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We define a gradient Ricci soliton to be rigid if it is a flat bundle $N \times_\Gamma \mathbb{R}^k$ where N is Einstein. It is known that not all gradient solitons are rigid. Here we offer several natural conditions on the curvature that characterize rigid gradient solitons. Other related results on rigidity of Ricci solitons are also explained in the last section.

1. Introduction

A Ricci soliton is a Riemannian metric together with a vector field (M, g, X) that satisfies

$$\operatorname{Ric} + \tfrac{1}{2} L_X g = \lambda g.$$

It is called *shrinking* when $\lambda > 0$, *steady* when $\lambda = 0$, and *expanding* when $\lambda < 0$. If $X = \nabla f$ the equation can also be written as

$$\operatorname{Ric} + \operatorname{Hess} f = \lambda g$$

and is called a *gradient* (*Ricci*) *soliton*. See [Cao 2006; Chow and Knopf 2004; Chow et al. 2006; Derdzinski 2008] for background on Ricci solitons and their connection to the Ricci flow; we remark that on a compact manifold Ricci solitons are always gradient solitons [Perelman 2002] and that every noncompact shrinking soliton is a gradient soliton [Naber 2007].

Clearly Einstein metrics are solitons with f trivial. Another interesting special case occurs when $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^n$. In this case

$$\operatorname{Hess} f = \lambda g$$

and therefore yields a gradient soliton where the background metric is flat. This example is called a *Gaussian*. Taking a product $N \times \mathbb{R}^k$ with N being Einstein with Einstein constant λ and $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^k$ yields a mixed gradient soliton. We can further take a quotient $N \times_\Gamma \mathbb{R}^k$, where Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$ (no translational components) to get a flat vector bundle

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over a base that is Einstein and with $f = \frac{\lambda}{2}d^2$ where d is the distance in the flat fibers to the base.

We say that a gradient soliton is *rigid* if it is of the type $N \times_\Gamma \mathbb{R}^k$ just described.

The goal of this paper is to determine when gradient solitons are rigid. For compact manifolds every steady or expanding compact soliton is rigid [Ivey 1993]. (In fact, at least in the steady case, this result seems to go back to Lichnerowicz; see the remark in [Bourguignon 1981, Section 3.10]). Moreover, in dimensions 2 [Hamilton 1988] and 3 [Ivey 1993] all shrinking compact solitons are rigid. The first nonrigid compact shrinking (Kähler) gradient solitons are in dimension 4 and were constructed by Koiso [1990]; see also [Cao 1996; Wang and Zhu 2004]. In any dimension compact shrinking solitons are rigid precisely when their scalar curvature is constant; see [Eminenti et al. 2008]. In fact, something a little more general is true.

Theorem 1.1. *A shrinking compact gradient soliton is rigid with trivial f if*

$$\int_M \mathrm{Ric}(\nabla f, \nabla f) \leq 0.$$

In the noncompact case Perel'man, building on work of Ivey and Hamilton, has shown that all 3-dimensional shrinking gradient solitons with bounded curvature are rigid [2003]. However, in higher dimensions, it is less clear how to detect rigidity. In fact there are expanding Ricci solitons with constant scalar curvature that are not rigid in the above sense. These spaces are left invariant metrics on nilpotent groups constructed by Lauret [2001] that are not gradient solitons. For other examples of noncompact gradient solitons with large symmetry groups see [Cao 1996; 1997; Feldman et al. 2003; Ivey 1994].

Note that if a soliton is rigid, then the "radial" curvatures vanish, that is,

$$R(\,\cdot\,, \nabla f)\nabla f = 0,$$

and the scalar curvature is constant. Conversely we just saw that constant scalar curvature or radial Ricci flatness:

$$\mathrm{Ric}(\nabla f, \nabla f) = 0$$

each implies rigidity on compact solitons. In the noncompact steady case it is not hard to see that constant scalar curvature also implies rigidity (see Proposition 3.2). For the expanding and shrinking case we prove the following.

Theorem 1.2. *A gradient soliton* $\mathrm{Ric} + \mathrm{Hess} f = \lambda g$ *is rigid if and only if it has constant scalar curvature and is radially flat*, *that is*,

$$\sec(E, \nabla f) = 0.$$

While radial flatness seems like a strong assumption, there are a number of weaker conditions that guarantee radial flatness.

Proposition 1.3. *The following conditions for a shrinking (or expanding) gradient soliton* $\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$ *all imply that the metric is radially flat and has constant scalar curvature.*

(1) *The scalar curvature is constant and* $\sec(E, \nabla f) \geq 0$ (*or* $\sec(E, \nabla f) \leq 0$).

(2) *The scalar curvature is constant and* $0 \leq \mathrm{Ric} \leq \lambda g$ (*or* $\lambda g \leq \mathrm{Ric} \leq 0$).

