

*Pacific  
Journal of  
Mathematics*

THE OPERATORS  $\partial$  AND  $\bar{\partial}$  OF A GENERALIZED COMPLEX  
STRUCTURE

ZHUO CHEN

Volume 242 No. 1

September 2009



## THE OPERATORS $\partial$ AND $\bar{\partial}$ OF A GENERALIZED COMPLEX STRUCTURE

ZHUO CHEN

**We prove that the  $\partial$  and  $\bar{\partial}$  operators introduced by Gualtieri for a generalized complex structure coincide with the  $\check{d}_*$  and  $\check{\partial}$  operators introduced by Alekseev and Xu for Evens–Lu–Weinstein modules of a Lie bialgebroid.**

### Introduction

Generalized complex structures [Hitchin 2003; Gualtieri 2003; Cavalcanti 2007] have been extensively studied recently due to their close connection with mirror symmetry. They include both symplectic and complex structures as extreme cases. Gualtieri [2003; 2007] defined the  $\partial$  and  $\bar{\partial}$  operators for any twisted generalized complex structure in the same way that these operators are defined in complex geometry. In fact, he proved that an  $H$ -twisted generalized complex structure  $\mathbb{J}$  determines an alternative grading of differential forms and a splitting  $d^H = \partial + \bar{\partial}$ , where  $d^H = d - H \wedge$  is the de Rham differential twisted by a closed three-form  $H$ .

A Lie bialgebroid, as introduced by Mackenzie and Xu [1994], is a pair of Lie algebroids  $(A, A^*)$  satisfying some compatibility condition; see also [Kosmann-Schwarzbach 1995]. They appear naturally in many places in Poisson geometry. In [Alekseev and Xu 2001], two differential operators  $\check{d}_*$  and  $\check{\partial}$  were introduced for Evens–Lu–Weinstein modules of a Lie bialgebroid, as follows.

We consider a pair of (real or complex) Lie algebroid structures on a vector bundle  $A$  and its dual  $A^*$ , and we assume that the (real or complex) line bundle  $\mathcal{L} = (\wedge^{\text{top}} A^* \otimes \wedge^{\text{top}} T^* M)^{1/2}$  exists. Then  $\mathcal{L}$  is a module over  $A^*$ , as discovered by Evens, Lu and Weinstein [Evens et al. 1999]. The Lie algebroid structures of  $A^*$  and  $A$  induce two natural differential operators  $\check{d}_* : \Gamma(\wedge^k A \otimes \mathcal{L}) \rightarrow \Gamma(\wedge^{k+1} A \otimes \mathcal{L})$  and  $\check{\partial} : \Gamma(\wedge^k A \otimes \mathcal{L}) \rightarrow \Gamma(\wedge^{k-1} A \otimes \mathcal{L})$ ; see Equations (2-1) through (2-7).

Since a generalized complex structure  $\mathbb{J}$  induces a (complex) Lie bialgebroid  $(L, \bar{L})$ , where  $L$  and  $\bar{L}$  are respectively the  $+i$  and  $-i$  eigenspaces of  $\mathbb{J}$ , it is tempting to investigate the relations between the operators  $\partial$ ,  $\bar{\partial}$ ,  $\check{d}_*$  and  $\check{\partial}$ . In this

---

*MSC2000:* primary 53C15, 53D17, 57R15, 58A10; secondary 15A66, 17B63, 17B66.

*Keywords:* generalized complex structure, Lie bialgebroid, modular cocycle.

This research was partially supported by NSFC, grant 10871007.

note, we show that  $\partial$  and  $\bar{\partial}$  essentially coincide with  $\check{d}_*$  and  $\check{\partial}$  respectively, under some natural isomorphisms.

## 1. Courant algebroids and Lie bialgebroids

In this article, all vector bundles are complex vector bundles. Likewise, Lie algebroids are always complex Lie algebroids.

A (complex) Courant algebroid consists of a vector bundle  $\pi : E \rightarrow M$ , a non-degenerate pseudometric  $\langle \cdot, \cdot \rangle$  on the fibers of  $\pi$ , a bundle map  $\rho : E \rightarrow T_{\mathbb{C}}M$ , called the anchor, and a  $\mathbb{C}$ -bilinear operation  $\circ$  on  $\Gamma(E)$  called the Dorfman bracket, which for all  $f \in C^\infty(M, \mathbb{C})$  and  $z_1, z_2, z_3 \in \Gamma(E)$  satisfy the relations

$$(1-1) \quad z_1 \circ (z_2 \circ z_3) = (z_1 \circ z_2) \circ z_3 + z_2 \circ (z_1 \circ z_3),$$

$$(1-2) \quad \rho(z_1 \circ z_2) = [\rho(z_1), \rho(z_2)],$$

$$(1-3) \quad z_1 \circ f z_2 = (\rho(x)f)z_2 + f(z_1 \circ z_2),$$

$$(1-4) \quad z_1 \circ z_2 + z_2 \circ z_1 = 2\mathcal{D}\langle z_1, z_2 \rangle,$$

$$(1-5) \quad \mathcal{D}f \circ z_1 = 0,$$

$$(1-6) \quad \rho(z_1)\langle z_2, z_3 \rangle = \langle z_1 \circ z_2, z_3 \rangle + \langle z_2, z_1 \circ z_3 \rangle,$$

where  $\mathcal{D} : C^\infty(M, \mathbb{C}) \rightarrow \Gamma(E)$  is the  $\mathbb{C}$ -linear map defined by  $\langle \mathcal{D}f, z_1 \rangle = \frac{1}{2}\rho(z_1)f$ .

The symmetric part of the Dorfman bracket is given by (1-4). The Courant bracket is defined as the skew-symmetric part  $[[z_1, z_2]] = \frac{1}{2}(z_1 \circ z_2 - z_2 \circ z_1)$  of the Dorfman bracket. Thus we have the relation  $z_1 \circ z_2 = [[z_1, z_2]] + \mathcal{D}\langle z_1, z_2 \rangle$ .

The definition of a Courant algebroid can be rephrased using the Courant bracket instead of the Dorfman bracket [Roytenberg 1999].

A Dirac structure is a smooth subbundle  $A \rightarrow M$  of the Courant algebroid  $E$ ; it is maximally isotropic with respect to the pseudometric and its space of sections is closed under (necessarily both) brackets. Thus a Dirac structure inherits a canonical Lie algebroid structure [Liu et al. 1997].

Let  $A \rightarrow M$  be a vector bundle. Assume that  $A$  and its dual  $A^*$  both carry a Lie algebroid structure with anchor maps  $a : A \rightarrow T_{\mathbb{C}}M$  and  $a_* : A^* \rightarrow T_{\mathbb{C}}M$ , brackets on sections

$$\Gamma(A) \otimes_{\mathbb{C}} \Gamma(A) \rightarrow \Gamma(A) : X \otimes Y \mapsto [X, Y],$$

$$\Gamma(A^*) \otimes_{\mathbb{C}} \Gamma(A^*) \rightarrow \Gamma(A^*) : \theta \otimes \zeta \mapsto [\theta, \zeta]_*,$$

and differentials  $d : \Gamma(\wedge^{\bullet} A^*) \rightarrow \Gamma(\wedge^{\bullet+1} A^*)$  and  $d_* : \Gamma(\wedge^{\bullet} A) \rightarrow \Gamma(\wedge^{\bullet+1} A)$ .

By [Kosmann-Schwarzbach 1995; Mackenzie and Xu 2000; 1994], this pair  $(A, A^*)$  of Lie algebroids is a Lie bialgebroid (or Manin triple) if  $d_*$  is a derivation of the Gerstenhaber algebra  $(\Gamma(\wedge^{\bullet} A), \wedge, [\cdot, \cdot])$  or, equivalently, if  $d$  is a derivation of the Gerstenhaber algebra  $(\Gamma(\wedge^{\bullet} A^*), \wedge, [\cdot, \cdot]_*)$ . The link between Courant and Lie bialgebroids is as follows.

