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We obtain weighted Sobolev interpolation inequalities on generalized John domains that include John domains (bounded or unbounded) for δ-doubling measures satisfying a weighted Poincaré inequality. These measures include ones arising from power weights $d(x, \partial\Omega)^\alpha$ and need not be doubling. As an application, we extend the Sobolev interpolation inequalities obtained by Caffarelli, Kohn and Nirenberg. We extend these inequalities to product spaces and give some applications on products $\Omega_1 \times \Omega_2$ of John domains for $A_p(\mathbb{R}^n \times \mathbb{R}^m)$ weights and power weights of the type $w(x, y) = \text{dist}(x, G_1)^\alpha \text{dist}(y, G_2)^\beta$, where $G_1 \subset \partial\Omega_1$ and $G_2 \subset \partial\Omega_2$. For certain cases, we obtain sharp conditions.

1. Introduction

Sobolev interpolation inequalities are useful tools in the study of solutions of certain partial differential equations; see [Caffarelli et al. 1982; Gutiérrez and Wheeden 1990; Fernandes 1991; Chua 1992]. These inequalities are indeed closely related to the Sobolev inequalities

$$\text{(P1)} \qquad \|f - f_{B,\mu}\|_{L^q_w(B)} \le A(B)\|\nabla f\|_{L^p_v(B)}$$

on a ball $B \subset \mathbb{R}^n$, where $f_{B,\mu} = \int_B f(x)d\mu/\mu(B)$ and in most cases $\mu = 1$ or $\mu = w$, and

$$A(B) = C|B|^{1/n} w(B)^{1/q} v(B)^{-1/p}.$$

There are many studies of such inequalities, for example, [Chanillo and Wheeden 1985; Sawyer and Wheeden 1992; Chiarenza and Frasca 1985], and they have been extended to domains other than cubes or balls, for example, in [Kufner 1985; Bojarski 1988; Iwaniec and Nolder 1985; Chua 1993; Hajłasz and Koskela 1998; Buckley and Koskela 1995]. Estimates of sharp constants have also been made

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in [Chua and Wheeden 2000; 2006; Acosta and Durán 2004] on convex domains. They have also been used to deduce Sobolev interpolation inequalities of the form

$$\text{(P2)} \quad \|\nabla^i f\|_{L^q_w(\mathbb{R}^n)} \le C\|f\|^{\alpha}_{L^r_{v_0}(\mathbb{R}^n)} \|\nabla^k f\|^{1-\alpha}_{L^p_v(\mathbb{R}^n)},$$

$$\text{(P3)} \quad \|\nabla^i f\|_{L^q_w(\Omega)} \le C(C_1\|f\|_{L^r_{v_0}(\Omega)})^{\alpha}\big(C_1\varepsilon_0^{-1}\|f\|_{L^r_{v_0}(\Omega)} + C_2\|\nabla^k f\|_{L^p_v(\Omega)}\big)^{1-\alpha}$$

for $0 \le i \le k-1$ and $0 < \alpha < 1$ by Caffarelli, Kohn and Nirenberg [1984], Gutiérrez and Wheeden [1991], and Chua [1994]. The inequality (P2) was also obtained by Brown and Hinton [1988; 1990] for some domains Ω in $\mathbb{R}^n$. Moreover, the author [Chua 1992; 2006] discussed (P2) and (P3) on (ε, ∞) domains (as introduced by Jones [1981]). Of course one cannot replace $\mathbb{R}^n$ in (P2) by bounded domains Ω; in that case (P3) seems to be a natural substitute of (P2). Indeed, Brown and Hinton discussed mostly the weighted interpolation inequalities in sum form, namely,

$$\text{(1-1)} \quad \|\nabla^i f\|_{L^q_w(\Omega)} \le C_1\varepsilon^{-\alpha}\|f\|_{L^r_{v_0}(\Omega)} + C_2\varepsilon^{1-\alpha}\|\nabla^k f\|_{L^p_v(\Omega)} \quad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

where $0 < \alpha < 1$. They were obtained from a basic Sobolev integral representation formula (in $\mathbb{R}^n$). It is easy to see that (P2) or (P3) is indeed equivalent to (1-1) with $\varepsilon_0 = \infty$ or $\varepsilon_0 < \infty$ respectively; see Remark 1.8(4) below.

It is well known that weighted Sobolev inequalities on cubes/balls imply that the inequalities will also hold on John domains [Chua 1993; Chua and Wheeden 2008; Hajłasz and Koskela 1998] under standard balance conditions on the weights; see also Theorem 2.11. However, even though it is well known that Sobolev interpolation (weighted or unweighted) inequalities hold for cubes, these inequalities have not been well studied on general domains. Indeed, it was only made known in [Chua 1995] that such inequalities remain true for Lipschitz domains when the weights involved satisfy some standard balance conditions, as they were shown to be special cases of a Boman-type domain introduced there.

In this paper, we will first define a generalization of John domains that clearly contains John domains. Surprisingly, it turns out that these generalized domains are equivalent to the domains introduced in [Chua 1995, Definition 1.2], and hence it is clear that the weighted Sobolev inequalities obtained in [Chua 1995] also hold on generalized John domains. Our generalized John domains include John domains and hence also Lipschitz domains; see Definition 1.2. Moreover, we will relax the standard doubling condition to just δ-doubling (see Definition 1.4). Note that power-type weights $d(\cdot, \Omega^c)^a$ for $a \ge 0$ will induce a δ-doubling measure on Ω (but we do not know whether it is doubling unless Ω is Lipschitz). We then extend our ideas to generalized John domains in product spaces. Meanwhile, as an application, we will discuss a Sobolev interpolation inequality that is an extension of the one obtained by Caffarelli, Kohn and Nirenberg [1984] and Lin [1986].

In what follows, $\langle H, d\rangle$ will always be a metric space. For any $x \in H$ and $r > 0$, recall that the metric balls are of the form $B(x, r) = \{y \in H : d(x, y) < r\}$ and call $B(x, r)$ the ball with center x and radius r. If $B = B(x, r)$ is a ball and c is a positive constant, we often use cB to denote $B(x, cr)$. We usually use $r(B)$ and x_B to denote the radius and center of a ball B. We say $\Omega \subset H$ satisfies the nonempty annuli property if $(\Omega \cap B(x, r)) \setminus B(x, r') \neq \varnothing$ for all $0 < r' < r$ and $x \in \Omega$ whenever Ω is not a subset of $B(x, r')$. The domains considered in this paper will always satisfy the nonempty annuli property. We say a family $\mathcal{F}$ of balls has bounded intercepts with bound K if each fixed ball in $\mathcal{F}$ intersects at most K balls in $\mathcal{F}$. Thus, if $\mathcal{F}$ consists of disjoint balls, then it has bounded intercepts with bound 1. If $\mathcal{F}$ has bounded intercepts with bound K, then $\mathcal{F} = \bigcup_{i=1}^{K} \mathcal{F}_i$ such that balls in each $\mathcal{F}_i$ are pairwise disjoint.

For $E, F \subset H$, we define

$$d(E, F) = \inf_{\substack{z_1 \in E \\ z_2 \in F}} d(z_1, z_2).$$

If $x \in H$, $F \subset H$, we define $d(x, F) = d(\{x\}, F)$. For a fixed $\Omega \subset H$ (here Ω is usually open) and when there is no danger of confusion, we will also write $d(x) = d(x, \Omega^c)$ and $d(E) = d(E, \Omega^c)$ for $x \in \Omega$ and $E \subset \Omega$. We also write $\operatorname{diam}(\Omega) = \sup\{d(x, y) : x, y \in \Omega\}$.

Let $\sigma, N > 1$. Recall that a Boman domain Ω in a metric space $\langle H, d\rangle$ has a covering W of balls such that

(1) $\sigma W = \{\sigma B\}_{B \in W}$ has bounded intercepts and $\sigma B \subset \Omega$;

(2) there exists a central ball $B^* \in W$ such that for any other ball $B \in W$, there exists a Boman chain connecting B to B^*, that is, a finite chain of balls $\{B_0 = B, B_1, \dots, B_K = B^*\} \subset W$ such that, for all i,
   - $B_i \cap B_{i+1}$ contains a ball B_i' such that $B_i \cup B_{i+1} \subset N_0 B_i'$ for some $N_0 > 1$ and
   - $B \subset N B_i$.

We will write $\Omega \in \mathcal{F}_d(\sigma, N)$, and sometimes we just write $\mathcal{F}(\sigma, N)$ when the choice of the metric is clear. We say W is a Boman cover of Ω and define $r(\Omega) = r(B^*)$. To reduce the number of constants involved, we will assume that $N_0 = N$ and that σB for any $B \in W$ intersects at most N balls in the family $\sigma W = \{\sigma B\}_{B \in W}$. Among the many examples of Boman domains in $\mathbb{R}^n$ are bounded Lipschitz domains, bounded (ε, ∞) domains, and John domains; see [Bojarski 1988; Iwaniec and Nolder 1985; Chua 1992; 1995].

Now, let us define John domains in a metric space.

Definition 1.1. Let $\Omega \subset H$ and $0 < c \leq 1$ (here c is usually < 1). We write $\Omega \in J(c)$ if there exists a "center" $x' \in \Omega$ such that for all $x \in \Omega$ with $x \neq x'$, there exists a map $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(l) = x'$ and such that

(1-2) $$d(\gamma(t_1), \gamma(t_2)) \le |t_2 - t_1| \quad \text{and} \quad d(\gamma(t)) = d(\gamma(t), \Omega^c) > ct$$

for all $t_1, t_2, t \in [0, l]$.

Clearly Ω is always open connected. We will usually say Ω is a John domain or a $J(c)$ domain. Though our definition may look different from the usual one [Martio and Sarvas 1979], it is essentially the same; see also [Acosta et al. 2006].

Under the assumption of the existence of a doubling measure, it is easy to see that a John domain (using a Whitney-type decomposition) is a Boman domain such that each Boman chain of balls $\{B_0, B_1, \dots, B_K = B^*\}$ can be chosen such that $r(B_i) \ge Cc_0^i r(B_0)$ for some fixed $c_0 > 1$ for all i. The converse is not obvious at all. It wasn't until 1995 that it was shown — by Buckley, Koskela and Lu [1996] — that a Boman domain is indeed a John domain when the domain satisfies a geodesic condition. We say that $\Omega \subset H$ satisfies the geodesic condition if, for any $z \in B(x, r) \subset \Omega$, there exists a $\gamma : [0, d(x, z)] \to B(x, r)$ such that

$$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0, d(x, z)].$$

Clearly, the metric balls that satisfy the geodesic condition are $J(1)$ domains. In most cases, domains considered in this paper will satisfy the geodesic condition.

We will now define a generalization of John domains.

Definition 1.2. Let $0 < c \le 1$ and $0 < M_0 \le \infty$. We write $\Omega \in J(c, M_0)$ if, given any $z \in \Omega$ and $0 < M < M_0$ with $d(z, \Omega^c) < M$, there exists $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = z$ and $d(\gamma(l), \Omega^c) \ge M$ and such that

(1-3) $$d(\gamma(t_1), \gamma(t_2)) \le |t_1 - t_2| \quad \text{and} \quad d(\gamma(t)) = d(\gamma(t), \Omega^c) > ct$$

for $t, t_1, t_2 \in [0, l]$. Of course, one can choose l such that $d(\gamma(l)) = M$. We will say Ω is a generalized John domain or $J(c, M_0)$ domain.

Our generalized John domains include John domains and the unbounded John domains introduced in [Väisälä 1989]; see also [Hurri-Syrjänen 1992]. It also includes uniform domains introduced in [Martio and Sarvas 1979]. We do not require a $J(c, M_0)$ domain to be connected; however, each of its connected component will be in $J(c, M_0)$.

Examples (unbounded generalized John domains in $\mathbb{R}^2$). (1) Let $x_k = (k, 0) \in \mathbb{R}^2$ for $k \in \mathbb{Z}$. Then $\Omega_N = \bigcup_{k \in \mathbb{Z}, |k| < N} B(x_k, 1 + 2^{-|k|-1}) \in J(1, 1)$ for $1 \le N \le \infty$, $\Omega_N \in J(c_N)$ for any positive number N, and $c_N \to 0$ as $N \to \infty$.

(2) Let $r > 0$ and let Ω be a convex domain in the Euclidean ball $B(0, r)$. Then

$$S = \{(x_1, x_2) \in \mathbb{R}^2 \setminus \Omega : |x_1| < 3r\} \in J(1, r).$$

See Proposition 2.24 for more examples.

Definition 1.3. Suppose Ω is an open subset of H and that $0 < \delta < 1$. A ball $B(x, r)$ will be called a δ-ball if $x \in \Omega$ and $r \le \delta d(x)$. It is easy to see that then $d(B) = d(B, \Omega^c) \ge (1 - \delta)r$. If B_1 and B_2 are intersecting δ-balls, then (by the triangle inequality)

$$d(B_1) \le (1 + 2/(1 - \delta))d(B_2) \tag{1-4}$$

and vice versa.

The concept of δ-ball has been introduced before; see for example [Sawyer and Wheeden 2006]. Balls in a Boman covering of a domain $\Omega \in \mathscr{F}(\sigma, N)$ are clearly δ-balls with $1/\sigma \le \delta < 1$.

We next define δ-doubling and doubling on Ω (usually open).

Definition 1.4. Let $\Omega \subset H$. We say μ is a doubling measure on Ω if there exists a doubling constant D_μ such that

$$\mu(2^k B) \le (D_\mu)^k \mu(B) \quad \text{for all } k \in \mathbb{N}$$

for any ball with center in Ω such that $r(B) \le \operatorname{diam}(\Omega)$. If the above is true for all balls in H, we will just say μ is doubling. Moreover, if $0 < \delta < 1$ and the above is true only for δ-balls B in Ω, then we will say μ is δ-doubling on Ω. Note that if $0 < \delta_1 < \delta_2 < 1$, then μ is δ_1-doubling if and only if it is δ_2-doubling. Clearly, if μ is δ-doubling on Ω then so is $\mu|_\Omega$, where $\mu|_\Omega(E) = \mu(E \cap \Omega)$. It is obvious that a doubling measure on Ω is always δ-doubling on Ω. Conversely, a δ-doubling measure on a John domain is also doubling on Ω since any ball with center in Ω with radius less than $\operatorname{diam}(\Omega)$ must contain a δ-ball of comparable size; see [Chua and Wheeden 2008, Proposition 2.2] or [Chua and Wheeden 2009] for details. Furthermore, if $\Omega \subset H_1 \times H_2$, where $\langle H_1, d_1\rangle$ and $\langle H_2, d_2\rangle$ are metric spaces, we say that μ is a product δ-doubling measure on Ω in $H_1 \times H_2$ if

$$\mu(2^k B_1 \times 2^k B_2)) \le (D_\mu)^k \mu(B_1 \times B_2) \quad \text{for all } k \in \mathbb{N}$$

for any product of balls $B_1 \times B_2 \subset H_1 \times H_2$ (that is, B_i is a metric ball in H_i for $i = 1, 2$) such that $B_1/\delta \times B_2/\delta \subset \Omega$. We define product doubling on Ω similarly.

If Ω satisfies the nonempty annuli property, then δ-doubling or doubling on Ω will imply reverse doubling (of the same type on Ω); see Proposition 2.8. If μ is doubling on H and $\Omega_0 \subset \Omega^c$, it is easy to see that any weight $d(x, \Omega_0)^\alpha$ will give rise to a δ-doubling measure $\mu_\alpha(E) = \int_E d(x, \Omega_0)^\alpha d\mu(x)$ on Ω for any $\alpha \ge 0$. It is clear that if μ_i is δ-doubling on $\Omega_i \subset H_i$ for $i = 1, 2$, then $\mu_1 \times \mu_2$ will be product δ-doubling on $\Omega_1 \times \Omega_2$ in $H_1 \times H_2$.

Most of the weights or measures studied in the previous papers were assumed to be at least doubling (on the whole space). In this paper we will relax these assumptions and consider merely δ-doubling measures on the domain involved. With the help of metrics d_λ (see Definition 1.9), we also manage to generalize our idea to study Sobolev interpolation inequalities on product spaces.

In what follows, C denotes various positive constants, which may differ even in a same string of estimates. We will use $C(\alpha, \beta, \dots)$ instead of C to emphasize when the constant depends only on $\alpha, \beta, \dots$. Also, $p' = p/(p-1)$ if $1 < p < \infty$ and $p' = \infty$ if $p = 1$. Next, for any open set $\Omega \subset \mathbb{R}^n$, we let $C^{k,1}_{\text{loc}}(\Omega)$ be the collection of all functions on Ω whose derivatives of degree $\le k$ exist and are locally Lipschitz continuous. If $f \in C^{0,1}_{\text{loc}}(\Omega)$ and $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, then for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $(x, y) \in \Omega$, we denote by $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ the partial gradients of f containing the x- and y-derivatives, respectively. If $f \in C^{1,1}_{\text{loc}}(\Omega)$, we denote by $\nabla^2_x f(x, y)$ the vector of all second order x-derivatives and by $\nabla_{xy} f(x, y)$ the vector of all y-derivatives of the x-derivatives of f.

If w is a measure on $\Omega \subset H$, we define for $1 \le p < \infty$

$$\|f\|_{L^p_w(\Omega)} = \left(\int_\Omega |f|^p dw\right)^{1/p} \quad \text{and} \quad \|f\|_{L^\infty_w(\Omega)} = \operatorname*{esssup}_{x\in\Omega} |f(x)|.$$

By a weight w, we always mean a nonnegative measurable function that is finite almost everywhere. We will also denote the measure arising from w by w, and sometimes we write dw instead of $w(x)dx$.

We now extend the Boman-type domain introduced in [Chua 1995] to metric spaces.

Definition 1.5. Let $1 < \sigma, N < \infty$ and $0 < \varepsilon_0 \le \infty$. Let Ω be a subset in a metric space $\langle H, d\rangle$. By $\Omega \in \mathcal{F}'_d(\sigma, N, \varepsilon_0)$, we mean that given any $0 < \varepsilon < \varepsilon_0$, there exist $\{\Omega_j\} \subset \mathcal{F}_d(\sigma, N)$ such that $\varepsilon \le r(\Omega_j) \le N\varepsilon$ for all j, and $\chi_\Omega \le \sum \chi_{\Omega_j} \le N\chi_\Omega$.

When there is no danger of confusion, we will write $\mathcal{F}'(\sigma, N, \varepsilon_0)$ instead of $\mathcal{F}'_d(\sigma, N, \varepsilon_0)$. We will show that these domains are just generalized John domains when the domain satisfies the geodesic condition; see Proposition 2.21. Our first theorem, extends [Chua 1995, Theorems 1.3 and 1.4] to the case of δ-doubling measures on the above domains in metric spaces.

Theorem 1.6. *Let Ω be a subset in a metric space $\langle H, d\rangle$. Let $A', A_0 > 0$, $\alpha, \beta \in \mathbb{R}$ and $\sigma, N > 1$. Let $0 < \varepsilon_0 \le \infty$, $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_0)$ and $1 \le p, r \le q < \infty$. Let $\delta = 1/\sigma$ and $1 \le \tau \le \sigma$. Let v, v_0, w, μ be measures such that w is δ-doubling on Ω and $dv_0 = \tilde{v}_0 d\mu$. Let*

$$\frac{w(B)^{1/q}}{\mu(B)} \|\tilde{v}_0^{-1/r}\|_{L^{r'}_\mu(B)} \le A_0 r(B)^{-\beta} \tag{1-5}$$

for all δ-balls B with $r(B) < N^2\varepsilon_0$.

