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We give a general definition of the exponents of a meromorphic connection ∇ on a holomorphic vector bundle $\mathscr{E}$ of rank n over a compact Riemann surface X. We prove that they can be computed as invariants of a vector bundle $\mathscr{E}_L$ canonically attached to $\mathscr{E}$, which we construct and call the *Levelt bundle* of $\mathscr{E}$, and whose degree (equal to the sum of the exponents) we estimate by upper and lower bounds (Fuchs' relations). We use this definition to construct, for every linear differential equation on a compact Riemann surface (with regular or irregular singularities), the *companion bundle* of the equation, a vector bundle endowed with a meromorphic connection that is equivalent to the given equation and has precisely the same singularities and the same set of exponents.

Introduction

Exponents of differential equations are complex numbers that describe the behavior of solutions in a neighborhood of a singular point. Consider a fuchsian differential equation with coefficients in $\mathbb{C}(z)$:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0.$$

Its exponents $(e_i^s)_{1\leqslant i\leqslant n}$ for $s \in \mathbb{P}^1(\mathbb{C})$, defined by Fuchs [1866], obey the celebrated *Fuchs relation*

$$\sum_{s\in\mathbb{P}^1(\mathbb{C})} \sum_{i=1}^{n} (e_i^s - (i-1)) = -n(n-1).$$

This relation, and several generalizations — on a Riemann surface in [Saito 1958] and for irregular singularities in [Bertrand and Beukers 1985; Chudnovsky and Chudnovsky 1984], for example — have been used in arithmetic, following ideas by C. L. Siegel, to give transcendence estimates for values of special functions

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on the basis of differential equations that they satisfy. Basically, differential equations and differential systems are equivalent, and both could be used for the same purpose. However, the theory of exponents for systems has lagged behind some 80 years, probably due to the absence of such a nice characterization of a regular singular point as provided by Fuchs' theorem for equations.

The counterexample given by A. A. Bolibrukh [1990b] to the strong Riemann–Hilbert problem (that is, with simple poles) is based on a Fuchs relation argument for differential systems, and has shown yet another application for this notion. Indeed, if

$$\frac{dX}{dz} = AX, \quad \text{where } A \in \mathrm{M}_n(\mathbb{C}(z)), \tag{1}$$

is a meromorphic differential system with regular singularities on $\mathbb{P}^1(\mathbb{C})$, Bolibrukh proves that its exponents $\mu_1^s, \dots, \mu_n^s$, defined by Levelt [1961], satisfy the relation $\sum_{s\in\mathbb{P}^1(\mathbb{C})} \sum_{i=1}^n \mu_i^s \leqslant 0$, and the equality is achieved exactly when the system has only simple poles; such systems are called *Fuchsian*.

Although both categories are usually treated as strictly equivalent, the sum of exponents for equations is a positive integer, while the sum of exponents for systems is negative. Hence, there cannot exist a differential system that is globally equivalent to a differential equation, in that it has the same exponents everywhere. In the regular case, such equivalent constructions (by way of a *companion matrix*) are always possible locally, as is well known. To perform this globally, one must glue these local data together; therefore we must introduce the framework of holomorphic vector bundles equipped with a meromorphic connection.

The first contribution of this paper is to bridge the gap between the two notions of exponents by constructing in a canonical way such a vector bundle (the *companion bundle* of the differential equation), which gives the correct local exponents at every singularity. In case the equation is fuchsian, the vector bundle that we get has been constructed by M. F. Singer and M. van der Put [2003], albeit from another perspective. In the irregular case, the exponents of the companion bundle are equal to the exponents $(\lambda_i^s)_{1\leqslant i\leqslant n}$ for $s \in \mathbb{P}^1(\mathbb{C})$, defined by D. Bertrand and G. Laumon [1985] as a generalization of Fuchs' exponents, for the arithmetical purposes explained above. In this last case, a ramification of the variable is usually required. For the construction of the companion bundle, it is thus necessary to replace the Riemann sphere with a compact covering $\pi : X \to \mathbb{P}^1(\mathbb{C})$, ramified over the singular points of the equation.

This requires therefore a precise definition of the exponents $\mu_1^{\mathscr{E},s}, \dots, \mu_n^{\mathscr{E},s}$ at any point $s \in X$ for a vector bundle $\mathscr{E}$ endowed with a meromorphic connection ∇ on a Riemann surface X. Our algebraic Definition 3, which extends the notion of exponents defined by Levelt for the regular case, involves the construction of a

special "well-behaved" *Levelt bundle* $\mathscr{E}_L$ of the bundle $\mathscr{E}$; see Theorem 1. This part of the article builds on local results from [Corel 2004], in which we defined the exponents of the linear differential system (1) at an *irregular* singularity $s \in \mathbb{P}^1(\mathbb{C})$.

First, we prove that these exponents are defined in a coordinate-invariant way (unlike for instance the *generalized exponents* defined by Van Hoeij [1997], whose Laurent polynomial part depends on the chosen local coordinate); consequently, we find that the Bertrand–Laumon exponents are well defined on a Riemann surface.

Second, we derive in Theorem 4 new Fuchs-like estimates for the sum of the exponents of a general meromorphic connection on a holomorphic bundle. Arithmetic applications therefore do not need to be restricted to actual solutions of differential equations, but can be used with horizontal sections of arbitrary meromorphic connections on any compact Riemann surface.

Finally, the definition of the Levelt bundle can shed some light on the Riemann–Hilbert problem and some of its generalizations. P. Deligne [1970] showed how to extend a holomorphic vector bundle $\mathscr{E}$, defined over the complement of a divisor $\mathscr{D}$ and endowed with a holomorphic connection ∇ having regular singularities on $\mathscr{D}$, into a logarithmic connection $(\overline{E}, \overline{\nabla})$ that has singularities on the divisor and is uniquely determined by a section of the natural projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. In this way, we get all logarithmic extensions of $\mathscr{E}$ with *nonresonant* residue; these are the *Deligne lattices*. This construction is sufficient to solve positively the weak Riemann–Hilbert problem (that is, the one with regular singularities). The strong Riemann–Hilbert problem however asks for a logarithmic bundle (with the prescribed monodromy) that is also trivial. Bolibrukh's counterexample requires therefore the knowledge of all the logarithmic extensions of a regular connection in order to prove that none is trivial. The Levelt bundles, in the regular case, correspond precisely to the missing logarithmic extensions, those for which we allow the residue to be *resonant*.

In the irregular case, the Levelt bundle is, roughly, the "closest" bundle with the property of being locally split in isotypical components, themselves logarithmic modulo an irregular part (we call this property "log-split"). Like in the regular case, the Levelt bundle happens to be the extension to the resonant case of a classical construction, here Malgrange's [1996] *canonical bundle*, which extends Deligne's to the irregular case. It is also a natural extension of the notion of "logarithmic bundle" as it is likewise characterized by a Fuchs relation (see Theorem 3)

$$\mathscr{E} \text{ is log-split if and only if } \quad \sum_{s\in X}\sum_{i=1}^{n} \mu_i^{\mathscr{E},s} = \deg \mathscr{E}. \tag{2}$$

In the formulation of Bolibrukh, Malek and Mitschi [Bolibruch et al. 2006], the generalized Riemann–Hilbert problem asks for a trivial bundle with minimal Poincaré ranks with respect to the unique connection satisfying some prescribed

"generalized monodromy data". As one expects, the reducibility of the monodromy is involved in the solubility, but more unexpectedly, ramification (or rather its absence) also is. Van der Put and Saito [2009] have already provided a counterexample to this version of the problem. However, it is not clear whether a bundle with minimal Poincaré ranks provides the most natural generalization of the logarithmic bundles required in the regular case. Indeed, a bundle fails to be logarithmic when the connection exhibits nilpotency (appearing under the form of an irregular part whose leading term is nilpotent) that can be suppressed by meromorphic gauge transforms. Requiring minimal Poincaré rank in the irregular case does not prevent the regular part of the connection from having a residual nilpotency hidden under the terms of the irregular part (this can result, for example, in a nondiagonalizable leading term). In terms of the Babbitt–Varadarajan decomposition of the connection into a regular connection and an irregular part [Babbitt and Varadarajan 1983; Corel 2004], the extension of the Fuchs relation for Fuchsian systems as formula (2) suggests that the equivalent of "logarithmic" in the irregular case requires minimal (that is, zero) Poincaré rank for the regular part, and diagonalizability for the irregular part of the connection, namely that the bundle be log-split. At any rate, it would be interesting to know if, in the unramified case at least, the variant of the generalized Riemann–Hilbert problem asking for a trivial log-split bundle admits a solution. On the other hand, the tools introduced here also apply for the minimum Poincaré rank case (Proposition 1), so interesting insights can be reasonably expected, although they would require more work, since the corresponding bundle has no direct characterization by exponents.

In Section 1, we prove that the exponents of a meromorphic connection on a holomorphic vector bundle can be defined by a global object, the *Levelt bundle*; we do this basically by gluing the local *Levelt lattices* constructed in [Corel 2004]. For these exponents, we establish new estimates based on the local Jordan structure of the residual maps of the connection. In Section 2, we construct the *companion bundle* of a differential equation, and in Section 3, we establish the global estimates.

