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MODULES FOR THE CORE OF EXTENDED AFFINE LIE ALGEBRAS OF TYPE A_1 WITH COORDINATES IN RANK 2 QUANTUM TORI

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We construct a Lie algebra L from rank 3 quantum tori and show that it is isomorphic to the core of extended affine Lie algebras of type A_1. Then we construct two classes — which turn out to be exhaustive — of irreducible $\mathbb{Z}$-graded highest weight L-modules and give necessary and sufficient conditions for these modules to have finite-dimensional homogeneous subspaces. As a consequence, we also determine all the irreducible $\mathbb{Z}$-graded L-modules with nonzero center and finite-dimensional homogeneous subspaces.

1. Introduction

Extended affine Lie algebras (EALAs), which were introduced in [Høegh-Krohn and Torrésani 1990] under the name of irreducible quasisimple Lie algebras, are higher-dimensional generalizations of affine Kac–Moody Lie algebras. Roughly speaking, they are complex Lie algebras that have a nondegenerate invariant form, a self-centralizing finite-dimensional ad-diagonalizable abelian subalgebra (that is, a Cartan subalgebra), a discrete irreducible root system, and ad-nilpotency of non-isotropic root spaces; see [Berman et al. 1996; Allison et al. 1997a; Allison et al. 1997b]. Prime examples of EALAs are toroidal Lie algebras, which are universal central extensions of $\dot{\mathfrak{g}} \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ (here $\dot{\mathfrak{g}}$ is a finite-dimensional simple Lie algebra); these were studied in [Frenkel 1985; Gao and Zeng 2006; Moody et al. 1990; Yamada 1989; Etingof and Frenkel 1994; Eswara Rao and Moody 1994; Berman and Cox 1994] and elsewhere. There are many EALAs that allow not only the Laurent polynomial algebra $\mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ as coordinate algebra but also quantum tori, Jordan tori and the octonian tori as coordinate algebras,

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depending on the type of the Lie algebra; see [Allison et al. 1997a; Berman et al. 1996; Berman et al. 1995; Allison and Gao 2001; Yoshii 1996]. The structure theory of the EALAs of type A_{d-1} is tied up with the Lie algebra $\mathfrak{gl}_d(\mathbb{C}) \otimes \mathbb{C}_Q$, where $\mathbb{C}_Q$ is the quantum torus. The quantum tori defined in [Manin 1991] are the noncommutative analogue of Laurent polynomial algebras. The universal central extension of the derivation Lie algebra of the rank 2 quantum torus is known as the q-analogue Virasoro-like algebra; see [Kirkman et al. 1994]. For representations of Lie algebras coordinatized by quantum tori, see [Jakobsen and Kac 1989; Berman and Szmigielski 1999; Gao 2000b; 2000a; Eswara Rao 2004; Rao 2003] and the references therein. For structure and representations of the q-analogue Virasoro-like algebra, see [Zhang and Zhao 1996; Jiang and Meng 1998; Rao and Zhao 2004; Lin and Tan 2006; 2008].

This paper is organized as follows. In Section 2, we first recall some concepts about quantum tori and EALAs of type A_1 with coordinates in rank 2 quantum tori. Next, we show that these EALAs are isomorphic to a Lie algebra L that is constructed from a special class of rank 3 quantum tori. Then we prove some basic propositions and reduce the classification of quasifinite irreducible $\mathbb{Z}$-graded L-modules to the classification of generalized highest weight modules and uniformly bounded modules. In Section 3, we construct two classes of irreducible $\mathbb{Z}$-graded highest weight L-modules, and give necessary and sufficient conditions for these modules to have finite-dimensional homogeneous subspaces. In Section 4, we prove generalized highest weight irreducible $\mathbb{Z}$-graded L-modules with finite-dimensional homogeneous subspaces must be highest (or lowest) weight modules; thus the modules constructed in Section 3 exhaust all generalized highest weight modules; see Theorem 4.3, our main theorem. As a consequence, we also complete the classification of irreducible $\mathbb{Z}$-graded L-modules with finite-dimensional homogeneous subspaces and nonzero center.

2. Basics

We use $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ to denote the sets of complex numbers, integers, nonnegative integers, and positive integers, respectively. We denote by $\mathbb{C}^*$ the nonzero complex numbers and by $\mathbb{Z}^{2*}$ the set $\mathbb{Z}^2 \setminus \{(0,0)\}$. All vector spaces we consider are over $\mathbb{C}$. As usual, if $u_1, u_2, \ldots, u_k$ are elements in vector spaces, we use $\langle u_1, \ldots, u_k \rangle$ to denote their linear span over $\mathbb{C}$. We let q be a nonzero complex number and suppose throughout that q is generic (that is, not a root of unity).

Now we recall the concept of quantum torus from [Manin 1991]. Let ν be a positive integer, and let $Q = (q_{ij})$ be a $\nu \times \nu$ matrix with elements in $\mathbb{C}^*$ such that $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ for $0 \le i, j \le \nu - 1$. A quantum torus associated to Q is the unital associative algebra $\mathbb{C}_Q[t_0^{\pm 1}, \ldots, t_{\nu-1}^{\pm 1}]$ (or, simply $\mathbb{C}_Q$) with generators

$t_0^{\pm 1}, \dots, t_{\nu-1}^{\pm 1}$ and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad \text{and} \quad t_i t_j = q_{ij} t_j t_i \quad \text{for all } 0 \le i, j \le \nu - 1.$$

Write $t^{\boldsymbol{m}} = t_0^{m_0} t_1^{m_1} \cdots t_{\nu-1}^{m_{\nu-1}}$ for $\boldsymbol{m} = (m_0, m_1, \dots, m_{\nu-1})$. Then

$$t^{\boldsymbol{m}} \cdot t^{\boldsymbol{n}} = \Bigl(\prod_{0 \le j \le i \le \nu-1} q_{ij}^{m_i n_j} \Bigr) t^{\boldsymbol{m}+\boldsymbol{n}}, \quad \text{where } \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^\nu.$$

If Q is a 2×2 matrix with $q_{21} = q$, we will simply write C_q instead of $\mathbb{C}_Q$.

Next we recall the construction of EALAs of type A_1 with coordinates in $\mathbb{C}_{q^2}$. Let E_{ij} be the 2×2 matrix with 1 at position (i, j) and 0 elsewhere. The Lie algebra $\tilde{\tau} = \mathfrak{gl}_2(\mathbb{C}_{q^2})$ is defined by the commutator

$$[E_{ij}(t^{\boldsymbol{m}}), E_{kl}(t^{\boldsymbol{n}})]_0 = \delta_{j,k} q^{2m_2 n_1} E_{il}(t^{\boldsymbol{m}+\boldsymbol{n}}) - \delta_{l,i} q^{2n_2 m_1} E_{kj}(t^{\boldsymbol{m}+\boldsymbol{n}}),$$

where $\boldsymbol{m} = (m_1, m_2)$ and $\boldsymbol{n} = (n_1, n_2)$ are in $\mathbb{Z}^2$. Thus the derived Lie subalgebra of $\tilde{\tau}$ is $\bar{\tau} = \mathrm{sl}_2(\mathbb{C}_{q^2}) \oplus \langle I(t^{\boldsymbol{m}}) \mid \boldsymbol{m} \in \mathbb{Z}^{2*} \rangle$, where $I = E_{11} + E_{22}$, since q is generic. The universal central extension of $\bar{\tau}$ is $\tau = \bar{\tau} \oplus \langle K_1, K_2 \rangle$ with Lie bracket

$$[X(t^{\boldsymbol{m}}), Y(t^{\boldsymbol{n}})] = [X(t^{\boldsymbol{m}}), Y(t^{\boldsymbol{n}})]_0 + \delta_{\boldsymbol{m}+\boldsymbol{n},0} q^{2m_2 n_1} (X, Y)(m_1 K_1 + m_2 K_2),$$

where K_1 and K_2 are central, $X(t^{\boldsymbol{m}}), Y(t^{\boldsymbol{n}}) \in \bar{\tau}$ and (X, Y) is the trace of XY. The Lie algebra τ is the core of the EALAs of type A_1 with coordinates in $\mathbb{C}_{q^2}$. If we add degree derivations d_1 and d_2 to τ, then $\tau \oplus \langle d_1, d_2 \rangle$ becomes an EALA since q is generic.

Now we construct our Lie algebra. Let

$$Q = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & q^{-1} \\ 1 & q & 1 \end{pmatrix}.$$

Let J be the two-sided ideal of $\mathbb{C}_Q$ generated by $t_0^2 - 1$. Define

$$\widetilde{L} = \mathbb{C}_Q / J = \langle t_0^i t_1^j t_2^k \mid i \in \mathbb{Z}_2,\ j, k \in \mathbb{Z} \rangle,$$

to be the quotient of $\mathbb{C}_Q$ by J, and identify t_0 with its image in $\widetilde{L}$. Then the derived Lie subalgebra of $\widetilde{L}$ is

$$\bar{L} = \langle t_0^{\bar{0}} t^{\boldsymbol{m}} \mid \boldsymbol{m} \in \mathbb{Z}^{2*} \rangle \oplus \langle t_0^{\bar{1}} t^{\boldsymbol{m}} \mid \boldsymbol{m} \in \mathbb{Z}^2 \rangle.$$

Now we define a central extension $L = \bar{L} \oplus \langle c_1, c_2 \rangle$ of $\bar{L}$ by the Lie bracket

$$\begin{aligned}[t_0^i t^{\boldsymbol{m}}, t_0^j t^{\boldsymbol{n}}] = \bigl((-1)^{m_1 j} q^{m_2 n_1} - (-1)^{i n_1} q^{m_1 n_2}\bigr) t_0^{i+j} t^{\boldsymbol{m}+\boldsymbol{n}} \\ + (-1)^{m_1 j} q^{m_2 n_1} \delta_{i+j,\bar{0}} \delta_{\boldsymbol{m}+\boldsymbol{n},0} (m_1 c_1 + m_2 c_2),\end{aligned}$$

where c_1 and c_2 are central and where i, j are in $\mathbb{Z}_2$, as are $\boldsymbol{m} = (m_1, m_2)$ and $\boldsymbol{n} = (n_1, n_2)$. One can easily see that $\langle t_0^{\bar{0}} t^{\boldsymbol{m}} \mid \boldsymbol{m} \in \mathbb{Z}^{2*}\rangle \oplus \langle c_1, c_2\rangle$ is a Lie subalgebra of L that is isomorphic to the q-analogue Virasoro-like algebra.

First we prove that the Lie algebra L is in fact isomorphic to the core of the EALAs of type A_1 with coordinates in $\mathbb{C}_{q^2}$.

Proposition 2.1. *The Lie algebra L is isomorphic to τ and the isomorphism is given by the linear extension of the map φ defined by*

$$\begin{aligned}
t_0^i t_1^{2m_1+1} t_2^{m_2} &\mapsto (-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}),\\
t_0^i t_1^{2m_1} t_2^{m_2} &\mapsto (-1)^i E_{11}(t_1^{m_1} t_2^{m_2}) + q^{-m_2} E_{22}(t_1^{m_1} t_2^{m_2}) + \delta_{i,\bar{1}}\delta_{m_1,0}\delta_{m_2,0}\tfrac{1}{2}K_1,\\
c_1 &\mapsto K_1,\\
c_2 &\mapsto 2K_2,
\end{aligned}$$

where $t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^i t_1^{2m_1} t_2^{m_2} \in L$.