(a) *If f and g are measurable functions on Ω for $\alpha \geq 0$, such that*

$$\|f - f_{B,\mu}\|_{L^q_w(B)} \leq A' r(B)^\alpha \|g\|_{L^p_v(\tau B)} \quad \text{for all } \delta\text{-balls } B \text{ with } r(B) < N^2 \varepsilon_0, \tag{1-6}$$

where $f_{B,\mu} = \int_B f \, d\mu / \mu(B)$, then

$$\|f\|_{L^q_w(\Omega)} \leq C(N, D_w, q)\big(A_0 \varepsilon^{-\beta} \|f\|_{L^r_{v_0}(\Omega)} + A' \varepsilon^\alpha \|g\|_{L^p_v(\Omega)}\big) \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{1-7}$$

(b) *Let $k \in \mathbb{N}$. Suppose $H = \mathbb{R}^n$ and d is a metric arising from a norm. Suppose μ is also δ-doubling on Ω. For any $f \in C^{k,1}_{\mathrm{loc}}(\Omega)$, $l \leq k$ and δ-ball B, let $P^l(B) f$ be the polynomial of degree $\leq l$ such that $\int_B D^\gamma(f - P^l(B) f) d\mu = 0$ for all $|\gamma| \leq l$. If $i \leq k$ for $i \in \mathbb{N} \cup \{0\}$, $k - i + \alpha \geq 0$,*

$$\frac{w(B)^{1/q}}{\mu(B)} \|f - P^k(B) f\|_{L^1_\mu(B)} \leq A' r(B)^{k+\alpha} \|\nabla^{k+1} f\|_{L^p_v(\tau B)}, \tag{1-8}$$

$$\|f - P^{k-i}(B) f\|_{L^q_w(B)} \leq A' r(B)^{k-i+\alpha} \|\nabla^{k-i+1} f\|_{L^p_v(\tau B)} \tag{1-9}$$

for all δ-balls B with $r(B) < N^2 \varepsilon_0$ and all $f \in C^{k,1}_{\mathrm{loc}}(\Omega)$, then

$$\|\nabla^i f\|_{L^q_w(\Omega)} \leq C(N, n, k, D_\mu, D_w, q) \times \big(A_0 \varepsilon^{-(\beta+i)} \|f\|_{L^r_{v_0}(\Omega)} + A' \varepsilon^{k-i+\alpha} \|\nabla^{k+1} f\|_{L^p_v(\Omega)}\big) \tag{1-10}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and all $f \in C^{k,1}_{\mathrm{loc}}(\Omega)$.

As most applications involve the Sobolev inequality (P1), we will prove a useful corollary of Theorem 1.6.

Corollary 1.7. *Let $\Omega \subset \mathbb{R}^n$. Let $\alpha, \beta \in \mathbb{R}$, $A', A_0, c_1 > 0$ and $\sigma, N > 1$. Let i and k be nonnegative integers with $i \leq k$. Let $0 < \varepsilon_0 \leq \infty$, $\Omega \in \mathscr{F}'(\sigma, N, \varepsilon_0)$ and $1 \leq p, r \leq q < \infty$. Let $\delta = 1/\sigma$. Let v, v_0, w, μ be measures such that w, μ, v are δ-doubling on Ω and $dv_0 = \tilde{v}_0 d\mu$. For all δ-balls B with $r(B) < N^2 \varepsilon$ and all $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$, suppose*

$$\|f - f_{B,\mu}\|_{L^q_w(B)} \leq A' r(B)^\alpha \|\nabla f\|_{L^p_v(B)}, \tag{1-11}$$

$$\frac{1}{\mu(B)} \|f - f_{B,\mu}\|_{L^1_\mu(B)} \leq c_1 \frac{r(B)}{v(B)^{1/p}} \|\nabla f\|_{L^p_v(B)}. \tag{1-12}$$

If, for all δ-balls B with $r(B) < N^2 \varepsilon_0$ and $k - i + \alpha \geq 0$,

$$\begin{gathered} \frac{w(B)^{1/q}}{\mu(B)} \|\tilde{v}_0^{-1/r}\|_{L^{r'}_\mu(B)} \leq A_0 r(B)^{-\beta}, \\ c_1 w(B)^{1/q} v(B)^{-1/p} r(B) \leq A' r(B)^\alpha, \end{gathered} \tag{1-13}$$

then, for all $\varepsilon \in (0, \varepsilon_0)$ *and all* $f \in C^{k,1}_{\mathrm{loc}}(\Omega)$,

$$\|\nabla^i f\|_{L^q_w(\Omega)} \le C(N, n, k, D_\upsilon, D_\mu, D_w, p, q) \times \left(A_0 \varepsilon^{-(\beta+i)} \|f\|_{L^r_{\upsilon_0}(\Omega)} + A' \varepsilon^{k-i+\alpha} \|\nabla^{k+1} f\|_{L^p_\upsilon(\Omega)}\right). \tag{1-14}$$

Remark 1.8. (1) [Chua 1995, Theorems 1.3 and 1.4] are indeed special cases of Theorem 1.6 with $H = \mathbb{R}^n$, $g = |\nabla f|$ and w a doubling weight. It was further assumed in [Chua 1995, Theorem 1.4], that $\mu = 1$ and $i = k$.

(2) (1-8) and (1-9) seem to be arbitrary. However, they are usually true under standard assumptions. For example, suppose

$$w(B)^{1/q} \upsilon(B)^{-1/p} r(B) \le C r(B)^\alpha, \tag{1-15}$$

$$\frac{1}{w(B)^{1/q}} \|f - f_{B,\mu}\|_{L^q_w(B)} \le \frac{C r(B)}{\upsilon(B)^{1/p}} \|\nabla f\|_{L^p_\upsilon(B)} \tag{1-16}$$

for all $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$ and δ-balls B. If υ and w are δ-doubling (on Ω), then (1-8) and (1-9) will hold with $\mu = w$. To see this, first observe that

$$\|f - f_{B,w}\|_{L^q_w(B)} \le \|f - f_{B,\mu}\|_{L^q_w(B)} + \|f_{B,\mu} - f_{B,w}\|_{L^q_w(B)} \le 2\|f - f_{B,\mu}\|_{L^q_w(B)}.$$

Hence by Hölder's inequality, we have

$$\frac{1}{w(B)} \|f - f_{B,w}\|_{L^1_w(B)} \le \frac{C r(B)}{\upsilon(B)^{1/p}} \|\nabla f\|_{L^p_\upsilon(B)}.$$

We can now apply Proposition 2.15 with $\mu = w$ to obtain

$$\|f - f_{B,w}\|_{L^p_\upsilon(B)} \le C r(B) \|\nabla f\|_{L^p_\upsilon(B)} \tag{1-17}$$

for all $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$ and δ-balls B. Taking $\mu = w$ and letting $P^k(B) f$ be as in Theorem 1.6, we have by iterations of (1-17)

$$\|\nabla(f - P^k(B) f)\|_{L^p_\upsilon(B)} \le C r(B)^k \|\nabla^{k+1} f\|_{L^p_\upsilon(B)}.$$

Hence,

$$\begin{aligned} \frac{1}{w(B)} \|f - P^k(B) f\|_{L^1_w(B)} &\le \frac{C r(B)}{\upsilon(B)^{1/p}} \|\nabla(f - P^k(B) f)\|_{L^p_\upsilon(B)} \\ &\le \frac{C r(B)^{k+1}}{\upsilon(B)^{1/p}} \|\nabla^{k+1} f\|_{L^p_\upsilon(B)}. \end{aligned}$$

The inequality (1-8) is now clear with $\mu = w$ and $\tau = 1$ by (1-15). Finally, (1-9) is also clear by (1-15), (1-16) and (1-17).

(3) In the theorem and corollary above, we have chosen $\delta = 1/\sigma$ for convenience: δ could be any constant such that $0 < \delta < 1$. Clearly the theorem above remains true if $\delta > 1/\sigma$. However, in case $\delta < 1/\sigma$, we will need Corollary 2.12.

(4) Interpolation inequalities of sum form such as (1-10), (1-14) or (1-7) are usually equivalent to interpolation inequalities of product form. For example, when $\alpha, \beta > 0$, inequality (1-7) is equivalent to

$$\|f\|_{L^q_w(\Omega)} \le C(A_0\|f\|_{L^r_{v_0}(\Omega)})^{\alpha/(\alpha+\beta)} \times (A'\|g\|_{L^p_v(\Omega)} + A_0\varepsilon_0^{-\beta-\alpha}\|f\|_{L^r_{v_0}(\Omega)})^{\beta/(\alpha+\beta)}. \tag{1-18}$$

Here is a short proof: Clearly, by the fact that arithmetic mean is larger than geometric mean, we know the right hand side of (1-18) is less than

$$C\big(\varepsilon^{-\beta}A_0\|f\|_{L^r_{v_0}(\Omega)} + \varepsilon^{\alpha}(A'\|g\|_{L^p_v(\Omega)} + A_0\varepsilon_0^{-\beta-\alpha}\|f\|_{L^r_{v_0}(\Omega)})\big) = C(\varepsilon^{-\beta}+\varepsilon_0^{-\beta})A_0\|f\|_{L^r_{v_0}(\Omega)} + C\varepsilon^{\alpha}A'\|g\|_{L^p_v(\Omega)}.$$

Inequality (1-7) is now clear since $\varepsilon \in (0, \varepsilon_0)$ and $\beta > 0$. Conversely, if (1-7) holds, its right side is less than

$$C\big(\varepsilon^{-\beta}A_0\|f\|_{L^r_{v_0}(\Omega)} + \varepsilon^{\alpha}(A'\|g\|_{L^p_v(\Omega)} + A_0\varepsilon_0^{-\beta-\alpha}\|f\|_{L^r_{v_0}(\Omega)})\big).$$

Inequality (1-18) is now clear by taking

$$\varepsilon = \left(\frac{A_0\|f\|_{L^r_{v_0}(\Omega)}}{A'\|g\|_{L^p_v(\Omega)} + A_0\varepsilon_0^{-\beta-\alpha}\|f\|_{L^r_{v_0}(\Omega)}}\right)^{1/(\alpha+\beta)},$$

which is possible since the above choice of ε is certainly less than ε_0. Note that if $\Omega \in \mathcal{F}'(\sigma, N, \infty)$, then $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_0)$ for all $\varepsilon_0 > 0$, and one could just let $\varepsilon_0 \to \infty$ in (1-18) to obtain (as C is independent of ε_0)

$$\|f\|_{L^q_w(\Omega)} \le C(A_0\|f\|_{L^r_{v_0}(\Omega)})^{\alpha/(\alpha+\beta)}(A'\|g\|_{L^p_v(\Omega)})^{\beta/(\alpha+\beta)}. \tag{1-19}$$

Indeed, it can also be obtained directly from (1-7) if we assume (1-7) holds for all $\varepsilon \in (0, \infty)$. Note that the equivalence of (1-7) and (1-19) is well known when $\varepsilon_0 = \infty$; see for example [Brown and Hinton 1990] or [Brown and Hinton 1988].

(5) Let μ, v and w be δ-doubling on $\Omega \subset \mathbb{R}^n$. Let $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_0)$, $0 < a < 1$ and $1 \le p \le q < \infty$. Suppose the normalized Sobolev inequality

$$\|f - f_{B,\mu}\|_{L^q_w(B)} \le Cw(B)^{1/q}v(B)^{-1/p}r(B)\|\nabla f\|_{L^p_v(B)} \tag{1-20}$$

holds for all δ-balls B with $r(B) < N^2\varepsilon_0$ and $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$.

If $v \in A_p(\mu)$ on all δ-balls in Ω, that is, $dv = \tilde{v}\,d\mu$ and

$$\frac{1}{\mu(B)}v(B)^{1/p}\|\tilde{v}^{-1/p}\|_{L^{p'}_\mu(B)} \le C,$$

for all δ-balls B in Ω, then

$$\|\nabla^i f\|_{L^q_w(\Omega)} \le C_1\big(\varepsilon^{-a(k+1)}\|f\|_{L^p_v(\Omega)} + \varepsilon^{(1-a)(k+1)}\|\nabla^{k+1}f\|_{L^p_v(\Omega)}\big) \tag{1-21}$$

for all functions $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$ and $\varepsilon \in (0, \varepsilon_0)$ if and only if

(1-22) $w(B)^{1/q} v(B)^{-1/p} \le C_2 r(B)^{-(k+1)a+i}$ for all δ-balls B with $r(B) < N^2\varepsilon_0$.

To see that the above is true, first note that by (1-20) and (1-22), we have (1-11) with $\alpha = 1-(k+1)a+i$. By Proposition 2.15, (1-20) implies

$$\|f - f_{B,\mu}\|_{L^p_v(B)} \le Cr(B)\|\nabla f\|_{L^p_v(B)} \tag{1-23}$$

for all δ-balls B with $r(B) < N^2\varepsilon_0$ and $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$. By the fact that $v \in A_p(\mu)$, (1-22) and (1-23), we have (1-12). Thus, in view of Corollary 1.7, we only need to show why the condition (1-22) is necessary. To this end, for simplicity, let us consider only metrics such that metric balls are cubes. For any δ-cube Q, there exists a polynomial P of degree $\ge k+1$ such that $D^\alpha P = 0$ on ∂Q for all $|\alpha| \le k$. Let $f = \chi_Q P$. Then $f \in C^{k,1}_{\mathrm{loc}}(\Omega)$. First by (1-21), we have

$$\begin{aligned}\|\nabla^i f\|_{L^q_w(\Omega)} &\le C_1 \|f\|^{1-a}_{L^p_v(\Omega)} (\varepsilon_0^{-k-1}\|f\|_{L^p_v(\Omega)} + \|\nabla^{k+1} f\|_{L^p_v(\Omega)})^a \\ &\le C_1 \|f\|^{1-a}_{L^p_v(\Omega)} (\varepsilon_0^{-a(k+1)}\|f\|^a_{L^p_v(\Omega)} + \|\nabla^{k+1} f\|^a_{L^p_v(\Omega)}).\end{aligned}$$

Hence,

$$\begin{aligned}\|\nabla^i P\|_{L^q_w(Q)} \le C_1 v(Q)^{(1-a)/p} \|P\|^{1-a}_{L^\infty(Q)} \\ \times \big(\varepsilon_0^{-a(k+1)} v(Q)^{a/p}\|P\|^a_{L^\infty(Q)} + v(Q)^{a/p}\|\nabla^{k+1}P\|^a_{L^\infty(Q)}\big).\end{aligned}$$

However, by Proposition 2.6 (Markov's inequality) and the fact that $D^\alpha P = 0$ on ∂Q for all $|\alpha| \le k$, we have

$$\begin{aligned}\|\nabla^{k+1} P\|_{L^\infty(Q)} &\le Cr(Q)^{-k-1}\|P\|_{L^\infty(Q)}, \\ Cr(Q)^{-i}\|P\|_{L^\infty(Q)} &\le \|\nabla^i P\|_{L^\infty(Q)}.\end{aligned}$$

But by (2-2), we have

$$\|\nabla^i P\|_{L^\infty(Q)} \le \frac{C}{w(Q)^{1/q}} \|\nabla^i P\|_{L^q_w(Q)}.$$

Hence

$$w(Q)^{1/q} r(Q)^{-i} \le Cv(Q)^{(1-a)/p}\big(\varepsilon_0^{-a(k+1)} v(Q)^{a/p} + v(Q)^{a/p} r(Q)^{-a(k+1)}\big).$$

Inequality (1-22) is now clear since $r(Q) < N^2\varepsilon_0$.

Let us now define a Boman-type domain in product spaces.

Definition 1.9. Let $\langle H_1, d_1\rangle$ and $\langle H_2, d_2\rangle$ be metric spaces. For each $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 > 0$, we will define a metric on $H_1 \times H_2$ by

$$d_\lambda((x, y), (u, v)) = \max\{d_1(x, u)/\lambda_1, d_2(y, v)/\lambda_2\}$$

for any $(u, v), (x, y) \in H_1 \times H_2$. Clearly d_λ defines a metric on $H_1 \times H_2$. Note that metric balls (with respect to d_λ) are just products of balls $I \times J \subset H_1 \times H_2$, with $r(I)/r(J) = \lambda_1/\lambda_2$. Let $\Omega \subset H_1 \times H_2$. We say $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_1, \varepsilon_2, H_1 \times H_2)$ if $\Omega \in \mathcal{F}'_{d_\lambda}(\sigma, N, 1)$ for all $\lambda = (\lambda_1, \lambda_2)$ such that $0 < \lambda_i < \varepsilon_i$ for $i = 1, 2$. In another words, for such λ_1 and λ_2, there exists $\{\Omega_j\} \subset \mathcal{F}_{d_\lambda}(\sigma, N)$ such that $\chi_\Omega \le \sum \chi_{\Omega_j} \le N\chi_\Omega$ and $1 \le r_{d_\lambda}(\Omega_j) \le N$.

In particular, when $H_1 \times H_2 = \mathbb{R}^n \times \mathbb{R}^m$, we will define

$$d_\lambda((x, y), (u, v)) = \max\{|x_i - u_i|/\lambda_1, |y_j - v_j|/\lambda_2 : 1 \le i \le n, 1 \le j \le m\}$$

for any $(u, v), (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Clearly, now d_λ arises from a norm. Note that now the metric balls (with respect to d_λ) are just parallelepipeds $I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$, (that is, I and J are cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively) with $r(I)/r(J) = \lambda_1/\lambda_2$.

In this paper, we will work on these domains, which include products of generalized John domains. Indeed, if $\Omega_i \in \mathcal{F}'(\sigma, N, \varepsilon_i)$ and $\Omega_i \subset H_i$ for $i = 1, 2$, then $\Omega_1 \times \Omega_2 \in \mathcal{F}'(\sigma, N', C\varepsilon_1, C\varepsilon_2, H_1 \times H_2)$; see Propositions 2.21 and 2.23. Our main theorem is about weighted interpolation inequalities on such domains.

Theorem 1.10. *Let $\langle H_1, d_1\rangle$ and $\langle H_2, d_2\rangle$ be metric spaces. Let $0 \le a_i, b_i \le 1$ such that $a_i + b_i \le 1$ for $i = 0, 1, 2$. Let $A_0, A_1, A_2 > 0$. Let $1 < \sigma, N < \infty$, $0 < \varepsilon_1, \varepsilon_2 \le \infty$ and $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_1, \varepsilon_2, H_1 \times H_2)$. Suppose $1 \le r_0, r_1, r_2 \le q < \infty$. Let $\delta = 1/\sigma$. Let μ, v_0, v_1, v_2 be measures, let $dv_0 = \tilde{v}_0 d\mu$, let w be a product δ-doubling measure on Ω in $H_1 \times H_2$, and let*

$$A_0(R) = w(R)^{1/q}\mu(R)^{-1}\|\tilde{v}_0^{-1/r_0}\|_{L^{r_0'}_\mu(R)}.$$

(a) If f, g_1 and g_2 are measurable functions on Ω such that

$$\|f - f_{R,\mu}\|_{L^q_w(R)} \le A_1(R)r(I)\|g_1\|_{L^{r_1}_{v_1}(R)} + A_2(R)r(J)\|g_2\|_{L^{r_2}_{v_2}(R)}, \tag{1-24}$$

$$A_i(R) \le A_i r(I)^{-a_i} r(J)^{-b_i} \quad \text{for } i = 0, 1, 2, \tag{1-25}$$

for all products $R = I \times J \subset H_1 \times H_2$ of balls I and J such that $\sigma R \subset \Omega$ with $r(I) < N^2\varepsilon_1$ and $r(J) < N^2\varepsilon_2$, then

$$\begin{aligned}\|f\|_{L^q_w(\Omega)} &\le C(q, D_w, N, \sigma)\\ &\times \big(A_0\lambda_1^{-a_0}\lambda_2^{-b_0}\|f\|_{L^{r_0}_{v_0}(\Omega)} + A_1\lambda_1^{1-a_1}\lambda_2^{-b_1}\|g_1\|_{L^{r_1}_{v_1}(\Omega)} + A_2\lambda_1^{-a_2}\lambda_2^{1-b_2}\|g_2\|_{L^{r_2}_{v_2}(\Omega)}\big)\end{aligned} \tag{1-26}$$

for all $\lambda_i \in (0, \varepsilon_i)$ for $i = 1, 2$.