(3) *The curvature tensor is harmonic.*

(4) $\mathrm{Ric} \geq 0$ (*or* $\mathrm{Ric} \leq 0$) *and* $\sec(E, \nabla f) = 0$.

Given Theorem 1.2 it is easy to see that rigid solitons also satisfy these conditions.

Condition (2) in Proposition 1.3 is very similar to a statement by Naber [2006], but our proof is quite different. The following result shows that, for shrinking solitons, the scalar curvature condition is in fact redundant. Thus we are offering an alternate proof for part of Naber's result.

Lemma 1.4 (Naber). *If M is a shrinking gradient Ricci soliton with* $0 \leq \mathrm{Ric} \leq \lambda g$ *then the scalar curvature is constant.*

There is an interesting relationship between this result and Perel'man's classification in dimension 3. The main part of the classification is to show that there are no noncompact shrinking gradient solitons with positive sectional curvature. Perel'man's proof has two parts: first he shows that such a metric has $\mathrm{scal} \leq 2\lambda$ and then he uses this fact, and the Gauss–Bonnet theorem, to arrive at a contradiction. If $\sec \geq 0$ and $\mathrm{scal} \leq 2\lambda$ then $\mathrm{Ric} \leq \lambda$. Therefore, Naber's lemma implies the following gap theorem which generalizes the second part of Perel'man's argument to higher dimensions.

Theorem 1.5. *If M^n is a shrinking gradient Ricci soliton with nonnegative sectional curvature and* $\mathrm{scal} \leq 2\lambda$ *then the universal cover of M is isometric to either* $\mathbb{R}^n$ *or* $S^2 \times \mathbb{R}^{n-2}$.

We also point out that, as an application of Theorem 1.2 and the techniques developed here, we can classify shrinking solitons with large symmetry and nonnegative curvature. We refer the reader to [Petersen and Wylie 2007a] for the discussion of this result.

Theorem 1.6. *All complete noncompact shrinking gradient solitons of cohomogeneity* 1 *with nonnegative Ricci curvature and* $\sec(E, \nabla f) \geq 0$ *are rigid.*

The key to the proof of Theorem 1.2 is an equation that in a fairly obvious way relates rigidity, radial curvatures, and scalar curvature:

$$\nabla_{\nabla f}\mathrm{Ric}+\mathrm{Ric}\circ(\lambda I-\mathrm{Ric})=R(\,\cdot\,,\nabla f)\nabla f+\tfrac{1}{2}\nabla.\nabla\mathrm{scal}.$$

In the context of condition (3) in Proposition 1.3 about harmonicity of the curvature there is a rather interesting connection with gradient solitons. Consider the exterior covariant derivative

$$d^{\nabla}:\Omega^{p}(M,TM)\to\Omega^{p+1}(M,TM)$$

for forms with values in the tangent bundle. The curvature can then be interpreted as the 2-form

$$R(X,Y)Z=\big((d^{\nabla}\circ d^{\nabla})(Z)\big)(X,Y)$$

and Bianchi's second identity as $d^{\nabla}R=0$. The curvature is harmonic if $d^{*}R=0$ where d^{*} is the adjoint of d^{∇}. If we think of Ric as a 1-form with values in TM then Bianchi's second identity implies

$$d^{\nabla}\mathrm{Ric}=-d^{*}R.$$

Thus the curvature tensor is harmonic if and only if the Ricci tensor is closed. This condition has been studied extensively as a generalization of being an Einstein metric (see [Besse 1987, Chapter 16]). It is also easy to see that it implies constant scalar curvature.

Next note that the condition for being a steady gradient soliton is the same as saying that the Ricci tensor is exact

$$\mathrm{Ric}=d^{\nabla}(-X)=-\nabla X.$$

Since the Ricci tensor is symmetric, this requires that X is locally a gradient field. The general gradient soliton equation

$$\mathrm{Ric}=d^{\nabla}(-X)+\lambda I$$

then appears to be a simultaneous generalization of being Einstein and exact. Thus Theorem 1.2 implies that rigid gradient solitons are precisely those metrics that satisfy all the generalized Einstein conditions.

Throughout the paper we also establish several other simple results that guarantee rigidity under slightly different assumptions on the curvature and geometry of the space.

2. Formulas

In this section we establish the general formulas that will used to prove the various rigidity results we are after. There are two sets of results. The most general and weakest for Ricci solitons and the more interesting and powerful for gradient solitons.

First we establish a general formula that leads to the Bochner formulas for Killing and gradient fields (see also [Poor 1981]).

