality, we may assume that

$$(3-17) \quad \hat{\varphi}(v_0) \leq \hat{\varphi}(x_0).$$

Also arguing as in [Motreanu et al. 2007, proof of Proposition 6], we can find  $\rho > 0$  small such that

$$(3-18) \quad \|v_0 - x_0\| > \rho \quad \text{and} \quad \hat{\varphi}(x_0) < \inf\{\hat{\varphi}(x) : \|x - x_0\| = \rho\} = \eta_\rho.$$

**Claim 3.7.**  $\hat{\varphi}$  satisfies the PS-condition.

*Proof of Claim 3.7.* Let  $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$  be a sequence such that

$$(3-19) \quad |\hat{\varphi}(x_n)| \leq M_4 \quad \text{for some } M_4 > 0, \text{ all } n \geq 1 \text{ and}$$

$$(3-20) \quad \hat{\varphi}'(x_n) \rightarrow 0 \quad \text{in } W^{-1,p'}(Z) \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \text{ as } n \rightarrow \infty.$$

From (3-20), we have

$$(3-21) \quad |\langle \hat{\varphi}'(x_n), u \rangle| \leq \varepsilon_n \|u\| \quad \text{for all } u \in W_0^{1,p}(Z) \text{ with } \varepsilon_n \downarrow 0 \\ \Rightarrow \left| \langle A(x_n), u \rangle - \int_Z \hat{f}(z, x_n) u \, dz \right| \leq \varepsilon_n \|u\| \text{ for all } u \in W_0^{1,p}(Z) \text{ with } \varepsilon_n \downarrow 0.$$

In (3-21) we choose  $u = -x_n^- \in W_0^{1,p}(Z)$ . Recalling the definition of  $\hat{f}(z, x)$ , we have

$$(3-22) \quad \|Dx_n^-\|_p^p - \int_Z \hat{f}_-(z, -x_n^-(z)) (-x_n^-(z)) \, dz \leq \varepsilon_n \|x_n^-\| \\ \Rightarrow \|Dx_n^-\|_p^p \leq c_3 \|x_n^-\| \quad \text{for some } c_3 > 0, \text{ all } n \geq 1 \\ \Rightarrow \{x_n^-\}_{n \geq 1} \subseteq W_0^{1,p}(Z) \text{ is bounded.}$$

From (3-19) and (3-22) we have

$$(3-23) \quad \frac{1}{p} \|Dx_n^+\|_p^p - \int_Z F_+(z, x_n^+(z)) \, dz \leq M_5 \quad \text{for some } M_5 > 0, \text{ all } n \geq 1 \\ \Rightarrow \varphi_+(x_n^+) \leq M_5 \quad \text{for all } n \geq 1.$$

But from Proposition 3.3, we know that  $\varphi_+$  is coercive. Hence, from (3-23) it follows that

$$(3-24) \quad \{x_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(Z) \text{ is bounded.}$$

From (3-22) and (3-24) we infer that  $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$  is bounded. So we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad x_n \rightarrow x \text{ in } L^r(Z).$$

From (3-21) we have

$$(3-25) \quad \left| \langle A(x_n), x_n - x \rangle - \int_Z \hat{f}(z, x_n)(x_n - x) dz \right| \leq \varepsilon_n \|x_n - x\|.$$

Note that

$$\int_Z \hat{f}(z, x_n)(x_n - x) dz \rightarrow 0$$

as  $n \rightarrow \infty$ . So from (3-25) we have

$$\lim_{n \rightarrow \infty} \langle A(x_n), x_n - x \rangle = 0 \Rightarrow x_n \rightarrow x \text{ in } W_0^{1,p}(Z) \text{ (see Proposition 2.4).}$$

Claim 3.7 follows. □

Then (3-17), (3-18) and Claim 3.7 permit the use of Theorem 2.1 (the Mountain Pass Theorem). Therefore, we obtain  $y_0 \in W_0^{1,p}(Z)$  such that

$$(3-26) \quad \hat{\varphi}'(y_0) = 0,$$

$$(3-27) \quad \hat{c} = \hat{\varphi}(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t)) \geq \eta_\rho > \hat{\varphi}(x_0) \geq \hat{\varphi}(v_0)$$

where  $\Gamma = \{\gamma \in C([0, 1], W_0^{1,p}(Z)) : \gamma(0) = v_0, \gamma(1) = x_0\}$ .