1. The Levelt bundle and the exponents

Let X be a compact Riemann surface and $\mathcal{D} = \sum_{x \in X} m_x x$ be a positive divisor on X with (finite) support $|\mathcal{D}| = \{x \in X \mid m_x \neq 0\}$. Let $\mathcal{O}_{\mathcal{D}}$ be the sheaf of meromorphic functions on X whose poles have order bounded by m_x at $x \in X$. For a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules (subsequently called a *vector bundle*) of finite rank on X, let $\mathcal{E}_{\mathcal{D}} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{D}}$. Let $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{\mathcal{D}}$ be a meromorphic connection with singular divisor $\mathcal{D}$ on a vector bundle $\mathcal{E}$ of rank n. In the sequel, we will always assume that $\mathcal{D}$ is the smallest possible. Sometimes for simplicity we'll just say "connection" for the pair $(\mathcal{E}, \nabla)$. The *Poincaré rank* of ∇ at $x \in X$ is the integer

$\mathfrak{p}_x(\nabla) = \max(0, m_x - 1)$. If $\mathfrak{p}_x(\nabla) = 0$, the sheaf $\mathscr{E}$ is said to be *logarithmic with respect to* ∇ *at* x. Let $\mathscr{S} = |\mathscr{D}|$ be the singular, and $\mathscr{S}_{\log} = \{x \in \mathscr{S} \mid \mathfrak{p}_x(\nabla) = 0\}$ the logarithmic singular sets of ∇. If $\mathscr{S}_{\log} \neq \varnothing$, then one can define the residue map $\operatorname{Res} \nabla \in \mathscr{E}nd(\mathscr{E}/\mathscr{E}_{-[\mathscr{S}_{\log}]})$, where $[\{x_1, \dots, x_m\}] = x_1 + \cdots + x_m$. If $x \in \mathscr{S} \setminus \mathscr{S}_{\log}$ is not logarithmic, the residue of $z^{\mathfrak{p}_x(\nabla)}\nabla$ for any local coordinate z induces the well-defined *polar map* $PM\,\nabla \in \mathbb{P}\mathscr{E}nd(\mathscr{E}/\mathscr{E}_{-[\mathscr{S}]})$ of ∇ over $\mathscr{E}$. For more details on these notions, see [Sabbah 2002].

1.1. ***Local maximal lattices.*** Let $x \in \mathscr{S}$. Let $\mathbb{O} = \widehat{\mathbb{O}_{X,x}}$ be the formal completion of the ring $\mathbb{O}_{X,x}$ endowed with its discrete valuation v with respect to its maximal ideal $\mathfrak{m}$, and let K be the fraction field of $\mathbb{O}$. The $\mathbb{O}$-module $\Lambda = \mathscr{E}_x \otimes_{\mathbb{O}_{X,x}} \mathbb{O}$ is a *lattice* in the K-vector space $V = \Lambda \otimes_{\mathbb{O}} K$, on which it induces an additive valuation $v_\Lambda(x) = \max\{k \in \mathbb{Z} \mid x \in \mathfrak{m}^k\Lambda\}$ for $x \in V$. Let $\mathbf{\Omega} = \mathbf{\Omega}_{X,x} \otimes_{\mathbb{O}_{X,x}} K$ and $\mathbf{\Omega}(k) = \{\omega \in \mathbf{\Omega} \mid v(\omega) \geqslant -k\}$. The germ of connection

$$\nabla_x : \mathscr{E}_x \to \mathscr{E}_x \otimes_{\mathbb{O}_{X,x}} \mathbf{\Omega}_{\mathscr{D},x}$$

extends as the map $\mathbf{\nabla} = \nabla_x \otimes \mathrm{id}_K + \mathrm{id}_V \otimes d : V \to V \otimes_K \mathbf{\Omega}$. The triple $(V, \Lambda, \mathbf{\nabla})$ is called the (*formal*) *local model* of $(\mathscr{E}, \nabla)$ at x.

Let z be a uniformizing parameter of K, and consider the derivation $\theta = z\frac{d}{dz}$. The Poincaré rank $\mathfrak{p}$ of ∇ at x satisfies $\mathfrak{p} = -v_\Lambda(\mathbf{\nabla}_\theta(\Lambda) + \mathfrak{m}\Lambda)$, where $v_\Lambda(M) = \inf_{x \in M} v_\Lambda(x)$. The map δ_Λ induced by $z^{\mathfrak{p}}\mathbf{\nabla}_\theta$ on $\overline{\Lambda} = \Lambda/\mathfrak{m}\Lambda$ is a representative of the germ $(PM\,\nabla)_x$. If $\mathfrak{p} = 0$, put $\delta_\Lambda = 0$. For any integer $\ell > 0$, we have $z^{\ell\mathfrak{p}}\mathbf{\nabla}_\theta^\ell(\Lambda) = (z^{\mathfrak{p}}\mathbf{\nabla}_\theta)^\ell \bmod \mathfrak{m}\Lambda$. The map induced by either on $\overline{\Lambda} = \Lambda/\mathfrak{m}\Lambda$ is equal to δ_Λ^ℓ. Let $\mathscr{K} = (k_1, \dots, k_n)$ be the (increasing) sequence of elementary divisors of a lattice $M \subset \Lambda$, and let $[\Lambda : M] = \sum_{i=1}^n (k_i - k_1)$ be its *index* with respect to Λ. A lattice M in V induces a natural flag in $\overline{\Lambda}$ as follows. Let

$$\mathfrak{m}\Lambda \subset M_k = (z^{-k}M \cap \Lambda) + \mathfrak{m}\Lambda \subset \Lambda \quad \text{for } k \in \mathbb{Z}.$$

The images $\overline{M}_k$ in the quotient space $\overline{\Lambda}$ induce a flag of $\mathbb{C}$-vector spaces

$$F_\Lambda(M) : 0 = \mathscr{F}_0 \subsetneq \mathscr{F}_1 \subsetneq \cdots \subsetneq \mathscr{F}_s = \overline{\Lambda},$$

where $\mathscr{F}_i = \overline{M}_{\mathbf{k}_i}$; here $\mathbf{k}_i$ is the i-th distinct elementary divisor of M.

Let $m(\mathbf{\nabla})$ be the *order of singularity* of $\mathbf{\nabla}$, that is, the minimum of the Poincaré ranks over all lattices. For any $\mathfrak{q} \geqslant m(\mathbf{\nabla})$, there exists a largest lattice of $\Lambda_\mathfrak{q} \subset \Lambda$ having Poincaré rank $\leqslant \mathfrak{q}$; see [Corel 2004, Corollary 2.2].

Proposition 1. *Let Λ be a lattice in V with Poincaré rank $\mathfrak{p}$. Let $\delta_i = \mathrm{rk}(\delta_\Lambda)^i$ and $\Delta_i = \delta_i - \delta_{i+1}$ for $i \geqslant 0$. For $m(\mathbf{\nabla}) \leqslant \mathfrak{q} \leqslant \mathfrak{p}$, the maximal lattice $\Lambda_\mathfrak{q} \subset \Lambda$ satisfies*

$$d \sum_{i \geqslant 1} \delta_i \leqslant [\Lambda : \Lambda_\mathfrak{q}] \leqslant \tfrac{1}{2}n(n-1)d - \tfrac{1}{2}\sum_{i \geqslant 0} \Delta_i(\Delta_i - 1), \quad \textit{where } d = \mathfrak{p} - \mathfrak{q}.$$

Let Λ and M be lattices in V, and let $\mathfrak{p}_\Lambda$ and $\mathfrak{p}_M$ be their respective Poincaré ranks with respect to the connection ∇.

Lemma 1. *For $k \in \mathbb{Z}$, one has $\delta_\Lambda(\overline{M}_k) \subset \overline{M}_\ell$, where $\ell = k + (\mathfrak{p}_M - \mathfrak{p}_\Lambda)$.*

Proof. Put $p = \mathfrak{p}_\Lambda$, let $m \in M$ such that $z^{-k}m \in \Lambda$ holds, and let $\lambda \in \Lambda$. Then,

$$z^p \nabla_\theta(z^{-k}m + z\lambda) = -kz^{p-k}m + z^{-k}z^p\nabla_\theta(m) + z(z^p\nabla_\theta(\lambda) + z^p\lambda).$$

By assumption, $z^{p-k}m \in z^{p-k}M \cap z^p\Lambda$ holds, and so does $z(z^p\nabla_\theta(\lambda)+z^p\lambda) \in \mathfrak{m}\Lambda$. By definition of Poincaré rank, we have $z^p\nabla_\theta(m) = z^{p-\mathfrak{p}_M}(z^{\mathfrak{p}_M}\nabla_\theta(m)) \in z^{p-\mathfrak{p}_M}M$, and, since $m \in z^k\Lambda$, we then get $z^p\nabla_\theta(m) \in z^k\Lambda$. Accordingly, we find that $z^{-k}z^p\nabla_\theta(m) \in z^{-k}(z^{p-\mathfrak{p}_M}M \cap z^k\Lambda) = z^{-(k+\mathfrak{p}_M-p)}M \cap \Lambda$. Consequently, we have $z^p\nabla_\theta(z^{-k}m + z\lambda) \in (z^{-(k+\mathfrak{p}_M-p)}M \cap \Lambda) + \mathfrak{m}\Lambda = M_{k+(\mathfrak{p}_M-p)}$, and the lemma is established by taking the quotient by $\mathfrak{m}\Lambda$. □