Lemma 2.1. *On a Riemannian manifold*

$$\operatorname{div}(L_X g)(X) = \tfrac{1}{2}\Delta|X|^2 - |\nabla X|^2 + \operatorname{Ric}(X, X) + D_X \operatorname{div} X.$$

When $X = \nabla f$ *is a gradient field we have*

$$(\operatorname{div} L_X g)(Z) = 2\operatorname{Ric}(Z, X) + 2D_Z \operatorname{div} X$$

or in $(1, 1)$*-tensor notation*

$$\operatorname{div}\nabla\nabla f = \operatorname{Ric}(\nabla f) + \nabla\Delta f.$$

Proof. We calculate with a frame that is parallel at p:

$$\begin{aligned}
&\operatorname{div}(L_X g)(X)\\
&\qquad = (\nabla_{E_i} L_X g)(E_i, X) = \nabla_{E_i}(L_X g(E_i, X)) - L_X g(E_i, \nabla_{E_i} X)\\
&\qquad = \nabla_{E_i}\big(g(\nabla_{E_i} X, X) + g(E_i, \nabla_X X)\big) - g(\nabla_{E_i} X, \nabla_{E_i} X) - g(E_i, \nabla_{\nabla_{E_i} X} X)\\
&\qquad = \Delta\tfrac{1}{2}|X|^2 + \nabla_{E_i} g(E_i, \nabla_X X) - |\nabla X|^2 - g(E_i, \nabla_{\nabla_{E_i} X} X)\\
&\qquad = \Delta\tfrac{1}{2}|X|^2 - |\nabla X|^2 + g(\nabla^2_{E_i, X} X, E_i)\\
&\qquad = \Delta\tfrac{1}{2}|X|^2 - |\nabla X|^2 + \operatorname{Ric}(X, X) + g(\nabla^2_{X, E_i} X, E_i)\\
&\qquad = \Delta\tfrac{1}{2}|X|^2 - |\nabla X|^2 + \operatorname{Ric}(X, X) + D_X \operatorname{div} X.
\end{aligned}$$

And when $Z \to \nabla_Z X$ is self-adjoint we have

$$\begin{aligned}
&(\operatorname{div} L_X g)(Z)\\
&\quad = (\nabla_{E_i} L_X g)(E_i, Z) = \nabla_{E_i}(L_X g(E_i, Z)) - L_X g(E_i, \nabla_{E_i} Z)\\
&\quad = \nabla_{E_i}\big(g(\nabla_{E_i} X, Z) + g(E_i, \nabla_Z X)\big) - g(\nabla_{E_i} X, \nabla_{E_i} Z) - g(E_i, \nabla_{\nabla_{E_i} Z} X)\\
&\quad = \nabla_{E_i}(g(\nabla_Z X, E_i)) + g(E_i, \nabla_{E_i}\nabla_Z X) - g(\nabla_{E_i} X, \nabla_{E_i} Z) - g(E_i, \nabla_{\nabla_{E_i} Z} X)\\
&\quad = 2g(\nabla^2_{E_i, Z} X, E_i) = 2\operatorname{Ric}(Z, X) + 2g(\nabla^2_{Z, E_i} X, E_i)\\
&\quad = 2\operatorname{Ric}(Z, X) + 2D_Z \operatorname{div} X.
\end{aligned}$$

□

Corollary 2.2. *If* X *is a Killing field*, *then*

$$\Delta\tfrac{1}{2}|X|^2 = |\nabla X|^2 - \operatorname{Ric}(X, X).$$

Proof. Use that $L_X g = 0 = \operatorname{div} X$ in the above formula. □

Corollary 2.3. *If X is a gradient field, then*

$$\Delta \tfrac{1}{2}|X|^2 = |\nabla X|^2 + D_X \operatorname{div} X + \operatorname{Ric}(X, X).$$

Proof. Let $Z = X$ in the second equation above and equate them to get the formula. □

We are now ready to derive formulas for Ricci solitons

$$\operatorname{Ric} + \tfrac{1}{2} L_X g = \lambda g.$$

Lemma 2.4. *A Ricci soliton satisfies*

$$\tfrac{1}{2}(\Delta - D_X)|X|^2 = |\nabla X|^2 - \lambda |X|^2.$$

Proof. The trace of the soliton equation says that

$$\operatorname{scal} + \operatorname{div} X = n\lambda$$

so

$$D_Z \operatorname{scal} = -D_Z \operatorname{div} X.$$

The contracted second Bianchi identity that forms the basis for Einstein's equations says that

$$D_Z \operatorname{scal} = 2 \operatorname{div} \operatorname{Ric}(Z).$$

Using $Z = X$ and the soliton equation then gives

$$\begin{aligned} -D_X \operatorname{div} X &= 2 \operatorname{div} \operatorname{Ric}(X) = -\operatorname{div}(L_X g)(X) \\ &= -\big(\tfrac{1}{2}\Delta|X|^2 - |\nabla X|^2 + \operatorname{Ric}(X, X) + D_X \operatorname{div} X\big). \end{aligned}$$