From (3-27) it is clear that  $y_0 \neq x_0$  and  $y_0 \neq v_0$ . We need to show that  $y_0$  is nontrivial (that is,  $y_0 \neq 0$ ). According to the minimax expression in (3-27), it suffices to produce  $\gamma_* \in \Gamma$  such that  $\hat{\varphi}|_{\gamma_*} < 0$ . Then

$$\hat{c} = \hat{\varphi}(y_0) < 0 = \hat{\varphi}(0)$$

and so  $y_0 \neq 0$ .

So our goal is to produce such a path  $\gamma_* \in \Gamma$ . Let

$$S = W_0^{1,p}(Z) \cap \partial B_1^{L^p}$$

be endowed with the relative  $W_0^{1,p}(Z)$ -topology and let

$$S_c = S \cap C_0^1(\bar{Z})$$

be endowed with the relative  $C_0^1(\bar{Z})$ -topology. Also let

$$\begin{aligned} \hat{\Gamma} &= \{\hat{\gamma} \in (C[-1, 1], S) : \hat{\gamma}(-1) = -u_1, \hat{\gamma}(1) = u_1\}, \\ \hat{\Gamma}_c &= \{\hat{\gamma} \in (C[-1, 1], S_c) : \hat{\gamma}(-1) = -u_1, \hat{\gamma}(1) = u_1\}. \end{aligned}$$

The density of  $S_c$  in  $S$  for the  $W_0^{1,p}(Z)$ -topology implies the density of  $\hat{\Gamma}_c$  in  $\hat{\Gamma}$  for the  $C([-1, 1], S)$ -topology. From (2-5) we see that given any  $\delta > 0$  we can find  $\hat{\gamma}_0 = \hat{\gamma}_0(\delta) \in \hat{\Gamma}_c$  such that

$$(3-28) \quad \max_{-1 \leq t \leq 1} \|D\hat{\gamma}_0(t)\|_p^p \leq \lambda_2 + \delta.$$

Since  $\hat{\gamma}_0 \in \hat{\Gamma}_c$ , we can find  $\varepsilon > 0$  small such that

$$\varepsilon |\hat{\gamma}_0(t)(z)| \leq \min\{\delta_0, -a\} \quad \text{for all } t \in T \text{ and all } z \in \bar{Z}.$$

Then for all  $t \in [0, 1]$ , we have

$$(3-29) \quad \begin{aligned} \hat{\varphi}(\varepsilon\hat{\gamma}_0(t)) &= \frac{\varepsilon^p}{p} \|D\hat{\gamma}_0(t)\|_p^p - \int_Z \hat{F}(z, \varepsilon\hat{\gamma}_0(t)(z)) dz \\ &\leq \frac{\varepsilon^p}{p} (\lambda_2 + \delta) - \frac{\varepsilon^p}{p} \xi_0 = \frac{\varepsilon^p}{p} (\lambda_2 + \delta - \xi_0) \end{aligned}$$

(see (3-28), hypothesis  $H_1(\text{vi})$  and recall that  $\|\hat{\gamma}_0(t)\|_p = 1$ ).

We choose

$$\delta < \xi_0 - \lambda_2$$

(hypothesis  $H_1(\text{vi})$ ). Then from (3-29) it follows that for  $\hat{\gamma}_0^\varepsilon = \varepsilon\hat{\gamma}_0$ , we have

$$(3-30) \quad \hat{\varphi}|_{\hat{\gamma}_0^\varepsilon} < 0.$$

Next we will produce a continuous path from  $\varepsilon u_1$  to  $x_0$  along which  $\hat{\varphi}$  is negative. Suppose that  $\{0, x_0\}$  are the only critical points of the functional  $\varphi_+$ . Otherwise, we have one more critical point of  $\varphi_+$ , which as before we can check that it is in  $\text{int}C_+$ . Hence it is also a critical point of  $\hat{\varphi}$  and thus it is a solution of (1-1). Therefore we have three constant sign solutions and we are done.