**Theorem 1.1** [Liu et al. 1997]. *There is a one-to-one correspondence between Lie bialgebroids and pairs of transversal Dirac structures in a Courant algebroid.*

More precisely, if the pair  $(A, A^*)$  is a Lie bialgebroid, then the vector bundle  $A \oplus A^* \rightarrow M$ , together with the pseudometric

$$(1-7) \quad \langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \frac{1}{2}(\xi_1(X_2) + \xi_2(X_1)),$$

the anchor map  $\rho = a + a_*$  (whose dual is given through  $\mathcal{D}f = df + d_*f$  for  $f \in C^\infty(M, \mathbb{C})$ ), and the Dorfman bracket

$$(1-8) \quad (X_1 + \xi_1) \circ (X_2 + \xi_2) = \\ ([X_1, X_2] + \mathcal{L}_{\xi_1} X_2 - \iota_{\xi_2}(d_* X_1)) + ([\xi_1, \xi_2]_* + \mathcal{L}_{X_1} \xi_2 - \iota_{X_2}(d \xi_1)),$$

is a Courant algebroid of which  $A$  and  $A^*$  are transverse Dirac structures. It is called the double of the Lie bialgebroid  $(A, A^*)$ . Here  $X_1$  and  $X_2$  denote arbitrary sections of  $A$ , and  $\xi_1$  and  $\xi_2$  arbitrary sections of  $A^*$ .

An important example is when  $A = T_{\mathbb{C}}M$  is the tangent bundle of a manifold  $M$  and  $A^* = T_{\mathbb{C}}^*M$  takes the trivial Lie algebroid structure. Then  $T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$  has the standard Courant algebroid structure. Severa and Weinstein [2001] observed that the Dorfman bracket on  $T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$  can be twisted by a closed three-form  $H \in Z^3(M)$ :

$$(x_1 + \eta_1) \circ_H (x_2 + \eta_2) = (x_1 + \eta_1) \circ (x_2 + \eta_2) + \iota_{x_2} \iota_{x_1} H \\ = [x_1, x_2] + \mathcal{L}_{x_1} \eta_2 - \mathcal{L}_{x_2} \eta_1 + \frac{1}{2} d(\eta_1 | x_2) + \iota_{x_2} \iota_{x_1} H.$$

And  $\circ_H$  defines a Courant algebroid structure on  $T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$ , using the same inner product and anchor. The corresponding Courant bracket is also twisted:

$$(1-9) \quad \llbracket x_1 + \eta_1, x_2 + \eta_2 \rrbracket_H \\ = \llbracket x_1 + \eta_1, x_2 + \eta_2 \rrbracket + \iota_{x_2} \iota_{x_1} H \\ = [x_1, x_2] + \mathcal{L}_{x_1} \eta_2 - \mathcal{L}_{x_2} \eta_1 + \frac{1}{2} d(\langle \eta_1 | x_2 \rangle - \langle \eta_2 | x_1 \rangle) + \iota_{x_2} \iota_{x_1} H.$$

## 2. Clifford modules and Dirac generating operators

Let  $V$  be a vector space of dimension  $r$  endowed with a nondegenerate symmetric bilinear form  $\langle \cdot, \cdot \rangle$ . Its Clifford algebra  $\mathcal{C}(V)$  is defined as the quotient of the tensor algebra  $\bigoplus_{k=0}^r V^{\otimes k}$  by the relations  $x \otimes y + y \otimes x = 2\langle x, y \rangle$ , with  $x, y \in V$ . It is naturally an associative  $\mathbb{Z}_2$ -graded algebra. Up to isomorphisms, there exists a unique irreducible module  $S$  of  $\mathcal{C}(V)$ , called spin representation [Chevalley 1997]. The vectors of  $S$  are called spinors.

**Example 2.1.** Let  $W$  be a vector space of dimension  $r$ . We can endow  $V = W \oplus W^*$  with the nondegenerate pairing defined in the same fashion as in Equation (1-7).

The representation of  $\mathcal{C}(V)$  on  $S = \bigoplus_{k=0}^r \bigwedge^k W$  defined by  $u \cdot w = u \wedge w$  and  $\xi \cdot w = \iota_\xi w$ , where  $u \in W$ ,  $\xi \in W^*$  and  $w \in S$ , is the spin representation. Note that  $S$  is  $\mathbb{Z}$ -graded and thus also  $\mathbb{Z}_2$ -graded.

Now let  $\pi : E \rightarrow M$  be a vector bundle endowed with a nondegenerate pseudo-metric  $\langle \cdot, \cdot \rangle$  on its fibers, and let  $\mathcal{C}(E) \rightarrow M$  be the associated bundle of Clifford algebras. Assume there exists a smooth vector bundle  $S \rightarrow M$  whose fiber  $S_m$  over a point  $m \in M$  is the spin module of the Clifford algebra  $\mathcal{C}(E)_m$ . Assume furthermore that  $S$  is  $\mathbb{Z}_2$ -graded, that is,  $S = S^0 \oplus S^1$ .

An operator  $O$  on  $\Gamma(S)$  is called even (or of degree 0) if  $O(S^i) \subset S^i$  and odd (or of degree 1) if  $O(S^i) \subset S^{i+1}$ . Here  $i \in \mathbb{Z}_2$ .

**Example 2.2.** If the vector bundle  $E$  decomposes as the direct sum  $A \oplus A^*$  of two transverse Lagrangian subbundles as in Example 2.1, then  $S = \bigwedge A$ . The multiplication by a function  $f \in C^\infty(M, \mathbb{C})$  is an even operator on  $\Gamma(S)$ , while the Clifford action of a section  $e \in \Gamma(E)$  is an odd operator on  $\Gamma(S)$ .

If  $O_1$  and  $O_2$  are operators of degree  $d_1$  and  $d_2$  respectively, then their commutator is the operator  $[O_1, O_2] = O_1 \circ O_2 - (-1)^{d_1 d_2} O_2 \circ O_1$ .

**Definition 2.3** [Alekseev and Xu 2001]. A Dirac generating operator for  $(E, \langle \cdot, \cdot \rangle)$  is an odd operator  $D$  on  $\Gamma(S)$  satisfying the following properties:

- (1)  $[D, f] \in \Gamma(E)$  for all  $f \in C^\infty(M, \mathbb{C})$ . This means that the operator  $[D, f]$  is the Clifford action of some section of  $E$ .
- (2)  $[[D, z_1], z_2] \in \Gamma(E)$  for all  $z_1, z_2 \in \Gamma(E)$ .
- (3) The square of  $D$  is multiplication by some function on  $M$ , that is,  $D^2$  is in  $C^\infty(M, \mathbb{C})$ .

Note that “deriving operators” of [Kosmann-Schwarzbach 2005] do not require assumption (3).

**Theorem 2.4** [Alekseev and Xu 2001]. *Let  $D$  be a Dirac generating operator for a vector bundle  $\pi : E \rightarrow M$ . Then there is a canonical Courant algebroid structure on  $E$ . The anchor  $\rho : E \rightarrow T_{\mathbb{C}}M$  is defined by  $\rho(z)f = 2\langle [D, f], z \rangle = [[D, f], z]$ , while the Dorfman bracket reads  $z_1 \circ z_2 = [[D, z_1], z_2]$ .*

We follow the same setup as in [Alekseev and Xu 2001; Chen and Stiénon 2009].

