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OBTAINING THE ONE-HOLED TORUS FROM PANTS: DUALITY IN AN SL(3, $\mathbb{C}$)-CHARACTER VARIETY

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The SL(3, $\mathbb{C}$)-representation variety $\mathfrak{R}$ of a free group $\mathtt{F}_r$ arises naturally by considering surface group representations for a surface with boundary. There is an SL(3, $\mathbb{C}$)-action on the coordinate ring of $\mathfrak{R}$. The geometric points of the subring of invariants of this action is an affine variety $\mathfrak{X}$. The points of $\mathfrak{X}$ parametrize isomorphism classes of completely reducible representations. The coordinate ring $\mathbb{C}[\mathfrak{X}]$ is a complex Poisson algebra with respect to a presentation of $\mathtt{F}_r$ imposed by the surface. In previous work, we have worked out the bracket on all generators when the surface is a three-holed sphere and when the surface is a one-holed torus. In this paper, we show how the symplectic leaves corresponding to these two different Poisson structures on $\mathfrak{X}$ relate to each other. In particular, they are symplectically dual at a generic point. Moreover, the topological gluing map that turns the three-holed sphere into the one-holed torus induces a rank-preserving Poisson map on $\mathbb{C}[\mathfrak{X}]$.

1. Introduction

In [Lawton 2009] we describe two competing Poisson structures on the variety of characters of SL(3, $\mathbb{C}$)-valued representations of a rank 2 free group. The purpose of this paper is to show that these two structures generically define symplectically dual symplectic leaves, and that a natural topological mapping nontrivially relates the two Poisson structures.

For the remainder of this section we briefly describe character varieties and their smooth stratum's foliation by complex symplectic submanifolds. In these terms we formulate our main theorems. In Section 2, we describe in further detail past results necessary to make sense of the discussion at hand. In particular, for the three-holed sphere and the one-holed torus, we explicitly review the algebraic structure and the Poisson structure of the character variety. Lastly, in Section 3 we restate and prove our main theorems.

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1.1. ***Algebraic structure of $\mathfrak{X}(\Sigma_{n,g})$.*** Let $\Sigma_{n,g}$ be a compact, connected, oriented surface of genus g with $n > 0$ open disks removed. If $g = 0$ we assume $n \geq 3$. Its fundamental group has the presentation

$$\pi_1(\Sigma_{n,g}, *) = \left\{\mathtt{x}_1, \mathtt{y}_1, \ldots, \mathtt{x}_g, \mathtt{y}_g, \mathtt{b}_1, \ldots, \mathtt{b}_n \mid \prod_{i=1}^{g}[\mathtt{x}_i, \mathtt{y}_i] \prod_{j=1}^{n} \mathtt{b}_j = 1\right\}.$$

The group $\mathtt{F}_r := \pi_1(\Sigma_{n,g}, *)$ is always free of rank $r = 2g + n - 1$ since $\Sigma_{n,g}$ retracts to a wedge of $2g + n - 1$ circles. Let $\mathfrak{G} = \mathrm{SL}(3, \mathbb{C})$ and let $\{\mathtt{g}_1, \ldots, \mathtt{g}_r\}$ be generators of $\mathtt{F}_r$. The representation variety $\mathfrak{R} = \mathrm{Hom}(\mathtt{F}_r, \mathfrak{G})$ is bijectively equivalent to $\mathfrak{G}^{\times r}$ given by evaluation as

$$\rho \mapsto (\rho(\mathtt{g}_1), \ldots, \rho(\mathtt{g}_r)),$$

and so inherits the structure of a smooth affine variety from $\mathfrak{G}$. The coordinate ring of $\mathfrak{G}$ is the complex polynomial ring in 9 indeterminates subject to the irreducible relation $\det(\mathbf{X}) - 1$, where $\mathbf{X} = (x_{ij})$ is a *generic matrix* and x_{ij} are the 9 indeterminates. There is a polynomial action of $\mathfrak{G}$ on the coordinate ring of $\mathfrak{R}$, denoted by $\mathbb{C}[\mathfrak{R}]$, by conjugation in r *generic matrices*; that is, for $g \in \mathfrak{G}$ and $f \in \mathbb{C}[\mathfrak{R}]$,

$$g \cdot f(\mathbf{X}_1, \ldots, \mathbf{X}_r) = f(g^{-1}\mathbf{X}_1 g, \ldots, g^{-1}\mathbf{X}_r g).$$

The results of [Procesi 1976] imply that the ring of invariants $\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$ is generated by $\{\mathrm{tr}(\mathbf{W}) \mid \mathtt{w} \in \mathtt{F}_r,\ |\mathtt{w}| \leq 6\}$. Here $\mathbf{W}$ is the word $\mathtt{w}$ in $\mathtt{F}_r$ with its letters replaced by generic matrices. Thus, $\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$ is a finitely generated domain, and so its geometric points are an irreducible algebraic set, $\mathfrak{X}(\Sigma_{n,g}) = \mathrm{Spec}(\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}) = \mathfrak{R} /\!\!/ \mathfrak{G}$, called the $\mathfrak{G}$*-character variety of* $\mathtt{F}_r$. The quotient notation just used means that it is a categorical quotient for the $\mathfrak{G}$-action; see [Dolgachev 2003; Mumford et al. 1994].