We set

$$a = m_+ < 0 = \varphi_+(0) = b.$$

From Proposition 3.3 we know that  $\varphi_+$  is coercive. Therefore,  $\varphi_+$  satisfies the PS-condition (it can be verified as in Claim 3.7 in the proof of Theorem 3.6). Apply Theorem 2.2 (the Second Deformation Theorem) to obtain a continuous deformation

$$h : [0, 1] \times (\varphi_+^b \setminus K_b^+) \rightarrow \varphi_+^b$$

where  $K_\theta^+ = \{x \in W_0^{1,p}(Z) : \varphi_+'(x) = 0, \varphi_+(x) = \theta\}$  for every  $\theta \in \mathbb{R}$ , such that

$$h(t, \cdot)|_{K_a^+} = \text{id}|_{K_a^+}$$



for all  $t \in [0, 1]$  and

$$(3-31) \quad h(1, \varphi_+^b \setminus K_b^+) \subseteq \varphi_+^a$$

$$(3-32) \quad \varphi_+(h(t, x)) \leq \varphi_+(h(s, x)) \quad \text{for all } 0 \leq s \leq t \leq 1, \text{ all } x \in \varphi_+^b \setminus K_b^+.$$

Let  $\gamma_+(t) = h(t, \varepsilon u_1)$ . Evidently  $\gamma_+ \in C([0, 1], W_0^{1,p}(Z))$ . Also

$$\begin{aligned} \gamma_+(0) &= h(0, \varepsilon u_1) = \varepsilon u_1 \quad (\text{since } h \text{ is a deformation}), \\ \gamma_+(1) &= h(1, \varepsilon u_1) = x_0 \quad (\text{see (3-31) and recall that } \varphi_+^a = \{x_0\}), \\ \varphi_+(\gamma_+(t)) &= \varphi_+(h(t, \varepsilon u_1)) \leq \varphi_+(h(0, \varepsilon u_1)) \\ &= \varphi_+(\varepsilon u_1) < 0 \quad (\text{see (3-32), (3-30) and } \hat{\varphi}|_{C_+} = \varphi_+|_{C_+}). \end{aligned}$$

Therefore, from the above we have that  $\gamma_+$  is a continuous path from  $\varepsilon u_1$  to  $x_0$  and

$$\varphi_+|_{\gamma_+} < 0.$$

Recall that  $f(z, x) \geq 0$  for a.a.  $z \in Z$ , all  $x \in \mathbb{R}_+$  and  $f(z, x) \leq 0$  for a.a.  $z \in Z$ , all  $x \in [a, 0]$ . So it follows that  $\hat{\varphi} \leq \varphi_+$  and hence

$$(3-33) \quad \hat{\varphi}|_{\gamma_+} < 0.$$

Finally, we produce a continuous path from  $-\varepsilon u_1$  to  $v_0$  along which  $\hat{\varphi}$  is negative. Again, we may assume that  $\{0, v_0\}$  are the only critical points of  $\hat{\varphi}_-$ . Otherwise, as before, we have a third nontrivial constant sign solution (in  $-\text{int}C_+$ ) of (1-1) and so we are done. We set

$$a = \hat{m}_- < 0 = \hat{\varphi}_-(0) = b.$$

The functional  $\hat{\varphi}_-$  is coercive; hence, it satisfies the PS-condition. We apply Theorem 2.2 (the Second Deformation Theorem) and obtain a continuous deformation

$$\hat{h} : [0, 1] \times (\hat{\varphi}_-^b \setminus \hat{K}_b^-) \rightarrow \hat{\varphi}_-^b$$