Let  $(A, [\cdot, \cdot], a)$  and  $(A^*, [\cdot, \cdot]_*, a_*)$  be a pair of Lie algebroids, where  $A$  is of rank  $r$  and the base manifold  $M$  is of dimension  $m$ . Then the line bundle  $\bigwedge^r A^* \otimes \bigwedge^m T_{\mathbb{C}}^* M$  is a module over the Lie algebroid  $A^*$  [Evens et al. 1999]:  $A$

section  $\alpha \in \Gamma(A^*)$  acts on  $\Gamma(\bigwedge^r A^* \otimes \bigwedge^m T_{\mathbb{C}}^* M)$  by

$$(2-1) \quad \begin{aligned} \nabla_{\alpha}(a_1 \wedge \cdots \wedge a_r \otimes \mu) \\ = \sum_{i=1}^r (a_1 \wedge \cdots \wedge [\alpha, a_i]_* \wedge \cdots \wedge a_r \otimes \mu) + a_1 \wedge \cdots \wedge a_r \otimes \mathcal{L}_{a_*(\alpha)} \mu. \end{aligned}$$

If it exists, the square root  $\mathcal{L} = (\bigwedge^r A^* \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2}$  of this line bundle is also a module over  $A^*$ . Here  $\mathcal{L}$  is a (complex) vector bundle whose square,  $\mathcal{L}^2 = \mathcal{L} \otimes \mathcal{L}$ , is isomorphic to  $\bigwedge^r A^* \otimes \bigwedge^m T_{\mathbb{C}}^* M$ . The  $A^*$ -module structure of  $\mathcal{L}$  is illustrated in [Evens et al. 1999, Proposition 4.3].

One can thus define a differential operator

$$(2-2) \quad \check{d}_* : \Gamma(\bigwedge^k A \otimes \mathcal{L}) \rightarrow \Gamma(\bigwedge^{k+1} A \otimes \mathcal{L}).$$

Similarly,  $(\bigwedge^r A \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2}$  is — provided it exists — a module over  $A$ . Hence we obtain a differential operator

$$(2-3) \quad \Gamma(\bigwedge^k A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2}) \rightarrow \Gamma(\bigwedge^{k+1} A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2}).$$

But the isomorphisms of vector bundles

$$(2-4) \quad \bigwedge^k A^* \cong \bigwedge^k A^* \otimes \bigwedge^{r-k} A^* \otimes \bigwedge^{r-k} A \cong \bigwedge^{r-k} A \otimes \bigwedge^r A^*$$

and

$$(2-5) \quad \bigwedge^r A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2} \cong (\bigwedge^r A^* \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2}$$

imply that

$$(2-6) \quad \begin{aligned} \bigwedge^k A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2} &\cong \bigwedge^{r-k} A \otimes \bigwedge^r A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2} \\ &\cong \bigwedge^{r-k} A \otimes (\bigwedge^r A^* \otimes \bigwedge^m T_{\mathbb{C}}^* M)^{1/2}. \end{aligned}$$

Therefore, one ends up with a differential operator

$$(2-7) \quad \check{\partial} : \Gamma(\bigwedge^k A \otimes \mathcal{L}) \rightarrow \Gamma(\bigwedge^{k-1} A \otimes \mathcal{L}).$$

**Theorem 2.5** [Chen and Stiénon 2009]. *The pair of Lie algebroids  $(A, A^*)$  is a Lie bialgebroid if and only if  $\check{D}^2 \in C^\infty(M, \mathbb{C})$ , that is, the square of the operator*

$$\check{D} = \check{d}_* + \check{\partial} : \Gamma(\bigwedge A \otimes \mathcal{L}) \rightarrow \Gamma(\bigwedge A \otimes \mathcal{L})$$

*is multiplication by some function  $\check{f} \in C^\infty(M, \mathbb{C})$ . Moreover  $\check{D}_*^2 = \check{f}$ , where  $\check{D}_* = \check{d} + \check{\partial}_*$  is defined analogously to  $\check{D}$  by exchanging the roles of  $A$  and  $A^*$ .*

### 3. Generalized complex geometry

In this section, we fix a real  $2n$ -dimensional manifold  $M$  and denote the tangent and cotangent bundle of  $M$  by  $T$  and  $T^*$ , respectively. Let  $T_{\mathbb{C}}$  and  $T_{\mathbb{C}}^*$  be respectively the complexification of  $T$  and  $T^*$ . The first vital ingredient in  $T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*$  is the natural pairing:

$$(3-1) \quad \langle x_1 + \eta_1, x_2 + \eta_2 \rangle = \frac{1}{2}(\langle x_1 | \eta_2 \rangle + \langle x_2 | \eta_1 \rangle) \quad \text{for all } x_i \in T_{\mathbb{C}} \text{ and } \eta_i \in T_{\mathbb{C}}^*.$$

Here on the right side,  $\langle x | \eta \rangle$  is the natural pairing between  $T_{\mathbb{C}}$  and  $T_{\mathbb{C}}^*$ .

Thus we have the Clifford algebra  $\mathcal{C}(T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*)$ , which acts on the spinor bundle  $\mathcal{M} \triangleq \bigoplus_{i=0}^{2n} \wedge^i T_{\mathbb{C}}^*$  via  $(x + \eta) \cdot \rho = \iota_x \rho + \eta \wedge \rho$  for all  $\rho \in \mathcal{M}$ .

Introduce a  $\mathbb{C}$ -linear map  $(\cdot)^T : \mathcal{M} \rightarrow \mathcal{M}$  by  $(\eta_1 \wedge \cdots \wedge \eta_j)^T = \eta_j \wedge \cdots \wedge \eta_1$ . The Mukai pairing  $(\cdot, \cdot) : \mathcal{M} \times \mathcal{M} \rightarrow \wedge^{2n}(T_{\mathbb{C}}^*)$  is defined by  $(\chi, \omega) = [\chi^T \wedge \omega]^{2n}$ , where  $[\ ]^{2n}$  indicates the top degree component of the product. Explicitly, if  $\chi = \sum_{i=0}^{2n} \chi_i$  and  $\omega = \sum_{i=0}^{2n} \omega_i$ , where  $\chi_i, \omega_i \in \wedge^i T_{\mathbb{C}}^*$ , then

$$(\chi, \omega) = \sum_{i=0}^{2n} (-1)^{i(i-1)/2} \chi_i \wedge \omega_{2n-i}.$$

For all  $\chi, \omega \in \mathcal{M}$  and  $\phi \in \wedge^2 T_{\mathbb{C}}^*$ , these properties are standard [Gualtieri 2003]:

$$(3-2) \quad (\chi, \omega) = (-1)^n (\omega, \chi),$$

$$(3-3) \quad (\phi \wedge \chi, \omega) + (\chi, \phi \wedge \omega) = 0.$$

Consider a real, closed 3-form  $H \in Z^3(M)$  and the twisted differential operator  $d^H = d + H \wedge (\cdot)$  it induces.

**Definition 3.1** [Gualtieri 2003; Cavalcanti 2006]. A twisted generalized complex structure with respect to  $H$  is determined by any of the following three equivalent objects:

- (i) A real automorphism  $\mathbb{J}$  of  $T \oplus T^*$  that squares to  $-1$ , is orthogonal with respect to the natural pairing (3-1), and has vanishing Nijenhuis tensor, meaning that for all  $z_1, z_2 \in \Gamma(T \oplus T^*)$ ,

$$N(z_1, z_2) \triangleq -[\mathbb{J}z_1, \mathbb{J}z_2]_H + \mathbb{J}[[\mathbb{J}z_1, z_2]]_H + \mathbb{J}[[z_1, \mathbb{J}z_2]]_H + [[z_1, z_2]]_H = 0.$$

Here  $[[\cdot, \cdot]]_H$  is the twisted Courant bracket defined in (1-9).

- (ii) A Dirac structure  $L \subset T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*$  that is twisted with respect to  $H$  and satisfies  $L \cap \bar{L} = \{0\}$ .
- (iii) A line subbundle  $N$  of  $\mathcal{M} = \wedge^\bullet(T_{\mathbb{C}}^*)$  generated at each point by a form  $u$ , such that  $L = \{X \in T_{\mathbb{C}} \oplus T_{\mathbb{C}}^* \mid X \cdot u = 0\}$  is maximally isotropic,  $(u, \bar{u}) \neq 0$ , and  $d^H u = e \cdot u$  for some  $e \in \Gamma(T \oplus T^*)$ .