1.2. ***The boundary map and foliation of the top stratum.*** The coordinate ring of $\mathfrak{G} /\!\!/ \mathfrak{G}$ is

$$\mathbb{C}[\mathfrak{G} /\!\!/ \mathfrak{G}] = \mathbb{C}[\mathrm{tr}(\mathbf{X}), \mathrm{tr}(\mathbf{X}^{-1})].$$

So $\mathfrak{G} /\!\!/ \mathfrak{G} = \mathbb{C}^2$, which we parametrize by coordinates $(\tau_{(1)}, \tau_{(-1)})$. We then define the boundary map

$$\mathfrak{b}_i : \mathfrak{X} = \mathfrak{R} /\!\!/ \mathfrak{G} = \mathrm{Hom}(\pi_1(\Sigma_{n,g}, *), \mathfrak{G}) /\!\!/ \mathfrak{G} \to \mathfrak{G} /\!\!/ \mathfrak{G}$$

by sending a representation class to the class corresponding to the restriction of ρ to the boundary $\mathtt{b}_i$, that is, $[\rho] \mapsto [\rho|_{\mathtt{b}_i}] = (\tau^i_{(1)}, \tau^i_{(-1)})$ Subsequently we define

$$\mathfrak{b}_{n,g} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n) : \mathfrak{X} = \mathfrak{G}^{\times r} /\!\!/ \mathfrak{G} \longrightarrow (\mathfrak{G} /\!\!/ \mathfrak{G})^{\times n}.$$

The map $\mathfrak{b}_{n,g}$ depends on the surface, not only its fundamental group. We refer to it as a *peripheral structure*, and the pair $(\mathfrak{X}, \mathfrak{b}_{n,g})$ as the *relative character variety*.

Let $\tau = ((\tau^1_{(1)}, \tau^1_{(-1)}), \ldots, (\tau^n_{(1)}, \tau^n_{(-1)})) \in \mathfrak{b}_{n,g}(\mathfrak{X}) \subset (\mathfrak{G} /\!\!/ \mathfrak{G})^n = \mathbb{C}^{2n}$ be a point in the image of the boundary map, and define $\mathfrak{L} = \mathfrak{b}_{n,g}^{-1}(\tau)$. Let $\mathscr{X}$ be the complement of the singular locus (a proper closed subvariety) in $\mathfrak{X}$. Thus $\mathscr{X}$ is a complex manifold dense in $\mathfrak{X}$. At regular values of $\mathfrak{b}_{n,g}$ (these are generic since $\mathfrak{b}_{n,g}$ is dominant), $\mathfrak{L}\cap\mathscr{X}$ is a submanifold of dimension $8r-8-2n=16(g-1)+6n$. It is shown in [Lawton 2009] that the union (over values of τ) of the *leaves*, $\mathscr{L}=\mathfrak{L}\cap\mathscr{X}$, foliate $\mathscr{X}$ by complex symplectic submanifolds, making $\mathscr{X}$ a complex Poisson manifold. This structure continuously extends over all of $\mathfrak{X}$.

1.3. ***The quotient map and main results.*** There are two orientable surfaces with Euler characteristic -1, the three-holed sphere and the one-holed torus. Both of these surfaces have fundamental groups free of rank 2.

Moreover, there is a natural topological quotient mapping $q:\Sigma_{3,0}\to\Sigma_{1,1}$ (independent of orientation) that maps the three-holed sphere (hereafter referred to as pants) to the one-holed torus. In [Lawton 2009] we work out explicitly (with respect to a coordinate system for $\mathfrak{X}$ and choices of orientation for the surfaces) the Poisson structures for the pants and the one-holed torus.

We now state our main theorems:

Theorem 1. *Depending on the choice of orientation,*

$$q^*:\mathbb{C}[\mathfrak{X}(\Sigma_{3,0})]\to\mathbb{C}[\mathfrak{X}(\Sigma_{1,1})]$$

is generically a rank-preserving Poisson (*anti*)*morphism.*

Let $\mathscr{L}(\Sigma_{3,0})$ and $\mathscr{L}(\Sigma_{1,1})$ be generic symplectic leaves of $\mathfrak{X}$.

Theorem 2. $\mathscr{L}(\Sigma_{3,0})$ *and* $\mathscr{L}(\Sigma_{1,1})$ *are generically transverse and so are symplectically dual to each other.*

2. Past results and background

In this section we very briefly review some of the results from [Lawton 2007; Lawton 2009] that we will need to prove our theorems.

2.1. ***Symplectic and Poisson structure on $\mathfrak{X}(\Sigma_{n,g})$.*** Guruprasad, Huebschmann, Jeffrey, and Weinstein [Guruprasad et al. 1997] showed ω (in the following commutative diagram) defines a symplectic form on the leaf $\mathscr{L}$ defined in the introduction.

$$\begin{tikzcd}
H^1(S,\partial S;\mathfrak{g}_{\mathrm{Ad}})\times H^1(S;\mathfrak{g}_{\mathrm{Ad}}) \arrow[r, "\cup"] & H^2(S,\partial S;\mathfrak{g}_{\mathrm{Ad}}\otimes\mathfrak{g}_{\mathrm{Ad}}) \arrow[d, "\mathrm{tr}_*"] \\
 & H^2(S,\partial S;\mathbb{C}) \arrow[d, "\cap[Z]"] \\
H^1_{\mathrm{par}}(S;\mathfrak{g}_{\mathrm{Ad}})\times H^1_{\mathrm{par}}(S;\mathfrak{g}_{\mathrm{Ad}}) \arrow[uu] \arrow[r, "\omega"] & H_0(S;\mathbb{C})=\mathbb{C}
\end{tikzcd}$$

With respect to this 2-form, we show in [Lawton 2009] that Goldman's proof [1984; 1986] of the Poisson bracket formula (a Lie bracket and derivation) generalizes directly to relative cohomology.