with the similar properties as before, where

$$\hat{K}_\theta^- = \{x \in W_0^{1,p}(Z) : \hat{\varphi}'_-(x) = 0, \hat{\varphi}_-(x) = \theta\}$$

for every  $\theta \in \mathbb{R}$ . We set  $\gamma_-(t) = \hat{h}(t, -\varepsilon u_1)$  and as we did for  $\gamma_+$ , we check that  $\gamma_-$  is a continuous path from  $-\varepsilon u_1$  to  $v_0$  such that

$$(3-34) \quad \hat{\varphi}_-|_{\gamma_-} < 0 \Rightarrow \hat{\varphi}|_{\gamma_-} < 0 \quad (\text{since } \hat{\varphi} \leq \hat{\varphi}_-; \text{ see hypothesis } H_1(v)).$$

We concatenate  $\gamma_-$ ,  $\hat{\gamma}_0^\varepsilon$  and  $\gamma_+$  and obtain  $\gamma_* \in \Gamma$  such that

$$\hat{\varphi}|_{\gamma_*} < 0 \quad (\text{see (3-30), (3-33), (3-34)}) \Rightarrow y_0 \neq 0.$$

From (3-26), we have

$$\begin{aligned}
 A(y_0) = \hat{N}(y_0) &\Rightarrow -\Delta_p y_0(z) = \hat{f}(z, y_0(z)) \quad \text{a.e. on } Z \\
 &\Rightarrow y_0 \in C_0^1(\bar{Z}) \setminus \{0\} \quad (\text{nonlinear regularity theory}).
 \end{aligned}$$

As in the proof of Proposition 3.5, acting with  $(a - y_0)^+ \in W_0^{1,p}(Z)$ , we obtain  $a \leq y_0$  and hence  $y_0$  is a nontrivial smooth solution of (1-1). □

### 4. The semilinear problem

In this section, we focus on the semilinear problem (that is,  $p = 2$ ). So, the problem under consideration is

$$\begin{aligned}
 (4-1) \quad &-\Delta x(z) = f(z, x(z)) \quad \text{a.e. on } Z, \\
 &x|_{\partial Z} = 0.
 \end{aligned}$$

By strengthening the regularity on  $f(z, \cdot)$  and using Morse theory, we can show that the problem has four nontrivial smooth solutions.

Now the nonlinearity hypotheses on  $f(z, x)$  are the following:

H<sub>2</sub>:  $f : Z \times \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $f(z, 0) = 0$  a.e. on  $Z$ .

- (i) For all  $x \in \mathbb{R}$ ,  $z \rightarrow f(z, x)$  is measurable.
- (ii) For almost all  $z \in Z$ ,  $x \rightarrow f(z, x)$  is  $C^1$ .
- (iii) For almost all  $z \in Z$  and all  $x \in \mathbb{R}$ ,

$$|f'_x(z, x)| \leq a(z) + c|x|^{r-2}$$

with  $a \in L^\infty(Z)_+$ ,  $c > 0$  and  $2 < r < 2^*$ .

- (iv) For almost all  $z \in Z$  and all  $x \geq 0$ ,  $f(z, x) \geq 0$  and

$$\lim_{x \rightarrow +\infty} \frac{2F(z, x)}{x^2} = \lambda_1 \quad \text{and} \quad \lim_{x \rightarrow +\infty} (2F(z, x) - \lambda_1 x^2) = -\infty,$$

both uniformly for almost all  $z \in Z$ .

- (v) There exists  $a < 0$  such that  $f(z, a) = 0$  a.e. on  $Z$  and  $f(z, x) \leq 0$  for a.a.  $z \in Z$  and all  $x \in [a, 0]$ .
- (vi) There exist  $\delta_0 > 0$  and an integer  $m \geq 2$  such that

$$\lambda_m \leq \frac{f(z, x)}{x} \leq \lambda_{m+1}$$

for a.a.  $z \in Z$ , all  $|x| \leq \delta_0$  and if  $m = 2$ , then in addition

$$\lambda_2 < f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x}$$

uniformly for almost all  $z \in Z$ .