Thus

$$\begin{aligned} \tfrac{1}{2}\Delta|X|^2 &= |\nabla X|^2 - \operatorname{Ric}(X, X) = |\nabla X|^2 + \tfrac{1}{2}(L_X g)(X, X) - \lambda|X|^2 \\ &= |\nabla X|^2 + \tfrac{1}{2} D_X |X|^2 - \lambda |X|^2, \end{aligned}$$

from which we get the equation. □

We now turn our attention to gradient solitons. In this case we can use (1, 1)-tensors and write the soliton equation as

$$\operatorname{Ric} + \nabla\nabla f = \lambda I$$

or in condensed form

$$\operatorname{Ric} + S = \lambda I, \qquad S = \nabla\nabla f.$$

With this notation we can now state and prove some interesting formulas for the scalar curvature of gradient solitons. The first and last are known (see [Chow et al. 2006]), while the middle ones seem to be new.

Lemma 2.5. *A gradient soliton satisfies*

$$\nabla \operatorname{scal} = 2\operatorname{Ric}(\nabla f),$$
$$\nabla_{\nabla f} S + S \circ (S - \lambda I) = -R(\cdot\,, \nabla f)\nabla f - \tfrac{1}{2}\nabla.\nabla \operatorname{scal},$$
$$\nabla_{\nabla f}\operatorname{Ric} + \operatorname{Ric} \circ (\lambda I - \operatorname{Ric}) = R(\cdot\,, \nabla f)\nabla f + \tfrac{1}{2}\nabla.\nabla \operatorname{scal},$$
$$\tfrac{1}{2}(\Delta - D_{\nabla f}) \operatorname{scal} = \tfrac{1}{2}\Delta_f \operatorname{scal} = \operatorname{tr}(\operatorname{Ric} \circ (\lambda I - \operatorname{Ric})).$$

Proof. We have the Bochner formula

$$\operatorname{div}(\nabla\nabla f) = \operatorname{Ric}(\nabla f) + \nabla \Delta f.$$

The trace of the soliton equation gives

$$\operatorname{scal} + \Delta f = n\lambda, \qquad \nabla \operatorname{scal} + \nabla \Delta f = 0.$$

while the divergence of the soliton equation gave us

$$\operatorname{div}\operatorname{Ric} + \operatorname{div}(\nabla\nabla f) = 0.$$

Together this yields

$$\nabla \operatorname{scal} = 2\operatorname{div}\operatorname{Ric} = -2\operatorname{div}(\nabla\nabla f) = -2\operatorname{Ric}(\nabla f) - 2\nabla\Delta f = -2\operatorname{Ric}(\nabla f) + 2\nabla \operatorname{scal}$$

and hence the first formula.

Using this one can immediately find a formula for the Laplacian of the scalar curvature. However our goal is to establish the second set of formulas. The last formula is then obtained by taking traces.

We use the equation

$$R(E, \nabla f)\nabla f = \nabla^2_{E,\nabla f}\nabla f - \nabla^2_{\nabla f, E}\nabla f.$$

The second term on the right hand side

$$\nabla^2_{\nabla f, E}\nabla f = (\nabla_{\nabla f} S)(E)$$

while the first can be calculated

$$\begin{aligned}
\nabla^2_{E,\nabla f}\nabla f &= -(\nabla_E \operatorname{Ric})(\nabla f) = -\nabla_E \operatorname{Ric}(\nabla f) + \operatorname{Ric}(\nabla_E \nabla f)\\
&= -\tfrac{1}{2}\nabla_E \nabla \operatorname{scal} + \operatorname{Ric} \circ S(E) = -\tfrac{1}{2}\nabla_E \nabla \operatorname{scal} + (\lambda I - S) \circ S(E)\\
&= -\tfrac{1}{2}\nabla_E \nabla \operatorname{scal} + \operatorname{Ric} \circ (\lambda I - \operatorname{Ric}).
\end{aligned}$$

This yields the set of formulas in the middle.

Taking traces in

$$\nabla_{\nabla f}\operatorname{Ric} + \operatorname{Ric} \circ (\lambda I - \operatorname{Ric}) = R(E, \nabla f)\nabla f + \tfrac{1}{2}\nabla_E \nabla \operatorname{scal}$$

yields

$$\nabla_{\nabla f} \operatorname{scal} + \operatorname{tr}(\operatorname{Ric} \circ (\lambda I - \operatorname{Ric})) = \operatorname{Ric}(\nabla f, \nabla f) + \tfrac{1}{2}\Delta \operatorname{scal}.$$

Since

$$\mathrm{Ric}(\nabla f, \nabla f) = \tfrac{1}{2} D_{\nabla f} \mathrm{scal},$$

we immediately get the last equation. □

If λ_i are the eigenvalues of the Ricci tensor, the last equation can be rewritten in several useful ways:

$$\begin{aligned} \tfrac{1}{2}\Delta_f \,\mathrm{scal} &= \mathrm{tr}\big(\mathrm{Ric} \circ (\lambda I - \mathrm{Ric})\big) = \sum \lambda_i(\lambda - \lambda_i) \\ &= -|\mathrm{Ric}|^2 + \lambda\, \mathrm{scal} \\ &= -\left|\mathrm{Ric} - \frac{1}{n}\,\mathrm{scal}\, g\right|^2 + \mathrm{scal}\left(\lambda - \frac{1}{n}\,\mathrm{scal}\right). \end{aligned}$$

3. Rigidity characterization

We start with a motivational appetizer on rigidity of gradient solitons.

