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ALGEBRAS OF TYPE  $A_1$  WITH COORDINATES IN RANK 2  
QUANTUM TORI**

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## MODULES FOR THE CORE OF EXTENDED AFFINE LIE ALGEBRAS OF TYPE $A_1$ WITH COORDINATES IN RANK 2 QUANTUM TORI

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**We construct a Lie algebra  $L$  from rank 3 quantum tori and show that it is isomorphic to the core of extended affine Lie algebras of type  $A_1$ . Then we construct two classes — which turn out to be exhaustive — of irreducible  $\mathbb{Z}$ -graded highest weight  $L$ -modules and give necessary and sufficient conditions for these modules to have finite-dimensional homogeneous subspaces. As a consequence, we also determine all the irreducible  $\mathbb{Z}$ -graded  $L$ -modules with nonzero center and finite-dimensional homogeneous subspaces.**

### 1. Introduction

Extended affine Lie algebras (EALAs), which were introduced in [Høegh-Krohn and Torr sani 1990] under the name of irreducible quasisimple Lie algebras, are higher-dimensional generalizations of affine Kac–Moody Lie algebras. Roughly speaking, they are complex Lie algebras that have a nondegenerate invariant form, a self-centralizing finite-dimensional ad-diagonalizable abelian subalgebra (that is, a Cartan subalgebra), a discrete irreducible root system, and ad-nilpotency of non-isotropic root spaces; see [Berman et al. 1996; Allison et al. 1997a; Allison et al. 1997b]. Prime examples of EALAs are toroidal Lie algebras, which are universal central extensions of  $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$  (here  $\mathfrak{g}$  is a finite-dimensional simple Lie algebra); these were studied in [Frenkel 1985; Gao and Zeng 2006; Moody et al. 1990; Yamada 1989; Etingof and Frenkel 1994; Eswara Rao and Moody 1994; Berman and Cox 1994] and elsewhere. There are many EALAs that allow not only the Laurent polynomial algebra  $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$  as coordinate algebra but also quantum tori, Jordan tori and the octonian tori as coordinate algebras,

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depending on the type of the Lie algebra; see [Allison et al. 1997a; Berman et al. 1996; Berman et al. 1995; Allison and Gao 2001; Yoshii 1996]. The structure theory of the EALAs of type  $A_{d-1}$  is tied up with the Lie algebra  $\mathfrak{gl}_d(\mathbb{C}) \otimes \mathbb{C}_Q$ , where  $\mathbb{C}_Q$  is the quantum torus. The quantum tori defined in [Manin 1991] are the noncommutative analogue of Laurent polynomial algebras. The universal central extension of the derivation Lie algebra of the rank 2 quantum torus is known as the  $q$ -analogue Virasoro-like algebra; see [Kirkman et al. 1994]. For representations of Lie algebras coordinatized by quantum tori, see [Jakobsen and Kac 1989; Berman and Szmigielski 1999; Gao 2000b; 2000a; Eswara Rao 2004; Rao 2003] and the references therein. For structure and representations of the  $q$ -analogue Virasoro-like algebra, see [Zhang and Zhao 1996; Jiang and Meng 1998; Rao and Zhao 2004; Lin and Tan 2006; 2008].

This paper is organized as follows. In Section 2, we first recall some concepts about quantum tori and EALAs of type  $A_1$  with coordinates in rank 2 quantum tori. Next, we show that these EALAs are isomorphic to a Lie algebra  $L$  that is constructed from a special class of rank 3 quantum tori. Then we prove some basic propositions and reduce the classification of quasifinite irreducible  $\mathbb{Z}$ -graded  $L$ -modules to the classification of generalized highest weight modules and uniformly bounded modules. In Section 3, we construct two classes of irreducible  $\mathbb{Z}$ -graded highest weight  $L$ -modules, and give necessary and sufficient conditions for these modules to have finite-dimensional homogeneous subspaces. In Section 4, we prove generalized highest weight irreducible  $\mathbb{Z}$ -graded  $L$ -modules with finite-dimensional homogeneous subspaces must be highest (or lowest) weight modules; thus the modules constructed in Section 3 exhaust all generalized highest weight modules; see Theorem 4.3, our main theorem. As a consequence, we also complete the classification of irreducible  $\mathbb{Z}$ -graded  $L$ -modules with finite-dimensional homogeneous subspaces and nonzero center.

## 2. Basics

We use  $\mathbb{C}$ ,  $\mathbb{Z}$ ,  $\mathbb{Z}_+$  and  $\mathbb{N}$  to denote the sets of complex numbers, integers, nonnegative integers, and positive integers, respectively. We denote by  $\mathbb{C}^*$  the nonzero complex numbers and by  $\mathbb{Z}^{2*}$  the set  $\mathbb{Z}^2 \setminus \{(0, 0)\}$ . All vector spaces we consider are over  $\mathbb{C}$ . As usual, if  $u_1, u_2, \dots, u_k$  are elements in vector spaces, we use  $\langle u_1, \dots, u_k \rangle$  to denote their linear span over  $\mathbb{C}$ . We let  $q$  be a nonzero complex number and suppose throughout that  $q$  is generic (that is, not a root of unity).

Now we recall the concept of quantum torus from [Manin 1991]. Let  $\nu$  be a positive integer, and let  $Q = (q_{ij})$  be a  $\nu \times \nu$  matrix with elements in  $\mathbb{C}^*$  such that  $q_{ii} = 1$  and  $q_{ij} = q_{ji}^{-1}$  for  $0 \leq i, j \leq \nu - 1$ . A quantum torus associated to  $Q$  is the unital associative algebra  $\mathbb{C}_Q[t_0^{\pm 1}, \dots, t_{\nu-1}^{\pm 1}]$  (or, simply  $\mathbb{C}_Q$ ) with generators

$t_0^{\pm 1}, \dots, t_{v-1}^{\pm 1}$  and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad \text{and} \quad t_i t_j = q_{ij} t_j t_i \quad \text{for all } 0 \leq i, j \leq v-1.$$

Write  $t^{\mathbf{m}} = t_0^{m_0} t_1^{m_1} \cdots t_{v-1}^{m_{v-1}}$  for  $\mathbf{m} = (m_0, m_1, \dots, m_{v-1})$ . Then

$$t^{\mathbf{m}} \cdot t^{\mathbf{n}} = \left( \prod_{0 \leq j \leq i \leq v-1} q_{ij}^{m_i n_j} \right) t^{\mathbf{m}+\mathbf{n}}, \quad \text{where } \mathbf{m}, \mathbf{n} \in \mathbb{Z}^v.$$

If  $Q$  is a  $2 \times 2$  matrix with  $q_{21} = q$ , we will simply write  $C_q$  instead of  $\mathbb{C}_Q$ .

Next we recall the construction of EALAs of type  $A_1$  with coordinates in  $\mathbb{C}_{q^2}$ . Let  $E_{ij}$  be the  $2 \times 2$  matrix with 1 at position  $(i, j)$  and 0 elsewhere. The Lie algebra  $\tilde{\tau} = \mathfrak{gl}_2(\mathbb{C}_{q^2})$  is defined by the commutator

$$[E_{ij}(t^{\mathbf{m}}), E_{kl}(t^{\mathbf{n}})]_0 = \delta_{j,k} q^{2m_2 n_1} E_{il}(t^{\mathbf{m}+\mathbf{n}}) - \delta_{l,i} q^{2n_2 m_1} E_{kj}(t^{\mathbf{m}+\mathbf{n}}),$$

where  $\mathbf{m} = (m_1, m_2)$  and  $\mathbf{n} = (n_1, n_2)$  are in  $\mathbb{Z}^2$ . Thus the derived Lie subalgebra of  $\tilde{\tau}$  is  $\bar{\tau} = \mathfrak{sl}_2(\mathbb{C}_{q^2}) \oplus \langle I(t^{\mathbf{m}}) \mid \mathbf{m} \in \mathbb{Z}^{2*} \rangle$ , where  $I = E_{11} + E_{22}$ , since  $q$  is generic. The universal central extension of  $\bar{\tau}$  is  $\tau = \bar{\tau} \oplus \langle K_1, K_2 \rangle$  with Lie bracket

$$[X(t^{\mathbf{m}}), Y(t^{\mathbf{n}})] = [X(t^{\mathbf{m}}), Y(t^{\mathbf{n}})]_0 + \delta_{\mathbf{m}+\mathbf{n}, 0} q^{2m_2 n_1} (X, Y)(m_1 K_1 + m_2 K_2),$$

where  $K_1$  and  $K_2$  are central,  $X(t^{\mathbf{m}}), Y(t^{\mathbf{n}}) \in \bar{\tau}$  and  $(X, Y)$  is the trace of  $XY$ . The Lie algebra  $\tau$  is the core of the EALAs of type  $A_1$  with coordinates in  $\mathbb{C}_{q^2}$ . If we add degree derivations  $d_1$  and  $d_2$  to  $\tau$ , then  $\tau \oplus \langle d_1, d_2 \rangle$  becomes an EALA since  $q$  is generic.

Now we construct our Lie algebra. Let

$$Q = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & q^{-1} \\ 1 & q & 1 \end{pmatrix}.$$

Let  $J$  be the two-sided ideal of  $\mathbb{C}_Q$  generated by  $t_0^2 - 1$ . Define

$$\tilde{L} = \mathbb{C}_Q / J = \langle t_0^i t_1^j t_2^k \mid i \in \mathbb{Z}_2, j, k \in \mathbb{Z} \rangle,$$

to be the quotient of  $\mathbb{C}_Q$  by  $J$ , and identify  $t_0$  with its image in  $\tilde{L}$ . Then the derived Lie subalgebra of  $\tilde{L}$  is

$$\bar{L} = \langle t_0^{\bar{0}} t^{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}^{2*} \rangle \oplus \langle t_0^{\bar{1}} t^{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}^2 \rangle.$$

Now we define a central extension  $L = \bar{L} \oplus \langle c_1, c_2 \rangle$  of  $\bar{L}$  by the Lie bracket

$$\begin{aligned} [t_0^i t^{\mathbf{m}}, t_0^j t^{\mathbf{n}}] &= ((-1)^{m_1 j} q^{m_2 n_1} - (-1)^{i n_1} q^{m_1 n_2}) t_0^{i+j} t^{\mathbf{m}+\mathbf{n}} \\ &\quad + (-1)^{m_1 j} q^{m_2 n_1} \delta_{i+j, \bar{0}} \delta_{\mathbf{m}+\mathbf{n}, 0} (m_1 c_1 + m_2 c_2), \end{aligned}$$

where  $c_1$  and  $c_2$  are central and where  $i, j$  are in  $\mathbb{Z}_2$ , as are  $\mathbf{m} = (m_1, m_2)$  and  $\mathbf{n} = (n_1, n_2)$ . One can easily see that  $\langle t_0^{\bar{0}} t^{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}^{2*} \rangle \oplus \langle c_1, c_2 \rangle$  is a Lie subalgebra of  $L$  that is isomorphic to the  $q$ -analogue Virasoro-like algebra.

First we prove that the Lie algebra  $L$  is in fact isomorphic to the core of the EALAs of type  $A_1$  with coordinates in  $\mathbb{C}_{q^2}$ .

**Proposition 2.1.** *The Lie algebra  $L$  is isomorphic to  $\tau$  and the isomorphism is given by the linear extension of the map  $\varphi$  defined by*

$$\begin{aligned} t_0^i t_1^{2m_1+1} t_2^{m_2} &\mapsto (-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}), \\ t_0^i t_1^{2m_1} t_2^{m_2} &\mapsto (-1)^i E_{11}(t_1^{m_1} t_2^{m_2}) + q^{-m_2} E_{22}(t_1^{m_1} t_2^{m_2}) + \delta_{i,\bar{1}} \delta_{m_1,0} \delta_{m_2,0} \frac{1}{2} K_1, \\ c_1 &\mapsto K_1, \\ c_2 &\mapsto 2K_2, \end{aligned}$$

where  $t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^i t_1^{2m_1} t_2^{m_2} \in L$ .

