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Let M be a complete noncompact manifold. We prove vanishing and finiteness results for harmonic p -forms on M , assuming both the curvature operator lower bound and the weighted Poincaré inequality on M .

1. Introduction

It is interesting to study the structures of noncompact complete manifolds, especially their topological properties at infinity. There are many results [Witten and Yau 1999; Cai and Galloway 1999; Wang 2001; 2001; Wan and Xin 2004] on the topology of conformally compact manifolds. Recently, by assuming that the Ricci curvature is bounded from below in terms of the dimension and of the first eigenvalue, Li and Wang [2001] obtained information on the topology of complete manifolds infinity and in some cases the metric structure of these manifolds, by proving a vanishing-type theorem of L^2 harmonic 1-forms. In his thesis, Lam [2007] generalized Li and Wang's result by relaxing the curvature assumptions. He proved that a manifold must have finitely many nonparabolic ends if a similar inequality between the Ricci curvature and the weight function in the weighted Poincaré inequality (see Definition 1.1) is valid outside a compact subset.

In this note, we will consider general harmonic p -forms. Working with a complete manifold M satisfying a weighted Poincaré inequality and a curvature operator lower bound expressed in terms of the dimension and the weight function, we prove vanishing and finiteness theorems for the L^d harmonic p -forms. Also, on an end of manifold with weighted p -Poincaré inequality, we prove that the Green's form satisfies a sharp decay estimate. Let us first recall some definitions.

Definition 1.1. Let M^m be an m -dimensional complete Riemannian manifold. We say that M^m satisfies a weighted Poincaré inequality [Li and Wang 2006] with a

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nonnegative weight function $\rho(x)$ if the inequality

$$\int_M \rho(x)\phi^2 \leq \int_M |\nabla\phi|^2 \quad \text{for } \phi \in C_0^\infty(M)$$

is valid for all compactly supported smooth functions $\phi \in C_0^\infty(M)$.

Definition 1.2. Let M^m be an m -dimensional complete Riemannian manifold. We say that M^m has property (\mathcal{P}_ρ) if a weighted Poincaré inequality is valid on M with some nonnegative weight function ρ and if the ρ -metric, defined by $ds_\rho^2 = \rho ds_M^2$, is complete.

Let $\lambda_1(M)$ denote the greatest lower bound of the spectrum of the Laplacian acting on L^2 functions. Then the variation principle for $\lambda_1(M)$ asserts the validity of the Poincaré inequality

$$\lambda_1(M) \int_M \phi^2 \leq \int_M |\nabla\phi|^2$$

for all compactly supported functions $\phi \in C_0^\infty(M)$. If $\lambda_1(M)$ is positive, then obviously M has property (\mathcal{P}_ρ) with $\rho(x) = \lambda_1(M)$. Property (\mathcal{P}_ρ) may be seen as a generalization of the assumption $\lambda_1(M) > 0$.

For harmonic p -forms, let $C_0^\infty(\wedge^p M)$ denote the space of smooth p -forms with compact support on M . Then we define property $(\mathcal{P}_{p,\rho})$ as follows.

Definition 1.3. Let M^m be an m -dimensional complete Riemannian manifold. We say that M^m has property $(\mathcal{P}_{p,\rho})$ if a weighted p -Poincaré inequality is valid on M with some nonnegative weight function ρ , that is,

$$\int_M \rho(x)|\phi|^2 \leq \int_M |d\phi|^2 + |\delta\phi|^2 \quad \text{for } \phi \in C_0^\infty(\wedge^p M),$$

and if the ρ -metric, defined by $ds_\rho^2 = \rho ds_M^2$, is complete.

If the greatest lower bound $\lambda_{1,p}$ of the p spectrum satisfies $\lambda_{1,p}(M) > 0$, then M has property $(\mathcal{P}_{p,\rho})$ with the weight function $\rho(x) = \lambda_{1,p}(M)$. Hence property $(\mathcal{P}_{p,\rho})$ can also be viewed as a generalization of the assumption that $\lambda_{1,p}(M) > 0$.

Throughout, we use $H_d^p(M^m)$ to denote the space of L^d harmonic p -forms, and $r_\rho(x)$ to denote the geodesic distance from some fixed point to x with respect to the metric ds_ρ^2 . Our main result is the following.