Proof. One can easily see that φ is a bijection. Thus we only need to prove that φ preserves Lie bracket. First we have

$$\begin{aligned}
&\big[(-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}), (-1)^j q^{-n_2} E_{12}(t_1^{n_1} t_2^{n_2}) + E_{21}(t_1^{n_1+1} t_2^{n_2})\big]\\
&\quad= \big((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{n_2(2m_1+1)}\big)\big((-1)^{i+j} E_{11}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\\
&\qquad\qquad + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\big)\\
&\qquad + \delta_{m_1+n_1+1,0}\delta_{m_2+n_2,0}(-1)^j q^{m_2(2n_1+1)}\big((-1)^{i+j}(m_1K_1 + m_2K_2)\\
&\qquad\qquad + (m_1+1)K_1 + m_2K_2\big)\\
&\quad= \big((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{n_2(2m_1+1)}\big)\big((-1)^{i+j} E_{11}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\\
&\qquad\qquad + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\big)\\
&\qquad + \delta_{i+j,\bar{0}}\delta_{m_1+n_1+1,0}\delta_{m_2+n_2,0}(-1)^j q^{m_2(2n_1+1)}((2m_1+1)K_1 + 2m_2K_2)\\
&\qquad + \delta_{i+j,\bar{1}}\delta_{m_1+n_1+1,0}\delta_{m_2+n_2,0}(-1)^j q^{m_2(2n_1+1)} K_1.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&[t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1+1} t_2^{n_2}]\\
&\qquad= \big((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{(2m_1+1)n_2}\big) t_0^{i+j} t_1^{2m_1+2n_1+2} t_2^{m_2+n_2}\\
&\qquad\quad + \delta_{i+j,\bar{0}}\delta_{2m_1+2n_1+2,0}\delta_{m_2+n_2,0}(-1)^j q^{m_2(2n_1+1)}((2m_1+1)c_1 + m_2c_2).
\end{aligned}$$

Thus

$$\varphi\big([t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1+1} t_2^{n_2}]\big) = \big[\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1+1} t_2^{n_2})\big].$$

Similarly, we have

$$\begin{aligned}
&\big[\varphi(t_0^i t_1^{2m_1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})\big] \\
&\quad = \big[(-1)^i E_{11}(t_1^{m_1} t_2^{m_2}) + q^{-m_2} E_{22}(t_1^{m_1} t_2^{m_2}), (-1)^j E_{11}(t_1^{n_1} t_2^{n_2}) + q^{-n_2} E_{22}(t_1^{n_1} t_2^{n_2})\big] \\
&\quad = (q^{2m_2n_1} - q^{2n_2m_1})\big((-1)^{i+j} E_{11}(t_1^{m_1+n_1} t_2^{m_2+n_2}) + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1} t_2^{m_2+n_2})\big) \\
&\qquad\qquad + \delta_{m_1+n_1,0}\delta_{m_2+n_2,0}\delta_{i+j,\bar{0}} q^{2m_2n_1}(2m_1K_1 + 2m_2K_2),
\end{aligned}$$

and

$$\begin{aligned}
[t_0^i t_1^{2m_1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}] &= (q^{2m_2n_1} - q^{2m_1n_2}) t_0^{i+j} t_1^{2m_1+2n_1} t_2^{m_2+n_2} \\
&\qquad + \delta_{i+j,\bar{0}}\delta_{m_1+n_1,0}\delta_{m_2+n_2,0} q^{2m_2n_1}(2m_1c_1 + m_2c_2).
\end{aligned}$$

Therefore

$$\big[\varphi(t_0^i t_1^{2m_1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})\big] = \varphi\big([t_0^i t_1^{2m_1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}]\big).$$

Finally, we have

$$\begin{aligned}
&\big[\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})\big] \\
&\quad = \big[(-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}), (-1)^j E_{11}(t_1^{n_1} t_2^{n_2}) + q^{-n_2} E_{22}(t_1^{n_1} t_2^{n_2})\big] \\
&\quad = ((-1)^j q^{2m_2n_1} - q^{n_2(2m_1+1)}) \\
&\qquad\qquad \cdot \big((-1)^{i+j} q^{-m_2-n_2} E_{12}(t_1^{m_1+n_1} t_2^{m_2+n_2}) + E_{21}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\big),
\end{aligned}$$

and

$$\big[t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}\big] = ((-1)^j q^{2m_2n_1} - q^{n_2(2m_1+1)}) t_0^{i+j} t_1^{2m_1+2n_1+1} t_2^{m_2+n_2}.$$

Thus

$$\big[\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})\big] = \varphi\big([t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}]\big). \qquad \square$$

Remark 2.2. This proof shows also that $\mathfrak{gl}_2(\mathbb{C}_{q^2}) \cong \widetilde{L}$ and $\bar{\tau} \cong \bar{L}$.

Next we will recall some concepts about $\mathbb{Z}$-graded L-modules. Fix a $\mathbb{Z}$-basis

$$\boldsymbol{m}_1 = (m_{11}, m_{12}) \quad \text{and} \quad \boldsymbol{m}_2 = (m_{21}, m_{22}) \in \mathbb{Z}^2.$$

If we define the degree of the elements in $\langle t_0^i t^{j\boldsymbol{m}_1 + k\boldsymbol{m}_2} \in L \mid i \in \mathbb{Z}_2, k \in \mathbb{Z}\rangle$ to be j and the degree of the elements in $\langle c_1, c_2\rangle$ to be zero, then L can be regarded as a $\mathbb{Z}$-graded Lie algebra with graded subspaces

$$L_j = \langle t_0^i t^{j\boldsymbol{m}_1 + k\boldsymbol{m}_2} \in L \mid i \in \mathbb{Z}_2, k \in \mathbb{Z}\rangle \oplus \delta_{j,0}\langle c_1, c_2\rangle,$$

so that $L = \bigoplus_{j\in\mathbb{Z}} L_j$. Setting $L_+ = \bigoplus_{j\in\mathbb{N}} L_j$ and $L_- = \bigoplus_{-j\in\mathbb{N}} L_j$, we have the triangular decomposition $L = L_- \oplus L_0 \oplus L_+$.

Definition 2.3. For any L-module V, if $V=\bigoplus_{m\in\mathbb{Z}} V_m$ with $L_j\cdot V_m\subset V_{m+j}$ for all $j,m\in\mathbb{Z}$, then V is called a $\mathbb{Z}$-graded L-module and V_m is called a homogeneous subspace of V with degree $m\in\mathbb{Z}$. The L-module V is called

(i) a quasifinite $\mathbb{Z}$-graded module if $\dim V_m<\infty$ for all $m\in\mathbb{Z}$;

(ii) a uniformly bounded module if there exists some $N\in\mathbb{N}$ such that $\dim V_m\le N$ for all $m\in\mathbb{Z}$;

(iii) a highest (respectively lowest) weight module if V is generated by some nonzero $v\in V_m$ such that $L_+\cdot v=0$ (respectively $L_-\cdot v=0$);

(iv) a generalized highest weight module with highest degree m (see for example [Su 2003]) if there is a $\mathbb{Z}$-basis $B=\{\boldsymbol{b}_1,\boldsymbol{b}_2\}$ of $\mathbb{Z}^2$ and a nonzero $v\in V_m$ such that V is generated by v, and $t_0^i t^{\boldsymbol{m}}\cdot v=0$ for all $\boldsymbol{m}\in\mathbb{Z}_+\boldsymbol{b}_1+\mathbb{Z}_+\boldsymbol{b}_2$ and $i\in\mathbb{Z}_2$;

(v) an irreducible $\mathbb{Z}$-graded module if V does not have any nontrivial $\mathbb{Z}$-graded submodules (see for example [Mathieu 1992]).

Denote by $\mathbb{O}_{\mathbb{Z}}$ the set of nontrivial quasifinite irreducible $\mathbb{Z}$-graded L-modules. From the definition, one sees that generalized highest weight modules contain highest and lowest weight modules as special cases. Since the central elements c_1 and c_2 of L act on irreducible graded modules V as scalars, we shall use the same symbols to denote these scalars.

Now we study the structure and representations of L_0. By the theory of Verma modules, the irreducible $\mathbb{Z}$-graded highest (or lowest) weight L-modules are classified by the characters of L_0.

Lemma 2.4. (1) *If m_{21} is an even integer, then L_0 is a Heisenberg Lie algebra.*

(2) *If m_{21} is an odd integer, then*

$$L_0=(\mathcal{A}+\mathcal{B})\oplus\langle m_{11}c_1+m_{12}c_2\rangle,$$

where $\mathcal{A}=\langle t_0^{\bar 0}t^{2j\boldsymbol{m}_2}, m_{21}c_1+m_{22}c_2 \mid j\in\mathbb{Z}\rangle$ is a Heisenberg Lie algebra and

$$\mathcal{B}=\langle t_0^{\bar 1}t^{j\boldsymbol{m}_2}, t_0^{\bar 0}t^{(2j+1)\boldsymbol{m}_2}, m_{21}c_1+m_{22}c_2 \mid j\in\mathbb{Z}\rangle,$$

which is isomorphic to the affine Lie algebra $A_1^{(1)}$ via the linear extension of the map ϕ defined by

$$\begin{aligned} t_0^{\bar 1}t^{2j\boldsymbol{m}_2} &\mapsto -q^{-2j^2m_{22}m_{21}}((E_{11}-E_{22})(x^j)+\tfrac{1}{2}K),\\ t_0^{i}t^{(2j+1)\boldsymbol{m}_2} &\mapsto q^{-\frac{1}{2}(2j+1)^2m_{22}m_{21}}((-1)^iE_{12}(x^j)+E_{21}(x^{j+1})),\\ m_{21}c_1+m_{22}c_2 &\mapsto K. \end{aligned}$$

Moreover, we have $[\mathcal{A},\mathcal{B}]=0$.

Proof. Statement (1) can be easily deduced from the definition of L_0.

To show $\mathcal{B} \cong A_1^{(1)}$ in case (2), we need to prove that ϕ preserves Lie bracket. Notice that

$$\begin{aligned}&\big[q^{-\frac{1}{2}(2j+1)^2 m_{22}m_{21}}((-1)^i E_{12}(x^j)+E_{21}(x^{j+1})),\\ &\quad q^{-\frac{1}{2}(2l+1)^2 m_{22}m_{21}}((-1)^k E_{12}(x^l)+E_{21}(x^{l+1}))\ \big]\\ &\qquad = q^{-\frac{1}{2}((2j+1)^2+(2l+1)^2)m_{22}m_{21}}\big(((-1)^i-(-1)^k)(E_{11}-E_{22})(x^{j+l+1})\\ &\qquad\qquad + \delta_{j+l+1,0}((-1)^i j+(-1)^k(j+1))K\big),\end{aligned}$$

and

$$\begin{aligned}&\big[t_0^i t^{(2j+1)\boldsymbol{m}_2}, t_0^k t^{(2k+1)\boldsymbol{m}_2}\big]\\ &\qquad = ((-1)^k-(-1)^i)q^{(2j+1)(2k+1)m_{22}m_{21}} t_0^{i+k} t^{(2j+2k+2)\boldsymbol{m}_2}\\ &\qquad\qquad + \delta_{i+k,\bar{0}}\delta_{j+k+1,0}(-1)^k q^{(2j+1)(2k+1)m_{22}m_{21}}(2j+1)(m_{21}c_1+m_{22}c_2).\end{aligned}$$

One sees that

$$\phi\big([t_0^i t^{(2j+1)\boldsymbol{m}_2}, t_0^k t^{(2k+1)\boldsymbol{m}_2}]\big) = \big[\phi(t_0^i t^{(2j+1)\boldsymbol{m}_2}), \phi(t_0^k t^{(2k+1)\boldsymbol{m}_2})\big].$$

From the facts that

$$\begin{aligned}&\big[-q^{-2j^2 m_{22}m_{21}}((E_{11}-E_{22})(x^j)+\tfrac{1}{2}K),\\ &\quad q^{-\frac{1}{2}(2l+1)^2 m_{22}m_{21}}((-1)^k E_{12}(x^l)+E_{21}(x^{l+1}))\big]\\ &\qquad = -q^{-\frac{1}{2}(4j^2+(2l+1)^2)m_{22}m_{21}}(2(-1)^k E_{12}(x^{l+j})-2E_{21}(x^{l+j+1}))\end{aligned}$$

and

$$\big[t_0^{\bar{1}} t^{2j\boldsymbol{m}_2}, t_0^k t^{(2l+1)\boldsymbol{m}_2}\big] = 2q^{2j(2l+1)m_{22}m_{21}} t_0^{k+\bar{1}} t^{(2j+2l+1)\boldsymbol{m}_2},$$

we have

$$\phi\big([t_0^{\bar{1}} t^{2j\boldsymbol{m}_2}, t_0^k t^{(2l+1)\boldsymbol{m}_2}]\big) = \big[\phi(t_0^{\bar{1}} t^{2j\boldsymbol{m}_2}), \phi(t_0^k t^{(2l+1)\boldsymbol{m}_2})\big].$$

Finally, we have

$$\begin{aligned}&\big[-q^{-2j^2 m_{22}m_{21}}((E_{11}-E_{22})(x^j)+\tfrac{1}{2}K), -q^{-2l^2 m_{22}m_{21}}((E_{11}-E_{22})(x^l)+\tfrac{1}{2}K)\big]\\ &\qquad = 2jq^{-2(j^2+l^2)m_{22}m_{21}}\delta_{j+l,0}K = 2jq^{4jl m_{22}m_{21}}\delta_{j+l,0}K,\end{aligned}$$

and

$$\big[t_0^{\bar{1}} t^{2j\boldsymbol{m}_2}, t_0^{\bar{1}} t^{2l\boldsymbol{m}_2}\big] = 2jq^{4jl m_{22}m_{21}}\delta_{j+l,0}(m_{21}c_1+m_{22}c_2).$$

Thus

$$\phi\big([t_0^{\bar{1}} t^{2j\boldsymbol{m}_2}, t_0^{\bar{1}} t^{2l\boldsymbol{m}_2}]\big) = \big[\phi(t_0^{\bar{1}} t^{2j\boldsymbol{m}_2}), \phi(t_0^{\bar{1}} t^{2l\boldsymbol{m}_2})\big].$$

This proves $\mathcal{B} \cong A_1^{(1)}$. The proof of the remaining claims is straightforward. □

Since the Lie subalgebra $\mathcal{B}$ of L_0 is isomorphic to the affine Lie algebra $A_1^{(1)}$, we need to collect some results from [Rao 1993] on the finite-dimensional irreducible modules of $A_1^{(1)}$.