Proposition 3.1. *A gradient soliton which is Einstein either has* $\mathrm{Hess} f = 0$ *or is a Gaussian.*

Proof. Assume that

$$\mu g + \mathrm{Hess}\, f = \lambda g.$$

If $\mu = \lambda$ then the Hessian vanishes. Otherwise we have that the Hessian is proportional to g. Multiplying f by a constant then leads us to a situation where

$$\mathrm{Hess}\, f = g.$$

This shows that f is a proper strictly convex function. By adding a suitable constant to f we also see that

$$r = \sqrt{f}$$

is a distance function from the unique minimum of f. It is now easy to see that the radial curvatures vanish and then that the space is flat (see also [Petersen 1998]). □

Next we dispense with rigidity for compact solitons.

Proof of Theorem 1.1. We have a Ricci soliton

$$\mathrm{Ric} + L_X g = \lambda g.$$

The Laplacian of X then satisfies

$$\Delta \tfrac{1}{2}|X|^2 = |\nabla X|^2 - \mathrm{Ric}(X, X).$$

The divergence theorem along with the assumption that

$$\int_M \mathrm{Ric}(\nabla f, \nabla f) \le 0$$

then shows that ∇X vanishes. In particular $L_X g = 0$. □

Note that the minimum principle applied to the similar formula

$$\tfrac{1}{2}(\Delta - D_X)|X|^2 = |\nabla X|^2 - \lambda |X|^2$$

gives a proof that every compact steady or expanding soliton is rigid.

Steady solitons are also easy to deal with:

Proposition 3.2. *A steady gradient soliton whose scalar curvature achieves its minimum is Ricci flat. Moreover, if f is not constant then it is a product of a Ricci flat manifold with $\mathbb{R}$.*

Proof. First we note that

$$\tfrac{1}{2}\Delta_f \operatorname{scal} = -\left|\operatorname{Ric} - \frac{1}{n}\operatorname{scal} g\right|^2 + \operatorname{scal}\left(\lambda - \frac{1}{n}\operatorname{scal}\right) = -\left|\operatorname{Ric} - \frac{1}{n}\operatorname{scal} g\right|^2 - \frac{1}{n}\operatorname{scal}^2 \leq 0.$$

Thus, by the strong minimum principle, if scal achieves its minimum then it is constant. Then $\Delta_f \operatorname{scal} = 0$, which implies that $\operatorname{scal} = 0$ and $\operatorname{Ric} = 0$. This shows that $\operatorname{Hess} f = 0$. Thus either f is constant or the manifold splits along the gradient of f. □

Note that the same argument also applies when we have an expanding soliton with nonnegative scalar curvature. In the general expanding or shrinking case the formula for the f-Laplacian of the scalar curvature gives the next result.

Proposition 3.3. *Assume that we have a gradient soliton*

$$\operatorname{Ric} + \operatorname{Hess} f = \lambda g$$

with constant scalar curvature and $\lambda \neq 0$. When $\lambda > 0$ we have $0 \leq \operatorname{scal} \leq n\lambda$. When $\lambda < 0$ we have $n\lambda \leq \operatorname{scal} \leq 0$. In either case the metric is Einstein when the scalar curvature equals either of the extreme values.

Proof. Again we have that

$$0 = \tfrac{1}{2}\Delta_f \operatorname{scal} = -\left|\operatorname{Ric} - \frac{1}{n}\operatorname{scal} g\right|^2 + \operatorname{scal}\left(\lambda - \frac{1}{n}\operatorname{scal}\right)$$

showing that

$$0 \leq \left|\operatorname{Ric} - \frac{1}{n}\operatorname{scal} g\right|^2 = \operatorname{scal}\left(\lambda - \frac{1}{n}\operatorname{scal}\right).$$

Thus $\operatorname{scal} \in [0, n\lambda]$ if the soliton is shrinking and the metric is Einstein if the scalar curvature takes on either of the boundary values. A similar analysis holds in the expanding case. □

Before proving the main characterization we study the conditions that guarantee radial flatness.

Proof of Proposition 1.3. Assume condition (1). Use the equations

$$0 = \tfrac{1}{2}\nabla_{\nabla f}\,\mathrm{scal} = \mathrm{Ric}(\nabla f, \nabla f) = \sum g\big(R(E_i, \nabla f)\nabla f, E_i\big)$$

to see that

$$g\big(R(E_i, \nabla f)\nabla f, E_i\big) = 0$$

if the radial curvatures are always nonnegative (nonpositive).