Theorem 1.4. For $m \geq 3$, let M^m be a complete noncompact Riemannian manifold with properties (\mathcal{P}_{ρ_0}) and $(\mathcal{P}_{p,\rho})$. Suppose the volume growth of M satisfies

$$\int_{B_\rho(2R) \setminus B_\rho(R)} \rho e^{-2(m-p-1)r_\rho} \leq cR$$

and the curvature operator K_p of M^m has the lower bound

$$K_p > -\frac{m-p}{m-p-1}\rho_0 \quad \text{on } M.$$

Then the space $H_d^p(M)$ is trivial for $2 \leq d \leq 2(m-p-1)/(m-p-2)$.

We will employ some of the arguments from [Li and Wang 2001] to prove the sharp decay estimate for the Green’s form in the next section, and prove the main theorem in Section 3.

2. Decay estimate

Let M^m be a complete manifold, and let Δ be the Hodge Laplace–Beltrami operator of M^m acting on differential p -forms. The Weitzenböck formula gives

$$\Delta = \nabla^2 - K_p,$$

where ∇^2 is the Bochner Laplacian and K_p is an endomorphism depending upon the curvature tensor of M^m . Using an orthonormal basis $\{\theta^1, \dots, \theta^m\}$ dual to $\{e_1, \dots, e_m\}$, one may express the curvature term K_p as

$$\langle K_p(w), w \rangle = \left\langle \sum_{j,k=1}^m \theta^k \wedge i_{e_j} R(e_k, e_j)w, w \right\rangle.$$

In particular, $\langle K_1(w), w \rangle = \text{Ric}(w^\sharp, w^\sharp)$, where w^\sharp is the vector dual to the form w . We say M^m has curvature lower bound k_p if for all p -forms w on M^m ,

$$\langle K_p(w), w \rangle \geq k_p |w|^2$$

We recall [Li 2000] that an end is simply an unbounded component of $M \setminus D$, where D is a compact smooth domain of M . Write $E(R) = E \cap B_q(R)$, and define $\partial E(R) = \partial E \cup (\partial B_q(R) \cap E)$. Let $\lambda_{1,p}(E(R))$ be the first eigenvalue of Δ for p -forms satisfying Dirichlet boundary conditions on $\partial E(R)$, that is, $\lambda_{1,p}(E) = \inf_{R>0, E(R) \neq \emptyset} \lambda_{1,p}(E(R))$; see [Donnelly 1984; Donnelly and Xavier 1984]. Therefore we have for all $w \in C_0^\infty(\wedge^p E)$

$$\lambda_{1,p}(E) \int |w|^2 \leq \int (|\nabla w|^2 + \langle K_p(w), w \rangle),$$

where $C_0^\infty(\wedge^p E)$ is the space of smooth p -forms with compact supported on the end E . If the p -spectrum $\lambda_{1,p}(E)$ is positive, then E has property $(\mathcal{P}_{p,\rho})$ with the weight function $\rho = \lambda_{1,p}(E)$.

In this section, we study the harmonic p -forms on the end E of a manifold with weighted p -Poincaré inequality and prove the following decay estimate. See [Donnelly 1984; Li and Wang 2001].

Lemma 2.1. *Let M be a complete noncompact manifold. If E is an end of M with the property $(\mathcal{P}_{p,\rho})$ for some nonnegative weight function ρ , then for any smooth harmonic p -form w satisfying*

$$(1) \quad \int_{E_\rho(2R) \setminus E_\rho(R)} \rho(x) \exp(-2r_\rho) |w|^2 dv = o(R),$$

we have

$$\int_{E_\rho(R)} \rho(x) \exp(2r_\rho) |w|^2 dv \leq CR,$$

$$\int_{E_\rho(R+1) \setminus E_\rho(R)} \rho(x) \exp(2r_\rho) |w|^2 dv \leq C$$

for all R sufficiently large and for some constant C depending on w and ρ .