The line bundle in (iii) is called the pure spinor line bundle corresponding to  $L$ .

**Remark 3.2.** See also [Alekseev and Xu 2001] for the relation between Dirac structures and Dirac generating operators.

To generalize the usual  $\partial$  and  $\bar{\partial}$  operators in complex geometry, Gualtieri [2003] introduced  $\partial$  and  $\bar{\partial}$  operators for any twisted generalized complex structure. We recall its construction briefly below.

Let  $\mathbb{J}$  be a twisted generalized complex structure, and let  $L \subset T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*$  be its  $+i$  eigenspace.  $L$  is a twisted Dirac structure and satisfies  $L \cap \bar{L} = \{0\}$ . We will regard  $\bar{L} = L^*$  by defining the canonical pairing between  $L$  and  $\bar{L}$  by

$$(3-4) \quad \langle X | \theta \rangle = 2\langle X, \theta \rangle \quad \text{for all } X \in L \text{ and } \theta \in \bar{L}.$$

The coefficient 2 is due to the natural pairing; see Equation (3-1).

Set  $N_0 = N$  and  $N_k = \wedge^k \bar{L} \cdot N$  for  $k = 1, \dots, 2n$ . Then  $\bar{N}_k = N_{2n-k}$  and specifically  $N_{2n} = \bar{N}$  is the pure spinor of  $\bar{L}$ . We have a decomposition

$$\mathcal{M} = N_0 \oplus N_1 \oplus \dots \oplus N_{2n}.$$

Gualtieri [2003] proves that one can decompose  $d^H = \partial + \bar{\partial}$ . Here

$$\partial : \Gamma(N_{\bullet}) \rightarrow \Gamma(N_{\bullet-1}), \quad \bar{\partial} : \Gamma(N_{\bullet}) \rightarrow \Gamma(N_{\bullet+1})$$

(or  $\Gamma(\bar{N}_{\bullet}) \rightarrow \Gamma(\bar{N}_{\bullet-1})$ ) are defined by

$$\partial(n_k) \triangleq pr_{N_{k-1}}(d^H n_k), \quad \bar{\partial}(n_k) \triangleq pr_{N_{k+1}}(d^H n_k) \quad \text{for all } n_k \in \Gamma(N_k).$$

Using the identification  $N_k = (\wedge^k \bar{L}) \otimes N$  (see (4-5)), Gualtieri observed that  $\bar{\partial}$  is determined by the rule

$$\bar{\partial}(W \otimes s) = (d_L W) \otimes s + (-1)^k W \otimes d^H s,$$

where  $W \in \Gamma(\wedge^k \bar{L})$  and  $s$  is a local nonvanishing section of  $N$ . It is clear from this description that  $N$  must be a module of the Lie algebroid  $L$ . Moreover, under the additional assumption that  $N$  admits a global nowhere vanishing section  $s$ , Gualtieri showed that  $d^H s = \bar{\partial} s = e \cdot s$ , where  $e \in C^\infty(\bar{L})$ , and  $e$  actually defines a cohomology class in the Lie algebroid cohomology  $H^1(L)$ .

#### 4. The main theorem

Given a generalized complex structure  $\mathbb{J}$  as above, it is clear that  $(L, \bar{L})$  is a Lie bialgebroid (regarding  $L^* = \bar{L}$  by (3-4)). We prove in Theorem 4.3 that the operators  $\check{d}_*$  and  $\check{\partial}$  for this particular situation are essentially  $\partial$  and  $\bar{\partial}$ .

We continue the notations in Section 3. The following lemma is easy.

**Lemma 4.1.** *For all  $W \in \wedge^j \bar{L}$  and  $X \in \wedge^i L$ , and for  $i \leq j \leq 2n$ , one has*

$$(4-1) \quad X \cdot W \cdot u = (-1)^{i(i-1)/2} (\iota_X W) \cdot u.$$

Here  $\iota$  denotes the generalized interior product defined by

$$(4-2) \quad \langle \iota_X W | Y \rangle = \langle W | X \wedge Y \rangle \quad \text{for all } Y \in \wedge^{j-i} L.$$

Let  $u$  be a nowhere vanishing local section of  $N$ . Assume that  $V \in \Gamma(\wedge^{2n} L)$  satisfies  $V \cdot \bar{u} = u$ . Hence  $\bar{V} \cdot u = \bar{u}$  and by (4-1),

$$(4-3) \quad \begin{aligned} \langle V | \bar{V} \rangle u &= (\iota_V \bar{V}) u = (-1)^n V \cdot \bar{V} \cdot u = (-1)^n u, \\ \langle V | \bar{V} \rangle &= (-1)^n. \end{aligned}$$

Here  $\langle V | \bar{V} \rangle$  is the natural pairing. Therefore the dual section of  $V$  is given by  $\Omega = (-1)^n \bar{V} \in \Gamma(\wedge^{2n} \bar{L})$ .

**Proposition 4.2** [Gualtieri 2003, Proposition 2.22]; also [Chevalley 1997, III.3.2].  
*The line bundle  $\mathcal{L} = (\wedge^{2n} \bar{L} \otimes \wedge^{2n} T_{\mathbb{C}}^*)^{1/2}$  and  $N_{2n} = \bar{N}$  are canonically isomorphic. The isomorphism can be explicitly described by*

$$(4-4) \quad \begin{aligned} \bar{N} \otimes \bar{N} &\rightarrow \mathcal{L}^2 = \wedge^{2n} \bar{L} \otimes \wedge^{2n} T_{\mathbb{C}}^*, \\ \omega_1 \otimes \omega_2 &\mapsto \Omega \otimes (V \cdot \omega_1, \omega_2). \end{aligned}$$

*This isomorphism does not depend on the choice of  $u$  and  $V$ .*

From now on we will identify  $\bar{N}$  with  $(\wedge^{2n} \bar{L} \otimes \wedge^{2n} T_{\mathbb{C}}^*)^{1/2}$ . As a consequence of the  $\bar{L}$ -module structure on the latter, we have two differential operators (see Equations (2-2) and (2-7))

$$\check{d}_* : (\wedge^{\bullet} L) \otimes \bar{N} \rightarrow (\wedge^{\bullet+1} L) \otimes \bar{N} \quad \text{and} \quad \check{\partial} : (\wedge^{\bullet} L) \otimes \bar{N} \rightarrow (\wedge^{\bullet-1} L) \otimes \bar{N}.$$

It is also shown in [Gualtieri 2003] that  $(\wedge^k \bar{L}) \otimes N \cong N_k$  and  $(\wedge^k L) \otimes \bar{N} \cong \bar{N}_k$  respectively by the two isomorphisms

$$(4-5) \quad I : (\wedge^k \bar{L}) \otimes N \rightarrow N_k, \quad W \otimes p \mapsto W \cdot p \quad \text{for all } W \in \wedge^k \bar{L}, p \in N,$$

$$(4-6) \quad \bar{I} : (\wedge^k L) \otimes \bar{N} \rightarrow \bar{N}_k, \quad X \otimes \bar{p} \mapsto X \cdot \bar{p} \quad \text{for all } X \in \wedge^k L, \bar{p} \in \bar{N}.$$

We now give our main theorem, whose proof is in Section 6.

