# Pacific Journal of Mathematics


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MOHAMED BEN AYED AND ABDELBAKI SELMI



Volume 242 No. 1 September 2009

PACIFIC JOURNAL OF MATHEMATICS
Vol. 242, No. 1, 2009

# CONCENTRATION PHENOMENA FOR A FOURTH-ORDER EQUATION ON $\mathbb{R}^n$

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**We consider the problem $\Delta^2 u = K(y)|u|^{8/(n-4)}u$ in $\mathbb{R}^n$ with $u$, $\Delta u \to 0$ as $|y| \to \infty$, where $K$ is a bounded and continuous function on $\mathbb{R}^n$, $n \geq 5$. Our aim is to construct infinitely many solutions which concentrate around $k$ points, $k \geq 2$, under some appropriate conditions on $K$. Moreover we prove that there is no solution which concentrates at one point.**

## 1. Introduction

In this paper, we consider the following problem:

$$(P_K) \qquad \begin{cases} \Delta^2 u = K(y)|u|^{8/(n-4)}u & y \in \mathbb{R}^n, \\ u \to 0, \ \ \Delta u \to 0 & \text{as } |y| \to +\infty, \end{cases}$$

where $n \geq 5$. The aim of this paper is to construct infinitely many solutions for $(P_K)$ under the condition that $K$ has a sequence of strictly local minimum points (respectively maximum points) moving to infinity. The solutions which we construct in this paper concentrate at $k$ points, $k \geq 2$, and when $K$ has a sequence of strictly local minimum points these solutions have to change sign and concentrate at two points each of which is a nearly local minimum point of $K$. When $K$ has a sequence of strictly local maximum points, solutions concentrating at $k$ points, $k \geq 2$ are constructed. These solutions are not necessary positive. However under an appropriate condition on $K$ we can prove that these constructed solutions are positive. Further we can perturb $K$ in $L^\infty$ norm to obtain another function $K_\varepsilon$ such that the problem $(P_{K_\varepsilon})$ has solutions which concentrate near $k$ fixed points, $k \geq 2$. We also explain why we do not have solutions which concentrate at one point.

In the past few decades, there has been a wide range of activity in the study of concentration phenomena for second-order elliptic equations involving critical Sobolev exponent; see for instance [Atkinson and Peletier 1987; Bahri et al. 1995; Ben Ayed et al. 2003; Brezis and Peletier 1989; Chabrowski and Yan 1999; del Pino et al. 2002; 2003; Han 1991; Micheletti and Pistoia 2003; Musso and Pistoia 2002;

*MSC2000:* primary 35J60, 35J65; secondary 58E05.
*Keywords:* sign-changing solution, critical Sobolev exponent, biharmonic operator.

Rey 1989; 1990; 1991; 1992; 1999] and the references therein. In sharp contrast to this, very little is known for equations involving the biharmonic operator. Our results extend to a fourth-order equation on $\mathbb{R}^n$ some results of [Yan 2000] that were previously known in the context of elliptic equations of second order. Compared with the second-order case, further difficulties have to be solved by delicate and careful estimates. Such estimates use the techniques developed by Bahri [1989] and Rey [1990].

To state our results, we fix some notation. Let $E$ be the closure of $C_c^\infty(\mathbb{R}^n)$ (the set of all smooth functions with compact support) equipped with the norm $\|\cdot\|$ and its inner product $\langle\, ,\rangle$ defined by

$$\text{(1-1)}\quad \|u\| = \left(\int_{\mathbb{R}^n} |\Delta u|^2\right)^{1/2}, \quad \langle u, v\rangle = \int_{\mathbb{R}^n} \Delta u \Delta v, \quad u, v \in E := \overline{C_c^\infty(\mathbb{R}^n)}.$$

We define the Sobolev constant by

$$\text{(1-2)}\quad S_n = \min \frac{\int_{\mathbb{R}^n} |\Delta u|^2}{\left(\int_{\mathbb{R}^n} |u|^{2n/(n-4)}\right)^{(n-4)/n}}, \qquad u \in L^{2n/(n-4)}(\mathbb{R}^n),\ \Delta u \in L^2(\mathbb{R}^n),\ u \neq 0.$$

For any $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_+^*$ we set

$$\text{(1-3)}\quad \delta_{x,\lambda}(y) = \frac{c_n \lambda^{(n-4)/2}}{(1+\lambda^2|y-x|^2)^{(n-4)/2}}, \qquad \text{with } c_n = \big(n(n-4)(n^2-4)\big)^{(n-4)/8}.$$

It is well known [Lin 1998] that $\delta_{x,\lambda}$ are the only solutions of

$$\text{(1-4)}\qquad \Delta^2 u = u^{(n+4)/(n-4)},\ u > 0 \text{ in } \mathbb{R}^n,$$

and are also the only minimizers of (1-2).

Let $k \in \mathbb{N}^*$, for $x_j = (x_{j_1}, \dots, x_{j_n}) \in \mathbb{R}^n$, $\lambda_j \in \mathbb{R}_+^*$, $j = 1, \dots, k$. Set

$$E_{x,\lambda,k} = \bigcap_{j=1}^{k} E_{x_j,\lambda_j},$$

where

$$\text{(1-5)}\quad E_{x_j,\lambda_j} = \left\{ v \in E,\ \langle \delta_{x_j,\lambda_j}, v\rangle = \left\langle \frac{\partial \delta_{x_j,\lambda_j}}{\partial \lambda_j}, v\right\rangle = \left\langle \frac{\partial \delta_{x_j,\lambda_j}}{\partial x_{j_i}}, v\right\rangle = 0,\ i \le n\right\},\ j \le k.$$

Now we state the main results of this paper.

**Theorem 1.1.** *Assume that $K$ is a bounded continuous function in $\mathbb{R}^n$ satisfying the following condition*:

$(H_1)$ *$K$ has a sequence of strictly local minimum points $z_j \in \mathbb{R}^n$ such that $|z_j| \to +\infty$ and in a small neighbourhood of each $z_j$, there are constants $K_j > 0$ and $\beta_j \in (n-4, n)$ such that*

$$K(y) = K_j + Q_j(y - z_j) + R_j(y - z_j), \tag{1-6}$$

*where $K_j$ satisfies $K_j \geq \eta$ for some constant $\eta > 0$, and $Q_j$ and $R_j$ satisfy*

$$a_0|y|^{\beta_j} \leq Q_j(y) \leq a_1|y|^{\beta_j} \quad \text{and} \quad R_j(y) = O(|y|^{\beta_j+\sigma}) \tag{1-7}$$

*for some constants $a_1 \geq a_0 > 0$ and $\sigma > 0$ independent of $j$.*

*Then for each small $\nu > 0$ and $z_{j_1}$ there exists another strictly local minimum point $z_{j_2}$, such that $(P_K)$ has a solution of the form*

$$u = \alpha_1 \delta_{x_{j_1}, \lambda_{j_1}} - \alpha_2 \delta_{x_{j_2}, \lambda_{j_2}} + v,$$

*where $(\alpha, x, \lambda, v)$ satisfies $v \in E_{x,\lambda,2}$ $\|v\| \leq \nu$, $x = (x_{j_1}, x_{j_2})$, $\lambda = (\lambda_{j_1}, \lambda_{j_2})$ and*

$$|z_{j_1} - z_{j_2}| > 1/\nu, \quad \|\alpha_i - K(y_{j_i})^{(4-n)/8}| \leq \nu, \quad |x_{j_i} - z_{j_i}| \leq \nu, \quad \lambda_{j_i} \geq 1/\nu, \quad i = 1, 2.$$

**Theorem 1.2.** *Assume that $K$ is a bounded continuous function in $\mathbb{R}^n$ satisfying the following condition*:

$(H_2)$ *$K$ has a sequence of strictly local maximum points $z_j \in \mathbb{R}^n$ such that $|z_j| \to \infty$ and in a small neighbourhood of each $z_j$, there are constants $K_j > 0$ and $\beta_j \in (n-4, n)$ such that*

$$K(y) = K_j - Q_j(y - z_j) + R_j(y - z_j), \tag{1-8}$$

*where $K_j$ satisfies $K_j \geq \eta$ for some constant $\eta > 0$, and $Q_j$ and $R_j$ satisfy* (1-7).

*Then for each small $\nu > 0$ and $z_{j_1}$ there exists another strictly local maximum point $z_{j_2}$, such that $(P_K)$ has a solution of the form*

$$u = \alpha_1 \delta_{x_{j_1}, \lambda_{j_1}} + \alpha_2 \delta_{x_{j_2}, \lambda_{j_2}} + v,$$

*where $(\alpha, x, \lambda, v)$ satisfies $v \in E_{x,\lambda,2}$, $\|v\| \leq \nu$, $x = (x_{j_1}, x_{j_2})$, $\lambda = (\lambda_{j_1}, \lambda_{j_2})$ and*

$$|z_{j_1} - z_{j_2}| > 1/\nu, \quad |\alpha_i - K(z_{j_i})^{(4-n)/8}| \leq \nu, \quad |x_{j_i} - z_{j_i}| \leq \nu, \quad \lambda_{j_i} \geq 1/\nu, \quad i = 1, 2.$$

**Remark 1.3.** (i) We can find some functions which satisfy the assumptions $(H_1)$ and $(H_2)$. Therefore the problem $(P_K)$ has at least four solutions given by Theorems 1.1 and 1.2. (In fact if $u$ is a solution of $(P_K)$ then $-u$ is another one).

(ii) To show that functions which satisfy the assumptions $(H_1)$ and $(H_2)$ exist, we can take some functions which are periodic in at least one variable and having one strictly local minimum point and one strictly local maximum point.

Observe that if $(P_K)$ has a solution then the problem $(P_{K o \tau_\zeta})$ has another one, where $\tau_\zeta : x \mapsto \zeta x$. The condition that $|z_j| \to +\infty$, in Theorems 1.1 and 1.2 is useful to get the distance $l = |z_1 - z_2|$ large enough in the proof of Theorems 1.1 and 1.2. Therefore for any two fixed points $z_1,\ z_2$, we can choose $\zeta$ small as desired such that $\frac{1}{\zeta}|z_1 - z_2|$ will be large as desired, hence the proof of Theorems 1.1 and 1.2 are valid. This leads to the following perturbed result.

**Theorem 1.4.** *Let $K$ be a bounded continuous function in $\mathbb{R}^n$. Then for any $\varepsilon > 0$, $x_0 \in \mathbb{R}^n$ satisfying $K(y) \geq \eta > 0$ for all $y \in \overline{B_\varepsilon(x_0)}$, $\nu > 0$ and any two different points $z_1,\ z_2 \in B_\varepsilon(x_0)$, we can find another continuous function $K_\varepsilon$ which satisfies $|K_\varepsilon - K|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon$, and $K_\varepsilon(y) = K(y)$ in $\mathbb{R}^n \setminus B_\varepsilon(x_0)$ such that the perturbed problem*

$$(P_{K_\varepsilon}) \qquad \begin{cases} \Delta^2 u = K_\varepsilon(y)|u|^{8/(n-4)}u, & y \in \mathbb{R}^n, \\ u \to 0,\ \Delta u \to 0, & \textit{as } |y| \to +\infty, \end{cases}$$

*satisfies one of the following statements*:

(1) $(P_{K_\varepsilon})$ *has a solution of the form* $u_\varepsilon = \alpha_1 \delta_{x_1,\lambda_1} - \alpha_2 \delta_{x_2,\lambda_2} + v$,

(2) $(P_{K_\varepsilon})$ *has a solution of the form* $u_\varepsilon = \alpha_1 \delta_{x_1,\lambda_1} + \alpha_2 \delta_{x_2,\lambda_2} + v$,

*where* $(\alpha, x, \lambda, v)$ *satisfies* $v \in E_{x,\lambda,2},\ x = (x_1, x_2),\ \lambda = (\lambda_1, \lambda_2)$ *and*

$$\|v\| \leq \nu,\ |\alpha_j - K(z_j)^{(4-n)/8}| \leq \nu, \quad |x_j - z_j| \leq \nu, \quad \lambda_j \geq 1/\nu, \quad j = 1, 2.$$

**Remark 1.5.** (i) We can perturb $K$ in $B_\varepsilon(x_0)$ and $B_\varepsilon(x_1)$ ($B_\varepsilon(x_0) \cap B_\varepsilon(x_1) = \varnothing$), so that the conclusions (1) and (2) of Theorem 1.4 hold at the same time.

(ii) Taking four different points $z_1, z_2, z_1'$ and $z_2'$ in $B_\varepsilon(x_0)$, we can choose $K_\varepsilon$ ($z_1$ and $z_2$ are two minimum points of $K_\varepsilon$, and $z_1'$ and $z_2'$ are two maximum points of $K_\varepsilon$) so that the conclusions (1) and (2) of Theorem 1.4 hold at the same time. Note that for (1), the concentration points $x_i$ are near $z_i$, but for (2), the concentration points $x_i$ are close to $z_i'$.

Note that in Theorems 1.1 and 1.2 we need some flatness of the function $K$ near the critical points of $K$. See $(H_1)$ and $(H_2)$. In these assumptions the constants $\beta_j$ are larger than $n-4$, however if $K$ is a $C^2$ function, we derive that $\beta_j \geq 2$. Furthermore if we assume that the critical points are nondegenerate, then near each local minimum point (respectively maximum point) of $K$, (1-6) (respectively (1-8)) holds with $\beta_j = 2$. We remark that $\beta_j = 2 \leq n-4$ if $n \geq 6$. Thus, this possibility is admissible only for $n = 5$. In this case we can improve the result of Theorem 1.2 in constructing some solutions with $k$ bubbles, $k \geq 2$. In fact, we have the following result.