Proof. Let ψ be a nonnegative cutoff function. Integration by parts gives

$$\begin{aligned} & \int_E |\nabla(\psi w)|^2 + \langle K_p(\psi w), \psi w \rangle \\ &= \int_E |\nabla \psi|^2 |w|^2 + 2\psi \langle w \nabla \psi, \nabla w \rangle + |\psi|^2 |\nabla w|^2 + \psi^2 \langle K_p(w), w \rangle \\ &= \int_E |\nabla \psi|^2 |w|^2 - \int_E \psi^2 \langle w, \Delta w \rangle_M \\ &= \int_E |\nabla \psi|^2 |w|^2. \end{aligned}$$

By property $(\mathcal{P}_{p,\rho})$,

$$\int_E \rho(x) \psi^2 |w|^2 dv \leq \int_E (|\nabla(\psi w)|^2 + \langle K_p(\psi w), \psi w \rangle),$$

so we have

$$(2) \quad \int_E \rho(x) \psi^2 |w|^2 dv \leq \int_E |\nabla \psi|^2 |w|^2 dv$$

for any cutoff function ψ on E . Let $\psi = \phi(r_\rho(x)) \exp(a(r_\rho(x)))$. Then

$$(3) \quad \int_E \phi^2 \exp(2a) |w|^2 dv \leq \int_E (|\nabla \phi|^2 + 2\langle \nabla \phi, \nabla a \rangle \phi + |\nabla a|^2 \phi^2) \exp(2a) |w|^2 dv.$$

Choose ϕ as the nonnegative cutoff function defined by

$$\phi(r_\rho(x)) = \begin{cases} r_\rho(x) - R_0 & \text{on } E_\rho(R_0 + 1) \setminus E_\rho(R_0), \\ 1 & \text{on } E_\rho(R) \setminus E_\rho(R_0 + 1), \\ (2R - r_\rho(x))/R & \text{on } E_\rho(2R) \setminus E_\rho(R), \\ 0 & \text{on } E \setminus E_\rho(2R), \end{cases}$$

and also choose $a(r_\rho(x))$ as

$$a(r_\rho(x)) = \begin{cases} \delta r_\rho(x) & \text{for } r_\rho \leq K/(1 + \delta), \\ K - r_\rho(x) & \text{for } r_\rho > K/(1 + \delta), \end{cases}$$

for some fixed $K > (R_0 + 1)(1 + \delta)$ and $0 < \delta < 1$. If $R \geq K/(1 + \delta)$, it is easy to check that

$$|\nabla\phi|^2(x) = \begin{cases} \rho(x) & \text{on } E_\rho(R_0 + 1) \setminus E_\rho(R_0), \\ 0 & \text{on } E_\rho(R) \setminus E_\rho(R_0 + 1), \\ \rho(x)/R^2 & \text{on } E_\rho(2R) \setminus E_\rho(R), \\ 0 & \text{on } E \setminus E_\rho(2R), \end{cases}$$

$$|\nabla a|^2(x) = \begin{cases} \delta^2 \rho(x) & \text{for } r \leq K/(1 + \delta), \\ \rho(x) & \text{for } r > K/(1 + \delta), \end{cases}$$

and then

$$\langle \nabla\phi, \nabla a \rangle(x) = \begin{cases} \delta\rho(x) & \text{on } E_\rho(R_0 + 1) \setminus E_\rho(R_0), \\ \rho(x)/R & \text{on } E_\rho(2R) \setminus E_\rho(R), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore (3) becomes

$$\begin{aligned} & \int_{E_\rho(2R)} \rho\phi^2 \exp(2a)|w|^2 dv \\ & \leq \int_{E_\rho(2R)} (|\nabla\phi|^2 + 2\langle \nabla\phi, \nabla a \rangle\phi + |\nabla a|^2\phi^2) \exp(2a)|w|^2 dv \\ & \leq \int_{E_\rho(R_0+1) \setminus E_\rho(R_0)} \rho \exp(2a)|w|^2 dv + \frac{1}{R^2} \int_{E_\rho(2R) \setminus E_\rho(R)} \rho \exp(2a)|w|^2 dv \\ & \quad + 2\delta \int_{E_\rho(R_0+1) \setminus E_\rho(R_0)} \rho\phi \exp(2a)|w|^2 dv + \frac{2}{R} \int_{E_\rho(2R) \setminus E_\rho(R)} \rho\phi \exp(2a)|w|^2 dv \\ & \quad + \delta^2 \int_{E_\rho(\frac{K}{1+\delta}) \setminus E_\rho(R_0)} \rho\phi^2 \exp(2a)|w|^2 dv + \int_{E_\rho(2R) \setminus E_\rho(\frac{K}{1+\delta})} \rho\phi^2 \exp(2a)|w|^2 dv. \end{aligned}$$