**Theorem 4.3.** *The following two diagrams are commutative.*

$$(4-7) \quad \begin{array}{ccc} (\wedge^k L) \otimes \bar{N} & \xrightarrow{\check{d}_*} & (\wedge^{k+1} L) \otimes \bar{N} \\ \downarrow I & & \downarrow I \\ \bar{N}_k & \xrightarrow{\partial} & \bar{N}_{k+1} \end{array}$$

$$(4-8) \quad \begin{array}{ccc} (\wedge^k L) \otimes \bar{N} & \xrightarrow{\bar{\partial}} & (\wedge^{k-1} L) \otimes \bar{N} \\ \bar{i} \downarrow & & \downarrow \bar{i} \\ \bar{N}_k & \xrightarrow{\bar{\partial}} & \bar{N}_{k-1} \end{array}$$

In retrospect, the existence of such a result has likely been suggested by earlier works of Gualtieri. In [2007], he constructed an  $L$ -module structure on  $N$  and an  $\bar{L}$ -module structure on  $\bar{N}$ , respectively by

$$(4-9) \quad \nabla_X p \triangleq X \cdot d^H p = X \cdot \bar{\partial} p,$$

$$(4-10) \quad \nabla_W \bar{p} \triangleq W \cdot d^H \bar{p} = W \cdot \partial \bar{p},$$

for all  $p \in \Gamma(N)$ ,  $X \in \Gamma(L)$  and  $W \in \Gamma(\bar{L})$ .

Here is a special situation of  $k = 0$  in diagram (4-7):

**Proposition 4.4.** *The above  $\bar{L}$ -module structure defined by (4-10) coincides with the  $\bar{L}$ -module structure defined by (2-1), under the isomorphism (4-4).*

## 5. Modular cocycles of Lie algebroids

In this section we establish a list of important identities valid in any Lie bialgebroid  $(A, A^*)$  and generalized complex structure; we use these in Section 6 to prove the statements of Section 4.

We continue the setup of Section 2: Let  $(A, [\cdot, \cdot], a)$  and  $(A^*, [\cdot, \cdot]_*, a_*)$  be a pair of rank- $r$  Lie algebroids over dimension- $m$  base manifold  $M$ .

Assume there exists a volume form  $s \in \Gamma(\wedge^m T_{\mathbb{C}}^* M)$  and a nowhere vanishing section  $\Omega \in \Gamma(\wedge^r A^*)$ , so that  $\mathcal{L}$  is the trivial line bundle over  $M$ . Let  $V \in \Gamma(\wedge^r A)$  be the section dual to  $\Omega$ , that is,  $\langle \Omega | V \rangle = 1$ . These induce bundle isomorphisms

$$(5-1) \quad \Omega^\sharp : \wedge^k A \rightarrow \wedge^{r-k} A^* : X \mapsto \iota_X \Omega,$$

$$(5-2) \quad V^\sharp : \wedge^k A^* \rightarrow \wedge^{r-k} A : \xi \mapsto \iota_\xi V.$$

Here we adopt similar conventions as that of Equation (4-2). These two operations are essentially inverse to each other:

$$(5-3) \quad (V^\sharp \circ \Omega^\sharp)(X) = (-1)^{k(r-1)} X \quad \text{for all } X \in \wedge^k A;$$

$$(5-4) \quad (\Omega^\sharp \circ V^\sharp)(\varphi) = (-1)^{k(r-1)} \varphi \quad \text{for all } \varphi \in \wedge^k A^*.$$

Consider the operator  $\partial$  dual to  $d$  with respect to  $V^\sharp$ , as defined by the diagram

$$(5-5) \quad \begin{array}{ccc} \Gamma(\wedge^k A^*) & \xrightarrow{V^\sharp} & \Gamma(\wedge^{r-k} A) \\ (-1)^k d \downarrow & & \downarrow \partial \\ \Gamma(\wedge^{k+1} A^*) & \xrightarrow{V^\sharp} & \Gamma(\wedge^{r-k-1} A), \end{array}$$

or by the relation

$$(5-6) \quad -V^\sharp d\alpha = (-1)^k \partial V^\sharp \alpha \quad \text{for all } \alpha \in \Gamma(\wedge^k A^*),$$

which, due to (5-3) and (5-4), can be rewritten as

$$(5-7) \quad \Omega^\sharp \partial \beta = (-1)^l d \Omega^\sharp \beta \quad \text{for all } \beta \in \Gamma(\wedge^l A).$$

The operator  $\partial$  is a Batalin–Vilkovisky operator for the Lie algebroid  $A$ ; see [Kosmann-Schwarzbach 2000; Koszul 1985; Xu 1999; Michéa and Novitchkov 2005]. Similarly, we have the operator  $\partial_*$  dual to  $d_*$ , as defined by

$$\begin{array}{ccc} \Gamma(\wedge^{r-k} A) & \xleftarrow{V^\sharp} & \Gamma(\wedge^k A^*) \\ (-1)^k d_* \downarrow & & \downarrow \partial_* \\ \Gamma(\wedge^{r-k+1} A) & \xleftarrow{V^\sharp} & \Gamma(\wedge^{k-1} A^*), \end{array}$$

or by the relation

$$d_* V^\sharp \alpha = (-1)^k V^\sharp \partial_* \alpha \quad \text{for all } \alpha \in \Gamma(\wedge^k A^*).$$

According to [Evens et al. 1999], there exists a unique  $X_0 \in \Gamma(A)$  such that

$$(5-8) \quad \mathcal{L}_\theta(\Omega \otimes s) = (\mathcal{L}_\theta \Omega) \otimes s + \Omega \otimes (\mathcal{L}_{a_*(\theta)} s) = \langle X_0 | \theta \rangle \Omega \otimes s \quad \text{for all } \theta \in \Gamma(A^*).$$

Similarly, there exists a unique  $\zeta_0 \in \Gamma(A^*)$  such that

$$(5-9) \quad \mathcal{L}_X(s \otimes V) = (\mathcal{L}_{a(X)} s) \otimes V + s \otimes (\mathcal{L}_X V) = \langle \zeta_0 | X \rangle s \otimes V \quad \text{for all } X \in \Gamma(A).$$

These sections  $X_0$  and  $\zeta_0$  are called *modular cocycles*, and their cohomology classes are called modular classes [Evens et al. 1999].

A simple computation yields the following formulas, which are also given in [Alekssev and Xu 2001].

**Proposition 5.1.** *With the above notations, the differential operators defined by Equations (2-2) and (2-7) are given respectively by*

$$(5-10) \quad \check{d}_*(X \otimes l) = (d_* X + \frac{1}{2} X_0 \wedge X) \otimes l$$

and

$$(5-11) \quad \check{\partial}(X \otimes l) = (-\partial X + \frac{1}{2}\iota_{\xi_0} X) \otimes l$$

for all  $X \in \Gamma(\wedge A)$  and  $l \in \Gamma(\mathcal{L})$ .

Hence the operator  $\check{D}$  in Theorem 2.5 reads

$$\check{D} = \check{d}_* + \check{\partial} = d_* - \partial + \frac{1}{2}(X_0 \wedge \cdot + \iota_{\xi_0}).$$

This construction of Dirac generating operators using modular cocycles appeared in [Alekseev and Xu 2001] and [Chen and Stiénon 2009].

Now we consider a twisted generalized complex structure  $\mathbb{J}$  on a  $2n$ -dimensional manifold  $M$  and let  $L$  and  $\bar{L}$  be respectively the  $+i$  and  $-i$  eigenspace of  $\mathbb{J}$ . Again we assume that  $u$  is a nowhere vanishing local section of  $N$ , the pure spinor bundle of  $L$ .

**Lemma 5.2** [Gualtieri 2007]. *There exists some  $e = x + \eta \in \Gamma(\bar{L})$  such that*

$$(5-12) \quad d^H u = \bar{\partial}u = du + H \wedge u = e \cdot u = \iota_x u + \eta \wedge u,$$

$$(5-13) \quad d^H \bar{u} = \partial \bar{u} = d\bar{u} + H \wedge \bar{u} = \bar{e} \cdot \bar{u} = \iota_{\bar{x}} \bar{u} + \bar{\eta} \wedge \bar{u}.$$

The main result in this section is the following.