**Theorem 1.6.** *Let $k \geq 2$ be a fixed integer. Assume $n = 5$ and $K$ is a bounded continuous function on $\mathbb{R}^5$ satisfying the following condition*:

$(H_2')$ *$K$ has a sequence of strictly local maximum points $z_j \in \mathbb{R}^5$ such that in a small neighbourhood of each $z_j$, $K$ is $C^3$ and we have $a_0 \leq K(z_j) \leq a_1$, $-a_1 \leq \Delta K(z_j) \leq -a_0 < 0$, for some $a_1 \geq a_0 > 0$. Moreover for any small $\tau > 0$, there is an $\eta = \eta(\tau) > 0$ such that $K(z_j) - K(y) \geq \eta$, for all $y \in \partial B_\tau(z_j)$. Furthermore, for any $L > 0$ and $z_{j_1}$, there exist $z_{j_2}, \dots, z_{j_k}$ such that $\min_{i \neq h} |z_{j_i} - z_{j_h}| \geq L$ and $\max_{i \neq h} |z_{j_i} - z_{j_h}| / \min_{i \neq h} |z_{j_i} - z_{j_h}| \leq C$, where $C > 0$ is a constant.*

*Then for each small $\nu > 0$ and $z_{j_1}$, we can find $k-1$ other strictly local maximum points $z_{j_2}, \dots, z_{j_k}$ such that $(P_K)$ has a solution of the form*

$$u = \sum_{i=1}^{k} \alpha_i \delta_{x_{j_i}, \lambda_{j_i}} + v,$$

*where $(\alpha, x, \lambda, v)$ satisfies $v \in E_{x,\lambda,k}$, $\|v\| \leq \nu$, $x = (x_{j_1}, \dots, x_{j_k})$, $\lambda = (\lambda_{j_1}, \dots, \lambda_{j_k})$ and for $i = 1, \dots, k$,*

$$|z_{j_i} - z_{j_h}| \geq 1/\nu,\ i \neq h, \quad |\alpha_i - K(z_{j_i})^{-1/8}| \leq \nu, \quad |x_{j_i} - z_{j_i}| \leq \nu, \quad \lambda_{j_i} \geq 1/\nu.$$

We remark that the proof of Theorem 1.6 is easier than the proof of Theorem 1.2. Indeed, assumption (1-8) also holds for Theorem 1.6 with $Q_j = D^2 K(z_j)$. Furthermore, all the $\beta_j$ are equal to 2. Hence some inequalities in the proof of Theorem 1.2 become equalities. However, we can obtain a more general result than Theorem 1.6 by assuming that $n \geq 5$ and in (1-8) all the constants $\beta_j$ are the same.

**Theorem 1.7.** *Let $n \geq 5$. Assume that $K$ is a bounded continuous function in $\mathbb{R}^n$ satisfying the following condition*:

$(H_2'')$ *$K$ has a sequence of strictly local maximum points $z_j \in \mathbb{R}^n$ such that $|z_j| \to \infty$ and there exists $\beta \in (n-4, n)$ such that in a small neighbourhood of each $z_j$, (1-8) and (1-7) are satisfied (here $\beta_j = \beta$). Furthermore, for any $L > 0$ and $z_{j_1}$, there exists $z_{j_2}, \dots, z_{j_k}$ such that $\min_{i \neq h} |z_{j_i} - z_{j_h}| \geq L$ and $\max_{i \neq h} |z_{j_i} - z_{j_h}| / \min_{i \neq h} |z_{j_i} - z_{j_h}| \leq C$, where $C > 0$ is a constant.*

*Then for each small $\nu > 0$ and $z_{j_1}$, we can find $k-1$ other strictly local maximum points $z_{j_2}, \dots, z_{j_k}$ such that $(P_K)$ has a solution of the form*

$$u = \sum_{i=1}^{k} \alpha_i \delta_{x_{j_i}, \lambda_{j_i}} + v,$$

*where* $(\alpha, x, \lambda, v)$ *satisfies* $v \in E_{x,\lambda,k}$, $\|v\| \leq \nu$, $x = (x_{j_1}, \dots, x_{j_k})$, $\lambda = (\lambda_{j_1}, \dots, \lambda_{j_k})$ *and*

$$|z_{j_i} - z_{j_h}| \geq 1/\nu,\ i \neq h,\ \ |\alpha_i - K(z_{j_i})^{(4-n)/8}| \leq \nu,\ \ |x_{j_i} - z_{j_i}| \leq \nu,\ \ \lambda_{j_i} \geq 1/\nu,\ \ i \leq k.$$

Using Theorem 1.6 we get the following perturbation result for the case $n = 5$.

**Theorem 1.8.** *Assume* $n = 5$. *Let* $K$ *be a bounded continuous function on* $\mathbb{R}^5$. *Then for any* $\varepsilon > 0$, $x_0 \in \mathbb{R}^5$ *satisfying* $K(y) \geq \eta > 0$ *for all* $y \in \overline{B_\varepsilon(x_0)}$, $\nu > 0$ *and any* $k$ *different points* $z_1, \dots, z_k \in B_\varepsilon(x_0)$, *with* $k \geq 2$, *we can find another continuous function* $K_\varepsilon$ *which satisfies* $|K_\varepsilon - K|_{L^\infty(\mathbb{R}^5)} \leq \varepsilon$, *and* $K_\varepsilon(y) = K(y)$ *in* $\mathbb{R}^5 \setminus B_\varepsilon(x_0)$ *such that* $(P_{K_\varepsilon})$ *has a solution of the form*

$$u = \sum_{j=1}^{k} \alpha_j \delta_{x_j, \lambda_j} + v,$$

*where* $(\alpha, x, \lambda, v)$ *satisfies* $v \in E_{x,\lambda,k}$, $x = (x_1, \dots, x_k)$, $\lambda = (\lambda_1, \dots, \lambda_k)$ *and*

$$\|v\| \leq \nu, \quad |\alpha_j - K(z_j)^{-1/8}| \leq \nu, \quad |x_j - z_j| \leq \nu, \quad \lambda_j \geq 1/\nu, \quad j = 1, \dots, k.$$

The constructed solutions, roughly speaking, concentrate at $k$ different points and in Theorems 1.1, 1.2, 1.6 and 1.7 the distance between different concentration points is very large, while in Theorems 1.4 and 1.8 the distance between different concentration points is fixed but $K$ is very steep on the concentration points.

Note that our solutions are not necessary positive. In fact, for the case of $\Delta$ instead of $\Delta^2$, we multiply the equation by the function $u^- = \max(0, -u)$ and we integrate on $\mathbb{R}^n$, so we are able to prove that the constructed solutions are positive. However, in our cases the function $u^-$ is not in the space $E$. To overcome this difficulty, we add another assumption on the function $K$. More precisely, we have:

**Theorem 1.9.** *In Theorems 1.2, 1.4-*(2), *1.6, 1.7 and 1.8, if we assume further that there exists a positive constant* $\eta_0$ *such that* $K \geq \eta_0 > 0$ *on* $\mathbb{R}^n$, *then the constructed solution is a positive function.*

Finally we give the following result which shows that $k \geq 2$ in our main results is necessary.

**Proposition 1.10.** *Assume that* $K(y)$ *is periodic in all variables and it satisfies* $\Delta K(x) < -c_0 < 0$ *for all global maximum points* $x$. *Then for any* $\alpha > 0$ *small, we have*

$$\operatorname{Sup}\{|u|_{L^\infty},\ u \text{ satisfies } (P_K),\ c \leq I(u) \leq c + \alpha\} < \infty,$$

*where* $c = \frac{2}{n} S_n^{n/4} / K_M^{(n-4)/4}$ *and* $K_M = \max_{\mathbb{R}^n} K(y)$.

The proof of our results is inspired by the methods of [Yan 2000]. As in [Bahri 1989; Bahri et al. 1995; Rey 1990] we first reduce the problem of finding a solution for $(P_K)$ to that of finding a critical point for a function defined in a finite dimensional domain.

Our paper is organized as follows. In Section 2 we give the proofs of Theorems 1.1, 1.2 and 1.4. Section 3 is devoted to the proofs of Theorems 1.6, 1.7 and 1.8. The proofs of Theorem 1.9 and Proposition 1.10 are given in Section 4. Some basic estimates needed in the proofs are presented in Appendices A and B.

## 2. Proofs of Theorems 1.1, 1.2 and 1.4

Our method is a variational one. Hence, we introduce the Euler Lagrange functional

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^n}|\Delta u|^2-\frac{n-4}{2n}\int_{\mathbb{R}^n}K(y)|u|^{2n/(n-4)},\quad u\in E:=\overline{C_c^\infty(\mathbb{R}^n)}. \tag{2-1}$$

Note that the critical points of $I$ are solutions of $(P_K)$ and inversely. Thus, to prove the theorems, we will construct some critical points of $I$. The constructed solutions concentrate at some critical points of $K$. Therefore, for $z_1$ and $z_2$ two critical points of $K$ and $\nu$ a small positive constant, we introduce the sets

$$D_{\nu,2}=\{\,(x,\lambda)\in(\mathbb{R}^n)^2\times\mathbb{R}^2,\ x_j\in\overline{B_\nu(z_j)},\ \lambda_j\ge 1/\nu,\ j=1,2\}, \tag{2-2}$$

$$M_{\nu,2}=\Big\{(\alpha,x,\lambda,v):(x,\lambda)\in D_{\nu,2},\ v\in E_{x,\lambda,2}\,,\ \sum_{j=1}^{2}\big|\alpha_j-K(z_j)^{(4-n)/8}\big|+\|v\|\le\nu\Big\}. \tag{2-3}$$

Our goal is to prove we can choose $(\alpha,x,\lambda,v)\in M_{\nu,2}$ so $u=\alpha_1\delta_{x_1,\lambda_1}+\kappa\alpha_2\delta_{x_2,\lambda_2}+v$ is a critical point of $I$, where $\kappa\in\{-1,1\}$. Since $|x_1-x_2|\ge d>0$ and the concentration $\lambda_i$'s are large, the interaction between $\delta_{x_1,\lambda_1}$ and $\delta_{x_2,\lambda_2}$ is very small. More precisely, using [Bahri 1989], it is equivalent to (with a multiplicative constant)

$$\varepsilon_{ij}=\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|x_i-x_j|^2\right)^{-(n-4)/2}\qquad\text{for } i\neq j. \tag{2-4}$$

*Proof of Theorem 1.1.* In this proof, we will assume that, near $z_1$ and $z_2$, $K$ satisfies (1-6) and (1-7). Let $J$ be the function defined by

$$J:M_{\nu,2}\to\mathbb{R},\quad (\alpha,x,\lambda,v)\mapsto I\left(\alpha_1\delta_{x_1,\lambda_1}-\alpha_2\delta_{x_2,\lambda_2}+v\right).$$

Note that $(\alpha,x,\lambda,v)\in M_{\nu,2}$ is a critical point of $J$ if and only if the function $u=\alpha_1\delta_{x_1,\lambda_1}-\alpha_2\delta_{x_2,\lambda_2}+v$ is a critical point of $I$. That means there exist $A_j$, $B_j$

and $C_{ji} \in \mathbb{R}$, $1 \le i \le n$ and $1 \le j \le 2$ such that

$$\frac{\partial J}{\partial \alpha_j} = 0, \tag{2-5}$$

$$\frac{\partial J}{\partial x_{j_i}} = B_j \Big\langle \frac{\partial^2 \delta_{x_j,\lambda_j}}{\partial \lambda_j \partial x_{j_i}}, v \Big\rangle + \sum_{h=1}^{n} C_{jh} \Big\langle \frac{\partial^2 \delta_{x_j,\lambda_j}}{\partial x_{j_h} \partial x_{j_i}}, v \Big\rangle, \tag{2-6}$$

$$\frac{\partial J}{\partial \lambda_j} = B_j \Big\langle \frac{\partial^2 \delta_{x_j,\lambda_j}}{\partial \lambda_j^2}, v \Big\rangle + \sum_{h=1}^{n} C_{jh} \Big\langle \frac{\partial^2 \delta_{x_j,\lambda_j}}{\partial x_{j_h} \partial \lambda_j}, v \Big\rangle, \tag{2-7}$$

$$\frac{\partial J}{\partial v} = \sum_{j=1}^{2} \Big( A_j \delta_{x_j,\lambda_j} + B_j \frac{\partial \delta_{x_j,\lambda_j}}{\partial \lambda_j} + \sum_{h=1}^{n} C_{jh} \frac{\partial \delta_{x_j,\lambda_j}}{\partial x_{j_h}} \Big), \tag{2-8}$$

where $x_{j_i}$ is the $i$–th component of $x_j$.

First we state the following proposition which allows us to reduce the original problem to a finite-dimensional problem and to show that the $v$-part of $u$ is negligible with respect to the concentration phenomenon.