Hence

$$\begin{aligned} (1 - \delta^2) \int_{E_\rho(\frac{K}{1+\delta}) \setminus E_\rho(R_0+1)} \rho \exp(2a)|w|^2 dv \\ \leq (\delta^2 + 2\delta + 1) \int_{E_\rho(R_0+1) \setminus E_\rho(R_0)} \rho \exp(2a)|w|^2 dv \\ \quad + \left(\frac{1}{R^2} + \frac{2}{R} \right) \int_{E_\rho(2R) \setminus E_\rho(R)} \rho \exp(2a)|w|^2 dv. \end{aligned}$$

By the definition of $a(x)$ and the growth estimate of w (see (1)), the last term on the right side tends to 0 as $R \rightarrow \infty$. Thus we obtain the estimate

$$\frac{1-\delta^2}{(\delta+1)^2} \int_{E_\rho(\frac{K}{1+\delta}) \setminus E_\rho(R_0+1)} \rho \exp(2\delta r_\rho) |w|^2 dv \leq \int_{E_\rho(R_0+1) \setminus E_\rho(R_0)} \rho \exp(2\delta r_\rho) |w|^2 dv.$$

Since the right side of this inequality is independent of K , by letting $K \rightarrow \infty$ we conclude that

$$(4) \quad \int_{E \setminus E_\rho(R_0+1)} \rho \exp(2\delta r_\rho) |w|^2 dv \leq C \quad \text{for some constant } 0 < C < \infty.$$

Next we improve this estimate by setting $a(r_\rho(x)) = r_\rho(x)$ in the preceding argument. For $R_0 < R_1 < R$, let us choose ϕ to be

$$\phi(x) = \begin{cases} (r_\rho(x) - R_0)/(R_1 - R_0) & \text{on } E_\rho(R_1) \setminus E_\rho(R_0), \\ (R - r_\rho(x))/(R - R_1) & \text{on } E_\rho(R) \setminus E_\rho(R_1). \end{cases}$$

The inequality (3) asserts that

$$\begin{aligned} \int_{E_\rho(R)} \rho \phi^2 \exp(2r_\rho) |w|^2 dv &\leq \int_{E_\rho(R)} |\nabla(\phi \exp(r_\rho))|^2 |w|^2 dv \\ &= \frac{1}{(R - R_1)^2} \int_{E_\rho(R) \setminus E_\rho(R_1)} \rho \exp(2r_\rho) |w|^2 dv \\ &\quad + \frac{1}{(R_1 - R_0)^2} \int_{E_\rho(R_1) \setminus E_\rho(R_0)} \rho \exp(2r_\rho) |w|^2 dv \\ &\quad - \frac{2}{(R - R_1)^2} \int_{E_\rho(R) \setminus E_\rho(R_1)} (R - r_\rho(x)) \rho \exp(2r_\rho) |w| k_2 dv \\ &\quad + \frac{2}{(R_1 - R_0)^2} \int_{E_\rho(R_1) \setminus E_\rho(R_0)} (r_\rho(x) - R_0) \rho \exp(2r_\rho) |w|^2 dv \\ &\quad + \int_{E_\rho(R)} \rho \phi^2 \exp(2r_\rho) |w|^2 dv. \end{aligned}$$

Then

$$\begin{aligned} \frac{2}{(R - R_1)^2} \int_{E_\rho(R) \setminus E_\rho(R_1)} (R - r_\rho(x)) \rho \exp(2r_\rho) |w|^2 dv \\ \leq \frac{1}{(R - R_1)^2} \int_{E_\rho(R) \setminus E_\rho(R_1)} \rho \exp(2r_\rho) |w|^2 dv \\ + \frac{1}{(R_1 - R_0)^2} \int_{E_\rho(R_1) \setminus E_\rho(R_0)} \rho \exp(2r_\rho) |w|^2 dv \\ + \frac{2}{(R_1 - R_0)} \int_{E_\rho(R_1) \setminus E_\rho(R_0)} \frac{r_\rho(x) - R_0}{R_1 - R_0} \rho \exp(2r_\rho) |w|^2 dv. \end{aligned}$$