Corollary 1. *Let M be a lattice having Poincaré rank $\mathfrak{p}_M < \mathfrak{p}_\Lambda$ and elementary divisors $(k_1, \dots, k_n)$ with respect to Λ. Let $F_\Lambda(M) : 0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_s = \overline{\Lambda}$ be the flag induced by M.*

(i) *The flag $F_\Lambda(M)$ satisfies $\delta_\Lambda(\mathcal{F}_i) \subset \mathcal{F}_{i-1}$ for all $1 \leqslant i \leqslant s$.*

(ii) *If $k_{i+1} - k_i < \mathfrak{p}_\Lambda - \mathfrak{p}_M$, the graduated map $\mathrm{gr}^{\mathcal{F}}_{i+1}(\delta_\Lambda) : \mathcal{F}_{i+1}/\mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1}$ is equal to 0.*

(iii) *For $\ell > 0$, let $t = \max\{1 \leqslant i \leqslant n \mid k_i - k_1 < \ell(\mathfrak{p}_\Lambda - \mathfrak{p}_M)\}$. Then $\overline{M}_t \subset \ker \delta^\ell_\Lambda$.*

Proof. The three assertions are consequences of Lemma 1. Indeed, if $\mathcal{F}_i = \overline{M}_{k_i}$, then $\delta_\Lambda(\mathcal{F}_i) \subset \overline{M}_{k_i+\mathfrak{p}_M-\mathfrak{p}_\Lambda} \subset \overline{M}_{k_i-1} = \overline{M}_{k_{i-1}} = \mathcal{F}_{i-1}$. The second is established by remarking that if $k_{i+1} - k_i < \mathfrak{p}_\Lambda - \mathfrak{p}_M$, then $k_{i+1} + \mathfrak{p}_M - \mathfrak{p}_\Lambda < k_i$, and therefore $k_{i+1} + \mathfrak{p}_M - \mathfrak{p}_\Lambda \leqslant k_i - 1$. Thus $\delta_\Lambda(\mathcal{F}_{i+1}) \subset \overline{M}_{k_i-1} = \mathcal{F}_{i-1}$. Finally, one has $\delta^\ell_\Lambda(\overline{M}_t) \subset \overline{M}_{k_t+\ell(\mathfrak{p}_M-\mathfrak{p}_\Lambda)} \subset \overline{M}_{k_1-1} = 0$, since $k_t - \ell(\mathfrak{p}_M - \mathfrak{p}_\Lambda) < k_1$ according to the definition of t; this establishes the third result. □

Let $\mathcal{K} = (k_1, \dots, k_n)$ be the elementary divisors of $\Lambda_\mathfrak{q}$ in Λ, and let $\mathcal{F}$ be the flag induced by $\Lambda_\mathfrak{q}$ in $\Lambda/\mathfrak{m}\Lambda$. Recall (see [Corel 2004, Proposition 4.2]) that $k_1 = 0$ and $\max_{i=1,\dots,n-1} k_{i+i} - k_i \leqslant \mathfrak{p} - \mathfrak{q} \leqslant k_n$. In this case, the converse of Corollary 1(ii) holds.

Lemma 2. *With the previous notations, $k_{i+1} - k_i = \mathfrak{p} - \mathfrak{q}$ if and only if $\mathrm{gr}^{\mathcal{F}}_{i+1}(\delta_\Lambda) \neq 0$.*

Proof. Let (e) be a basis of Λ and (ε) a basis of $\Lambda_\mathfrak{q}$ such that the matrix of the basis change from (e) to (ε) is z^K where $K = \mathrm{diag}(k_1 I_{n_1}, \dots, k_s I_{n_s})$, by considering the elementary divisors with multiplicities. Recall that $n_i = \dim \mathcal{F}_i - \dim \mathcal{F}_{i-1}$. This means to consider the sequence $t_1 < \cdots < t_s$ such that $n_j = t_j - t_{j-1} + 1$ and

$$0 = k_1 = \cdots = k_{t_1} < k_{t_1+1} = \cdots = k_{t_2} < \cdots < k_{t_{s-1}+1} = \cdots = k_{t_s} = k_n.$$

Let $A = \mathrm{Mat}(\nabla_\theta, (e)) = A_{-\mathfrak{p}} z^{-\mathfrak{p}} + \cdots$. The matrix $A_{-\mathfrak{p}}$ has the block-triangular form

$$A_{-\mathfrak{p}} = \begin{pmatrix} 0_{n_1} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1s} \\ 0 & 0_{n_2} & \cdots & \mathcal{A}_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0_{n_s} \end{pmatrix}. \tag{3}$$

Let $\ell \in \{1, \dots, n-1\}$ such that $k_\ell > 0$, and consider the $\tilde{\mathcal{K}} = (\tilde{k}_1, \dots, \tilde{k}_n)$, where $\tilde{k}_i = k_i$ for $i \leqslant \ell$ and $\tilde{k}_j = k_j - 1$ for $j \geqslant \ell + 1$. Since $\Lambda_{\mathfrak{q}}$ is maximal, there exists (i_0, j_0) such that $v(A_{i_0, j_0}) + \tilde{k}_{j_0} - \tilde{k}_{i_0} < -\mathfrak{q}$. We have

$$\tilde{k}_j - \tilde{k}_i = \begin{cases} k_j - k_i & \text{if } i \leqslant \ell \text{ and } j \leqslant \ell, \text{ or } i \geqslant \ell + 1 \text{ and } j \geqslant \ell + 1, \\ k_j - k_i + 1 & \text{if } i \geqslant \ell + 1 \text{ and } j \leqslant \ell, \\ k_j - k_i - 1 & \text{if } i \leqslant \ell \quad \text{and } j \geqslant \ell + 1. \end{cases}$$

Thus, we get $i_0 \leqslant \ell < \ell + 1 \leqslant j_0$, and accordingly $v(A_{i_0, j_0}) + k_{j_0} - k_{i_0} = -\mathfrak{q}$. Assume that $k_{\ell+1} - k_\ell = \mathfrak{p} - \mathfrak{q}$. Since $k_{j_0} - k_{i_0} \geqslant k_{\ell+1} - k_\ell$, we have

$$-\mathfrak{p} \leqslant v(A_{i_0, j_0}) \leqslant -k_{j_0} + k_{i_0} - \mathfrak{q} \leqslant -\mathfrak{p};$$

hence $v(A_{i_0, j_0}) = -\mathfrak{p}$. Therefore $(A_{-\mathfrak{p}})_{i_0, j_0} \neq 0$ with $k_{i_0} = k_\ell$ and $k_{j_0} = k_{\ell+1}$. Let u be the index such that $k_\ell = k_{t_u}$. Then, in the block structure of $A_{-\mathfrak{p}}$ we have $\mathrm{rk}\, \mathcal{A}_{u,u+1} \geqslant 1$. This implies that $\mathcal{A}_{u,u+1} \neq 0$ if and only if $k_{t_{u+1}} - k_{t_u} = \mathfrak{p} - \mathfrak{q}$. But $\mathcal{A}_{u,u+1}$ is nothing but the matrix of the graduated map $\mathrm{gr}^{\mathcal{F}}_{u+1}(\delta_\Lambda)$. □

Let $\ell \geqslant 0$ be an integer. Let $a_\ell = \max\{1 \leqslant i \leqslant n \mid k_i < \ell(\mathfrak{p} - \mathfrak{q})\}$ and $a_0 = 0$, and let

$$\mathcal{J}_\ell = \{1 \leqslant i \leqslant n - \ell \mid k_{i+\ell} = k_i + \ell(\mathfrak{p} - \mathfrak{q})\} \text{ and } g_\ell = |\mathcal{J}_\ell|.$$

Corollary 2. *For any integer $\ell \geqslant 0$, we have $g_\ell \leqslant \delta_\ell \leqslant n - a_\ell$.*

Proof. The second inequality is a direct consequence of Corollary 1. Indeed, one has $k_1 = 0$ and $a_\ell = \dim \overline{M}_{a_\ell} \leqslant \dim \ker \delta^\ell$. For the first inequality, note that $\mathrm{rk}\, \delta_\Lambda \geqslant \sum_{i=0}^{s-1} \mathrm{rk}\, \mathrm{gr}^{\mathcal{F}}_{i+1}(\delta_\Lambda)$. By Lemma 2, we get

$$\delta_1 \geqslant \left|\{1 \leqslant i \leqslant n - 1 \mid k_{i+1} = k_i + \mathfrak{p} - \mathfrak{q}\}\right| = g_1.$$

Since δ_Λ is strictly upper block-triangular, we have $\mathrm{rk}\, \delta^\ell_\Lambda \geqslant \sum_{i=0}^{s-\ell} \mathrm{rk}\, \mathrm{gr}^{\mathcal{F}}_{i+\ell}(\delta^\ell_\Lambda)$ for $\ell > 0$. Now, $\mathrm{gr}^{\mathcal{F}}_{i+\ell}(\delta^\ell_\Lambda) = \mathrm{gr}^{\mathcal{F}}_{i+\ell}(\delta_\Lambda) \circ \mathrm{gr}^{\mathcal{F}}_{i+\ell-1}(\delta_\Lambda) \circ \cdots \circ \mathrm{gr}^{\mathcal{F}}_{i+1}(\delta_\Lambda)$. If $i \in \mathcal{J}_\ell$, then $\dim \mathcal{F}_{i+1}/\mathcal{F}_i = \cdots = \dim \mathcal{F}_{i+\ell}/\mathcal{F}_{i+\ell-1} = 1$. Since $\mathrm{gr}^{\mathcal{F}}_k(\delta_\Lambda)$ is nonzero for $i + 1 \leqslant k \leqslant i + \ell$, the map $\mathrm{gr}^{\mathcal{F}}_{i+\ell}(\delta^\ell_\Lambda)$ is also nonzero, and $\mathrm{rk}\, \delta^\ell_\Lambda \geqslant g_\ell$ follows. □

1.2. ***The proof of Proposition 1.*** The proof relies on two combinatorial results.