**Proposition 2.1.** *Assume that near $z_1$ and $z_2$, $K$ satisfies* (1-6) *and* (1-7). *There exists $\nu_0 > 0$, such that for each $\nu \in (0, \nu_0]$ and $(x, \lambda) \in D_{\nu,2}$, there exists a unique $(\alpha(x, \lambda), v(x, \lambda)) \in \mathbb{R}^2 \times E_{x,\lambda,2}$ such that* (2-5) *and* (2-8) *are satisfied and we have the estimate*

$$\sum_{j=1}^{2} \Big| \alpha_j - \frac{1}{K(x_j)^{(n-4)/8}} \Big| + \|v\| = O\Big( \sum_{j=1}^{2} \Big( |x_j - z_j|^{\beta_j} + \frac{1}{\lambda_j^{\inf(\beta_j,(n+4)/2)}} \Big) + \varepsilon_{12}^{1/2+\tau} \Big), \tag{2-9}$$

*where $\tau > 0$ is a constant. Moreover the function $(x, \lambda) \mapsto \big(\alpha(x, \lambda), v(x, \lambda)\big)$ is $C^1$.*

*Proof.* Let $w = (\overline{\alpha}, v) \in \mathbb{R}^2 \times E_{x,\lambda,2}$, $\overline{\alpha} = (\overline{\alpha}_1, \overline{\alpha}_2)$, $\overline{\alpha_j} = \alpha_j - K(x_j)^{(4-n)/8}$, $j = 1, 2$, and let

$$\mathcal{J}(x, \lambda, w) = J(\alpha, x, \lambda, v), \quad w \in \mathbb{R}^2 \times E_{x,\lambda,2},$$
$$H_{x,\lambda,2} = \frac{1}{K(x_1)^{(n-4)/8}} \delta_{x_1,\lambda_1} - \frac{1}{K(x_2)^{(n-4)/8}} \delta_{x_2,\lambda_2}.$$

As in [Bahri 1989] (see also [Rey 1990]), we expand $\mathcal{J}(x, \lambda, w)$ at $w = 0$. We get

$$\mathcal{J}(x, \lambda, w) = J(\alpha_0, x, \lambda, 0) + F(w) + \frac{1}{2} Q(w) + R(w),$$
$$\text{where } \alpha_0 = \Big( \frac{1}{K(x_1)^{(n-4)/8}}, \frac{1}{K(x_2)^{(n-4)/8}} \Big)$$

and where

$$F(w)=\langle H_{x,\lambda,2},\overline{\alpha}_1\delta_{x_1,\lambda_1}-\overline{\alpha}_2\delta_{x_2,\lambda_2}\rangle \\ -\int_{\mathbb{R}^n}K|H_{x,\lambda,2}|^{8/(n-4)}H_{x,\lambda,2}(\overline{\alpha}_1\delta_{x_1,\lambda_1}-\overline{\alpha}_2\delta_{x_2,\lambda_2}+v),$$

$$Q(w)=\|\overline{\alpha}\delta_{x_1,\lambda_1}-\overline{\alpha}_2\delta_{x_2,\lambda_2}\|^2+\|v\|^2 \\ -\frac{n+4}{n-4}\int_{\mathbb{R}^n}K|H_{x,\lambda,2}|^{8/(n-4)}(\overline{\alpha}_1\delta_{x_1,\lambda_1}-\overline{\alpha}_2\delta_{x_2,\lambda_2}+v)^2$$

and $R(w)$ satisfies

$$R(w)=O\bigl(\|w\|^{\min(3,2n/(n-4))}\bigr),\quad R'(w)=O\bigl(\|w\|^{\min(2,(n+4)/(n-4))}\bigr),\\ R''(w)=O\bigl(\|w\|^{\min(1,8/(n-4))}\bigr).$$

It is clear that $F$ is a continuous linear form on $\mathbb{R}^2\times E_{x,\lambda,2}$ which is equipped with the $\mathbb{R}^2\times E$ scalar product. Therefore there exists a unique $f\in\mathbb{R}^2\times E_{x,\lambda,2}$ such that $F(w)=\langle f,w\rangle$. Furthermore, $Q$ is a continuous quadratic form over $\mathbb{R}^2\times E_{x,\lambda,2}$. It satisfies

$$Q(w)=-\gamma\sum_{i=1}^2(1+o(1))\overline{\alpha}_i{}^2+\|v\|^2-\frac{n+4}{n-4}\sum_{i=1}^2\int_{\mathbb{R}^n}\delta_{x_i,\lambda_i}^{8/(n-4)}v^2+o(\|v\|^2),$$

where $\gamma$ is a positive constant. Now using [Ben Ayed and El Mehdi 2007] we know that the quadratic form

$$v\mapsto\|v\|^2-\frac{n+4}{n-4}\sum_{i=1}^2\int_{\mathbb{R}^n}\delta_{x_i,\lambda_i}^{8/(n-4)}v^2, \tag{2-10}$$

is positive definite on the space $E_{x,\lambda,2}$. Hence it is clear that $Q$ is an invertible quadratic form. Therefore from the implicit function theorem, we derive the existence of a $C^1$ map which to $(x,\lambda)\in D_{\nu,2}$, $\nu<\nu_0$ ($\nu_0$ small enough) associates $w(x,\lambda)=(\overline{\alpha}(x,\lambda),v(x,\lambda))\in\mathbb{R}^2\times E_{x,\lambda,2}$ such that

$$\|w(x,\lambda)\|=O(\|f\|). \tag{2-11}$$

Moreover, for $\alpha_j(x,\lambda)=\overline{\alpha}_j(x,\lambda)+K(x_j)^{(4-n)/8}$, we have that $(\alpha(x,\lambda),v(x,\lambda))$ satisfies (2-5) and (2-8) for certain $A_j$, $B_j$ $C_{ji}$, $i=1,\dots,n$, $j=1,2$. It remains to estimate $\|f\|$. Using Lemmas A.2 and A.3 we derive

$$\|f\|=O\biggl(\sum_{j=1}^2\biggl(|x_j-z_j|^{\beta_j}+\frac{1}{\lambda_j^{\inf(\beta_j,(n+4)/2)}}\biggr)+\varepsilon_{12}^{1/2+\tau}\biggr),$$

where $\tau>0$ is a constant. From (2-11) the same estimate holds for $\|w(x,\lambda)\|$. $\square$

Without loss of generality, suppose $z_{j_1} = z_1$ and $z_2$ is another local minimum point of $K$ with $l = |z_2 - z_1|$ is large enough. Define

$$L_1 = l^{(n-4)\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)},$$
$$L_2 = l^{(n-4)\beta_1/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)},$$

where $\beta_j,\ j = 1, 2$ is the constant defined in (1-7).

Let $(x, \lambda) \mapsto \big(\alpha(x, \lambda), v(x, \lambda)\big)$ be the function defined in Proposition 2.1. We consider the problem

$$\sup\{J\big(\alpha(x, \lambda), x, \lambda, v(x, \lambda)\big),\ (x, \lambda) \in S_{\nu,2}\}, \tag{2-12}$$

where

$$S_{\nu,2} = \{(x, \lambda) \in D_{\nu,2},\ \lambda_j \in [\gamma_1 L_j, \gamma_2 L_j],\ j = 1, 2\}, \tag{2-13}$$

$\gamma_1 > 0$ is a small constant and $\gamma_2 > 0$ is a large constant, which will be determined later. Since $S_{\nu,2}$ is a compact set, it follows that the problem (2-12) has a maximizer $(\bar{x}, \bar{\lambda}) \in S_{\nu,2}$. We will prove that for $\nu$ small enough, there exists $l_0 > 0$ such that if $l = |z_2 - z_1| > l_0$, the maximizer $(\bar{x}, \bar{\lambda})$ is an interior point of $S_{\nu,2}$. Hence $(\bar{x}, \bar{\lambda})$ is a critical point of $J\big(\alpha(x, \lambda), x, \lambda, v(x, \lambda)\big)$.

By Proposition 2.1 and Lemma A.4 we have for any $(x, \lambda) \in S_{\nu,2}$,

$$\begin{aligned} J\big(\alpha(x, \lambda), x, \lambda, v(x, \lambda)\big) &= \mathcal{J}(x, \lambda, w) \\ &= \mathcal{J}(x, \lambda, 0) + O\big(\|f\|\|w\| + \|w\|^2\big) \\ &= \sum_{j=1}^{2} I\big(K(x_j)^{-(n-4)/8}\delta_{x_j,\lambda_j}\big) + \frac{D\varepsilon_{12}}{K(x_1)^{(n-4)/8}K(x_2)^{(n-4)/8}} \\ &\qquad + O\bigg(\sum_{j=1}^{2}\bigg(|x_j - z_j|^{2\beta_j} + \frac{1}{\lambda_j^{\inf(2\beta_j, n+4)}}\bigg) + \varepsilon_{12}^{1+\tau}\bigg), \end{aligned} \tag{2-14}$$

where $D > 0$ is a constant depending on $n$ only.

On the other hand, using $(H_1)$, a computation shows that

$$\begin{aligned} I(K(x_j)^{-(n-4)/8}\delta_{x_j,\lambda_j}) &= \bigg(\frac{1}{2}\frac{1}{K(x_j)^{(n-4)/4}} - (n-4)/(2n)\frac{K(z_j)}{K(x_j)^{n/4}}\bigg)S_n^{n/4} \\ &\qquad - \frac{n-4}{2n}\frac{1}{K(x_j)^{n/4}}\int_{\mathbb{R}^n} Q_j\bigg(\frac{y}{\lambda_j} + x_j - z_j\bigg)\delta_{0,1}^{2n/(n-4)} \\ &\qquad + O\bigg(|x_j - z_j|^{\beta_j+\sigma} + \frac{1}{\lambda_j^{\beta_j+\sigma}}\bigg). \end{aligned} \tag{2-15}$$

Hence

$$I(K(z_j)^{-(n-4)/8}\delta_{z_j,\lambda_j}) \tag{2-16}$$
$$= \frac{2S_n^{n/4}}{nK(z_j)^{(n-4)/4}} - \frac{n-4}{2n}\frac{1}{K(z_j)^{n/4}}\int_{\mathbb{R}^n} Q_j\left(\frac{y}{\lambda_j}\right)\delta_{0,1}^{2n/(n-4)} + O\left(\frac{1}{\lambda_j^{\beta_j+\sigma}}\right)$$
$$\geq \frac{2S_n^{n/4}}{nK(z_j)^{(n-4)/4}} - \frac{C}{\lambda_j^{\beta_j}} + O\left(\frac{1}{\lambda_j^{\beta_j+\sigma}}\right).$$

At this time we will proceed in two steps.

**Step 1.** We claim that $|\bar{x}_j - z_j| < C/\bar{\lambda}_j$, if $l$ is large enough.

Using the fact that $J\big(\alpha(\bar{x},\bar{\lambda}),\bar{x},\bar{\lambda},v(\bar{x},\bar{\lambda})\big) \geq J\big(\alpha(z,\bar{\lambda}),z,\bar{\lambda},v(z,\bar{\lambda})\big)$ together with (2-14), (2-15) and (2-16) we obtain

$$\sum_{j=1}^{2}\left(\frac{1}{2}\frac{1}{K(\bar{x}_j)^{(n-4)/4}} - \frac{n-4}{2n}\frac{K(z_j)}{K(\bar{x}_j)^{n/4}}\right)S_n^{n/4} - \frac{n-4}{2n}\frac{1}{K(\bar{x}_j)^{n/4}}\int_{\mathbb{R}^n} Q_j\left(\frac{y}{\bar{\lambda}_j} + \bar{x}_j - z_j\right)\delta_{0,1}^{2n/(n-4)} \tag{2-17}$$
$$\geq \sum_{j=1}^{2}\frac{2S_n^{n/4}}{nK(z_j)^{(n-4)/4}} - \sum_{j=1}^{2}\frac{C}{\bar{\lambda}_j^{\beta_j}} - \frac{D\varepsilon_{12}}{K(\bar{x}_1)^{(n-4)/8}K(\bar{x}_2)^{(n-4)/8}} - \frac{D\varepsilon_{12}}{K(z_1)^{(n-4)/8}K(z_2)^{(n-4)/8}}$$
$$+ O\left(\sum_{j=1}^{2}\left(|\bar{x}_j - z_j|^{\beta_j+\sigma} + \frac{1}{\bar{\lambda}_j^{\beta_j+\sigma}}\right) + \varepsilon_{12}^{1+\tau}\right).$$

Now by $(H_1)$ a computation shows that

$$\frac{1}{2}\frac{1}{K(\bar{x}_j)^{(n-4)/4}} - \frac{n-4}{2n}\frac{K(z_j)}{K(\bar{x}_j)^{n/4}} = \frac{2}{nK(z_j)^{(n-4)/4}} + O\big(|\bar{x}_j - z_j|^{2\beta_j}\big), \tag{2-18}$$

$$Q_j\left(\frac{y}{\bar{\lambda}_j} + \bar{x}_j - z_j\right) \geq a_0\left|\frac{y}{\bar{\lambda}_j} + \bar{x}_j - z_j\right|^{\beta_j} \geq a_0|\bar{x}_j - z_j|^{\beta_j} - c\frac{|y|^{\beta_j}}{\bar{\lambda}_j^{\beta_j}}, \tag{2-19}$$

where $a_0$ and $c$ are some positive constants. Therefore (2-17), (2-18) and (2-19) imply

$$\sum_{j=1}^{2}|\bar{x}_j - z_j|^{\beta_j} = O\left(\sum_{j=1}^{2}\left(|\bar{x}_j - z_j|^{\beta_j+\sigma} + \frac{1}{\bar{\lambda}_j^{\beta_j}}\right) + \varepsilon_{12}\right).$$

Since $\bar{x}_j \in B_\nu(z_j)$, it follows that for $\nu$ small enough

$$|\bar{x}_j - z_j| = O\Big(\Big(\sum_{i=1}^{2} \frac{1}{\bar{\lambda}_i^{\beta_i}} + \varepsilon_{12}\Big)^{1/\beta_j}\Big). \tag{2-20}$$

On the other hand since $\bar{\lambda}_j \in [\gamma_1 L_j, \gamma_2 L_j]$, one has

$$\frac{1}{\bar{\lambda}_j^{\beta_j}} = O\big(l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}\big), \quad j = 1, 2 \tag{2-21}$$

$$\varepsilon_{12} = O\big(l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}\big). \tag{2-22}$$

Then (2-20), (2-21) and (2-22) imply

$$|\bar{x}_j - z_j| = O\big(l^{-((n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2))(1/\beta_j)}\big) = O\Big(\frac{1}{\bar{\lambda}_j}\Big),$$

and the claim follows.