On the other hand, for any fixed $0 < t < R - R_1$,

$$\begin{aligned} \frac{t}{(R - R_1)^2} \int_{E_\rho(R-t) \setminus E_\rho(R_1)} \rho \exp(2r_\rho) |w|^2 dv \\ \leq \frac{1}{(R - R_1)^2} \int_{E_\rho(R) \setminus E_\rho(R_1)} (R - r_\rho(x)) \rho \exp(2r_\rho) |w|^2 dv, \end{aligned}$$

we deduce that

$$\begin{aligned} (5) \quad \frac{2t}{(R - R_1)^2} \int_{E_\rho(R-t) \setminus E_\rho(R_1)} \rho \exp(2r_\rho) |w|^2 dv \\ \leq \left(\frac{1}{(R_1 - R_0)^2} + \frac{2}{R_1 - R_0} \right) \int_{E_\rho(R_1) \setminus E_\rho(R_0)} \rho \exp(\sqrt{2}r_\rho) |w|^2 dv \\ + \frac{1}{(R - R_1)^2} \int_{E_\rho(R) \setminus E_\rho(R_1)} \rho \exp(\sqrt{2}r_\rho) |w|^2 dv. \end{aligned}$$

Observe that if $R_1 = R_0 + 1$, if $t = 1$, and if

$$g(R) = \int_{E_\rho(R) \setminus E_\rho(R_0+1)} \rho \exp(2r_\rho) |w|^2 dv,$$

then the inequality (5) can be written as

$$g(R - 1) \leq C_1 R^2 + \frac{1}{2} g(R),$$

where $C_1 = 2 \int_{E_\rho(R_0+1) \setminus E_\rho(R_0)} \rho \exp(2r_\rho) |w|^2 dv$ is independent of R . Iterating this inequality, we show that for any positive integer k , $R \geq 1$, and constant C_2 ,

$$\begin{aligned} g(R) &\leq C_1 \sum_{i=1}^k \frac{(R+i)^2}{2^{i-1}} + 2^{-k} g(R+k) \\ &\leq C_1 R^2 \sum_{i=1}^{\infty} \frac{(1+i)^2}{2^{i-1}} + 2^{-k} g(R+k) \leq C_2 R^2 + 2^{-k} g(R+k). \end{aligned}$$

However, the previous estimate in (4) asserts that

$$\int_E \rho \exp(2\delta r_\rho) |w|^2 dv \leq C \quad \text{for any } \delta < 1.$$

This implies that

$$\begin{aligned} g(R+k) &= \int_{E_\rho(R+2k) \setminus E_\rho(R_0+1)} \rho \exp(2r_\rho) |w|^2 \\ &\leq \exp(2(1-\delta)(R+k)) \int_{E_\rho(R+k) \setminus E_\rho(R_0+1)} \rho \exp(2\delta r_\rho) |w|^2 \\ &\leq C \exp(2(1-\delta)(R+k)). \end{aligned}$$

Hence $2^{-k}g(R+k) \rightarrow 0$ as $k \rightarrow \infty$ by choosing $2(1-\delta) < \ln 2$. This proves the estimate that $g(R) \leq C_2 R^2$. Adjusting the constant, we have

$$(6) \quad \int_{E_\rho(R)} \rho \exp(2r_\rho) |w|^2 dv \leq C_3 R^2 \quad \text{for all } R \geq R_0.$$

Using inequality (5) again and choosing $R_1 = R_0 + 1$, $R > 2R_1$ and $t = R/2$, we conclude that

$$R \int_{E_\rho(R/2) \setminus E_\rho(R_0+1)} \rho \exp(2r_\rho) |w|^2 dv \leq C_4 R^2 + 2 \int_{E_\rho(R) \setminus E_\rho(R_0+1)} \rho \exp(2r_\rho) |w|^2 dv.$$

However, applying the estimate (6) to the second term on the right side, we have

$$\int_{E_\rho(R/2) \setminus E_\rho(R_0+1)} \rho \exp(2r_\rho) |w|^2 dv \leq C_5 R.$$

Therefore

$$(7) \quad \int_{E_\rho(R)} \rho \exp(2r_\rho) |w|^2 dv \leq C_6 R \quad \text{for } R \geq R_0.$$