**Remark 4.1.** Evidently hypotheses  $H_2$ (ii)–(iii) imply that we can find  $\xi_0 > 0$  large such that for almost all  $z \in Z$ ,  $x \rightarrow \xi_0 x + f(z, x)$  is nondecreasing on  $[a, 0]$ .

**Example 4.2.** The following function  $f(x)$  satisfies hypotheses  $H_2$ . Again for the sake of simplicity we drop the  $z$ -dependence:

$$f(x) = \begin{cases} x^2 + x & \text{if } x \leq 0, \\ c(x - x^2) & \text{if } x \in [0, 1], \\ \lambda_1 x - (\lambda_1 + c) \ln x - \lambda_1 & \text{if } x \geq 1, \end{cases}$$

with  $\lambda_m \leq c \leq \lambda_{m+1}$ ,  $m \geq 3$ .

**Theorem 4.3.** *If hypothesis  $H_2$  hold then problem (4-1) has at least four nontrivial solutions*

$$x_0 \in \text{int}C_+, \quad v_0 \in -\text{int}C_+, \quad \text{and} \quad y_0, u_0 \in C_0^1(\bar{Z}).$$

*Proof.* From Theorem 3.6, we already have three nontrivial smooth solutions

$$x_0 \in \text{int}C_+, \quad v_0 \in -\text{int}C_+, \quad y_0 \in C_0^1(\bar{Z}).$$

From Propositions 3.4 and 3.5 we know that  $x_0, v_0$  are both local minimizers of  $\hat{\varphi}$ . Therefore

$$(4-2) \quad C_k(\hat{\varphi}, x_0) = C_k(\hat{\varphi}, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0$$

(see [Chang 1993, p. 33] and [Mawhin and Willem 1989, p. 175]).

Hypothesis  $H_2$ (vi) and [Li et al. 2001, Proposition 1.1] imply that

$$(4-3) \quad C_k(\hat{\varphi}, 0) = \delta_{k,d_m}\mathbb{Z}$$

for all  $k \geq 0$ , where

$$d_m = \dim \bigoplus_{i=1}^m E(\lambda_i)$$

( $E(\lambda_i)$  is the eigenspace corresponding to the eigenvalue  $\lambda_i$ ). From the proof of Theorem 3.6, we know that  $y_0 \in C_0^1(\bar{Z}) \setminus \{0\}$  was obtained via the use of the mountain pass theorem (see Theorem 2.1). Moreover, as before, using the maximum principle of Vázquez, we obtain  $a < y_0(z)$  for all  $z \in \bar{Z}$ . Note that  $\hat{\varphi} \in C^{2-0}(H_0^1(Z))$ . The fact that  $\hat{\varphi}$  is not necessarily  $C^2$  does not allow the direct use of well-known results from Morse theory. We overcome this inconvenience by approximating  $\hat{\varphi}$  with a  $C^2$ -functional  $\varphi_0$ , keeping the essential properties intact. Note that the nonlinearity  $f(z, \cdot)$  need not be  $C^1$  only at  $x = a$ . So we approximate  $\hat{f}(z, x)$  by a Caratheodory function  $f_0(z, x)$  which is  $C^1$  in the  $x$ -variable, differs from  $\hat{f}(z, \cdot)$  only near  $a$  and for a given  $\varepsilon > 0$ , we have

$$\int_Z \sup_{-r \leq x \leq r} |\hat{f}(z, x) - f_0(z, x)| dz < \varepsilon \quad \text{for all } r > 0.$$

We let  $\varphi_0$  be the  $C^2$ -functional corresponding to  $f_0(z, x)$ .