*Proof.* One can easily see that  $\varphi$  is a bijection. Thus we only need to prove that  $\varphi$  preserves Lie bracket. First we have

$$\begin{aligned} & [(-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}), (-1)^j q^{-n_2} E_{12}(t_1^{n_1} t_2^{n_2}) + E_{21}(t_1^{n_1+1} t_2^{n_2})] \\ &= ((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{n_2(2m_1+1)}) ((-1)^{i+j} E_{11}(t_1^{m_1+n_1+1} t_2^{m_2+n_2}) \\ &\quad + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})) \\ &\quad + \delta_{m_1+n_1+1,0} \delta_{m_2+n_2,0} (-1)^j q^{m_2(2n_1+1)} ((-1)^{i+j} (m_1 K_1 + m_2 K_2) \\ &\quad + (m_1 + 1) K_1 + m_2 K_2) \\ &= ((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{n_2(2m_1+1)}) ((-1)^{i+j} E_{11}(t_1^{m_1+n_1+1} t_2^{m_2+n_2}) \\ &\quad + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})) \\ &\quad + \delta_{i+j,\bar{0}} \delta_{m_1+n_1+1,0} \delta_{m_2+n_2,0} (-1)^j q^{m_2(2n_1+1)} ((2m_1 + 1) K_1 + 2m_2 K_2) \\ &\quad + \delta_{i+j,\bar{1}} \delta_{m_1+n_1+1,0} \delta_{m_2+n_2,0} (-1)^j q^{m_2(2n_1+1)} K_1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & [t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1+1} t_2^{n_2}] \\ &= ((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{(2m_1+1)n_2}) t_0^{i+j} t_1^{2m_1+2n_1+2} t_2^{m_2+n_2} \\ &\quad + \delta_{i+j,\bar{0}} \delta_{2m_1+2n_1+2,0} \delta_{m_2+n_2,0} (-1)^j q^{m_2(2n_1+1)} ((2m_1 + 1) c_1 + m_2 c_2). \end{aligned}$$

Thus

$$\varphi([t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1+1} t_2^{n_2}]) = [\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1+1} t_2^{n_2})].$$

Similarly, we have

$$\begin{aligned} & [\varphi(t_0^i t_1^{2m_1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})] \\ &= [(-1)^i E_{11}(t_1^{m_1} t_2^{m_2}) + q^{-m_2} E_{22}(t_1^{m_1} t_2^{m_2}), (-1)^j E_{11}(t_1^{n_1} t_2^{n_2}) + q^{-n_2} E_{22}(t_1^{n_1} t_2^{n_2})] \\ &= (q^{2m_2 n_1} - q^{2n_2 m_1}) ((-1)^{i+j} E_{11}(t_1^{m_1+n_1} t_2^{m_2+n_2}) + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1} t_2^{m_2+n_2})) \\ &\quad + \delta_{m_1+n_1,0} \delta_{m_2+n_2,0} \delta_{i+j,\bar{0}} q^{2m_2 n_1} (2m_1 K_1 + 2m_2 K_2), \end{aligned}$$

and

$$\begin{aligned} [t_0^i t_1^{2m_1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}] &= (q^{2m_2 n_1} - q^{2m_1 n_2}) t_0^{i+j} t_1^{2m_1+2n_1} t_2^{m_2+n_2} \\ &\quad + \delta_{i+j,\bar{0}} \delta_{m_1+n_1,0} \delta_{m_2+n_2,0} q^{2m_2 n_1} (2m_1 c_1 + m_2 c_2). \end{aligned}$$

Therefore

$$[\varphi(t_0^i t_1^{2m_1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})] = \varphi([t_0^i t_1^{2m_1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}]).$$

Finally, we have

$$\begin{aligned} & [\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})] \\ &= [(-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}), (-1)^j E_{11}(t_1^{n_1} t_2^{n_2}) + q^{-n_2} E_{22}(t_1^{n_1} t_2^{n_2})] \\ &= ((-1)^j q^{2m_2 n_1} - q^{n_2(2m_1+1)}) \\ &\quad \cdot ((-1)^{i+j} q^{-m_2-n_2} E_{12}(t_1^{m_1+n_1} t_2^{m_2+n_2}) + E_{21}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})), \end{aligned}$$

and

$$[t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}] = ((-1)^j q^{2m_2 n_1} - q^{n_2(2m_1+1)}) t_0^{i+j} t_1^{2m_1+2n_1+1} t_2^{m_2+n_2}.$$

Thus

$$[\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})] = \varphi([t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}]). \quad \square$$

**Remark 2.2.** This proof shows also that  $\mathfrak{gl}_2(\mathbb{C}_{q^2}) \cong \tilde{L}$  and  $\bar{\tau} \cong \bar{L}$ .

Next we will recall some concepts about  $\mathbb{Z}$ -graded  $L$ -modules. Fix a  $\mathbb{Z}$ -basis

$$\mathbf{m}_1 = (m_{11}, m_{12}) \quad \text{and} \quad \mathbf{m}_2 = (m_{21}, m_{22}) \in \mathbb{Z}^2.$$

If we define the degree of the elements in  $\langle t_0^i t_1^{jm_1+km_2} \in L \mid i \in \mathbb{Z}_2, k \in \mathbb{Z} \rangle$  to be  $j$  and the degree of the elements in  $\langle c_1, c_2 \rangle$  to be zero, then  $L$  can be regarded as a  $\mathbb{Z}$ -graded Lie algebra with graded subspaces

$$L_j = \langle t_0^i t_1^{jm_1+km_2} \in L \mid i \in \mathbb{Z}_2, k \in \mathbb{Z} \rangle \oplus \delta_{j,0} \langle c_1, c_2 \rangle,$$

so that  $L = \bigoplus_{j \in \mathbb{Z}} L_j$ . Setting  $L_+ = \bigoplus_{j \in \mathbb{N}} L_j$  and  $L_- = \bigoplus_{-j \in \mathbb{N}} L_j$ , we have the triangular decomposition  $L = L_- \oplus L_0 \oplus L_+$ .

**Definition 2.3.** For any  $L$ -module  $V$ , if  $V = \bigoplus_{m \in \mathbb{Z}} V_m$  with  $L_j \cdot V_m \subset V_{m+j}$  for all  $j, m \in \mathbb{Z}$ , then  $V$  is called a  $\mathbb{Z}$ -graded  $L$ -module and  $V_m$  is called a homogeneous subspace of  $V$  with degree  $m \in \mathbb{Z}$ . The  $L$ -module  $V$  is called

- (i) a quasifinite  $\mathbb{Z}$ -graded module if  $\dim V_m < \infty$  for all  $m \in \mathbb{Z}$ ;
- (ii) a uniformly bounded module if there exists some  $N \in \mathbb{N}$  such that  $\dim V_m \leq N$  for all  $m \in \mathbb{Z}$ ;
- (iii) a highest (respectively lowest) weight module if  $V$  is generated by some nonzero  $v \in V_m$  such that  $L_+ \cdot v = 0$  (respectively  $L_- \cdot v = 0$ );
- (iv) a generalized highest weight module with highest degree  $m$  (see for example [Su 2003]) if there is a  $\mathbb{Z}$ -basis  $B = \{\mathbf{b}_1, \mathbf{b}_2\}$  of  $\mathbb{Z}^2$  and a nonzero  $v \in V_m$  such that  $V$  is generated by  $v$ , and  $t_0^i t^{\mathbf{m}} \cdot v = 0$  for all  $\mathbf{m} \in \mathbb{Z}_+ \mathbf{b}_1 + \mathbb{Z}_+ \mathbf{b}_2$  and  $i \in \mathbb{Z}_2$ ;
- (v) an irreducible  $\mathbb{Z}$ -graded module if  $V$  does not have any nontrivial  $\mathbb{Z}$ -graded submodules (see for example [Mathieu 1992]).

Denote by  $\mathbb{C}_{\mathbb{Z}}$  the set of nontrivial quasifinite irreducible  $\mathbb{Z}$ -graded  $L$ -modules. From the definition, one sees that generalized highest weight modules contain highest and lowest weight modules as special cases. Since the central elements  $c_1$  and  $c_2$  of  $L$  act on irreducible graded modules  $V$  as scalars, we shall use the same symbols to denote these scalars.

Now we study the structure and representations of  $L_0$ . By the theory of Verma modules, the irreducible  $\mathbb{Z}$ -graded highest (or lowest) weight  $L$ -modules are classified by the characters of  $L_0$ .

**Lemma 2.4.** (1) *If  $m_{21}$  is an even integer, then  $L_0$  is a Heisenberg Lie algebra.*

(2) *If  $m_{21}$  is an odd integer, then*

$$L_0 = (\mathcal{A} + \mathcal{B}) \oplus \langle m_{11}c_1 + m_{12}c_2 \rangle,$$

where  $\mathcal{A} = \langle t_0^{\bar{0}} t^{2jm_2}, m_{21}c_1 + m_{22}c_2 \mid j \in \mathbb{Z} \rangle$  is a Heisenberg Lie algebra and

$$\mathcal{B} = \langle t_0^{\bar{1}} t^{jm_2}, t_0^{\bar{0}} t^{(2j+1)m_2}, m_{21}c_1 + m_{22}c_2 \mid j \in \mathbb{Z} \rangle,$$

which is isomorphic to the affine Lie algebra  $A_1^{(1)}$  via the linear extension of the map  $\phi$  defined by

$$\begin{aligned} t_0^{\bar{1}} t^{2jm_2} &\mapsto -q^{-2j^2 m_{22} m_{21}} ((E_{11} - E_{22})(x^j) + \frac{1}{2}K), \\ t_0^{\bar{0}} t^{(2j+1)m_2} &\mapsto q^{-\frac{1}{2}(2j+1)^2 m_{22} m_{21}} ((-1)^j E_{12}(x^j) + E_{21}(x^{j+1})), \\ m_{21}c_1 + m_{22}c_2 &\mapsto K. \end{aligned}$$

Moreover, we have  $[\mathcal{A}, \mathcal{B}] = 0$ .

*Proof.* Statement (1) can be easily deduced from the definition of  $L_0$ .

To show  $\mathcal{B} \cong A_1^{(1)}$  in case (2), we need to prove that  $\phi$  preserves Lie bracket. Notice that

$$\begin{aligned} & \left[ q^{-\frac{1}{2}(2j+1)^2 m_{22} m_{21}} ((-1)^i E_{12}(x^j) + E_{21}(x^{j+1})), \right. \\ & \left. q^{-\frac{1}{2}(2l+1)^2 m_{22} m_{21}} ((-1)^k E_{12}(x^l) + E_{21}(x^{l+1})) \right] \\ &= q^{-\frac{1}{2}((2j+1)^2 + (2l+1)^2) m_{22} m_{21}} \left( ((-1)^i - (-1)^k) (E_{11} - E_{22})(x^{j+l+1}) \right. \\ & \quad \left. + \delta_{j+l+1,0} ((-1)^i j + (-1)^k (j+1)) K \right), \end{aligned}$$

and

$$\begin{aligned} & [t_0^i t^{(2j+1)m_2}, t_0^k t^{(2k+1)m_2}] \\ &= ((-1)^k - (-1)^i) q^{(2j+1)(2k+1)m_{22} m_{21}} t_0^{i+k} t^{(2j+2k+2)m_2} \\ & \quad + \delta_{i+k,0} \delta_{j+k+1,0} (-1)^k q^{(2j+1)(2k+1)m_{22} m_{21}} (2j+1)(m_{21} c_1 + m_{22} c_2). \end{aligned}$$

One sees that

$$\phi([t_0^i t^{(2j+1)m_2}, t_0^k t^{(2k+1)m_2}]) = [\phi(t_0^i t^{(2j+1)m_2}), \phi(t_0^k t^{(2k+1)m_2})].$$

From the facts that

$$\begin{aligned} & \left[ -q^{-2j^2 m_{22} m_{21}} ((E_{11} - E_{22})(x^j) + \frac{1}{2}K), \right. \\ & \left. q^{-\frac{1}{2}(2l+1)^2 m_{22} m_{21}} ((-1)^k E_{12}(x^l) + E_{21}(x^{l+1})) \right] \\ &= -q^{-\frac{1}{2}(4j^2 + (2l+1)^2) m_{22} m_{21}} (2(-1)^k E_{12}(x^{l+j}) - 2E_{21}(x^{l+j+1})) \end{aligned}$$

and

$$[t_0^{\bar{1}} t^{2jm_2}, t_0^k t^{(2l+1)m_2}] = 2q^{2j(2l+1)m_{22} m_{21}} t_0^{k+\bar{1}} t^{(2j+2l+1)m_2},$$

we have

$$\phi([t_0^{\bar{1}} t^{2jm_2}, t_0^k t^{(2l+1)m_2}]) = [\phi(t_0^{\bar{1}} t^{2jm_2}), \phi(t_0^k t^{(2l+1)m_2})].$$

Finally, we have

$$\begin{aligned} & \left[ -q^{-2j^2 m_{22} m_{21}} ((E_{11} - E_{22})(x^j) + \frac{1}{2}K), -q^{-2l^2 m_{22} m_{21}} ((E_{11} - E_{22})(x^l) + \frac{1}{2}K) \right] \\ &= 2jq^{-2(j^2+l^2)m_{22} m_{21}} \delta_{j+l,0} K = 2jq^{4jl m_{22} m_{21}} \delta_{j+l,0} K, \end{aligned}$$

and

$$[t_0^{\bar{1}} t^{2jm_2}, t_0^{\bar{1}} t^{2lm_2}] = 2jq^{4jl m_{22} m_{21}} \delta_{j+l,0} (m_{21} c_1 + m_{22} c_2).$$

Thus

$$\phi([t_0^{\bar{1}} t^{2jm_2}, t_0^{\bar{1}} t^{2lm_2}]) = [\phi(t_0^{\bar{1}} t^{2jm_2}), \phi(t_0^{\bar{1}} t^{2lm_2})].$$

This proves  $\mathcal{B} \cong A_1^{(1)}$ . The proof of the remaining claims is straightforward.  $\square$

Since the Lie subalgebra  $\mathcal{B}$  of  $L_0$  is isomorphic to the affine Lie algebra  $A_1^{(1)}$ , we need to collect some results from [Rao 1993] on the finite-dimensional irreducible modules of  $A_1^{(1)}$ .