Let Σ be an oriented surface with boundary, and $\alpha, \beta \in \pi_1(\Sigma, *)$. Let $\alpha \cap \beta$ be the set of (transverse) double point intersections of α and β. Let $\epsilon(p, \alpha, \beta)$ be the oriented intersection number at $p \in \alpha \cap \beta$, and let $\alpha_p \in \pi_1(\Sigma, p)$ be the curve α based at p.

In these terms the bracket is defined on $\mathbb{C}[\mathfrak{X}]$ by

$$\{\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta))\} = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta)\big(\mathrm{tr}(\rho(\alpha_p \beta_p)) - \tfrac{1}{3}\,\mathrm{tr}(\rho(\alpha))\,\mathrm{tr}(\rho(\beta))\big).$$

See [Lawton 2009, Sections 3 and 4] for further details.

2.2. ***Algebraic structure of $\mathfrak{X}(\Sigma_{3,0})$ and $\mathfrak{X}(\Sigma_{1,1})$.*** We now review the algebraic structure of $\mathbb{C}[\mathfrak{X}]$ for the pants and one-holed torus and the corresponding Poisson structures in those cases. Details are available in [Lawton 2007; 2009].

Let

$$\mathbb{C}[t_{(1)}, t_{(-1)}, t_{(2)}, t_{(-2)}, t_{(3)}, t_{(-3)}, t_{(4)}, t_{(-4)}, t_{(5)}]$$

be a freely generated complex polynomial ring, and let

$$R = \mathbb{C}[t_{(1)}, t_{(-1)}, t_{(2)}, t_{(-2)}, t_{(3)}, t_{(-3)}, t_{(4)}, t_{(-4)}].$$

Define the ring homomorphism

$$R[t_{(5)}] \xrightarrow{\Pi} \mathbb{C}[\mathfrak{X}]$$

by

$$\begin{aligned}
t_{(1)} &\mapsto \mathrm{tr}(\mathbf{X}_1), & t_{(-1)} &\mapsto \mathrm{tr}(\mathbf{X}_1^{-1}),\\
t_{(2)} &\mapsto \mathrm{tr}(\mathbf{X}_2), & t_{(-2)} &\mapsto \mathrm{tr}(\mathbf{X}_2^{-1}),\\
t_{(3)} &\mapsto \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2), & t_{(-3)} &\mapsto \mathrm{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2^{-1}),\\
t_{(4)} &\mapsto \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}), & t_{(-4)} &\mapsto \mathrm{tr}(\mathbf{X}_1^{-1}\mathbf{X}_2),\\
t_{(5)} &\mapsto \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_1^{-1}\mathbf{X}_2^{-1}).
\end{aligned}$$

It can be shown using trace equations that Π is surjective, and hence

$$R[t_{(5)}]/\ker(\Pi) \cong \mathbb{C}[\mathfrak{X}].$$

The Krull dimension of $\mathfrak{X}$ is 8 since generic orbits are 8-dimensional. Hence, $\ker(\Pi)$ is nonzero and principal.

Let $\mathbb{S}$ be the formal sum of the elements in the group generated by the permutations (in cycle notation)

$$(1, 2)(-1, -2)(4, -4) \quad \text{and} \quad (1, -1)(3, -4)(-3, 4)$$

acting on the indices of the generators of $R[t_{(5)}]/\ker(\Pi)$. The action is induced by the following elements of the $\mathrm{Out}(\mathtt{F}_2)$:

$$\mathfrak{t}=\begin{cases}\mathtt{x}_1\mapsto\mathtt{x}_2\\ \mathtt{x}_2\mapsto\mathtt{x}_1\end{cases}\qquad\text{and}\qquad \mathfrak{i}_1=\begin{cases}\mathtt{x}_1\mapsto\mathtt{x}_1^{-1}\\ \mathtt{x}_2\mapsto\mathtt{x}_2.\end{cases}$$

The group generated has order 8 and is isomorphic to the dihedral group D_4. In [Lawton 2007, Theorem 8 and Corollary 15] we show this:

Theorem 3. (1) *$\mathfrak{X}=\mathfrak{G}^{\times 2}/\!\!/\mathfrak{G}$ is a degree* 6 *hypersurface in* $\mathbb{C}^9$.

(2) $\ker(\Pi)=(t_{(5)}^2-Pt_{(5)}+Q)$, *where* $P,Q\in R$.

(3) *There is a D_4-equivariant surjection* (*submersion*) $m:\mathfrak{X}\to\mathbb{C}^8$, *which is generically* 2-*to*-1.

(4) *P and Q are given by*

$$P=\mathbb{S}\big(\tfrac{1}{8}\big(t_{(1)}t_{(-1)}t_{(2)}t_{(-2)}-4t_{(1)}t_{(-2)}t_{(-4)}+2t_{(1)}t_{(-1)}+2t_{(3)}t_{(-3)}\big)\big)-3,$$

$$\begin{aligned}Q=\mathbb{S}\big(\tfrac{1}{8}\big(&2t_{(-2)}t_{(-1)}^2t_{(1)}^2t_{(2)}+4t_{(1)}^2t_{(2)}^2t_{(3)}-4t_{(1)}^3t_{(-2)}t_{(2)}\\&-8t_{(-4)}t_{(-2)}t_{(-1)}t_{(1)}^2-4t_{(4)}t_{(3)}t_{(2)}t_{(1)}t_{(-2)}+8t_{(1)}t_{(3)}t_{(-4)}^2\\&+8t_{(-4)}t_{(1)}t_{(2)}^2-8t_{(3)}^2t_{(2)}t_{(1)}+4t_{(4)}t_{(-3)}t_{(2)}^2+t_{(-2)}t_{(-1)}t_{(2)}t_{(1)}\\&+t_{(-3)}t_{(-4)}t_{(3)}t_{(4)}+4t_{(-3)}t_{(-1)}t_{(3)}t_{(1)}+4t_{(1)}^3+4t_{(3)}^3\\&+12t_{(-4)}t_{(-2)}t_{(1)}-12t_{(-4)}t_{(2)}t_{(3)}-12t_{(1)}t_{(-1)}-12t_{(3)}t_{(-3)}\big)\big)+9.\end{aligned}$$