Let $\nu > 0$, and let $\underline{a} = (a_1, \ldots, a_\nu)$ be a finite sequence of nonzero distinct numbers. For $1 \le i \le \nu$, let V_i be finite-dimensional irreducible sl_2-modules, and let $\boldsymbol{v} := (v_1 \otimes \cdots \otimes v_\nu) \in V_1 \otimes \cdots \otimes V_\nu$. We then define an $A_1^{(1)}$-module $V(\underline{a}) = V_1 \otimes V_2 \otimes \cdots \otimes V_\nu$ by setting

$$X(x^j)\cdot \boldsymbol{v} = \sum_{i=1}^{\nu} a_i^j v_1 \otimes \cdots \otimes (X\cdot v_i) \otimes \cdots \otimes v_\nu \quad \text{and} \quad K\cdot \boldsymbol{v} = 0$$

for $X \in \mathrm{sl}_2$ and $j \in \mathbb{Z}$. Clearly $V(\underline{a})$ is a finite-dimensional irreducible $A_1^{(1)}$-module. For any $Q(x) \in \mathbb{C}[x^{\pm 1}]$, we have $X(Q(x))\cdot (V_1 \otimes \cdots \otimes V_\nu) = 0$ for all $X \in \mathrm{sl}_2$ if and only if $\prod_{i=1}^{\nu}(x - a_1) \mid Q(x)$. Now by Lemma 2.4(2), if m_{21} is an odd integer, we can define a finite-dimensional irreducible L_0-module $V(\underline{a}, \psi) = V_1 \otimes \cdots \otimes V_\nu$ by

$$
\begin{aligned}
t_0^{\bar 0} t^{2j\boldsymbol{m}_2}\cdot \boldsymbol{v} &= \psi(t_0^{\bar 0} t^{2j\boldsymbol{m}_2})\cdot (v_1 \otimes \cdots \otimes v_\nu),\\
t_0^{\bar 1} t^{2j\boldsymbol{m}_2}\cdot \boldsymbol{v} &= -q^{-2j^2 m_{22} m_{21}} \sum_{i=1}^{\nu} a_i^j v_1 \otimes \cdots \otimes ((E_{11} - E_{22})\cdot v_i) \otimes \cdots \otimes v_\nu,\\
t_0^{i} t^{(2j+1)\boldsymbol{m}_2}\cdot \boldsymbol{v} &= q^{-\frac{1}{2}(2j+1)^2 m_{22} m_{21}} \Big((-1)^i \sum_{i=1}^{\nu} a_i^j v_1 \otimes \cdots \otimes (E_{12}\cdot v_i) \otimes \cdots \otimes v_\nu \\
&\qquad + \sum_{i=1}^{\nu} a_i^{j+1} v_1 \otimes \cdots \otimes (E_{21}\cdot v_i) \otimes \cdots \otimes v_\nu \Big),\\
(m_{21} c_1 + m_{22} c_2)\cdot \boldsymbol{v} &= 0
\end{aligned}
$$

for $j \in \mathbb{Z}$ and $i \in \mathbb{Z}_2$. Here ψ is a linear function over $\mathcal{A}$.

Theorem 2.5 [Rao 1993, Theorem 2.14]. *Let V be a finite-dimensional irreducible $A_1^{(1)}$-module. Then V is isomorphic to $V(\underline{a})$ for some finite-dimensional irreducible* sl_2*-modules $V_1, \ldots, V_\nu$ and a finite sequence $\underline{a} = (a_1, \ldots, a_\nu)$ of nonzero distinct numbers.*

This theorem and Lemma 2.4 implies another:

Theorem 2.6. *Let m_{21} be an odd integer, and let V be a finite-dimensional irreducible L_0-module. Then V is isomorphic to $V(\underline{a}, \psi)$, where $V_1, \ldots, V_\nu$ are finite-dimensional irreducible* sl_2*-modules, $\underline{a} = (a_1, \ldots, a_\nu)$ is a finite sequence of nonzero distinct numbers, and ψ is a linear function over $\mathcal{A}$.*

Remark 2.7. Let m_{21} be an odd integer, and let $V(\underline{a}, \psi)$ be a finite-dimensional irreducible L_0-modules defined as above. One can see that for any $k \in \mathbb{Z}_2$,

$$\Big(\sum_{i=1}^{n} b_i q^{\frac{1}{2}(2i+1)^2 m_{22} m_{21}} t_0^k t^{(2i+1)\boldsymbol{m}_2}\Big) \cdot (V_1 \otimes \cdots \otimes V_\nu) = 0 \text{ and}$$

$$\Big(\sum_{i=1}^{n} b_i q^{2i^2 m_{22} m_{21}} t_0^{\bar{1}} t^{2i\boldsymbol{m}_2}\Big) \cdot (V_1 \otimes \cdots \otimes V_\nu) = 0$$

if and only if $\prod_{i=1}^{\nu}(x - a_1) \mid (\sum_{i=1}^{n} b_i x^i)$.

Proposition 2.8. *If V is an irreducible $\mathbb{Z}$-graded L-module, then V is a generalized highest weight module or a uniformly bounded module.*

Proof. Let $V = \bigoplus_{m \in \mathbb{Z}} V_m$. We first prove that if there exists a $\mathbb{Z}$-basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of $\mathbb{Z}^2$ and a homogeneous vector $v \neq 0$ such that $t_0^i t^{\boldsymbol{b}_1} \cdot v = t_0^i t^{\boldsymbol{b}_2} \cdot v = 0$ for all $i \in \mathbb{Z}_2$, then V is a generalized highest weight module.

For $A \subset \mathbb{Z}^2$, we denote by t^A the set $\{t^a \mid a \in A\}$.

By assumption, one can deduce that $t_0^i t^{\mathbb{N}\boldsymbol{b}_1 + \mathbb{N}\boldsymbol{b}_2} \cdot v = 0$ for all $i \in \mathbb{Z}_2$. Thus for the $\mathbb{Z}$-basis $\{\boldsymbol{m}_1 = 3\boldsymbol{b}_1 + \boldsymbol{b}_2, \boldsymbol{m}_2 = 2\boldsymbol{b}_1 + \boldsymbol{b}_2\}$ of $\mathbb{Z}^2$ we have $t_0^i t^{\mathbb{Z}_+\boldsymbol{m}_1 + \mathbb{Z}_+\boldsymbol{m}_2} v = 0$ for all $i \in \mathbb{Z}_2$, so that V meets the definition of generalized highest weight module.

We can prove our proposition. Suppose that V is not a generalized highest weight module. For any $m \in \mathbb{Z}$, consider the maps

$$t_0^{\bar{0}} t^{-m\boldsymbol{m}_1 + \boldsymbol{m}_2} : V_m \mapsto V_0, \qquad t_0^{\bar{1}} t^{-m\boldsymbol{m}_1 + \boldsymbol{m}_2} : V_m \mapsto V_0,$$
$$t_0^{\bar{0}} t^{(1-m)\boldsymbol{m}_1 + \boldsymbol{m}_2} : V_m \mapsto V_1, \qquad t_0^{\bar{1}} t^{(1-m)\boldsymbol{m}_1 + \boldsymbol{m}_2} : V_m \mapsto V_1.$$

Since $\{-m\boldsymbol{m}_1 + \boldsymbol{m}_2, (1-m)\boldsymbol{m}_1 + \boldsymbol{m}_2\}$ is a $\mathbb{Z}$-base of $\mathbb{Z}^2$, one can check that

$$\ker t_0^{\bar{0}} t^{-m\boldsymbol{m}_1 + \boldsymbol{m}_2} \cap \ker t_0^{\bar{0}} t^{(1-m)\boldsymbol{m}_1 + \boldsymbol{m}_2} \cap \ker t_0^{\bar{1}} t^{-m\boldsymbol{m}_1 + \boldsymbol{m}_2} \cap \ker t_0^{\bar{1}} t^{(1-m)\boldsymbol{m}_1 + \boldsymbol{m}_2} = \{0\},$$

Therefore $\dim V_m \leq 2 \dim V_0 + 2 \dim V_1$. So V is a uniformly bounded module. □

3. The highest weight irreducible $\mathbb{Z}$-graded L-modules

In this section, V is a finite-dimensional irreducible L_0-module; V becomes a $(L_0 + L_+)$-module if we put $L_+ v = 0$ for all $v \in V$. Then we obtain an induced L-module

$$\overline{M}^+(V, \boldsymbol{m}_1, \boldsymbol{m}_2) = \operatorname{Ind}_{L_0 + L_+}^{L} V = U(L) \otimes_{U(L_0 + L_+)} V \simeq U(L_-) \otimes V,$$

where $U(L)$ is the universal enveloping algebra of L. If we set V to be the homogeneous subspace of $\overline{M}^+(V, \boldsymbol{m}_1, \boldsymbol{m}_2)$ with degree 0, then $\overline{M}^+(V, \boldsymbol{m}_1, \boldsymbol{m}_2)$ becomes a $\mathbb{Z}$-graded L-module in a natural way. Obviously, $\overline{M}^+(V, \boldsymbol{m}_1, \boldsymbol{m}_2)$ has a unique maximal proper submodule J that trivially intersects with V. So we obtain an irreducible $\mathbb{Z}$-graded highest weight L-module

$$M^+(V, \boldsymbol{m}_1, \boldsymbol{m}_2) = \overline{M}^+(V, \boldsymbol{m}_1, \boldsymbol{m}_2)/J.$$

We can write it as $M^+(V, \boldsymbol{m}_1, \boldsymbol{m}_2) = \bigoplus_{i\in\mathbb{Z}_+} V_{-i}$, where V_{-i} is the homogeneous subspace of degree $-i$. Since L_- is generated by L_{-1}, and L_+ is generated by L_1, we see by the construction of $M^+(V, \boldsymbol{m}_1, \boldsymbol{m}_2)$ that

$$L_{-1}V_{-i} = V_{-i-1} \quad \text{for all } i \in \mathbb{Z}_+, \tag{3-1}$$

and for a homogeneous vector $v \in V_i$ with $i < 0$,

$$L_1 \cdot v = 0 \quad \text{implies} \quad v = 0. \tag{3-2}$$

Similarly, V gives rise to an irreducible lowest weight $\mathbb{Z}$-graded L-module $M^-(V, \boldsymbol{m}_1, \boldsymbol{m}_2)$.

If $m_{21} \in \mathbb{Z}$ is even, then L_0 is a Heisenberg Lie algebra by Lemma 2.4. It is well known from the representation theory of this algebra that V must be a one-dimensional module $\mathbb{C}v_0$, and there is a linear function ψ over L_0 such that

$$t_0^i t^{j\boldsymbol{m}_2} \cdot v_0 = \psi(t_0^i t^{j\boldsymbol{m}_2}) \cdot v_0 \quad \text{and} \quad \psi(m_{21}c_1 + m_{22}c_2) = 0 \quad \text{for all } i \in \mathbb{Z}_2,\ j \in \mathbb{Z}.$$

In this case, we denote the corresponding highest and lowest weight irreducible $\mathbb{Z}$-graded L-modules by $M^+(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$ and $M^-(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$, respectively.

If m_{21} is an odd integer, then by Theorem 2.6, V must be isomorphic to $V(\underline{a}, \psi)$, and we denote the corresponding highest and lowest weight irreducible $\mathbb{Z}$-graded L-modules by $M^+(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$ and $M^-(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$, respectively.

The L-modules $M^+(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$ and $M^+(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$ are quasifinite only for certain $\underline{a}$ and ψ, which we shall now determine.