Let  $\nu > 0$ , and let  $\underline{a} = (a_1, \dots, a_\nu)$  be a finite sequence of nonzero distinct numbers. For  $1 \leq i \leq \nu$ , let  $V_i$  be finite-dimensional irreducible  $\mathfrak{sl}_2$ -modules, and let  $\mathbf{v} := (v_1 \otimes \cdots \otimes v_\nu) \in V_1 \otimes \cdots \otimes V_\nu$ . We then define an  $A_1^{(1)}$ -module  $V(\underline{a}) = V_1 \otimes V_2 \otimes \cdots \otimes V_\nu$  by setting

$$X(x^j) \cdot \mathbf{v} = \sum_{i=1}^{\nu} a_i^j v_1 \otimes \cdots \otimes (X \cdot v_i) \otimes \cdots \otimes v_\nu \quad \text{and} \quad K \cdot \mathbf{v} = 0$$

for  $X \in \mathfrak{sl}_2$  and  $j \in \mathbb{Z}$ . Clearly  $V(\underline{a})$  is a finite-dimensional irreducible  $A_1^{(1)}$ -module. For any  $Q(x) \in \mathbb{C}[x^{\pm 1}]$ , we have  $X(Q(x)) \cdot (V_1 \otimes \cdots \otimes V_\nu) = 0$  for all  $X \in \mathfrak{sl}_2$  if and only if  $\prod_{i=1}^{\nu} (x - a_i) \mid Q(x)$ . Now by Lemma 2.4(2), if  $m_{21}$  is an odd integer, we can define a finite-dimensional irreducible  $L_0$ -module  $V(\underline{a}, \psi) = V_1 \otimes \cdots \otimes V_\nu$  by

$$\begin{aligned} t_0^{\bar{0}} t^{2jm_2} \cdot \mathbf{v} &= \psi(t_0^{\bar{0}} t^{2jm_2}) \cdot (v_1 \otimes \cdots \otimes v_\nu), \\ t_0^{\bar{1}} t^{2jm_2} \cdot \mathbf{v} &= -q^{-2j^2 m_{22} m_{21}} \sum_{i=1}^{\nu} a_i^j v_1 \otimes \cdots \otimes ((E_{11} - E_{22}) \cdot v_i) \otimes \cdots \otimes v_\nu, \\ t_0^i t^{(2j+1)m_2} \cdot \mathbf{v} &= q^{-\frac{1}{2}(2j+1)^2 m_{22} m_{21}} \left( (-1)^i \sum_{i=1}^{\nu} a_i^j v_1 \otimes \cdots \otimes (E_{12} \cdot v_i) \otimes \cdots \otimes v_\nu \right. \\ &\quad \left. + \sum_{i=1}^{\nu} a_i^{j+1} v_1 \otimes \cdots \otimes (E_{21} \cdot v_i) \otimes \cdots \otimes v_\nu \right), \\ (m_{21} c_1 + m_{22} c_2) \cdot \mathbf{v} &= 0 \end{aligned}$$

for  $j \in \mathbb{Z}$  and  $i \in \mathbb{Z}_2$ . Here  $\psi$  is a linear function over  $\mathcal{A}$ .

**Theorem 2.5** [Rao 1993, Theorem 2.14]. *Let  $V$  be a finite-dimensional irreducible  $A_1^{(1)}$ -module. Then  $V$  is isomorphic to  $V(\underline{a})$  for some finite-dimensional irreducible  $\mathfrak{sl}_2$ -modules  $V_1, \dots, V_\nu$  and a finite sequence  $\underline{a} = (a_1, \dots, a_\nu)$  of nonzero distinct numbers.*

This theorem and Lemma 2.4 implies another:

**Theorem 2.6.** *Let  $m_{21}$  be an odd integer, and let  $V$  be a finite-dimensional irreducible  $L_0$ -module. Then  $V$  is isomorphic to  $V(\underline{a}, \psi)$ , where  $V_1, \dots, V_\nu$  are finite-dimensional irreducible  $\mathfrak{sl}_2$ -modules,  $\underline{a} = (a_1, \dots, a_\nu)$  is a finite sequence of nonzero distinct numbers, and  $\psi$  is a linear function over  $\mathcal{A}$ .*

**Remark 2.7.** Let  $m_{21}$  be an odd integer, and let  $V(\underline{a}, \psi)$  be a finite-dimensional irreducible  $L_0$ -modules defined as above. One can see that for any  $k \in \mathbb{Z}_2$ ,

$$\left( \sum_{i=1}^n b_i q^{\frac{1}{2}(2i+1)^2 m_{22} m_{21}} t_0^k t^{\bar{1} (2i+1) m_2} \right) \cdot (V_1 \otimes \cdots \otimes V_n) = 0 \quad \text{and}$$

$$\left( \sum_{i=1}^n b_i q^{2i^2 m_{22} m_{21}} t_0^{\bar{1}} t^{2i m_2} \right) \cdot (V_1 \otimes \cdots \otimes V_n) = 0$$

if and only if  $\prod_{i=1}^n (x - a_i) \mid (\sum_{i=1}^n b_i x^i)$ .

**Proposition 2.8.** *If  $V$  is an irreducible  $\mathbb{Z}$ -graded  $L$ -module, then  $V$  is a generalized highest weight module or a uniformly bounded module.*

*Proof.* Let  $V = \bigoplus_{m \in \mathbb{Z}} V_m$ . We first prove that if there exists a  $\mathbb{Z}$ -basis  $\{\mathbf{b}_1, \mathbf{b}_2\}$  of  $\mathbb{Z}^2$  and a homogeneous vector  $v \neq 0$  such that  $t_0^i t^{\mathbf{b}_1} \cdot v = t_0^i t^{\mathbf{b}_2} \cdot v = 0$  for all  $i \in \mathbb{Z}_2$ , then  $V$  is a generalized highest weight module.

For  $A \subset \mathbb{Z}^2$ , we denote by  $t^A$  the set  $\{t^a \mid a \in A\}$ .

By assumption, one can deduce that  $t_0^i t^{\mathbb{N}\mathbf{b}_1 + \mathbb{N}\mathbf{b}_2} \cdot v = 0$  for all  $i \in \mathbb{Z}_2$ . Thus for the  $\mathbb{Z}$ -basis  $\{\mathbf{m}_1 = 3\mathbf{b}_1 + \mathbf{b}_2, \mathbf{m}_2 = 2\mathbf{b}_1 + \mathbf{b}_2\}$  of  $\mathbb{Z}^2$  we have  $t_0^i t^{\mathbb{Z}\mathbf{m}_1 + \mathbb{Z}\mathbf{m}_2} v = 0$  for all  $i \in \mathbb{Z}_2$ , so that  $V$  meets the definition of generalized highest weight module.

We can prove our proposition. Suppose that  $V$  is not a generalized highest weight module. For any  $m \in \mathbb{Z}$ , consider the maps

$$\begin{aligned} t_0^{\bar{0}} t^{-m\mathbf{m}_1 + \mathbf{m}_2} : V_m &\mapsto V_0, & t_0^{\bar{1}} t^{-m\mathbf{m}_1 + \mathbf{m}_2} : V_m &\mapsto V_0, \\ t_0^{\bar{0}} t^{(1-m)\mathbf{m}_1 + \mathbf{m}_2} : V_m &\mapsto V_1, & t_0^{\bar{1}} t^{(1-m)\mathbf{m}_1 + \mathbf{m}_2} : V_m &\mapsto V_1. \end{aligned}$$

Since  $\{-m\mathbf{m}_1 + \mathbf{m}_2, (1-m)\mathbf{m}_1 + \mathbf{m}_2\}$  is a  $\mathbb{Z}$ -base of  $\mathbb{Z}^2$ , one can check that

$$\ker t_0^{\bar{0}} t^{-m\mathbf{m}_1 + \mathbf{m}_2} \cap \ker t_0^{\bar{0}} t^{(1-m)\mathbf{m}_1 + \mathbf{m}_2} \cap \ker t_0^{\bar{1}} t^{-m\mathbf{m}_1 + \mathbf{m}_2} \cap \ker t_0^{\bar{1}} t^{(1-m)\mathbf{m}_1 + \mathbf{m}_2} = \{0\},$$

Therefore  $\dim V_m \leq 2 \dim V_0 + 2 \dim V_1$ . So  $V$  is a uniformly bounded module.  $\square$

### 3. The highest weight irreducible $\mathbb{Z}$ -graded $L$ -modules

In this section,  $V$  is a finite-dimensional irreducible  $L_0$ -module;  $V$  becomes a  $(L_0 + L_+)$ -module if we put  $L_+ v = 0$  for all  $v \in V$ . Then we obtain an induced  $L$ -module

$$\bar{M}^+(V, \mathbf{m}_1, \mathbf{m}_2) = \text{Ind}_{L_0 + L_+}^L V = U(L) \otimes_{U(L_0 + L_+)} V \simeq U(L_-) \otimes V,$$

where  $U(L)$  is the universal enveloping algebra of  $L$ . If we set  $V$  to be the homogeneous subspace of  $\bar{M}^+(V, \mathbf{m}_1, \mathbf{m}_2)$  with degree 0, then  $\bar{M}^+(V, \mathbf{m}_1, \mathbf{m}_2)$  becomes a  $\mathbb{Z}$ -graded  $L$ -module in a natural way. Obviously,  $\bar{M}^+(V, \mathbf{m}_1, \mathbf{m}_2)$  has a

unique maximal proper submodule  $J$  that trivially intersects with  $V$ . So we obtain an irreducible  $\mathbb{Z}$ -graded highest weight  $L$ -module

$$M^+(V, \mathbf{m}_1, \mathbf{m}_2) = \overline{M}^+(V, \mathbf{m}_1, \mathbf{m}_2)/J.$$

We can write it as  $M^+(V, \mathbf{m}_1, \mathbf{m}_2) = \bigoplus_{i \in \mathbb{Z}_+} V_{-i}$ , where  $V_{-i}$  is the homogeneous subspace of degree  $-i$ . Since  $L_-$  is generated by  $L_{-1}$ , and  $L_+$  is generated by  $L_1$ , we see by the construction of  $M^+(V, \mathbf{m}_1, \mathbf{m}_2)$  that

$$(3-1) \quad L_{-1}V_{-i} = V_{-i-1} \quad \text{for all } i \in \mathbb{Z}_+,$$

and for a homogeneous vector  $v \in V_i$  with  $i < 0$ ,

$$(3-2) \quad L_1 \cdot v = 0 \quad \text{implies} \quad v = 0.$$

Similarly,  $V$  gives rise to an irreducible lowest weight  $\mathbb{Z}$ -graded  $L$ -module  $M^-(V, \mathbf{m}_1, \mathbf{m}_2)$ .