**Step 2.** We claim that $\bar{\lambda}_j \in (\gamma_1 L_j, \gamma_2 L_j)$.

Write $\bar{\lambda}_j = t_j L_j,\ j = 1, 2$. Since $\beta_j > n-4$, we see that there exists $(t_{01}, t_{02}) \in \mathbb{R}^2$ with $t_{0j} > 0$ large enough such that

$$\sum_{j=1}^{2} \frac{C'}{t_{0j}^{\beta_j}} - \frac{D}{t_{01}^{(n-4)/2} t_{02}^{(n-4)/2} K(z_1)^{(n-4)/8} K(z_2)^{(n-4)/8}} < -c_0 < 0. \tag{2-23}$$

Let $\lambda_{0j} = t_{0j} L_j,\ j = 1, 2$. Then (2-14) and (2-16) imply

$$\begin{aligned} J(\alpha(z, \lambda_0), z, \lambda_0, v(z, \lambda_0)) &\geq \sum_{j=1}^{2} \frac{2S_n^{n/4}}{nK(z_j)^{(n-4)/4}} - \sum_{j=1}^{2} \frac{C}{\lambda_{0j}^{\beta_j}} + O\Big(\sum_{j=1}^{2} \frac{1}{\lambda_{0j}^{\beta_j+\sigma}}\Big) \\ &\quad + \frac{D\varepsilon_{12}}{K(z_1)^{(n-4)/8} K(z_2)^{(n-4)/8}} + O\Big(\sum_{j=1}^{2} \frac{1}{\lambda_{0j}^{\inf(2\beta_j, n+4)}} + \varepsilon_{12}^{1+\tau}\Big). \end{aligned}$$

Then using (2-23), we obtain

$$\begin{aligned} J\big(\alpha(z, \lambda_0), z, \lambda_0, v(z, \lambda_0)\big) &\geq \sum_{j=1}^{2} \frac{2S_n^{n/4}}{nK(z_j)^{(n-4)/4}} + c_0' l^{(-(n-4)\beta_1\beta_2)/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)} \\ &\quad + O\Big(\sum_{j=1}^{2} \frac{1}{\lambda_j^{\beta_j+\sigma}} + \varepsilon_{12}^{1+\tau}\Big). \end{aligned} \tag{2-24}$$

On the other hand by (2-14), (2-15), (2-18) together with the fact $|\bar{x}_j - z_j| < C/\bar{\lambda}_j$, we get

$$J\big(\alpha(\bar{x},\bar{\lambda}),\bar{x},\bar{\lambda},v(\bar{x},\bar{\lambda})\big) \tag{2-25}$$
$$\begin{aligned}&\le \sum_{j=1}^{2}\frac{2S_n^{n/4}}{nK(z_j)^{(n-4)/4}} \\ &\qquad -\frac{n-4}{2n}\frac{1}{K(\bar{x}_j)^{n/4}}\frac{a_0}{\bar{\lambda}_j^{\beta}}\int_{\mathbb{R}^n}\big|y+\bar{\lambda}_j(\bar{x}_j-z_j)\big|^{\beta_j}\delta_{0,1}^{2n/(n-4)} \\ &\qquad +\frac{D\varepsilon_{12}}{K(\bar{x}_1)^{(n-4)/8}K(\bar{x}_2)^{(n-4)/8}}+O\bigg(\sum_{j=1}^{2}\frac{1}{\bar{\lambda}_j^{\beta_j+\sigma}}+\varepsilon_{12}^{1+\tau}\bigg) \\ &\le \frac{D\varepsilon_{12}}{K(\bar{x}_1)^{(n-4)/8}K(\bar{x}_2)^{(n-4)/8}}+\sum_{j=1}^{2}\frac{2S_n^{n/4}}{nK(z_j)^{(n-4)/4}}-\frac{C}{\bar{\lambda}_j^{\beta_j}} \\ &\qquad +O\bigg(\frac{1}{\bar{\lambda}_j^{\beta_j+\sigma}}+\varepsilon_{12}^{1+\tau}\bigg).\end{aligned}$$

Combining $J\big(\alpha(\bar{x},\bar{\lambda}),\bar{x},\bar{\lambda},v(\bar{x},\bar{\lambda})\big) \ge J\big(\alpha(z,\lambda_0),z,\lambda_0,v(z,\lambda_0)\big)$ with Equations (2-24) and (2-25) we obtain

$$\sum_{j=1}^{2}\frac{C}{\bar{\lambda}_j^{\beta_j}}+\frac{D\varepsilon_{12}}{K(\bar{x}_1)^{(n-4)/8}K(\bar{x}_2)^{(n-4)/8}}+O\bigg(\sum_{j=1}^{2}\frac{1}{\bar{\lambda}_j^{\beta_j+\sigma}}+\varepsilon_{12}^{1+\tau}\bigg) \ge c_0'' l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}. \tag{2-26}$$

If we take $\nu$ small enough such that $|\bar{x}_1-\bar{x}_2| > l/2$, we get

$$\begin{aligned}\frac{1}{\bar{\lambda}_j^{\beta_j+\sigma}} &\le C\nu^{\sigma} l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}, \quad j=1,2,\\ \varepsilon_{12}^{1+\tau} &\le C\nu^{\tau(n-4)} l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}.\end{aligned}$$

Then (2-26) implies

$$\sum_{j=1}^{2}\frac{C}{\bar{\lambda}_j^{\beta_j}}-\frac{D\varepsilon_{12}}{(K(\bar{x}_1)K(\bar{x}_2))^{(n-4)/8}} \le \big(c_1\nu^{\sigma}+c_2\nu^{\tau(n-4)}-c_0''\big) l^{(4-n)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}. \tag{2-27}$$

Since $c_1\nu^\sigma + c_2\nu^{\tau(n-4)}$ tends to zero as $\nu$ goes to zero, we can choose $\nu$ small enough such that $c_1\nu^\sigma + c_2\nu^{\tau(n-4)} < c_0''/2$ and (2-27) becomes

$$\sum_{j=1}^{2} \frac{C}{\bar\lambda_j^{\beta_j}} - \frac{D\varepsilon_{12}}{K(\bar x_1)^{(n-4)/8}K(\bar x_2)^{(n-4)/8}} \leq \frac{-c_0''}{2} l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}. \tag{2-28}$$

First, assume that $\bar\lambda_1 = \gamma_1 L_1$. Then

$$\varepsilon_{12} = \frac{1+o(1)}{(\bar\lambda_1\bar\lambda_2|x_1-x_2|^2)^{(n-4)/2}} = \frac{1+o(1)}{(\gamma_1 t_2)^{(n-4)/2}} l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}$$
$$\leq \frac{1+o(1)}{\gamma_1{}^{n-4}} l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}.$$

The last inequality follows from the fact that $\bar\lambda_2 = t_2 L_2 \in [\gamma_1 L_2, \gamma_2 L_2]$. Then

$$\sum_{j=1}^{2} \frac{C}{\bar\lambda_j^{\beta_j}} - \frac{D\varepsilon_{12}}{K(\bar x_1)^{(n-4)/8}K(\bar x_2)^{(n-4)/8}} \geq \left(\frac{C}{\gamma_1^{\beta_1}} - \frac{C'}{\gamma_1^{n-4}}\right) l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}. \tag{2-29}$$

Since $\beta_1 > n-4$ we see that, $C/\gamma_1^{\beta_1} - C'/\gamma_1^{n-4}$ tends to infinity as $\gamma_1$ tends to zero. So we can choose $\gamma_1$ small enough such that $C/\gamma_1^{\beta_1} - C'/\gamma_1^{n-4} \geq k_0 > 0$. Hence (2-29) implies

$$\sum_{j=1}^{2} \frac{C}{\bar\lambda_j^{\beta_j}} - \frac{D\varepsilon_{12}}{K(\bar x_1)^{(n-4)/8}K(\bar x_2)^{(n-4)/8}} \geq k_0 l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}. \tag{2-30}$$

Combining (2-28) and (2-30), we obtain a contradiction.

Now, assume that $\bar\lambda_1 = \gamma_2 L_1$. Then

$$\varepsilon_{12} = \frac{1+o(1)}{(\bar\lambda_1\bar\lambda_2|x_1-x_2|^2)^{(n-4)/2}} = \frac{1+o(1)}{(\gamma_2 t_2)^{(n-4)/2}} l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}$$
$$\leq \frac{1+o(1)}{(\gamma_1\gamma_2)^{(n-4)/2}} l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)},$$

since $\lambda_2 = t_2 L_2 \in [\gamma_1 L_2, \gamma_2 L_2]$. It follows that

$$\sum_{j=1}^{2} \frac{C}{\bar{\lambda}_j^{\beta_j}} - (D\varepsilon_{12})/K(x_1)^{(n-4)/8} K(x_2)^{(n-4)/8} \geq -C'' \frac{1+o(1)}{(\gamma_1\gamma_2)^{(n-4)/2}} l^{-(n-4)\beta_1\beta_2/(\beta_1\beta_2-(\beta_1+\beta_2)(n-4)/2)}. \tag{2-31}$$

Combining (2-28) and (2-31) we get

$$-C'' \frac{1+o(1)}{(\gamma_1\gamma_2)^{(n-4)/2}} \leq \frac{-C_0''}{2}.$$

Now since $(1+o(1))/(\gamma_1\gamma_2)^{(n-4)/2}$ tends to zero as $\gamma_2$ tends to infinity, we derive a contradiction. The same argument can be applied to $\bar{\lambda}_2$ and the claim follows.

Since $(\bar{x}, \bar{\lambda})$ is an interior point of $S_{\nu,2}$ maximizing $J(\alpha(x,\lambda), x, \lambda, v(x,\lambda))$ on $S_{\nu,2}$, it follows that

$$u := \alpha_1(\bar{x},\bar{\lambda})\delta_{\bar{x}_1,\bar{\lambda}_1} - \alpha_2(\bar{x},\bar{\lambda})\delta_{\bar{x}_2,\bar{\lambda}_2} + v(\bar{x},\bar{\lambda}),$$

is a critical point of $J$. Hence our theorem follows. □

*Proof of Theorem 1.2.* In this proof, we will assume that near $z_1$ and $z_2$, $K$ satisfies (1-8) and (1-7). Let

$$J : M_{\nu,2} \to \mathbb{R}, \quad (\alpha, x, \lambda, v) \mapsto I\left(\alpha_1\delta_{x_1,\lambda_1} + \alpha_2\delta_{x_2,\lambda_2} + v\right).$$

As in Proposition 2.1 we get a $C^1$ map $(\alpha(x,\lambda), v(x,\lambda))$ such that

$$\frac{\partial J}{\partial \alpha_j} = 0, \ j = 1,2 \quad \text{and} \quad \frac{\partial J}{\partial v} = \sum_{j=1}^{2}\left(A_j\delta_{x_j,\lambda_j} + B_j \frac{\partial \delta_{x_j,\lambda_j}}{\partial \lambda_j} + \sum_{h=1}^{n} C_{jh} \frac{\partial \delta_{x_j,\lambda_j}}{\partial x_{jh}}\right),$$

for certain $A_j$, $B_j$ and $C_{ij} \in \mathbb{R}^n$, $i = 1,\dots,n$, $j = 1,2$. Moreover the estimate (2-9) holds. Then replacing the problem (2-12) by

$$\inf\{J\big(\alpha(x,\lambda), x, \lambda, v(x,\lambda)\big), \ (x,\lambda) \in S_{\nu,2}\}, \tag{2-32}$$

where $S_{\nu,2}$ is defined in (2-13), and following the proof of Theorem 1.1, our result follows. Note that there are some changes in the proof taking account of the sign behind the function $Q_j$ and the new problem (2-32) instead of (2-12). □

*Proof of Theorem 1.4.* We begin by proving Claim (1). Let $\tau > 0$ be small enough so that $B_{2\tau}(z_1) \cap B_{2\tau}(z_2) = \phi$. For a fixed $\beta \in (n-4, n)$, we define

$$K_\varepsilon(y) = \begin{cases} K(y), & \text{if } y \in \mathbb{R}^n \setminus \bigcup_{j=1}^{2} B_{2\eta\tau}(z_j), \\ K(z_j) + (1/\eta^\beta)|y - z_j|^\beta, & \text{if } y \in B_{\eta\tau}(z_j), \ j = 1,2, \end{cases}$$

where $\eta > 0$ is a small constant. Since $\tau$ is small and $K$ is continuous, for each $y \in B_{\eta\tau}(z_j)$, we have

$$|K_\varepsilon(y) - K(y)| = |K(z_j) + \frac{1}{\eta^\beta}|y - z_j|^\beta - K(y)| \\ \leq |K(y) - K(z_j)| + \tau^\beta \leq \tau + \tau^\beta \leq C\tau < \varepsilon. \tag{2-33}$$

In $\bigcup_{j=1}^2 \big(B_{2\eta\tau}(z_j) \diagdown B_{\eta\tau}(z_j)\big)$, $K_\varepsilon$ can be continuously extended such that (2-33) is satisfied. Then consider the problem

$$\begin{cases} \Delta^2 u = K_\varepsilon(y)|u|^{8/(n-4)}u, & y \in \mathbb{R}^n, \\ u \to 0, \ \Delta u \to 0, & \text{as } |y| \to +\infty. \end{cases} \tag{2-34}$$

Let $w(y) = \eta^{(n-4)/2} u(\eta y)$. Then $w$ satisfies

$$\begin{cases} \Delta^2 w = K_\varepsilon^*(y)|w|^{8/(n-4)}w, & y \in \mathbb{R}^n, \\ w \to 0, \ \Delta w \to 0, & \text{as } |y| \to +\infty, \end{cases} \tag{2-35}$$

where $K_\varepsilon^*(y) = K_\varepsilon(\eta y)$. Let $z_j^* = z_j/\eta$, $j = 1, 2$. For any $y \in B_\tau(z_j^*)$, we have

$$\begin{aligned} K_\varepsilon^*(y) = K_\varepsilon(\eta y) &= K(z_j) + \frac{1}{\eta^\beta}|\eta y - z_j|^\beta \\ &= K_\varepsilon(z_j) + \frac{1}{\eta^\beta}|\eta y - z_j|^\beta \qquad (\text{since } K_\varepsilon(z_j) = K(z_j)) \\ &= K_\varepsilon(\eta z_j^*) + \frac{1}{\eta^\beta}|\eta y - z_j|^\beta \\ &= K_\varepsilon^*(z_j^*) + \frac{1}{\eta^\beta}|\eta y - z_j|^\beta. \end{aligned}$$

Thus $K_\varepsilon^*(y) > K_\varepsilon(z_j^*)$, for all $y \in B_\tau(z_j^*) \setminus \{z_j^*\}$. Hence $z_1^*$ and $z_2^*$ are two strictly local minimum points of $K_\varepsilon^*(y)$ with $|z_1^* - z_2^*| = |z_1 - z_2|/\eta$. Moreover

$$K_\varepsilon^*(y) = K_\varepsilon^*(z_j^*) + |y - z_j^*|^\beta \quad \text{for all } y \in B_\tau(z_j^*).$$

Then arguing as in Theorem 1.1 we see that for any $\nu > 0$, we can choose $\eta > 0$ small enough so that (2-35) has a solution of the form

$$w = \alpha_1 \delta_{x_1^*, \lambda_1^*} - \alpha_2 \delta_{x_2^*, \lambda_2^*} + v^*$$

where $v^* \in E_{x*, \lambda^*, 2}$, $\|v^*\| < \nu$ and for $j = 1, 2$, $|\alpha_j - 1/K_\varepsilon^*(z_j^*)^{(n-4)/8}| < \nu$, $|x_j^* - z_j^*| < \nu$, $1/\lambda_j^* > \nu$. We deduce that (2-34) has a solution of the form $u = \alpha_1 \delta_{x_1, \lambda_1} - \alpha_2 \delta_{x_2, \lambda_2} + v$ where $v(y) = \eta^{-(n-4)/2} v^*(y/\eta)$, $x_j = \eta x_j^*$ and $\lambda_j = \lambda_j^*/\eta$, and it is easy to check that $u$ satisfies the desired properties.