For later use, we obtain from the definition of L the equations

$$\begin{aligned} \big[t_0^j t^{\boldsymbol{m}_1+k\boldsymbol{m}_2}, t_0^r t^{-\boldsymbol{m}_1+s\boldsymbol{m}_2} t^{i\boldsymbol{m}_2}\big] &= q^{i(-m_{12}+sm_{22})m_{21}}\big[t_0^j t^{\boldsymbol{m}_1+k\boldsymbol{m}_2}, t_0^r t^{-\boldsymbol{m}_1+(s+i)\boldsymbol{m}_2}\big] \\ &= q^{-m_{11}m_{12}-km_{11}m_{22}+sm_{12}m_{21}+ksm_{21}m_{22}}(-1)^{r(m_{11}+km_{21})} \\ &\quad \cdot \big((1-(-1)^{(j+r)m_{11}+(kr+js+ji)m_{21}} q^{(k+s+i)\alpha}) t_0^{j+r} t^{(k+s)\boldsymbol{m}_2} t^{i\boldsymbol{m}_2} \\ &\qquad + \delta_{k+s+i,0}\delta_{j+r,\bar{0}} q^{-(k+s)^2 m_{21}m_{22}}((m_{11}+km_{21})c_1 + (m_{12}+km_{22})c_2)\big), \end{aligned} \tag{3-3}$$

and

$$\begin{aligned} \big[t_0^s t^{k\boldsymbol{m}_2} t^{i\boldsymbol{m}_2}, t_0^r t^{-\boldsymbol{m}_1+j\boldsymbol{m}_2}\big] &= q^{kim_{22}m_{21}}\big[t_0^s t^{(k+i)\boldsymbol{m}_2}, t_0^r t^{-\boldsymbol{m}_1+j\boldsymbol{m}_2}\big] \\ &= q^{km_{22}(-m_{11}+jm_{21})}(-1)^{(rk+ri)m_{21}}(q^{-i\alpha} - (-1)^{sm_{11}+(rk+ri+sj)m_{21}} q^{k\alpha}) \\ &\quad \cdot t_0^{r+s} t^{-\boldsymbol{m}_1+(k+j)\boldsymbol{m}_2} t^{i\boldsymbol{m}_2}. \end{aligned} \tag{3-4}$$

Here $\alpha = m_{11}m_{22} - m_{12}m_{21} \in \{\pm 1\}$,

Lemma 3.1. *Let m_{21} be an even integer. Then $M^{\pm}(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathcal{O}_{\mathbb{Z}}$ if and only if there exists a polynomial $P(t^{\boldsymbol{m}_2}) = \sum_{i=0}^{n} a_i t^{i\boldsymbol{m}_2} \in \mathbb{C}[t^{\boldsymbol{m}_2}]$ with $a_0 a_n \neq 0$ such that for $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_2$,*

$$\psi\big(t_0^j t^{k\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) - (-1)^j q^{k\alpha} t_0^j t^{k\boldsymbol{m}_2} P(q^\alpha t^{\boldsymbol{m}_2}) + \delta_{j,\bar{0}} a_{-k} q^{-k^2 m_{21} m_{22}} \beta\big) = 0, \tag{3-5}$$

where $\alpha = m_{11}m_{22} - m_{12}m_{21} \in \{\pm 1\}$ and $\beta = m_{11}c_1 + m_{12}c_2$, and where $a_k = 0$ if $k \notin \{0, 1, \dots, n\}$.

Proof. Since m_{21} is an even integer and $m_{11}m_{22} - m_{12}m_{21} \in \{\pm 1\}$, we see m_{11} is an odd integer.

We first prove the forward implication. Since $\dim V_{-1} < \infty$, there exist an integer s and a polynomial $P(t^{\boldsymbol{m}_2}) = \sum_{i=0}^{n} a_i t^{i\boldsymbol{m}_2} \in \mathbb{C}[t^{\boldsymbol{m}_2}]$ with $a_0 a_n \neq 0$ such that

$$t_0^{\bar{0}} t^{-\boldsymbol{m}_1 + s\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) \cdot v_0 = 0.$$

Applying $t_0^j t^{\boldsymbol{m}_1 + k\boldsymbol{m}_2}$ for any $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_2$ to this equation, we have

$$0 = t_0^j t^{\boldsymbol{m}_1 + k\boldsymbol{m}_2} \cdot t_0^{\bar{0}} t^{-\boldsymbol{m}_1 + s\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) \cdot v_0 = \sum_{i=0}^{n} \big[t_0^j t^{\boldsymbol{m}_1 + k\boldsymbol{m}_2}, a_i t_0^{\bar{0}} t^{-\boldsymbol{m}_1 + s\boldsymbol{m}_2} t^{i\boldsymbol{m}_2}\big] \cdot v_0.$$

Thus, by (3-3), we have

$$\begin{aligned} 0 &= \psi\Big(\sum_{i=0}^{n} a_i \big((1 - (-1)^j q^{(k+s+i)\alpha}) t_0^j t^{(k+s)\boldsymbol{m}_2} t^{i\boldsymbol{m}_2} + \delta_{k+s+i,0} \delta_{j,\bar{0}} q^{-(k+s)^2 m_{21} m_{22}} \beta\big)\Big) \\ &= \psi\big(t_0^j t^{(k+s)\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) - (-1)^j q^{(k+s)\alpha} t_0^j t^{(k+s)\boldsymbol{m}_2} P(q^\alpha t^{\boldsymbol{m}_2}) \\ &\qquad + a_{-k-s} \delta_{j,\bar{0}} q^{-(k+s)^2 m_{21} m_{22}} \beta\big). \end{aligned}$$

This proves necessity.

We next prove sufficiency.

Claim 1. For any $s \in \mathbb{Z}_+$, there is a polynomial $P_s(t^{\boldsymbol{m}_2}) = \sum_{i \in \mathbb{Z}} a_{s,i} t^{i\boldsymbol{m}_2} \in \mathbb{C}[t^{\boldsymbol{m}_2}]$ such that, for all $r \in \mathbb{Z}_2$ and $k \in \mathbb{Z}$,

$$\begin{aligned} \big(t_0^r t^{k\boldsymbol{m}_2} P_s(t^{\boldsymbol{m}_2}) - (-1)^r q^{k\alpha} t_0^r t^{k\boldsymbol{m}_2} P_s(q^\alpha t^{\boldsymbol{m}_2}) + \delta_{r,\bar{0}} a_{s,-k} q^{-k^2 m_{21} m_{22}} \beta\big) \cdot V_{-s} &= 0, \\ t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_s(t^{\boldsymbol{m}_2}) \cdot V_{-s} &= 0. \end{aligned}$$

We prove the claim by induction on s. For $s = 0$, the first equation holds with $P_0(t^{\boldsymbol{m}_2}) = P(t^{\boldsymbol{m}_2})$, where P is as in the proof of necessity, and by (3-2), the second equation proved by proceeding as in the forward direction.

Now suppose the claim holds for s. For $s + 1$, the equations in the claim are equivalent, for all $r \in \mathbb{Z}_2$ and $k \in \mathbb{Z}$, to

$$\begin{aligned} \big(t_0^r Q(t^{\boldsymbol{m}_2}) - (-1)^r t_0^r Q(q^\alpha t^{\boldsymbol{m}_2}) + \delta_{r,\bar{0}} a_Q \beta\big) \cdot V_{-s} &= 0, \\ t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) \cdot V_{-s} &= 0 \end{aligned} \tag{3-6}$$

for any $Q(t^{\boldsymbol{m}_2}) \in \mathbb{C}[t^{\pm\boldsymbol{m}_2}]$ with $P_s(t^{\boldsymbol{m}_2}) \mid Q(t^{\boldsymbol{m}_2})$, where a_Q is the constant term of $Q(t^{\boldsymbol{m}_2})$.

Let $P_{s+1}(t^{\boldsymbol{m}_2}) = P_s(q^{\alpha} t^{\boldsymbol{m}_2}) P_s(t^{\boldsymbol{m}_2}) P_s(q^{-\alpha} t^{\boldsymbol{m}_2})$. Then we have

$$P_s(t^{\boldsymbol{m}_2}) \mid P_{s+1}(t^{\boldsymbol{m}_2}), \quad P_s(t^{\boldsymbol{m}_2}) \mid P_{s+1}(q^{\alpha} t^{\boldsymbol{m}_2}), \quad P_s(t^{\boldsymbol{m}_2}) \mid P_{s+1}(q^{-\alpha} t^{\boldsymbol{m}_2}).$$

For any $p, r \in \mathbb{Z}_2$ and $j, k \in \mathbb{Z}$, by induction and (3-4), we have

$$\begin{aligned}
&\big(t_0^r t^{k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) - (-1)^r q^{k\alpha} t_0^r t^{k\boldsymbol{m}_2} P_{s+1}(q^{\alpha} t^{\boldsymbol{m}_2}) + \delta_{r,\bar{0}} a_{s+1,-k} q^{-k^2 m_{21} m_{22}} \beta\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot t_0^p t^{-\boldsymbol{m}_1 + j\boldsymbol{m}_2} \cdot V_{-s} \\
&\quad = \big(t_0^r t^{k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) - (-1)^r q^{k\alpha} t_0^r t^{k\boldsymbol{m}_2} P_{s+1}(q^{\alpha} t^{\boldsymbol{m}_2}) \\
&\qquad\qquad\qquad + \delta_{r,\bar{0}} a_{s+1,-k} q^{-k^2 m_{21} m_{22}} \beta, t_0^p t^{-\boldsymbol{m}_1 + j\boldsymbol{m}_2}\big) \cdot V_{-s} \\
&\quad = q^{-k m_{22} m_{11} + k j m_{22} m_{21}} \big(t_0^{r+p} t^{-\boldsymbol{m}_1 + (k+j)\boldsymbol{m}_2} \big(P_{s+1}(q^{-\alpha} t^{\boldsymbol{m}_2}) - 2(-1)^r q^{k\alpha} P_{s+1}(t^{\boldsymbol{m}_2}) \\
&\qquad\qquad\qquad + q^{2k\alpha} P_{s+1}(q^{\alpha} t^{\boldsymbol{m}_2})\big)\big) \cdot V_{-s} \\
&\quad = 0.
\end{aligned}$$

Thus, by (3-1) and (3-2), we obtain

$$\begin{aligned}
\text{(3-7)}\quad &\big(t_0^r t^{k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) - (-1)^r q^{k\alpha} t_0^r t^{k\boldsymbol{m}_2} P_{s+1}(q^{\alpha} t^{\boldsymbol{m}_2}) \\
&\qquad\qquad + \delta_{r,\bar{0}} a_{s+1,-k} q^{-k^2 m_{21} m_{22}} \beta\big) \cdot V_{-s-1} = 0.
\end{aligned}$$

This proves the first equation in the claim for $i = s+1$.

Using (3-3), (3-6) and induction, we deduce for any $l, k, j \in \mathbb{Z}$ and $n, r, p \in \mathbb{Z}_2$ that

$$\begin{aligned}
&t_0^n t^{\boldsymbol{m}_1 + l\boldsymbol{m}_2} \cdot t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot t_0^p t^{-\boldsymbol{m}_1 + j\boldsymbol{m}_2} \cdot V_{-s} \\
&\quad = \big[t_0^n t^{\boldsymbol{m}_1 + l\boldsymbol{m}_2}, t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2})\big] \cdot t_0^p t^{-\boldsymbol{m}_1 + j\boldsymbol{m}_2} \cdot V_{-s} \\
&\qquad\qquad + t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot t_0^n t^{\boldsymbol{m}_1 + l\boldsymbol{m}_2} \cdot t_0^p t^{-\boldsymbol{m}_1 + j\boldsymbol{m}_2} \cdot V_{-s} \\
&\quad = (-1)^r q^{-m_{11} m_{12} + k m_{12} m_{21} - l m_{11} m_{22} + l k m_{21} m_{22}} \\
&\qquad \cdot \big(t_0^{n+r} t^{(l+k)\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) - (-1)^{n+r} q^{(k+l)\alpha} t_0^{n+r} t^{(l+k)\boldsymbol{m}_2} P_{s+1}(q^{\alpha} t^{\boldsymbol{m}_2}) \\
&\qquad\qquad + a_{s+1,-l-k} \delta_{r+n,\bar{0}} q^{-(l+k)^2 m_{21} m_{22}} \beta\big) \cdot V_{-s} \\
&\quad = 0,
\end{aligned}$$

since $t_0^n t^{\boldsymbol{m}_1 + l\boldsymbol{m}_2} \cdot t_0^p t^{-\boldsymbol{m}_1 + j\boldsymbol{m}_2} \cdot V_{-s} \in V_{-s}$. Hence by (3-2),

$$t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot t_0^p t^{-\boldsymbol{m}_1 + j\boldsymbol{m}_2} \cdot V_{-s} = 0 \quad \text{for all } r, p \in \mathbb{Z}_2 \text{ and } k, j \in \mathbb{Z}.$$

Thus, by (3-1), $t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot V_{-s-1} = 0$, which implies the second equation in the claim for $i = s+1$. Therefore the claim follows by induction.

