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We prove the Liouville theorem for the mean field equation (also called the conformal curvature equation) in $\mathbb{R}^2$, an a priori bound for solutions of the mean field equation on the negative part of indefinite nonlinearity, and the symmetry property of mean field equation on an annulus with zero Dirichlet boundary condition.

We study the mean field equation (also called the conformal curvature equation) in a smooth domain D of the plane $\mathbb{R}^2$; that is, we study

$$-\Delta u = K(x)e^u \quad \text{for } x \in D, \tag{1}$$

where $K = K(x)$ is a smooth function on $\bar{D}$. Usually $K(x)$ is assumed to be positive, but here we allow $K(x)e^u$ to change sign; then (1) is said to have indefinite nonlinearity. This equation has received much attention in recent years for its rich physical and geometrical content, such as its relation with the Nirenberg problem on S^2 in geometry and with Chern–Simons–Higgs theory [Yang 2001] in gauge theory. See [Cheng and Ni 1991] and [Lin 2007] for deep results on the mean field equations and related topics. See also [Ma and Wei 2001] and [Tarantello 2004].

Using the boundary blow-up method [Du and Ma 2001], we have the following Liouville theorem for the Liouville equation on $\mathbb{R}^2$; see [Liouville 1853].

Theorem 1. *Let $H(x)$ be a positive smooth function on $\mathbb{R}^2$. Assume that there are positive constants $C > 0$, $R_0 > 0$, and $\beta \in [0, 2)$ such that $\inf_{|x|\le R} H(x) \ge CR^{-\beta}$ for all $R > R_0$. Then there is no $u \in C^2(\mathbb{R}^2)$ satisfying the Liouville equation*

$$\Delta u = H(x)e^u \quad \textit{for } x = (x_1, x_2) \in \mathbb{R}^2. \tag{2}$$

When $H(x) = 1$, this result with the extra assumption $\int_{\mathbb{R}^2} e^u < \infty$ was proved in [Li 1999] by the spherical averaging method. Actually, this result can be easily derived from [Osserman 1957] and [Keller 1957]. We present it here because it

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is useful in local estimates with indefinite nonlinearity. Using the method used to prove Theorem 1, we have this:

Theorem 2. *Let u be a solution of (1) with $D = B_R(0)$. Assume that there is a positive constant $C_0 > 0$ such that $K(x) \le -C_0$ in the ball $B_R(0)$. Then there is a constant $C = C(R, \sup_{B_R} K)$ such that $u(x) \le C$ in $B_{R/2}(0)$.*

Following the classic paper [Gidas et al. 1979], Chen and Li [2008] proved this result on S^2 by the moving plane method. Our method is simpler.

Let $a > 1$ be a fixed constant. Let $A = \{x \in \mathbb{R}^2 : a^{-1} < |x| < a\}$ be an annulus. Using the moving sphere method as in [Chen and Li 1995; Padilla 1997], we have:

Theorem 3. *For $x \in \mathbb{R}^2$, let $r = |x|$. Suppose the function $K(x)$ of (1) is continuous in $\bar{A}$ and satisfies $K(x) < 0$ for $r < 1$, and $K(\lambda^2 x/r^2) - K(x) < 0$ for every $\lambda > 1$ and $r > \lambda$. Assume $u \in C^2(A) \cap C(\bar{A})$ satisfies (1) in A with the Dirichlet boundary condition $u = 0$ on ∂A.*

Then

$$u(x) \ge u(x/r^2) - 4\log r \quad \textit{for all points } x \textit{ satisfying } 1 \le |x| \le a, \tag{3}$$

and either

$$\partial_r u(z) = -2/|z| \quad \textit{for some point } z \textit{ satisfying } 1 < |z| < a, \tag{4}$$

or else

$$\partial_r u(x) > -2/r \quad \textit{for all points } x \textit{ satisfying } 1 < |x| < a. \tag{5}$$

In the proof of Theorem 3, we may use the notation $u = u(r)$ since the angular variable plays no role.

Padilla [1997] used a similar method to show radial symmetry of solutions for other nonlinear Dirichlet problems on higher-dimensional annuli.

Proof of Theorem 1. Choose any fixed point $x_0 \in \mathbb{R}^2$, and let $R > 0$. Let $r = |x - x_0|$ and let $\epsilon > 0$. Define

$$M(x) := M_{R,\epsilon}(x) = 2\log \frac{2R}{\sqrt{\epsilon}(\mathbb{R}^2 - r^2)} \quad \text{for } r \le R.$$

The metric $g = e^{M(x)}dx^2$ on $B_R = B_R(x_0)$ is the Poincaré metric with scalar curvature -2ϵ. Hence $\Delta M = 2\epsilon e^M$ for $x \in B_R$ and $M(x) \to \infty$ as $x \to \partial B_R$. Choose $\epsilon = CR^{-\beta}/2$, where $R > R_0$. By our assumption on H, we have $\Delta u \ge 2\epsilon e^u$ in B_R.