Lemma 3. *Let $T = (t_1, \dots, t_n)$ be an increasing sequence of integers such that $t_1 = 0$ and $t_{i+1} - t_i \leqslant 1$. Let $J_\ell = \{1 \leqslant i \leqslant n-\ell \mid t_{i+\ell} = t_i + \ell\}$ and $c_\ell = |J_\ell \setminus J_{\ell+1}|$. Then we have*

$$\frac{1}{2}\sum_{j\geqslant 0} c_j(c_j-1) \leqslant \sum_{i\in J_0\setminus J_1}(n-i) \leqslant c_0\sum_{j\geqslant 0} c_j - \frac{1}{2}\sum_{j\geqslant 0} c_j(c_j-1).$$

Proof. Let $J_0 \setminus J_1 = \{h_1, \dots, h_{c_0}\}$ with $h_0 = 0 < h_1 < \cdots < h_{c_0} = n$, and let $R = \sum_{i\in J_0\setminus J_1}(n-i)$. Clearly, $R = \sum_{i=1}^{c_0}(n-h_i) = \sum_{i=0}^{c_0} i(h_{i+1}-h_i)$. The interval $I = [0, n]$ is cut into c_0 segments of respective lengths $h_{i+1}-h_i$ for $0 \leqslant i \leqslant c_0-1$. By construction, there are $|\{1 \leqslant i \leqslant c_0 \mid h_i - h_{i-1} = j\}| = c_{j-1} - c_j$ segments of length j. The quantity R is the sum of these lengths weighted by their index of appearance along the interval I; therefore it is minimal (respectively maximal) when the segments are ordered by decreasing (respectively increasing) length. Thus, we get the bounds

$$\sum_{j\geqslant 1} j \sum_{i=c_j}^{c_{j-1}-1} i \leqslant R \leqslant \sum_{j\geqslant 1} j \sum_{i=c_0-c_{j-1}}^{c_0-c_j-1} i.$$

A direct computation yields the claimed result. □

For an integer sequence $(g_i)_{i\geqslant 0}$, let $(\Delta g_i)_{i\geqslant 0}$ be defined by $\Delta g_i = g_i - g_{i+1}$. Let us say that $(g_i)_{i\geqslant 0}$ is *strongly decreasing* if both $(g_i)_{i\geqslant 0}$ and $(\Delta g_i)_{i\geqslant 0}$ are nonincreasing. Let $\Delta g = \sum_{i\geqslant 0}\Delta g_i(\Delta g_i - 1)$. Note that

$$\Delta g = g_0^2 - 2\sum_{j\geqslant 0} g_{j+1}\Delta g_i. \tag{4}$$

Lemma 4. *Let $(g_i)_{i\geqslant 0}$ and $(\delta_i)_{i\geqslant 0}$ be two integer strongly decreasing sequences such that $g_0 = \delta_0$ and $\delta_i \geqslant g_i$ for all $i \geqslant 0$. Then $\Delta g \geqslant \Delta\delta$.*

Proof. By formula (4), one has $h = \frac{1}{2}(\Delta g - \Delta\delta) = \sum_{i\geqslant 1}(\delta_i - g_i)(\Delta\delta_{i-1} - \Delta g_i)$. Write $\Delta\delta_{i-1} - \Delta g_i = (\delta_{i-1} - g_{i-1}) - (\delta_i - g_i) + \Delta g_{i-1} - \Delta g_i$. Substituting in h, we straightforwardly get $h = \sum_{i\geqslant 1}(\delta_i - g_i)(\Delta g_{i-1} - \Delta\delta_i)$. Summing the two expressions for h, we get $2h = \sum_{i\geqslant 1}(\delta_i - g_i)(\Delta g_{i-1} - \Delta g_i + \Delta\delta_{i-1} - \Delta\delta_i) \geqslant 0$; hence $\Delta g \geqslant \Delta\delta$. □

Proof of Proposition 1. Let us consider the sequence $\mathcal{K} = (k_1, \dots, k_n)$ of elementary divisors of $\Lambda_{\mathfrak{q}}$ with respect to Λ and let $d = \mathfrak{p} - \mathfrak{q}$. Recall that

$$[\Lambda : \Lambda_{\mathfrak{q}}] = \dim_{\mathbb{C}} \Lambda/\Lambda_{\mathfrak{q}} = k_1 + \cdots + k_n;$$

see [Corel 2004, Lemma 2.3, page 1371]. Define ℓ_0 as the largest ℓ such that $a_\ell < n$, so that $1 \leqslant a_1 \leqslant \cdots \leqslant a_{\ell_0} < n = a_{\ell_0+1}$. By definition of a_i, we get

$\sum_{i=a_j+1}^{a_{j+1}} k_i \geqslant jd(a_{j+1}-a_j)$. According to Corollary 2, we have

$$[\Lambda:\Lambda_{\mathrm{q}}]=\sum_{j=0}^{\ell_0}\sum_{i=a_j+1}^{a_{j+1}} k_i \geqslant \sum_{j=0}^{\ell_0} j(a_{j+1}-a_j)d=\sum_{j=1}^{\ell_0}(n-a_j)d \geqslant d\sum_{j=1}^{\ell_0}\delta_j,$$

which proves the left side of the proposition. For $1 \leqslant i \leqslant n-1$, let $d_i = k_{i+1}-k_i$. Since $d_i \leqslant d-1$ whenever $i \notin \mathcal{J}_1$, we get

$$\sum_{i=1}^{n} k_i = \sum_{i=1}^{n-1}(n-i)d_i \leqslant \tfrac{1}{2}n(n-1)d - \sum_{i\notin\mathcal{J}_1}(n-i).$$

We wish to minimize $R=\sum_{i\in\mathcal{J}_0\setminus\mathcal{J}_1}(n-i)$. Construct the following sequence $T=(t_1,\dots,t_n)$ defined by induction as

$$\begin{cases} t_1 = 0, \\ t_{i+1} = t_i + [(k_{i+1}-k_i)/d], \quad \text{where } [x] \text{ denotes the integer part of } x. \end{cases}$$

The sequence T satisfies the assumptions of Lemma 3 with $c_i = \Delta g_i$; hence

$$R \geqslant \frac{1}{2}\sum_{j\geqslant 0}\Delta g_j(\Delta g_j - 1). \tag{5}$$

Let n_i be the number of Jordan blocks of size i in the Jordan normal form of δ. We have

$$\delta_i = \sum_{j\geqslant i+1}(j-i)n_j \quad \text{and} \quad \Delta_i = \delta_i - \delta_{i+1} = \sum_{j\geqslant i+1} n_j \quad \text{for all } i \geqslant 0. \tag{6}$$

Accordingly, we have $\delta_i \geqslant \delta_{i+1}$ and $\Delta_i \geqslant \Delta_{i+1}$ for all $i \geqslant 0$. Let $\Gamma_i = g_i - g_{i+1}$. Both $g_i \geqslant g_{i+1}$ and $\Gamma_i \geqslant \Gamma_{i+1}$ hold, whereas Corollary 2 ensures that $g_i \leqslant \delta_i$. Both sequences are strongly decreasing, and $g_0 = n = \delta_0$. Therefore, by Lemma 4, we get $\Gamma = \sum_{j\geqslant 0}\Gamma_j(\Gamma_j - 1) \geqslant \Delta = \sum_{j\geqslant 0}\Delta_j(\Delta_j - 1)$. According to (5), we have $R \geqslant \Gamma$, so $R \geqslant \Delta$, thereby establishing the right side of the proposition. □

1.3. ***The Levelt bundle.*** Let $(\mathscr{E},\nabla)$ be a meromorphic connection, and $(V,\Lambda,\mathbf{\nabla})$ be the local model at $x\in X$. Let $\mathbf{\nabla} = \nabla^{\mathrm{reg}} + \omega$ be the formal *Babbitt–Varadarajan decomposition* of $\mathbf{\nabla}$ into a regular formal connection ∇^{reg} and the K-linear map $\omega : V \to V \otimes_K \mathbf{\Omega}$ that we introduced in [Corel 2004, Theorem 3]. By a slight abuse of terminology, we will act as if ω (which we call an *irregular part of* ∇ *at* x from now on) were an endomorphism of V with coefficients in $\mathbf{\Omega}$, and speak of its eigenspaces, eigenvalues, and so on. Although the Babbitt–Varadarajan decomposition depends on the choice of a local coordinate, we can retrieve from it the following invariants of ∇ at x:

- the *ramification index* $\rho_x(\nabla)\in\mathbb{N}$, equal to the splitting degree of ω;

- the *Katz rank* $\kappa_x(\nabla) = \max(-v_x(\omega_1) - 1, \dots, -v_x(\omega_n) - 1) \in \mathbb{Q}$;
- the *Malgrange irregularity* $i_x(\nabla) = \sum_{i=1}^{n}(-v_x(\omega_i) - 1) \in \mathbb{Q}$.