(b) Suppose also that $H_1 \times H_2 = \mathbb{R}^n \times \mathbb{R}^m$, that μ is a product δ-doubling measure on Ω in $\mathbb{R}^n \times \mathbb{R}^m$, and that $d\mu = d\mu_1 \times d\mu_2$, where μ_1 and μ_2 are measures on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Assume

$$\|g - g_{I,\mu_1}\|_{L^1_{\mu_1}(I)} \le c_2 r(I)\|\nabla g\|_{L^1_{\mu_1}(I)} \quad \text{where } c_2 \ge 1 \tag{1-27}$$

for all cubes $I \subset \mathbb{R}^n$ and $g \in C^{0,1}(I)$, that is, g is Lipschitz continuous on I. If the inequalities (1-25),

$$\|f - f_{R,\mu}\|_{L^q_w(R)} \le \textit{right side}, \tag{1-28}$$

$$\frac{w(R)^{1/q}}{\mu(R)}\|f - f_{R,\mu}\|_{L^1_\mu(R)} \le \textit{right side}, \tag{1-29}$$

where

$$\textit{right side} = A_1(R)r(I)\|\nabla_x f\|_{L^{r_1}_{v_1}(R)} + A_2(R)r(J)\|\nabla_y f\|_{L^{r_2}_{v_2}(R)},$$

for all $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$ and parallelepipeds $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $\sigma R \subset \Omega$, with $r(I) < N^2\varepsilon_1$ and $r(J) < N^2\varepsilon_2$, then

$$\begin{aligned}\|\nabla_x f\|_{L^q_w(\Omega)} &\le C(m, n, D_w, D_\mu, N, q, \sigma)\\ &\quad\times c_2\big(A_1\lambda_1^{1-a_1}\lambda_2^{-b_1}\|\nabla_x^2 f\|_{L^{r_1}_{v_1}(\Omega)}\\ &\qquad + A_2\lambda_1^{-a_2}\lambda_2^{1-b_2}\|\nabla_{xy} f\|_{L^{r_2}_{v_2}(\Omega)} + A_0\lambda_1^{-1-a_0}\lambda_2^{-b_0}\|f\|_{L^{r_0}_{v_0}(\Omega)}\big)\end{aligned} \tag{1-30}$$

for all $\lambda_i \in (0, \varepsilon_i)$, $i = 1, 2$, and $f \in C^{1,1}_{\mathrm{loc}}(\Omega)$. Of course, a similar inequality holds for $\|\nabla_y f\|_{L^q_w(\Omega)}$, in which the roles of x and y are interchanged under similar assumptions.

Remark 1.11. (1) If $a_0 = a_2 = 0$, then $\|f\|_{L^q_w(\Omega)}$ is independent of $\|g_1\|_{L^{r_1}_{v_1}(\Omega)}$ in (1-26).

(2) If $b_1 = b_0 = 0$, then $\|f\|_{L^q_w(\Omega)}$ and $\|\nabla_x f\|_{L^q_w(\Omega)}$ are independent of $\|g_2\|_{L^{r_2}_{v_2}(\Omega)}$ and $\|\nabla_{xy} f\|_{L^{r_2}_{v_2}(\Omega)}$, respectively (in (1-26) and (1-30)).

(3) Again, (1-26) or (1-30) are equivalent to interpolation inequalities of product form when $a_i = a$ and $b_i = b$ for $i = 0, 1, 2$; for example, (1-26) is equivalent to

$$\begin{aligned}\|f\|_{L^q_w(\Omega)} &\le C(A_0\|f\|_{L^{r_0}_{v_0}(\Omega)})^{1-a-b}\big(A_0\varepsilon_1^{-1}\|f\|_{L^{r_0}_{v_0}(\Omega)} + A_1\|g_1\|_{L^{r_1}_{v_1}(\Omega)}\big)^a\\ &\qquad\times\big(A_0\varepsilon_2^{-1}\|f\|_{L^{r_0}_{v_0}(\Omega)} + A_2\|g_2\|_{L^{r_2}_{v_2}(\Omega)}\big)^b.\end{aligned} \tag{1-31}$$

To see this, note that by the fact that geometric mean is less than arithmetic mean, the right side of the above is less than

$$\begin{aligned}&C\big(\lambda_1^{-a}\lambda_2^{-b}A_0\|f\|_{L^{r_0}_{v_0}(\Omega)} + \lambda_1^{1-a}\lambda_2^{-b}(A_0\varepsilon_1^{-1}\|f\|_{L^{r_0}_{v_0}(\Omega)} + A_1\|g_1\|_{L^{r_1}_{v_1}(\Omega)})\\ &\qquad\qquad + \lambda_1^{-a}\lambda_2^{1-b}(A_0\varepsilon_2^{-1}\|f\|_{L^{r_0}_{v_0}(\Omega)} + A_2\|g_2\|_{L^{r_2}_{v_2}(\Omega)})\big)\\ &\quad = C\big((\lambda_1^{-a}\lambda_2^{-b} + \lambda_1^{1-a}\lambda_2^{-b}\varepsilon_1^{-1} + \lambda_1^{-a}\lambda_2^{1-b}\varepsilon_2^{-1})A_0\|f\|_{L^{r_0}_{v_0}(\Omega)}\\ &\qquad\qquad + \lambda_1^{1-a}\lambda_2^{-b}A_1\|g_1\|_{L^{r_1}_{v_1}(\Omega)} + \lambda_1^{-a}\lambda_2^{1-b}A_2\|g_2\|_{L^{r_2}_{v_2}(\Omega)}\big).\end{aligned}$$

It is now easy to see that (1-26) holds. Conversely, it suffices to see that (when $a_i = a$, $b_i = b$ for $i = 0, 1, 2$) the right side of (1-26) is less than the above. Then (1-31) can then be obtained with

$$\lambda_i = \left(\frac{A_0\|f\|_{L^{r_0}_{v_0}(\Omega)}}{\varepsilon_i^{-1}A_0\|f\|_{L^{r_0}_{v_0}(\Omega)} + A_i\|g_i\|_{L^{r_i}_{v_i}(\Omega)}}\right)^{1/2} \quad \text{for } i = 1, 2.$$

(4) If either $\varepsilon_1 = \infty$ or $\varepsilon_2 = \infty$, all we need to do is to let $\varepsilon_1 \to \infty$ or $\varepsilon_2 \to \infty$ in (1-31).

(5) If $a_i = a,\ b_i = b,\ A_i = A_0,\ v_i = v$ and $r_i = r$ for $i = 0, 1, 2$, it is necessary that

$$w(R)^{1/q}v(R)^{-1/r} \le CA_0 r(I)^{-a} r(J)^{-b} \tag{1-32}$$

in order for (1-26) to hold for all $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$ with $g_1 = |\nabla_x f|$ and $g_2 = |\nabla_y f|$. In fact, for any parallelepiped $R = I \times J \subset \sigma R \subset \Omega$ such that $r(I) < N^2\varepsilon_1$ and $r(J) < N^2\varepsilon_2$, by choosing an appropriate function similar to Remark 1.8(5), we have by (1-31) (using the fact that w is δ-doubling),

$$\begin{aligned} w(R)^{1/q} &\le C(A_0 v(R)^{1/r})^{1-a-b}\big(A_0\varepsilon_1^{-1}v(R)^{1/r} + CA_0 r(I)^{-1}v(R)^{1/r}\big)^a \\ &\qquad\qquad\qquad \times \big(A_0\varepsilon_2^{-1}v(R)^{1/r} + CA_0 r(J)^{-1}v(R)^{1/r}\big)^b \\ &= CA_0 v(R)^{1/r}\left(\frac{1+\varepsilon_1 r(I)^{-1}}{\varepsilon_1}\right)^a\left(\frac{1+\varepsilon_2 r(J)^{-1}}{\varepsilon_2}\right)^b \\ &\le CA_0 v(R)^{1/r} r(I)^{-a} r(J)^{-b}, \end{aligned}$$

since $\varepsilon_2 r(J)^{-1}$ and $\varepsilon_1 r(I)^{-1} \ge 1/N^2$. It is now clear that we have (1-32).

(6) Some necessary conditions for the Sobolev inequalities in product spaces have been obtained in [Shi and Torchinsky 1993; Fefferman and Stein 1982; Lu and Wheeden 1998; Chua 1999]; see also Proposition 2.16.

2. Preliminaries

First, let us state a useful lemma on polynomials. Its proof is a simple modification of that of [Strömberg and Torchinsky 1989, Chapter 3, Lemma 7].

Lemma 2.1. *Let Ω be a closed convex set in $\mathbb{R}^n$. If p is a polynomial of degree k such that*

$$M = |p(x_0)| = \max_{x\in\Omega}|p(x)|,\ x_0 \in \Omega,$$

then

$$|p(x_0 + t(x - x_0))| \ge \frac{M}{2^k} \text{ for all } 0 \le t \le 1/(2k)^k \text{ and } x \in \Omega. \tag{2-1}$$

We will write $\Omega^{x_0} = \{x_0 + t(x - x_0) : 0 \le t \le 1/(2k)^k,\ x \in \Omega\}$. *Clearly*, $\Omega^{x_0} \subset \Omega$.

Remark 2.2. If d is a metric arising from a norm in $\mathbb{R}^n$ and $\Omega = \overline{B(x_1, r)}$ with $x_0 \in \partial B(x_1, r)$, then

$$B(z_0, \frac{r}{(2k)^k}) \subset \Omega^{x_0} \quad \text{where } z_0 = x_0 + \frac{r}{(2k)^k}(x_1 - x_0).$$

Note that $B(x_1, r) \subset B(z_0, ((2(2k)^k - 1)/(2k)^k)r) \subset 2(2k)^k B(z_0, r/(2k)^k)$.

It is now easy to prove the following proposition. From now on, in Euclidean space, we will only be interested in those metrics arising from norms.

Proposition 2.3. *Let* $k \in \mathbb{N}$, $p_0 > 0$ *and a polynomial* p *of degree less than* k. *Let* $B = B(x, r)$ *be a norm ball in* $\mathbb{R}^n$. *If* $\mu(B) \le C_\mu \mu(B')$ *for any ball* $B' \subset B$ *such that* $B \subset 4(2k)^k B'$, *then*

$$\|p\|_{L^\infty(B)} \le 2^k \left(\frac{C_\mu}{\mu(B)}\right)^{1/p_0} \|p\|_{L^{p_0}_\mu(B)}. \tag{2-2}$$

Proof. It suffices to see that if $|p(x_0)| = \|p\|_{L^\infty(B)}$, $x_0 \in \overline{B}$, then there exists $B' = B(x_1, r_1) \subset B \subset 2B(x_1, r_1)$ with $x_0 \in \partial B'$. By the previous remark, we know it contains a ball B_0 such that $B' \subset 2(2k)^k B_0$ with $|p| \ge |p(x_0)|/2^k$ on B_0.

Remark 2.4. It follows that if μ is a δ-doubling measure on Ω, p is a polynomial of degree $\le k$, and B is a δ-ball, then

$$\|p\|_{L^\infty(B)} \le \frac{2^k C_\mu}{\mu(B)} \|p\|_{L^1_\mu(B)}, \tag{2-3}$$

where C_μ is the constant in the previous proposition. Clearly, C_μ depends only on the doubling constant D_μ and k.

Also, we have the following simple property about polynomials.

Lemma 2.5 [Chua 1992, Theorem 2.2]. *Let* $\gamma > 0$. *Let* B *be a metric ball in Euclidean space with the metric arising from a norm and let* E *be a* (*Lebesgue*) *measurable set in* B *with* $| E |> \gamma \mid B \mid$. *If* p *is a polynomial of degree* $\le k$, *then*

$$\| p \|_{L^\infty(E)} \ge C \| p \|_{L^\infty(B)},$$

where C *depends only on* γ, k *and the choice of norm. Moreover*, *if we consider only metric of the form* d_λ *as in Definition 1.9*, *then indeed* C *is also independent of* $\lambda = (\lambda_1, \lambda_2)$.

Note that though [Chua 1992, Theorem 2.2] is only proved for cubes, it is easy to see that the proof also works for norm balls. Moreover, if we consider only metric of the form d_λ as in Definition 1.9, then indeed C is also independent of $\lambda = (\lambda_1, \lambda_2)$ as there is a simple one-to-one linear transformation between their respective unit norm balls with center at the origin.

Now, let us state Markov's inequality; see for example [Bos and Milman 1993, Theorem 1.1] and [Bos and Milman 1995].

Proposition 2.6. *Let p be any polynomial in $\mathbb{R}^n$ of degree less than k. Then*

$$\|D^\alpha p\|_{L^\infty(Q)} \le k^{2|\alpha|} r(Q)^{-|\alpha|} \|p\|_{L^\infty(Q)} \quad \textit{for all cubes } Q \textit{ in } \mathbb{R}^n.$$

Our next theorem concerns the projection of functions into polynomials and is just an extension of [Chua 2005, Proposition 2.4] to δ-doubling measure.

Proposition 2.7. *Let d be a metric arising from a norm on $\mathbb{R}^n$. Let $0 < \delta < 1$. For any $k \in \mathbb{N}$, δ-ball $B \subset \Omega \subset \mathbb{R}^n$ and δ-doubling measure μ on Ω, there exists a projection $\pi_k^\mu(B) : L^1_\mu(B) \to \mathcal{P}_{k-1}$ (space of polynomials of degree $< k$) such that*

$$\|\pi_k^\mu(B) f\|_{L^\infty(B)} \le \frac{C(k, n, D_\mu)}{\mu(B)} \|f\|_{L^1_\mu(B)}.$$

Proof. First note that $\mathcal{P}_{k-1}$ is a finite-dimensional vector space over $\mathbb{R}^n$ and that $\int_B p_1 p_2 \, d\mu$ defines an inner product on $\mathcal{P}_{k-1}$. There is an orthonormal basis $\{\varphi_1, \varphi_2, \dots, \varphi_m\} \subset \mathcal{P}_{k-1}$ with respect to this inner product. Then $\|\varphi_i\|_{L^2_\mu(B)} = 1$ and

$$p(x) = \sum_{i=1}^m \varphi_i(x) \int_B p(y)\varphi_i(y) d\mu \quad \text{if } p \in \mathcal{P}_{k-1}.$$

We now define

$$\pi_k^\mu(B) f(x) = \sum_{i=1}^m \varphi_i(x) \int_B f(y)\varphi_i(y) d\mu \quad \text{for } f \in L^1_\mu(B).$$

It is clear that $\pi_k^\mu(B)$ is a projection to $\mathcal{P}_{k-1}$. Next, by (2-3) and Hölder's inequality, we have

$$\|\varphi_i\|_{L^\infty(B)} \le \frac{2^k C_\mu}{\mu(B)^{1/2}} \|\varphi_i\|_{L^2_\mu(B)} = 2^k C_\mu / \mu(B)^{1/2}.$$

It is now clear that

$$\|\pi_k^\mu(B) f\|_{L^\infty(B)} \le \sum_{i=1}^m \|\varphi_i\|_{L^\infty(B)} \|\varphi_i\|_{L^\infty(B)} \|f\|_{L^1_\mu(B)} \le \frac{m(2^k C_\mu)^2}{\mu(B)} \|f\|_{L^1_\mu(B)}. \quad \square$$

The next theorem is slightly different from [Wheeden 1993, page 269].

Proposition 2.8. *Let $0 < \delta < 1$ and $N \ge 1$. Let $\Omega \subset H$ such that Ω satisfies the nonempty annuli property. If μ is a δ-doubling measure on Ω, then there exist $C_1, C_2, D_1, D_2 > 0$ depending only on N and the doubling constant D_μ of μ, such that*

$$C_1 (r(B)/r(\tilde{B}))^{D_1} \le \mu(B)/\mu(\tilde{B}) \le C_2 (r(B)/r(\tilde{B}))^{D_2} \tag{2-4}$$

for all δ-balls B and $\tilde{B}$ such that $B \subset N\tilde{B}$. If μ is doubling on Ω, then (2-4) *will also hold for all balls B and $\tilde{B}$ with centers in Ω such that $B \subset N\tilde{B}$ and* $r(B), r(\tilde{B}) \le \operatorname{diam}(\Omega)$.

Proof. We will only prove the first part, that is, the case when μ is δ-doubling. We use a simple modification of argument used in [Wheeden 1993]. Since the left inequality follows immediately from standard argument, we will only show the right one. We first show that there exists a $\beta > 1$ such that $\mu(B) \ge \beta\mu(B/4)$ for all δ-balls B. Suppose $x \in \Omega$ and $s > 0$. Assume that $B(x, 4s)$ is a δ-ball. By the nonempty annuli property, there exists $y \in \Omega$ such that $2s \le d(x, y) < 3s$. Thus if $z \in B(y, s)$, we have

$$2s \le d(x, y) \le d(x, z) + d(z, y) < d(x, z) + s.$$

Hence $d(x, z) > s$, and clearly $B(y, s) \cap B(x, s) = \varnothing$. Also, for $z \in B(y, s)$,

$$d(x, z) \le d(x, y) + d(z, y) < 3s + s = 4s.$$

Hence $B(y, s) \subset B(x, 4s)$. Now, since μ is a measure, we know

$$\mu(B(x, 4s)) \ge \mu(B(y, s)) + \mu(B(x, s)) \tag{2-5}$$

since the two balls are disjoint and inside $B(x, 4s)$. Also, $B(x, 4s) \subset B(y, 7s)$. Using the δ-doubling property of μ on Ω, we have

$$\mu(B(x, 4s)) \le \mu(B(y, 7s)) \le (D_\mu)^3\mu(B(y, s)) = (1/\eta)\mu(B(y, s)).$$

Hence by (2-5), we have

$$\mu(B(x, 4s)) \ge \eta\mu(B(x, 4s)) + \mu(B(x, s)).$$

Thus

$$\mu(B(x, 4s)) \ge (1 - \eta)^{-1}\mu(B(x, s)) = \beta\mu(B(x, s)).$$

It is now easy to see that there exists $D_2 > 0$ such that

$$\mu(B(x, r))/\mu(B(x, \tilde{r})) \le C(r/\tilde{r})^{D_2} \quad \text{for } r \le \tilde{r} \text{ and } B(x, \tilde{r}) \text{ a } \delta\text{-ball},$$

where C is an absolute constant. Suppose now $B = B(x, r) \subset N\tilde{B} = B(y, N\tilde{r})$ and that B and $\tilde{B}$ are both δ-balls. Then $B(y, (N + 1)\tilde{r}) \supset B(x, \tilde{r})$ and

$$\mu(B(y, \tilde{r}))/\mu(B(x, r)) \ge C(N, D_\mu)\mu(B(x, \tilde{r}))/\mu(B(x, r)) \ge C_2(\tilde{r}/r)^{D_2}.$$

The first part of the proposition is now clear. □

Now, let us define the Hardy–Littlewood maximal function with respect to a doubling measure w on a given set Ω.

Definition 2.9. Let $\Omega \subset H$ and let f be a function on Ω. We define

$$M_w^\Omega f(x) = \sup \frac{1}{w(B)} \int_{B\cap\Omega} |f|\,dw \quad \text{for } x \in \Omega,$$

where the supremum is taken over all balls B with center x.

By a proof similar to the one used for the usual Hardy–Littlewood maximal function and the Vitali type covering lemma, it can be shown when w is doubling on Ω that

$$w\{x \in \Omega : M_w^\Omega f(x) > t\} \le (C(D_w)/t)\|f\|_{L_w^1(\Omega)} \quad \text{for all } t > 0.$$

On the other hand, it is obvious that $\|M_w^\Omega f\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)}$. By a standard interpolation argument, we see that

$$\|M_w^\Omega f\|_{L_w^p(\Omega)} \le C(D_w, p)\|f\|_{L_w^p(\Omega)} \quad \text{if } 1 < p < \infty.$$

The next lemma is similar to [Chua 1993, Lemma 2.5] and is an extension of [Iwaniec and Nolder 1985, Lemma 4] and [Bojarski 1988, Lemma 4.2].