Let $w = u - M$. Note that $\Delta w \ge 2\epsilon(e^u - e^M) = C(x)w$ for $x \in B_R$, where $C(x) = 2\epsilon(e^u - e^M)/(u - M) > 0$. Note that $w(x) \to -\infty$ as $x \to \partial B_R$. Hence by maximum principle we have $w \le 0$ in B_R. Then we have

$$u(x_0) \le M(x_0) = 2\log \frac{2}{\sqrt{\epsilon}R} \to -\infty \quad \text{as } R \to \infty,$$

which contradicts that $u(x_0)$ is finite. $\square$

Proof of Theorem 2. Without loss of generality we may let $C_0 = 1$. Recall that $\Delta u = (-K)e^u \geq e^u$ in B_R. Let $w = u - M_{R,1}$. Then as in our previous proof, $\Delta w \geq C(x)w$ for $x \in B_R$. Then by maximum principle again, we have $w(x) \leq 0$ for $x \in B_R$, that is, $u(x) \leq M_{R,1}(x)$ for $x \in B_R$. Hence

$$u(x) \leq 2\log(8/(3R)) \quad \text{for } x \in B_{R/2}. \qquad \square$$

This result is a local version of one obtained in [Chen and Li 2008] by the moving sphere method, which we will now use.

Proof of Theorem 3. Given $\lambda \in (1, a)$, let $T_\lambda = \partial B_\lambda$ and $\Sigma_\lambda = \{x \in \mathbb{R}^2 : \lambda < |x| < a\}$.

For $x \in \Sigma_\lambda$, let $x^\lambda = \lambda^2 x/|x|^2$. Note that

$$|x| > |x^\lambda| = \lambda^2/|x| > 1/|x| \quad \text{on } \Sigma_\lambda.$$

Recall that the Kelvin transform $v(x)$ for the function $u(x)$ outside the unit ball $B := B_1(0) \subset \mathbb{R}^2$ is $v(x) = u(x/|x|^2) - 4\log|x|$.

For any $\lambda > 0$, we define

$$v_\lambda(x) = u(\lambda^2 x/|x|^2) + 4\log\lambda - 4\log|x|.$$

Noting that $\Delta(u(\lambda^2 x/|x|^2)) = (\lambda^4/|x|^4)\Delta u(\lambda^2 x/|x|^2)$, we have

$$-\Delta v_\lambda = K(\lambda^2 x/|x|^2)e^{v_\lambda}.$$

Let $w_\lambda(x) = u(x) - v_\lambda(x)$ on Σ_λ. Then we have

$$-\Delta w_\lambda = K(\lambda^2 x/|x|^2)C(x,\lambda)w_\lambda + (K(x) - K(\lambda^2 x/|x|^2))e^u \quad \text{on } \Sigma_\lambda,$$

where

$$C(x,\lambda) := (e^{u(x)} - e^{v_\lambda(x)})/(u(x) - v_\lambda(x)) > 0.$$

By assumption, we have $(K(\lambda^2 x/|x|^2) - K(x))e^u \leq 0$ on Σ_λ, and then

$$-\Delta w_\lambda \geq K(\lambda^2 x/|x|^2)C(x,\lambda)w_\lambda \quad \text{on } \Sigma_\lambda. \tag{6}$$

Note that $w_\lambda = 0$ on T_λ. We remark that for $\lambda = 1$, we have $w_1 \geq 0$ on $\partial\Sigma_1$. By the maximum principle and Hopf's boundary point lemma, we have $w_1 > 0$ in Σ_1, which is (3), and $\partial_r w_1 > 0$ on $|x| = 1$. We now assume that (4) is not true.

We claim that $\partial_r w_\lambda > 0$ on T_λ for all $1 < \lambda < a$.

This claim is true for λ near 1 by continuity. Then by using the standard moving sphere method and Hopf's boundary point lemma, we need only to show that for $\lambda \in [1, a)$, there is a neighborhood U_λ of T_λ in Σ_λ such that

$$w_\lambda > 0 \text{ in } U_\lambda \quad \text{and} \quad \partial_r w_\lambda > 0 \text{ on } T_\lambda. \tag{7}$$

and that this neighborhood depends continuously on λ. In fact, let

$$\lambda_1 = \sup\{\lambda \in [1, a) : w_{\tilde{\lambda}} > 0 \text{ for all } \tilde{\lambda} \leq \lambda \text{ and } x \in U_{\tilde{\lambda}}\}.$$

If $\lambda_1 < a$, by continuity, we have $w_{\lambda_1} \geq 0$ in T_{λ_1} and $\partial_r w_{\lambda_1}|_{r=\lambda_1} \geq 0$ in T_{λ_1}. By our assumption that (4) is not true, we must have $\partial_r w_{\lambda_1}|_{r=\lambda_1} > 0$ in T_{λ_1}. By this, we conclude that there is some $\epsilon > 0$ and a neighborhood U_λ of T_λ in Σ_λ such that $w_\lambda > 0$ in U_λ for all $\lambda \in [\lambda_1, \lambda_1 + \epsilon)$; this contradicts the definition of λ_1.

Hence, $\lambda_1 = a$. Setting $r = \lambda > 1$ in the second inequality of (7), we have

$$\partial_r u(r) > -2/r \quad \text{on } \Sigma_1.$$

This is the desired inequality (5). □

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