If  $m_{21} \in \mathbb{Z}$  is even, then  $L_0$  is a Heisenberg Lie algebra by Lemma 2.4. It is well known from the representation theory of this algebra that  $V$  must be a one-dimensional module  $\mathbb{C}v_0$ , and there is a linear function  $\psi$  over  $L_0$  such that

$$t_0^i t^{jm_2} \cdot v_0 = \psi(t_0^i t^{jm_2}) \cdot v_0 \quad \text{and} \quad \psi(m_{21}c_1 + m_{22}c_2) = 0 \quad \text{for all } i \in \mathbb{Z}_2, j \in \mathbb{Z}.$$

In this case, we denote the corresponding highest and lowest weight irreducible  $\mathbb{Z}$ -graded  $L$ -modules by  $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$  and  $M^-(\psi, \mathbf{m}_1, \mathbf{m}_2)$ , respectively.

If  $m_{21}$  is an odd integer, then by Theorem 2.6,  $V$  must be isomorphic to  $V(\underline{a}, \psi)$ , and we denote the corresponding highest and lowest weight irreducible  $\mathbb{Z}$ -graded  $L$ -modules by  $M^+(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2)$  and  $M^-(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2)$ , respectively.

The  $L$ -modules  $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$  and  $M^+(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2)$  are quasifinite only for certain  $\underline{a}$  and  $\psi$ , which we shall now determine.

For later use, we obtain from the definition of  $L$  the equations

$$(3-3) \quad \begin{aligned} [t_0^j t^{m_1+km_2}, t_0^r t^{-m_1+sm_2} t^{im_2}] &= q^{i(-m_{12}+sm_{22})m_{21}} [t_0^j t^{m_1+km_2}, t_0^r t^{-m_1+(s+i)m_2}] \\ &= q^{-m_{11}m_{12}-km_{11}m_{22}+sm_{12}m_{21}+ksm_{21}m_{22}} (-1)^{r(m_{11}+km_{21})} \\ &\quad \cdot ((1 - (-1)^{(j+r)m_{11}+(kr+js+ji)m_{21}} q^{(k+s+i)\alpha}) t_0^{j+r} t^{(k+s)m_2} t^{im_2} \\ &\quad + \delta_{k+s+i,0} \delta_{j+r,0} q^{-(k+s)^2 m_{21} m_{22}} ((m_{11} + km_{21})c_1 + (m_{12} + km_{22})c_2)), \end{aligned}$$

and

$$(3-4) \quad \begin{aligned} [t_0^s t^{km_2} t^{im_2}, t_0^r t^{-m_1+jm_2}] &= q^{kim_{22}m_{21}} [t_0^s t^{(k+i)m_2}, t_0^r t^{-m_1+jm_2}] \\ &= q^{km_{22}(-m_{11}+jm_{21})} (-1)^{(rk+ri)m_{21}} (q^{-i\alpha} - (-1)^{sm_{11}+(rk+ri+sj)m_{21}} q^{k\alpha}) \\ &\quad \cdot t_0^{r+s} t^{-m_1+(k+j)m_2} t^{im_2}. \end{aligned}$$

Here  $\alpha = m_{11}m_{22} - m_{12}m_{21} \in \{\pm 1\}$ ,

**Lemma 3.1.** *Let  $m_{21}$  be an even integer. Then  $M^\pm(\psi, \mathbf{m}_1, \mathbf{m}_2) \in \mathbb{C}_{\mathbb{Z}}$  if and only if there exists a polynomial  $P(t^{m_2}) = \sum_{i=0}^n a_i t^{im_2} \in \mathbb{C}[t^{m_2}]$  with  $a_0 a_n \neq 0$  such that for  $k \in \mathbb{Z}$  and  $j \in \mathbb{Z}_2$ ,*

$$(3-5) \quad \psi(t_0^j t^{km_2} P(t^{m_2}) - (-1)^j q^{k\alpha} t_0^j t^{km_2} P(q^\alpha t^{m_2}) + \delta_{j, \bar{0}} a_{-k} q^{-k^2 m_{21} m_{22}} \beta) = 0,$$

where  $\alpha = m_{11} m_{22} - m_{12} m_{21} \in \{\pm 1\}$  and  $\beta = m_{11} c_1 + m_{12} c_2$ , and where  $a_k = 0$  if  $k \notin \{0, 1, \dots, n\}$ .

*Proof.* Since  $m_{21}$  is an even integer and  $m_{11} m_{22} - m_{12} m_{21} \in \{\pm 1\}$ , we see  $m_{11}$  is an odd integer.

We first prove the forward implication. Since  $\dim V_{-1} < \infty$ , there exist an integer  $s$  and a polynomial  $P(t^{m_2}) = \sum_{i=0}^n a_i t^{im_2} \in \mathbb{C}[t^{m_2}]$  with  $a_0 a_n \neq 0$  such that

$$t_0^{\bar{0}} t^{-m_1 + s m_2} P(t^{m_2}) \cdot v_0 = 0.$$

Applying  $t_0^j t^{m_1 + k m_2}$  for any  $k \in \mathbb{Z}$  and  $j \in \mathbb{Z}_2$  to this equation, we have

$$0 = t_0^j t^{m_1 + k m_2} \cdot t_0^{\bar{0}} t^{-m_1 + s m_2} P(t^{m_2}) \cdot v_0 = \sum_{i=0}^n [t_0^j t^{m_1 + k m_2}, a_i t_0^{\bar{0}} t^{-m_1 + s m_2} t^{im_2}] \cdot v_0.$$

Thus, by (3-3), we have

$$\begin{aligned} 0 &= \psi \left( \sum_{i=0}^n a_i \left( (1 - (-1)^j q^{(k+s+i)\alpha} \right) t_0^j t^{(k+s)m_2} t^{im_2} + \delta_{k+s+i, \bar{0}} \delta_{j, \bar{0}} q^{-(k+s)^2 m_{21} m_{22}} \beta \right) \\ &= \psi \left( t_0^j t^{(k+s)m_2} P(t^{m_2}) - (-1)^j q^{(k+s)\alpha} t_0^j t^{(k+s)m_2} P(q^\alpha t^{m_2}) \right. \\ &\quad \left. + a_{-k-s} \delta_{j, \bar{0}} q^{-(k+s)^2 m_{21} m_{22}} \beta \right). \end{aligned}$$

This proves necessity.

We next prove sufficiency.

**Claim 1.** For any  $s \in \mathbb{Z}_+$ , there is a polynomial  $P_s(t^{m_2}) = \sum_{i \in \mathbb{Z}} a_{s,i} t^{im_2} \in \mathbb{C}[t^{m_2}]$  such that, for all  $r \in \mathbb{Z}_2$  and  $k \in \mathbb{Z}$ ,

$$\begin{aligned} (t_0^r t^{k m_2} P_s(t^{m_2}) - (-1)^r q^{k\alpha} t_0^r t^{k m_2} P_s(q^\alpha t^{m_2}) + \delta_{r, \bar{0}} a_{s, -k} q^{-k^2 m_{21} m_{22}} \beta) \cdot V_{-s} &= 0, \\ t_0^r t^{-m_1 + k m_2} P_s(t^{m_2}) \cdot V_{-s} &= 0. \end{aligned}$$

We prove the claim by induction on  $s$ . For  $s = 0$ , the first equation holds with  $P_0(t^{m_2}) = P(t^{m_2})$ , where  $P$  is as in the proof of necessity, and by (3-2), the second equation proved by proceeding as in the forward direction.

Now suppose the claim holds for  $s$ . For  $s + 1$ , the equations in the claim are equivalent, for all  $r \in \mathbb{Z}_2$  and  $k \in \mathbb{Z}$ , to

$$(3-6) \quad \begin{aligned} (t_0^r Q(t^{m_2}) - (-1)^r t_0^r Q(q^\alpha t^{m_2}) + \delta_{r, \bar{0}} a_Q \beta) \cdot V_{-s} &= 0, \\ t_0^r t^{-m_1 + k m_2} Q(t^{m_2}) \cdot V_{-s} &= 0 \end{aligned}$$

for any  $Q(t^{m_2}) \in \mathbb{C}[t^{\pm m_2}]$  with  $P_s(t^{m_2}) \mid Q(t^{m_2})$ , where  $a_Q$  is the constant term of  $Q(t^{m_2})$ .

Let  $P_{s+1}(t^{m_2}) = P_s(q^\alpha t^{m_2})P_s(t^{m_2})P_s(q^{-\alpha} t^{m_2})$ . Then we have

$$P_s(t^{m_2}) \mid P_{s+1}(t^{m_2}), \quad P_s(t^{m_2}) \mid P_{s+1}(q^\alpha t^{m_2}), \quad P_s(t^{m_2}) \mid P_{s+1}(q^{-\alpha} t^{m_2}).$$

For any  $p, r \in \mathbb{Z}_2$  and  $j, k \in \mathbb{Z}$ , by induction and (3-4), we have

$$\begin{aligned} & (t_0^r t^{km_2} P_{s+1}(t^{m_2}) - (-1)^r q^{k\alpha} t_0^r t^{km_2} P_{s+1}(q^\alpha t^{m_2}) + \delta_{r, \bar{0}} a_{s+1, -k} q^{-k^2 m_{21} m_{22}} \beta) \\ & \qquad \qquad \qquad \cdot t_0^p t^{-m_1 + jm_2} \cdot V_{-s} \\ &= (t_0^r t^{km_2} P_{s+1}(t^{m_2}) - (-1)^r q^{k\alpha} t_0^r t^{km_2} P_{s+1}(q^\alpha t^{m_2}) \\ & \qquad \qquad \qquad + \delta_{r, \bar{0}} a_{s+1, -k} q^{-k^2 m_{21} m_{22}} \beta, t_0^p t^{-m_1 + jm_2}) \cdot V_{-s} \\ &= q^{-km_{22} m_{11} + kjm_{22} m_{21}} (t_0^{r+p} t^{-m_1 + (k+j)m_2} (P_{s+1}(q^{-\alpha} t^{m_2}) - 2(-1)^r q^{k\alpha} P_{s+1}(t^{m_2}) \\ & \qquad \qquad \qquad + q^{2k\alpha} P_{s+1}(q^\alpha t^{m_2}))) \cdot V_{-s} \\ &= 0. \end{aligned}$$

Thus, by (3-1) and (3-2), we obtain

$$(3-7) \quad (t_0^r t^{km_2} P_{s+1}(t^{m_2}) - (-1)^r q^{k\alpha} t_0^r t^{km_2} P_{s+1}(q^\alpha t^{m_2}) + \delta_{r, \bar{0}} a_{s+1, -k} q^{-k^2 m_{21} m_{22}} \beta) \cdot V_{-s-1} = 0.$$

This proves the first equation in the claim for  $i = s + 1$ .

Using (3-3), (3-6) and induction, we deduce for any  $l, k, j \in \mathbb{Z}$  and  $n, r, p \in \mathbb{Z}_2$  that

$$\begin{aligned} & t_0^n t^{m_1 + lm_2} \cdot t_0^r t^{-m_1 + km_2} P_{s+1}(t^{m_2}) \cdot t_0^p t^{-m_1 + jm_2} \cdot V_{-s} \\ &= [t_0^n t^{m_1 + lm_2}, t_0^r t^{-m_1 + km_2} P_{s+1}(t^{m_2})] \cdot t_0^p t^{-m_1 + jm_2} \cdot V_{-s} \\ & \qquad \qquad \qquad + t_0^r t^{-m_1 + km_2} P_{s+1}(t^{m_2}) \cdot t_0^n t^{m_1 + lm_2} \cdot t_0^p t^{-m_1 + jm_2} \cdot V_{-s} \\ &= (-1)^r q^{-m_{11} m_{12} + km_{12} m_{21} - lm_{11} m_{22} + lkm_{21} m_{22}} \\ & \qquad \cdot (t_0^{n+r} t^{(l+k)m_2} P_{s+1}(t^{m_2}) - (-1)^{n+r} q^{(k+l)\alpha} t_0^{n+r} t^{(l+k)m_2} P_{s+1}(q^\alpha t^{m_2}) \\ & \qquad \qquad \qquad + a_{s+1, -l-k} \delta_{r+n, \bar{0}} q^{-(l+k)^2 m_{21} m_{22}} \beta) \cdot V_{-s} \\ &= 0, \end{aligned}$$

since  $t_0^n t^{m_1 + lm_2} \cdot t_0^p t^{-m_1 + jm_2} \cdot V_{-s} \in V_{-s}$ . Hence by (3-2),

$$t_0^r t^{-m_1 + km_2} P_{s+1}(t^{m_2}) \cdot t_0^p t^{-m_1 + jm_2} \cdot V_{-s} = 0 \quad \text{for all } r, p \in \mathbb{Z}_2 \text{ and } k, j \in \mathbb{Z}.$$

Thus, by (3-1),  $t_0^r t^{-m_1 + km_2} P_{s+1}(t^{m_2}) \cdot V_{-s-1} = 0$ , which implies the second equation in the claim for  $i = s + 1$ . Therefore the claim follows by induction.