Lemma 2.10. *Let $\Omega \subset H$, and let w be a doubling measure on Ω. Let $\{B_\alpha\}_{\alpha\in I}$ be an arbitrary family of balls with center in Ω and with radius less than* $\operatorname{diam}(\Omega)$. *If $\{a_\alpha\}_{\alpha\in I}$ is a family of nonnegative real numbers, then for $1 \le p < \infty$ and $N \ge 1$, we have*

$$\Big\|\sum_\alpha a_\alpha \chi_{NB_\alpha}\Big\|_{L_w^p(\Omega)} \le C(D_w, p, N)\Big\|\sum_\alpha a_\alpha \chi_{B_\alpha}\Big\|_{L_w^p(\Omega)}.$$

Sketch of the proof. We follow approach in [Chua 1993] (which in turn follows the approach in [Iwaniec and Nolder 1985]); however, in the case $1 < p < \infty$, we now make use of M_w^Ω instead of the usual weighted Hardy–Littlewood maximal functions. The case $p = 1$ follows immediately from the fact that w is doubling on Ω; hence $w(NB_\alpha) \le C(N, D_w)w(B_\alpha)$ since B_α are balls with center in Ω and with radius less than $\operatorname{diam}(\Omega)$. □

Retracing the proof of [Chua 1993, Theorem 1.5] using Lemma 2.10, inequality (2-3) and Lemma 2.5, we find that the following is true since δ-doubling measures on a John domain are also doubling on the domain [Chua and Wheeden 2008, Proposition 2.2]; see also [Chua and Wheeden 2008, Theorem 2.9].

Theorem 2.11. *Let $\sigma, N > 1$, $1 \le q < \infty$, $\delta = 1/\sigma$ and $k \in \mathbb{N}$. Let $\Omega \subset H$. Let $\Omega \in \mathcal{F}(\sigma, N)$, and let W be a corresponding Boman cover with center ball B^*. Let f be a function on Ω, and let w be a δ-doubling measure on Ω with doubling constant D_w. If there is a constant $a(f, B)$ associated to any $B \in W$, then*

$$\|f - a(f, B^*)\|_{L_w^q(\Omega)}^q \le C(q, D_w, N) \sum_{B\in W} \|f - a(f, B)\|_{L_w^q(B)}^q. \tag{2-6}$$

In case the metric space is $\mathbb{R}^n$ with any metric d arising from a norm and if there is a polynomial $P(f, B)$ of degree $\le k$ associated to any $B \in W$, then

$$\|f - P(f, B^*)\|^q_{L^q_w(\Omega)} \le C(n, k, q, D_w, N, d) \sum_{B \in W} \|f - P(f, B)\|^q_{L^q_w(B)}. \tag{2-7}$$

If the metric d is of the form d_λ as in Definition 1.9, then, just as in Lemma 2.5, the constant above is independent of $\lambda = (\lambda_1, \lambda_2)$.

Corollary 2.12. *Let $N, \tau > 1$ and $0 < \delta < 1$. Let $\Omega \subset H$, and let w be a δ-doubling measure on Ω. Suppose Ω satisfies the geodesic condition and the nonempty annuli property. If f and g are measurable functions on Ω such that*

$$\|f - f_{B,\mu}\|_{L^q_w(B)} \le A(B) \|g\|_{L^p_v(\tau B)} \quad \text{for all } \delta\text{-balls } B \text{ with } \tau B \subset \Omega \text{ and } r(B) < N^2 \varepsilon_0, \tag{2-8}$$

where $A(\tilde{B}) \le \bar{A} A(B)$ for all δ-balls B and $\tilde{B}$ with $\tilde{B} \subset B$, then

$$\|f - f_{B,\mu}\|_{L^q_w(B)} \le C(\tau, q, D_w) \bar{A} A(B) \|g\|_{L^p_v(B)} \quad \text{for all } \delta\text{-balls } B \text{ in } \Omega \text{ with } r(B) < N^2 \varepsilon_0\ . \tag{2-9}$$

Proof. Any ball $B \subset \Omega$ is a $J(1)$ domain. Hence $B \in \mathcal{F}(\tau, N_0)$ with some constant $N_0 = C(\tau) > 1$; see Proposition 2.21 and the remark after the proof.

For each δ-ball B such that $r(B) \le N^2 \varepsilon_0$, let W be a Boman cover of B. Then we have by Theorem 2.11 that

$$\begin{aligned} \|f - f_{B,\mu}\|_{L^q_w(B)} &\le C(D_w, \tau, q) \Big(\sum_{\tilde{B} \in W} \|f - f_{\tilde{B},\mu}\|^q_{L^q_w(\tilde{B})} \Big)^{1/q} \\ &\le C \Big(\sum_{\tilde{B} \in W} A(\tilde{B}) \|g\|^q_{L^p_w(\tau \tilde{B})} \Big)^{1/q} \le C(D_w, \tau, q) \bar{A} A(B) \|g\|_{L^p_w(B)} \end{aligned}$$

since $q \ge p$ and $\sum \chi_{\tau \tilde{B}} \le N_0 \chi_B$. $\square$

Next, a consequence of [Franchi et al. 2003, Theorems 1 and 2, Corollary 3]:

Proposition 2.13. *Let $1 < q < \infty$ and $c_0 \ge 1$. Let $0 < \delta < 1$. Let $\Omega \subset H$, and let Ω satisfy the geodesic condition and the nonempty annuli property. Let f be a measurable function defined on a δ-ball B_0 and let 'a' be a nonnegative set function on all balls B in B_0. Let μ and w be δ-doubling measures on Ω. Suppose that for any metric ball B in B_0,*

$$\frac{1}{\mu(B)} \|f - f_{B,\mu}\|_{L^1_\mu(B)} \le a(B) \tag{2-10}$$

such that there exists $0 < \theta < 1$ *with*

$$\sum_{B \in \mathscr{C}} (a(B)^q w(B))^\theta \leq (c_0 a(B_0)^q w(B_0))^\theta \tag{2-11}$$

for any collection $\mathscr{C}$ *of disjoint balls* B *in* B_0. *Then, for all* $t > 0$,

$$w\{x \in B_0 : |f(x) - f_{B_0,\mu}| > t\} \leq C(D_w, \theta) c_0 a(B_0)^q w(B_0)/t^q. \tag{2-12}$$

If (2-10) *and* (2-11) *hold with* $a(B) = b(B, f)$ *for all* $f \in \mathfrak{F}$ *with* $'b'$, *and* $\mathfrak{F}$ *satisfies* [Franchi et al. 2003, (H8)—(H13) on pages 524–525], *then we also have the strong-type inequality*

$$\|f - f_{B_0,\mu}\|_{L^q_w(B_0)} \leq C(D_w, \theta) c_0 b(B_0, f) \quad \textit{for all } f \in \mathfrak{F}. \tag{2-13}$$

Remark 2.14. (1) By Proposition 2.8, μ and w are reverse doubling (on δ-balls).

(2) Similar theorems of this form have been discussed in [Hajłasz and Koskela 1998; 2000; Franchi et al. 1998; Chua 2001]. A more extensive discussion can be found in [Chua and Wheeden 2008].

Proposition 2.15. *Let* $1 \leq p < \infty$ *and* $A_0 > 0$. *Let* $\Omega \subset H$. *Suppose* Ω *satisfies the geodesic condition and the nonempty annuli property. Let* $0 < \delta < 1$. *Let* f *and* g *be measurable functions on* Ω. *If* μ *and* υ *are* δ-*doubling measures on* Ω *such that*

$$\frac{1}{\mu(B)} \|f - f_{B,\mu}\|_{L^1_\mu(B)} \leq c_0 \frac{r(B)}{\upsilon(B)^{1/p}} \|g\|_{L^p_\upsilon(B)} \quad \textit{for all } \delta\textit{-balls } B,$$

then

$$\|f - f_{B,\mu}\|_{L^p_\upsilon(B)} \leq C(p, D_\upsilon) c_0 r(B) \|g\|_{L^p_\upsilon(B)} \quad \textit{for all } \delta\textit{-balls } B. \tag{2-14}$$

Proof. First, since υ is δ-doubling on Ω, by Proposition 2.8, there exist $k > 1$ and a constant C, both depending on p and D_υ, such that

$$\left(\frac{\upsilon(B)}{\upsilon(\tilde{B})}\right)^{1-1/k} \geq C(p, D_\upsilon) \left(\frac{r(B)}{r(\tilde{B})}\right)^p \quad \text{for all } \delta\text{-balls } B, \tilde{B} \text{ such that } B \subset \tilde{B}.$$

Let $a(B) = c_0 (r(B)/\upsilon(B)^{1/p}) \|g\|_{L^p_\upsilon(B)}$. If $\theta = 1/k$ and $q = kp$, then for any collection $\mathscr{C}$ of disjoint balls in the ball $\tilde{B}$, we have

$$\begin{aligned}
\sum_{B \in \mathscr{C}} (a(B)^q \upsilon(B))^\theta &= \sum_{B \in \mathscr{C}} c_0^p \frac{r(B)^p}{\upsilon(B)^{1-1/k}} \|g\|^p_{L^p_\upsilon(B)} \\
&\leq C(p, D_\upsilon) c_0^p \frac{r(\tilde{B})^p}{\upsilon(\tilde{B})^{1-1/k}} \sum_{B \in \mathscr{C}} \|g\|^p_{L^p_\upsilon(B)} \\
&\leq C(p, D_\upsilon) c_0^p a(\tilde{B})^p \upsilon(\tilde{B})^{1/k}.
\end{aligned}$$

Since $0 < p < kp = q$, it follows from Proposition 2.13 that

$$\|f - f_{B_0,\mu}\|_{L^p_v(B_0)} \le C(p, D_v)c_0 r(B_0)\|g\|_{L^p_v(B)}. \qquad \square$$

Next, we will prove some Sobolev inequalities on parallelepipeds.

Proposition 2.16. *Suppose $1 \le r_1, r_2 < q < \infty$. Let $0 < \delta < 1$ and $\sigma = 1/\delta$. Let w be a product δ-doubling measure on $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, and let v_1 and v_2 be nonnegative weights on $\mathbb{R}^{n+m}$. Let*

$$A_i(R) = w(R)^{1/q}\|v_i^{-1/r_i}\|_{L^{r_i'}(R)}|R|^{-1} \quad \textit{for } i = 1, 2.$$

If $A_i(R)r(I) \le \bar{A}A_i(\tilde{R})r(\tilde{I})$ for $i = 1, 2$ for all parallelepipeds $R = I \times J \subset \tilde{R} = \tilde{I} \times \tilde{J} \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $\sigma\tilde{R} \subset \Omega$ and $r(I)/r(\tilde{I}) = r(J)/r(\tilde{J})$, then for all parallelepipeds $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $\sigma R \subset \Omega$ and $f \in C^{0,1}(R)$,

$$\begin{aligned}(2\text{-}15)\quad &\|f - f_R\|_{L^q_w(R)} \\ &\le C(m, n, q, D_w)\bar{A}\big(A_1(R)r(I)\|\nabla_x f\|_{L^{r_1}_{v_1}(R)} + A_2(R)r(J)\|\nabla_y f\|_{L^{r_2}_{v_2}(R)}\big),\end{aligned}$$

where $f_R = \int_R f/|R|$.

In particular, if $1 \le r_1, r_2, p < \infty$ such that $1/p \ge 1/r_i - 1/(m+n)$ for $i = 1, 2$, then

$$\begin{aligned}(2\text{-}16)\quad &\|f - f_R\|_{L^p(R)} \\ &\le C(m, n, p)\big(r(I)|R|^{1/p-1/r_1}\|\nabla_x f\|_{L^{r_1}(R)} + r(J)|R|^{1/p-1/r_2}\|\nabla_y f\|_{L^{r_2}(R)}\big)\end{aligned}$$

holds for all $f \in C^{0,1}(R)$ and all parallelepipeds $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$.

Proof. This result is probably quite well known. However, as we are unable to find a suitable reference, we will give a quick sketch here.

If $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$ is a parallelepiped, then, noting that $r(I)$ and $r(J)$ are equal to half of the edge lengths of the cubes I and J, respectively, we have

$$\|f - f_R\|_{L^1(R)} \le \sqrt{n}r(I)\|\nabla_x f\|_{L^1(R)} + \sqrt{m}r(J)\|\nabla_y f\|_{L^1(R)};$$

see [Lu and Wheeden 1998, page 148] or [Chua and Wheeden 2006, Theorem 1.3]. Hence by Hölder's inequality, we have

$$\begin{aligned}(2\text{-}17)\quad \frac{1}{|R|}\|f - f_R\|_{L^1(R)} \le C(m, n)\big(&r(I)A_1(R)w(R)^{-1/q}\|\nabla_x f\|_{L^{r_1}_{v_1}(R)} \\ &+ r(J)A_2(R)w(R)^{-1/q}\|\nabla_y f\|_{L^{r_2}_{v_2}(R)}\big).\end{aligned}$$

For any fixed parallelepiped $R_0 = I_0 \times J_0 \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $\sigma R_0 \subset \Omega$, let $l(J_0)/l(I_0) = \eta$. Consider the metric d_λ with $\lambda = (1, \eta)$ as in Definition 1.9. Let $\mathscr{C}$ be any collection of disjoint metric balls (with respect to d_λ) in R_0. Then

$r(J)/r(I) = r(J_0)/r(I_0) = \eta$ for any $R = I \times J \in \mathscr{C}$. Let $r = \max\{r_1, r_2\}$. Check that

$$\Big(\sum_{R\in\mathscr{C}} w(R)^{-r/q}\big(r(I)A_1(R)\|\nabla_x f\|_{L^{r_1}_{v_1}(R)}+r(J)A_2(R)\|\nabla_y f\|_{L^{r_2}_{v_2}(R)}\big)^r w(R)^{r/q}\Big)^{1/r}$$
$$\leq \Big(\sum_R (r(I)A_1(R))^r \|\nabla_x f\|^r_{L^{r_1}_{v_1}(R)}\Big)^{1/r} + \Big(\sum_R (l(J)A_2(R))\|\nabla_y f\|^r_{L^{r_2}_{v_2}(R)}\Big)^{1/r}$$
$$\leq \bar{A}\big(r(I_0)A_1(R_0)\|\nabla_x f\|_{L^{r_1}_{v_1}(R_0)}+r(J)A_2(R_0)\|\nabla_y f\|_{L^{r_2}_{v_2}(R_0)}\big).$$

Let

$$b(R, f) = w(R)^{-1/q}\big(r(I)A_1(R)\|\nabla_x f\|_{L^{r_1}_{v_1}(R)} + r(J)A_2(R)\|\nabla_y f\|_{L^{r_2}_{v_2}(R)}\big),$$

and $\mathfrak{F} = C^{0,1}(R)$. It then follows from Proposition 2.13 that (2-15) holds since $b(R, f)$ and $C^{0,1}(R)$ satisfy [Franchi et al. 2003, (H8)–(H13)]. The constants involved are independent of the ratio $r(I_0)/r(J_0)$, since w is product δ-doubling. This proves the first part of the proposition.

Let $q > 1$ be such that $1/q = 1/r - 1/(m+n)$, where $r = \max\{r_1, r_2\}$. Clearly $q > r$. Let $w = v_1 = v_2 = 1$. It is then easy to check that $A_i(R)r(I) \leq A_i(\tilde{R})r(\tilde{I})$ for $i = 1, 2$ and for all parallelepipeds $R = I \times J \subset \tilde{R} = \tilde{I} \times \tilde{J} \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $r(I)/r(\tilde{I}) = r(J)/r(\tilde{J})$. Hölder's inequality completes the proof, since $q \geq p$. □

Remark 2.17. We are unsure whether the above can also be proved by using [Franchi et al. 1998, Theorem 3.1] when $w \in A_\infty$; see [Chua 1999, Theorem 2.4]. This is because the former's proof only mentions that the constant is independent of the ball and the function f, while we need our constant to be independent of the ratio η of the parallelepiped.

The following proposition gives a simple extension of facts from [Sawyer and Wheeden 1992, page 843], which concern the construction of a crude notion of dyadic cubes in a metric space.

Proposition 2.18. *Let $\langle H, d\rangle$ be a metric space. For each $k \in \mathbb{Z}$, there exists a collection of balls $\{B^k_i = B(x^k_i, 3^k)\}_i$ in H such that*

(1) *for each k, $H = \bigcup_i B^k_i$ and every ball of radius 3^{k-1} is inside at least one B^k_i;*

(2) *the balls $\hat{B}^k_i = B(x^k_i, 3^{k-1})$ are disjoint in i for each k, that is, for every k, $\hat{B}^k_j \cap \hat{B}^k_i = \varnothing$ if $i \neq j$; and*

(3) *if $0 < \delta < 1$ and there exists a δ-doubling measure on an open set $\Omega \subset H$, then the subcollection of $\{B^k_i = B(x^k_i, 3^k)\}_i$ consisting of δ-balls in Ω has bounded intercepts with bound depending only on the doubling constant of the measure.*

Proof. For (1) and (2), see [Sawyer and Wheeden 1992, pages 843–844], where these were proved for a homogeneous space (see also [Coifman and Weiss 1971]). For a doubling measure, (3) is proved in [Sawyer and Wheeden 1992, page 844] by using a standard volume argument; a similar argument works for a δ-doubling measure. □

Lemma 2.19. *Let $\Omega \subset H$, $0 < \delta < 1$ and $N > 1$. Suppose there is a δ-doubling measure μ on Ω. If $\mathcal{F} = \{B_i\}$ is a family of bounded (say K) intersecting δ-balls in Ω such that $\lambda^{-1} r(B_j) \le r(B_i) \le \lambda r(B_j)$ for all $B_i, B_j \in \mathcal{F}$ whenever the intersection $NB_i \cap NB_j$ is nonempty. Then the family $\{NB_i\}_{B_i \in \mathcal{F}}$ also has bounded intercepts with bound $C(\lambda, N, D_\mu)K$, where D_μ is the doubling constant of μ.*

Proof. Fix a ball B_0. First, $NB_i \cap NB_0 \neq \varnothing$ will imply $B_i \subset C_1 B_0 = C(\lambda, N)B_0$. However, by standard volume arguments, there can be at most $C_2 = C(\lambda, N, D_\mu)$ disjoint balls B_i with $r(B_i) \ge r(B_0)/\lambda$ in $C_1 B_0$ since $\mu(C_1 B_0) \le C(\lambda, N, D_\mu)\mu(B_i)$ for all $B_i \in C_1 B_0$ and $B_i \in \mathcal{F}$. It is now easy to see that there are at most $C_2 K$ balls B_i from $\mathcal{F}$ containing in $C_1 B_0$. Thus, the family $\{NB_i\}_{B_i \in \mathcal{F}}$ of balls must have bounded intercepts. □

Proposition 2.18 yields a special Whitney-type decomposition of an open set in a metric space. Coifman and Weiss [1971] obtained a similar statement on homogeneous space (assuming μ is doubling instead of δ-doubling).

Proposition 2.20. *Let Ω be an open set in a metric space $\langle H, d\rangle$ with a δ-doubling measure μ on Ω. Let $\tau \ge 1$ and $0 < \delta < 1/\tau$. Then Ω has a covering of δ-balls $W = \{B_\alpha\}$ such that*

(a) $r(B_\alpha) \le \delta d(B_\alpha) \le 3^2 r(B_\alpha)$;

(b) *for every $\tau \ge 1$ that satisfies $\tau\delta < 1$, there is a constant K, depending only on τ, δ and the doubling constant D_μ, such that each ball τB_i intersects at most K balls τB_j; and*

(c) *the family of balls $\widetilde{W} = \{B/3 : B \in W\}$ is also a cover of Ω.*

Proof. For each $k \in \mathbb{Z}$, there is a set of balls of radius 3^k as in Proposition 2.18. Let $\widetilde{W}$ be the collection of balls $B_i^k = B(x_i^k, 3^k)$ such that

$$3^{k+1} \le \delta d(B_i^k) < 3^{k+3}. \tag{2-18}$$

For any $x \in \Omega$, there exists a $k' \in \mathbb{Z}$ such that $3^{k'} \le \delta d(x) < 3^{k'+1}$. Let $k = k' - 2$. By Proposition 2.18, there exists i such that $B(x, 3^{k-1}) \subset B_i^k = B(x_i^k, 3^k)$. Moreover,

$$d(B_i^k) \le d(x) \le 3^{k'+1}/\delta = 3^{k+3}/\delta,$$
$$d(B_i^k) \ge d(x_i^k) - 3^k \ge d(x) - d(x, x_i^k) - 3^k \ge 3^{k+1}/\delta.$$

Thus $B_i^k \in \widetilde{W}$ and it is clear that it covers Ω.