To prove Claim (2), we take

$$K_\varepsilon(y) = \begin{cases} K(y), & \text{if } y \in \mathbb{R}^n \setminus \bigcup_{j=1}^2 B_{2\eta\tau}(z_j), \\ K(z_j) - (1/\eta^\beta)|y - z_j|^\beta, & \text{if } y \in B_{\eta\tau}(z_j),\ j = 1, 2, \end{cases}$$

where $\tau$ and $\beta$ are defined as in the proof of Claim (1). Finally, following the previous proof, Claim (2) follows. □

## 3. Proofs of Theorems 1.6, 1.7 and 1.8

*Proof of Theorem 1.6.* Let $z_1, \dots, z_k$ be $k$ different strictly local maximum points of $K$ such that

$$l := \min_{i \neq j} |z_i - z_j| \text{ is large} \quad \text{and} \quad \max_{i \neq j} |z_i - z_j|/l \text{ is bounded.} \tag{3-1}$$

Note that this choice is possible using the assumption of the theorem.

As in the previous section, we introduce the sets

$$D_{\nu,k} = \{\, (x, \lambda),\ x_j \in \overline{B_\nu(z_j)},\ \lambda_j \geq 1/\nu,\ j = 1, \dots, k\},$$

$$M_{\nu,k} = \left\{ (\alpha, x, \lambda, v) : (x, \lambda) \in D_{\nu,k},\ v \in E_{x,\lambda,k},\ \sum_{j=1}^{k} \left|\alpha_j - K(z_j)^{-1/8}\right| + \|v\| \leq \nu \right\},$$

and our functional will be

$$J : M_{\nu,k} \to \mathbb{R}, \quad (\alpha, x, \lambda, v) \mapsto I\left( \sum_{i=1}^{k} \alpha_i \delta_{x_i,\lambda_i} + v \right).$$

As before, we start by giving the estimate of the $v$-part and the $\alpha$-variables. Using Lemmas A.5 and A.6, we obtain similarly to Proposition 2.1 the following result:

**Proposition 3.1.** *Assume that $K$ is a $C^2$ function. Then there exists $\nu_0 > 0$, such that for each $\nu \in (0, \nu_0]$ and $(x, \lambda) \in D_{\nu,k}$, there exists a unique $(\alpha(x, \lambda), v(x, \lambda)) \in \mathbb{R}^k \times E_{x,\lambda,k}$ such that* (2-5) *and* (2-8) *are satisfied.* (*We remark that the sum in* (2-8) *will be from* 1 *to $k$*). *We note that the function $(x, \lambda) \mapsto \big(\alpha(x, \lambda), v(x, \lambda)\big)$ is a $C^1$ map. Moreover we have*

$$\sum_{j=1}^{k} \left| \alpha_j - \frac{1}{K(x_j)^{1/8}} \right| + \|v\| = O\left( \sum_{j=1}^{k} \left( \frac{|\nabla K(x_j)|}{\lambda_j} + \frac{1}{\lambda_j{}^2} \right) + \sum_{i \neq j} \varepsilon_{ij}^{1/2+\tau} \right),$$

*where $\tau$ is a positive constant.*

We then consider the problem

$$\inf\{J\big(\alpha(x, \lambda), x, \lambda, v(x, \lambda)\big),\ (x, \lambda) \in S_{\nu,k}\}, \tag{3-2}$$

where $S_{\nu,k} := \{(x,\lambda) \in D_{\nu,k},\ \lambda_j \in [\gamma_1 l, \gamma_2 l]\}$ and $\gamma_2 > \gamma_1 > 0$ are two constants to be determined later. Since $S_{\nu,k}$ is a compact set it follows that the problem (3-2) has a minimizer $(\bar{x},\bar{\lambda}) \in S_{\nu,k}$. We will prove that $(\bar{x},\bar{\lambda})$ is an interior point of $S_{\nu,k}$ and thus a critical point of $J\big(\alpha(x,\lambda), x, \lambda, v(x,\lambda)\big)$. For this we proceed in two steps.

**Step 1.** We claim that $\bar{x}_j \in B_\nu(z_j)$ if $l := \min_{i\neq j} |z_i - z_j|$ is large enough. By Proposition 3.1 and Lemma A.7 we have

$$
\begin{aligned}
(3\text{-}3)\quad & J\big(\alpha(x,\lambda), x, \lambda, v(x,\lambda)\big) \\
&= I(H_{x,\lambda,k}) + O\bigg(\sum_{j=1}^{k}\bigg(\frac{|\nabla K(x_j)|^2}{{\lambda_j}^2} + \frac{1}{{\lambda_j}^4}\bigg) + \sum_{i\neq j}\varepsilon_{ij}^{1+\tau}\bigg) \\
&= \frac{2}{5}\sum_{j=1}^{k}\frac{S_5^{5/4}}{K(x_j)^{1/4}} - \frac{1}{10}\sum_{j=1}^{k}\frac{B\Delta K(x_j)}{\lambda_j^2 K(x_j)^{5/4}} - \sum_{i\neq j}\frac{D\varepsilon_{ij}}{K(x_i)^{1/8}K(x_j)^{1/8}} \\
&\qquad + O\bigg(\sum_{j=1}^{k}\bigg(\frac{|\nabla K(x_j)|^2}{{\lambda_j}^2}\bigg) + \sum_{i\neq j}\varepsilon_{ij}^{1+\tau}\bigg) + o\bigg(\sum_{j=1}^{k}\frac{1}{{\lambda_j}^2}\bigg).
\end{aligned}
$$

Using the fact that $J(\alpha(\bar{x},\bar{\lambda}),\bar{x},\bar{\lambda},v(\bar{x},\bar{\lambda})) \le J(\alpha(z,\bar{\lambda}),z,\bar{\lambda},v(z,\bar{\lambda}))$ together with (3-3), we derive

$$
0 \le \sum_{j=1}^{k}\bigg(\frac{1}{K(\bar{x}_j)^{1/4}} - \frac{1}{K(z_j)^{1/4}}\bigg) \le C\bigg(\sum_{j=1}^{k}\frac{1}{{\bar{\lambda}_j}^2} + \sum_{i\neq j}\varepsilon_{ij}\bigg).
$$

It follows that

$$
(3\text{-}4)\qquad 0 \le K(z_j) - K(\bar{x}_j) \to 0 \quad \text{as } l \to +\infty.
$$

Assume $\bar{x}_j \in \partial B_\nu(z_j)$. By hypothesis $(H_2')$ we have $K(z_j) - K(\bar{x}_j) \ge \eta(\nu) > 0$, which contradicts (3-4). Hence $\bar{x}_j \in B_\nu(z_j)$ if $l$ is large enough.

**Step 2.** We claim that $\bar{\lambda}_j \in (\gamma_1 l, \gamma_2 l)$ if $\gamma_1$ is small enough and $\gamma_2$ is large enough. Consider the function

$$
f(t) = -\frac{B}{10}\sum_{j=1}^{k}\frac{\Delta K(x_j)}{K(x_j)^{5/4}}{t_j}^8 - D\sum_{i\neq j}\frac{a_{ij}t_i^2t_j^2}{K(x_i)^{1/8}K(x_j)^{1/8}},
$$

where $a_{ij} = l/|x_i - x_j|$. Since each $x_i$ is close to $z_i$, from (3-1), we get that each $a_{ij}$ is bounded below and above and $\Delta K(x_i) < -c < 0$ for each $i$. Hence, it is easy to check that $f(t)$ has a global minimizer $t^* = ({t_1}^*, \dots, {t_k}^*) \in \mathbb{R}^k$. Moreover there are constants $b_2 > b_1 > 0$ such that $b_1 \le |{t_j}^*| \le b_2$ for any global minimizer $t^*$ of $f(t)$ and $j = 1, \dots, k$. Indeed since $f(t) \to +\infty$ as $|t| \to +\infty$ we deduce $|t^*| \le b_2$ for some constant $b_2$. On the other hand, we have $\min_{\mathbb{R}^k} f(t) < -c' < 0$.

Indeed, we have $l = \min_{i \neq j} |z_i - z_j|$, without loss of generality, we may assume that $l = |z_1 - z_2|$, which implies that $a_{12}$ satisfies $1/2 \leq a_{12} < 2$, then

$$\begin{aligned} f(\theta, \theta, 0, \dots, 0) &= -\frac{B}{10}\left(\frac{\Delta K(x_1)}{K(x_1)^{5/4}} + \frac{\Delta K(x_2)}{K(x_2)^{5/4}}\right)\theta^8 - \frac{2Da_{12}}{K(x_1)^{1/8}K(x_2)^{1/8}}\theta^4 \\ &= (B'\theta^4 - D')\theta^4, \end{aligned}$$

where $B'$ and $D'$ are some positive constants independent of $l$. Since $B'\theta^4 - D'$ tends to $-D'$ as $\theta$ tends to 0, we see that there exists $\eta > 0$ such that if $|\theta| < \eta$ we have $(B'\theta^4 - D')\theta^4 \leq -(D'/2)\eta^4$. It follows $\min_{\mathbb{R}^k} f(t) \leq f(\theta, \theta, 0, \dots, 0) \leq -(D'/2)\eta^4$. We deduce since $f(t) \to 0$ as $|t| \to 0$ that $|t^*| \geq b' > 0$. Now since $\frac{\partial f}{\partial t_j}(t_1{}^*, \dots, t_k{}^*) = 0$, we get

$$-\frac{B}{5}\frac{\Delta K(x_j)}{K(x_j)^{9/8}}t_j{}^{*6} = D\sum_{i=1,\ i\neq j}^{k}\frac{a_{ij}t_i^{*2}}{K(x_i)^{1/8}}. \tag{3-5}$$

Therefore, if $|t_j{}^*|$ is small for some $j$, then from (3-5) and the fact that $a_{ij} \geq c > 0$ for each $i \neq j$, $|t_i{}^*|$ is also small for $i = 1, \dots, k$. We obtain a contradiction.

So the function

$$-\frac{B}{10}\sum_{j=1}^{k}\frac{\Delta K(x_j)}{K(x_j)^{5/4}}\frac{1}{\theta_j{}^2} - D\sum_{i\neq j}\frac{a_{ij}}{K(x_i)^{1/8}K(x_j)^{1/8}\theta_i{}^{1/2}\theta_j{}^{1/2}}, \tag{3-6}$$

has a global minimizer $\theta^* = (\theta_1{}^*, \dots, \theta_k{}^*)$ and there are constants $b_2' > b_1' > 0$ such that $b_2' > \theta_j{}^* > b_1' > 0,\ j = 1, \dots, k$ for any global minimizer $\theta^* = (\theta_1{}^*, \dots, \theta_k{}^*)$. Denote $\bar\lambda_j = \theta_j l,\ \lambda^* = \theta_j{}^* l$. Using the fact that $J(\alpha(\bar x, \bar\lambda), \bar x, \bar\lambda, v(\bar x, \bar\lambda)) \leq J(\alpha(\bar x, \bar\lambda^*), \bar x, \bar\lambda^*, v(\bar x, \bar\lambda^*))$ together with (3-3), we derive

$$\begin{aligned} -\frac{B}{10}\sum_{j=1}^{k}&\frac{\Delta K(\bar x_j)}{K(x_j)^{5/4}\theta_j{}^2} - D\sum_{i\neq j}\frac{a_{ij}}{K(\bar x_i)^{1/8}K(\bar x_j)^{1/8}\theta_i^{1/2}\theta_j^{1/2}} \\ &\leq -\frac{B}{10}\sum_{j=1}^{k}\frac{\Delta K(\bar x_j)}{K(x_j)^{5/4}\theta_j^{*2}} - D\sum_{i\neq j}\frac{a_{ij}}{K(\bar x_i)^{1/8}K(\bar x_j)^{1/8}\theta_i^{*1/2}\theta_j^{*1/2}} \\ &\qquad + O\left(\sum_{j=1}^{k}\frac{|\nabla K(\bar x_j)|^2}{\theta_j^{*2}}\right) + o\left(\sum_{j=1}^{k}\frac{1}{\theta_j^{*2}}\right). \end{aligned}$$

Hence, $\theta_j$ will tend to one of the global minimum points of the function defined by (3-6). As a result, if $\gamma_1 > 0$ is small enough and $\gamma_2 > 0$ is large enough, $\bar\lambda_j = \theta_j l \in (\gamma_1 l, \gamma_2 l)$.