From the second equation of the claim and (3-1), we see that

$$\dim V_{-s-1} \leq 2 \deg(P_{s+1}(t^{m_2})) \cdot \dim V_s \quad \text{for all } s \in \mathbb{Z}_+,$$

where  $\deg$  means ‘‘degree of’’. Hence  $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2) \in \mathbb{C}_\mathbb{Z}$ .

The statement for  $M^-(\psi, \mathbf{m}_1, \mathbf{m}_2)$  is proved similarly. □

**Theorem 3.2.** *Let  $m_{21}$  be an even integer. Then  $M^\pm(\psi, \mathbf{m}_1, \mathbf{m}_2) \in \mathbb{C}_\mathbb{Z}$  if and only if there exist*

$$b_{10}^{(j)}, b_{11}^{(j)}, \dots, b_{1s_1}^{(j)}, b_{20}^{(j)}, b_{21}^{(j)}, \dots, b_{2s_2}^{(j)}, \dots, b_{r0}^{(j)}, b_{r1}^{(j)}, \dots, b_{rs_r}^{(j)} \in \mathbb{C}$$

for  $j \in \mathbb{Z}_2$  and  $\alpha_1, \dots, \alpha_r \in \mathbb{C}^*$  such that for any  $i \in \mathbb{Z}^*$  and  $j \in \mathbb{Z}_2$ ,

$$\begin{aligned} \psi(t_0^j t^{im_2}) &= \frac{\sum_{\ell=1}^r (\sum_{k=0}^{s_\ell} b_{\ell k}^{(j)} i^k) \alpha_\ell^i}{(1 - (-1)^j q^{i\alpha}) q^{(1/2)i^2 m_{21} m_{22}}}, \\ \psi(\beta) &= b_{10}^{(0)} + b_{20}^{(0)} + \dots + b_{r0}^{(0)}, \\ \psi(t_0^{\bar{1}} t^0) &= \frac{1}{2}(b_{10}^{(1)} + b_{20}^{(1)} + \dots + b_{r0}^{(1)}), \quad \text{and} \quad \psi(m_{21}c_1 + m_{22}c_2) = 0, \end{aligned}$$

where  $\alpha = m_{11}m_{22} - m_{21}m_{12} \in \{\pm 1\}$  and  $\beta = m_{11}c_1 + m_{12}c_2$ .

*Proof.* We first prove necessity. Let  $f_{j,i} = \psi((1 - (-1)^j q^{i\alpha}) q^{\frac{1}{2}i^2 m_{21} m_{22}} t_0^j t^{im_2})$  for  $j \in \mathbb{Z}_2$  and  $i \in \mathbb{Z}^*$ . Also let  $f_{0,0} = \psi(\beta)$  and  $f_{1,0} = \psi(2t_0^{\bar{1}} t^0)$ . By Lemma 3.1 there exist complex numbers  $a_0, a_1, \dots, a_n$  with  $a_0 a_n \neq 0$  such that

$$(3-8) \quad \sum_{i=0}^n a_i q^{-\frac{1}{2}i^2 m_{21} m_{22}} f_{j,k+i} = 0 \quad \text{for all } k \in \mathbb{Z} \text{ and } j \in \mathbb{Z}_2.$$

Let  $b_i = a_i q^{-(1/2)i^2 m_{21} m_{22}}$ . Suppose  $\alpha_1, \dots, \alpha_r$  are distinct roots of the equation  $\sum_{i=0}^n b_i x^i = 0$  with respective multiplicities  $s_1 + 1, \dots, s_r + 1$ . By a well-known combinatorial formula, we see that for  $j \in \mathbb{Z}_2$  there exist

$$b_{10}^{(j)}, b_{11}^{(j)}, \dots, b_{1s_1}^{(j)}, \dots, b_{r0}^{(j)}, b_{r1}^{(j)}, \dots, b_{rs_r}^{(j)} \in \mathbb{C}$$

such that  $f_{j,i} = \sum_{\ell=1}^r (\sum_{k=0}^{s_\ell} b_{\ell k}^{(j)} i^k) \alpha_\ell^i$  for all  $i \in \mathbb{Z}$ . The equations of the theorem follow.

We now prove sufficiency. For  $j \in \mathbb{Z}_2$  and  $i \in \mathbb{Z}^*$ , set

$$\begin{aligned} Q(x) &= \prod_{i=1}^r (x - \alpha_i)^{s_i+1} = \sum_{i=1}^n b_i x^i \in \mathbb{C}[x], \\ f_{j,i} &= (1 - (-1)^j q^{i\alpha}) q^{(1/2)i^2 m_{21} m_{22}} \psi(t_0^j t^{im_2}), \end{aligned}$$

and set  $f_{0,0} = \psi(\beta)$  and  $f_{1,0} = 2\psi(t_0^{\bar{1}} t^0)$ . Then with  $b_i$  and  $a_i$  related as before, we deduce that (3-8) holds. Thus (3-5) holds for  $P(t^{m_2}) = \sum_{i=0}^n a_i t^{im_2}$ . Sufficiency now follows by using Lemma 3.1. □

**Lemma 3.3.** *If  $m_{21}$  is an odd integer, then  $M^+(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2) \in \mathbb{C}_{\mathbb{Z}}$  if and only if there exists a polynomial  $P(t^{\mathbf{m}_2}) = \sum_{i=0}^n a_i t^{2im_2} \in \mathbb{C}[t^{\mathbf{m}_2}]$  with  $a_0 a_n \neq 0$  such that for any  $k \in \mathbb{Z}$  and  $v \in V_0$ ,*

$$(3-9) \quad (t_0^{\bar{0}} t^{2km_2} P(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2km_2} P(q^\alpha t^{\mathbf{m}_2}) + a_{-k} q^{-4k^2 m_{21} m_{22}} \beta) \cdot v = 0,$$

$$(3-10) \quad t_0^{\bar{0}} t^{(2k+1)m_2} P(t^{\mathbf{m}_2}) \cdot v = t_0^{\bar{0}} t^{(2k+1)m_2} P(q^\alpha t^{\mathbf{m}_2}) \cdot v = 0,$$

$$(3-11) \quad t_0^{\bar{1}} t^{km_2} P(t^{\mathbf{m}_2}) \cdot v = t_0^{\bar{1}} t^{km_2} P(q^\alpha t^{\mathbf{m}_2}) \cdot v = 0,$$

where  $a_k = 0$  if  $k \notin \{0, 1, \dots, n\}$ , and  $\alpha = m_{11} m_{22} - m_{12} m_{21}$  and  $\beta = m_{11} c_1 + m_{12} c_2$ .

*Proof.* First necessity. Since  $V_0$  is a finite-dimensional irreducible  $L_0$ -module, we have  $V_0 \cong V(\underline{a}, \psi)$  as  $L_0$ -modules by Theorem 2.6. Since  $\mathcal{H} = \langle t_0^{\bar{1}} t^{2km_2} \mid k \in \mathbb{Z} \rangle$  is an abelian Lie subalgebra of  $L_0$ , we can choose a common eigenvector  $v_0 \in V_0$  of  $\mathcal{H}$ . First we prove the following claim.

**Claim 2.** There is a polynomial  $P_e(t^{\mathbf{m}_2}) = \sum_{i=0}^n a_i t^{2im_2}$  with  $a_n a_0 \neq 0$  such that

$$(3-12) \quad \begin{aligned} & (t_0^{\bar{0}} t^{2km_2} Q(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2km_2} Q(q^\alpha t^{\mathbf{m}_2}) + a_Q \beta) \cdot v_0 = 0, \\ & (t_0^{\bar{1}} t^{2km_2} Q(t^{\mathbf{m}_2}) - (-1)^{m_{11}} q^{2k\alpha} t_0^{\bar{1}} t^{2km_2} Q(q^\alpha t^{\mathbf{m}_2})) \cdot v_0 = 0, \\ & (t_0^{\bar{0}} t^{(2k+1)m_2} Q(t^{\mathbf{m}_2}) - q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)m_2} Q(q^\alpha t^{\mathbf{m}_2})) \cdot v_0 = 0, \\ & (t_0^{\bar{1}} t^{(2k+1)m_2} Q(t^{\mathbf{m}_2}) - (-1)^{m_{11}} q^{(2k+1)\alpha} t_0^{\bar{1}} t^{(2k+1)m_2} Q(q^\alpha t^{\mathbf{m}_2})) \cdot v_0 = 0 \end{aligned}$$

for all  $k \in \mathbb{Z}$  and  $Q(t^{\mathbf{m}_2}) \in \mathbb{C}[t^{\pm 2m_2}]$  with  $P_e(t^{\mathbf{m}_2}) \mid Q(t^{\mathbf{m}_2})$ , where  $a_Q$  is the constant term of  $t^{2km_2} Q(t^{\mathbf{m}_2})$ .

To prove the claim, since  $\dim V_{-1} < \infty$ , there exist an integer  $s$  and a polynomial  $P_e(t^{\mathbf{m}_2}) = \sum_{i=0}^n a_i t^{2im_2} \in \mathbb{C}[t^{\mathbf{m}_2}]$  with  $a_0 a_n \neq 0$  such that

$$(3-13) \quad t_0^{\bar{0}} t^{-m_1 + 2sm_2} P_e(t^{\mathbf{m}_2}) \cdot v_0 = 0.$$

Applying  $t_0^{\bar{0}} t^{m_1 + 2km_2}$  for any  $k \in \mathbb{Z}$  to the above equation, we have

$$\begin{aligned} 0 &= t_0^{\bar{0}} t^{m_1 + 2km_2} \cdot t_0^{\bar{0}} t^{-m_1 + 2sm_2} P_e(t^{\mathbf{m}_2}) \cdot v_0 \\ &= \sum_{i=0}^n a_i [t_0^{\bar{0}} t^{m_1 + 2km_2}, q^{2im_{21}(-m_{12} + 2sm_{22})} t_0^{\bar{0}} t^{-m_1 + 2(s+i)m_2}] \cdot v_0 \\ &= q^{-m_{11} m_{12} - 2km_{22} m_{11} + 2sm_{12} m_{21} + 4ksm_{21} m_{22}} \\ &\quad \cdot (t_0^{\bar{0}} t^{2(k+s)m_2} P_e(t^{\mathbf{m}_2}) - q^{2(s+k)\alpha} t_0^{\bar{0}} t^{2(k+s)m_2} P_e(q^\alpha t^{\mathbf{m}_2}) \\ &\quad \quad \quad + a_{-k-s} q^{-4(k+s)^2 m_{21} m_{22}} \beta) \cdot v_0. \end{aligned}$$

Now applying  $t_0^{\bar{1}}t^{m_1+2km_2}$  for any  $k \in \mathbb{Z}$  to (3-13), we have

$$\begin{aligned}
 0 &= t_0^{\bar{1}}t^{m_1+2km_2} \cdot t_0^{\bar{0}}t^{-m_1+2sm_2} P_e(t^{m_2}) \cdot v_0 \\
 &= \sum_{i=0}^n a_i [t_0^{\bar{1}}t^{m_1+2km_2}, q^{2im_{21}(-m_{12}+2sm_{22})} t_0^{\bar{0}}t^{-m_1+2(s+i)m_2}] \cdot v_0 \\
 &= q^{-m_{11}m_{12}-2km_{22}m_{11}+2sm_{12}m_{21}+4ksm_{21}m_{22}} \\
 &\quad \cdot (t_0^{\bar{1}}t^{2(k+s)m_2} P_e(t^{m_2}) - (-1)^{m_{11}} q^{2(s+k)\alpha} t_0^{\bar{1}}t^{2(k+s)m_2} P_e(q^\alpha t^{m_2})) \cdot v_0.
 \end{aligned}$$

By applying  $t_0^{\bar{0}}t^{m_1+(2k+1)m_2}$  and  $t_0^{\bar{1}}t^{m_1+(2k+1)m_2}$  to (3-13) one respectively gets that

$$\begin{aligned}
 0 &= t_0^{\bar{0}}t^{m_1+(2k+1)m_2} \cdot t_0^{\bar{0}}t^{-m_1+2sm_2} P_e(t^{m_2}) \cdot v_0 \\
 &= q^{-m_{11}m_{12}-(2k+1)m_{11}m_{22}+2sm_{12}m_{21}+2s(2k+1)m_{21}m_{22}} \\
 &\quad \cdot (t_0^{\bar{0}}t^{(2k+2s+1)m_2} P_e(t^{m_2}) - q^{(2k+2s+1)\alpha} t_0^{\bar{0}}t^{(2k+2s+1)m_2} P_e(q^\alpha t^{m_2})) \cdot v_0, \\
 0 &= t_0^{\bar{1}}t^{m_1+(2k+1)m_2} \cdot (t_0^{\bar{0}}t^{-m_1+2sm_2} P_e(t^{m_2})) \cdot v_0 \\
 &= q^{-m_{11}m_{12}-(2k+1)m_{11}m_{22}+2sm_{12}m_{21}+2s(2k+1)m_{21}m_{22}} \\
 &\quad \cdot (t_0^{\bar{1}}t^{(2k+2s+1)m_2} P_e(t^{m_2}) - (-1)^{m_{11}} q^{(2k+2s+1)\alpha} t_0^{\bar{0}}t^{(2k+2s+1)m_2} P_e(q^\alpha t^{m_2})) \cdot v_0.
 \end{aligned}$$

So we obtain the four equations of Claim 2.