From the second equation of the claim and (3-1), we see that

$$\dim V_{-s-1} \le 2 \deg(P_{s+1}(t^{\boldsymbol{m}_2})) \cdot \dim V_s \quad \text{for all } s \in \mathbb{Z}_+,$$

where deg means "degree of". Hence $M^+(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathbb{O}_{\mathbb{Z}}$.

The statement for $M^-(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$ is proved similarly. □

Theorem 3.2. *Let m_{21} be an even integer. Then $M^{\pm}(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathbb{O}_{\mathbb{Z}}$ if and only if there exist*

$$b_{10}^{(j)}, b_{11}^{(j)}, \dots, b_{1s_1}^{(j)}, b_{20}^{(j)}, b_{21}^{(j)}, \dots, b_{2s_2}^{(j)}, \dots, b_{r0}^{(j)}, b_{r1}^{(j)}, \dots, b_{rs_r}^{(j)} \in \mathbb{C}$$

for $j \in \mathbb{Z}_2$ and $\alpha_1, \dots, \alpha_r \in \mathbb{C}^*$ such that for any $i \in \mathbb{Z}^*$ and $j \in \mathbb{Z}_2$,

$$\psi(t_0^j t^{i\boldsymbol{m}_2}) = \frac{\sum_{\ell=1}^{r} \bigl(\sum_{k=0}^{s_1} b_{\ell k}^{(j)} i^k\bigr)\alpha_\ell^i}{(1-(-1)^j q^{i\alpha}) q^{(1/2)i^2 m_{21} m_{22}}},$$
$$\psi(\beta) = b_{10}^{(0)} + b_{20}^{(0)} + \dots + b_{r0}^{(0)},$$
$$\psi(t_0^{\bar{1}} t^{\boldsymbol{0}}) = \tfrac{1}{2}(b_{10}^{(1)} + b_{20}^{(1)} + \dots + b_{r0}^{(1)}), \quad \text{and} \quad \psi(m_{21}c_1 + m_{22}c_2) = 0,$$

where $\alpha = m_{11}m_{22} - m_{21}m_{12} \in \{\pm 1\}$ and $\beta = m_{11}c_1 + m_{12}c_2$.

Proof. We first prove necessity. Let $f_{j,i} = \psi((1-(-1)^j q^{i\alpha}) q^{\frac{1}{2}i^2 m_{21} m_{22}} t_0^j t^{i\boldsymbol{m}_2})$ for $j \in \mathbb{Z}_2$ and $i \in \mathbb{Z}^*$. Also let $f_{0,0} = \psi(\beta)$ and $f_{1,0} = \psi(2t_0^1 t^{\boldsymbol{0}})$. By Lemma 3.1 there exist complex numbers $a_0, a_1, \dots, a_n$ with $a_0 a_n \neq 0$ such that

$$\sum_{i=0}^{n} a_i q^{-\frac{1}{2}i^2 m_{21} m_{22}} f_{j,k+i} = 0 \quad \text{for all } k \in \mathbb{Z} \text{ and } j \in \mathbb{Z}_2. \tag{3-8}$$

Let $b_i = a_i q^{-(1/2)i^2 m_{21} m_{22}}$. Suppose $\alpha_1, \dots, \alpha_r$ are distinct roots of the equation $\sum_{i=0}^{n} b_i x^i = 0$ with respective multiplicities $s_1+1, \dots, s_r+1$. By a well-known combinatorial formula, we see that for $j \in \mathbb{Z}_2$ there exist

$$b_{10}^{(j)}, b_{11}^{(j)}, \dots, b_{1s_1}^{(j)}, \dots, b_{r0}^{(j)}, b_{r1}^{(j)}, \dots, b_{rs_r}^{(j)} \in \mathbb{C}$$

such that $f_{j,i} = \sum_{\ell=1}^{r} \bigl(\sum_{k=0}^{s_1} b_{\ell k}^{(j)} i^k\bigr)\alpha_\ell^i$ for all $i \in \mathbb{Z}$. The equations of the theorem follow.

We now prove sufficiency. For $j \in \mathbb{Z}_2$ and $i \in \mathbb{Z}^*$, set

$$Q(x) = \prod_{i=1}^{r} (x-\alpha_i)^{s_i+1} = \sum_{i=1}^{n} b_i x^i \in \mathbb{C}[x],$$
$$f_{j,i} = (1-(-1)^j q^{i\alpha}) q^{(1/2)i^2 m_{21} m_{22}} \psi(t_0^j t^{i\boldsymbol{m}_2}),$$

and set $f_{0,0} = \psi(\beta)$ and $f_{1,0} = 2\psi(t_0^1 t^{\boldsymbol{0}})$. Then with b_i and a_i related as before, we deduce that (3-8) holds. Thus (3-5) holds for $P(t^{\boldsymbol{m}_2}) = \sum_{i=0}^{n} a_i t^{i\boldsymbol{m}_2}$. Sufficiency now follows by using Lemma 3.1. □

Lemma 3.3. *If m_{21} is an odd integer, then $M^+(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathcal{O}_{\mathbb{Z}}$ if and only if there exists a polynomial $P(t^{\boldsymbol{m}_2}) = \sum_{i=0}^n a_i t^{2i\boldsymbol{m}_2} \in \mathbb{C}[t^{\boldsymbol{m}_2}]$ with $a_0 a_n \neq 0$ such that for any $k \in \mathbb{Z}$ and $v \in V_0$,*

$$\big(t_0^{\bar{0}} t^{2k\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\boldsymbol{m}_2} P(q^\alpha t^{\boldsymbol{m}_2}) + a_{-k} q^{-4k^2 m_{21} m_{22}} \beta\big) \cdot v = 0, \tag{3-9}$$

$$t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) \cdot v = t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} P(q^\alpha t^{\boldsymbol{m}_2}) \cdot v = 0, \tag{3-10}$$

$$t_0^{\bar{1}} t^{k\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) \cdot v = t_0^{\bar{1}} t^{k\boldsymbol{m}_2} P(q^\alpha t^{\boldsymbol{m}_2}) \cdot v = 0, \tag{3-11}$$

where $a_k = 0$ if $k \notin \{0, 1, \dots, n\}$, and $\alpha = m_{11}m_{22} - m_{12}m_{21}$ and $\beta = m_{11}c_1 + m_{12}c_2$.

Proof. First necessity. Since V_0 is a finite-dimensional irreducible L_0-module, we have $V_0 \cong V(\underline{a}, \psi)$ as L_0-modules by Theorem 2.6. Since $\mathcal{H} = \langle t_0^{\bar{1}} t^{2k\boldsymbol{m}_2} \mid k \in \mathbb{Z}\rangle$ is an abelian Lie subalgebra of L_0, we can choose a common eigenvector $v_0 \in V_0$ of $\mathcal{H}$. First we prove the following claim.

Claim 2. There is a polynomial $P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) = \sum_{i=0}^n a_i t^{2i\boldsymbol{m}_2}$ with $a_n a_0 \neq 0$ such that

$$\begin{aligned}
&\big(t_0^{\bar{0}} t^{2k\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\boldsymbol{m}_2} Q(q^\alpha t^{\boldsymbol{m}_2}) + a_Q \beta\big) \cdot v_0 = 0,\\
&\big(t_0^{\bar{1}} t^{2k\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) - (-1)^{m_{11}} q^{2k\alpha} t_0^{\bar{1}} t^{2k\boldsymbol{m}_2} Q(q^\alpha t^{\boldsymbol{m}_2})\big) \cdot v_0 = 0,\\
&\big(t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) - q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} Q(q^\alpha t^{\boldsymbol{m}_2})\big) \cdot v_0 = 0,
\end{aligned}$$

$$\big(t_0^{\bar{1}} t^{(2k+1)\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) - (-1)^{m_{11}} q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} Q(q^\alpha t^{\boldsymbol{m}_2})\big) \cdot v_0 = 0 \tag{3-12}$$

for all $k \in \mathbb{Z}$ and $Q(t^{\boldsymbol{m}_2}) \in \mathbb{C}[t^{\pm 2\boldsymbol{m}_2}]$ with $P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) \mid Q(t^{\boldsymbol{m}_2})$, where a_Q is the constant term of $t^{2k\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2})$.

To prove the claim, since $\dim V_{-1} < \infty$, there exist an integer s and a polynomial $P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) = \sum_{i=0}^n a_i t^{2i\boldsymbol{m}_2} \in \mathbb{C}[t^{\boldsymbol{m}_2}]$ with $a_0 a_n \neq 0$ such that

$$t_0^{\bar{0}} t^{-\boldsymbol{m}_1 + 2s\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) \cdot v_0 = 0. \tag{3-13}$$

Applying $t_0^{\bar{0}} t^{\boldsymbol{m}_1 + 2k\boldsymbol{m}_2}$ for any $k \in \mathbb{Z}$ to the above equation, we have

$$\begin{aligned}
0 &= t_0^{\bar{0}} t^{\boldsymbol{m}_1 + 2k\boldsymbol{m}_2} \cdot t_0^{\bar{0}} t^{-\boldsymbol{m}_1 + 2s\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) \cdot v_0\\
&= \sum_{i=0}^n a_i \big[t_0^{\bar{0}} t^{\boldsymbol{m}_1 + 2k\boldsymbol{m}_2}, q^{2i m_{21}(-m_{12} + 2s m_{22})} t_0^{\bar{0}} t^{-\boldsymbol{m}_1 + 2(s+i)\boldsymbol{m}_2}\big] \cdot v_0\\
&= q^{-m_{11}m_{12} - 2k m_{22} m_{11} + 2s m_{12} m_{21} + 4ks m_{21} m_{22}}\\
&\quad \cdot \big(t_0^{\bar{0}} t^{2(k+s)\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) - q^{2(s+k)\alpha} t_0^{\bar{0}} t^{2(k+s)\boldsymbol{m}_2} P_{\mathrm{e}}(q^\alpha t^{\boldsymbol{m}_2})\\
&\qquad\qquad + a_{-k-s} q^{-4(k+s)^2 m_{21} m_{22}} \beta\big) \cdot v_0.
\end{aligned}$$

Now applying $t_0^{\bar{1}} t^{\boldsymbol{m}_1+2k\boldsymbol{m}_2}$ for any $k \in \mathbb{Z}$ to (3-13), we have

$$\begin{aligned} 0 &= t_0^{\bar{1}} t^{\boldsymbol{m}_1+2k\boldsymbol{m}_2} \cdot t_0^{\bar{0}} t^{-\boldsymbol{m}_1+2s\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) \cdot v_0 \\ &= \textstyle\sum_{i=0}^{n} a_i [t_0^{\bar{1}} t^{\boldsymbol{m}_1+2k\boldsymbol{m}_2}, q^{2im_{21}(-m_{12}+2sm_{22})} t_0^{\bar{0}} t^{-\boldsymbol{m}_1+2(s+i)\boldsymbol{m}_2}] \cdot v_0 \\ &= q^{-m_{11}m_{12}-2km_{22}m_{11}+2sm_{12}m_{21}+4ksm_{21}m_{22}} \\ &\qquad \cdot \big(t_0^{\bar{1}} t^{2(k+s)\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) - (-1)^{m_{11}} q^{2(s+k)\alpha} t_0^{\bar{1}} t^{2(k+s)\boldsymbol{m}_2} P_{\mathrm{e}}(q^{\alpha} t^{\boldsymbol{m}_2})\big) \cdot v_0. \end{aligned}$$

By applying $t_0^{\bar{0}} t^{\boldsymbol{m}_1+(2k+1)\boldsymbol{m}_2}$ and $t_0^{\bar{1}} t^{\boldsymbol{m}_1+(2k+1)\boldsymbol{m}_2}$ to (3-13) one respectively gets that

$$\begin{aligned} 0 &= t_0^{\bar{0}} t^{\boldsymbol{m}_1+(2k+1)\boldsymbol{m}_2} \cdot t_0^{\bar{0}} t^{-\boldsymbol{m}_1+2s\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) \cdot v_0 \\ &= q^{-m_{11}m_{12}-(2k+1)m_{11}m_{22}+2sm_{12}m_{21}+2s(2k+1)m_{21}m_{22}} \\ &\qquad \cdot \big(t_0^{\bar{0}} t^{(2k+2s+1)\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) - q^{(2k+2s+1)\alpha} t_0^{\bar{0}} t^{(2k+2s+1)\boldsymbol{m}_2} P_{\mathrm{e}}(q^{\alpha} t^{\boldsymbol{m}_2})\big) \cdot v_0, \\ 0 &= t_0^{\bar{1}} t^{\boldsymbol{m}_1+(2k+1)\boldsymbol{m}_2} \cdot (t_0^{\bar{0}} t^{-\boldsymbol{m}_1+2s\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2})) \cdot v_0 \\ &= q^{-m_{11}m_{12}-(2k+1)m_{11}m_{22}+2sm_{12}m_{21}+2s(2k+1)m_{21}m_{22}} \\ &\qquad \cdot \big(t_0^{\bar{1}} t^{(2k+2s+1)\boldsymbol{m}_2} P_{\mathrm{e}}(t^{\boldsymbol{m}_2}) - (-1)^{m_{11}} q^{(2k+2s+1)\alpha} t_0^{\bar{0}} t^{(2k+2s+1)\boldsymbol{m}_2} P_{\mathrm{e}}(q^{\alpha} t^{\boldsymbol{m}_2})\big) \cdot v_0. \end{aligned}$$

So we obtain the four equations of Claim 2.