From Steps 1 and 2, $(\bar x, \bar\lambda)$ is an interior point of $S_{\nu,k}$ and thus it is a critical point of the function $J\big(\alpha(x, \lambda), x, \lambda, v(x, \lambda)\big)$. □

*Proof of Theorem 1.7.* As in the proof of Theorem 1.6, let $z_1, \dots, z_k$ be $k$ different strictly local maximum points of $K$ satisfying (3-1). Define

$$L = l^{(n-4)/(\beta-n+4)},$$

where $\beta$ is defined in $(H_2'')$. As in Proposition 2.1, we get a map $(\alpha(x,\lambda), v(x,\lambda))$ which is $C^1$ such that

$$\frac{\partial J}{\partial \alpha_j} = 0,\ j = 1, \dots, k \ \text{ and } \ \frac{\partial J}{\partial v} = \sum_{j=1}^{k} \left( A_j \delta_{x_j,\lambda_j} + B_j \frac{\partial \delta_{x_j,\lambda_j}}{\partial \lambda_j} + \sum_{h=1}^{n} C_{jh} \frac{\partial \delta_{x_j,\lambda_j}}{\partial x_{jh}} \right).$$

Moreover we have the estimate

$$\sum_{j=1}^{k} \left| \alpha_j - \frac{1}{K(x_j)^{(n-4)/8}} \right| + \|v\| = O\left( \sum_{j=1}^{k} \left( |x_j - z_j|^{\beta_j} + \frac{1}{\lambda_j^{\inf(\beta_j,(n+4)/2)}} \right) + \sum_{i \neq j} \varepsilon_{ij}^{1/2+\tau} \right),$$

where $\tau > 0$ is a constant. We consider the problem

$$\inf\{ J\big(\alpha(x,\lambda), x, \lambda, v(x,\lambda)\big),\ (x,\lambda) \in S_{\nu,k} \}, \tag{3-7}$$

where

$$S_{\nu,k} = \{(x,\lambda) \in D_{\nu,k},\ \lambda_j \in [\gamma_1 L, \gamma_2 L],\ j = 1, \dots k\},$$

$\gamma_1 > 0$ is a small constant and $\gamma_2 > 0$ is a large constant. Then arguing as in the proof of Theorem 1.2, Theorem 1.7 follows. □

*Proof of Theorem 1.8.* Let $\eta > 0$. As in Theorem 1.4 we define $K_\varepsilon(y) = K(z_j) - (1/\eta^2)|y - z_j|^2$, for $y \in B_{\tau\eta}(z_j)$, $j = 1, \dots, k$ with a suitable extension of $K_\varepsilon(y)$ into $\mathbb{R}^5 \backslash \bigcup_{j=1}^{k} B_{\tau\eta}(z_j)$. Then using Theorem 1.6 and arguing as in the proof of Theorem 1.4, we see that the perturbed problem $(P_{K_\varepsilon})$ has a solution concentrating at the $k$ given points $z_j$ if $\eta > 0$ is small enough. □

## 4. Proofs of Theorem 1.9 and Proposition 1.10

*Proof of Theorem 1.9.* Let $u$ be a solution of $(P_K)$ of the form $u = \sum_{i \le k} \alpha_i \delta_{x_i,\lambda_i} + v$. We will argue as in [Ben Ayed et al. 2005]. Let $u = u^+ - u^-$, $u^+ = \max(0,u)$, $u^- = \max(0,-u)$. Then we have $|u^-| < |v|$ and $(u^-)^{(n+4)/(n-4)} \in L^{2n/(n+4)}$. Let us introduce $w$ satisfying

$$\Delta^2 w = -K(u^-)^{(n+4)/(n-4)},\ w, \Delta w \to 0 \quad \text{as } |y| \to +\infty. \tag{4-1}$$

Using a regularity argument, we derive that $w \in D^{1,2}(\mathbb{R}^n)$. Furthermore, since $K \ge 0$ by the maximum principle, $w \le 0$. Multiplying (4-1) by $w$ and integrating on $\mathbb{R}^n$, we obtain

$$\|w\|^2 = \int_{\mathbb{R}^n} \Delta^2 w \cdot w = -\int_{\mathbb{R}^n} K(u^-)^{(n+4)/(n-4)} w \le c_1 \|w\| |u^-|_{L^{2n/(n-4)}}^{(n+4)/(n-4)}$$

so that, we have either $\|w\| = 0$ and it follows $u^- = 0$ or $\|w\| \neq 0$ and therefore

$$\|w\| \leq c_1 |u^-|^{(n+4)/(n-4)}_{L^{2n/(n-4)}}. \tag{4-2}$$

Now, in view of the fact that $u$ is a solution of $(P_K)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta^2 w u = \int_{\mathbb{R}^n} w \Delta^2 u &= \int_{\mathbb{R}^n} K|u|^{8/(n-4)} u w \\ &= -\int_{u \leq 0} K(u^-)^{(n+4)/(n-4)} w + \int_{u \geq 0} K(u^+)^{(n+4)/(n-4)} w \\ &\leq -\int_{u \leq 0} K(u^-)^{(n+4)/(n-4)} w, \quad \text{(since } w \leq 0,\ K \geq 0) \\ &\leq -\int_{\mathbb{R}^n} K(u^-)^{(n+4)/(n-4)} w = \int_{\mathbb{R}^n} \Delta^2 w \cdot w = \|w\|^2.\end{aligned} \tag{4-3}$$

On another hand, using the fact that $K \geq \eta_0 > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta^2 w u &= -\int_{\mathbb{R}^n} K(u^-)^{(n+4)/(n-4)} u \\ &= \int_{\mathbb{R}^n} K(u^-)^{2n/(n-4)} \geq c_2 |u^-|^{2n/(n-4)}_{L^{2n/(n-4)}}.\end{aligned} \tag{4-4}$$

From (4-2), (4-3) and (4-4), we deduce

$$c_2 |u^-|^{2n/(n-4)}_{L^{2n/(n-4)}} \leq \|w\|^2 \leq \eta_0 |u^-|^{2n/(n-4)}_{L^{2n/(n-4)}}.$$

Now since, $|u^-|_{L^{2n/(n-4)}}$ is small enough, we derive a contradiction, and the case $\|w\| \neq 0$ cannot occur. Therefore $u^- = 0$ on $\mathbb{R}^n$, and the strong maximum principle implies that $u > 0$. □

*Proof of Proposition 1.10.* We proceed by contradiction. Assume that there exists a sequence of solutions $u_m$ of $(P_K)$ such that $|u_m|_{L^\infty} \to +\infty$ and $I(u_m) \to c$ as $m \to +\infty$. Denote $|u_m|_{L^\infty} = \mu_m^{(n-4)/2}$ and let $x'_m \in \mathbb{R}^n$ be a maximum point of $u_m$. Since $K(y)$ is periodic in all variables, by translation we may assume that $x'_m$ is bounded and thus we may assume that $x'_m \to x_0$ as $m \to +\infty$. Set

$$w_m(y) = \frac{1}{\mu_m^{(n-4)/2}} u_m\left(\frac{y}{\mu_m} + x'_m\right).$$

Then $w_m$ satisfies

$$\begin{cases} \Delta^2 w_m = K(y/\mu_m + x'_m)|w_m|^{8/(n-4)} w_m, & y \in \mathbb{R}^n, \\ u_m \to 0,\ \Delta u_m \to 0, & \text{as } |y| \to +\infty, \\ w_m(0) = 1. \end{cases} \tag{4-5}$$

By the $L^p$ estimate, we see that $w_m$ converges weakly in $E$ and converges in $C^4_{\mathrm{loc}}(\mathbb{R}^n)$ to a function $w_0 \in E$ satisfying

$$\begin{cases} \Delta^2 w_0 = K(x_0)|w_0|^{8/(n-4)} w_0, & \text{in } \mathbb{R}^n, \\ w_0 \to 0,\ \ \Delta w_0 \to 0, & \text{as } |y| \to +\infty. \end{cases} \tag{4-6}$$

Let $t_0 = K(x_0)^{8/(n-4)} w_0$. Then $t_0$ satisfies

$$\begin{cases} \Delta^2 t_0 = |t_0|^{8/(n-4)} t_0, & \text{in } \mathbb{R}^n \\ t_0 \to 0,\ \ \Delta t_0 \to 0, & \text{as } |y| \to +\infty. \end{cases} \tag{4-7}$$

We have

$$\|t_0\|^2 = K(x_0)^{(n-4)/n} \|w_0\|^2 \geq S_n^{n/4}. \tag{4-8}$$

Observe that

$$\begin{aligned} \frac{2}{n}\int_{\mathbb{R}^n} |\Delta w_m|^2 &= \frac{1}{2}\int_{\mathbb{R}^n} |\Delta w_m|^2 - \frac{n-4}{2n}\int_{\mathbb{R}^n} K\left(\frac{y}{\mu_m} + x'_m\right)|w_m|^{2n/(n-4)} \\ &= \frac{1}{2}\int_{\mathbb{R}^n} |\Delta u_m|^2 - \frac{n-4}{2n}\int_{\mathbb{R}^n} K(z)|u_m|^{2n/(n-4)} \\ &= I(u_m) \to c := \frac{2}{n}\frac{S_n^{n/4}}{K_M^{(n-4)/4}} \quad \text{as } m \to +\infty. \end{aligned} \tag{4-9}$$

Since $\liminf \|w_m\| \geq \|w_0\|$, it follows that

$$c = \frac{2}{n}\frac{S_n^{n/4}}{K_M^{(n-4)/4}} = \frac{2}{n}\int_{\mathbb{R}^n} |\Delta w_m|^2 + o(1) \geq \frac{2}{n}\int_{\mathbb{R}^n} |\Delta w_0|^2 + o(1), \tag{4-10}$$

which implies

$$\|t_0\|^2 \leq \left(\frac{K(x_0)}{K_M}\right)^{(n-4)/4} S_n^{n/4}. \tag{4-11}$$

Hence from (4-8) and (4-11), we get $\|t_0\|^2 = S_n^{n/4}$ and $K(x_0) = K_M$, that is, $x_0$ is a global maximum point of $K$. Therefore $S_n$ is achieved with $t_0$, which implies the existence of $a_0$, $\lambda_0$ such that $t_0 = \delta_{a_0,\lambda_0}$. From (4-8), (4-9) and (4-10), we have

$$\frac{2}{n}\|w_m\|^2 + o(1) \geq \frac{2}{n}\|w_0\|^2 \geq \frac{2}{n}\frac{S_n^{n/4}}{K(x_0)^{(n-4)/4}} = c = \frac{2}{n}\|w_m\|^2 + o(1).$$

It follows that $\|w_m\| \to \|w_0\|$ as $m \to +\infty$ and then $w_m$ converges strongly to $w_0$. Hence

$$\|u_m - K_M^{(4-n)/8}\delta_{y_m,\xi_m}\| \to 0 \quad \text{as } m \to +\infty,$$

with $y_m \to x_0$, $\xi_m \to +\infty$. Then, following the same idea as in [Bahri 1989; Bahri and Coron 1988; Rey 1990], we can write

$$u_m = \alpha_m \delta_{x_m,\lambda_m} + v_m, \tag{4-12}$$

where $v_m \in E_{x_m,\lambda_m}$, $\|v_m\| \to 0$, $\alpha_m \to K_M^{(4-n)/8}$, $x_m \to x_0$ and $\lambda_m \to +\infty$, as $m \to +\infty$. Next, we will give an estimate of $v_m$ defined in (4-12). We have by multiplying $\Delta^2 u_m = K|u_m|^{8/(n-4)} u_m$ by $v_m$ and integrating

$$\begin{aligned}\|v_m\|^2 &= \int_{\mathbb{R}^n} K(y)|u_m|^{8/(n-4)} u_m v_m \\ &= \frac{n+4}{n-4}\int_{\mathbb{R}^n} K(y)(\alpha_m \delta_{x_m,\lambda_m})^{8/(n-4)} v_m^2 \\ &\qquad\qquad + O\biggl(\int_{\mathbb{R}^n} K(y)\delta_{x_m,\lambda_m}^{(n+4)/(n-4)} v_m + \|v_m\|^{2+\tau}\biggr)\end{aligned}$$

where $\tau > 0$ is a constant. It follows since $\alpha_m = 1/K_M^{(n-4)/8} + o(1)$

$$\begin{aligned}&(1+o(1))\|v_m\|^2 \\ &\qquad = \frac{n+4}{n-4}\int_{\mathbb{R}^n} \frac{K(y)}{K_M}\delta_{x_m,\lambda_m}^{8/(n-4)} v_m^2 + O\biggl(\int_{\mathbb{R}^n} K(y)\delta_{x_m,\lambda_m}^{(n+4)/(n-4)} v_m\biggr).\end{aligned} \tag{4-13}$$

Since $v_m \in E_{x_m,\lambda_m}$, a computation using Holder's inequality and Sobolev embedding theorem shows that

$$\int_{\mathbb{R}^n} K(y)\delta_{x_m,\lambda_m}^{(n+4)/(n-4)} v_m = O\biggl(\frac{|\nabla K(x_m)|}{\lambda_m} + \frac{1}{\lambda_m^2}\biggr)\|v_m\|.$$

Then (4-13) implies

$$(1+o(1))\|v_m\|^2 \le \frac{n+4}{n-4}\int_{\mathbb{R}^n} \delta_{x_m,\lambda_m}^{8/(n-4)} v_m^2 + O\biggl(\frac{|\nabla K(x_m)|}{\lambda_m} + \frac{1}{\lambda_m^2}\biggr)\|v_m\|.$$

Since the quadratic form defined by (2-10) is positive definite, we derive the estimate

$$\|v_m\| = O\biggl(\frac{|\nabla K(x_m)|}{\lambda_m} + \frac{1}{\lambda_m^2}\biggr). \tag{4-14}$$

Multiplying equation $\Delta^2 u_m = K(y)|u_m|^{8/(n-4)} u_m$ by $\partial\delta_{x_m,\lambda_m}/\partial\lambda_m$ and integrating, we obtain

$$\int_{\mathbb{R}^n} \Delta u_m \Delta \frac{\partial \delta_{x_m,\lambda_m}}{\partial\lambda_m} = \int_{\mathbb{R}^n} K(y)|u_m|^{8/(n-4)} u_m \frac{\partial \delta_{x_m,\lambda_m}}{\partial\lambda_m}. \tag{4-15}$$

Since $v_m \in E_{x_m,\lambda_m}$, we have

$$\int_{\mathbb{R}^n} \Delta u_m \Delta \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} = 0. \tag{4-16}$$

On the other hand

$$\begin{aligned}\int_{\mathbb{R}^n} K(y)|u_m|^{8/(n-4)} u_m \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} &= \int_{\mathbb{R}^n} K(y)(\alpha_m \delta_{x_m,\lambda_m})^{(n+4)/(n-4)} \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} \\ &\quad + \frac{n+4}{n-4} \int_{\mathbb{R}^n} K(y)(\alpha_m \delta_{x_m,\lambda_m})^{8/(n-4)} v_m \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} \\ &\quad + O\biggl( \int_{|\alpha_m \delta_{x_m,\lambda_m}| \le |v_m|} |v_m|^{(n+4)/(n-4)} \left| \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} \right| \\ &\qquad + \int_{|v_m| \le |\alpha_m \delta_{x_m,\lambda_m}|} \delta_{x_m,\lambda_m}^{(12-n)/(n-4)} |v_m|^2 \left| \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} \right| \biggr).\end{aligned} \tag{4-17}$$

Using the fact that $\alpha_m = 1/K_M^{(n-4)/8} + o(1)$ together with Lemma B.3, we derive

$$\int_{\mathbb{R}^n} K(y)(\alpha_m \delta_{x_m,\lambda_m})^{(n+4)/(n-4)} \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} = \frac{-B \Delta K(x_m)}{\lambda_m^3} + o\left(\frac{1}{\lambda_m^3}\right). \tag{4-18}$$