We are ready to prove the theorem by using (7). Setting $t = 2$ and $R_1 = R - 4$ in (5), we have

$$\begin{aligned} & \int_{E_\rho(R-2) \setminus E_\rho(R-4)} \rho \exp(2r_\rho) |w|^2 dv \\ & \leq \left(\frac{4}{(R-R_0-4)^2} + \frac{8}{R-R_0-4} \right) \int_{E_\rho(R-4) \setminus E_\rho(R_0)} \rho \exp(2r_\rho) |w|^2 dv \\ & \quad + \frac{1}{4} \int_{E_\rho(R) \setminus E_\rho(R-4)} \rho \exp(2r_\rho) |w|^2 dv. \end{aligned}$$

According to (7), the first term of the right side is bounded by a constant. Hence this inequality can be rewritten as

$$\int_{E_\rho(R-2) \setminus E_\rho(R-4)} \rho \exp(2r_\rho) |w|^2 dv \leq C_7 + \frac{1}{2} \int_{E_\rho(R) \setminus E_\rho(R-4)} \rho \exp(2r_\rho) |w|^2 dv.$$

Iterating this inequality k times, we have

$$\begin{aligned} & \int_{E_\rho(R+2) \setminus E_\rho(R)} \rho \exp(2r_\rho) |w|^2 dv \\ & \leq C_7 \sum_{i=0}^{k-1} 2^{-i} + \frac{1}{2^k} \int_{E_\rho(R+2(k+1)) \setminus E_\rho(R)} \rho \exp(2r_\rho) |w|^2 dv. \end{aligned}$$

Using (7) again, we conclude the second term is bounded by

$$\frac{1}{2^k} \int_{E_\rho(R+2(k+1)) \setminus E_\rho(R)} \rho \exp(2r_\rho) |w|^2 dv \leq \frac{C(R + 2(k + 1))}{2^k},$$

and the upper bound tends to zero as $k \rightarrow \infty$. Hence

$$\int_{E_\rho(R+2) \setminus E_\rho(R)} \rho \exp(2r_\rho) |w|^2 dv \leq C_8$$

for some constant $C_8 > 0$ independent of R . □

Corollary 2.2. *Let M be a complete manifold. If E is an end of M with positive $\lambda(E)$, where $\lambda(E)$ is equal to either $\lambda_{1,p}(E)$ or $\lambda_1(E) + K_p$, then for any smooth harmonic p -form w satisfying*

$$\int_{E(2R) \setminus E(R)} \exp(-2\sqrt{\lambda(E)}r) |w|^2 dv = o(R),$$

we have

$$\begin{aligned} \int_{E(R)} \exp(2\sqrt{\lambda(E)}r) |w|^2 dv &\leq CR, \\ \int_{E(R+1) \setminus E(R)} \exp(2\sqrt{\lambda(E)}r) |w|^2 dv &\leq C, \end{aligned}$$

for all R sufficiently large.

3. Vanishing and finiteness theorems of harmonic p -forms

Let w be a harmonic p -form on an m -dimensional manifold M . Then w satisfies the Kato inequality [Wan and Xin 2004; Calderbank et al. 2000; Herzlich 2000]

$$|\nabla w|^2 \geq \frac{m - p + 1}{m - p} |\nabla |w||^2,$$

and equality holds if and only if there exists a 1-form α with $\alpha \wedge w = 0$ such that

$$(8) \quad \nabla w = \alpha \otimes w - \frac{1}{m + 1 - p} \sum_{j=1}^m \theta^j \otimes (\theta^j \wedge i_{\alpha^\sharp} w),$$

where $\{\theta^1, \dots, \theta^m\}$ is an orthonormal basis for the cotangent bundle and α^\sharp is the vector dual to α .

Now we are ready to prove vanishing and finiteness theorems for harmonic p -forms using the decay estimate Lemma 2.1 and the Kato inequality. To simplify our statement, we will assume the function ρ is bounded in the rest of the section.