2.3. ***Poisson structures of $\mathfrak{X}(\Sigma_{3,0})$ and $\mathfrak{X}(\Sigma_{1,1})$.*** For our purposes, a Poisson variety is an affine variety $\mathfrak{X}$ over $\mathbb{C}$ that is endowed with a Lie bracket $\{\cdot,\cdot\}$ on its coordinate ring $\mathbb{C}[\mathfrak{X}]$ and that acts as a formal derivation (that is, it satisfies the Leibniz rule). On the smooth strata of $\mathfrak{X}$ (denoted by $\mathscr{X}$), it makes $\mathscr{X}$ a complex Poisson manifold in the usual sense (by Stone–Weierstrass). For any such Poisson bracket, there exists an exterior bivector field $\mathfrak{a}\in\Lambda^2(T\mathscr{X})$ whose restriction to symplectic leaves is given by the symplectic form as $\{f,g\}=\omega(H_g,H_f)$ (where $H_f=\{f,\cdot\}$ is called the Hamiltonian vector field). If $f,g\in\mathbb{C}[\mathfrak{X}]$, then with respect to interior multiplication, $\{f,g\}=\mathfrak{a}\cdot df\otimes dg$. In local coordinates $(z_1,\dots,z_k)$, it takes the form

$$\mathfrak{a}=\sum_{i,j}\mathfrak{a}_{i,j}\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial z_j},$$

and so

$$\begin{aligned}\{f,g\}&=\sum_{i,j}\Big(\mathfrak{a}_{i,j}\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial z_j}\Big)\cdot\Big(\frac{\partial f}{\partial z_i}dz_i\otimes\frac{\partial g}{\partial z_j}dz_j\Big)\\&=\sum_{i,j}\mathfrak{a}_{i,j}\Big(\frac{\partial f}{\partial z_i}\frac{\partial g}{\partial z_j}-\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial z_i}\Big).\end{aligned}$$

Denote the bivector associated to $\mathfrak{X}(\Sigma_{n,g})$ by $\mathfrak{a}(\Sigma_{n,g})$. In the case of the pants or the one-holed torus, there are 9 generating functions of $\mathbb{C}[\mathfrak{X}]$; they are $t_{(\pm i)}$ for $1 \leq i \leq 4$ and $t_{(5)}$. Since the bivector is a Lie bracket and a derivation, its formulation is in these terms. Let $\mathfrak{a}_{i,j} = \{t_{(i)}, t_{(j)}\}$. In [Lawton 2009, Corollary 26, Theorem 29] we show the following two structure theorems.

Theorem 4. *The Poisson bivector on $\mathfrak{X}(\Sigma_{3,0})$ is*

$$\mathfrak{a}(\Sigma_{3,0}) = (P - 2t_{(5)})\frac{\partial}{\partial t_{(4)}} \wedge \frac{\partial}{\partial t_{(-4)}} + (1-\mathfrak{i})\left(\mathfrak{a}_{4,5}\frac{\partial}{\partial t_{(4)}} \wedge \frac{\partial}{\partial t_{(5)}}\right),$$

where $\mathfrak{a}_{4,5} = (\partial/\partial t_{(-4)})(Q - t_{(5)}P)$ *and* $\mathfrak{i} = \mathfrak{i}_1\mathfrak{t}\mathfrak{i}_1\mathfrak{t}$ *is the mapping* $\mathtt{x}_i \mapsto \mathtt{x}_i^{-1}$.

In D_4 define $\mathfrak{i}_2 = \mathfrak{t}\mathfrak{i}_1\mathfrak{t}$, the mapping that sends $\mathtt{x}_2 \mapsto \mathtt{x}_2^{-1}$.

Additionally, define the elements $\Sigma_1 = 1 + \mathfrak{i} - \mathfrak{i}_1 - \mathfrak{i}_2$ and $\Sigma_2 = 1 + \mathfrak{i} - \mathfrak{t} - \mathfrak{i}\mathfrak{t}$ of the group ring of D_4. Note that $\frac{1}{2}\Sigma_1\Sigma_2 = 1 + \mathfrak{i} - \mathfrak{i}_1 - \mathfrak{i}_2 - \mathfrak{t} - \mathfrak{i}\mathfrak{t} + \mathfrak{i}_1\mathfrak{t} + \mathfrak{i}_2\mathfrak{t}$.