On the other hand, we can choose an integer  $s$  and a polynomial  $P_o(t^{m_2}) = \sum_{i=0}^n a_i t^{2im_2} \in \mathbb{C}[t^{m_2}]$  with  $a_0 a_n \neq 0$  such that  $t_0^{\bar{0}}t^{-m_1+(2s+1)m_2} P_o(t^{m_2}) \cdot v_0 = 0$ , since  $\dim V_{-1} < \infty$ . Thus by a calculation similar to the proof of Claim 2, we can deduce the following claim.

**Claim 3.** There is a polynomial  $P_o(t^{m_2}) = \sum_{i=0}^n a_i t^{2im_2}$  with  $a_n a_0 \neq 0$  such that

$$\begin{aligned}
 &(t_0^{\bar{0}}t^{2km_2} Q(t^{m_2}) - q^{2k\alpha} t_0^{\bar{0}}t^{2km_2} Q(q^\alpha t^{m_2}) + a_Q \beta) \cdot v_0 = 0, \\
 &(t_0^{\bar{1}}t^{2km_2} Q(t^{m_2}) - (-1)^{m_{11}+1} q^{2k\alpha} t_0^{\bar{1}}t^{2km_2} Q(q^\alpha t^{m_2})) \cdot v_0 = 0, \\
 &(t_0^{\bar{0}}t^{(2k+1)m_2} Q(t^{m_2}) - q^{(2k+1)\alpha} t_0^{\bar{0}}t^{(2k+1)m_2} Q(q^\alpha t^{m_2})) \cdot v_0 = 0, \\
 (3-14) \quad &(t_0^{\bar{1}}t^{(2k+1)m_2} Q(t^{m_2}) - (-1)^{m_{11}+1} q^{(2k+1)\alpha} t_0^{\bar{1}}t^{(2k+1)m_2} Q(q^\alpha t^{m_2})) \cdot v_0 = 0
 \end{aligned}$$

for all  $k \in \mathbb{Z}$  and  $Q(t^{m_2}) \in \mathbb{C}[t^{\pm 2m_2}]$  with  $P_o(t^{m_2}) \mid Q(t^{m_2})$ , where  $a_Q$  is the constant term of  $t^{2km_2} Q(t^{m_2})$ .

Let  $P(t^{m_2}) = \sum_{i=0}^n a_i t^{2im_2}$  be the product of  $P_o(t^{m_2})$  and  $P_e(t^{m_2})$ . We see that both (3-12) and (3-14) hold for  $P(t^{m_2})$ . Thus one can directly deduce that both (3-9) and (3-11) hold for  $P(t^{m_2})$  and  $v_0 \in V_0$ . Since  $v_0$  is an eigenvector of  $t_0^{\bar{1}}$ , we have

$$\begin{aligned}
 0 &= t_0^{\bar{1}} \cdot t_0^{\bar{1}}t^{(2k+1)m_2} P(t^{m_2}) \cdot v_0 \\
 &= [t_0^{\bar{1}}, t_0^{\bar{1}}t^{(2k+1)m_2} P(t^{m_2})] \cdot v_0 = 2t_0^{\bar{0}}t^{(2k+1)m_2} P(t^{m_2}) \cdot v_0,
 \end{aligned}$$

and

$$\begin{aligned} 0 &= t_0^{\bar{1}} \cdot t_0^{\bar{1}} t^{(2k+1)m_2} P(q^\alpha t^{m_2}) \cdot v_0 \\ &= [t_0^{\bar{1}}, t_0^{\bar{1}} t^{(2k+1)m_2} P(q^\alpha t^{m_2})] \cdot v_0 = 2t_0^{\bar{0}} t^{(2k+1)m_2} P(q^\alpha t^{m_2}) \cdot v_0, \end{aligned}$$

which implies (3-10) for  $P(t^{m_2})$  and  $v_0$ .

From the definition of  $L_0$ , one easily deduces that if (3-9)–(3-11) hold for any  $v \in V$ , then they also hold for  $t_0^s t^{km_2} \cdot v$  for all  $\forall s \in \mathbb{Z}/2\mathbb{Z}$  and  $k \in \mathbb{Z}$ . This completes the proof of necessity since  $V_0$  is an irreducible  $L_0$ -module.

Now sufficiency.

**Claim 4.** For any  $s \in \mathbb{Z}_+$ , there is a polynomial  $P_s(t^{m_2}) = \sum_{j \in \mathbb{Z}} a_{s,j} t^{2jm_2} \in \mathbb{C}[t^{2m_2}]$  such that for all  $r \in \mathbb{Z}_2$  and  $k \in \mathbb{Z}$ ,

$$\begin{aligned} (t_0^{\bar{0}} t^{2km_2} P_s(t^{m_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2km_2} P_s(q^\alpha t^{m_2}) + a_{s,-k} q^{-4k^2 m_{21} m_{22}} \beta) \cdot V_{-s} &= 0, \\ t_0^{\bar{0}} t^{(2k+1)m_2} P_s(t^{m_2}) \cdot V_{-s} &= t_0^{\bar{1}} t^{km_2} P_s(t^{m_2}) \cdot V_{-s} = 0, \\ t_0^r t^{-m_1 + km_2} P_s(t^{m_2}) \cdot V_{-s} &= 0. \end{aligned}$$

We prove this claim by induction on  $s$ . By assumption and the definition of the  $L_0$ -module  $V_0$ , the claim holds for  $s = 0$  with  $P_0(t^{m_2}) = P(t^{m_2})$ . Suppose it holds for  $s$ , and consider it for  $s + 1$ .

The equations in the claim are equivalent, for all  $r \in \mathbb{Z}_2$  and  $k \in \mathbb{Z}$ , to

$$(3-15) \quad \begin{aligned} (t_0^{\bar{0}} Q(t^{m_2}) - t_0^{\bar{0}} Q(q^\alpha t^{m_2}) + a_Q \beta) \cdot V_{-s} &= 0, \\ t_0^{\bar{0}} t^{(2k+1)m_2} Q(t^{m_2}) \cdot V_{-s} &= t_0^{\bar{1}} t^{km_2} Q(t^{m_2}) \cdot V_{-s} = 0, \\ t_0^r t^{-m_1 + km_2} Q(t^{m_2}) \cdot V_{-s} &= 0 \end{aligned}$$

for any  $Q(t^{m_2}) \in \mathbb{C}[t^{\pm 2m_2}]$  with  $P_s(t^{m_2}) \mid Q(t^{m_2})$ , where  $a_Q$  is the constant term of  $Q(t^{m_2})$ .

Let  $P_{s+1}(t^{m_2}) = P_s(q^\alpha t^{m_2}) P_s(t^{m_2}) P_s(q^{-\alpha} t^{m_2})$ . For any  $p, r \in \mathbb{Z}_2$  and  $j, k \in \mathbb{Z}$ , using induction and (3-15) we have

$$\begin{aligned} &(t_0^{\bar{0}} t^{2km_2} P_{s+1}(t^{m_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2km_2} P_{s+1}(q^\alpha t^{m_2}) + a_{s+1,-k} q^{-4k^2 m_{21} m_{22}} \beta) \cdot V_{-s} \\ &= [t_0^{\bar{0}} t^{2km_2} P_{s+1}(t^{m_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2km_2} P_{s+1}(q^\alpha t^{m_2}) \\ &\quad + a_{s+1,-k} q^{-k^2 m_{21} m_{22}} \beta, t_0^p t^{-m_1 + jm_2}] \cdot V_{-s} \\ &= q^{2km_{22}(-m_{11} + jm_{21})} \\ &\quad \cdot (t_0^p t^{-m_1 + (2k+j)m_2} (P_{s+1}(q^{-\alpha} t^{m_2}) - 2q^{2k\alpha} P_{s+1}(t^{m_2}) + q^{4k\alpha} P_{s+1}(q^\alpha t^{m_2}))) \cdot V_{-s}, \end{aligned}$$

which is equal to zero. Thus, by (3-1), we obtain that

$$(t_0^{\bar{0}} t^{2km_2} P_{s+1}(t^{m_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2km_2} P_{s+1}(q^\alpha t^{m_2}) + a_{s+1,-k} q^{-4k^2 m_{21} m_{22}} \beta) \cdot V_{-s-1} = 0.$$

Similarly, one can prove that for all  $k \in \mathbb{Z}$

$$t_0^{\bar{0}} t^{(2k+1)m_2} P_{s+1}(t^{m_2}) \cdot V_{-s-1} = t_0^{\bar{1}} t^{km_2} P_{s+1}(t^{m_2}) \cdot V_{-s-1} = 0.$$

This proves the first two equations of the claim for  $s + 1$ .

Using the two equations just above and induction, we deduce that for any  $l, k \in \mathbb{Z}$  and  $n, r \in \mathbb{Z}_2$ ,

$$\begin{aligned} & t_0^n t^{m_1+lm_2} \cdot t_0^r t^{-m_1+km_2} P_{s+1}(t^{m_2}) \cdot V_{-s-1} \\ &= [t_0^n t^{m_1+lm_2}, t_0^r t^{-m_1+km_2} P_{s+1}(t^{m_2})] \cdot V_{-s-1} \\ & \quad + t_0^r t^{-m_1+km_2} P_{s+1}(t^{m_2}) \cdot t_0^n t^{m_1+lm_2} \cdot V_{-s-1} \\ &= (-1)^{r(m_{11}+lm_{21})} q^{-m_{11}m_{12}+km_{12}m_{21}-lm_{11}m_{22}+lkm_{21}m_{22}} \\ & \quad \cdot (t_0^{n+r} t^{(l+k)m_2} P_{s+1}(t^{m_2}) - (-1)^{(n+r)m_{11}+nk+r} q^{(k+l)\alpha} t_0^{n+r} t^{(l+k)m_2} P_{s+1}(q^\alpha t^{m_2}) \\ & \quad + a_{s+1,i} \delta_{k+l+2i,0} \delta_{r+n,\bar{0}} q^{-(l+k)^2 m_{21} m_{22}} \beta) \cdot V_{-s-1}, \end{aligned}$$

which is equal to zero. Hence, by (3-2),

$$t_0^r t^{-m_1+km_2} P_{s+1}(t^{m_2}) \cdot V_{-s-1} = 0$$

for all  $r \in \mathbb{Z}_2$  and  $k \in \mathbb{Z}$ , which implies the third equation in the claim for  $s + 1$ . Therefore the claim follows by induction.