Now let $W = \{B_\alpha = 3\tilde{B}_\alpha : \tilde{B}_\alpha \in \widetilde{W}\}$. Clearly B_α are δ-balls and (a) and (c) hold.

Next, we will show (b). By Proposition 2.18, $\widetilde{W}_k = \{\tilde{B}_j \in \widetilde{W} : r(B_j) = 3^k\}$ has bounded intercepts for each k. It then follows from Lemma 2.19 that the set $\tau W_k = \{3\tau\tilde{B}_j = \tau B_j : \tilde{B}_j \in \widetilde{W}_k\}$ also has bounded intercepts. Finally, note that if $\tau B_i \cap \tau B_j \neq \varnothing$ and $B_i, B_j \in W$, then by (1-4) and (a), we have $r(B_i) \sim r(B_j)$. It is then easy to see that (b) holds. □

We can now show that any open set that satisfies the geodesic condition is a Boman domain if and only if it is a John domain. Also, the notion of generalized John domains is the same as the Boman-type domain introduced in Definition 1.5.

Proposition 2.21. *Let* $0 < \delta < 1$. *Let* $\Omega \subset H$ *such that* Ω *satisfies the geodesic condition. Suppose* μ *is a* δ*-doubling measure on* Ω. *If* $\Omega \in J(c)$ *for* $0 < c < 1$, *then there exist* $\sigma, N > 1$ *such that* $\Omega \in \mathcal{F}(\sigma, N)$, *where* N *depends only on* c *and* D_μ. *Conversely, if* $\Omega \in \mathcal{F}(\sigma, N)$, *then* $\Omega \in J(c)$ *with* c *depending only on* N *and* D_μ. *If* $\Omega \in J(c, M_0)$ *for* $M_0 > 0$, *then* $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_0)$ *with* ε_0 *depending only on* M_0. *Finally, if* $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_0)$, *then* $\Omega \in J(c, M_0)$, *where* $M_0 = 4\varepsilon_0$ *and* $c = C(N, D_\mu)$.

Proof. The first part of the proof follows the proof in [Buckley et al. 1996]. First, by Proposition 2.20, Ω has a Whitney-type decomposition W. For convenience, we will take $\delta = 1/2$ in Proposition 2.20, though of course we could also choose $\delta < 1/2$. By Definition 1.3, we know for $B_1, B_2 \in W$ that if B_1 intersects B_2, then $d(B_1) \sim d(B_2)$ and hence $r(B_1) \sim r(B_2)$ by Proposition 2.20(a). If $\Omega \in J(c)$, it is easy to see that this family W of δ-balls provides a Boman covering of Ω and hence $\Omega \in \mathcal{F}(\sigma, N)$, with $\sigma = 2$ and N depending only on the doubling constant D_μ and c. We will choose a ball that contains the center x', the *center ball* B^*. Any Boman chain (connecting B to B^*) can be just taken to be an appropriate subfamily of balls (in W) along the curve connecting the center of B to $x' \in B^*$ (this curve exists by the c-John condition). By balls along the curve, we mean balls $B \in W$ such that $B/3$ intersects the curve.

Next, suppose $\Omega \in \mathcal{F}(\sigma, N)$ and Ω satisfies the geodesic condition. Then, similar to the proof in [Buckley et al. 1996], it suffices to find a number K such that if $\{B_0, B_1, \ldots, B_N = B'\}$ is a Boman chain with $N \geq K$, then there exists a B_j with $r(B_j) \geq 2r(B_0)$. To find such a constant K, note that $r(B_i) \geq r(B_0)/N$ because $B_0 \subset NB_i$ and $\{B_i\}$ has bounded intercepts with bound depending on the doubling constant. Suppose there are M_0 disjoint balls B_i with $r(B_i) < 2r(B_0)$ and $B_0 \subset NB_i$. Then $B_i \subset C_1(N)B_0$. However, by δ-doubling,

$$\mu(B_i) \geq C(N, D_\mu)\mu(B_0) \quad \text{and} \quad \mu(C_1(N)B_0) \leq C(N, D_\mu)\mu(B_0).$$

It is then clear that $M_0 \le C(N, D_\mu)$, and hence K depends only on N and D_μ. Thus, by the argument of [Buckley et al. 1996], we know that $\Omega \in J(c)$ with x_{B^*} the center of B^* and c depending only on D_μ and N. It is now easy to see that if $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_0)$, then $\Omega \in J(c, M_0)$ with $M_0 = 4\varepsilon_0$ and c depending only on D_μ and N. Indeed, for any $M < 4\varepsilon_0$ and $d(z) < M$ with $z \in \Omega$, we have $z \in \Omega_j \subset \Omega$ with $\Omega_j \in \mathcal{F}(\sigma, N)$ and $M/4 \le r(\Omega_j) \le NM/4$. We know from the above that $\Omega_j \in J(c)$ (here c depends only on N and D_μ) with x_{B_j} the center of the center ball B_j of Ω_j. Hence $d(x_{B_j}, \Omega_j^c) = d(B_j, \Omega_j^c) + r(B_j) \ge 4r(B_j) \ge M$. Now, since $\Omega_j \in J(c)$ for all $x \in \Omega_j$, there exists $\gamma : [0, l] \to \Omega_j$ satisfying (1-2). It is then clear that $\Omega \in J(c, M_0)$ since $d(\gamma(t), \Omega^c) \ge d(\gamma(t), \Omega_j^c)$.

Conversely, suppose $\Omega \in J(c, M_0)$. For each $M < M_0$, we will show that Ω can be decomposed into union of bounded overlapping $J(c/3)$ domains that contain a center ball of size about M. Let us fix a set W of δ-balls with $\delta = 1/2$ as in Proposition 2.20. We will now consider two types of balls $B \in W$ with $d(B) > M/2$.

Type 1: $W_1 = \{B \in W : d(B) > 2M\}$. For any fixed ball $B \in W_1$, we will find a finite number of disjoint balls $\{Q_i(B)\}_{i \in I(B)}$ with center in B and of radius $M/3$, such that $B \subset \bigcup 3Q_i(B)$. Such family of balls exists since there is a δ-doubling measure on Ω. Each of these $3Q_i(B)$ is a δ-ball and the family $\{3Q_i(B)\}$ (for each fixed B) has bounded intercepts by standard volume argument, since there is a δ-doubling measure on Ω. Each of these balls $3Q_i(B)$ is clearly a domain in $J(1)$ by the assumed geodesic condition.

Type 2: $W_2 = \{B \in W : M/2 < d(B) \le 2M\}$. For any $B \in W_2$ and $0 < \tau \le c$, let $\Omega_B = \Omega_B(\tau)$ be the set of $y \in \Omega$ such that there exists $\gamma : [0, l] \to \Omega$ with $\gamma(0) = y$ and $\gamma(l) = x_B$ (the center of B) and such that

$$d(\gamma(t_1), \gamma(t_2)) \le |t_1 - t_2| \quad \text{and} \quad d(\gamma(t)) > \tau t \quad \text{for all } t, t_1, t_2 \in [0, l]. \tag{2-19}$$

Since $d(x_B) \le d(B) + r(B) \le Cr(B)$, it is clear that $\Omega_B(\tau) \subset C(\tau)B$. We will now check that $\Omega_B = \Omega_B(\tau)$ is a $J(\tau)$ domain.

It is easy to see that Ω_B is open since Ω satisfies the geodesic condition. Hence it suffices to observe that if γ satisfies (2-19), then $\Gamma_\gamma \subset \Omega_B$, where

$$\Gamma_\gamma = \{z : d(z, \gamma(t)) \le \tau t,\ 0 \le t \le l\}.$$

So suppose $d(z, \gamma(t_0)) = t_1 \le \tau t_0$. Since Ω satisfies the geodesic condition, we can define $\gamma_z : [0, l - (t_0 - t_1)] \to \Omega$ by

$$\gamma_z(t) = \begin{cases} \text{a geodesic with } \gamma_z(0) = z \text{ and } \gamma_z(t_1) = \gamma(t_0) & \text{if } t \in [0, t_1], \\ \gamma(t + (t_0 - t_1)) & \text{if } t \in [t_1, l - (t_0 - t_1)]. \end{cases}$$

It is easy to see that γ_z satisfies (2-19). We will now fix $\tau = c/3$.

We will now show for any $x \in \Omega$ that either $x \in \Omega_B(c/3)$ for some $B \in W_2$ or $x \in B$ with $d(B) > M/2$. Note that if $d(x) \geq M$ and $B \in W$ contains x, then $d(B)+2r(B) > d(x) \geq M$ and hence $d(B) > M/2$. Thus, we only need to look at those $x \in \Omega$ with $d(x) < M$. By definition, we know for any $x \in \Omega$ with $d(x) < M$, there is a a curve $\gamma : [0, l] \to \Omega$ connecting x to x' that satisfies condition (2-19) with $\tau = c$ and $d(x') = M$. Let $x' \in B$ with $B \in W$. Then $B \in W_2$. It is clear that $x \in \Omega_B(c/3)$ if $x' = x_B$. If $x' \neq x_B$, then, since Ω satisfies the geodesic condition, x' can be connected to x_B by a geodesic in the ball B. We can then extend γ from $[0, l]$ to $[0, l + d(x_B, x')]$ by attaching the geodesic on $[l, l + d(x_B, x')]$. Recall that $cl < d(\gamma(l)) = M$ and $r(B) \leq d(B)/2$, and hence for $t \in [l, l+d(x_B, x')]$, we have

$$d(\gamma(t)) \geq d(B) = \tfrac{2}{3}d(B) + \tfrac{1}{3}d(B) > \tfrac{1}{3}M + \tfrac{1}{3}d(B) \geq c\tfrac{1}{3}(l + 2r(B)) \geq \tfrac{1}{3}ct$$

since $d(x_B, x') < r(B)$. Thus, $x \in \Omega_B(\tau) = \Omega_B$ with $\tau = c/3$. Hence Ω is a union of domains in $\{\Omega_B : B \in W_2\}$ and $\{3Q_i(B) : i \in I(B), B \in W_1\}$.

It is clear that Ω_B or $3Q_i(B)$ has a center ball with radius more than CM and less than $C(c)M$. Moreover, it follows from the first part that each of these domains is in $\mathcal{F}(2, N)$, with N depending only on c.

Finally, it can be checked that they have bounded overlaps. First, let us show that $\mathcal{F}_1 = \{3Q_i(B) : i \in I_B, B \in W_1\}$ has bounded intercepts. We will first check that $\mathcal{F}_0 = \{Q_i(B) : i \in I_B, B \in W_1\}$ has bounded intercepts. Let us fix a ball $Q_0(B_1)$ in the family. Note that if $Q_i(B_j)$ intersects $Q_0(B_1)$, then $3Q_0(B_1) \supset Q_i(B_j)$ and $\frac{3}{2}B_j$ will intersect $\frac{3}{2}B_1$. But, by Proposition 2.20(b), there are at most $C_2 = C(D_\mu)$ balls in $\{\frac{3}{2}B_j : B \in W_1\}$ that intersect $\frac{3}{2}B_1$. Moreover, for each fixed $B_j \in W_1$, a ball $Q_0(B_1)$ intersects at most $C_3 = C(D_\mu)$ balls from $\{Q_k(B_j) : k \in I(B_j)\}$ since they are disjoint balls in $\frac{3}{2}B_j$ (again by the standard volume argument). It is now clear that $Q_0(B_1)$ will intersect at most $C_2C_3 = C(D_\mu)$ balls in the family $\mathcal{F}_0 = \{Q_i(B) : i \in I_B, B \in W_1\}$. It then follows from Lemma 2.19 that $\mathcal{F}_1$ has bounded intercepts (with bound $K_1 = C(D_\mu)$).

Next, note that $\Omega_B \subset C_4B = C(c)B$. Hence if Ω_{B_1} intersects Ω_{B_2}, then C_4B_2 intersects C_4B_1. Since the family W_2 has bounded intercepts, by Lemma 2.19, the family $\mathcal{F}_2 = \{C_4B_i : B_i \in W_1\}$ also has bounded intercepts. Thus, any fixed Ω_{B_1} intersects at most $K_2 = C(D_\mu, c)$ domains in $\{\Omega_{B_i}, B_i \in W_2\}$.

To complete the proof, we need to show that any Ω_{B_1} intersects a bounded number of sets $3Q_i(B_j)$ and any $3Q_0(B_2)$ intersects at most a bounded number of sets Ω_{B_i}. We will only prove the first part. By Lemma 2.19, it suffices to show that the family $\{B_1\} \cup \mathcal{F}_0$ has bounded intercepts since $r(B_1) \sim M$ and $\Omega_{B_1} \subset C_4B_1$. Since $\mathcal{F}_0$ is already known to have bounded intercepts, it suffices to show that B_1 intersects at most a bounded number of balls from $\mathcal{F}_0$. But B_1 intersects B_α in $\mathcal{F}_0$ provided $B_\alpha \subset C_5B_1 = C(D_\mu)B_1$. Hence there can be at most $C_6 = C(D_\mu)$ disjoint balls from $\mathscr{F}_0$ in C_5B_1 (again by the volume argument). Thus, B_1 can intersect at most $K_1C_6 = C(D_\mu)$ balls from $\mathscr{F}_0$. □

Remark 2.22. The above proof also shows that $\Omega \in J(c)$ implies $\Omega \in \mathscr{F}(\sigma, N)$ for any $\sigma \geq 2$ if we choose $\delta = 1/\sigma$. However, N will then depend also on σ.

We now show that products of generalized John domains are a special class of generalized John domains.

Proposition 2.23. *Let $\langle H_i, d_i\rangle$ be metric spaces for $i = 1, 2$. Let $\sigma, N > 1$ and $0 < \varepsilon_1, \varepsilon_2 \leq \infty$. Let $\Omega_i \subset H_i$ such that $\Omega_i \in \mathscr{F}'(\sigma, N, \varepsilon_i)$ for $i = 1, 2$. If both Ω_i satisfy the geodesic condition, then $\Omega_1 \times \Omega_2 \in \mathscr{F}'(\sigma, N', C\varepsilon_1, C\varepsilon_2, H_1 \times H_2)$.*

Proof. By the previous proposition, it suffices to show that if $\Omega_i \in J(c, \varepsilon_i)$ for $i = 1, 2$, then for any metric d_λ defined in Definition 1.9 with $\lambda = (\lambda_1, \lambda_2)$ and $0 < \lambda_i < \varepsilon_i$, we have $\Omega_1 \times \Omega_2 \in J(c, 1)$ with respect to d_λ. So let $(x_1, x_2) \in \Omega_1 \times \Omega_2$. Then since $\Omega_i \in J(c, \varepsilon_i)$, there exist $\gamma_i : [0, l_i] \to \Omega_i$ for $i = 1, 2$ such that

$$d_i(\gamma_i(t), \gamma_i(s)) \leq |t - s| \quad \text{and} \quad d_i(\gamma_i(t), \Omega_i^c) > ct \quad \text{for } t, s \in [0, l_i]$$

and $d(\gamma_i(l_i), \Omega_i^c) \geq \lambda_i$ for $i = 1, 2$. Without loss of generality, we may assume $l_1/\lambda_1 \leq l_2/\lambda_2$. Define

$$\gamma(t) = \begin{cases} (\gamma_1(\lambda_1 t), \gamma_2(\lambda_2 t)) & \text{if } t \leq l_1/\lambda_1, \\ (\gamma_1(l_1), \gamma_2(\lambda_2 t)) & \text{if } t \in [l_1/\lambda_1, l_2/\lambda_2]. \end{cases}$$

It is now easy to see that

$$d_\lambda(\gamma(t), \gamma(s)) \leq |t - s| \quad \text{and} \quad d_\lambda(\gamma(t), (\Omega_1 \times \Omega_2)^c) > ct \quad \text{for all } t, s \in [0, l_2/\lambda_2].$$

Moreover, it is clear that $d_\lambda(\gamma(l_2/\lambda_2)) \geq 1$. □

Finally, let us show that if a point is removed from a generalized John domain, then it is still a generalized John domain under an additional, mild assumption.

Proposition 2.24. *Let $z \in \Omega \subset H$ and suppose $\Omega \in J(c, M_0)$, where $0 < c < 1$ and $0 < M_0 \leq \infty$. Suppose Ω satisfies the geodesic condition and the nonempty annuli property. Suppose there exists a c_0 satisfying $0 < c_0 < 1$ and the mild property that*

- *for any two points $x_1, x_2 \in B(z, d(z)/2) \subset \Omega$ such that $d(x_1, z) \leq d(x_2, z)$, there exists $\eta : [0, l'] \to B(z, d(z)/2)$ such that $d(\eta(t), z)$ is nondecreasing, $\eta(0) = x_1$, $\eta(l') = x_2$, and*

$$c_0|t_1 - t_2| \leq d(\eta(t_1), \eta(t_2)) \leq |t_1 - t_2| \quad \textit{for all } t_1, t_2 \in [0, l'].$$

Then $\Omega \setminus \{z\} \in J(cc_0/3, M_0/3)$.

Proof. For convenience, let $d(z)=\inf\{d(x,z):x\in\Omega^c\}=2\varepsilon$. Suppose $0<M<M_0$. Let us consider two cases.

Case (i): $2\varepsilon\geq 2M/3$. For any $x\in\Omega\setminus\{z\}$, if $x\in B(z,\varepsilon)$, we need only connect x to any point of the boundary of $B(z,\varepsilon)$ using the assumed η. Note that the boundary of $B(z,\varepsilon)$ is nonempty by the geodesic condition and the nonempty annuli property. Thus, we may assume $x\notin B(z,\varepsilon)$ and we only need to consider the case $d(x)<M/3$.

By definition, there exists $\gamma:[0,l]\to\Omega$ such that $\gamma(0)=x$ and $d(\gamma(l))=M$, and

$$d(\gamma(t_1),\gamma(t_2))\leq|t_1-t_2|\ \text{ and }\ d(\gamma(t))>ct\quad\text{for all } t,t_1,t_2\in[0,l].$$

If $\gamma[0,l]\cap B(z,\varepsilon)=\varnothing$, then $d(\gamma(t),z)\geq M/3>cc_0t/3$ for all $t\in[0,l]$, since $l\leq M/c$. Now suppose $\gamma[0,l]\cap B(z,\varepsilon)\neq\varnothing$. Then there exists a $t'\in[0,l]$ such that $d(\gamma(t'),z)=\varepsilon\geq M/3$, and we may assume $\gamma[0,t']\cap B(z,\varepsilon)=\varnothing$. It is then clear that $d(\gamma(t),z)\geq M/3>cc_0t/3$ for all $t\in[0,t']$. Note that $d(\gamma(t'))\geq M/3$.

Case (ii): Now suppose $2\varepsilon<2M/3$. Again, let $x\in\Omega\setminus\{z\}$ such that $d(x)<M/3$ or $d(x,z)<M/3$. In any case, $d(x)<M$. Hence, by definition, there exists $\gamma:[0,l]\to\Omega$ such that $\gamma(0)=x$ and $d(\gamma(l))=M$, and

$$d(\gamma(t_1),\gamma(t_2))\leq|t_1-t_2|\ \text{ and }\ d(\gamma(t))>ct\quad\text{for all } t,t_1,t_2\in[0,l].$$

We will now consider two subcases.

Subcase (a): $\gamma[0,l]\cap B(z,\varepsilon)=\varnothing$. Here it is clear that $d(\gamma(t),z)>ct/3$ for all t. Indeed,

$$\begin{aligned} d(\gamma(t),z)&\geq d(\gamma(t))-d(z)>ct-2\varepsilon\geq ct/3 &&\text{if } t\geq 3\varepsilon/c,\\ d(\gamma(t),z)&\geq\varepsilon>ct/3 &&\text{if } t<3\varepsilon/c.\end{aligned}$$

Note that $d(\gamma(l),z)\geq d(\gamma(l))-d(z)\geq M/3$. This proves this subcase.