Then after doing 28 calculations, while observing symmetry, we conclude:

Theorem 5. *The Poisson bivector field on $\mathfrak{X}(\Sigma_{1,1})$ is*

$$\mathfrak{a}(\Sigma_{1,1}) = \Sigma_1\left(\mathfrak{a}_{1,2}\frac{\partial}{\partial t_{(1)}} \wedge \frac{\partial}{\partial t_{(2)}}\right) + \Sigma_2\left(\mathfrak{a}_{3,4}\frac{\partial}{\partial t_{(3)}} \wedge \frac{\partial}{\partial t_{(4)}}\right) + \tfrac{1}{2}\Sigma_1\Sigma_2\left(\mathfrak{a}_{1,3}\frac{\partial}{\partial t_{(1)}} \wedge \frac{\partial}{\partial t_{(3)}} + \mathfrak{a}_{1,-3}\frac{\partial}{\partial t_{(1)}} \wedge \frac{\partial}{\partial t_{(-3)}}\right),$$

where

$$\begin{aligned}
\mathfrak{a}_{1,2} &= t_{(3)} - \tfrac{1}{3}t_{(1)}t_{(2)},\\
\mathfrak{a}_{1,3} &= \tfrac{2}{3}t_{(1)}t_{(3)} - t_{(-1)}t_{(2)} + t_{(-4)},\\
\mathfrak{a}_{1,-3} &= -t_{(-2)} + \tfrac{1}{3}t_{(1)}t_{(-3)},\\
\mathfrak{a}_{3,4} &= -t_{(1)}^2 + t_{(-1)} - t_{(-4)}t_{(-2)} - t_{(2)}t_{(-3)} + t_{(-1)}t_{(2)}t_{(-2)} - \tfrac{1}{3}t_{(3)}t_{(4)}.
\end{aligned}$$

Comment 6. The orientations chosen on these surfaces are opposite. Our presentation of the pants has the boundary on the outside whereas the one-holed torus has the same boundary (after projection) on the inside. Since the orientations of the boundaries are the same, the surfaces are "inside-out" with respect to each other, and so the orientations are reversed. Consequently, if the quotient mapping taking the pants to the one-holed torus is to preserve orientations, one of the above two bivectors must be multiplied by -1.

3. Obtaining the torus from pants

Let $q : \Sigma_{3,0} \to \Sigma_{1,1}$ be the quotient map given by identifying two of the boundaries (call them b_1 and b_2). Let x_0 be a fixed base point. Let $x_1 \in b_1$ and $x_2 \in b_2$ be also fixed, where $q(x_1) = q(x_2)$.

Then the third boundary b_3 in $\Sigma_{3,0}$ is homotopic to $(b_1b_2)^{-1}$ in $\pi_1(\Sigma_{3,0}, x_0)$.

Let γ_1 and γ_2 be paths from x_0 to x_1 and x_0 to x_2 respectively. The image $q(\gamma_1\gamma_2^{-1}) := \beta$ is a nontrivial based loop in $\Sigma_{1,1}$. Moreover, $(\gamma_1\gamma_2^{-1})b_1(\gamma_1\gamma_2^{-1})^{-1}$ is homotopic to b_2^{-1}, since $\gamma_1^{-1}b_1\gamma_1$ is homotopic to $\gamma_2^{-1}b_2^{-1}\gamma_2$ in $\Sigma_{1,1}$.

Therefore,

$$q_\sharp : \pi_1(\Sigma_{3,0}, x_0) = \langle b_1, b_2, b_3 \mid b_1b_2b_3 = 1\rangle \to \pi_1(\Sigma_{1,1}, x_0) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta]\gamma = 1\rangle$$

is injective and given by

$$b_1 \mapsto \alpha, \quad b_2 \mapsto \beta\alpha^{-1}\beta^{-1}, \quad b_3 \mapsto [\alpha, \beta]^{-1}.$$

Consequently, $q_* : \mathfrak{X}(\Sigma_{1,1}) \to \mathfrak{X}(\Sigma_{3,0})$ is given by

$$[(A, B)] \mapsto [(A, BA^{-1}B^{-1})],$$

and $q^* : \mathbb{C}[\mathfrak{X}(\Sigma_{3,0})] \to \mathbb{C}[\mathfrak{X}(\Sigma_{1,1})]$ is given by $f \mapsto f \circ q_*$.

To be concrete, we write the assignments that determine q^*:

$$
\begin{aligned}
&t_{(1)} \mapsto t_{(1)}, \qquad t_{(-1)} \mapsto t_{(-1)}, \quad t_{(2)} \mapsto t_{(-1)}, \quad t_{(-2)} \mapsto t_{(1)},\\
&t_{(3)} \mapsto t_{(5)}, \qquad t_{(-3)} \mapsto \mathrm{tr}(A^{-1}BAB^{-1}) = P - t_{(5)},\\
&t_{(4)} \mapsto \mathrm{tr}(ABAB^{-1})\\
&\quad = t_{(-4)}t_{(-2)} + t_{(-1)} + t_{(-3)}t_{(2)} - t_{(-1)}t_{(2)}t_{(-2)} + t_{(3)}t_{(4)},\\
&t_{(-4)} \mapsto \mathrm{tr}(A^{-1}BA^{-1}B^{-1})\\
&\quad = t_{(-4)}t_{(-3)} + t_{(1)} + t_{(3)}t_{(-2)} - t_{(1)}t_{(2)}t_{(-2)} + t_{(2)}t_{(4)},\\
&t_{(5)} \mapsto \mathrm{tr}(ABA^{-1}B^{-1}A^{-1}BAB^{-1})\\
&\quad = t_{(-3)}^3 - 2t_{(-2)}t_{(-1)}t_{(-3)}^2 + t_{(-4)}t_{(2)}t_{(-3)}^2 + t_{(1)}t_{(4)}t_{(-3)}^2 + t_{(-2)}^2t_{(-1)}^2t_{(-3)}\\
&\qquad - t_{(-2)}t_{(1)}t_{(2)}^2t_{(-3)} + t_{(-1)}t_{(4)}^2t_{(-3)} + t_{(-4)}^2t_{(-2)}t_{(-3)} - t_{(-4)}t_{(-1)}t_{(-3)}\\
&\qquad + t_{(-2)}^2t_{(1)}t_{(-3)} - t_{(-4)}t_{(-2)}t_{(-1)}t_{(2)}t_{(-3)} + 2t_{(1)}t_{(2)}t_{(-3)} + t_{(-2)}t_{(2)}t_{(3)}t_{(-3)}\\
&\qquad - 3t_{(3)}t_{(-3)} + t_{(2)}^2t_{(4)}t_{(-3)} - 3t_{(-2)}t_{(4)}t_{(-3)} - t_{(-2)}t_{(-1)}t_{(1)}t_{(4)}t_{(-3)}\\
&\qquad + t_{(-4)}t_{(3)}t_{(4)}t_{(-3)} + t_{(-2)}^3 + t_{(4)}^3 + t_{(-2)}^2t_{(-1)}t_{(1)}t_{(2)}^2 - 2t_{(-2)}t_{(1)}t_{(4)}^2\\
&\qquad + t_{(2)}t_{(3)}t_{(4)}^2 + 2t_{(-4)}t_{(-2)}t_{(1)} - t_{(-2)}^3t_{(-1)}t_{(1)} - 3t_{(-2)}t_{(2)} - t_{(-4)}t_{(-2)}^2t_{(1)}t_{(2)}\\
&\qquad - t_{(-2)}t_{(-1)}t_{(1)}t_{(2)} + t_{(-4)}t_{(-2)}^2t_{(3)} + 2t_{(-2)}t_{(-1)}t_{(3)} - t_{(-2)}^2t_{(-1)}t_{(2)}t_{(3)}\\
&\qquad + t_{(-2)}^2t_{(1)}^2t_{(4)} - t_{(-2)}t_{(-1)}t_{(2)}^2t_{(4)} + t_{(-2)}t_{(3)}^2t_{(4)} - 3t_{(-4)}t_{(4)} + t_{(-2)}^2t_{(-1)}t_{(4)}\\
&\qquad + t_{(-4)}t_{(-2)}t_{(2)}t_{(4)} + 2t_{(-1)}t_{(2)}t_{(4)} - t_{(1)}t_{(3)}t_{(4)} - t_{(-2)}t_{(1)}t_{(2)}t_{(3)}t_{(4)} + 3.
\end{aligned}
$$

These last three identities follow from recursive trace reduction formulas; see [Lawton 2007; 2008].

Let $\{\cdot,\cdot\}_1$ be the bracket corresponding to $\Sigma_{1,1}$ and let $\{\cdot,\cdot\}_3$ be the bracket corresponding to $\Sigma_{3,0}$. Let $\mathfrak{Q}$ be the image of q^*. We now prove Theorem 1.

Theorem 7. (1) *$\mathfrak{Q}$ is a Poisson subalgebra of* $\mathbb{C}[\mathfrak{X}(\Sigma_{1,1})]$.

(2) *q^* is an anti-Poisson morphism*; *that is*, $\{q^*(f), q^*(g)\}_1 = -q^*(\{f, g\}_3)$.

(3) *at a generic point of* $\mathfrak{X}$, $\operatorname{rank}(\mathfrak{a}(\Sigma_{1,1})|_{\mathfrak{Q}}) = \operatorname{rank}(\mathfrak{a}(\Sigma_{3,0}))$.

Proof. First we note that (2) implies (1).

To prove (2), since q^* is an algebra morphism and the bracket is a derivation, it is enough to verify it on all generators of the algebra. One can use the explicit form of the mapping q^* and the explicit form of the bivectors to verify the result. However, since q preserves transversality of cycles, double points, and does not affect orientation, it follows that for any two cycles α and β in $\Sigma_{3,0}$ used in computing the bivector $\mathfrak{a}(\Sigma_{3,0})$, we have

$$(1)\quad q^*\big(\{\operatorname{tr}(\rho(\alpha)), \operatorname{tr}(\rho(\beta))\}_3\big) = \sum_{q(p)\in q(\alpha)\cap q(\beta)} \epsilon(q(p), q(\alpha), q(\beta))\big(\operatorname{tr}(\rho(q(\alpha)_{q(p)}q(\beta)_{q(p)})) - \tfrac{1}{3}\operatorname{tr}(\rho(q(\alpha)))\operatorname{tr}(\rho(q(\beta)))\big).$$

However, as noted in Section 2.3, the intersection numbers $\epsilon(q(p), q(\alpha), q(\beta))$ and $\epsilon(p, \alpha, \beta)$ must be reversed since the bracket computations of [Lawton 2009] are with respect to opposite orientations on the surfaces $\Sigma_{3,0}$ and $\Sigma_{1,1}$.

Thus Equation (1) is exactly $-\{\operatorname{tr}(q(\alpha)), \operatorname{tr}(q(\beta))\}_1$, as was to be shown.

To prove (3) we first note that the rank of a bivector is the rank of the antisymmetric matrix of functions $(\mathfrak{a}_{ij})$. Then, from (2), there are only three nonzero coefficients to $\mathfrak{a}(\Sigma_{1,1})$ after restricting to the image of q^*. Namely, $t_{(5)}$ and $P - t_{(5)}$ are Casimirs for $\{\cdot,\cdot\}_1$, and since the mapping is Poisson and $t_{(\pm1)}$ are fixed, it follows that they are Casimirs in the images since they are Casimirs in the preimage. Thus we are left with the image generators $q^*(t_{(j)})$ for $j = 4, -4, 5$. Since the bivector on the Poisson subalgebra is exactly the induced one, we explicitly formulate the bivector matrix and compute its rank, finding it generically 2. In particular, the matrix has the form

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

where

$$a = \{q^*(t_{(4)}), q^*(t_{(-4)})\}_1, \quad b = \{q^*(t_{(4)}), q^*(t_{(5)})\}_1, \quad c = \{q^*(t_{(-4)}), q^*(t_{(5)})\}_1.$$