Here v_x is the valuation defined by $(\mathcal{O}_X)_x$ (extended to formal ramified 1-forms in the natural way). We'll use the same notation for the local model at a given point.

Definition 1. Let $(\mathcal{E}, \nabla)$ be a meromorphic connection. Let $x \in X$, let $(V, \Lambda, \boldsymbol{\nabla})$ be the local model of $(\mathcal{E}, \nabla)$, and let $\boldsymbol{\nabla} = \nabla^{\mathrm{reg}} + \omega$ be a Babbitt–Varadarajan decomposition of ∇ at x. The vector bundle $\mathcal{E}$ is *log-split at x* if

(i) $\rho_x(\nabla) = 1$, that is, ∇ is unramified at x;

(ii) $\nabla^{\mathrm{reg}}(\Lambda) \subset \Lambda \otimes_{\mathcal{O}} \boldsymbol{\Omega}(1)$, that is, Λ is logarithmic with respect to ∇^{reg};

(iii) $\bigoplus_{i=1}^{s} \Lambda \cap V_i = \Lambda$, where $V_1, \dots, V_s$ the eigenspaces of ω.

We will next use a classical fact; see [Van der Put and Singer 2003, page 166].

Lemma 5. *Let $\mathcal{E}$ be endowed with a meromorphic connection ∇. For any family of lattices $M(x) \in \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} K(\mathcal{O}_{X,x})$ for x in a discrete set $\mathcal{S}$, there exists a unique vector bundle $\mathcal{E}^M$ such that*

$$(\mathcal{E}^M)_x = \begin{cases} \mathcal{E}_x & \text{if } x \notin \mathcal{S}, \\ M(x) & \text{if } x \in \mathcal{S} \end{cases}$$

and a canonical meromorphic connection on $\mathcal{E}^M$ that coincides with ∇ over $X \setminus \mathcal{S}$. We still denote it by ∇ although it may have additional singularities on $\mathcal{S}$.

Let $\Xi(\mathcal{E})$ denote the set of bundles obtained from $\mathcal{E}$ in this way. A natural extension of the local notion of *index* recalled in Section 1.1 to the set $\Xi(\mathcal{E})$ is

$$[\mathcal{E} : \mathcal{F}] = \deg \mathcal{E} - \deg \mathcal{F} - n \sum_{x \in X} v_{\mathcal{E}_x}(\mathcal{F}_x) \quad \text{for } \mathcal{F} \in \Xi(\mathcal{E}).$$

Theorem 1. *Let $\mathcal{E}$ be a holomorphic vector bundle equipped with a connection ∇ that is unramified everywhere. There exists a unique maximal subsheaf $\mathcal{E}_L$ of locally free $\mathcal{O}_X$-modules of rank n of $\mathcal{E}$ that is log-split everywhere. The sheaf $\mathcal{E}_L$ is called the* Levelt bundle *of the vector bundle $\mathcal{E}$.*

Proof. At every singularity x, we can construct the formal *Levelt lattice*

$$\hat{L} = (\mathcal{E}_x \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}_{X,x}})_L$$

of the formalized stalk of $\mathcal{E}$ at x; see [Corel 2004, Proposition 4.1 and Definition 14, page 1384]. The Levelt bundle $\mathcal{E}_L$ is then simply constructed as the extension of $\mathcal{E}$ by the family of lattices $L_x = \hat{L} \cap (\mathcal{E}_x \otimes_{\mathcal{O}_X} K(\mathcal{O}_{X,x}))$ for $x \in \mathcal{S}$. □

Definition 2. Let $\tilde{X}$ be a compact Riemann surface. A nonconstant holomorphic map $\tilde{X} \xrightarrow{\pi} X$ is called a ∇-*admissible cover* of X if the pull-back connection $\pi^*\nabla$ on $\pi^*\mathcal{E}$ over $\tilde{X}$ is unramified everywhere.

Proposition 2. *There exists a ∇-admissible cover of X for any connection ∇.*

Proof. Let $\mathcal{S} = \{x_1, \dots, x_p\}$ be the singular set of ∇ on X, and let $\rho_i = \rho_{x_i}(\nabla)$ be the ramification index at x_i as in Definition 1. Let $m = \operatorname{lcm}(\rho_1, \dots, \rho_p)$, and let g be a cycle of maximal length m in the symmetric group S_m. If p is even, let $g_i = g^{(-1)^{k-1}}$ for $1 \leqslant i \leqslant p$. If $p = 2q+1$ is odd, embed S_m into S_{2m} as the subgroup fixing the last m terms (in the standard action on $\{1, \dots, 2m\}$). Let $\tilde{g}$ be the cycle acting as g on those last m terms. Define now $g_i = g^{(-1)^{k-1}}$ for $1 \leqslant i \leqslant 2q-1$, and put $g_{p-1} = \tilde{g}$ and $g_p = (g\tilde{g})^{-1}$. By construction, we have $g_1 \cdots g_p = 1$, and each permutation g_i is an indecomposable cycle of length m, except at most one which appears as the product of two disjoint cycles of length m. By Riemann's existence theorem (see [Lando and Zvonkin 2004, Theorem 1.8.14 and Remark 1.8.15, pages 74–75]), there exist a compact Riemann surface $\tilde{X}$ and a holomorphic covering $\tilde{X} \xrightarrow{\pi} X$, ramified over $\mathcal{S} = \{x_1, \dots, x_p\}$, such that the order of ramification at all points of $\pi^{-1}(\mathcal{S})$ is equal to m, which is divisible by the corresponding ρ_i. Therefore, the covering map π is ∇-admissible. Note that the degree of π is m in the even case and $2m$ in the odd one, and moreover that each fiber over $\mathcal{S}$ is of cardinality 1, except maybe at one point, where it is at most 2. $\square$

At the cost of introducing an *apparent* ramification point $x_0 \notin \mathcal{S}$, one can impose that the ramification of π be *equal* to the ramification of the connection, as follows. For $\rho, m \in \mathbb{N}$, let the *regular (ρ, m)-permutation* in $S_{\rho m}$ be defined as the product $g = c_1 \cdots c_m$ of the m disjoint cycles

$$c_k = \begin{pmatrix} k\rho+1 & \cdots & (k+1)\rho \\ k\rho+2 & \cdots & k\rho+1 \end{pmatrix}$$

of length ρ. Let $N = \operatorname{lcm}(\rho_1, \dots, \rho_p)$, and for all $1 \leqslant i \leqslant p$, let $m_i = N/\rho_i$. Define $g_i \in S_N$ as the regular (ρ_i, m_i)-permutation in S_N. The group $G = \langle g_i \rangle$ generated by these permutations acts transitively since N is the lcm of the cycle lengths. For an arbitrary point $x_0 \notin \mathcal{S}$, put $g_0 = (g_1 \cdots g_p)^{-1}$. Let $\tilde{X} \xrightarrow{\pi} X$ be the ramified cover attached to the data $(x_0, \dots, x_p, g_0, \dots, g_p)$, as in the proof of Proposition 2. In this case, the fiber $\pi^{-1}(x_i)$ for $i \geqslant 1$ consists of exactly m_i points, each of ramification index ρ_i. For further reference, we call this special covering the *minimal ∇-admissible covering with apparent ramification at x_0*.