Subcase (b): $\gamma[0,l]\cap B(z,\varepsilon)\neq\varnothing$. Let

$$t_0=\inf\{t\in[0,l]:\gamma(t)\in B(z,\varepsilon)\}\quad\text{and}\quad t_0'=\sup\{t\in[0,l]:\gamma(t)\in B(z,\varepsilon)\}.$$

Note that $d(\gamma(t_0'),z)=\varepsilon$ as $\gamma(l)\notin B(z,\varepsilon)$ because $d(\gamma(l))=M$. By assumption, there exists $\eta:[0,l']\to\Omega$ with $\eta(0)=\gamma(t_0)$ and $\eta(l')=\gamma(t_0')$ such that

$$c_0|t_1-t_2|\leq d(\eta(t_1),\eta(t_2))\leq|t_1-t_2|\quad\text{for all } t_1,t_2\in[0,l']$$

and $d(\eta(t),z)$ is nondecreasing. Note that $t_0+l'\leq t_0+(t_0'-t_0)/c_0\leq t_0'/c_0$ and

$$c_0t\leq d(\eta(t),\eta(0))\leq d(\eta(t),z)+d(\eta(0),z)\leq 2d(\eta(t),z),$$

and hence $d(\eta(t),z)\geq c_0t/2$.

We define

$$\bar{\gamma} : [0, \bar{l}] \to \Omega, \quad t \mapsto \begin{cases} \gamma(t) & \text{if } t \in [0, t_0], \\ \eta(t - t_0) & \text{if } t \in [t_0, t_0 + l'], \\ \gamma(t - l' + (t_0' - t_0)) & \text{if } t \in [t_0 + l', \bar{l}], \end{cases}$$

where $\bar{l} = l + l' - t_0' + t_0$.

First, for $t \in [0, t_0]$, similar to Subcase (a), it is easy to see that $d(\gamma(t), z)) > ct/3$.

Next, consider $t \in [t_0, t_0 + l']$. Note the following.

- $d(\bar{\gamma}(t)) \geq \varepsilon > cc_0(t_0 + l')/3 \geq cc_0 t/3$. This is because
$$t_0 + l' \leq t_0'/c_0 < \frac{d(\gamma(t_0'))}{cc_0} \leq \frac{d(\gamma(t_0'), z) + d(z)}{cc_0} \leq \frac{3\varepsilon}{cc_0}.$$
- $d(\bar{\gamma}(t), z) > cc_0 t/3$. If $t_0 = 0$ this is because $d(\bar{\gamma}(t), z) = d(\eta(t), z) \geq c_0 t/2$. On the other hand, if $t_0 > 0$, then
$$d(\bar{\gamma}(t), z) \geq d(\gamma(t_0), z) = \varepsilon > cc_0(t + l')/3 \geq cc_0 t/3.$$

Next, if $t \in [t_0 + l', \bar{l}]$, then since $c_0 l' \leq (t_0' - t_0)$ and $t \geq l'$, we have

$$d(\bar{\gamma}(t)) = d(\gamma(t - l' + t_0' - t_0)) > c(t - l' + t_0' - t_0) \geq c(t - (1 - c_0)l') \geq cc_0 t \geq cc_0 t/3.$$

If $t \geq 3\varepsilon/(cc_0)$, then

$$d(\bar{\gamma}(t), z) \geq d(\bar{\gamma}(t) - d(z) > c(t - l + t_0' - t_0) - 2\varepsilon \geq cc_0 t - 2\varepsilon \geq cc_0 t/3.$$

On the other hand, when $t < 3\varepsilon/(cc_0)$, it is clear that $d(\bar{\gamma}(t), z) \geq \varepsilon > cc_0 t/3$. Finally, note that $d(\bar{\gamma}(\bar{l}), z) \geq d(\bar{\gamma}(\bar{l})) - d(z) \geq M - 2\varepsilon > M/3$. □

Remark 2.25. If $\Omega \in J(c, M_0)$ for $\Omega \subset \mathbb{R}^n$, it follows from Proposition 2.24 that $\Omega \setminus \{z\} \in J(c'/3, M_0/3)$ with $c' = c/C$, where $C > 1$ is an absolute constant.

3. Proofs of the main results

Proof of Theorem 1.6. The proof of part (a) is just a simple modification of the proofs of [Chua 1995, Theorems 1.3 and 1.4].

Given any $\varepsilon \in (0, \varepsilon_0)$, there exists a set $\{\Omega_j\} \subset \mathcal{F}(\sigma, N)$ such that $\bigcup \Omega_j = \Omega$, $\sum \chi_{\Omega_j} \leq N$ and $\varepsilon \leq r(\Omega_j) \leq N\varepsilon$. For each Ω_j, let Q_j be the central ball in Ω_j, and let W_j be its corresponding Boman cover. Then by Theorem 2.11,

$$\begin{aligned} \|f - f_{Q_j,\mu}\|_{L^q_w(\Omega_j)} &\leq C(D_w, q, N)\Big(\sum_{B \in W_j} \|f - f_{B,\mu}\|^q_{L^q_w(B)}\Big)^{1/q} \\ &\leq C A' r(Q_j)^\alpha \Big(\sum \|g\|^q_{L^p_v(\tau B)}\Big)^{1/q} \quad \text{(by (1-6), since } \alpha \geq 0) \\ &\leq C(D_w, q, N) A' r(Q_j)^\alpha \|g\|_{L^p_v(\Omega_j)} \end{aligned}$$

since $q \geq p$ and $\sum_{B \in W_j} \chi_{\tau B} \leq N \chi_{\Omega_j}$. Hence, by the triangle inequality,

$$\begin{aligned}
\|f\|_{L^q_w(\Omega_j)} &\leq \|f_{Q_j,\mu}\|_{L^q_w(\Omega_j)} + \|f - f_{Q_j,\mu}\|_{L^q_w(\Omega_j)} \\
&\leq \|f_{Q_j,\mu}\|_{L^q_w(\Omega_j)} + C(q, D_w, N) A' r(Q_j)^\alpha \|g\|_{L^p_v(\Omega_j)} \\
&\leq \frac{w(\Omega_j)^{1/q}}{\mu(Q_j)} \int_{Q_j} |f| \tilde{v}_0^{1/r} \tilde{v}_0^{-1/r} \, d\mu + C A' r(\Omega_j)^\alpha \|g\|_{L^p_v(\Omega_j)} \\
&\leq C(D_w, q) \frac{w(Q_j)^{1/q}}{\mu(Q_j)} \|\tilde{v}_0^{-1/r}\|_{L^{r'}_\mu(Q_j)} \|f\|_{L^r_{v_0}(Q_j)} + C A' r(\Omega_j)^\alpha \|g\|_{L^p_v(\Omega_j)} \\
&\qquad\qquad \text{(by Hölder's inequality and the fact that } w \text{ is } \delta\text{-doubling)} \\
&\leq C A_0 r(Q_j)^{-\beta} \|f\|_{L^r_{v_0}(\Omega_j)} + C A' r(\Omega_j)^\alpha \|g\|_{L^p_v(\Omega_j)} \quad \text{(by (1-5))} \\
&\leq C(N, D_w, q) (A_0 \varepsilon^{-\beta} \|f\|_{L^r_{v_0}(\Omega_j)} + C A' \varepsilon^\alpha \|g\|_{L^p_v(\Omega_j)}).
\end{aligned}$$

Hence by the fact that $(\sum |a_j + b_j|^q)^{1/q} \leq (\sum |a_j|^q)^{1/q} + (\sum |b_j|^q)^{1/q}$,

$$\begin{aligned}
\Bigl(\sum \|f\|^q_{L^q_w(\Omega_j)}\Bigr)^{1/q} &\leq C A_0 \varepsilon^{-\beta} \Bigl(\sum \|f\|^q_{L^r_{v_0}(\Omega_j)}\Bigr)^{1/q} + C A' \varepsilon^\alpha \Bigl(\sum \|g\|^q_{L^p_v(\Omega_j)}\Bigr)^{1/q} \\
&\leq C A_0 \varepsilon^{-\beta} \Bigl(\sum \|f\|^r_{L^r_{v_0}(\Omega_j)}\Bigr)^{1/r} + C A' \varepsilon^\alpha \Bigl(\sum \|g\|^p_{L^p_v(\Omega_j)}\Bigr)^{1/p},
\end{aligned}$$

since $r, p \leq q$. Thus

$$\|f\|_{L^q_w(\Omega)} \leq C(N, D_w, q) (A_0 \varepsilon^{-\beta} \|f\|_{L^r_{v_0}(\Omega)} + A' \varepsilon^\alpha \|g\|_{L^p_v(\Omega)}).$$

This completes part (a). For part (b), even though we use essentially the ideas of [Chua 1995], this time the proof is more tricky because we do not assume $\mu = 1$ and $i = k$. First by Proposition 2.6, the triangle inequality, and Proposition 2.7, we have (like [Chua 1995])

$$\begin{aligned}
&\|\nabla^i P^k(Q_j) f\|_{L^q_w(Q_j)} \\
&\quad \leq C(n, k) r(Q_j)^{-i} w(Q_j)^{1/q} \|P^k(Q_j) f\|_{L^\infty(Q_j)} \\
&\quad \leq C r(Q_j)^{-i} w(Q_j)^{1/q} \\
&\qquad\qquad \times (\|\pi^\mu_{k+1}(Q_j) f\|_{L^\infty(Q_j)} + \|\pi^\mu_{k+1}(Q_j) f - P^k(Q_j) f\|_{L^\infty(Q_j)}) \\
&\quad = C r(Q_j)^{-i} w(Q_j)^{1/q} \\
&\qquad\qquad \times (\|\pi^\mu_{k+1}(Q_j) f\|_{L^\infty(Q_j)} + \|\pi^\mu_{k+1}(Q_j)(f - P^k(Q_j) f)\|_{L^\infty(Q_j)}) \\
&\quad \leq C(n, k, D_\mu) w(Q_j)^{1/q} r(Q_j)^{-i} \mu(Q_j)^{-1} \\
&\qquad\qquad \times (\|f\|_{L^1_\mu(Q_j)} + \|f - P^k(Q_j) f\|_{L^1_\mu(Q_j)})
\end{aligned}$$

$$\le C\big(w(Q_j)^{1/q} r(Q_j)^{-i} \mu(Q_j)^{-1} \|\tilde{v}_0^{-1/r}\|_{L^{r'}_\mu(Q_j)} \|f\|_{L^r_{v_0}(Q_j)} + A' r(Q_j)^{k-i+\alpha} \|\nabla^{k+1} f\|_{L^p_v(\tau Q_j)}\big)$$

(by Hölder's inequality and (1-8))

$$\le C\big(A_0 r(Q_j)^{-i-\beta} \|f\|_{L^r_{v_0}(Q_j)} + A' r(Q_j)^{k-i+\alpha} \|\nabla^{k+1} f\|_{L^p_v(\Omega_j)}\big)$$

by (1-5). Also, since w is δ-doubling and $\Omega_j \subset N Q_j$, we have by the triangle inequality, Lemma 2.5, (2-3) and Theorem 2.11 that

$$\begin{aligned}
\|\nabla^i f\|_{L^q_w(\Omega_j)} &\le \|\nabla^i P^k(Q_j) f\|_{L^q_w(\Omega_j)} + \|\nabla^i (f - P^k(Q_j) f)\|_{L^q_w(\Omega_j)} \\
&\le C(N, D_w, k, q, n) \|\nabla^i P^k(Q_j) f\|_{L^q_w(Q_j)} + \|\nabla^i (f - P^k(Q_j) f)\|_{L^q_w(\Omega_j)} \\
&\le C\Big(\|\nabla^i P^k(Q_j) f\|_{L^q_w(Q_j)} + \Big(\sum_{B \in W_j} \|\nabla^i (f - P^k(B) f)\|^q_{L^q_w(B)}\Big)^{1/q}\Big).
\end{aligned}$$

Hence by (1-9) and the penultimate calculation above, (recall that $k - i + \alpha \ge 0$)

$$\begin{aligned}
\|\nabla^i f\|_{L^q_w(\Omega_j)} &\le C(n, k, D_w, D_\mu, q, N) \\
&\qquad \times (A_0 r(Q_j)^{-\beta} \|f\|_{L^r_{v_0}(Q_j)} + A' r(Q_j)^{k-i+\alpha} \|\nabla^{k+1} f\|_{L^p_v(\Omega_j)}) \\
&\le C A_0 \varepsilon^{-\beta-i} \|f\|_{L^r_{v_0}(\Omega_j)} + C A' \varepsilon^{k-i+\alpha} \|\nabla^{k+1} f\|_{L^p_v(\Omega_j)}.
\end{aligned}$$

Thus, just as before, we have

$$\begin{aligned}
\|\nabla^i f\|_{L^q_w(\Omega)} &\le C(n, k, D_w, D_\mu, N, q) \\
&\qquad \times (A_0 \varepsilon^{-\beta-i} \|f\|_{L^r_{v_0}(\Omega)} + A' \varepsilon^{k-i+\alpha} \|\nabla^{k+1} f\|_{L^p_v(\Omega)}). \qquad \square
\end{aligned}$$

Proof of Corollary 1.7. We will show that both (1-9) and (1-8) hold with $\tau = 1$. Let B be a δ-ball such that $r(B) < N^2 \varepsilon_0$. First note that (1-11) implies (2-14) with $g = |\nabla f|$ by Proposition 2.15; hence we have

$$\begin{aligned}
&\frac{w(B)^{1/q}}{\mu(B)} \|f - P^k(B) f\|_{L^1_\mu(B)} \\
&\qquad \le c_1 \frac{r(B) w(B)^{1/q}}{v(B)^{1/p}} \|\nabla (f - P^k(B) f)\|_{L^p_v(B)} && \text{(by (1-12))} \\
&\qquad \le C(D_v, p) c_1 \frac{r(B)^{k+1} w(B)^{1/q}}{v(B)^{1/p}} \|\nabla^{k+1} f\|_{L^p_v(B)} \\
&\qquad\qquad && \text{(by repeated application of (2-14))} \\
&\qquad \le C(D_v, p) A' r(B)^{\alpha+k} \|\nabla^{k+1} f\|_{L^p_v(B)}, && \text{(by (1-13))}
\end{aligned}$$

and for any $l \le k$,

$$\begin{aligned}
&\|f - P^l(B)f\|_{L^q_w(B)} \\
&\qquad \le A' r(B)^\alpha \|\nabla(f - P^l(B)f\|_{L^p_v(B)} \qquad \text{(by (1-11))} \\
&\qquad \le C(D_v, p) A' r(B)^{\alpha+l} \|\nabla^{l+1} f\|_{L^p_v(B)} \quad \text{(by repeated application of (2-14)).}
\end{aligned}$$

The corollary is now a clear consequence of Theorem 1.6. □

Proof of Theorem 1.10. Given any $\lambda_i \in (0, \varepsilon_i)$ for $i = 1, 2$, there exist $\{\Omega_j\}$ such that $\bigcup \Omega_j = \Omega$ and $\Omega_j \in \mathcal{F}_{d_\lambda}(\sigma, N)$, where $\lambda = (\lambda_1, \lambda_2)$, and also such that $\sum \chi_{\Omega_j} \le N \chi_\Omega$ and $1 \le r_{d_\lambda}(\Omega_j) \le N$. For each fixed j, let W_j be a Boman cover of Ω_j (with respect to d_λ). Let $R_j = I_j \times J_j$ be the central ball (with respect to d_λ) in Ω_j. Then recall that $\lambda_1 \le r(I_j) \le N\lambda_1$ and $\lambda_2 \le r(J_j) \le N\lambda_2$. Let $\eta = \lambda_2/\lambda_1$. Then $r(J_\alpha)/r(I_\alpha) = \eta$ for all $R_\alpha = I_\alpha \times J_\alpha \in W_j$. Recall that by Theorem 2.11 with $d = d_\lambda$, we have

$$\begin{aligned}
&(3\text{-}1) \quad \|f - f_{R_j,\mu}\|_{L^q_w(\Omega_j)} \\
&\le C(N, q, D_w) \Big(\sum_{R_\alpha \in W_j} \|f - f_{R_\alpha,\mu}\|^q_{L^q_w(R_\alpha)} \Big)^{1/q} \\
&\le C \Big(\sum_{R_\alpha \in W_j} \big(A_1(R_\alpha) r(I_\alpha) \|g_1\|_{L^{r_1}_{v_1}(R_\alpha)} + A_2(R_\alpha) r(J_\alpha) \|g_2\|_{L^{r_2}_{v_2}(R_\alpha)} \big)^q \Big)^{1/q} \\
&\le C \Big(\sum_{R_\alpha \in W_j} A_1(R_\alpha) r(I_\alpha) \|g_1\|^q_{L^{r_1}_{v_1}(R_\alpha)} \Big)^{1/q} + C \Big(\sum_{R_\alpha \in W_j} A_2(R_\alpha) r(J_\alpha) \|g_2\|^q_{L^{r_2}_{v_2}(R_\alpha)} \Big)^{1/q} \\
&\le C A_1 r(I_j)^{1-a_1-b_1} \eta^{-b_1} \Big(\sum_{R_\alpha \in W_j} \|g_1\|^q_{L^{r_1}_{v_1}(R_\alpha)} \Big)^{1/q} \\
&\qquad\qquad + C A_2 r(I_j)^{1-a_2-b_2} \eta^{1-b_2} \Big(\sum_{R_\alpha \in W_j} \|g_2\|^q_{L^{r_2}_{v_2}(R_\alpha)} \Big)^{1/q} \\
&\le C(N, q, D_w) \\
&\qquad \times \big(A_1 r(I_j)^{1-a_1} r(J_j)^{-b_1} \|g_1\|_{L^{r_1}_{v_1}(\Omega_j)} + A_2 r(I_j)^{-a_2} r(J_j)^{1-b_2} \|g_2\|_{L^{r_2}_{v_2}(\Omega_j)} \big).
\end{aligned}$$

Above, the second inequality follows from (1-24). The fourth follows from (1-25) since $R_\alpha = I_\alpha \times J_\alpha \subset NR_j = NI_j \times NJ_j$ for all $R_\alpha \in W_j$ and $r(J_\alpha)/r(I_\alpha) = r(J_j)/r(I_j) = \eta$. The last follows since $q \ge r_1, r_2$ and $\sum_{R_\alpha \in W_j} \chi_{R_\alpha} \le N \chi_{\Omega_j}$. Also, by Hölder's inequality and the facts that w is product δ-doubling on Ω in $H_1 \times H_2$ and that $A_0(R_j) = w(R_j)^{1/q} \mu(R)^{-1} \|\tilde{v}_0^{-1/r_0}\|_{L^{r_0'}_\mu(R_j)}$, we have

$$\begin{aligned}
\|f_{R_j,\mu}\|_{L^q_w(\Omega_j)} &\le \frac{w(\Omega_j)^{1/q}}{\mu(R_j)} \int_{R_j} |f| \, d\mu \\
&\le C(D_w, q) A_0(R_j) \|f\|_{L^{r_0}_{v_0}(\Omega_j)} \le C A_0 r(I_j)^{-a_0} r(J_j)^{-b_0} \|f\|_{L^{r_0}_{v_0}(\Omega_j)}
\end{aligned}$$

by (1-25). Hence by the triangle inequality, we have

$$\|f\|_{L^q_w(\Omega_j)} \le C(D_w, q, N)\big(A_0\lambda_1^{-a_0}\lambda_2^{-b_0}\|f\|_{L^{r_0}_{v_0}(\Omega_j)} + A_1\lambda_1^{1-a_1}\lambda_2^{-b_1}\|g_1\|_{L^{r_1}_{v_1}(\Omega_j)} + A_2\lambda_1^{-a_2}\lambda_2^{1-b_2}\|g_2\|_{L^{r_2}_{v_2}(\Omega_j)}\big).$$

Next by the previous estimates and the fact that $q \ge r_0, r_1, r_2$, we have

$$\begin{aligned}\|f\|_{L^q_w(\Omega)} &\le \Big(\sum_j \|f\|^q_{L^q_v(\Omega_j)}\Big)^{1/q}\\ &\le CA_0\lambda_1^{-a_0}\lambda_2^{-b_0}\Big(\sum_j \|f\|^{r_0}_{L^{r_0}_{v_0}(\Omega_j)}\Big)^{1/r_0} + CA_1\lambda_1^{1-a_1}\lambda_2^{-b_1}\Big(\sum_j \|g_1\|^{r_1}_{L^{r_1}_{v_1}(\Omega_j)}\Big)^{1/r_1}\\ &\qquad + CA_2\lambda_1^{-a_2}\lambda_2^{1-b_2}\Big(\sum_j \|g_2\|^{r_2}_{L^{r_2}_{v_2}(\Omega_j)}\Big)^{1/r_2}\\ &\le C(D_w, N, q)\big(A_0\lambda_1^{-a_0}\lambda_2^{-b_0}\|f\|_{L^{r_0}_{v_0}(\Omega)} + A_1\lambda_1^{1-a_1}\lambda_2^{-b_1}\|g_1\|_{L^{r_1}_{v_1}(\Omega)}\\ &\qquad + A_2\lambda_1^{-a_2}\lambda_2^{1-b_2}\|g_2\|_{L^{r_2}_{v_2}(\Omega)}\big).\end{aligned}$$

This proves part (a).