Next, a computation using Holder's inequality, Sobolev embedding theorem shows that

$$\int_{\mathbb{R}^n} K(y)(\alpha_m \delta_{x_m,\lambda_m})^{8/(n-4)} v_m \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} = O\left( \frac{|\nabla K(x_m)|}{\lambda_m^2} + \frac{1}{\lambda_m^3} \right) \|v_m\|, \tag{4-19}$$

$$\int_{|\alpha_m \delta_{x_m,\lambda_m}| \le |v_m|} |v_m|^{(n+4)/(n-4)} \left| \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} \right| = O\left( \frac{\|v_m\|^{2n/(n-4)}}{\lambda_m} \right), \tag{4-20}$$

$$\int_{|v_m| \le |\alpha_m \delta_{x_m,\lambda_m}|} \delta_{x_m,\lambda_m}^{(12-n)/(n-4)} |v_m|^2 \left| \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} \right| = O\left( \frac{\|v_m\|^2}{\lambda_m^2} \right). \tag{4-21}$$

From (4-17)–(4-21) and (4-14) we get

$$\int_{\mathbb{R}^n} K(y)|u_m|^{8/(n-4)} u_m \frac{\partial \delta_{x_m,\lambda_m}}{\partial \lambda_m} = \frac{-B \Delta K(x_m)}{\lambda_m^3} + o\left(\frac{1}{\lambda_m^3}\right). \tag{4-22}$$

Then (4-15), (4-16) and (4-22) imply

$$\frac{-B \Delta K(x_m)}{\lambda_m^3} + o\left(\frac{1}{\lambda_m^3}\right) = 0 \tag{4-23}$$

which contradicts the fact $\Delta K(x_m) \to \Delta K(x_0) \neq 0$. This ends the proof of Proposition 1.10. □

## Appendix A

In this section we will focus on the estimates needed in the proof of Theorem 1.1. Hence we will assume that $(H_1)$ holds. Note that the same program is needed for Theorem 1.2. There are some changes in the formula but the proofs are the same. We have to take account of the form of $H_{x,\lambda,2}$ and the behavior of the function $K$ near the critical point.

**Lemma A.1.** *For any $x \in B_\nu(z_j)$ and $v \in E_{x,\lambda}$ we have*

$$\int_{\mathbb{R}^n} K(y)\delta_{x,\lambda}^{(n+4)/(n-4)} v = O\left(|x-z_j|^{\beta_j} + \frac{1}{\lambda^{\inf(\beta_j,(n+4)/2)}}\right)\|v\|.$$

*Proof.* Since $v \in E_{x,\lambda}$, we have

$$\begin{aligned}(\text{A-1})\quad &\int_{\mathbb{R}^n} K(y)\delta_{x,\lambda}^{(n+4)/(n-4)} v \\ &= \int_{B_\varrho(x)} \big(K(y)-K(x)\big)\delta_{x,\lambda}^{(n+4)/(n-4)} v + \int_{B_\varrho^c(x)} \big(K(y)-K(x)\big)\delta_{x,\lambda}^{(n+4)/(n-4)} v.\end{aligned}$$

Using Holder's inequality and Sobolev imbedding theorem we compute

$$(\text{A-2})\qquad \int_{B_\varrho^c(x)} \big(K(y)-K(x)\big)\delta_{x,\lambda}^{(n+4)/(n-4)} v = O\left(\frac{\|v\|}{\lambda^{(n+4)/2}}\right),$$

$$(\text{A-3})\qquad \int_{B_\varrho(x)} \big(K(y)-K(x)\big)\delta_{x,\lambda}^{(n+4)/(n-4)} v = O\left(|x-z_j|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}}\right)\|v\|,$$

by (1-6) and (1-7). Then the lemma follows from (A-1), (A-2) and (A-3). □

**Lemma A.2.** *For any $(x,\lambda) \in D_{\nu,2}$ and $v \in E_{x,\lambda,2}$ we have*

$$\begin{aligned}(\text{A-4})\quad &\int_{\mathbb{R}^n} K(y)|H_{x,\lambda,2}|^{8/(n-4)} H_{x,\lambda,2} v \\ &= O\left(\sum_{j=1}^{2}\left(|x_j-z_j|^{\beta_j} + \frac{1}{\lambda_j^{\inf(\beta_j,(n+4)/2)}}\right) + \varepsilon_{12}^{1/2+\tau}\right)\|v\|,\end{aligned}$$

*where $\tau$ is a positive constant.*

*Proof.* For $p > 1$, there exists $C(p) > 1$ such that for any $a, b \in R_+$, we have

$$(\text{A-5})\ \big||a-b|^{p-1}(a-b) - a^p + b^p\big| \le \begin{cases} C(p)a^{p/2}b^{p/2}, & \text{if } p \le 2,\\ C(p)(a^{p-1}b + ab^{p-1}), & \text{if } p > 2.\end{cases}$$

From (A-5) we see that

$$\text{(A-6)}\quad \int_{\mathbb{R}^n} K(y)|H_{x,\lambda,2}|^{8/(n-4)} H_{x,\lambda,2} v$$
$$= \int_{\mathbb{R}^n} \frac{1}{K(x_1)^{(n+4)/8}} K(y)\delta_{x_1,\lambda_1}^{(n+4)/(n-4)} v - \int_{\mathbb{R}^n} \frac{1}{K(x_2)^{(n+4)/8}} K(y)\delta_{x_2,\lambda_2}^{(n+4)/(n-4)} v$$
$$+ \begin{cases} O\left(\int_{\mathbb{R}^n} (\delta_{x_1,\lambda_1}\delta_{x_2,\lambda_2})^{(n+4)/(2(n-4))}|v|\right), & \text{if } n \geq 12, \\ O\left(\sum_{i\neq j}\int_{\mathbb{R}^n} \delta_{x_i,\lambda_i}^{8/(n-4)}\delta_{x_j,\lambda_j}|v|\right), & \text{if } n < 12. \end{cases}$$

By Holder's inequality, the Sobolev embedding theorem and Lemma B.2 we have

$$\text{(A-7)}\quad \int_{\mathbb{R}^n} (\delta_{x_1,\lambda_1}\delta_{x_2,\lambda_2})^{(n+4)/(2(n-4))}|v| = O\left(\varepsilon_{12}^{(n+4)/(2(n-4))}(\log \varepsilon_{12}^{-1})^{(n+4)/(2n)}\right)\|v\|$$

$$\text{(A-8)}\quad \int_{\mathbb{R}^n} \delta_{x_i,\lambda_i}^{8/(n-4)}\delta_{x_j,\lambda_j}|v| = O\left(\varepsilon_{12}(\log \varepsilon_{12}^{-1})^{(n-4)/n}\right)\|v\|, \text{ for } i \neq j.$$

The lemma then follows from (A-6), (A-7), (A-8) and Lemma A.1. □

**Lemma A.3.**

$$\langle H_{x,\lambda,2}, \alpha_1\delta_{x_1,\lambda_1} - \alpha_2\delta_{x_2,\lambda_2}\rangle - \int_{\mathbb{R}^n} K(y)|H_{x,\lambda,2}|^{8/(n-4)} H_{x,\lambda,2}(\alpha_1\delta_{x_1,\lambda_1} - \alpha_2\delta_{x_2,\lambda_2})$$
$$= O\left(\sum_{j=1}^{2}\left(|x_j - z_j|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}}\right) + \varepsilon_{12}\right).$$

*Proof.* By Lemma B.1 we have

$$\text{(A-9)}\quad \langle H_{x,\lambda,2}, \alpha_1\delta_{x_1,\lambda_1} - \alpha_2\delta_{x_2,\lambda_2}\rangle = \frac{\alpha_1 S_n^{n/4}}{K(x_1)^{(n-4)/8}} - \frac{\alpha_2 S_n^{n/4}}{K(x_2)^{(n-4)/8}} + O\left(\varepsilon_{12}\right),$$

where $S_n$ is defined by (1-2). On the other hand it is easy to get

$$\text{(A-10)}\quad \int_{\mathbb{R}^n} K(y)|H_{x,\lambda,2}|^{8/(n-4)} H_{x,\lambda,2}(\alpha_1\delta_{x_1,\lambda_1} - \alpha_2\delta_{x_2,\lambda_2})$$
$$= \frac{\alpha_1}{K(x_1)^{(n+4)/8}}\int_{\mathbb{R}^n} K(y)\delta_{x_1,\lambda_1}^{2n/(n-4)} - \frac{\alpha_2}{K(x_2)^{(n+4)/8}}\int_{\mathbb{R}^n} K(y)\delta_{x_2,\lambda_2}^{2n/(n-4)} + O\left(\varepsilon_{12}\right).$$

Now,

$$\text{(A-11)}\quad \int_{\mathbb{R}^n} K(y)\delta_{x_j,\lambda_j}^{2n/(n-4)} = K(x_j)S_n^{n/4} + \int_{\mathbb{R}^n} \left(K(y) - K(x_j)\right)\delta_{x_j,\lambda_j}^{2n/(n-4)}.$$

Since $K$ is bounded, it is easy to check that

$$\int_{B^c_\varrho(x_j)} \big(K(y)-K(x_j)\big)\delta^{2n/(n-4)}_{x_j,\lambda_j} = O\left(\frac{1}{\lambda_j^n}\right). \tag{A-12}$$

On the other hand by using (1-6) and (1-7), we compute

$$\int_{B_\varrho(x_j)} \big(K(y)-K(x_j)\big)\delta^{2n/(n-4)}_{x_j,\lambda_j} = O\left(\frac{1}{\lambda_j^{\beta_j}} + |x_j-z_j|^{\beta_j}\right). \tag{A-13}$$

The lemma follows from (A-9)–(A-13). □

**Lemma A.4.** *There exists a constant $\tau > 0$ such that*

$$I(H_{x,\lambda,2}) = \sum_{j=1}^{2} I\left(\frac{1}{K(x_j)^{(n-4)/8}}\delta_{x_j,\lambda_j}\right) + \frac{D\varepsilon_{12}}{K(x_1)^{(n-4)/8}K(x_2)^{(n-4)/8}} \\ + O\left(\sum_{j=1}^{2}\left(|x_j-z_j|^{2\beta_j} + \frac{1}{\lambda_j^{2n}}\right) + \varepsilon_{12}^{1+\tau}\right).$$

*Proof.* The proof follows immediately from the fact that $K$ is bounded, (A-5), Lemmas B.1, B.2, and from

$$\int_{B^c_\varrho(x_i)} K(y)\delta^{(n+4)/(n-4)}_{x_i,\lambda_i}\delta_{x_j,\lambda_j} \\ \le c\left(\int_{B^c_\varrho(x_i)} \delta^{n/(n-4)}_{x_i,\lambda_i}\delta^{n/(n-4)}_{x_j,\lambda_j}\right)^{(n-4)/n}\left(\int_{B^c_\varrho(x_i)} \delta^{2n/(n-4)}_{x_i,\lambda_i}\right)^{4/n} \\ \le \frac{c}{\lambda_i^4}\,\varepsilon_{ij}\big(\log\varepsilon_{ij}^{-1}\big)^{(n-4)/n}. \tag{A-14}$$

In the ball $B_\varrho(x_i)$, by (1-6) and (1-7), we have

$$\int_{B_\varrho(x_i)} K(y)\delta^{(n+4)/(n-4)}_{x_i,\lambda_i}\delta_{x_j,\lambda_j} \\ = K(z_i)\int_{B_\varrho(x_i)} \delta^{(n+4)/(n-4)}_{x_i,\lambda_i}\delta_{x_j,\lambda_j} + \int_{B_\varrho(x_i)} Q_i(y-z_i)\delta^{(n+4)/(n-4)}_{x_i,\lambda_i}\delta_{x_j,\lambda_j} \\ + \int_{B_\varrho(x_i)} R_i(y-z_i)\delta^{(n+4)/(n-4)}_{x_i,\lambda_i}\delta_{x_j,\lambda_j}. \tag{A-15}$$

We compute

$$\frac{K(z_i)}{K(x_i)}\int_{B_\varrho(x_i)} \delta^{(n+4)/(n-4)}_{x_i,\lambda_i}\delta_{x_j,\lambda_j} = D\varepsilon_{ij} + O\big(|x_i-z_i|^{2\beta_i} + \varepsilon_{ij}^{1+\tau}\big), \tag{A-16}$$

$$\text{(A-17)}\quad \int_{B_\varrho(x_i)} Q_i(y-z_i)\delta_{x_i,\lambda_i}^{(n+4)/(n-4)}\delta_{x_j,\lambda_j} = O\biggl(|x_i-z_i|^{2\beta_i}+\frac{1}{\lambda_i^{2n}}+\varepsilon_{ij}^{1+\tau}\biggr),$$

$$\text{(A-18)}\quad \int_{B_\varrho(x_i)} R_i(y-z_i)\delta_{x_i,\lambda_i}^{(n+4)/(n-4)}\delta_{x_j,\lambda_j} = O\biggl(|x_i-z_i|^{2\beta_i+2\sigma}+\frac{1}{\lambda_i^{2n}}+\varepsilon_{ij}^{1+\tau}\biggr).$$

This completes the proof. □

In the following, we will focus in dimension five and we will assume that $K$ is a $C^2$ function. Hence for each $x\in\mathbb{R}^5$, we can expand $K$ near $x$ and we obtain

$$K(y)=K(x)+\nabla K(x)(y-x)+\frac{1}{2}D^2K(x)(y-x,\,y-x)+o(\|y-x\|^2).$$

Moreover, we have $\|D^2K(x)\|$ is bounded.