Proposition 3. *Let $(\mathcal{E}, \nabla)$ be an unramified connection on X, and let $\mathcal{E}_L$ be the Levelt bundle of $\mathcal{E}$. For any covering map $\tilde{X} \xrightarrow{\pi} X$, we have $(\pi^*\mathcal{E})_L = \pi^*(\mathcal{E}_L)$.*

Proof. We only need to prove the local version of this fact. Let $\tilde{X}$ be a compact Riemann surface, and let $\tilde{X} \xrightarrow{\pi} X$ be a nonconstant holomorphic map. For $\tilde{x} \in \tilde{X}$, the fraction field H of the formal local ring $\mathcal{O}_H = \widehat{\mathcal{O}_{\tilde{X},\tilde{x}}}$ is a finite extension of the fraction field K of $\mathcal{O} = \widehat{\mathcal{O}_{X,x}}$ of degree $m = e_{\tilde{x}}(\pi)$. The local model (W, M, Δ) of $(\pi^*\mathcal{E}, \pi^*\nabla)$ satisfies $W = V \otimes_K H$ and $M = \Lambda \otimes_{\mathcal{O}} \mathcal{O}_H$. In other terms, (W, M, Δ)

is a degree m finite extension of (V, Λ, ∇). Assume that ∇ is unramified at x. The stalk $\Lambda_L = (\mathscr{E}_L)_x$ is the Levelt lattice of $\Lambda = \mathscr{E}_x$. We want to prove that the Levelt lattice $M_L = ((\pi^*\mathscr{E})_L)_{\tilde{x}}$ of M satisfies $M_L = \Lambda_L \otimes_{\mathbb{C}} \mathbb{C}_H$. We first assume that ∇ is regular. The Levelt lattice Λ_L is then simply the largest logarithmic sublattice of Λ. Suppose that $(\Lambda_L)_H \subsetneq M_L$. The lattice $L^- = M_L \cap V$ is the largest sublattice of M_L (with the standard embedding $V \hookrightarrow W$) that is defined over $\mathbb{C}$, and it clearly satisfies $\nabla(L^-) \subset L^- \otimes_{\mathbb{C}} \Omega(1)$. By maximality of the Levelt lattice, we get $\Lambda_L = L^-$. By duality, the smallest $\mathbb{C}$-rational lattice L^+ containing M_L also satisfies $\nabla(L^+) \subset L^+ \otimes_{\mathbb{C}} \Omega(1)$. Therefore, we get the inclusions $L^- = \Lambda_L \subset L^+ \subset \Lambda$. Since Λ_L is maximal, we conclude that $L^- = L^+ = \Lambda_L$; hence $M_L = (\Lambda_L)_H$ and we get the desired contradiction. If ∇ is not regular, let $\boldsymbol{\nabla} = \nabla^r + \omega$ be a Babbitt–Varadarajan decomposition of $\boldsymbol{\nabla}$, and let $V_H = \bigoplus_{i=1}^s V_i$ be the direct sum of eigenspaces of ω. According to [Corel 2004, proof of Proposition 4.1], the Levelt lattice M_L satisfies $M_L = \bigoplus_{i=1}^s M_L(\nabla^{\mathrm{reg}}) \cap V_i$, where $M_L(\nabla^{\mathrm{reg}})$ is the Levelt lattice of M with respect to the regular connection ∇^{reg}. The first part of the proof then yields the claimed result. □

Assume $\mathscr{E}$ is log-split with respect to the connection ∇. Let $x \in X$ and let ω be an irregular part of ∇ at x. The local model $(V, \Lambda, \boldsymbol{\nabla})$ of $\mathscr{E}$ at x satisfies

$$(\boldsymbol{\nabla} - \omega)(\Lambda) \subset \Lambda \otimes_{\mathbb{C}} \Omega(1).$$

Therefore $\langle \boldsymbol{\nabla} - \omega, \theta \rangle$ induces a linear map Γ on the $\mathbb{C}$-vector space $\Lambda/\mathfrak{m}\Lambda$ of dimension n, which is the *residue* of the regular connection ∇^{reg} with respect to Λ, and is independent of the irregular part ω.

Definition 3. The *exponents* $\mu_1^{\mathscr{E},x}, \dots, \mu_n^{\mathscr{E},x}$ at x of the connection $(\mathscr{E}, \nabla)$ are

(1) the eigenvalues of the map Γ defined just above if $\mathscr{E}$ is log-split and

(2) otherwise, the numbers

$$\mu_i^{\mathscr{E},x} = \frac{1}{e_y(\pi)} \mu_i^{\mathscr{E}_L(\pi),y} \quad \text{for } 1 \leqslant i \leqslant n \text{ for any } \ y \in \pi^{-1}(x),$$

where $\tilde{X} \xrightarrow{\pi} X$ is any ∇-admissible ramified cover.

By definition, the Levelt bundle $\mathscr{E}_L(\pi) = (\pi^*\mathscr{E})_L$ of $\pi^*\mathscr{E}$ is log-split with respect to $\pi^*\nabla$. According to Proposition 3, the previous definition makes sense.

2. The companion bundles of a differential equation

2.1. ***The Fuchs bundle.*** Let $\mathcal{S}$ be a discrete subset of X and $U_0 = X \setminus \mathcal{S}$. For any $x_i \in \mathcal{S}$ there exists an open neighborhood U_i of x_i such that $U_i \cap \mathcal{S} = \{x_i\}$ and $U_i \cap U_j = \varnothing$ for nonzero $i \neq j$. The open covering $\mathfrak{U} = (U_0, \dots, U_p)$ of X will be called an *adapted covering* of X (for $\mathcal{S}$).

Let X be a compact Riemann surface, and let $K = \Gamma(X, \mathcal{M}_X)$ be the function field of X. The set $\mathrm{Der}(K/\mathbb{C}) = \Gamma(X, \Theta_X)$ of $\mathbb{C}$-derivations of K and its dual $\Omega^1_{K/\mathbb{C}} = \Gamma(X, \mathcal{M}^1_X)$ of differential forms are respectively the global sections of the sheaves of meromorphic sections of the holomorphic tangent and cotangent bundle of X. For any nontrivial derivation $\tau \in \mathrm{Der}(K/\mathbb{C})$, let ω_τ be the unique differential form such that $\langle \omega_\tau, \tau \rangle = 1$. Let $\mathfrak{D}_X = K[\tau]$. We consider a monic differential operator

$$L = \tau^n + a_1 \tau^{n-1} + \cdots + a_n \in \mathfrak{D}_X. \tag{7}$$

The singular set $\mathcal{S}$ of (7) is the reunion of the supports of the divisor of τ and the polar divisors of $a_1, \ldots, a_n$.

We consider an adapted covering $(U_0, \ldots, U_p)$ of X such that there is a local coordinate t_i on U_i for $i > 0$. Locally around x_i, there exists a meromorphic function $f_i \in \Gamma(U_i, \mathcal{M}_X)$ such that $f_i \tau = t_i \frac{d}{dt_i}$. We can assume, at the cost of shrinking U_i, that f_i is holomorphic and does not vanish on $U_i \setminus \{x_i\}$. Put $\theta_i = t_i \frac{d}{dt_i}$. The operator $L_i = f_i^n L$ can be written on U_i in the form

$$L_i = \theta_i^n + b_1^i \theta_i^{n-1} + \cdots + b_n^i.$$

Define differential systems defined over the subsets U_i by

$$\text{(S)} \qquad \begin{array}{lll} (0) & df = \Omega_0 f \text{ on } U_0 & \text{with } \Omega_0 = A_{L(\tau)} \otimes \omega_\tau, \\ (i) & df = \Omega_i f \text{ on } U_i \text{ for } i > 0 & \text{with } \Omega_i = A_{L_i(\theta_i)} \otimes dt_i/t_i, \end{array}$$

where A_L is the *companion matrix* of the operator L.

Lemma 6. *The systems (S) can be glued together as a holomorphic vector bundle $\mathfrak{F}$ endowed with a meromorphic connection ∇ on X, having exactly $\mathcal{S}$ as singular set. We say that the pair $(\mathfrak{F}, \nabla)$ is the* Fuchs bundle *of the operator L.*

Proof. By construction of the adapted covering (U_i) of X, only the intersections $U_{0i} = U_0 \cap U_i$ are nonempty. Let Q_i be the matrix whose i-th column represents the coefficients of θ_i in the basis $(1, \tau, \ldots, \tau^{n-1})$. Also by construction, the matrix $P_i = {}^t Q_i^{-1}$ satisfies the relation $A_{L_i(\theta_i)} = (f_i/t_i) P_i^{-1} A_{L(\tau)} P_i - P_i^{-1} \theta_i P_i$. Restricting if necessary the open sets U_i, the matrix P_i is holomorphically invertible in the set $U_i \setminus \{x_i\} = U_{0i}$. Tensoring by $dt_i/t_i = (t_i/f_i)\omega_\tau$, we get

$$\Omega_i = P_i^{-1} \Omega_0 P_i - P_i^{-1} dP_i. \tag{8}$$

The collection $\mathfrak{P} = (P_i)_{i=0}^p$ is a (Čech) cocycle in $Z^1(\mathfrak{U}, \mathrm{GL}_n(\mathcal{O}_X))$. Let $\mathbf{F}$ be a holomorphic vector bundle on X having $\mathfrak{P}$ as cocycle, with $\mathfrak{F}$ its sheaf of sections. Then (8) defines a meromorphic connection ∇ on $\mathfrak{F}$ with singularities only in $\mathcal{S}$. $\square$

2.2. ***The companion bundle.*** Let $(\mathfrak{F}, \nabla)$ be the Fuchs bundle of a differential operator $L \in \mathfrak{D}_X$ with singular set $\mathcal{S}$. Let $\tilde{X} \xrightarrow{\pi} X$ be a ∇-admissible ramified cover of X. We construct a holomorphic vector bundle $\mathfrak{C}(\pi)$ on $\tilde{X}$ as follows.