From the third equation of the claim and (3-1), we see that

$$\dim V_{-s-1} \leq 2 \deg(P_{s+1}(t^{m_2})) \cdot \dim V_s \quad \text{for all } s \in \mathbb{Z}_+,$$

Hence  $M^+(V(\underline{a}, \psi), \underline{m}_1, \underline{m}_2) \in \mathbb{O}_{\mathbb{Z}}$ . □

**Theorem 3.4.** *Let  $m_{21}$  be an odd integer. Then  $M^+(\underline{a}, \psi, \underline{m}_1, \underline{m}_2) \in \mathbb{O}_{\mathbb{Z}}$  if and only if there exist  $b_{10}, b_{11}, \dots, b_{1s_1}, b_{20}, b_{21}, \dots, b_{2s_2}, \dots, b_{r0}, b_{r1}, \dots, b_{rs_r} \in \mathbb{C}$ , and  $\alpha_1, \dots, \alpha_r \in \mathbb{C}^*$  such that for any  $i \in \mathbb{Z}^*$  and  $j \in \mathbb{Z}_2$ ,*

$$\begin{aligned} \psi(t_0 t^{2im_2}) &= \frac{\sum_{\ell=1}^r (\sum_{k=0}^{s_1} b_{\ell k} i^k) \alpha_{\ell}^i}{(1 - q^{2i\alpha}) q^{2i^2 m_{21} m_{22}}}, \\ \psi(\beta) &= b_{10} + b_{20} + \dots + b_{r0}, \quad \text{and} \quad \psi(m_{21}c_1 + m_{22}c_2) = 0, \end{aligned}$$

where  $\alpha = m_{11}m_{22} - m_{21}m_{12} \in \{\pm 1\}$ .

*Proof.* First necessity. Let  $f_i = \psi((1 - q^{2i\alpha}) q^{2i^2 m_{21} m_{22}} t_0^{\bar{0}} t^{2im_2})$  for  $i \in \mathbb{Z}^*$  and let  $f_0 = \psi(\beta)$ . By Lemma 3.3, there exist complex numbers  $a_0, a_1, \dots, a_n$  with  $a_0 a_n \neq 0$  such that  $\sum_{i=0}^n a_i q^{-2i^2 m_{21} m_{22}} f_{k+i} = 0$  for all  $k \in \mathbb{Z}$ . Thus, by using the techniques of the proof of Theorem 3.2, we deduce necessity.

Now sufficiency. Set

$$Q(x) = \left( \prod_{i=1}^r (x - \alpha_i)^{s_i+1} \right) \left( \prod_{j=1}^v (x - a_j) \right) \left( \prod_{j=1}^v (x - q^{2\alpha} a_j) \right) =: \sum_{i=1}^n b_i x^i,$$

and  $f_i = \psi((1 - q^{2i\alpha})q^{2i^2m_{21}m_{22}}t_0^{\bar{0}}t^{2im_2})$  for all  $i \in \mathbb{Z}^*$ . Set  $f_0 = \psi(\beta)$ . Then one can easily verify that

$$(3-16) \quad \sum_{i=0}^n b_i f_{k+i} = 0 \quad \text{for all } k \in \mathbb{Z}.$$

Meanwhile, we have  $(\prod_{j=1}^v (x - a_j)) \mid x^k Q(x)$  and  $(\prod_{j=1}^v (x - a_j)) \mid x^k Q(q^{2\alpha}x)$  for any  $k \in \mathbb{Z}$ , which implies for all  $s \in \mathbb{Z}_2$  that

$$(3-17) \quad \sum_{i=1}^n b_i q^{\frac{1}{2}(2i+2k+1)^2 m_{22} m_{21}} t_0^s t^{(2i+2k+1)m_2} \cdot V_0 = 0,$$

$$(3-18) \quad \sum_{i=1}^n b_i q^{2i\alpha} q^{\frac{1}{2}(2i+2k+1)^2 m_{22} m_{21}} t_0^s t^{(2i+2k+1)m_2} \cdot V_0 = 0$$

and, by Remark 2.7,

$$(3-19) \quad \sum_{i=1}^n b_i q^{2(i+k)^2 m_{22} m_{21}} t_0^{\bar{1}} t^{2(i+k)m_2} \cdot V_0 = 0,$$

$$(3-20) \quad \sum_{i=1}^n b_i q^{2i\alpha} q^{2(i+k)^2 m_{22} m_{21}} t_0^{\bar{1}} t^{2(i+k)m_2} \cdot V_0 = 0.$$

Let  $b'_i = q^{2i^2 m_{21} m_{22}} b_i$  for  $0 \leq i \leq n$  and  $P(x) = \sum_{i=1}^n b'_i x^i$ . By (3-16) and the construction of  $V(a, \psi)$ , we have

$$\begin{aligned} & (t_0^{\bar{0}} t^{2km_2} P(t^{2m_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2km_2} P(q^{2\alpha} t^{2m_2}) + b'_{-k} q^{-4k^2 m_{21} m_{22}} \beta) \cdot V_0 \\ &= q^{-2k^2 m_{21} m_{22}} \psi \left( \sum_{i=1}^n b_i (1 - q^{2(k+i)\alpha}) q^{2(k+i)^2 m_{22} m_{21}} t_0^{\bar{0}} t^{2(k+i)m_2} + b_{-k} \beta \right) \cdot V_0 \\ &= q^{-2k^2 m_{21} m_{22}} \sum_{i=1}^n b_i f_{k+i} \cdot V_0 = 0, \end{aligned}$$

which implies (3-9). Similarly, we have, for any  $k \in \mathbb{Z}$ ,

$$\begin{aligned} t_0^s t^{(2k+1)m_2} P(t^{2m_2}) \cdot V_0 &= \sum_{i=1}^n b_i q^{(2i^2+4ki+2i)m_{21}m_{22}} t_0^s t^{(2k+2i+1)m_2} \cdot V_0 \\ &= q^{-2k^2-2k-\frac{1}{2}} \sum_{i=1}^n b_i q^{\frac{1}{2}(2k+2i+1)^2 m_{21}m_{22}} t_0^s t^{(2k+2i+1)m_2} \cdot V_0 \end{aligned}$$

and

$$\begin{aligned} t_0^s t^{(2k+1)m_2} P(q^{2\alpha} t^{2m_2}) \cdot V_0 &= \sum_{i=1}^n b_i q^{2i\alpha + (2i^2 + 4ki + 2i)m_{21}m_{22}} t_0^s t^{(2k+2i+1)m_2} \cdot V_0 \\ &= q^{-2k^2 - 2k - \frac{1}{2}} \sum_{i=1}^n b_i q^{2i\alpha} q^{\frac{1}{2}(2k+2i+1)^2 m_{21}m_{22}} t_0^s t^{(2k+2i+1)m_2} \cdot V_0 \end{aligned}$$

which then vanish by (3-17) and (3-18), respectively. Now one can easily deduce the equations  $t_0^{\bar{1}} t^{2km_2} P(t^{2m_2}) \cdot V_0 = 0$  and  $t_0^{\bar{1}} t^{2km_2} P(q^{2\alpha} t^{2m_2}) \cdot V_0 = 0$  by using (3-19) and (3-20), respectively. Therefore (3-9)–(3-11) hold for  $P(t^{2m_2}) = \sum_{i=1}^n b'_i t^{2im_2}$ . Thus  $M^+(a, \psi, m_1, m_2) \in \mathbb{C}_{\mathbb{Z}}$  by Lemma 3.3.  $\square$

**Remark 3.5.** A linear function  $\psi$  over  $L_0$  of the form described in Theorem 3.2 is called an exp-polynomial function over  $L_0$ ; a linear function  $\psi$  over  $\mathcal{A}$  of the form described in Theorem 3.4 is called an exp-polynomial function over  $\mathcal{A}$ .

#### 4. Classification of generalized highest weight irreducible $\mathbb{Z}$ -graded $L$ -modules

**Lemma 4.1.** *Suppose  $V$  is a nontrivial irreducible generalized highest weight  $\mathbb{Z}$ -graded  $L$ -module corresponding to a  $\mathbb{Z}$ -basis  $B = \{\mathbf{b}_1, \mathbf{b}_2\}$  of  $\mathbb{Z}^2$ .*

- (1) *For any  $v \in V$ , there is some  $p \in \mathbb{N}$  such that  $t_0^i t^{m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2} \cdot v = 0$  for all  $m_1, m_2 \geq p$  and  $i \in \mathbb{Z}_2$ .*
- (2) *For any nonzero  $v \in V$ ,  $m_1, m_2 > 0$  and  $i \in \mathbb{Z}_2$ , we have  $t_0^i t^{-m_1 \mathbf{b}_1 - m_2 \mathbf{b}_2} \cdot v \neq 0$ .*

*Proof.* Assume that  $v_0$  is a generalized highest weight vector corresponding to the  $\mathbb{Z}$ -basis  $B = \{\mathbf{b}_1, \mathbf{b}_2\}$  of  $\mathbb{Z}^2$ .

(1) By the irreducibility of  $V$  and the Poincaré–Birkhoff–Witt theorem, there exists a  $u \in U(L)$  such that  $v = u \cdot v_0$ , where  $u$  is a linear combination of elements of the form

$$u_n = (t_0^{k_1} t^{i_1 \mathbf{b}_1 + j_1 \mathbf{b}_2}) \cdot (t_0^{k_2} t^{i_2 \mathbf{b}_1 + j_2 \mathbf{b}_2}) \cdots (t_0^{k_n} t^{i_n \mathbf{b}_1 + j_n \mathbf{b}_2),$$

where  $\cdot$  denotes the product in  $U(L)$ . Thus, we may assume  $u = u_n$ . Take

$$p_1 = - \sum_{i_s < 0} i_s + 1 \quad \text{and} \quad p_2 = - \sum_{j_s < 0} j_s + 1.$$

By induction on  $n$ , one gets that  $t_0^k t^{i \mathbf{b}_1 + j \mathbf{b}_2} \cdot v = 0$  for any  $k \in \mathbb{Z}_2$ ,  $i \geq p_1$  and  $j \geq p_2$ , which gives the result with  $p = \max\{p_1, p_2\}$ .

(2) Suppose there is a nonzero  $v \in V$ , an  $i \in \mathbb{Z}_2$  and  $m_1, m_2 > 0$  such that  $t_0^i t^{-m_1 \mathbf{b}_1 - m_2 \mathbf{b}_2} \cdot v = 0$ . Let  $p$  be as in the proof of (1). Then for all  $j \in \mathbb{Z}_2$ ,

$$t_0^i t^{-m_1 \mathbf{b}_1 - m_2 \mathbf{b}_2}, t_0^j t^{\mathbf{b}_1 + p(m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2)}, t_0^j t^{\mathbf{b}_2 + p(m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2)}$$

act trivially on  $v$ . These elements generate the Lie algebra  $L$ . So  $V$  is a trivial module, a contradiction.  $\square$

**Lemma 4.2.** *If  $V \in \mathbb{C}_{\mathbb{Z}}$  is a generalized highest weight  $L$ -module corresponding to the  $\mathbb{Z}$ -basis  $B = \{\mathbf{b}_1, \mathbf{b}_2\}$  of  $\mathbb{Z}^2$ , then  $V$  must be a highest or lowest weight module.*

*Proof.* Suppose  $V$  is a generalized highest weight module corresponding to the  $\mathbb{Z}$ -basis  $\{\mathbf{b}_1 = b_{11}\mathbf{m}_1 + b_{12}\mathbf{m}_2, \mathbf{b}_2 = b_{21}\mathbf{m}_1 + b_{22}\mathbf{m}_2\}$  of  $\mathbb{Z}^2$ . By shifting the index of  $V_i$  if necessary, we can suppose the highest degree of  $V$  is 0. Let  $a = b_{11} + b_{21}$  and  $\wp(V) = \{m \in \mathbb{Z} \mid V_m \neq 0\}$ . We may assume  $a \neq 0$ : In fact, if  $a = 0$ , we can choose  $\mathbf{b}'_1 = 3\mathbf{b}_1 + \mathbf{b}_2$  and  $\mathbf{b}'_2 = 2\mathbf{b}_1 + \mathbf{b}_2$ . Then  $V$  is a generalized highest weight  $\mathbb{Z}$ -graded module corresponding to the  $\mathbb{Z}$ -basis  $\{\mathbf{b}'_1, \mathbf{b}'_2\}$  of  $\mathbb{Z}^2$ . Replacing  $\mathbf{b}_1$  and  $\mathbf{b}_2$  by  $\mathbf{b}'_1$  and  $\mathbf{b}'_2$  gives  $a \neq 0$ .

Now we prove that  $V$  is a highest weight module if  $a > 0$ . Let

$$\mathcal{A}_i = \{j \in \mathbb{Z} \mid i + aj \in \wp(V)\} \quad \text{for all } 0 \leq i < a.$$

Then there is  $m_i \in \mathbb{Z}$  such that  $\mathcal{A}_i = \{j \in \mathbb{Z} \mid j \leq m_i\}$  or  $\mathcal{A}_i = \mathbb{Z}$  by Lemma 4.1(2).