On the other hand, we can choose an integer s and a polynomial $P_{\mathrm{o}}(t^{\boldsymbol{m}_2}) = \sum_{i=0}^{n} a_i t^{2i\boldsymbol{m}_2} \in \mathbb{C}[t^{\boldsymbol{m}_2}]$ with $a_0 a_n \neq 0$ such that $t_0^{\bar{0}} t^{-\boldsymbol{m}_1+(2s+1)\boldsymbol{m}_2} P_{\mathrm{o}}(t^{\boldsymbol{m}_2}) \cdot v_0 = 0$, since $\dim V_{-1} < \infty$. Thus by a calculation similar to the proof of Claim 2, we can deduce the following claim.

Claim 3. There is a polynomial $P_{\mathrm{o}}(t^{\boldsymbol{m}_2}) = \sum_{i=0}^{n} a_i t^{2i\boldsymbol{m}_2}$ with $a_n a_0 \neq 0$ such that

$$\begin{aligned} \big(t_0^{\bar{0}} t^{2k\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\boldsymbol{m}_2} Q(q^{\alpha} t^{\boldsymbol{m}_2}) + a_Q \beta\big) \cdot v_0 &= 0, \\ \big(t_0^{\bar{1}} t^{2k\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) - (-1)^{m_{11}+1} q^{2k\alpha} t_0^{\bar{1}} t^{2k\boldsymbol{m}_2} Q(q^{\alpha} t^{\boldsymbol{m}_2})\big) \cdot v_0 &= 0, \\ \big(t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) - q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} Q(q^{\alpha} t^{\boldsymbol{m}_2})\big) \cdot v_0 &= 0, \\ \big(t_0^{\bar{1}} t^{(2k+1)\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2}) - (-1)^{m_{11}+1} q^{(2k+1)\alpha} t_0^{\bar{1}} t^{(2k+1)\boldsymbol{m}_2} Q(q^{\alpha} t^{\boldsymbol{m}_2})\big) \cdot v_0 &= 0 \end{aligned} \tag{3-14}$$

for all $k \in \mathbb{Z}$ and $Q(t^{\boldsymbol{m}_2}) \in \mathbb{C}[t^{\pm 2\boldsymbol{m}_2}]$ with $P_{\mathrm{o}}(t^{\boldsymbol{m}_2}) \mid Q(t^{\boldsymbol{m}_2})$, where a_Q is the constant term of $t^{2k\boldsymbol{m}_2} Q(t^{\boldsymbol{m}_2})$.

Let $P(t^{\boldsymbol{m}_2}) = \sum_{i=0}^{n} a_i t^{2i\boldsymbol{m}_2}$ be the product of $P_{\mathrm{o}}(t^{\boldsymbol{m}_2})$ and $P_{\mathrm{e}}(t^{\boldsymbol{m}_2})$. We see that both (3-12) and (3-14) hold for $P(t^{\boldsymbol{m}_2})$. Thus one can directly deduce that both (3-9) and (3-11) hold for $P(t^{\boldsymbol{m}_2})$ and $v_0 \in V_0$. Since v_0 is an eigenvector of $t_0^{\bar{1}}$, we have

$$\begin{aligned} 0 &= t_0^{\bar{1}} \cdot t_0^{\bar{1}} t^{(2k+1)\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) \cdot v_0 \\ &= \big[t_0^{\bar{1}}, t_0^{\bar{1}} t^{(2k+1)\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2})\big] \cdot v_0 = 2 t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} P(t^{\boldsymbol{m}_2}) \cdot v_0, \end{aligned}$$

and

$$\begin{aligned}0&=t_0^{\bar 1}\cdot t_0^{\bar 1}t^{(2k+1)\boldsymbol{m}_2}P(q^\alpha t^{\boldsymbol{m}_2})\cdot v_0\\&=\big[t_0^{\bar 1},t_0^{\bar 1}t^{(2k+1)\boldsymbol{m}_2}P(q^\alpha t^{\boldsymbol{m}_2})\big]\cdot v_0=2t_0^{\bar 0}t^{(2k+1)\boldsymbol{m}_2}P(q^\alpha t^{\boldsymbol{m}_2})\cdot v_0,\end{aligned}$$

which implies (3-10) for $P(t^{\boldsymbol{m}_2})$ and v_0.

From the definition of L_0, one easily deduces that if (3-9)–(3-11) hold for any $v\in V$, then they also hold for $t_0^s t^{k\boldsymbol{m}_2}\cdot v$ for all $\forall s\in\mathbb{Z}/2\mathbb{Z}$ and $k\in\mathbb{Z}$. This completes the proof of necessity since V_0 is an irreducible L_0-module.

Now sufficiency.

Claim 4. For any $s\in\mathbb{Z}_+$, there is a polynomial $P_s(t^{\boldsymbol{m}_2})=\sum_{j\in\mathbb{Z}}a_{s,j}t^{2j\boldsymbol{m}_2}\in\mathbb{C}[t^{2\boldsymbol{m}_2}]$ such that for all $r\in\mathbb{Z}_2$ and $k\in\mathbb{Z}$,

$$\begin{aligned}\big(t_0^{\bar 0}t^{2k\boldsymbol{m}_2}P_s(t^{\boldsymbol{m}_2})-q^{2k\alpha}t_0^{\bar 0}t^{2k\boldsymbol{m}_2}P_s(q^\alpha t^{\boldsymbol{m}_2})+a_{s,-k}q^{-4k^2m_{21}m_{22}}\beta\big)\cdot V_{-s}=0,\\ t_0^{\bar 0}t^{(2k+1)\boldsymbol{m}_2}P_s(t^{\boldsymbol{m}_2})\cdot V_{-s}=t_0^{\bar 1}t^{k\boldsymbol{m}_2}P_s(t^{\boldsymbol{m}_2})\cdot V_{-s}=0,\\ t_0^r t^{-\boldsymbol{m}_1+k\boldsymbol{m}_2}P_s(t^{\boldsymbol{m}_2})\cdot V_{-s}=0.\end{aligned}$$

We prove this claim by induction on s. By assumption and the definition of the L_0-module V_0, the claim holds for $s=0$ with $P_0(t^{\boldsymbol{m}_2})=P(t^{\boldsymbol{m}_2})$. Suppose it holds for s, and consider it for $s+1$.

The equations in the claim are equivalent, for all $r\in\mathbb{Z}_2$ and $k\in\mathbb{Z}$, to

$$\begin{aligned}\big(t_0^{\bar 0}Q(t^{\boldsymbol{m}_2})-t_0^{\bar 0}Q(q^\alpha t^{\boldsymbol{m}_2})+a_Q\beta\big)\cdot V_{-s}=0,\\ t_0^{\bar 0}t^{(2k+1)\boldsymbol{m}_2}Q(t^{\boldsymbol{m}_2})\cdot V_{-s}=t_0^{\bar 1}t^{k\boldsymbol{m}_2}Q(t^{\boldsymbol{m}_2})\cdot V_{-s}=0,\\ t_0^r t^{-\boldsymbol{m}_1+k\boldsymbol{m}_2}Q(t^{\boldsymbol{m}_2})\cdot V_{-s}=0\end{aligned}\tag{3-15}$$

for any $Q(t^{\boldsymbol{m}_2})\in\mathbb{C}[t^{\pm 2\boldsymbol{m}_2}]$ with $P_s(t^{\boldsymbol{m}_2})\mid Q(t^{\boldsymbol{m}_2})$, where a_Q is the constant term of $Q(t^{\boldsymbol{m}_2})$.

Let $P_{s+1}(t^{\boldsymbol{m}_2})=P_s(q^\alpha t^{\boldsymbol{m}_2})P_s(t^{\boldsymbol{m}_2})P_s(q^{-\alpha}t^{\boldsymbol{m}_2})$. For any $p,r\in\mathbb{Z}_2$ and $j,k\in\mathbb{Z}$, using induction and (3-15) we have

$$\begin{aligned}&\big(t_0^{\bar 0}t^{2k\boldsymbol{m}_2}P_{s+1}(t^{\boldsymbol{m}_2})-q^{2k\alpha}t_0^{\bar 0}t^{2k\boldsymbol{m}_2}P_{s+1}(q^\alpha t^{\boldsymbol{m}_2})+a_{s+1,-k}q^{-4k^2m_{21}m_{22}}\beta\big)\cdot\\&\qquad\qquad t_0^p t^{-\boldsymbol{m}_1+j\boldsymbol{m}_2}\cdot V_{-s}\\&=\big[t_0^{\bar 0}t^{2k\boldsymbol{m}_2}P_{s+1}(t^{\boldsymbol{m}_2})-q^{2k\alpha}t_0^{\bar 0}t^{2k\boldsymbol{m}_2}P_{s+1}(q^\alpha t^{\boldsymbol{m}_2})\\&\qquad\qquad+a_{s+1,-k}q^{-k^2m_{21}m_{22}}\beta,t_0^p t^{-\boldsymbol{m}_1+j\boldsymbol{m}_2}\big]\cdot V_{-s}\\&=q^{2km_{22}(-m_{11}+jm_{21})}\\&\quad\cdot\big(t_0^p t^{-\boldsymbol{m}_1+(2k+j)\boldsymbol{m}_2}\big(P_{s+1}(q^{-\alpha}t^{\boldsymbol{m}_2})-2q^{2k\alpha}P_{s+1}(t^{\boldsymbol{m}_2})+q^{4k\alpha}P_{s+1}(q^\alpha t^{\boldsymbol{m}_2})\big)\big)\cdot V_{-s},\end{aligned}$$

which is equal to zero. Thus, by (3-1), we obtain that

$$\big(t_0^{\bar 0}t^{2k\boldsymbol{m}_2}P_{s+1}(t^{\boldsymbol{m}_2})-q^{2k\alpha}t_0^{\bar 0}t^{2k\boldsymbol{m}_2}P_{s+1}(q^\alpha t^{\boldsymbol{m}_2})+a_{s+1,-k}q^{-4k^2m_{21}m_{22}}\beta\big)\cdot V_{-s-1}=0.$$

Similarly, one can prove that for all $k \in \mathbb{Z}$

$$t_0^{\bar{0}} t^{(2k+1)\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot V_{-s-1} = t_0^{\bar{1}} t^{k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot V_{-s-1} = 0.$$

This proves the first two equations of the claim for $s+1$.

Using the two equations just above and induction, we deduce that for any $l, k \in \mathbb{Z}$ and $n, r \in \mathbb{Z}_2$,

$$\begin{aligned}
&t_0^n t^{\boldsymbol{m}_1 + l\boldsymbol{m}_2} \cdot t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot V_{-s-1} \\
&= \big[t_0^n t^{\boldsymbol{m}_1 + l\boldsymbol{m}_2}, t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2})\big] \cdot V_{-s-1} \\
&\qquad\qquad + t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot t_0^n t^{\boldsymbol{m}_1 + l\boldsymbol{m}_2} \cdot V_{-s-1} \\
&= (-1)^{r(m_{11} + l m_{21})} q^{-m_{11}m_{12} + k m_{12} m_{21} - l m_{11} m_{22} + l k m_{21} m_{22}} \\
&\quad \cdot \Big(t_0^{n+r} t^{(l+k)\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) - (-1)^{(n+r)m_{11} + nk + rl} q^{(k+l)\alpha} t_0^{n+r} t^{(l+k)\boldsymbol{m}_2} P_{s+1}(q^\alpha t^{\boldsymbol{m}_2}) \\
&\qquad\qquad + a_{s+1,i} \delta_{k+l+2i,0} \delta_{r+n,\bar{0}} q^{-(l+k)^2 m_{21} m_{22}} \beta\Big) \cdot V_{-s-1},
\end{aligned}$$

which is equal to zero. Hence, by (3-2),

$$t_0^r t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} P_{s+1}(t^{\boldsymbol{m}_2}) \cdot V_{-s-1} = 0$$

for all $r \in \mathbb{Z}_2$ and $k \in \mathbb{Z}$, which implies the third equation in the claim for $s+1$. Therefore the claim follows by induction.