Let $(\pi^*\mathfrak{F}, \pi^*\nabla)$ be the pull-back of the Fuchs bundle. Let $\tilde{x} \in \pi^{-1}(\mathcal{S})$ and consider the local model (W, M, Δ) of $(\pi^*\mathfrak{F}, \pi^*\nabla)$ at $\tilde{x}$. Let K and H be the field of fractions of $\mathbb{O}_{X,\pi(\tilde{x})}$ and $\mathbb{O}_{\tilde{X},\tilde{x}}$, respectively. The degree of the extension $[\hat{H} : \hat{K}] = [H : K]$ is divisible by the ramification index of ∇. Therefore W admits a basis (e) in which any irregular part is diagonal. Let $\mathcal{M}$ be the $\mathbb{O}_{\hat{H}}$-lattice spanned by (e) in W. We define a local lattice $M(\tilde{x})$ at $\tilde{x}$ by putting $M(\tilde{x}) = \mathcal{M} \cap (\pi^*\mathfrak{F})_{\tilde{x}}$.

Definition 4. Let the *companion bundle of* L with respect to the admissible ramified cover π be the pair $(\mathfrak{C}(\pi), \nabla^\pi)$, where $\mathfrak{C}(\pi)$ is the continuation of the pull-back bundle $(\pi^*\mathfrak{F}, \pi^*\nabla)$ by the family of local lattices $(M(\tilde{x}))_{x \in \pi^{-1}(\mathcal{S})}$ and ∇^π is the extension of the pull-back connection $\pi^*\nabla$ to $\mathfrak{C}(\pi)$ according to Lemma 5.

Note 1. When L is fuchsian, the Fuchs bundle of L coincides with the companion bundle $\mathfrak{C}(\mathrm{id}_X)$. However, the companion bundle of an irregular operator does not generally come from a bundle defined on X — it is not even equal to the Levelt bundle $(\mathfrak{F})_L(\pi)$ of the Fuchs bundle.

The definition of Fuchs' exponents has been extended by D. Bertrand and G. Laumon [1985] in relation with arithmetic transcendence problems.

Theorem 2. *Let $X = \mathbb{P}^1(\mathbb{C})$ and $L \in \mathbb{C}(z)[\frac{d}{dz}]$. The exponents $\mu_j^{\mathfrak{C}(\pi),\tilde{x}_i}(\nabla^\pi)$ of ∇^π with respect to the companion bundle $\mathfrak{C}(\pi)$ of an admissible ramified cover π satisfy*

$$\mu_j^{\mathfrak{C}(\pi),\tilde{x}_i}(\nabla^\pi) = e_{\tilde{x}_i}(\pi)\lambda_j^{\pi(\tilde{x}_i)}(L),$$

where the $\lambda_j^{\pi(\tilde{x}_i)}(L)$ are the Bertrand–Laumon exponents of L.

Proof. If the operator L is fuchsian, the Fuchs bundle $\mathfrak{F}$ is logarithmic and the exponents $\mu_j^{\mathfrak{F},x_i}(\nabla)$ of ∇ with respect to $\mathfrak{F}$ are equal to the eigenvalues of the residue of Ω_i. By construction, the characteristic polynomial of $\mathrm{Res}_{x_i}\,\Omega_i$ is indeed equal to the indicial polynomial $P_L^{x_i}$. If L is not fuchsian, the bundle $\mathfrak{C}(\pi)$ is nevertheless log-split. At any $\tilde{x} \in \mathcal{S}(\nabla^\pi)$, consider the formal local model $(V, \Lambda, \mathbf{\nabla})$ of $(\mathfrak{C}(\pi), \nabla^\pi)$. There is an isomorphism of $\mathfrak{D}_{\hat{H}}$-modules

$$\mathfrak{D}_{\hat{H}}/\mathfrak{D}_{\hat{H}}L \simeq \bigoplus_{i=1}^{t} \mathfrak{D}_{\hat{H}}/\mathfrak{D}_{\hat{H}}L_i(\theta_\zeta - \eta_i),$$

where L_i are formal fuchsian operators and $\eta_i \in (1/\zeta)\mathbb{C}[(1/\zeta)]$, where ζ is a uniformizing parameter of H, and $\theta_\zeta = \zeta\frac{d}{d\zeta}$; see [Robba 1980, théorème 3.2]. There exists an H-basis (e) of V and a local parameter ζ of $\tilde{X}$ such that the matrix

of the irregular part ω of ∇ attached to ζ at $\tilde{x}$ is the diagonal matrix

$$\mathrm{Mat}(\omega_H, (e)) = \mathrm{diag}(\omega_1 I_{n_1}, \dots, \omega_s I_{n_t}), \quad \text{where } \omega_i = \eta_i \otimes (dz/z).$$

As a consequence of [Corel 2004, proof of Theorem 3, page 1383], the map ω_H is in fact defined as the tensor extension of a map ω defined over K. Explicitly, $\mathrm{Mat}(\boldsymbol{\nabla} - \omega, (e)) = \mathrm{diag}(B_{L_1} \otimes (d\zeta/\zeta), \dots, B_{L_t} \otimes (d\zeta/\zeta))$. By definition, the exponents $\mu_j^{\mathfrak{E}(\pi),\tilde{x}_i}(\nabla^\pi)$ are the eigenvalues of $\mathrm{Res}_{\zeta=0}\,\mathrm{Mat}(\boldsymbol{\nabla} - \omega, (e))$, and therefore the roots of the characteristic polynomials of

$$\mathrm{Res}_{\zeta=0}\, B_{L_i} = e_{\tilde{x}_i}(\pi)\,\mathrm{Res}_{z=0}\, B_{L_i}.$$

But this matches exactly the definition of the exponents of the operator L; see [Bertrand and Beukers 1985]. □

Corollary 3. *The definition of the Bertrand–Laumon exponents of a differential operator $L \in \mathfrak{D}_X$ holds on any Riemann surface X, and so does Theorem 2 if X is compact.*

Indeed, the companion bundle and its exponents are independent of the local parameter. This answers a question raised originally by G. Laumon.

3. Fuchs relations

Theorem 3 (Fuchs relation). *Let $\mathscr{E}$ be a holomorphic vector bundle equipped with a meromorphic connection ∇. The sum of the exponents $\mu_i^{\mathscr{E},x}$ of ∇ with respect to $\mathscr{E}$ at all points $x \in X$ satisfies the relations*

$$\sum_{x\in X}\sum_{i=1}^{n} \mu_i^{\mathscr{E},x} = \frac{\deg \mathscr{E}_L(\pi)}{\deg \pi} \leqslant \deg \mathscr{E},$$

where $\mathscr{E}_L(\pi)$ is the Levelt bundle of $\mathscr{E}$ in any admissible ramified cover $\tilde{X} \xrightarrow{\pi} X$. Moreover, equality is achieved if and only if $\mathscr{E}$ is log-split.

Proof. Assume that $\mathscr{E}$ is log-split with respect to ∇. Let $\bigwedge^n \nabla$ be the connection induced by ∇ on the determinant line bundle $\bigwedge^n \mathscr{E}$. A line bundle is always log-split with respect to a connection on it, so Definition 3 gives $\sum_{i=1}^n \mu_i^{\mathscr{E},x} = \mathrm{Res}_x \bigwedge^n \nabla$. By Bott's [1967] residue formula for line bundles, we get $\sum_{x\in X} \mathrm{Res}_x \bigwedge^n \nabla = \deg \bigwedge^n \mathscr{E} = \deg \mathscr{E}$ and hence the result. In the general case, let $\tilde{X} \xrightarrow{\pi} X$ be a ∇-admissible cover. By definition, $\mu_i^{\mathscr{E},x} = \mu_i^{\mathscr{E}_L(\pi),\tilde{x}}/e_{\tilde{x}}(\pi)$ for any $\tilde{x} \in \pi^{-1}(x)$. Therefore, we get

$$\sum_{\tilde{x}\in\pi^{-1}(x)}\sum_{i=1}^{n} \mu_i^{\mathscr{E}_L(\pi),\tilde{x}} = \sum_{\tilde{x}\in\pi^{-1}(x)} e_{\tilde{x}}(\pi)\sum_{i=1}^{n} \mu_i^{\mathscr{E},x} = \deg \pi \sum_{i=1}^{n} \mu_i^{\mathscr{E},x}.$$

Since $\mathscr{E}_L(\pi)$ is log-split, we get $\sum_{\tilde{x}\in\tilde{X}}\sum_{i=1}^{n}\mu_i^{\mathscr{E}_L(\pi),\tilde{x}}=\deg\mathscr{E}_L(\pi)$.