However any matrix of this form has rank 2 as long as all three of a, b, c are not 0, in which case the rank is 0. By direct calculation one sees that all polynomials a, b, c are in terms of *only* algebraically independent generators, and so none of a, b, c are generically 0 on $\mathfrak{X}$. So generically the rank is 2, and the rank of $\{\cdot,\cdot\}_3$ is two since the rank is equal to the dimension of a symplectic leaf. Hence, the mapping is rank-preserving as long as it is not completely degenerate (which is generically the case). □

Comment 8. Equation (1) used in the argument above shows much more. For any two surfaces Σ_{n_1,g_1} and Σ_{n_2,g_2} with $n_1 > n_2 > 0$ and $\chi(\Sigma_{n_1,g_1}) = \chi(\Sigma_{n_2,g_2})$, there is a quotient map (identifying pairs of boundary components) $q : \Sigma_{n_1,g_1} \to \Sigma_{n_2,g_2}$ that gives an injection on fundamental groups and therefore gives a map of coordinate rings $q^* : \mathbb{C}[\mathfrak{X}(\Sigma_{n_1,g_1})] \to \mathbb{C}[\mathfrak{X}(\Sigma_{n_2,g_2})]$. This is true not only for $\mathrm{SL}(3, \mathbb{C})$ but for any complex algebraic reductive Lie group $\mathfrak{G}$. The argument above shows that if the orientations of the surfaces correspond to each other, then q^* is a Poisson mapping and if the orientations are opposite, then it is an anti-Poisson morphism. Therefore the image of q^* is generally a Poisson subalgebra of the codomain of q^*.

It does not seem clear whether rank-preserving is a general property or not.

See [Fock and Rosly 1999] for more about induced Poisson mappings (in the context of the moduli of flat connections on an n-holed surface).

Let $\mathscr{L}(\Sigma_{n,g})$ be a generic symplectic leaf of $\mathfrak{X}(\Sigma_{n,g})$. We now prove Theorem 2.

Theorem 9. *$\mathscr{L}(\Sigma_{3,0})$ and $\mathscr{L}(\Sigma_{1,1})$ are generically transverse.*

Proof. The mapping $m : \mathfrak{X} \to \mathbb{C}^8$ from Theorem 3 is given by

$$(t_{(1)}, t_{(-1)}, \ldots, t_{(4)}, t_{(-4)}, t_{(5)}) \mapsto (t_{(1)}, t_{(-1)}, \ldots, t_{(4)}, t_{(-4)}).$$

It is surjective, and since the first eight generators are algebraically independent, it is submersive as well.

This immediately implies that the mapping $\mathfrak{b}_{3,0} : \mathfrak{X}(\Sigma_{3,0}) \to \mathbb{C}^6$ given by

$$(t_{(1)}, t_{(-1)}, \ldots, t_{(4)}, t_{(-4)}, t_{(5)}) \mapsto (t_{(1)}, t_{(-1)}, \ldots, t_{(3)}, t_{(-3)})$$

is likewise surjective and submersive. Consequently, $\mathscr{L}(\Sigma_{3,0}) = \mathfrak{b}_{3,0}^{-1}(\mathfrak{b}_{3,0}(x))$ has dimension 2 for any $x \in \mathscr{X}$. Moreover, we can locally parametrize this leaf by the coordinates $(t_{(4)}, t_{(-4)})$ since the other six coordinates $t_{(\pm i)}$ for $i = 1, 2, 3$ are held constant and $t_{(5)}$ is then determined by the defining relation $t_{(5)}^2 - P t_{(5)} + Q$. In particular, flows through these coordinates determine a dimension 2 subspace $T_*\mathscr{L}(\Sigma_{3,0}) \subset T_*\mathscr{X}$ of the tangent space.

Now consider the mapping $\mathfrak{b}_{1,1} : \mathfrak{X}(\Sigma_{1,1}) \to \mathbb{C}^2$ given by

$$(t_{(1)}, t_{(-1)}, \ldots, t_{(4)}, t_{(-4)}, t_{(5)}) \mapsto (t_{(5)}, P - t_{(5)}).$$

Note that this is in fact the correct mapping since $P = \mathrm{tr}([A, B]) + \mathrm{tr}([B, A])$; see [Lawton 2007]. It is shown in [Lawton 2009] that the boundary mapping is always surjective if $g > 0$.

However, $\mathfrak{b}_{1,1}$ may not be everywhere submersive, in particular, in the case when $P^2 - 4Q$ vanishes. However, there is an open dense set of $\mathfrak{X}$ where $d\mathfrak{b}_{1,1}$ is onto (call it $\mathcal{U}$), since $\mathfrak{b}_{1,1}$ is surjective and regular; see [Lawton 2009]. We may assume that $\mathcal{U} \subset \mathcal{X}$.

Take any $u \in \mathcal{U}$. [Lawton 2009] shows that the leaves $\mathcal{L}_1 := \mathfrak{b}_{1,1}^{-1}(\mathfrak{b}_{1,1}(u)) \cap \mathcal{X}$ and $\mathcal{L}_3 := \mathfrak{b}_{3,0}^{-1}(\mathfrak{b}_{3,0}(u)) \cap \mathcal{X}$ are complex symplectic manifolds of dimensions 6 and 2, respectively. Consequently, these leaves are *properly* transverse; that is, $\dim \mathcal{L}_1 + \dim \mathcal{L}_3 = \dim \mathcal{X}$.