From the third equation of the claim and (3-1), we see that

$$\dim V_{-s-1} \le 2 \deg(P_{s+1}(t^{\boldsymbol{m}_2})) \cdot \dim V_s \quad \text{for all } s \in \mathbb{Z}_+,$$

Hence $M^+(V(\underline{a}, \psi), \boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathcal{O}_{\mathbb{Z}}$. □

Theorem 3.4. *Let m_{21} be an odd integer. Then $M^+(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathcal{O}_{\mathbb{Z}}$ if and only if there exist $b_{10}, b_{11}, \dots, b_{1s_1}, b_{20}, b_{21}, \dots, b_{2s_2}, \dots, b_{r0}, b_{r1}, \dots, b_{rs_r} \in \mathbb{C}$, and $\alpha_1, \dots, \alpha_r \in \mathbb{C}^*$ such that for any $i \in \mathbb{Z}^*$ and $j \in \mathbb{Z}_2$,*

$$\psi(t_0 t^{2i\boldsymbol{m}_2}) = \frac{\sum_{\ell=1}^r \big(\sum_{k=0}^{s_1} b_{\ell k} i^k\big) \alpha_\ell^i}{(1 - q^{2i\alpha}) q^{2i^2 m_{21} m_{22}}},$$
$$\psi(\beta) = b_{10} + b_{20} + \cdots + b_{r0}, \quad \text{and} \quad \psi(m_{21}c_1 + m_{22}c_2) = 0,$$

where $\alpha = m_{11}m_{22} - m_{21}m_{12} \in \{\pm 1\}$.

Proof. First necessity. Let $f_i = \psi((1 - q^{2i\alpha}) q^{2i^2 m_{21} m_{22}} t_0^{\bar{0}} t^{2i\boldsymbol{m}_2})$ for $i \in \mathbb{Z}^*$ and let $f_0 = \psi(\beta)$. By Lemma 3.3, there exist complex numbers $a_0, a_1, \dots, a_n$ with $a_0 a_n \ne 0$ such that $\sum_{i=0}^n a_i q^{-2i^2 m_{21} m_{22}} f_{k+i} = 0$ for all $k \in \mathbb{Z}$. Thus, by using the techniques of the proof of Theorem 3.2, we deduce necessity.

Now sufficiency. Set

$$Q(x) = \Big(\prod_{i=1}^r (x - \alpha_i)^{s_i + 1}\Big)\Big(\prod_{j=1}^\nu (x - a_j)\Big)\Big(\prod_{j=1}^\nu (x - q^{2\alpha} a_j)\Big) =: \sum_{i=1}^n b_i x^i,$$

and $f_i=\psi((1-q^{2i\alpha})q^{2i^2m_{21}m_{22}}t_0^{\bar{0}}t^{2i\boldsymbol{m}_2})$ for all $i\in\mathbb{Z}^*$. Set $f_0=\psi(\beta)$. Then one can easily verify that

$$\sum_{i=0}^{n} b_i f_{k+i}=0 \quad \text{for all } k\in\mathbb{Z}. \tag{3-16}$$

Meanwhile, we have $\left(\prod_{j=1}^{\nu}(x-a_j)\right)\mid x^kQ(x)$ and $\left(\prod_{j=1}^{\nu}(x-a_j)\right)\mid x^kQ(q^{2\alpha}x)$ for any $k\in\mathbb{Z}$, which implies for all $s\in\mathbb{Z}_2$ that

$$\sum_{i=1}^{n} b_i q^{\frac{1}{2}(2i+2k+1)^2m_{22}m_{21}}t_0^{s}t^{(2i+2k+1)\boldsymbol{m}_2}\cdot V_0=0, \tag{3-17}$$

$$\sum_{i=1}^{n} b_i q^{2i\alpha}q^{\frac{1}{2}(2i+2k+1)^2m_{22}m_{21}}t_0^{s}t^{(2i+2k+1)\boldsymbol{m}_2}\cdot V_0=0 \tag{3-18}$$

and, by Remark 2.7,

$$\sum_{i=1}^{n} b_i q^{2(i+k)^2m_{22}m_{21}}t_0^{\bar{1}}t^{2(i+k)\boldsymbol{m}_2}\cdot V_0=0, \tag{3-19}$$

$$\sum_{i=1}^{n} b_i q^{2i\alpha}q^{2(i+k)^2m_{22}m_{21}}t_0^{\bar{1}}t^{2(i+k)\boldsymbol{m}_2}\cdot V_0=0. \tag{3-20}$$

Let $b_i'=q^{2i^2m_{21}m_{22}}b_i$ for $0\le i\le n$ and $P(x)=\sum_{i=1}^{n} b_i'x^i$. By (3-16) and the construction of $V(\underline{a},\psi)$, we have

$$\begin{aligned}
&\left(t_0^{\bar{0}}t^{2k\boldsymbol{m}_2}P(t^{2\boldsymbol{m}_2})-q^{2k\alpha}t_0^{\bar{0}}t^{2k\boldsymbol{m}_2}P(q^{2\alpha}t^{2\boldsymbol{m}_2})+b_{-k}'q^{-4k^2m_{21}m_{22}}\beta\right)\cdot V_0\\
&\qquad=q^{-2k^2m_{21}m_{22}}\psi\left(\sum_{i=1}^{n} b_i(1-q^{2(k+i)\alpha})q^{2(k+i)^2m_{22}m_{21}}t_0^{\bar{0}}t^{2(k+i)\boldsymbol{m}_2}+b_{-k}\beta\right)\cdot V_0\\
&\qquad=q^{-2k^2m_{21}m_{22}}\sum_{i=1}^{n} b_i f_{k+i}\cdot V_0=0,
\end{aligned}$$

which implies (3-9). Similarly, we have, for any $k\in\mathbb{Z}$,

$$\begin{aligned}
t_0^{s}t^{(2k+1)\boldsymbol{m}_2}P(t^{2\boldsymbol{m}_2})\cdot V_0&=\sum_{i=1}^{n} b_i q^{(2i^2+4ki+2i)m_{21}m_{22}}t_0^{s}t^{(2k+2i+1)\boldsymbol{m}_2}\cdot V_0\\
&=q^{-2k^2-2k-\frac{1}{2}}\sum_{i=1}^{n} b_i q^{\frac{1}{2}(2k+2i+1)^2m_{21}m_{22}}t_0^{s}t^{(2k+2i+1)\boldsymbol{m}_2}\cdot V_0
\end{aligned}$$

and

$$t_0^s t^{(2k+1)\boldsymbol{m}_2} P(q^{2\alpha} t^{2\boldsymbol{m}_2}) \cdot V_0 = \sum_{i=1}^{n} b_i q^{2i\alpha + (2i^2+4ki+2i)m_{21}m_{22}} t_0^s t^{(2k+2i+1)\boldsymbol{m}_2} \cdot V_0$$
$$= q^{-2k^2-2k-\frac{1}{2}} \sum_{i=1}^{n} b_i q^{2i\alpha} q^{\frac{1}{2}(2k+2i+1)^2 m_{21}m_{22}} t_0^s t^{(2k+2i+1)\boldsymbol{m}_2} \cdot V_0$$

which then vanish by (3-17) and (3-18), respectively. Now one can easily deduce the equations $t_0^{\bar{1}} t^{2k\boldsymbol{m}_2} P(t^{2\boldsymbol{m}_2}) \cdot V_0 = 0$ and $t_0^{\bar{1}} t^{2k\boldsymbol{m}_2} P(q^{2\alpha} t^{2\boldsymbol{m}_2}) \cdot V_0 = 0$ by using (3-19) and (3-20), respectively. Therefore (3-9)–(3-11) hold for $P(t^{2\boldsymbol{m}_2}) = \sum_{i=1}^{n} b_i' t^{2i\boldsymbol{m}_2}$. Thus $M^+(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathcal{O}_{\mathbb{Z}}$ by Lemma 3.3. □

Remark 3.5. A linear function ψ over L_0 of the form described in Theorem 3.2 is called an exp-polynomial function over L_0; a linear function ψ over $\mathcal{A}$ of the form described in Theorem 3.4 is called an exp-polynomial function over $\mathcal{A}$.

4. Classification of generalized highest weight irreducible $\mathbb{Z}$-graded L-modules

Lemma 4.1. *Suppose V is a nontrivial irreducible generalized highest weight $\mathbb{Z}$-graded L-module corresponding to a $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of $\mathbb{Z}^2$.*

(1) *For any $v \in V$, there is some $p \in \mathbb{N}$ such that $t_0^i t^{m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2} \cdot v = 0$ for all $m_1, m_2 \geq p$ and $i \in \mathbb{Z}_2$.*

(2) *For any nonzero $v \in V$, $m_1, m_2 > 0$ and $i \in \mathbb{Z}_2$, we have $t_0^i t^{-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2} \cdot v \neq 0$.*

Proof. Assume that v_0 is a generalized highest weight vector corresponding to the $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of $\mathbb{Z}^2$.

(1) By the irreducibility of V and the Poincaré–Birkhoff–Witt theorem, there exists a $u \in U(L)$ such that $v = u \cdot v_0$, where u is a linear combination of elements of the form

$$u_n = (t_0^{k_1} t^{i_1\boldsymbol{b}_1 + j_1\boldsymbol{b}_2}) \cdot (t_0^{k_2} t^{i_2\boldsymbol{b}_1 + j_1 2\boldsymbol{b}_2}) \cdots (t_0^{k_n} t^{i_n\boldsymbol{b}_1 + j_n\boldsymbol{b}_2}),$$

where $\cdot$ denotes the product in $U(L)$. Thus, we may assume $u = u_n$. Take

$$p_1 = -\sum_{i_s<0} i_s + 1 \quad \text{and} \quad p_2 = -\sum_{j_s<0} j_s + 1.$$

By induction on n, one gets that $t_0^k t^{i\boldsymbol{b}_1 + j\boldsymbol{b}_2} \cdot v = 0$ for any $k \in \mathbb{Z}_2$, $i \geq p_1$ and $j \geq p_2$, which gives the result with $p = \max\{p_1, p_2\}$.

(2) Suppose there is a nonzero $v \in V$, an $i \in \mathbb{Z}_2$ and $m_1, m_2 > 0$ such that $t_0^i t^{-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2} \cdot v = 0$. Let p be as in the proof of (1). Then for all $j \in \mathbb{Z}_2$,

$$t_0^i t^{-m_1\boldsymbol{b}_1 - m_2\boldsymbol{b}_2}, \; t_0^j t^{\boldsymbol{b}_1 + p(m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2)}, \; t_0^j t^{\boldsymbol{b}_2 + p(m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2)}$$

act trivially on v. These elements generate the Lie algebra L. So V is a trivial module, a contradiction. □

Lemma 4.2. *If $V \in \mathbb{O}_{\mathbb{Z}}$ is a generalized highest weight L-module corresponding to the $\mathbb{Z}$-basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of $\mathbb{Z}^2$, then V must be a highest or lowest weight module.*

Proof. Suppose V is a generalized highest weight module corresponding to the $\mathbb{Z}$-basis $\{\boldsymbol{b}_1 = b_{11}\boldsymbol{m}_1 + b_{12}\boldsymbol{m}_2, \boldsymbol{b}_2 = b_{21}\boldsymbol{m}_1 + b_{22}\boldsymbol{m}_2\}$ of $\mathbb{Z}^2$. By shifting the index of V_i if necessary, we can suppose the highest degree of V is 0. Let $a = b_{11} + b_{21}$ and $\wp(V) = \{m \in \mathbb{Z} \mid V_m \neq 0\}$. We may assume $a \neq 0$: In fact, if $a = 0$, we can choose $\boldsymbol{b}'_1 = 3\boldsymbol{b}_1 + \boldsymbol{b}_2$ and $\boldsymbol{b}'_2 = 2\boldsymbol{b}_1 + \boldsymbol{b}_2$. Then V is a generalized highest weight $\mathbb{Z}$-graded module corresponding to the $\mathbb{Z}$-basis $\{\boldsymbol{b}'_1, \boldsymbol{b}'_2\}$ of $\mathbb{Z}^2$. Replacing $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ by $\boldsymbol{b}'_1$ and $\boldsymbol{b}'_2$ gives $a \neq 0$.