**Proposition 5.3.** *Let  $V \in \Gamma(\wedge^{2n} L)$  such that  $V \cdot \bar{u} = u$ . Then the modular cocycle of  $L$  with respect to the top form  $V$  and the volume form  $s = (u, \bar{u})$  is  $2e$ , where  $e \in \Gamma(\bar{L})$  is given by Lemma 5.2.*

*Similarly, the modular cocycle of  $\bar{L}$  with respect to  $\Omega = (-1)^n \bar{V} \in \Gamma(\wedge^{2n} \bar{L})$  and  $s$  is  $2\bar{e}$ .*

Before the proof, we need a couple of identities and lemmas. Since  $L$  is a Lie algebroid and  $L^* = \bar{L}$ , we have the differential

$$d_L : \Gamma(\wedge^\bullet \bar{L}) \rightarrow \Gamma(\wedge^{\bullet+1} \bar{L}).$$

We also have the equality

$$(5-14) \quad \begin{aligned} \bar{\partial}(W \cdot u) &= (d_L W) \cdot u + (-1)^k W \cdot \bar{\partial}u \\ &= (d_L W) \cdot u + (-1)^k (W \wedge e) \cdot u \\ &= (d_L W + e \wedge W) \cdot u \quad \text{for all } W \in \Gamma(\wedge^k \bar{L}), \end{aligned}$$

which encodes the  $L$ -module structure on  $N$  defined by Equation (4-9).

Similarly, one has

$$(5-15) \quad \begin{aligned} \partial(X \cdot \bar{u}) &= (d_{\bar{L}} X) \cdot \bar{u} + (-1)^i X \cdot \partial \bar{u} \\ &= (d_{\bar{L}} X + \bar{e} \wedge X) \cdot \bar{u} \quad \text{for all } X \in \Gamma(\wedge^i L). \end{aligned}$$

**Lemma 5.4.** *For any  $X = a + \zeta \in \Gamma(L)$ , we have*

$$(5-16) \quad (\mathcal{L}_X \bar{V}) \cdot u = \bar{\partial}(X \cdot \bar{u}) - \langle e | X \rangle \bar{u},$$

$$(5-17) \quad \mathcal{L}_a u = \langle e | X \rangle u - (d\zeta + \iota_a H) \wedge u,$$

$$(5-18) \quad \mathcal{L}_a \bar{u} = \bar{\partial}(X \cdot \bar{u}) + (d_{\bar{L}} X) \cdot \bar{u} - (d\zeta + \iota_a H) \wedge \bar{u}.$$

*Proof.* A basic fact is that

$$(5-19) \quad 0 = X \cdot u = \iota_a u + \zeta \wedge u$$

for any  $X = a + \zeta \in \Gamma(L)$ . Hence

$$\begin{aligned} \bar{\partial}(X \cdot \bar{u}) &= \bar{\partial}(X \cdot \bar{V} \cdot u) = \bar{\partial}((\iota_X \bar{V}) \cdot u) \\ &= (d_L \iota_X \bar{V}) \cdot u - (\iota_X \bar{V} \wedge e) \cdot u \quad (\text{by (5-14)}) \\ &= (d_L \iota_X \bar{V} + \iota_X d_L \bar{V}) \cdot u + (\langle e | X \rangle \bar{V}) \cdot u \\ &= (\mathcal{L}_X \bar{V}) \cdot u + \langle e | X \rangle \bar{u}. \end{aligned}$$

This proves (5-16). For (5-17), we have

$$\begin{aligned} \mathcal{L}_a u &= \iota_a d u + d \iota_a u \\ &= \iota_a (\iota_x u + \eta \wedge u - H \wedge u) - d(\zeta \wedge u) \quad (\text{by (5-12) and (5-19)}) \\ &= -\iota_x \iota_a u + \langle a | \eta \rangle u - \eta \wedge \iota_a u - \iota_a H \wedge u + H \wedge \iota_a u - d\zeta \wedge u + \zeta \wedge d u \\ &= \iota_x (\zeta \wedge u) + \langle a | \eta \rangle u + (\eta - H) \wedge (\zeta \wedge u) \\ &\quad - \iota_a H \wedge u - d\zeta \wedge u + \zeta \wedge (\iota_x u + \eta \wedge u - H \wedge u) \quad (\text{by (5-12) and (5-14)}) \\ &= (\langle x | \zeta \rangle + \langle a | \eta \rangle) u - \iota_a H \wedge u - d\zeta \wedge u. \end{aligned}$$

To prove (5-18), we observe that, on one hand

$$\begin{aligned} d^H(X \cdot \bar{u}) &= \bar{\partial}(X \cdot \bar{u}) + \partial(X \cdot \bar{u}) \\ &= \bar{\partial}(X \cdot \bar{u}) + (d_{\bar{L}} X) \cdot \bar{u} - (X \wedge \bar{e}) \cdot \bar{u} \quad (\text{by (5-15)}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d^H(X \cdot \bar{u}) &= d(\iota_a \bar{u} + \zeta \wedge \bar{u}) + H \wedge (X \cdot \bar{u}) \\ &= d \iota_a \bar{u} + d\zeta \wedge \bar{u} - \zeta \wedge d\bar{u} + H \wedge (X \cdot \bar{u}) \\ &= (d \iota_a \bar{u} + \iota_a d\bar{u}) + d\zeta \wedge \bar{u} - (\iota_a + \zeta \wedge) d\bar{u} + H \wedge (X \cdot \bar{u}) \\ &= \mathcal{L}_a \bar{u} + d\zeta \wedge \bar{u} - X \cdot (\bar{e} \cdot \bar{u} - H \wedge \bar{u}) + H \wedge (X \cdot \bar{u}) \quad (\text{by (5-13)}) \\ &= \mathcal{L}_a \bar{u} + d\zeta \wedge \bar{u} - (X \wedge \bar{e}) \cdot \bar{u} + X \cdot (H \wedge \bar{u}) + H \wedge (X \cdot \bar{u}) \\ &= \mathcal{L}_a \bar{u} + d\zeta \wedge \bar{u} - (X \wedge \bar{e}) \cdot \bar{u} + (\iota_a + \zeta \wedge)(H \wedge \bar{u}) + H \wedge (\iota_a \bar{u} + \zeta \wedge \bar{u}) \\ &= \mathcal{L}_a \bar{u} + d\zeta \wedge \bar{u} - (X \wedge \bar{e}) \cdot \bar{u} + (\iota_a H) \wedge \bar{u}. \quad \square \end{aligned}$$

**Lemma 5.5** [Cavalcanti 2006]. *The Mukai pairing vanishes in  $N_i \times N_k$ , unless  $i + k = 2n$ , in which case it is nondegenerate.*

*Proof of Proposition 5.3.* For an  $X = a + \zeta \in \Gamma(L)$ , we assume that  $\bar{\partial}(X \cdot \bar{u}) = f\bar{u}$  for some function  $f \in C^\infty(M, \mathbb{C})$ . Then (5-16) implies that  $\mathcal{L}_X \bar{V} = (f - \langle e | X \rangle) \bar{V}$ . Since the pairing between  $V$  and  $\bar{V}$  is a constant (see (4-3)),

$$(5-20) \quad \mathcal{L}_X V = (\langle e | X \rangle - f)V.$$

According to (5-17) and (5-18), we also have

$$\begin{aligned} \mathcal{L}_{\rho_L(X)} s &= \mathcal{L}_a(u, \bar{u}) = (\mathcal{L}_a u, \bar{u}) + (u, \mathcal{L}_a \bar{u}) \\ &= (\langle e | X \rangle u - (d\zeta + \iota_a H) \wedge u, \bar{u}) + (u, f\bar{u} + (d_{\bar{L}} X) \cdot \bar{u} - (d\zeta + \iota_a H) \wedge \bar{u}) \\ &= (\langle e | X \rangle + f)(u, \bar{u}) = (\langle e | X \rangle + f)s. \end{aligned}$$

In the last step, we have applied (3-3) and Lemma 5.5. In turn, we get

$$\mathcal{L}_X V \otimes s + V \otimes \mathcal{L}_{\rho_L(X)} s = 2\langle e | X \rangle V \otimes s.$$