Set  $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ . We want to prove  $\mathcal{A}_i \neq \mathbb{Z}$  for all  $0 \leq i < a$ . Otherwise, by shifting the index of  $\mathcal{A}_i$  if necessary, we may assume  $\mathcal{A}_0 = \mathbb{Z}$ . Thus we can choose a nonzero  $v_j \in V_{aj}$  for any  $j \in \mathbb{Z}$ . By Lemma 4.1(1), there is a  $p_{v_j} > 0$  with

$$(4-1) \quad t_0^k t^{s_1 \mathbf{b}_1 + s_2 \mathbf{b}_2} \cdot v_j = 0 \quad \text{for all } s_1, s_2 > p_{v_j} \text{ and } k \in \mathbb{Z}_2.$$

Choose  $\{k_j \in \mathbb{N} \mid j \in \mathbb{N}\}$  and  $v_{k_j} \in V_{ak_j}$  such that

$$(4-2) \quad k_{j+1} > k_j + p_{v_{k_j}} + 2.$$

If we can prove that  $\{t_0^{\bar{0}} t^{-k_j \mathbf{b}} \cdot v_{k_j} \mid j \in \mathbb{N}\} \subset V_0$  is a set of linearly independent vectors, then we will have a contradiction that proves the desired result.

Indeed, for any  $r \in \mathbb{N}$ , there exists  $a_r \in \mathbb{N}$  such that  $t_0^0 t^{x\mathbf{b} + \mathbf{b}_1} v_{k_r} = 0$  for all  $x \geq a_r$  by Lemma 4.1(1). On the other hand, we know that  $t_0^0 t^{x\mathbf{b} + \mathbf{b}_1} \cdot v_{k_r} \neq 0$  for any  $x < -1$  by Lemma 4.1(2). Thus we can choose  $s_r \geq -2$  such that

$$(4-3) \quad t_0^{\bar{0}} t^{s_r \mathbf{b} + \mathbf{b}_1} \cdot v_{k_r} \neq 0 \quad \text{and} \quad t_0^{\bar{0}} t^{x\mathbf{b} + \mathbf{b}_1} \cdot v_{k_r} = 0 \quad \text{for all } x > s_r.$$

By (4-2) we have  $k_r + s_r - k_j > p_{v_{k_j}}$  for all  $1 \leq j < r$ . Hence by (4-1) we know that for all  $1 \leq j < r$ ,

$$\begin{aligned} & t_0^{\bar{0}} t^{(k_r + s_r)\mathbf{b} + \mathbf{b}_1} \cdot t_0^{\bar{0}} t^{-k_j \mathbf{b}} \cdot v_{k_j} \\ &= \left[ t_0^{\bar{0}} t^{(k_r + s_r)\mathbf{b} + \mathbf{b}_1}, t_0^{\bar{0}} t^{-k_j \mathbf{b}} \right] \cdot v_{k_j} \\ &= q^{-k_j((k_r + s_r)(b'_{12} + b'_{22}) + b'_{12})(b'_{11} + b'_{21})} (1 - q^{k_j(b'_{12}b'_{21} - b'_{11}b'_{22})}) t_0^{\bar{0}} t^{(k_r + s_r - k_j)\mathbf{b} + \mathbf{b}_1} \cdot v_{k_j} \\ &= 0, \end{aligned}$$

where

$$\begin{aligned} b'_{11} &= b_{11}m_{11} + b_{12}m_{21}, & b'_{12} &= b_{11}m_{12} + b_{12}m_{22}, \\ b'_{21} &= b_{21}m_{11} + b_{22}m_{21}, & b'_{22} &= b_{21}m_{12} + b_{22}m_{22}. \end{aligned}$$

Now by (4-2) and (4-3), one gets

$$\begin{aligned} & t_0^{\bar{0}} t^{(k_r+s_r)\mathbf{b}+\mathbf{b}_1} \cdot t_0^{\bar{0}} t^{-k_r\mathbf{b}} \cdot v_{k_r} \\ &= [t_0^{\bar{0}} t^{(k_r+s_r)\mathbf{b}+\mathbf{b}_1}, t_0^{\bar{0}} t^{-k_r\mathbf{b}}] \cdot v_{k_r} \\ &= q^{-k_r((k_r+s_r)(b'_{12}+b'_{22})+b'_{12})(b'_{11}+b'_{21})} (1 - q^{k_r(b'_{12}b'_{21}-b'_{11}b'_{22})}) t_0^{\bar{0}} t^{s_r\mathbf{b}+\mathbf{b}_1} \cdot v_{k_r} \neq 0. \end{aligned}$$

Hence if  $\sum_{j=1}^n \lambda_j t_0^{\bar{0}} t^{-k_j\mathbf{b}} \cdot v_{k_j} = 0$  then  $\lambda_n = \lambda_{n-1} = \dots = \lambda_1 = 0$  by the arbitrariness of  $r$ . So we see that the coefficients of  $\lambda_j$  form a set of linearly independent vectors, which contradicts that  $V \in \mathbb{C}_{\mathbb{Z}}$ . Therefore, for any  $0 \leq i < a$ , there is a  $m_i \in \mathbb{Z}$  such that  $\mathcal{A}_i = \{j \in \mathbb{Z} \mid j \leq m_i\}$ , which implies that  $V$  is a highest weight module since  $\wp(V) = \bigcup_{i=0}^{a-1} \mathcal{A}_i$ .

Similarly, one can prove  $V$  is a lowest weight module if  $a < 0$ . □

From Lemma 4.2 and the results in Section 3, we get our main theorem:

**Theorem 4.3.**  *$V$  is a quasifinite irreducible  $\mathbb{Z}$ -graded  $L$ -module if and only if one of the following statements hold:*

- (1)  *$V$  is a uniformly bounded module.*
- (2) *If  $m_{21}$  is an even integer, then there exists an exp-polynomial function  $\psi$  over  $L_0$  such that*

$$V \cong M^+(\psi, \mathbf{m}_1, \mathbf{m}_2) \quad \text{or} \quad V \cong M^-(\psi, \mathbf{m}_1, \mathbf{m}_2).$$

- (3) *If  $m_{21}$  is an odd integer, then there exist an exp-polynomial function  $\psi$  over  $\mathcal{A}$ , a finite sequence of nonzero distinct numbers  $\underline{a} = (a_1, \dots, a_\nu)$  and some finite-dimensional irreducible  $\mathfrak{sl}_2$ -modules  $V_1, \dots, V_\nu$  such that*

$$V \cong M^+(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2) \quad \text{or} \quad V \cong M^-(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2).$$

**Corollary 4.4.** *If  $V$  is a quasifinite irreducible  $\mathbb{Z}$ -graded  $L$ -module with nontrivial center, then one of the following statements must hold:*

- (1) *If  $m_{21}$  is an even integer, then there exists an exp-polynomial function  $\psi$  over  $L_0$  such that*

$$V \cong M^+(\psi, \mathbf{m}_1, \mathbf{m}_2) \quad \text{or} \quad V \cong M^-(\psi, \mathbf{m}_1, \mathbf{m}_2).$$

- (2) *If  $m_{21}$  is an odd integer, then there exist an exp-polynomial function  $\psi$  over  $\mathcal{A}$ , a finite sequence of nonzero distinct numbers  $\underline{a} = (a_1, \dots, a_\nu)$  and some finite-dimensional irreducible  $\mathfrak{sl}_2$  modules  $V_1, \dots, V_\nu$  such that*

$$V \cong M^+(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2) \quad \text{or} \quad V \cong M^-(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2).$$

*Proof.* By Theorem 4.3, we only need to show that  $V$  is not a uniformly bounded module. From the definition of the Lie algebra  $L$ , we see that

$$\mathfrak{H}_i = \langle t_0^{\bar{0}} t^{km_i}, m_{i1}c_1 + m_{i2}c_2 \mid k \in \mathbb{Z}^* \rangle \quad \text{for } i = 1, 2$$

are Heisenberg Lie algebras. Now  $m_{21}c_1 + m_{22}c_2$  must be zero since  $V$  is a quasifinite irreducible  $\mathbb{Z}$ -graded  $L$ -module. Thus, by assumption, we have that  $m_{11}c_1 + m_{12}c_2 \neq 0$  since  $\{\mathbf{m}_1, \mathbf{m}_2\}$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^2$ . Therefore,  $V$  is not a uniformly bounded module by a well-known result from the representation theory of the Heisenberg Lie algebra.  $\square$

**Theorem 4.5.** *The modules  $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$  and  $M^-(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2)$  are uniformly bounded only when they are trivial.*

*Proof.* Set  $V \cong M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$  or  $V \cong M^-(\underline{a}, \psi, \mathbf{m}_1, \mathbf{m}_2)$ , and suppose  $V$  is not trivial. Also set  $V = \bigoplus_{k \in \mathbb{Z}_+} V_{-k}$ . By nontriviality, there exist  $v_0 \in V_0$ ,  $k \in \mathbb{Z}^*$  and  $l \in \mathbb{Z}_2$  such that  $t_0^l t^{km_2} \cdot v_0 \neq 0$ . Thus

$$\begin{aligned} t_0^{\bar{0}} t^{m_1} \cdot t_0^l t^{-m_1+km_2} \cdot v_0 &= [t_0^{\bar{0}} t^{m_1}, t_0^l t^{-m_1+km_2}] v_0 \\ &= ((-1)^{lm_{11}} q^{m_{12}(-m_{11}+km_{21})} - q^{m_{11}(-m_{12}+km_{22})}) t_0^l t^{km_2} \cdot v_0, \end{aligned}$$

which is nonzero; this implies that  $t_0^l t^{-m_1+km_2} \cdot v_0 \neq 0$ .

Next, we prove that if  $0 \neq v_{-m} \in V_{-m}$  then  $t_0^{\bar{0}} t^{-m_1} \cdot v_{-m} \neq 0$ . Suppose  $t_0^{\bar{0}} t^{-m_1} \cdot v_{-m} = 0$  for some  $0 \neq v_{-m} \in V_{-m}$ . From the construction of  $V$ , we know that  $t_0^l t^{(m+1)m_1 \pm m_2}$  also act trivially on  $v_{-m}$  for any  $l \in \mathbb{Z}_2$ . Since  $L$  is generated by the set  $\{t_0^{\bar{0}} t^{-m_1}, t_0^l t^{(m+1)m_1 \pm m_2} \mid l = \bar{0}, \bar{1}\}$ , we see  $V$  is a trivial module, a contradiction.

Set

$$\mathcal{A}_n = \{(t_0^{\bar{0}} t^{-m_1})^j \cdot t_0^l t^{(-n+j)m_1+km_2} \cdot v_0 \mid 0 \leq j < n\} \subset V_{-n} \quad \text{for all } n \in \mathbb{N}.$$

Now we prove that  $\mathcal{A}_n$  is a linearly independent set of vectors. If

$$\sum_{j=0}^{n-1} \lambda_j (t_0^{\bar{0}} t^{-m_1})^j t_0^l t^{(-n+j)m_1+km_2} \cdot v_0 = 0,$$

then for any  $0 \leq i < n-1$  we have

$$\begin{aligned} 0 &= q^{n(n-i)m_{11}m_{12}-k(n-i)m_{12}m_{21}} t_0^{\bar{0}} t^{(n-i)m_1} \cdot \sum_{j=0}^{n-1} \lambda_j (t_0^{\bar{0}} t^{-m_1})^j \cdot t_0^l t^{(-n+j)m_1+km_2} \cdot v_0 \\ &= \sum_{j=0}^i \lambda_j q^{j(n-i)m_{11}m_{12}} ((-1)^{l(n-i)m_{11}} - q^{k(n-i)\alpha}) (t_0^{\bar{0}} t^{-m_1})^j \cdot t_0^l t^{(j-i)m_1+km_2} \cdot v_0, \end{aligned}$$

where  $\alpha = m_{11}m_{22} - m_{12}m_{21}$ ; this implies  $\lambda_0 = \cdots = \lambda_{n-1} = 0$ . Hence  $\mathcal{A}_n$  is a set of linear independent vectors in  $V_{-n}$  and thus  $\dim V_{-n} \geq n$ . Since  $n$  was arbitrary,  $V$  is not a uniformly bounded module.  $\square$

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