Theorem 3.1. For $m \geq 3$, let M^m be a complete noncompact Riemannian manifold with properties (\mathcal{P}_{ρ_0}) and $(\mathcal{P}_{p,\rho})$. Suppose the volume growth of M satisfies

$$\int_{B_\rho(2R) \setminus B_\rho(R)} \rho e^{-2(m-p-1)r_\rho} \leq c R$$

and the curvature operator K_p of M^m has the lower bound

$$K_p \geq -\frac{m-p}{m-p-1} \rho_0 \quad \text{on } M.$$

Then any harmonic p -form w in L^d with $2 \leq d \leq 2(m-p-1)/(m-p-2)$ must either vanish or satisfy Equation (8).

Proof. The theorem is obviously true if $d = 2$ as $(\mathcal{P}_{p,\rho})$ holds on M . So we assume $d > 2$. Let w be a smooth harmonic p -form. By the Kato inequality, the Bochner formula becomes $|w| \Delta |w| \geq \frac{1}{m-p} |\nabla |w||^2 + K_p |w|^2$.

Let $g = |w|^{(m-p-1)/(m-p)}$. Then this inequality can be rewritten as

$$(9) \quad \Delta g \geq \frac{m-p-1}{m-p} K_p g.$$

We first show that g satisfies the integral estimate $\int_{B_\rho(2R) \setminus B_\rho(R)} \rho g^2 \leq CR$. To see this, using the Schwarz inequality, we have

$$(10) \quad \int_{B_\rho(2R) \setminus B_\rho(R)} \rho g^2 \leq \left(\int_{B_\rho(2R) \setminus B_\rho(R)} \rho \exp(2r_\rho) |w|^2 \right)^{m-p-1/(m-p)} \cdot \left(\int_{B_\rho(2R) \setminus B_\rho(R)} \rho \exp(-2(m-p-1)r_\rho) \right)^{1/(m-p)}.$$

By the volume growth condition, the second term on the right side satisfies

$$\int_{B_\rho(2R) \setminus B_\rho(R)} \rho \exp(-2(m-p-1)r_\rho) \leq c R.$$

On the other hand, for $a = d/(d-2)$, we have

$$\begin{aligned} \int_{B_\rho(2R) \setminus B_\rho(R)} \rho \exp(-2r_\rho) |w|^2 &\leq \left(\int_{B_\rho(2R) \setminus B_\rho(R)} \rho^a \exp(-2a r_\rho) \right)^{1/a} \\ &\quad \cdot \left(\int_{B_\rho(2R) \setminus B_\rho(R)} |w|^d \right)^{2/d} \\ &\leq C \left(\int_{B_\rho(2R) \setminus B_\rho(R)} \rho \exp(-2(m-p-1)r_\rho) \right)^{1/a} \\ &\leq CR^{1/a}, \end{aligned}$$

since w is in L^d . Now according to Lemma 2.1, one has

$$\int_{B_\rho(2R) \setminus B_\rho(R)} \rho \exp(2r_\rho) |w|^2 \leq CR.$$

Then (10) can be written as $\int_{B_\rho(2R) \setminus B_\rho(R)} \rho g^2 \leq CR$.

To finish the proof of the theorem, note that for a cutoff function ϕ , we have

$$\int_M \rho_0 \phi^2 g^2 \leq \int_M |\nabla(\phi g)|^2 = \int_M |\nabla\phi|^2 g^2 + 2\phi g \langle \nabla\phi, \nabla g \rangle + |\phi|^2 |\nabla g|^2.$$

Also,

$$\int_M 2\phi g \langle \nabla\phi, \nabla g \rangle = \int_M g \langle \nabla\phi^2, \nabla g \rangle = - \int_M \phi^2 |\nabla g|^2 - \int_M \phi^2 g \Delta g.$$

Therefore

$$\int_M \rho_0 \phi^2 g^2 \leq \int_M |\nabla\phi|^2 g^2 - \int_M \phi^2 g \Delta g,$$

or in other words,

$$(11) \quad \int_M \phi^2 g (\rho_0 g + \Delta g) \leq \int_M |\nabla\phi|^2 g^2.$$

Let us now choose $\phi = \phi(r_\rho)$ to satisfy the properties that

$$\phi = \begin{cases} 1 & \text{on } B_\rho(R), \\ 0 & \text{on } M \setminus B_\rho(2R), \end{cases}$$

and

$$|\phi'(t)| \leq 2R^{-1} \quad \text{for } R \leq t \leq 2R.$$

Then

$$\int_{B_\rho(2R)} \phi^2 g^2 \left(\rho_0 + \frac{m-p-1}{m-p} K_p \right) \leq CR^{-2} \int_{B_\rho(2R) \setminus B_\rho(R)} \rho g^2.$$