Assume condition (2). Then

$$0 = \tfrac{1}{2}\Delta_f\,\mathrm{scal} = \mathrm{tr}(\mathrm{Ric}\circ(\lambda I - \mathrm{Ric})).$$

The assumptions on the Ricci curvature imply that $\mathrm{Ric}\circ(\lambda I - \mathrm{Ric})$ is a nonnegative operator. Thus

$$\mathrm{Ric}\circ(\lambda I - \mathrm{Ric}) = 0.$$

This shows that the only possible eigenvalues for Ric and $\nabla\nabla f$ are 0 and λ.

To establish radial flatness we then use that the formula

$$\nabla_{\nabla f}\mathrm{Ric} + \mathrm{Ric}\circ(\lambda I - \mathrm{Ric}) = R(\cdot\,, \nabla f)\nabla f + \tfrac{1}{2}\nabla_{\cdot}\nabla\,\mathrm{scal}$$

is reduced to

$$R(\cdot\,, \nabla f)\nabla f = \nabla_{\nabla f}\mathrm{Ric} = -\nabla^2_{\nabla f,\cdot}\nabla f.$$

Next pick a field E such that $\nabla_E\nabla f = 0$ then

$$g(\nabla^2_{\nabla f,E}\nabla f, E) = g(\nabla_{\nabla f}\nabla_E\nabla f, E) - g(\nabla_{\nabla_{\nabla f}E}\nabla f, E) = -g(\nabla_E\nabla f, \nabla_{\nabla f}E) = 0$$

and when $\nabla_E\nabla f = \lambda E$,

$$\begin{aligned} g(\nabla^2_{\nabla f,E}\nabla f, E) &= g(\nabla_{\nabla f}\nabla_E\nabla f, E) - g(\nabla_{\nabla_{\nabla f}E}\nabla f, E) \\ &= \lambda g(\nabla_{\nabla f}E, E) - g(\nabla_E\nabla f, \nabla_{\nabla f}E) \\ &= \lambda g(\nabla_{\nabla f}E, E) - \lambda g(E, \nabla_{\nabla f}E) = 0. \end{aligned}$$

Finally if $\nabla_E\nabla f = 0$ and $\nabla_F\nabla f = \lambda F$ then

$$g(\nabla^2_{\nabla f,E}\nabla f, F) = g(\nabla_{\nabla f}\nabla_E\nabla f, F) - g(\nabla_{\nabla_{\nabla f}E}\nabla f, F) = -g(\nabla_E\nabla f, \nabla_{\nabla f}F) = 0.$$

Thus

$$g\big(R(E_i, \nabla f)\nabla f, E_j\big) = 0$$

when E_i is an eigenbasis.

Assume condition (3). Use the soliton equation to see that

$$(\nabla_X\mathrm{Ric})(Y, Z) - (\nabla_Y\mathrm{Ric})(X, Z) = -g(R(X, Y)\nabla f, Z).$$

Using the second Bianchi identity we also get that

$$(\nabla_X\mathrm{Ric})(Y, Z) - (\nabla_Y\mathrm{Ric})(X, Z) = \mathrm{div}R(X, Y, Z) = 0$$

since the curvature is harmonic. Thus $R(X,Y)\nabla f=0$. In particular the metric is radially flat.

Assume condition (4). If $\sec(E,\nabla f)=0$ then

$$g(\mathrm{Ric}(\nabla f),\nabla f)=0.$$

From elementary linear algebra, for a nonnegative (or nonpositive) definite, self-adjoint operator T,

$$\langle Tv,v\rangle=0 \quad\Rightarrow\quad Tv=0.$$

Thus if Ric is nonnegative or nonpositive then radial flatness implies

$$0=2\mathrm{Ric}(\nabla f)=\nabla\,\mathrm{scal}\,. \qquad\square$$

We now turn our attention to the main theorem. To prepare the way we show:

Proposition 3.4. *Assume that we have a gradient soliton*

$$\mathrm{Ric}+\mathrm{Hess}\,f=\lambda g$$

with constant scalar curvature $\lambda\neq 0$ and a nontrivial f. For a suitable constant α

$$f+\alpha=\frac{\lambda}{2}r^2,$$

where r is a smooth function whenever $\nabla f\neq 0$ and satisfies

$$|\nabla r|=1.$$

Proof. Observe that

$$\tfrac{1}{2}\nabla(\mathrm{scal}+|\nabla f|^2)=\mathrm{Ric}(\nabla f)+\nabla_{\nabla f}\nabla f=\lambda\nabla f,$$

which shows

$$\mathrm{scal}+|\nabla f|^2-2\lambda f=\mathrm{const}.$$

By adding a suitable constant to f we can then assume that

$$|\nabla f|^2=2\lambda f.$$

Thus f has the same sign as λ and the same zero locus as its gradient. If we define r such that

$$f=\frac{\lambda}{2}r^2$$

then

$$\nabla f=\lambda r\nabla r \quad\text{and}\quad 2\lambda f=|\nabla f|^2=\lambda^2r^2|\nabla r|^2=2\lambda f|\nabla r|^2. \qquad\square$$

This allows us to establish our characterization of rigid gradient solitons.