Now we prove that V is a highest weight module if $a > 0$. Let

$$\mathcal{A}_i = \{j \in \mathbb{Z} \mid i + aj \in \wp(V)\} \quad \text{for all } 0 \le i < a.$$

Then there is $m_i \in \mathbb{Z}$ such that $\mathcal{A}_i = \{j \in \mathbb{Z} \mid j \le m_i\}$ or $\mathcal{A}_i = \mathbb{Z}$ by Lemma 4.1(2).

Set $\boldsymbol{b} = \boldsymbol{b}_1 + \boldsymbol{b}_2$. We want to prove $\mathcal{A}_i \neq \mathbb{Z}$ for all $0 \le i < a$. Otherwise, by shifting the index of $\mathcal{A}_i$ if necessary, we may assume $\mathcal{A}_0 = \mathbb{Z}$. Thus we can choose a nonzero $v_j \in V_{aj}$ for any $j \in \mathbb{Z}$. By Lemma 4.1(1), there is a $p_{v_j} > 0$ with

$$t_0^k t^{s_1\boldsymbol{b}_1 + s_2\boldsymbol{b}_2} \cdot v_j = 0 \quad \text{for all } s_1, s_2 > p_{v_j} \text{ and } k \in \mathbb{Z}_2. \tag{4-1}$$

Choose $\{k_j \in \mathbb{N} \mid j \in \mathbb{N}\}$ and $v_{k_j} \in V_{ak_j}$ such that

$$k_{j+1} > k_j + p_{v_{k_j}} + 2. \tag{4-2}$$

If we can prove that $\{t_0^{\bar{0}} t^{-k_j\boldsymbol{b}} \cdot v_{k_j} \mid j \in \mathbb{N}\} \subset V_0$ is a set of linearly independent vectors, then we will have a contradiction that proves the desired result.

Indeed, for any $r \in \mathbb{N}$, there exists $a_r \in \mathbb{N}$ such that $t_0^0 t^{x\boldsymbol{b}+\boldsymbol{b}_1} v_{k_r} = 0$ for all $x \ge a_r$ by Lemma 4.1(1). On the other hand, we know that $t_0^0 t^{x\boldsymbol{b}+\boldsymbol{b}_1} \cdot v_{k_r} \neq 0$ for any $x < -1$ by Lemma 4.1(2). Thus we can choose $s_r \ge -2$ such that

$$t_0^{\bar{0}} t^{s_r\boldsymbol{b}+\boldsymbol{b}_1} \cdot v_{k_r} \neq 0 \quad \text{and} \quad t_0^{\bar{0}} t^{x\boldsymbol{b}+\boldsymbol{b}_1} \cdot v_{k_r} = 0 \quad \text{for all } x > s_r. \tag{4-3}$$

By (4-2) we have $k_r + s_r - k_j > p_{v_{k_j}}$ for all $1 \le j < r$. Hence by (4-1) we know that for all $1 \le j < r$,

$$\begin{aligned} &t_0^{\bar{0}} t^{(k_r+s_r)\boldsymbol{b}+\boldsymbol{b}_1} \cdot t_0^{\bar{0}} t^{-k_j\boldsymbol{b}} \cdot v_{k_j} \\ &\quad = \big[t_0^{\bar{0}} t^{(k_r+s_r)\boldsymbol{b}+\boldsymbol{b}_1}, t_0^{\bar{0}} t^{-k_j\boldsymbol{b}}\big] \cdot v_{k_j} \\ &\quad = q^{-k_j((k_r+s_r)(b'_{12}+b'_{22})+b'_{12})(b'_{11}+b'_{21})} (1 - q^{k_j(b'_{12}b'_{21}-b'_{11}b'_{22})}) t_0^{\bar{0}} t^{(k_r+s_r-k_j)\boldsymbol{b}+\boldsymbol{b}_1} \cdot v_{k_j} \\ &\quad = 0, \end{aligned}$$

where

$$b'_{11} = b_{11}m_{11} + b_{12}m_{21}, \qquad b'_{12} = b_{11}m_{12} + b_{12}m_{22},$$
$$b'_{21} = b_{21}m_{11} + b_{22}m_{21}, \qquad b'_{22} = b_{21}m_{12} + b_{22}m_{22}.$$

Now by (4-2) and (4-3), one gets

$$\begin{aligned} &t_0^{\bar{0}} t^{(k_r+s_r)\boldsymbol{b}+\boldsymbol{b}_1} \cdot t_0^{\bar{0}} t^{-k_r \boldsymbol{b}} \cdot v_{k_r} \\ &= \big[t_0^{\bar{0}} t^{(k_r+s_r)\boldsymbol{b}+\boldsymbol{b}_1}, t_0^{\bar{0}} t^{-k_r \boldsymbol{b}}\big] \cdot v_{k_r} \\ &= q^{-k_r((k_r+s_r)(b'_{12}+b'_{22})+b'_{12})(b'_{11}+b'_{21})} (1 - q^{k_r(b'_{12}b'_{21} - b'_{11}b'_{22})}) t_0^{\bar{0}} t^{s_r \boldsymbol{b}+\boldsymbol{b}_1} \cdot v_{k_r} \neq 0. \end{aligned}$$

Hence if $\sum_{j=1}^{n} \lambda_j t_0^{\bar{0}} t^{-k_j \boldsymbol{b}} \cdot v_{k_j} = 0$ then $\lambda_n = \lambda_{n-1} = \cdots = \lambda_1 = 0$ by the arbitrariness of r. So we see that the coefficients of λ_j form a set of linearly independent vectors, which contradicts that $V \in \mathbb{O}_{\mathbb{Z}}$. Therefore, for any $0 \le i < a$, there is a $m_i \in \mathbb{Z}$ such that $\mathscr{A}_i = \{j \in \mathbb{Z} \mid j \le m_i\}$, which implies that V is a highest weight module since $\wp(V) = \bigcup_{i=0}^{a-1} \mathscr{A}_i$.

Similarly, one can prove V is a lowest weight module if $a < 0$. □

From Lemma 4.2 and the results in Section 3, we get our main theorem:

Theorem 4.3. *V is a quasifinite irreducible $\mathbb{Z}$-graded L-module if and only if one of the following statements hold*:

(1) *V is a uniformly bounded module.*

(2) *If m_{21} is an even integer, then there exists an exp-polynomial function ψ over L_0 such that*

$$V \cong M^+(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \quad or \quad V \cong M^-(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2).$$

(3) *If m_{21} is an odd integer, then there exist an exp-polynomial function ψ over $\mathscr{A}$, a finite sequence of nonzero distinct numbers $\underline{a} = (a_1, \dots, a_\nu)$ and some finite-dimensional irreducible sl_2-modules $V_1, \dots, V_\nu$ such that*

$$V \cong M^+(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \quad or \quad V \cong M^-(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2).$$

Corollary 4.4. *If V is a quasifinite irreducible $\mathbb{Z}$-graded L-module with nontrivial center, then one of the following statements must hold*:

(1) *If m_{21} is an even integer, then there exists an exp-polynomial function ψ over L_0 such that*

$$V \cong M^+(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \quad or \quad V \cong M^-(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2).$$

(2) *If m_{21} is an odd integer, then there exist an exp-polynomial function ψ over $\mathscr{A}$, a finite sequence of nonzero distinct numbers $\underline{a} = (a_1, \dots, a_\nu)$ and some finite-dimensional irreducible sl_2 modules $V_1, \dots, V_\nu$ such that*

$$V \cong M^+(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2) \quad or \quad V \cong M^-(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2).$$

Proof. By Theorem 4.3, we only need to show that V is not a uniformly bounded module. From the definition of the Lie algebra L, we see that

$$\mathcal{H}_i = \langle t_0^{\bar{0}} t^{k\boldsymbol{m}_i}, m_{i1}c_1 + m_{i2}c_2 \mid k \in \mathbb{Z}^* \rangle \quad \text{for } i = 1, 2$$

are Heisenberg Lie algebras. Now $m_{21}c_1 + m_{22}c_2$ must be zero since V is a quasifinite irreducible $\mathbb{Z}$-graded L-module. Thus, by assumption, we have that $m_{11}c_1 + m_{12}c_2 \neq 0$ since $\{\boldsymbol{m}_1, \boldsymbol{m}_2\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Therefore, V is not a uniformly bounded module by a well-known result from the representation theory of the Heisenberg Lie algebra. □

Theorem 4.5. *The modules $M^+(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$ and $M^-(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$ are uniformly bounded only when they are trivial.*

Proof. Set $V \cong M^+(\psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$ or $V \cong M^+(\underline{a}, \psi, \boldsymbol{m}_1, \boldsymbol{m}_2)$, and suppose V is not trivial. Also set $V = \bigoplus_{k\in\mathbb{Z}_+} V_{-k}$. By nontriviality, there exist $v_0 \in V_0$, $k \in \mathbb{Z}^*$ and $l \in \mathbb{Z}_2$ such that $t_0^l t^{k\boldsymbol{m}_2} \cdot v_0 \neq 0$. Thus

$$\begin{aligned} t_0^{\bar{0}} t^{\boldsymbol{m}_1} \cdot t_0^l t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} \cdot v_0 &= [t_0^{\bar{0}} t^{\boldsymbol{m}_1}, t_0^l t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2}] v_0 \\ &= ((-1)^{l m_{11}} q^{m_{12}(-m_{11}+km_{21})} - q^{m_{11}(-m_{12}+km_{22})}) t_0^l t^{k\boldsymbol{m}_2} \cdot v_0, \end{aligned}$$

which is nonzero; this implies that $t_0^l t^{-\boldsymbol{m}_1 + k\boldsymbol{m}_2} \cdot v_0 \neq 0$.

Next, we prove that if $0 \neq v_{-m} \in V_{-m}$ then $t_0^{\bar{0}} t^{-\boldsymbol{m}_1} \cdot v_{-m} \neq 0$. Suppose $t_0^{\bar{0}} t^{-\boldsymbol{m}_1} \cdot v_{-m} = 0$ for some $0 \neq v_{-m} \in V_{-m}$. From the construction of V, we know that $t_0^l t^{(m+1)\boldsymbol{m}_1 \pm \boldsymbol{m}_2}$ also act trivially on v_{-m} for any $l \in \mathbb{Z}_2$. Since L is generated by the set $\{t_0^{\bar{0}} t^{-\boldsymbol{m}_1}, t_0^l t^{(m+1)\boldsymbol{m}_1 \pm \boldsymbol{m}_2} \mid l = \bar{0}, \bar{1}\}$, we see V is a trivial module, a contradiction.

Set

$$\mathcal{A}_n = \left\{ (t_0^{\bar{0}} t^{-\boldsymbol{m}_1})^j \cdot t_0^l t^{(-n+j)\boldsymbol{m}_1 + k\boldsymbol{m}_2} \cdot v_0 \mid 0 \leq j < n \right\} \subset V_{-n} \quad \text{for all } n \in \mathbb{N}.$$

Now we prove that $\mathcal{A}_n$ is a linearly independent set of vectors. If

$$\sum_{j=0}^{n-1} \lambda_j (t_0^{\bar{0}} t^{-\boldsymbol{m}_1})^j t_0^l t^{(-n+j)\boldsymbol{m}_1 + k\boldsymbol{m}_2} \cdot v_0 = 0,$$

then for any $0 \leq i < n-1$ we have

$$\begin{aligned} 0 &= q^{n(n-i)m_{11}m_{12} - k(n-i)m_{12}m_{21}} t_0^{\bar{0}} t^{(n-i)\boldsymbol{m}_1} \cdot \sum_{j=0}^{n-1} \lambda_j (t_0^{\bar{0}} t^{-\boldsymbol{m}_1})^j \cdot t_0^l t^{(-n+j)\boldsymbol{m}_1 + k\boldsymbol{m}_2} \cdot v_0 \\ &= \sum_{j=0}^{i} \lambda_j q^{j(n-i)m_{11}m_{12}} ((-1)^{l(n-i)m_{11}} - q^{k(n-i)\alpha}) (t_0^{\bar{0}} t^{-\boldsymbol{m}_1})^j \cdot t_0^l t^{(j-i)\boldsymbol{m}_1 + k\boldsymbol{m}_2} \cdot v_0, \end{aligned}$$

where $\alpha = m_{11}m_{22} - m_{12}m_{21}$; this implies $\lambda_0 = \cdots = \lambda_{n-1} = 0$. Hence $\mathcal{A}_n$ is a set of linear independent vectors in V_{-n} and thus dim $V_{-n} \geq n$. Since n was arbitrary, V is not a uniformly bounded module. □

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