We now show $\dim \mathcal{L}_1 \cap \mathcal{L}_3 = 0$ and $\mathcal{L}_1 \cap \mathcal{L}_3 \neq \varnothing$. At an intersection point, $P = t_{(5)} + t_{(-5)} := C$ and $Q = t_{(5)} t_{(-5)} := D$ and $t_{(\pm 1)}, t_{(\pm 2)}, t_{(\pm 3)}$ are all fixed. Moreover, solving $P = C$ for $t_{(4)}$ generically gives

$$(2)\quad t_{(4)} \mapsto \frac{1}{t_{(-4)} - t_{(-1)} t_{(2)}} \big(C + t_{(-4)} t_{(-2)} t_{(1)} - t_{(-1)} t_{(1)} - t_{(-2)} t_{(2)} + t_{(-3)} t_{(1)} t_{(2)} \\ - t_{(-2)} t_{(-1)} t_{(1)} t_{(2)} - t_{(-3)} t_{(3)} + t_{(-2)} t_{(-1)} t_{(3)} + 3 \big).$$

Now, substituting this into $Q - D = 0$ gives a monic degree six polynomial in the variable $t_{(-4)}$ since the degree in $t_{(4)}$ of Q is 3. So the intersection is nonempty and of dimension 0 (at most 6 discrete points) if $t_{(-4)} - t_{(-1)} t_{(2)} \neq 0$. Otherwise, setting $t_{(-4)} = t_{(-1)} t_{(2)}$ and substituting this into $Q - D = 0$ gives a monic degree 3 polynomial in $t_{(4)}$. Either way, the intersection is nonempty and of dimension 0.

We claim that the tangent space to $\mathcal{L}_3$ locally can be determined by flows through $\{t_{(1)}, t_{(-1)}, t_{(2)}, t_{(-2)}, t_{(3)}, t_{(-3)}\}$ by solving for $t_{(4)}$ and $t_{(-4)}$ in terms of $t_{(\pm 1)}, \ldots, t_{(\pm 3)}$ on an open subset since both P and Q are constant on $\mathcal{L}_3$.

Explicitly, substituting Equation (2) into $Q - D = 0$, where $t_{(\pm i)}$ for $1 \leq i \leq 3$ are now not fixed, locally and generically gives $t_{(-4)}$, and subsequently $t_{(4)}$, as functions of $t_{(\pm i)}$ for $1 \leq i \leq 3$. Thus the flows through $t_{(\pm i)}$ for $1 \leq i \leq 3$ determine a full-dimensional tangent space to $\mathcal{L}_3$ whenever $t_{(-4)} - t_{(-1)} t_{(2)} \neq 0$.

Switching the roles of $t_{(4)}$ and $t_{(-4)}$ gives a like result at any point at which $t_{(4)} - t_{(1)} t_{(-2)} \neq 0$.

However, the tangent space to any point in $\mathcal{L}_1$ is given by the kernel to the mapping $M := (\partial f_i / \partial t_{(\pm j)})$, where

$$f_1 = t_{(5)}^2 - P t_{(5)} + Q, \quad f_2 = t_{(5)} - a, \quad f_3 = P - t_{(5)} - b,$$

and $C = a + b$ and $D = ab$; see [Harris 1992]. This follows since these three functions define the leaf as an algebraic set cut out of $\mathbb{C}^9$. At any smooth point in the leaf, the dimension of the kernel is 6. So whenever $t_{(4)} - t_{(1)} t_{(-2)} \neq 0$ and $t_{(-4)} - t_{(-1)} t_{(2)} \neq 0$ using $P = C$, $Q = D$, and $t_{(5)} = a$ from above, this matrix has entries that are rational functions of $t_{(\pm 1)}, \ldots, t_{(\pm 3)}$ alone. On the other hand, if any smooth point also satisfies $t_{(4)} - t_{(1)} t_{(-2)} = 0$ and $t_{(-4)} - t_{(-1)} t_{(2)} = 0$, then solving for $t_{(\pm 4)}$ again gives M as a matrix in these six variables. Thus the flows through these six coordinate functions always determine the tangent space at a smooth point of $\mathcal{L}_1$.

Consequently, the span of the flows through $\{t_{(4)}, t_{(-4)}\}$ and $\{t_{(\pm 1)}, \dots, t_{(\pm 3)}\}$ generically and locally give full-dimensional tangent spaces to $\mathscr{L}_3$ and $\mathscr{L}_1$, respectively, at an intersection point u. However, collectively they span a full-dimensional tangent space to $\mathscr{X}$ since they are globally independent. Hence, $T_u\mathscr{L}_1 + T_u\mathscr{L}_3 = T_u\mathscr{X}$ for any point in $u \in \mathscr{U}$.

Compounded with the fact that the leaves are properly transverse and trivially intersect, we conclude that $T_u\mathscr{X} = T_u\mathscr{L}_1 \oplus T_u\mathscr{L}_3$ for any $u \in \mathscr{U}$; that is, the leaves are generically transverse. □

We thus conclude that the tangent spaces $T_u\mathscr{X}$ are symplectic with respect to the product form. This does not imply that $\mathscr{X}$ is complex symplectic since the form may not be closed. We call two symplectic submanifolds symplectically dual if their tangent spaces are symplectic duals to each other with respect to this form.

Corollary 10. *The symplectic leaves of $\mathscr{X}$ are generically symplectically dual.*

Comment 11. This sort of phenomena is not general. For $\mathrm{SL}(2, \mathbb{C})$, the leaves $\mathscr{L}(\Sigma_{3,0})$ and $\mathscr{L}(\Sigma_{1,1})$ are respectively of dimension 0 and 2 and the variety $\mathfrak{X}$ is of dimension 3, so there is not transversality.

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