Proof of Theorem 1.2. We consider the case where $\lambda > 0$ as the other case is similar aside from some sign changes.

Using the condensed version of the soliton equation

$$\mathrm{Ric} + S = \lambda I, \qquad S = \nabla\nabla f,$$

we have

$$\nabla_{\nabla f} S + S \circ (S - \lambda I) = 0, \qquad \nabla_{\nabla f} \mathrm{Ric} + \mathrm{Ric} \circ (\lambda I - \mathrm{Ric}) = 0.$$

Assume that $f = \frac{1}{2}\lambda r^2$, where r is a nonnegative distance function. The minimum set

$$N = \{x : f(x) = 0\}$$

for f is also characterized as

$$N = \{x \in M : \nabla f(x) = 0\}.$$

This shows that $S \circ (S - \lambda I) = 0$ on N.

When $r > 0$ we note that the smallest eigenvalue for S is always absolutely continuous and therefore satisfies the differential equation

$$D_{\nabla f} \mu_{\min} = \mu_{\min}(\lambda - \mu_{\min}).$$

We claim that $\mu_{\min} \geq 0$. Using $r > 0$ as an independent coordinate and $\nabla f = \lambda r \nabla r$ yields

$$\partial_r \mu_{\min} = \frac{1}{\lambda r} \mu_{\min}(\lambda - \mu_{\min}).$$

This equation can be solved by separation of variables. In particular, $\mu_{\min} \to -\infty$ in finite time provided $\mu_{\min} < 0$ somewhere. This contradicts smoothness of f. Thus we can conclude that $\mu_{\min} \geq 0$ and hence that f is convex.

Now that we know f is convex the minimum set N must be totally convex. We also know that on N the eigenvalues of $\nabla\nabla f$ can only be 0 and λ. Thus their multiplicities are constant since the scalar curvature is constant. Using that the rank of $\nabla\nabla f$ is constant we see that N is a submanifold whose tangent space is given by $\ker(\nabla\nabla f)$. This in turn shows that N is a totally geodesic Einstein submanifold.

Note that when $\lambda > 0$ the minimum set N is in fact compact as it must be an Einstein manifold with Einstein constant λ.

The normal exponential map

$$\exp : \upsilon(N) \to M$$

follows the integral curves for ∇f or ∇r and is therefore a diffeomorphism.

Using the fundamental equations (see [Petersen 1998]) we see that the metric is completely determined by the fact that it is radially flat and that N is totally geodesic. From this it follows that the bundle is flat and hence of the type $N \times_\Gamma \mathbb{R}^k$.

Alternately note that radial flatness shows that all Jacobi fields along geodesics tangent to ∇f must be of the form

$$J = E + tF$$

where E and F are parallel. This also yields the desired vector bundle structure. □

4. Other results

In this section we discuss some further applications of the formulas derived above. For gradient solitons there is a naturally associated measure

$$dm = e^{-f}\, d\mathrm{vol}_g$$

which makes the operator $\Delta_f = \Delta - \nabla f$ self-adjoint. Namely the following identity holds for compactly supported functions:

$$\int_M \Delta_f(\phi)\psi\, dm = -\int_M \langle \nabla\phi, \nabla\psi\rangle\, dm = \int_M \phi\Delta_f(\psi)\, dm.$$

The measure dm also plays an important role in Perel'man's entropy formulas for the Ricci flow [2002]. Yau [1976] proves that on a complete Riemannian manifold any L^α, positive, subharmonic function is constant. The argument depends solely on using integration by parts and picking a clever test function ϕ. Therefore, the argument completely generalizes to the measure dm and operator Δ_f. Note that this idea was first used for solitons by Naber. Specifically the following Liouville theorem holds (see [Petersen and Wylie 2007b]).

Theorem 4.1 (Yau). *Any nonnegative real valued function u with $\Delta_f(u)(x) \geq 0$ which satisfies the condition*

$$\lim_{r\to\infty}\left(\frac{1}{r^2}\int_{B(p,r)} u^\alpha\, dm\right) = 0 \tag{4-1}$$

for some $\alpha > 1$ is constant.