The right side of this tends to zero as $R \rightarrow \infty$. Since $K_p \geq -(m-p)\rho_0/(m-p-1)$, we conclude that g must be identically zero, or

$$(12) \quad \Delta g = \frac{m-p-1}{m-p} K_p g.$$

This in particular implies that w must satisfy (8). □

Next we prove the finiteness theorem for the space of harmonic p -forms if the curvature lower bound only holds on $M \setminus B_q(R_0)$, where $B_q(R_0)$ is a geodesic ball in M .

Theorem 3.2. For $m \geq 3$, let M^m be a complete noncompact Riemannian manifold with properties (\mathcal{P}_{ρ_0}) and $(\mathcal{P}_{p,\rho})$. Suppose the volume growth of M satisfies

$$\int_{B_\rho(2R) \setminus B_\rho(R)} \rho e^{-2(m-p-1)r_\rho} \leq c R$$

and the curvature operator K_p of M^m has the lower bound

$$K_p \geq -\left(\frac{m-p}{m-p-1} - \epsilon\right)\rho_0 \quad \text{on } M \setminus B_q(R_0).$$

Then $\dim H_d^p(M) \leq C(m, p, \epsilon, B_q(R_0))$ with $2 \leq d \leq 2(m-p-1)/(m-p-2)$.

Proof. According to the proof of the vanishing Theorem 3.1, for each $w \in H_d^p$, function $g = |w|^{2(m-p-1)/(m-p)}$ satisfies the estimate

$$\int_{B_\rho(2R) \setminus B_\rho(R)} g^2 \leq C R.$$

Also, the Bochner formula together with the curvature assumption implies that the function g satisfies the differential inequality

$$\Delta g \geq \left(\frac{m-p-1}{m-p}\epsilon - \rho_0\right)g \quad \text{on } M \setminus B_q(R_0).$$

Let ϕ be a cutoff function satisfying

$$\phi = \begin{cases} 0 & \text{on } B_q(R_0), \\ 1 & \text{on } B_\rho(R) \setminus B_q(2R_0), \\ 0 & \text{on } M \setminus B_\rho(2R), \end{cases}$$

$$|\nabla \phi| \leq C R_0^{-1} \quad \text{on } B_q(2R_0) \setminus B_q(R_0),$$

$$|\nabla \phi| \leq C \sqrt{\rho} R^{-1} \quad \text{on } B_\rho(2R) \setminus B_\rho(R)$$

for some constant $C > 0$.

Since

$$\int_M \rho_0 \phi^2 g^2 \leq \int_M |\nabla(\phi g)|^2 = \int_M |\nabla \phi|^2 g^2 + 2\phi g \langle \nabla \phi, \nabla g \rangle + |\phi|^2 |\nabla g|^2$$

and

$$\int_M 2\phi g \langle \nabla \phi, \nabla g \rangle = \int_M g \langle \nabla \phi^2, \nabla g \rangle = - \int_M \phi^2 |\nabla g|^2 - \int_M \phi^2 g \Delta g,$$

we conclude

$$\int_M \rho_0 \phi^2 g^2 \leq \int_M |\nabla \phi|^2 g^2 - \int_M \phi^2 g \Delta g.$$

Hence, we have

$$\int_{M \setminus B_q(R_0)} \phi^2 g(\rho_0 g + \Delta g) \leq C R_0^{-2} \int_{B_q(2R_0) \setminus B_q(R_0)} g^2 + C R^{-2} \int_{B_\rho(2R) \setminus B_\rho(R)} \rho g^2.$$

Let $R \rightarrow \infty$. Then

$$\frac{m-p-1}{m-p} \epsilon \int_{M \setminus B_q(2R_0)} g^2 \leq C R_0^{-2} \int_{B_q(2R_0) \setminus B_q(R_0)} g^2.$$

In particular,

$$(13) \quad \int_{B_q(3R_0)} g^2 \leq \left(1 + \frac{C}{\epsilon R_0^2}\right) \int_{B_q(2R_0)} g^2.$$

It is now standard to conclude [Li 1980] that $\dim H_d^p \leq C$. □

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