We now prove part (b). Again, let $R_j = I_j \times J_j$ be the central ball in Ω_j. For any fixed $f \in C^{1,1}_{\text{loc}}(\Omega)$, let $g(x) = \int_{J_j} f(x, y)\,d\mu_2(y)$. Then

$$\frac{\partial g}{\partial x_i}(x) = \int_{J_j} \frac{\partial f}{\partial x_i}(x, y)\,d\mu_2(y) \quad \text{for } i = 1, 2, \ldots, n. \tag{3-2}$$

Let $P(g) = P(I_j)g$ be the polynomial of degree ≤ 1 in $\mathbb{R}^n$ such that

$$\begin{aligned}\int_{I_j} P(g)\,d\mu_1(x) &= \int_{I_j} g\,d\mu_1(x),\\ \int_{I_j} \frac{\partial g}{\partial x_i}\,d\mu_1(x) &= \int_{I_j} \frac{\partial P(g)}{\partial x_i}\,d\mu_1(x) \quad \text{for } i = 1, 2, \cdots, n.\end{aligned}$$

Note that

$$\begin{aligned}\int_{J_j}\int_{I_j} f\,d\mu &= \int_{J_j}\int_{I_j} \frac{1}{\mu_2(J_j)} P(g)\,d\mu,\\ \int_{J_j}\int_{I_j} \frac{\partial f}{\partial x_i}\,d\mu &= \int_{J_j}\int_{I_j} \frac{1}{\mu_2(J_j)}\frac{\partial P(g)}{\partial x_i}\,d\mu \quad \text{for } i = 1, 2, \ldots, n.\end{aligned} \tag{3-3}$$

Also, if $Pf = P(R_j)f$ is the polynomial in $\mathbb{R}^{n+m}$ of degree ≤ 1 such that

$$\int_{R_j} D^\gamma(f - Pf)\,d\mu = 0 \quad \text{for all } |\gamma| \le 1,$$

then $\int_{J_j} P(R_j) f(x,y)\,d\mu_2(y) = P(g)(x)$ by (3-2) and (3-3).

Next observe that

$$\begin{aligned}
&\frac{1}{\mu_2(J_j)}\|\nabla_x Pg\|_{L^q_w(R_j)}\\
&\le \frac{w(R_j)^{1/q}}{\mu_2(J_j)}\|\nabla_x Pg\|_{L^\infty(R_j)} = \frac{w(R_j)^{1/q}}{\mu_2(J_j)}\|\nabla_x Pg\|_{L^\infty(I_j)}\\
&\le C(n) r(I_j)^{-1}\frac{w(R_j)^{1/q}}{\mu_2(J_j)}\|Pg\|_{L^\infty(I_j)} \qquad \text{(by Proposition 2.6)}\\
&\le Cr(I_j)^{-1}\frac{w(R_j)^{1/q}}{\mu_2(J_j)}\|Pg-\pi_1^{\mu_1}g\|_{L^\infty(I_j)} + Cr(I_j)^{-1}\frac{w(R_j)^{1/q}}{\mu_2(J_j)}\|\pi_1^{\mu_1}g\|_{L^\infty(I_j)}\\
&\text{(by the triangle inequality, where } \pi_1^{\mu_1} \text{ is the projection given in Proposition 2.7)}\\
&= Cr(I_j)^{-1}\frac{w(R_j)^{1/q}}{\mu_2(J_j)}\|\pi_1^{\mu_1}(g-Pg)\|_{L^\infty(I_j)} + Cr(I_j)^{-1}\frac{w(R_j)^{1/q}}{\mu_2(J_j)}\|\pi_1^{\mu_1}g\|_{L^\infty(I_j)}\\
&\le C(D_{\mu_1},n) r(I_j)^{-1}\mu(R_j)^{-1}w(R_j)^{1/q}\big(\|g-Pg\|_{L^1_{\mu_1}(I_j)} + \|g\|_{L^1_{\mu_1}(I_j)}\big)\\
&\qquad\qquad \text{(by Proposition 2.7)}\\
&\le C(D_{\mu_1},n)\big(c_2 w(R_j)^{1/q}\mu(R_j)^{-1}\|\nabla_x(g-Pg)\|_{L^1_{\mu_1}(I_j)}\\
&\qquad\qquad + r(I_j)^{-1}w(R_j)^{1/q}\mu(R_j)^{-1}\|g\|_{L^1_{\mu_1}(I_j)}\big) \quad \text{(by (1-27))}\\
&\le C(D_{\mu_1},n) w(R_j)^{1/q}\mu(R_j)^{-1}\Big(c_2\Big\|\int_{J_j}|\nabla_x(f-Pf)|\,d\mu_2\Big\|_{L^1_{\mu_1}(I_j)}\\
&\qquad\qquad + r(I_j)^{-1}\Big\|\int_{J_j} f\,d\mu_2\Big\|_{L^1_{\mu_1}(I_j)}\Big)\\
&\le Cw(R_j)^{1/q}\mu(R_j)^{-1}\big(c_2\|\nabla_x(f-Pf)\|_{L^1_\mu(R_j)} + r(I_j)^{-1}\|f\|_{L^1_\mu(R_j)}\big)\\
&\qquad\qquad \text{(by Fubini's theorem)}\\
&\le Cc_2\big(A_1(R_j)r(I_j)\|\nabla_x^2 f\|_{L^{r_1}_{v_1}(R_j)} + A_2(R_j)r(J_j)\|\nabla_{xy}f\|_{L^{r_2}_{v_2}(R_j)}\big)\\
&\qquad\qquad + Cr(I_j)^{-1}A_0(R_j)\|f\|_{L^{r_0}_{v_0}(R_j)} \quad \text{(by (1-29) and Hölder's inequality)}\\
&\le C(D_\mu,n)c_2\big(A_1\lambda_1^{1-a_1}\lambda_2^{-b_1}\|\nabla_x^2 f\|_{L^{r_1}_{v_1}(\Omega_j)} + A_2\lambda_1^{-a_2}\lambda_2^{1-b_2}\|\nabla_{xy}f\|_{L^{r_2}_{v_2}(\Omega_j)}\\
&\qquad\qquad + A_0\lambda_1^{-1-a_0}\lambda_2^{-b_0}\|f\|_{L^{r_0}_{v_0}(\Omega_j)}\big).
\end{aligned}$$

The last inequality follows from (1-25) and the fact that $R_j \subset \Omega_j$. Hence by the triangle inequality, by the facts that w is product δ-doubling and $\Omega_j \subset NR_j$, and by (2-2) and Lemma 2.5, we have

$$\begin{aligned}
\|\nabla_x f\|_{L^q_w(\Omega_j)} &\le \Big\|\nabla_x\Big(f - \frac{1}{\mu_2(J_j)}Pg\Big)\Big\|_{L^q_w(\Omega_j)} + \frac{1}{\mu_2(J_j)}\|\nabla_x Pg\|_{L^q_w(\Omega_j)}\\
&\le \Big\|\nabla_x\Big(f - \frac{1}{\mu_2(J_j)}Pg\Big)\Big\|_{L^q_w(\Omega_j)} + \frac{C(D_w,N,q)}{\mu_2(J_j)}\|\nabla_x Pg\|_{L^q_w(R_j)}.
\end{aligned}$$

Recall that

$$\int_{R_j} \frac{\partial f}{\partial x_i} d\mu = \int_{R_j} \frac{1}{\mu_2(J_j)} \frac{\partial P(g)}{\partial x_i} d\mu \quad \text{for } i = 1, 2, \dots, n.$$

Hence by (1-28) and Theorem 2.11, similar to (3-1) we have

$$\begin{aligned}\Big\| \nabla_x \big(f - \frac{1}{\mu_2(J_j)} Pg\big)\big) \Big\|_{L^q_w(\Omega_j)} &\le C(D_w, N, q, n, m) \\ &\times \big(A_1 r(I_j)^{1-a_1} r(J_j)^{-b_1} \|\nabla_x^2 f\|_{L^{r_1}_{v_1}(\Omega_j)} + A_2 r(I_j)^{-a_2} r(J_j)^{1-b_2} \|\nabla_{xy} f\|_{L^{r_2}_{v_2}(\Omega_j)} \big).\end{aligned}$$

Thus by the previous estimates, we have

$$\begin{aligned}\|\nabla_x f\|_{L^q_w(\Omega_j)} \le C(D_w, N, q, n, m) c_2 & \\ \times \big(A_1 \lambda_1^{1-a_1} \lambda_2^{-b_1} \|\nabla_x^2 f\|_{L^{r_1}_{v_1}(\Omega_j)} &+ A_2 \lambda_1^{-a_2} \lambda_2^{1-b_2} \|\nabla_{xy} f\|_{L^{r_2}_{v_2}(\Omega_j)} \\ &+ A_0 \lambda_1^{-1-a_0} \lambda_2^{-b_0} \|f\|_{L^{r_0}_{v_0}(\Omega_j)} \big).\end{aligned}$$

Finally, by arguments similar to the ones used to estimate $\|f\|_{L^q_w(\Omega)}$, we have

$$\begin{aligned}\|\nabla_x f\|_{L^q_w(\Omega)} \le C(D_w, N, q, n, m) c_2 & \\ \times \big(A_1 \lambda_1^{1-a_1} \lambda_2^{-b_1} \|\nabla_x^2 f\|_{L^{r_1}_{v_1}(\Omega)} &+ A_2 \lambda_1^{-a_2} \lambda_2^{1-b_2} \|\nabla_{xy} f\|_{L^{r_2}_{v_2}(\Omega)} \\ &+ A_0 \lambda_1^{-1-a_0} \lambda_2^{-b_0} \|f\|_{L^{r_0}_{v_0}(\Omega)} \big). \quad \square\end{aligned}$$

4. Applications

The next theorem is an extension of [Chua 1995, Corollary 3.1]. We are able to obtain a stronger result simply because Theorem 1.6 is stronger; see Remark 1.8(1).

Theorem 4.1. *Let σ, $N > 1$ and $0 < \varepsilon_0 \le \infty$. Let $\Omega \subset \mathbb{R}^n$ such that $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_0)$. Let $c_1 \ge 1$ and $c_2, c_3, D_1 > 0$. Let $1 \le p, r \le q < \infty$. Let v and w be δ-doubling measures on Ω such that*

$$\begin{aligned} w(B)^{1/q} v(B)^{-1/p} &\le c_1 r(B)^{\tau-1}, \\ w(B) &\ge c_2 r(B)^{D_1}, \end{aligned} \tag{4-1}$$

$$\|f - f_{B,w}\|_{L^q_w(B)} \le c_3 w(B)^{1/q} v(B)^{-1/p} r(B) \|\nabla f\|_{L^p_v(B)} \tag{4-2}$$

for all $f \in C^{0,1}_{\text{loc}}(\Omega)$, all δ-balls B, and $k \in \mathbb{N} \cup \{0\}$. Let $i, k \in \mathbb{N} \cap \{0\}$ with $i \le k$. Let

(i) $a = k - i + \tau + \min\{t/q - s/p, 0\} \ge 0$,
$b = i + D_1(1/r - 1/q) - \min\{t/q - u/r, 0\}$;

(ii) *$t/q \le s/p$ and $t/q \le u/r$ if Ω is unbounded.*

Let $G \subset \Omega^c$ and $\rho(x) = d(x, G)$. Let $\rho_G = \sup_{x\in\Omega} \rho(x)$. Let $w^t(E) = \int_E \rho(x)^t dw$ be δ-doubling, and define w^u and v^s similarly. Then for all $f \in C^{k,1}_{\mathrm{loc}}(\Omega)$, we have

$$\|\nabla^i f\|_{L^q_{w^t}(\Omega)} \le C(N, \sigma, n, p, q, r, s, t, u, D_w, D_v, c_2, c_3)c_1 \times \left(\varepsilon^{-b}\rho_G^{\max\{t/q-u/r,0\}}\|f\|_{L^r_{w^u}(\Omega)} + \varepsilon^a \rho_G^{\max\{t/q-s/p,0\}}\|\nabla^{k+1} f\|_{L^p_{v^s}(\Omega)}\right) \tag{4-3}$$

for all $\varepsilon \in (0, \varepsilon_0)$. (When $\rho_G = \infty$, we will use the convention $\infty^0 = 1$.)

Proof of Theorem 4.1. The proof goes like that of [Chua 1995, Corollary 3.1], except that it was assumed there that $i = k$ and $v = w = 1$. For convenience, let us use the metric

$$d(x, y) = \max\{|x_i - y_i| : i = 1, \dots, n\}.$$

Let $\delta = 1/\sigma$. If B is a metric ball (which is indeed a cube) in Ω such that $\sigma B \subset \Omega$, then B is a δ-ball in Ω and we know that

$$\begin{aligned} d(B, G) \le d(x, G) = \rho(x) &\le d(B, G) + 2r(B) \\ &\le (1 + 2/(\sigma - 1))d(B, G) \quad \text{for all } x \in B. \end{aligned} \tag{4-4}$$

Thus

$$\begin{aligned} &d(B, G)^\alpha w(B) \le w^\alpha(B) = \int_B \rho(x)^\alpha dw \le C(\alpha, \sigma)d(B, G)^\alpha w(B) && \text{for } \alpha \ge 0, \\ &d(B, G)^\alpha w(B) \ge w^\alpha(B) \ge C(\alpha, \sigma)d(B, G)^\alpha w(B) && \text{for } \alpha < 0. \end{aligned}$$

Similarly, we have $v^s(B) \sim d(B, G)^s v(B)$. Again, since $r(B) \le d(B, G)/(\sigma - 1)$, we have

$$\begin{aligned} \frac{w^t(B)^{1/q}}{w(B)}\|\rho^{-u/r}\|_{L^{r'}_w(B)} &\le C(q, r, \sigma, t, u)d(B, G)^{t/q-u/r} w(B)^{1/q-1/r} \\ &\le C(q, r, \sigma, t, u, c_2)\rho_G^{\max\{t/q-u/r,0\}} r(B)^{D_1(1/q-1/r)+\min\{t/q-u/r,0\}} \\ &\le C(q, r, \sigma, t, u, c_2)\rho_G^{\max\{t/q-u/r,0\}} r(B)^{-b+i}. \end{aligned}$$

Inequality (1-5) is now clear with $\mu = w$, $v_0 = w^u$ and $\beta = b - i$. We now need to establish (1-8) and (1-9). To this end, first note that by (4-2), Hölder's inequality and Proposition 2.15, we have

$$\|f - f_{B,w}\|_{L^p_v(B)} \le c_3 C(D_v, p)r(B)\|\nabla f\|_{L^p_v(B)} \quad \text{for all } \delta\text{-balls } B.$$

Hence by (4-2), letting $P^k(B)f$ be as in (1-8) with $\mu = w$, we have

$$\frac{1}{w(B)}\|f - P^k(B)f\|_{L^1_w(B)} \le C(D_v, p)(c_3 r(B))^{k+1}\|\nabla^{k+1} f\|_{L^p_v(B)}. \tag{4-5}$$

Since

$$w^t(B)^{1/q} v^s(B)^{-1/p} r(B) \le C(t,s,\sigma,p,q) c_1 r(B)^\tau d(B,G)^{t/q-s/p}$$
$$\le C(t,s,\sigma,p,q) c_1 r(B)^{a-k+i} \rho_G^{\max\{t/q-s/p,0\}},$$

it is now easy to see (again by (4-4)) that

$$\frac{w^t(B)^{1/q}}{w(B)} \|f - P^k(B)f\|_{L^1_w(B)}$$
$$\le C(D_v,p,t,s,\sigma,q,c_3) c_1 r(B)^{a+i} \rho_G^{\max\{t/q-s/p,0\}} \|\nabla^{k+1} f\|_{L^p_{v^s}(B)}.$$

Next, by (4-2), (4-5) and (4-4), we have

$$\|f - P^{k-i}(B)f\|_{L^q_{w^t}(B)}$$
$$\le C(D_v,p,t,s,\sigma,q,c_3) c_1 r(B)^a \rho_G^{\max\{t/q-s/p,0\}} \|\nabla^{k-i+1} f\|_{L^p_{v^s}(B)}$$

for all $f \in C^{0,1}_{\text{loc}}(\Omega)$ and any δ-ball B. These establish that (1-8) and (1-9) hold with $\mu = w,\ v = v^s,\ w = w^t$ and $a = k - i + \alpha$, $b = i + \beta$. The theorem is now clear by Theorem 1.6(b). □

Remark 4.2. (1) Clearly, w^t is δ-doubling on Ω when $t \ge 0$.

(2) In case $w = v = 1$, we know $D_1 = n$ and $\tau = 1 - n/p + n/q$. Thus when $1/q \ge 1/p - 1/n$, we have by the nonweighted Poincaré inequality and Theorem 4.1

$$\text{(4-6)} \quad \|\nabla^i f\|_{L^q_{\rho^t}(\Omega)} \le C(N,\sigma,p,q,n,s,t,u,r)$$
$$\times \left(\varepsilon^{-b} \rho_G^{\max\{t/q-u/r,0\}} \|f\|_{L^r_{\rho^u}(\Omega)} + \varepsilon^a \rho_G^{\max\{t/q-s/p,0\}} \|\nabla^{k+1} f\|_{L^p_{\rho^s}(\Omega)}\right)$$

for all $\varepsilon \in (0, \varepsilon_0)$, where ρ^s, ρ^t and ρ^u are the measures arising from the weights $\rho(x)^s$, $\rho(x)^t$ and $\rho(x)^u$, respectively.

(3) In [Chua 1995, Corollary 3.1], it was assumed that ρ^t is doubling and $k = i$. We have only assumed here that ρ^t is δ-doubling. Note that ρ^t is certainly δ-doubling when $t \ge 0$.

(4) In the case that $a, b > 0$, under the assumption of Theorem 4.1, we have

$$\text{(4-7)} \quad \|\nabla^i f\|_{L^q_{w^t}(\Omega)} \le C(N,n,\sigma,p,q,r,s,t,u,D_w,D_v,c_2,c_3) c_1$$
$$\times \left(K_1 \|f\|_{L^r_{w^u}(\Omega)}\right)^{a/(a+b)} \left(\varepsilon_0^{-a-b} K_1 \|f\|_{L^r_{w^u}(\Omega)} + K_2 \|\nabla^{k+1} f\|_{L^p_{v^s}(\Omega)}\right)^{b/(a+b)},$$

where $K_1 = \rho_G^{\max\{t/q-u/r,0\}}$ and $K_2 = \rho_G^{\max\{t/q-s/p,0\}}$.

(5) If $0 \le \alpha \le a,\ 0 \le \beta \le b$ and $\varepsilon_0 < \infty$, then it is clear that (4-6) will still hold with a and b be replaced by α and β, respectively.