Define

$$\Omega_{u,C} = \{x : u(x) \geq C\}.$$

If we only have a bound on the f-Laplacian on $\Omega_{u,C}$ then we can apply the L^α Liouville theorem to prove:

Corollary 4.2. *If $\Delta_f(u)(x) \geq 0$ for all $x \in \Omega_{u,C}$ and u satisfies* (4-1) *then either u is constant or $u \leq C$.*

Proof. Apply Theorem 4.1 to the function $(u - C)_+ = \max\{u - C, 0\}$. Then $(u - C)_+$ is constant which implies either $u \leq C$ or u is constant. □

One can also derive upper bounds on the growth of the measure dm from the inequality $\mathrm{Ric}+\mathrm{Hess}\, f \geq \lambda g$; see [Morgan 2005; Wei and Wylie 2007]. In particular, when $\lambda > 0$, the measure is bounded above by a Gaussian measure. Combining this estimate with the L^α maximum principle gives the following strong Liouville theorem for shrinking gradient Ricci solitons.

Corollary 4.3 [Wei and Wylie 2007]. *If M is a complete manifold satisfying*

$$\mathrm{Ric}+\mathrm{Hess}\, f \geq \lambda g$$

for $\lambda > 0$ and u is a real valued function such that

$$\Delta_f(u) \geq 0 \qquad \textit{and} \qquad u(x) \leq K e^{\beta d(p,x)^2}$$

for some $\beta < \frac{\lambda}{2}$ then u is constant.

A similar result, under the additional assumption that Ric is bounded above, is proven in [Naber 2006]. In fact, one can see immediately from the equation

$$\Delta_f(\mathrm{scal}) = \sum \lambda_i(\lambda - \lambda_i)$$

that if $0 \leq \mathrm{Ric} \leq \lambda$ for a shrinking soliton then scal is bounded, nonnegative, and has $\Delta_f(\mathrm{scal}) \geq 0$. Therefore it is constant and we have Lemma 1.4. Using the Liouville theorem also gives a different proof of the following well-known fact for shrinking gradient solitons.

Theorem 4.4. *If* scal *is bounded*, *then*

$$0 \leq \inf_M \mathrm{scal} \leq n\lambda.$$

Moreover, *if* $\mathrm{scal} \geq n\lambda$ *then M is Einstein.*

Proof. First suppose that $\mathrm{scal} \geq n\lambda$. By the Cauchy–Schwarz inequality,

$$\Delta_f(\mathrm{scal}) = -|\mathrm{Ric}|^2 + \lambda\, \mathrm{scal} \leq -\frac{\mathrm{scal}^2}{n} + \lambda\, \mathrm{scal} \leq \mathrm{scal}\Big(\lambda - \frac{\mathrm{scal}}{n}\Big).$$

So that $\Delta_f(\mathrm{scal}) \leq 0$. Let K be the upper bound on scal then the function

$$u = K - \mathrm{scal}$$

is bounded, nonnegative, and has $\Delta_f(u) \geq 0$. So by Corollary 4.3 scal is constant and thus must be Einstein.

To see the other inequality consider that on $\Omega_0 = \{x : \mathrm{scal}(x) \leq 0\}$,

$$\Delta_f(\mathrm{scal}) \leq 0,$$

so applying Corollary 4.2 to $-\mathrm{scal}$ gives the result. $\square$

For steady and expanding gradient solitons we can also apply the L^α Liouville theorem to the equation, $\Delta_f(|\nabla f|^2) \geq 0$.

Theorem 4.5. *Let $\alpha > 2$. If M is a steady or expanding soliton with*

$$\limsup_{r\to\infty} \frac{1}{r^2} \int_{B(p,r)} |\nabla f|^\alpha e^{-f} \, d\mathrm{vol}_g = 0, \tag{4-2}$$

then M is Einstein.

We think of Theorem 4.5 as a gap theorem for the quantity

$$\int_{B(p,r)} |\nabla f|^\alpha e^{-f} \, d\mathrm{vol}_g$$

since, if M is Einstein, the quantity is zero.

For steady solitons $\mathrm{scal} + |\nabla f|^2$ is constant so if the scalar curvature is bounded then so is $|\nabla f|$ and (4-2) is equivalent to the measure dm growing subquadratically. Therefore, we have:

Corollary 4.6. *If M is a steady Ricci soliton with bounded scalar curvature and*

$$\lim_{r\to\infty} \frac{1}{r^2} \int_{B(p,r)} e^{-f} \, d\mathrm{vol}_g = 0.$$

Then M is Ricci flat.

We note the relation of this result to the theorem proved by the second author and Wei that if $\mathrm{Ric} + \mathrm{Hess}\, f \geq 0$ and f is bounded then the growth of $e^{-f}\, d\mathrm{vol}_g$ is at least linear [2007]. Since Ricci flat manifolds have at least linear volume growth Corollary 4.6 implies that steady Ricci solitons with bounded scalar curvature also have at least linear dm-volume growth. There are Ricci flat manifolds with linear volume growth so Corollary 4.6 can be viewed as a gap theorem for the growth of dm on gradient steady solitons.

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