This proves the first claim. By symmetry, we also have

$$\mathcal{L}_W \bar{V} \otimes \bar{s} + \bar{V} \otimes \mathcal{L}_{\rho_{\bar{L}}(W)} \bar{s} = 2\langle \bar{e} | W \rangle \bar{V} \otimes \bar{s}$$

for all  $W \in \Gamma(\bar{L})$ . By (3-2), we know  $\bar{s} = (\bar{u}, u) = (-1)^n(u, \bar{u}) = (-1)^n s$ . Thus

$$\mathcal{L}_W \Omega \otimes s + \Omega \otimes \mathcal{L}_{\rho_{\bar{L}}(W)} s = 2\langle \bar{e} | W \rangle \Omega \otimes s,$$

which implies that  $2\bar{e}$  is the modular cocycle of  $\bar{L}$  with respect to  $\Omega$  and  $s$ .  $\square$

## 6. Proof of the main theorem

*Proof of Proposition 4.4.*  $\bar{N}$  has an induced  $\bar{L}$ -module structure arising from the  $\bar{L}$ -module  $\mathcal{L}$ ; see (2-1). According to the second statement of Proposition 5.3, we know that this module structure is determined by the equation

$$\mathcal{L}_W \bar{u} = \langle \bar{e} | W \rangle \bar{u} \quad \text{for all } W \in \Gamma(\bar{L}).$$

This coincides with the standard  $\bar{L}$ -module structure defined by (4-10) because

$$W \cdot \partial \bar{u} = W \cdot \bar{e} \cdot \bar{u} = \langle \bar{e} | W \rangle \bar{u},$$

by Lemma 4.1.  $\square$

*Proof of Theorem 4.3.* By Proposition 5.3 and Equations (5-10) and (5-11), we conclude that

$$\begin{aligned} \check{d}_*(X \otimes \bar{u}) &= (d_{\bar{L}} X + \bar{e} \wedge X) \otimes \bar{u}, \\ \check{\partial}(X \otimes \bar{u}) &= (-\partial X + \iota_e X) \otimes \bar{u} \end{aligned}$$

for all  $X \in \Gamma(\bigwedge^i L)$ .

Comparing with the expression for  $\partial$  in (5-15), we immediately know that diagram (4-7) is commutative. To prove the commutativity of diagram (4-8), it suffices to prove that

$$(6-1) \quad \bar{\partial}(X \cdot \bar{u}) = (-\partial X + \iota_e X) \cdot \bar{u} \quad \text{for all } X \in \Gamma(\wedge^i L).$$

In fact, by (5-14),

$$\begin{aligned} \text{left side of (6-1)} &= \bar{\partial}(X \cdot \bar{V} \cdot u) = (-1)^{i(i-1)/2} \bar{\partial}((\iota_X \bar{V}) \cdot u) \\ &= (-1)^{i(i-1)/2} (d_L \iota_X \bar{V} + e \wedge \iota_X \bar{V}) \cdot u. \end{aligned}$$

We also have

$$(\iota_e X) \cdot \bar{u} = e \cdot X \cdot \bar{V} \cdot u = (-1)^{i(i-1)/2} e \cdot (\iota_X \bar{V}) \cdot u = (-1)^{i(i-1)/2} (e \wedge \iota_X \bar{V}) \cdot u.$$

By (5-3) and (5-7),

$$\begin{aligned} \partial X &= (-1)^{(i-1)(2n-1)} V^\# \Omega^\# \partial X \\ &= (-1)^{(i-1)(2n-1)+i} V^\# d_L \Omega^\# X = -V^\# d_L \Omega^\# X. \end{aligned}$$

Hence

$$\begin{aligned} -(\partial X) \cdot \bar{u} &= (V^\# d_L \Omega^\# X) \cdot \bar{u} \\ &= (-1)^{(2n-i+1)(2n-i)/2} (d_L \iota_X \Omega) \cdot V \cdot \bar{u} = (-1)^{i(i-1)/2} (d_L \iota_X \bar{V}) \cdot u. \end{aligned}$$

This proves (6-1), and the proof of Theorem 4.3 is thus completed.  $\square$

## 7. Some corollaries

The first obvious result is that, by the isomorphisms

$$\wedge^k L \otimes \mathcal{L} \cong (\wedge^k L) \cdot \bar{N} = \bar{N}_k = N_{2n-k},$$

the Dirac generating operator constructed by Theorem 2.5 for  $E = L \oplus \bar{L}$  is exactly

$$\check{D} = \check{d}_* + \check{d} = \partial + \bar{\partial} = d^H,$$

and specifically  $\check{f} = \check{D}^2 = 0$ .

In Section 5, we defined for any Lie bialgebroid  $(A, A^*)$  a pair of operators  $d_*$  and  $\partial$  on  $\Gamma(\wedge A)$ , and similarly  $d$  and  $\partial_*$  on  $\Gamma(\wedge A^*)$ . Let  $D = d_* + \partial$  and  $D_* = d + \partial_*$ . Their squares yield the pair of Laplacian operators

$$(7-1) \quad \Delta = D^2 = d_* \partial + \partial d_* : \Gamma(\wedge^k A) \rightarrow \Gamma(\wedge^k A),$$

$$(7-2) \quad \Delta_* = D_*^2 = d \partial_* + \partial_* d : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^k A^*).$$

See [Alekseev and Xu 2001].

**Theorem 7.1** [Chen and Stiénon 2009, Theorem 3.4]. *If  $(A, A^*)$  is a Lie bialgebroid, then*

$$\begin{aligned}\Delta &= \frac{1}{2}(\mathcal{L}_{X_0} + \mathcal{L}_{\xi_0}) : \Gamma(\wedge A) \rightarrow \Gamma(\wedge A), \\ \Delta_* &= \frac{1}{2}(\mathcal{L}_{X_0} + \mathcal{L}_{\xi_0}) : \Gamma(\wedge A^*) \rightarrow \Gamma(\wedge A^*),\end{aligned}$$

where  $X_0$  and  $\xi_0$  are modular cocycles defined by (5-8) and (5-9).

As an immediate corollary, we have:

**Corollary 7.2.** *Let  $(L, \bar{L})$  be the Lie bialgebroid coming from a twisted generalized complex structure  $\mathbb{J}$ . The Laplacian operators  $\Delta$  and  $\Delta_*$  defined by (7-1) and (7-2) are given by*

$$\begin{aligned}\Delta f &= \Delta_* f = \frac{1}{2} pr_T(e + \bar{e})(f) && \text{for all } f \in C^\infty(M, \mathbb{C}), \\ \Delta X &= (\bar{e} + \frac{1}{2}e) \circ_H X - \frac{i}{2}\mathbb{J}(e \circ_H X) && \text{for all } X \in \Gamma(L), \\ \Delta_* W &= (e + \frac{1}{2}\bar{e}) \circ_H W + \frac{i}{2}\mathbb{J}(\bar{e} \circ_H W) && \text{for all } W \in \Gamma(\bar{L}).\end{aligned}$$

*Proof.* For  $e \in \Gamma(\bar{L})$  and  $\bar{e} \in \Gamma(L)$ , the Lie derivations  $\mathcal{L}_e$  and  $\mathcal{L}_{\bar{e}}$  on  $\Gamma(L)$  are given by

$$\mathcal{L}_e X = pr_L(e \circ_H X) \quad \text{and} \quad \mathcal{L}_{\bar{e}} = \bar{e} \circ_H X \quad \text{for all } X \in \Gamma(L).$$

The projections of  $T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*$  to  $L$  and  $\bar{L}$  are given respectively by

$$pr_L(z) = \frac{1}{2}(z - i\mathbb{J}z) \quad \text{and} \quad pr_{\bar{L}}(z) = \frac{1}{2}(z + i\mathbb{J}z) \quad \text{for all } z \in T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*.$$

The claim then follows directly from Theorem 7.1.  $\square$

It is well known [Mackenzie and Xu 1994] that, if  $a$  and  $a_*$  denote the anchor maps of a Lie bialgebroid  $(A, A^*)$ , the bundle map

$$(7-3) \quad \pi^\# = a \circ (a_*)^* : T_{\mathbb{C}}^* M \rightarrow T_{\mathbb{C}} M$$

defines a (complex) Poisson structure on  $M$ .