(6) Conditions (i) and (ii) of Theorem 4.1 when $w = v = 1$ (hence $\tau = 1 - n/p + n/q$ and $D_1 = n$) are also necessary when $\varepsilon_0 = \infty$. Indeed, suppose (4-6) holds with $a + b = \lambda = k + 1 + n(1/r - 1/p) + u/r - s/p > 0$. Then

$$\|\nabla^i f\|_{L^q_{\rho^t}(\Omega)} \le C(\|f\|_{L^r_{\rho^u}(\Omega)})^{a/\lambda}(\|\nabla^{k+1} f\|_{L^p_{\rho^s}(\Omega)})^{b/\lambda} \quad \text{for all } f \in C^{0,1}_{\mathrm{loc}}(\Omega).$$

It is well known that $1/q \ge 1/p - 1/n$; see for example [Caffarelli et al. 1984] or [Lin 1986]. For any δ-ball B in Ω, let f be as in Remark 1.8(5). Then

$$r(B)^{-i}|B|^{1/q}d(B,G)^{t/q} \le C|B|^{a/(r\lambda)}d(B,G)^{ua/(r\lambda)}(r(B)^{-k-1}|B|^{1/p}d(B,G)^{s/p})^{b/\lambda}.$$

Thus, we must have (letting $a' = a/\lambda$ and $b' = b/\lambda$)

$$r(B)^{b'(k+1)-i+n(1/q-a'/r-b'/p)}d(B,G)^{t/q-a'u/r-b's/p} \le C$$

for all δ-balls in Ω. However, since Ω is unbounded, $t/q \le a'u/r + b's/p$ and hence $t/q \le u/r$ and $t/q \le s/p$. Next, since $d(B,G) \ge d(B) \ge (\sigma - 1)r(B)$, we have

$$b'(k+1) - i + n(1/q - a'/r - b'/p) + t/q - a'u/r - b's/p = 0$$

and hence $(\lambda/q)(n + t - qi) = (a/r)(n + u) + (b/p)(n + s - (k+1)p)$.

We now note that $(n+u)/r \ne (n + s - p(k+1))/p$, since otherwise

$$(n + t - qi)/q = (n + u)/r = (n + s - p(k+1))/p$$

and hence $\lambda = 0$, which is impossible.

It is then clear that a and b are as in condition (i).

The following Sobolev interpolation inequality, an application of Theorem 4.1, extends the one obtained by Caffarelli, Kohn and Nirenberg [1984] and Lin [1986].

Theorem 4.3. *Let $\sigma, N > 1$ and $0 < \varepsilon_0 \le \infty$. Let $\Omega \subset \mathbb{R}^n$ and $\Omega \in \mathscr{F}'(\sigma, N, \varepsilon_0)$. Let $i, k \in \mathbb{N}$ such that $i \le k$. Suppose $t > -n$, $1 \le p, r \le q < \infty$, $1/q \ge 1/p - 1/n$ and that the conditions* (i) *and* (ii) *of Theorem 4.1 hold with $\tau = 1 - n/p + n/q$ and $D_1 = n$. Then* (4-6) *holds with $G = \{0\}$.*

Proof. We will apply Theorem 4.1 with $v = w = 1$. Inequality (4-2) holds since $1/q \ge 1/p - 1/n$. Also, (4-1) holds with $\tau = 1 - n/p + n/q$ and $D_1 = n$. Also, it is well known that $|x|^t$ is doubling for $t > -n$. It is hence clear that the measure arising from the weight $\rho(x)^t = |x|^t$ is δ-doubling on Ω. The theorem now follows immediately from the fact that $\Omega \setminus \{0\} \in \mathscr{F}'(\sigma, N', \varepsilon_0')$, with N' and ε_0' depending on N and ε_0, respectively, by Propositions 2.24 and 2.21. □

Remark 4.4. (1) Lin [1986] has also dealt with fractional derivatives.

(2) Inequality (4-7) has been obtained by Caffarelli, Kohn and Nirenberg [1984] when $\Omega = \mathbb{R}^n$. We have only considered the case $q \geq p, r$ here. However, we have no restriction on s and u. Also, Chua [2005] made some observations about sharp conditions when $\Omega = \mathbb{R}^n$.

(3) For C^∞ functions with compact support in Ω, Gurka and Opic [1991] have also obtained similar results on bounded Lipschitz domains Ω for measures arising from powers of distance weight $d(x, \partial\Omega)$.

(4) We can of course extend our idea to obtain various weighted Sobolev interpolation inequalities with distance type weights. For example, G can be also chosen as a line segment or image of a Lipschitz map (if we know that $\Omega \setminus G$ is still a generalized John domain). Some sufficient conditions for such distance-type weights to be doubling were been discussed in [Chua 1995].

We now discuss interpolation inequalities for power weights on product spaces.

Theorem 4.5. *Let $0 \leq a_i, b_i \leq 1$ such that $a_i + b_i \leq 1$ for $i = 0, 1, 2$. Let $\sigma, N > 1$ and $1 \leq r_0, r_1, r_2 \leq q < \infty$. Let $0 < \varepsilon_1, \varepsilon_2 \leq \infty$ and let $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_1, \varepsilon_2, \mathbb{R}^n \times \mathbb{R}^m)$. Let $G_1 \subset \mathbb{R}^n$ and $G_2 \subset \mathbb{R}^m$, with $(G_1 \times \mathbb{R}^m) \cap \Omega$ and $(\mathbb{R}^n \times G_2) \cap \Omega$ both empty. Let $w(x, y) = d(x, G_1)^\alpha d(y, G_2)^\beta$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Assume w is product δ-doubling on Ω in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose $v_i(x, y) = d(x, G_1)^{\alpha_i} d(y, G_2)^{\beta_i}$ for $i = 0, 1, 2$. Let*

$$\rho_1 = \sup\{d(x, G_1) : (x, y) \in \Omega \text{ for some } y \in \mathbb{R}^m\},$$
$$\rho_2 = \sup\{d(y, G_2) : (x, y) \in \Omega \text{ for some } x \in \mathbb{R}^n\}.$$

Suppose these conditions hold:

(i) $\alpha/q - \alpha_i/r_i \leq 0$ *if* $\rho_1 = \infty$ *and* $\beta/q - \beta_i/r_i \leq 0$ *when* $\rho_2 = \infty$ *for* $i = 0, 1, 2$.

(ii) $\min\{\alpha/q - \alpha_i/r_i, 0\} + \min\{\beta/q - \beta_i/r_i, 0\} + (m + n)(1/q - 1/r_i) \geq -1$ *for* $i = 0, 1, 2$.

(iii) $-a_i = n(1/q - 1/r_i) + \min\{\alpha/q - \alpha_i/r_i, 0\}$,
$-b_i = m(1/q - 1/r_i) + \min\{\beta/q - \beta_i/r_i, 0\}$ *for* $i = 0, 1, 2$,

Then (*again we will use the convention* $\infty^0 = 1$)

$$\begin{aligned}(4\text{-}8)\quad \|f\|_{L^q_w(\Omega)} \leq C\big(&\lambda_1^{-a_0}\lambda_2^{-b_0}\rho_1^{\max\{\alpha/q-\alpha_0/r_0,0\}}\rho_2^{\max\{\beta/q-\beta_0/r_0,0\}}\|f\|_{L^{r_0}_{v_0}(\Omega)}\\ &+\lambda_1^{1-a_1}\lambda_2^{-b_1}\rho_1^{\max\{\alpha/q-\alpha_1/r_1,0\}}\rho_2^{\max\{\beta/q-\beta_1/r_1,0\}}\|\nabla_x f\|_{L^{r_1}_{v_1}(\Omega)}\\ &+\lambda_1^{-a_2}\lambda_2^{1-b_2}\rho_1^{\max\{\alpha/q-\alpha_2/r_2,0\}}\rho_2^{\max\{\beta/q-\beta_2/r_2,0\}}\|\nabla_y f\|_{L^{r_2}_{v_2}(\Omega)}\big)\end{aligned}$$

for all $\lambda_i \in (0, \varepsilon_i)$ *with* $i = 1, 2$, *and* $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$. *Also*

$$\begin{aligned}
&\|\nabla_x f\|_{L^q_w(\Omega)} \\
&\quad\le C\big(\lambda_1^{1-a_1}\lambda_2^{-b_1}\rho_1^{\max\{\alpha/q-\alpha_1/r_1,0\}}\rho_2^{\max\{\beta/q-\beta_1/r_1,0\}}\|\nabla_x^2 f\|_{L^{r_1}_{v_1}(\Omega)} \\
&\qquad\quad+\lambda_1^{-a_2}\lambda_2^{1-b_2}\rho_1^{\max\{\alpha/q-\alpha_2/r_2,0\}}\rho_2^{\max\{\beta/q-\beta_2/r_2,0\}}\|\nabla_{xy} f\|_{L^{r_2}_{v_2}(\Omega)} \\
&\qquad\qquad+\lambda_1^{-1-a_0}\lambda_2^{-b_0}\rho_1^{\max\{\alpha/q-\alpha_0/r_0,0\}}\rho_2^{\max\{\beta/q-\beta_0/r_0,0\}}\|f\|_{L^{r_0}_{v_0}(\Omega)}\big)
\end{aligned}\tag{4-9}$$

for all $\lambda_i \in (0, \varepsilon_i)$ *with* $i = 1, 2$, *and* $f \in C^{1,1}_{\mathrm{loc}}(\Omega)$. *All the constants above depend only on* $q, r_i, m, n, \alpha, \alpha_i, \beta, \beta_i, \sigma$ *and* N.

Remark 4.6. (1) If $D = \Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^m$ in Theorem 4.5, we may take $G_1 \subset \Omega_1^c$ and $G_2 \subset \Omega_2^c$.

(2) The distances $d(x, G_1)$ and $d(y, G_2)$ can be any distance functions arising from any norm on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Of course, all norms on Euclidean space are equivalent. For convenience, we will use our previous norm, namely, $d(x, u) = \max\{|u_i - x_i| : i = 1, 2, \dots, n\}$ for any $x, u \in \mathbb{R}^n$ (or $\mathbb{R}^m$).

(3) Clearly, if $\alpha, \beta \ge 0$, then w is product δ-doubling on Ω in $\mathbb{R}^n \times \mathbb{R}^m$.

Proof of Theorem 4.5. We will use Theorem 1.10 with $\mu = 1$. First, for any parallelepiped $R = I \times J \subset \Omega$ such that $\sigma R \subset \Omega$, we have

$$\begin{aligned}
d(I, G_1) + 2r(I) \ge d(x, G_1) \ge d(I, G_1) \ge (\sigma - 1)r(I),\\
d(J, G_2) + r(J) \ge d(y, G_2) \ge d(J, G_2) \ge (\sigma - 1)r(J)
\end{aligned}$$

for any $(x, y) \in I \times J$. Hence $w(x, y)$ is comparable to $d(I, G_1)^\alpha d(J, G_2)^\beta$ for any $(x, y) \in R$ and hence $w(I \times J)$ is comparable to $d(I, G_1)^\alpha d(J, G_2)^\beta |I||J|$. Similar estimates can be obtained for v_0, v_1 and v_2.

For any Lipschitz continuous function f on R, we have by Proposition 2.16,

$$\begin{aligned}
&\|f - f_R\|_{L^q(R)}\\
&\quad\le C(m, n, q)\big(r(I)|R|^{1/q-1/r_1}\|\nabla_x f\|_{L^{r_1}(R)} + r(J)|R|^{1/q-1/r_2}\|\nabla_y f\|_{L^{r_2}(R)}\big)
\end{aligned}$$

for all parallelepipeds $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$, where $f_R = \int_R f/|R|$.

It is then clear that for any parallelepiped $R = I \times J \subset \Omega$ such that $\sigma R \subset \Omega$, we have

$$\begin{aligned}
&\|f - f_R\|_{L^q_w(R)} \le C(m, n, q)\\
&\times\big(K_1 r(I)^{1+n(1/q-1/r_1)} r(J)^{m(1/q-1/r_1)} d(I, G_1)^{\alpha/q-\alpha_1/r_1} d(J, G_2)^{\beta/q-\beta_1/r_1}\|\nabla_x f\|_{L^{r_1}_{v_1}(R)}\\
&\quad+ K_2 r(I)^{n(1/q-1/r_2)} r(J)^{1+m(1/q-1/r_2)} d(I, G_1)^{\alpha/q-\alpha_2/r_2} d(J, G_2)^{\beta/q-\beta_2/r_2}\|\nabla_y f\|_{L^{r_2}_{v_2}(R)}\big),
\end{aligned}$$

where $K_i = C(\sigma, q, \alpha, \beta, \alpha_i, \beta_i, r_i)$. We now let

$$A_i(R) = r(I)^{n(1/q-1/r_i)+\min\{\alpha/q-\alpha_i/r_i,0\}} r(J)^{m(1/q-1/r_i)+\min\{\beta/q-\beta_i/r_i,0\}} \times \rho_1^{\max\{\alpha/q-\alpha_i/r_i,0\}} \rho_2^{\max\{\beta/q-\beta_i/r_i,0\}}$$

for $i = 1, 2$. Also, recall that

$$\begin{aligned} A_0(R) &= v(R)^{1/q} |R|^{-1} \|v_0^{-1/r_0}\|_{L^{r_0'}(R)} \\ &\le C d(I, G_1)^{\alpha/q-\alpha_0/r_0} d(J, G_2)^{\beta/q-\beta_0/r_0} |I|^{1/q-1/r_0} |J|^{1/q-1/r_0} \\ &\le C \rho_1^{\max\{\alpha/q-\alpha_0/r_0,0\}} \rho_2^{\max\{\beta/q-\beta_0/r_0,0\}} \\ &\qquad \times r(I)^{n(1/q-1/r_0)+\min\{\alpha/q-\alpha_0/r_0,0\}} r(J)^{m(1/q-1/r_0)+\min\{\beta/q-\beta_0/r_0,0\}}. \end{aligned}$$

It is now easy to see that (1-25) holds with $A_i = K_i \rho_1^{\max\{\alpha/q-\alpha_i/r_i,0\}} \rho_2^{\max\{\beta/q-\beta_i/r_i,0\}}$. Moreover, we have

$$\|f - f_R\|_{L^q_w(R)} \le C K_1 A_1(R) l(I) \|\nabla_x f\|_{L^{r_1}_{v_1}(R)} + C K_2 A_2(R) l(J) \|\nabla_y f\|_{L^{r_2}_{v_2}(R)}$$

for all parallelepipeds $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $\sigma R \subset \Omega$. This establishes (1-28) with $\mu = 1$. Similarly, (1-29) is clear.

Finally, (1-27) clearly holds by the unweighted Poincaré inequality.

The theorem now follows from Theorem 1.10. □

Next, let us prove an interesting case when we have A_p weights on $\mathbb{R}^n \times \mathbb{R}^m$; see [Chua 1999] or [Fefferman and Stein 1982] for definitions.

Theorem 4.7. *Let $1 < p < \infty$ and $1 < \sigma, N, 0 < \varepsilon_1, \varepsilon_2 < \infty$. If $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$ and $\Omega \in \mathcal{F}'(\sigma, N, \varepsilon_1, \varepsilon_2, \mathbb{R}^n \times \mathbb{R}^m)$ is a bounded set, then there exists an $\varepsilon > 0$ depending only on the weight w, such that for any $p < q \le p + \varepsilon$ there exist $0 < h_1, h_2 < 1$ with $h_1 + h_2 \le 1$ such that*

$$\|f\|_{L^q_w(\Omega)} \le C w(R_0)^{1/q-1/p} r(I_0)^{h_1} r(J_0)^{h_2} \times \big(\lambda_1^{-h_1} \lambda_2^{-h_2} \|f\|_{L^p_w(\Omega)} + \lambda_1^{1-h_1} \lambda_2^{-h_2} \|\nabla_x f\|_{L^p_w(\Omega)} + \lambda_1^{-h_1} \lambda_2^{1-h_2} \|\nabla_y f\|_{L^p_w(\Omega)}\big)$$

for all $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$ and

$$\|\nabla_x f\|_{L^q_w(\Omega)} \le C w(R_0)^{1/q-1/p} l(I_0)^{h_1} l(J_0)^{h_2} \times \big(\lambda_1^{-1-h_1} \lambda_2^{-h_2} \|f\|_{L^p_w(\Omega)} + \lambda_1^{1-h_1} \lambda_2^{-h_2} \|D_x^2 f\|_{L^q_w(\Omega)} + \lambda_1^{-h_1} \lambda_2^{1-h_2} \|\nabla_{xy} f\|_{L^p_w(\Omega)}\big)$$

for all $f \in C^{1,1}_{\mathrm{loc}}(\Omega)$, where $R_0 = I_0 \times J_0$ is any parallelepiped that contains Ω.

Proof. First note that since $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$, by the Hölder inequality we have

$$\left(\frac{w(R)}{w(\tilde{R})}\right)^{1/p} \left(\frac{|R|}{|\tilde{R}|}\right)^{-1} \left(\frac{w^{-1/(p-1)}(R)}{w^{-1/(p-1)}(\tilde{R})}\right)^{1/p'} \le C$$

for all parallelepipeds $R \subset \tilde{R} \subset \mathbb{R}^n \times \mathbb{R}^m$. Next since w is reverse doubling on $\mathbb{R}^n \times \mathbb{R}^m$, there exists an $\eta > 0$ such that $w(R)/w(\tilde{R}) \geq C(|R|/|\tilde{R}|)^\eta$ for all parallelepipeds $R \subset \tilde{R} \subset \mathbb{R}^n \times \mathbb{R}^m$. Hence if we choose $\varepsilon > 0$ such that $(m+n)\eta(1/p-1/(p+\varepsilon)) \leq 1$, then for any $p < q \leq p+\varepsilon$, there exist $0 < h_1, h_2 < 1$ with $h_1 + h_2 \leq 1$, such that

$$\left(\frac{w(R)}{w(\tilde{R})}\right)^{1/q} \left(\frac{|I|}{|\tilde{I}|}\right)^{h_1/n-1} \left(\frac{|J|}{|\tilde{J}|}\right)^{h_2/m-1} \left(\frac{w^{-1/(p-1)}(R)}{w^{-1/(p-1)}(\tilde{R})}\right)^{1/p'} \leq C. \tag{4-10}$$

for all parallelepipeds $R = I \times J \subset \tilde{R} = \tilde{I} \times \tilde{J} \subset \mathbb{R}^n \times \mathbb{R}^m$. Let

$$A_i(R) = w(R)^{1/q} |R|^{-1} (w^{-1/(p-1)}(R))^{1/p'}.$$

Since $h_1 + h_2 \leq 1$, if $r(J)/r(I) = r(\tilde{I})/r(\tilde{J})$, we have

$$\frac{A_i(R)}{A_i(\tilde{R})} \frac{r(I)}{r(\tilde{I})} \leq \frac{A_i(R)}{A_i(\tilde{R})} \left(\frac{r(I)}{r(\tilde{I})}\right)^{h_1+h_2} = \frac{A_i(R)}{A_i(\tilde{R})} \left(\frac{r(I)}{r(\tilde{I})}\right)^{h_1} \left(\frac{r(J)}{r(\tilde{J})}\right)^{h_2} \leq C$$

by (4-10).

Hence by Proposition 2.16, we have

$$\begin{aligned} &\|f - f_R\|_{L^q_w(R)} \\ &\quad \leq C w(R)^{1/q} |R|^{-1} (w^{-1/(p-1)}(R))^{1/p'} (r(I) \|\nabla_x f\|_{L^p_w(R)} + r(J) \|D_y f\|_{L^p_w(R)}), \end{aligned}$$

for all $f \in C^{0,1}_{\mathrm{loc}}(\Omega)$ and parallelepipeds $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$. Furthermore, if $R_0 = I_0 \times J_0$ is any parallelepiped that contains Ω, then

$$A_i(R) \leq C w(R_0)^{1/q-1/p} r(I_0)^{h_1} r(J_0)^{h_2} r(I)^{-h_1} r(J)^{-h_2}$$

for $i = 0, 1, 2$ and all parallelepipeds $R = I \times J \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $\sigma R \subset \Omega$.

The result now follows from Theorem 1.10. □

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