**Lemma A.5.** *For any $x\in D_{\nu,k}$ and $v\in E_{x,\lambda,k}$, we have*

$$\text{(A-19)}\qquad \int_{\mathbb{R}^5} K(y)H_{x,\lambda,k}^9 v = O\biggl(\sum_{j=1}^{k}\biggl(\frac{|\nabla K(x_j)|}{\lambda_j}+\frac{1}{\lambda_j^2}\biggr)+\sum_{i\neq j}\varepsilon_{ij}^{1/2+\tau}\biggr)\|v\|$$

*where $\tau>0$ is a constant.*

*Proof.* As in (A-6) and using Lemma B.2, we have

$$\text{(A-20)}\quad \int_{\mathbb{R}^5} K(y)H_{x,\lambda,k}^9 v = \sum_{j=1}^{k}\int_{\mathbb{R}^5} K(y)\frac{1}{K(x_j)^{9/8}}\delta_{x_j,\lambda_j}^9 v + O\biggl(\sum_{i\neq j}\varepsilon_{ij}^{1/2+\tau}\biggr)\|v\|.$$

For the integral in the right hand side of (A-20), we follow the proof of Lemma A.1. But here we cannot use $(H_1)$. In fact, (A-1) and (A-2) hold. It remains to compute

$$\text{(A-21)}\quad \int_{B_\varrho(x_j)}(K(y)-K(x_j))\delta_{x_j,\lambda_j}^9 v = \int_{B_\varrho(x_j)}\nabla K(x_j)(y-x_j)\delta_{x_j,\lambda_j}^9 v + O\biggl(\int_{B_\varrho(x_j)}|y-x_j|^2\delta_{x_j,\lambda_j}^9|v|\biggr).$$

Now, by using Holder's inequality and the Sobolev imbedding theorem, we have

$$\text{(A-22)}\qquad \int_{B_\varrho(x_j)}\nabla K(x_j)(y-x_j)\delta_{x_j,\lambda_j}^9 v = O\biggl(\frac{|\nabla K(x_j)|}{\lambda_j}\|v\|\biggr),$$

$$\text{(A-23)}\qquad \int_{B_\varrho(x_j)}|y-x_j|^2\delta_{x_j,\lambda_j}^9 v = O\biggl(\frac{\|v\|}{\lambda_j^2}\biggr).$$

Then the lemma follows from (A-1), (A-2), (A-21), (A-22) and (A-23). □

**Lemma A.6.** *We have*

$$\langle H_{x,\lambda,k}, \delta_{x_j,\lambda_j}\rangle - \int_{\mathbb{R}^5} K(y) H^9_{x,\lambda,k}\delta_{x_j,\lambda_j} = O\biggl(\sum_{j=1}^{k} \frac{1}{\lambda_j^2} + \sum_{i\neq j} \varepsilon_{ij}\biggr). \tag{A-24}$$

*Proof.* Similarly to (A-9) and (A-10), we have

$$\langle H_{x,\lambda,k}, \delta_{x_j,\lambda_j}\rangle = \frac{S_5^{5/4}}{K(x_j)^{1/8}} + O\biggl(\sum_{i\neq j} \varepsilon_{ij}\biggr), \tag{A-25}$$

$$\int_{\mathbb{R}^5} K(y) H^9_{x,\lambda,k}\delta_{x_j,\lambda_j} = \frac{1}{K(x_j)^{9/8}} \int_{\mathbb{R}^5} K(y)\delta^{10}_{x_j,\lambda_j} + O\biggl(\sum_{i\neq j} \varepsilon_{ij}\biggr). \tag{A-26}$$

Since $K$ is a $C^2$ function, then expanding $K$ around $x_j$ and using the evenness of $\delta_{x_j,\lambda_j}$ with respect to $y - x_j$, we get

$$\int_{\mathbb{R}^5} K(y)\delta^{10}_{x_j,\lambda_j} = K(x_j) S_5^{5/4} + O\biggl(\frac{1}{\lambda_j^2}\biggr). \tag{A-27}$$

From (A-25), (A-26) and (A-27), the lemma follows. □

**Lemma A.7.**

$$I(H_{x,\lambda,k}) = \frac{2}{5}\sum_{j=1}^{k} \frac{S_5^{5/4}}{K(x_j)^{1/4}} - \frac{1}{10}\sum_{j=1}^{k} \frac{1}{K(x_j)^{5/4}} \frac{B\,\Delta K(x_j)}{\lambda_j^2} - \sum_{i\neq j} \frac{D\varepsilon_{ij}}{K(x_i)^{1/8} K(x_j)^{1/8}} + o\biggl(\sum_{j=1}^{k} \frac{1}{\lambda_j^2}\biggr) + O\biggl(\sum_{i\neq j} \varepsilon_{ij}^{1+\tau}\biggr),$$

*where* $B = \frac{1}{5}\int_{\mathbb{R}^5} |x|^2 \delta^{10}_{0,1}$.

*Proof.* We have

$$I(H_{x,\lambda,k}) = \frac{1}{2}\|H_{x,\lambda,k}\|^2 - \frac{1}{10}\int_{\mathbb{R}^5} K(y) H^{10}_{x,\lambda,k}. \tag{A-28}$$

First by Lemma B.1, one has

$$\|H_{x,\lambda,2}\|^2 = \sum_{j=1}^{k} \frac{S_5^{5/4}}{K(x_j)^{1/4}} + 2\sum_{i\neq j} \frac{D\varepsilon_{ij}}{K(x_i)^{1/8} K(x_j)^{1/8}} + O\biggl(\sum_{i<j} \varepsilon_{ij}^{1+\tau}\biggr). \tag{A-29}$$

Second using Lemma B.2, we get

$$\text{(A-30)} \quad \int_{\mathbb{R}^5} K(y) H^{10}_{x,\lambda,k} = \sum_{j=1}^{k} \frac{1}{K(x_j)^{5/4}} \int_{\mathbb{R}^5} K(y)\delta^{10}_{x_j,\lambda_j} + 10 \sum_{i\neq j} \frac{1}{K(x_i)^{9/8} K(x_j)^{1/8}} \int_{\mathbb{R}^5} K(y)\delta^{9}_{x_i,\lambda_i}\delta_{x_j,\lambda_j} + O\biggl(\sum_{i\neq j} \varepsilon_{ij}^{1+\tau}\biggr).$$

Now since $K$ is a $C^2$ function, by expanding $K$ around $x_j$ and using the evenness of $\delta_{x_j,\lambda_j}$ with respect $y - x_j$, we compute

$$\text{(A-31)} \quad \int_{\mathbb{R}^5} K(y)\delta^{10}_{x_j,\lambda_j} = K(x_j) S_5^{5/4} + \frac{\Delta K(x_j)}{\lambda_j^2}\frac{1}{5}\int_{\mathbb{R}^5} \frac{|x|^2}{(1+|x|^2)^5} + o\biggl(\frac{1}{\lambda_j^2}\biggr).$$

We have also by expanding $K$ around $x_i$ and using Lemmas B.1 and B.2

$$\text{(A-32)} \quad \int_{\mathbb{R}^5} K(y)\delta^{9}_{x_i,\lambda_i}\delta_{x_j,\lambda_j} = K(x_i) D\varepsilon_{ij} + O\biggl(\frac{1}{\lambda_i^5} + \varepsilon_{ij}^{1+\tau}\biggr).$$

It is easy to see that the lemma follows from (A-28)–(A-32). □

## Appendix B

A computation similar to the one performed in [Bahri 1989] shows that, for $i \neq j$, if the interaction $\varepsilon_{ij}$ is small and the concentration $\lambda_i$ are large, then we have the following lemmas:

**Lemma B.1.**

$$\text{(B-1)} \quad \int_{\mathbb{R}^n} \delta^{(n+4)/(n-4)}_{x_i,\lambda_i}\delta_{x_j,\lambda_j} = D\varepsilon_{ij} + O\bigl(\varepsilon_{ij}^{(n-2)/(n-4)}\bigr),$$

$$\textit{where } D = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+4)/2}}.$$

**Lemma B.2.** *Let* $\alpha > 1$, $\beta > 1$ *be such that* $\alpha+\beta = 2n/(n-4)$ *and let* $\theta = \inf(\alpha, \beta)$. *Then we have*

$$\text{(B-2)} \quad \int_{\mathbb{R}^n} \delta^{\alpha}_{x_i,\lambda_i}\delta^{\beta}_{x_j,\lambda_j} = O\Bigl(\varepsilon_{ij}^{\theta}\bigl(\log(\varepsilon_{ij}^{-1})\bigr)^{(n-4)\theta/n}\Bigr).$$

**Lemma B.3.** *If* $K$ *is a* $C^2$ *function near the concentration point* $x$, *then*

$$\text{(B-3)} \quad \int_{\mathbb{R}^n} K(y)\delta^{(n+4)/(n-4)}_{x,\lambda}\frac{\partial \delta_{x,\lambda}}{\partial \lambda} = -\frac{n-4}{2n} c_2 \frac{\Delta K(x)}{\lambda^3} + o\biggl(\frac{1}{\lambda^3}\biggr).$$

## References

[Atkinson and Peletier 1987] F. V. Atkinson and L. A. Peletier, "Elliptic equations with nearly critical growth", *J. Differential Equations* **70**:3 (1987), 349–365. MR 89e:35054 Zbl 0657.35058

[Bahri 1989] A. Bahri, *Critical points at infinity in some variational problems*, Pitman Research Notes in Math. Ser. **182**, Longman Scientific & Technical, Harlow, 1989. MR 91h:58022 Zbl 0676.58021

[Bahri and Coron 1988] A. Bahri and J.-M. Coron, "On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain", *Comm. Pure Appl. Math.* **41**:3 (1988), 253–294. MR 89c:35053 Zbl 0649.35033

[Bahri et al. 1995] A. Bahri, Y. Li, and O. Rey, "On a variational problem with lack of compactness: the topological effect of the critical points at infinity", *Calc. Var. Partial Differential Equations* **3**:1 (1995), 67–93. MR 98c:35049 Zbl 0814.35032

[Ben Ayed and El Mehdi 2007] M. Ben Ayed and K. El Mehdi, "The Paneitz curvature problem on lower-dimensional spheres", *Ann. Global Anal. Geom.* **31**:1 (2007), 1–36. MR 2007k:53038 Zbl 05126708

[Ben Ayed et al. 2003] M. Ben Ayed, K. El Mehdi, O. Rey, and M. Grossi, "A nonexistence result of single peaked solutions to a supercritical nonlinear problem", *Commun. Contemp. Math.* **5**:2 (2003), 179–195. MR 2004k:35140 Zbl 1066.35035

[Ben Ayed et al. 2005] M. Ben Ayed, K. El Mehdi, and M. Hammami, "Some existence results for a Paneitz type problem via the theory of critical points at infinity", *J. Math. Pures Appl.* (9) **84**:2 (2005), 247–278. MR 2006b:35075 Zbl 02164964

[Brezis and Peletier 1989] H. Brezis and L. A. Peletier, "Asymptotics for elliptic equations involving critical growth", pp. 149–192 in *Partial differential equations and the calculus of variations, Vol. I*, edited by F. Colombini et al., Progr. Nonlinear Differential Equations Appl. **1**, Birkhäuser, Boston, 1989. MR 91a:35030 Zbl 0685.35013

[Chabrowski and Yan 1999] J. Chabrowski and S. Yan, "Concentration of solutions for a nonlinear elliptic problem with nearly critical exponent", *Topol. Methods Nonlinear Anal.* **13**:2 (1999), 199–233. MR 2001h:35049 Zbl 0956.35048

[Han 1991] Z.-C. Han, "Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8**:2 (1991), 159–174. MR 92c:35047 Zbl 0729.35014

[Lin 1998] C.-S. Lin, "A classification of solutions of a conformally invariant fourth order equation in $\mathbf{R}^n$", *Comment. Math. Helv.* **73**:2 (1998), 206–231. MR 99c:35062 Zbl 0933.35057

[Micheletti and Pistoia 2003] A. M. Micheletti and A. Pistoia, "Existence of blowing-up solutions for a slightly subcritical or a slightly supercritical non-linear elliptic equation on $\mathbb{R}^n$", *Nonlinear Anal.* **52**:1 (2003), 173–195. MR 2003i:35098 Zbl 1023.35037

[Musso and Pistoia 2002] M. Musso and A. Pistoia, "Multispike solutions for a nonlinear elliptic problem involving the critical Sobolev exponent", *Indiana Univ. Math. J.* **51**:3 (2002), 541–579. MR 2003g:35079 Zbl 1074.35037

[del Pino et al. 2002] M. del Pino, P. Felmer, and M. Musso, "Multi-peak solutions for super-critical elliptic problems in domains with small holes", *J. Differential Equations* **182**:2 (2002), 511–540. MR 2003h:35086 Zbl 1014.35028

[del Pino et al. 2003] M. del Pino, P. Felmer, and M. Musso, "Two-bubble solutions in the super-critical Bahri–Coron's problem", *Calc. Var. Partial Differential Equations* **16**:2 (2003), 113–145. MR 2004a:35079 Zbl 1142.35421

[Rey 1989] O. Rey, "Proof of two conjectures of H. Brézis and L. A. Peletier", *Manuscripta Math.* **65**:1 (1989), 19–37. MR 90i:35109 Zbl 0708.35032

[Rey 1990] O. Rey, "The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent", *J. Funct. Anal.* **89**:1 (1990), 1–52. MR 91b:35012 Zbl 0786.35059

[Rey 1991] O. Rey, "Blow-up points of solutions to elliptic equations with limiting nonlinearity", *Differential Integral Equations* **4**:6 (1991), 1155–1167. MR 92i:35056 Zbl 0830.35043

[Rey 1992] O. Rey, "Concentration of solutions to elliptic equations with critical nonlinearity", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9**:2 (1992), 201–218. MR 93e:35040 Zbl 0761.35034

[Rey 1999] O. Rey, "The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension 3", *Adv. Differential Equations* **4**:4 (1999), 581–616. MR 2000d:35080 Zbl 0952.35051

[Yan 2000] S. Yan, "Concentration of solutions for the scalar curvature equation on $\mathbf{R}^N$", *J. Differential Equations* **163**:2 (2000), 239–264. MR 2001g:35086 Zbl 0954.35058

Received July 10, 2008.

MOHAMED BEN AYED
DÉPARTEMENT DE MATHÉMATIQUES
FACULTÉ DES SCIENCES DE SFAX
ROUTE SOUKRA
SFAX
TUNISIA
Mohamed.Benayed@fss.rnu.tn

ABDELBAKI SELMI
DÉPARTEMENT DE MATHÉMATIQUES
FACULTÉ DES SCIENCES DE BIZERTE
ZARZOUNA 7021
BIZERTE
TUNISIA
Abdelbaki.Selmi@fsb.rnu.tn