Then, exploiting the continuity of the Morse critical groups in the  $C^1$ -norm (see [Chang 2005, p. 337]), we have

$$(4-4) \quad C_k(\hat{\varphi}|_{C_0^1(\bar{Z})}, y_0) = C_k(\varphi_0|_{C_0^1(\bar{Z})}, y_0) \quad \text{for all } k \geq 0.$$

But from [Liu and Wu 2002], we know that, for all  $k \geq 0$ ,

$$(4-5) \quad C_k(\hat{\varphi}, y_0) = C_k(\hat{\varphi}|_{C_0^1(\bar{Z})}, y_0) \quad \text{and} \quad C_k(\hat{\varphi}_0, y_0) = C_k(\varphi_0|_{C_0^1(\bar{Z})}, y_0).$$

From (4-4) and (4-5), we infer that

$$(4-6) \quad C_k(\hat{\varphi}, y_0) = C_k(\varphi_0, y_0) \quad \text{for all } k \geq 0.$$

But  $\varphi_0 \in C^2(H_0^1(Z))$  and  $\varphi_0''(y_0)$  is a Fredholm operator. Suppose that the Morse index of  $\varphi_0$  at  $y_0$  is zero. Then

$$(4-7) \quad \|Du\|_2^2 \geq \int_Z mu^2 dz$$

for all  $u \in H_0^1(Z)$ , where  $m(z) = f_0'(z, y_0(z))$  and  $m \in L^\infty(Z)$ . Let  $u \in \ker \varphi_0''(y_0)$ . Then

$$(4-8) \quad -\Delta u(z) = m(z)u(z) \quad \text{a.e. on } Z, \quad u|_{\partial Z} = 0.$$

If  $m^+ = 0$  then clearly (4-8) has only the trivial solution. If  $m^+ \neq 0$  then from the variational characterization of the principal eigenvalue  $\hat{\lambda}_1(m)$  of  $(-\Delta, H_0^1(Z), m)$ , we have

$$\lambda_1(m) \geq 1 \quad (\text{see (4-7)}) \Rightarrow \dim \ker \varphi_0''(y_0) \leq 1 \quad (\text{see (4-8)}).$$

So we apply [Mawhin and Willem 1989, Corollary 8.5, p. 195] and obtain:

$$(4-9) \quad C_k(\varphi_0, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0 \Rightarrow C_k(\hat{\varphi}, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0 \quad (\text{see (4-6)}).$$

**Claim 4.4.**  $\hat{\varphi}$  is coercive.

*Proof.* We argue indirectly. So suppose that Claim 4.4 is not true. Then we can find  $\{x_n\}_{n \geq 1} \subseteq H_0^1(Z)$  and  $M_6 > 0$  such that

$$(4-10) \quad \|x_n\| \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \hat{\varphi}(x_n) \leq M_6 \quad \text{for all } n \geq 1.$$

Then, recalling the definition of  $\hat{f}(z, x)$ , we have

$$\varphi_+(x_n) = \frac{1}{2} \|Dx_n\|_2^2 - \int_Z F_+(z, x_n) dz \leq M_7 \quad \text{for some } M_7 > 0, \quad \text{all } n \geq 1,$$

which contradicts Proposition 3.3 (see (4-10)). □

Using Claim 4.4 and directly from the definition of the critical groups of  $\hat{\varphi}$  at infinity, we have

$$(4-11) \quad C_k(\hat{\varphi}, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0.$$

Suppose that  $\{0, x_0, v_0, y_0\}$  are all the critical points of  $\hat{\varphi}$ . Then from (2-1) with  $t = -1$ , we have

$$2(-1)^0 + (-1)^{d_m} + (-1)^1 = (-1)^0 \quad (\text{see (4-2), (4-3), (4-9), (4-11)}) \Rightarrow (-1)^{d_m} = 0,$$

a contradiction. So there is a fourth nontrivial critical point  $u_0 \in H_0^1(Z)$  of  $\hat{\varphi}$ . We can show that  $u_0 \geq a$ ; hence, it solves (4-1) and by regularity theory  $u_0 \in C_0^1(\bar{Z})$ .  $\square$

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