In particular, for the Lie bialgebroid  $(L, \bar{L})$  coming from the twisted generalized complex structure  $\mathbb{J}$ , the map  $-i\pi$  is a real Poisson structure. In fact, up to a factor of 2, it is given by [Barton and Stiénon 2008; Gualtieri 2007]

$$(7-4) \quad P(\zeta, \eta) = \langle \mathbb{J}\zeta, \eta \rangle.$$

Let us briefly recall the definition of the modular vector field of a Poisson manifold  $(M, \pi)$  from [Weinstein 1997]. Let  $\omega \in \Omega^{\text{top}}(M)$  be a volume form. The modular vector field with respect to  $\omega$  is the derivation  $X_\omega$  of the algebra of functions  $C^\infty(M)$  characterized by

$$(7-5) \quad \mathcal{L}_{\pi^\#(df)}\omega = X_\omega(f)\omega.$$

For the Poisson structure induced from a Lie bialgebroid, the relation between modular cocycles and modular vector fields is as follows.

**Lemma 7.3** [Chen and Stiénon 2009, Corollary 3.8]. *Suppose  $M$  is an orientable manifold with volume form  $s \in \Omega^{\text{top}}(M)$ , and let  $(A, A^*)$  be a real Lie bialgebroid over  $M$  with associated Poisson bivector  $\pi$  defined by (7-3). Then the modular vector field of the Poisson manifold  $(M, \pi)$  with respect to  $s$  is*

$$(7-6) \quad X_s = \frac{1}{2}(a_*(\zeta_0) - a(X_0)),$$

where  $\zeta_0$  and  $X_0$  are modular cocycles defined by (5-8) and (5-9) (choosing arbitrary  $V$  and  $\Omega$ ).

As an immediate consequence, we obtain [Gualtieri 2007, Proposition 3.27].

**Corollary 7.4.** *The modular vector field of the Poisson structure  $P$  defined in (7-4) is given by  $\frac{i}{2} pr_{T_{\mathbb{C}}}(e - \bar{e})$  with respect to the volume form  $s = (u, \bar{u})$ .*

### Acknowledgments

I thank Camille Laurent-Gengoux, Yi Lin and Ping Xu for useful discussions and comments.

### References

- [Alekseev and Xu 2001] A. Alekseev and P. Xu, “Derived brackets and Courant algebroids”, unpublished manuscript, 2001, Available at <http://www.math.psu.edu/ping/anton-final.pdf>.
- [Barton and Stiénon 2008] J. Barton and M. Stiénon, “Generalized complex submanifolds”, *Pacific J. Math.* **236**:1 (2008), 23–44. MR 2398985 Zbl 1155.53330
- [Cavalcanti 2006] G. R. Cavalcanti, “The decomposition of forms and cohomology of generalized complex manifolds”, *J. Geom. Phys.* **57**:1 (2006), 121–132. MR 2008k:53091 Zbl 1106.57019
- [Cavalcanti 2007] G. Cavalcanti, *Introduction to generalized complex geometry*, Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2007. MR 2008k:53180 Zbl 1144.53090
- [Chen and Stiénon 2009] Z. Chen and M. Stiénon, “Dirac generating operators and Manin triples”, *J. London Math. Soc.* **79**:2 (2009), 399–421.
- [Chevalley 1997] C. Chevalley, *Collected works, II: The algebraic theory of spinors and Clifford algebras*, edited by P. Cartier and C. Chevalley, Springer, Berlin, 1997. MR 99f:01028 Zbl 0899.01032
- [Evens et al. 1999] S. Evens, J.-H. Lu, and A. Weinstein, “Transverse measures, the modular class and a cohomology pairing for Lie algebroids”, *Quart. J. Math. Oxford Ser. (2)* **50**:200 (1999), 417–436. MR 2000i:53114 Zbl 0968.58014
- [Gualtieri 2003] M. Gualtieri, *Generalized complex geometry*, thesis, Oxford University, 2003. arXiv math/0401221
- [Gualtieri 2007] M. Gualtieri, “Generalized complex geometry”, preprint, 2007. arXiv math/0703298v2
- [Hitchin 2003] N. Hitchin, “Generalized Calabi–Yau manifolds”, *Q. J. Math.* **54**:3 (2003), 281–308. MR 2004h:32024 Zbl 1076.32019

- [Kosmann-Schwarzbach 1995] Y. Kosmann-Schwarzbach, “Exact Gerstenhaber algebras and Lie bialgebroids”, *Acta Appl. Math.* **41**:1-3 (1995), 153–165. MR 97i:17021 Zbl 0837.17014
- [Kosmann-Schwarzbach 2000] Y. Kosmann-Schwarzbach, “Modular vector fields and Batalin–Vilkovisky algebras”, pp. 109–129 in *Poisson geometry* (Warsaw, 1998), edited by J. Grabowski and P. Urbański, Banach Center Publ. **51**, Polish Acad. Sci., Warsaw, 2000. MR 2001d:17026 Zbl 1018.17020
- [Kosmann-Schwarzbach 2005] Y. Kosmann-Schwarzbach, “Quasi, twisted, and all that ... in Poisson geometry and Lie algebroid theory”, pp. 363–389 in *The breadth of symplectic and Poisson geometry*, edited by J. E. Marsden and T. S. Ratiu, Progr. Math. **232**, Birkhäuser, Boston, MA, 2005. MR 2005g:53157
- [Koszul 1985] J.-L. Koszul, “Crochet de Schouten–Nijenhuis et cohomologie”, pp. 257–271 in *Élie Cartan et les mathématiques d’aujourd’hui* (Lyon, 1984), Astérisque **1985**, Numero Hors Serie, Société Mathématique de France, Paris, 1985. MR 88m:17013 Zbl 0615.58029
- [Liu et al. 1997] Z.-J. Liu, A. Weinstein, and P. Xu, “Manin triples for Lie bialgebroids”, *J. Differential Geom.* **45**:3 (1997), 547–574. MR 98f:58203 Zbl 0885.58030
- [Mackenzie and Xu 1994] K. C. H. Mackenzie and P. Xu, “Lie bialgebroids and Poisson groupoids”, *Duke Math. J.* **73**:2 (1994), 415–452. MR 95b:58171 Zbl 0844.22005
- [Mackenzie and Xu 2000] K. C. H. Mackenzie and P. Xu, “Integration of Lie bialgebroids”, *Topology* **39**:3 (2000), 445–467. MR 2001b:53104 Zbl 0961.58009
- [Michéa and Novitchkov 2005] S. Michéa and G. Novitchkov, “BV-generators and Lie algebroids”, *Internat. J. Math.* **16**:10 (2005), 1175–1191. MR 2007c:17029 Zbl 1106.17028
- [Roytenberg 1999] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, thesis, University of California at Berkeley, 1999. arXiv math/9910078
- [Ševera and Weinstein 2001] P. Ševera and A. Weinstein, “Poisson geometry with a 3-form background”, *Progr. Theoret. Phys. Suppl.* **144** (2001), 145–154. MR 2005e:53132 Zbl 1029.53090
- [Weinstein 1997] A. Weinstein, “The modular automorphism group of a Poisson manifold”, *J. Geom. Phys.* **23**:3-4 (1997), 379–394. MR 98k:58095 Zbl 0902.58013
- [Xu 1999] P. Xu, “Gerstenhaber algebras and BV-algebras in Poisson geometry”, *Comm. Math. Phys.* **200**:3 (1999), 545–560. MR 2000b:17025 Zbl 0941.17016

Received December 2, 2008. Revised April 15, 2009.

ZHUO CHEN  
TSINGHUA UNIVERSITY  
DEPARTMENT OF MATHEMATICS  
BEIJING 100084  
CHINA  
chenzhuott@gmail.com