Now assume that ∇ is unramified. The exact sequence $0\to\mathscr{E}_L\to\mathscr{E}\to\mathfrak{P}\to 0$ yields the following relation for the dimensions of the cohomology groups:

$$h^0(X,\mathscr{E}_L)-h^0(X,\mathscr{E})+h^0(X,\mathfrak{P})-h^1(X,\mathscr{E}_L)+h^1(X,\mathscr{E})-h^1(X,\mathfrak{P})=0. \tag{9}$$

By the Riemann–Roch theorem for a vector bundle $\mathscr{F}$ [Gunning 1967, page 64], we have $h^0(X,\mathscr{F})-h^1(X,\mathscr{F})=\deg\mathscr{F}+n(1-g)$, where g is the genus of X. The sheaf $\mathfrak{P}$ is a skyscraper sheaf with support on $\mathscr{S}$; thus $H^1(X,\mathfrak{P})=0$. From (9), we get

$$\deg\mathscr{E}-\deg\mathscr{E}_L=h^0(X,\mathfrak{P})=\sum_{x\in\mathscr{S}}\dim_{\mathbb{C}}(\mathscr{E}/\mathscr{E}_L)_x=\sum_{x\in\mathscr{S}}[\mathscr{E}_x:(\mathscr{E}_x)_L]\geqslant 0 \tag{10}$$

and hence the inequality. So, $\deg\mathscr{E}=\deg\mathscr{E}_L$ if and only if $\mathscr{E}_x=(\mathscr{E}_x)_L$ for all $x\in\mathscr{S}$, that is, $\mathscr{E}=\mathscr{E}_L$. The ramified case follows since $\deg\pi\,\deg\mathscr{E}=\deg\pi^*\mathscr{E}$. $\square$

This result shows that log-split is a sensible extension to the irregular case of the classical notion of logarithmic vector bundle for a regular connection. The maximality property stated in Theorem 1 together with relation (10) yield the following consequence, and show that the exponents also measure how far the vector bundle $\mathscr{E}$ is from being log-split.

Corollary 4. *For any ∇-admissible covering π of X, and any log-split bundle $\mathscr{F}\in\Xi(\pi^*\mathscr{E})$, one has*

$$0\leqslant d_\nabla(\mathscr{E})=\deg\mathscr{E}-\sum_{x\in X}\sum_{i=1}^{n}\mu_i^{\mathscr{E},x}\leqslant\frac{1}{\deg\pi}[\pi^*\mathscr{E}:\mathscr{F}],$$

and the minimum is attained only for the Levelt bundle $\mathscr{E}_L(\pi)$ for a given covering π. The nonnegative rational number $d_\nabla(\mathscr{E})$ will be called the ∇-defect *of the bundle $\mathscr{E}$.*

Corollary 5. *Let $\mathfrak{C}(\pi)$ be the companion bundle of a differential operator $L\in\mathfrak{D}_X$ with respect to a ramified cover π of X. The degree* $\deg\mathfrak{C}(\pi)$ *satisfies the relation*

$$\deg\mathfrak{C}(\pi)=\left(\tfrac{1}{2}n(n-1)(p+2g-2)+\tfrac{1}{2}i(\operatorname{End}L)\right)\deg\pi,$$

where g is the genus of X and $i(\operatorname{End}L)=\sum_{x\in X}i_x(\operatorname{End}L)$ is the total irregularity of the operator $\operatorname{End}L$ *induced by L on* $\operatorname{End}\mathfrak{D}_X$. *Therefore, a companion bundle is never trivial except if L is regular with $p=2$ singularities over $X=\mathbb{P}^1(\mathbb{C})$.*

Proof. We have $\sum_{\tilde{x}\in\tilde{X}}\sum_{j=1}^{n}\mu_j^{\mathfrak{C}(\pi),\tilde{x}}(\nabla)=\deg\mathfrak{C}(\pi)$ since the bundle $\mathfrak{C}(\pi)$ is log-split with respect to ∇. On the other hand, the generalized Fuchs relation on a

Riemann surface (see [Saito 1958] or [Chudnovsky and Chudnovsky 1984]) asserts that

$$\sum_{x\in\mathcal{S}(L)}\Big(\sum_{j=1}^{n}\lambda_j^x(L)-\tfrac{1}{2}i_x(\operatorname{End}L)\Big)=(p+2g-2)\tfrac{1}{2}n(n-1).$$

A vector bundle $\mathcal{F}$ is trivial only if it has degree 0. The only case when this can happen for $\mathfrak{C}(\pi)$ is for a fuchsian operator on $\mathbb{P}^1(\mathbb{C})$ with $p=2$, and it does indeed since the Fuchs bundle over $\mathbb{P}^1(\mathbb{C})$ is isomorphic to $\bigoplus_{i=0}^{n-1}\mathcal{O}(-i(p-2))$; see [Van der Put and Singer 2003, Proposition 6.14]. □

Note 2. In the case of $\mathbb{P}^1(\mathbb{C})$, Bolibrukh [1990a] constructs a Fuchsian system from a fuchsian operator L, whose exponents μ_i^s coincide with the exponents λ_i^s of L, except at one point x, where $\mu_i^x=\lambda_i^x-i(p-2)$ for $1\leqslant i\leqslant n$. This system can be understood as a meromorphic trivialization of the Fuchs bundle $\mathfrak{F}$ of L, which is holomorphic outside x and moreover logarithmic at x, the correction to the exponents at x corresponding precisely to the type of the bundle.

3.1. *The local index of the Levelt bundle.*

Definition 5. Let $f\in\operatorname{End}(\mathbb{C}^n)$, let $\{\lambda_1,\dots,\lambda_t\}$ be the distinct eigenvalues of f, and let $E_i=\ker(f-\lambda_i\operatorname{id}_{\mathbb{C}^n})^n$ be the characteristic space corresponding to λ_i. Put finally $\rho_i^j=\operatorname{rk}(f|_{E_i}-\lambda_i\operatorname{id}_{E_i})^j$. We define the *Jordan invariant* $J(f)$ and the *kernel invariant* $K(f)$ of f as the quantities

$$J(f)=\sum_{d\geqslant1}\tfrac{1}{2}d(d-1)n_d=\sum_{i=1}^{t}\sum_{j\geqslant1}\rho_i^j\quad\text{and}\quad K(f)=\frac{1}{2}\sum_{i=1}^{t}\sum_{j\geqslant0}\Delta_i^j(\Delta_i^j-1),$$

where n_d is the number of Jordan blocks of dimension d in the Jordan normal form of f, and $\Delta_i^j=\rho_i^j-\rho_i^{j+1}$.

Let $(V,\Lambda,\boldsymbol{\nabla})$ be the local model of $(\mathscr{E},\nabla)$ at a point $x\in\mathcal{S}(\nabla)$, with corresponding field K, ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. Let Λ_L be the Levelt lattice of Λ. Take a representative $\delta=\delta_\Lambda$ of $(PM\,\nabla)_x$, and let $E_1,\dots,E_t\subset\Lambda/\mathfrak{m}\Lambda$ be the characteristic spaces corresponding to the different eigenvalues of δ. By Sibuya's lemma (see [Babbitt and Varadarajan 1983, Lemma 1, page 42]), the space V is a direct sum of uniquely defined $\boldsymbol{\nabla}$-stable sub-K-vector spaces $\{W_1,\dots,W_t\}$ of V, such that $(W_i\cap\Lambda)/\mathfrak{m}\Lambda=E_i$. Let $\nabla_i=\boldsymbol{\nabla}|_{W_i}$ and $\delta_i=\delta|_{W_i}$. The formal connections $(W_i,\nabla_i)_{1\leqslant i\leqslant t}$ form the *Sibuya splitting* of $(\mathscr{E},\nabla)$ at x. Note that we have $\bigoplus_{i=1}^t W_i\cap\Lambda=\Lambda$ and $\Lambda_L=\bigoplus_{i=1}^t(W_i\cap\Lambda)_L$ [Corel 2004, page 1389].

Proposition 4. *If the connection ∇ is unramified at x, the index $[\Lambda:\Lambda_L]$ of the Levelt lattice Λ_L of Λ satisfies*

$$(\mathfrak{p}-\kappa)J(\delta)\leqslant\sum_{i=1}^{t}(\mathfrak{p}-\kappa_i)J(\delta_i)\leqslant[\Lambda:\Lambda_L]\leqslant\tfrac{1}{2}n(n-1)\mathfrak{p}-\tfrac{1}{2}i_x(\mathscr{E}nd\,\nabla)-K(\delta),$$

where $\mathfrak{p} = \mathfrak{p}_x(\mathscr{E}nd\,\nabla)$, $\kappa = \kappa_x(\mathscr{E}nd\,\nabla)$ *and* $\kappa_i = \kappa(\mathrm{End}\,\nabla_i)$.

Proof. Since ∇ is unramified at x, the Katz rank κ is an integer and we have

$$[\Lambda : \Lambda_L] = [\Lambda : \Lambda_\kappa] + [\Lambda_\kappa : \Lambda_L]. \tag{11}$$

On the other hand, by Sibuya's lemma, we have $[\Lambda : \Lambda_L] = \sum_{i=1}^t [\Lambda_i : (W_i \cap \Lambda)_L]$. We can therefore restrict to the case where the map δ has only one eigenvalue λ. Let $\omega_i = \varphi_i dz/z$ be the eigenvalues of the irregular part of ∇ at x attached to a local coordinate z. For $Q = a_k z^k + \cdots + a_\ell z^\ell \in \mathbb{C}[z, z^{-1}]$, put $v(Q) = k$ and $\deg Q = \ell$. If $\lambda \neq 0$, then there exists a unique $P \in (1/z)\mathbb{C}[1/z]$ of largest degree such that $v(\varphi_i - P) > v(\varphi_i)$ for all i. If $\lambda = 0$, it can also happen that the polynomial P is defined. Otherwise, put $P = 0$. The map $\nabla' = \nabla - P dz/z$ is also a connection on V. According to [Corel 2004, Lemma 4.2], the Levelt lattices $\Lambda_L(\nabla)$ and $\Lambda_L(\nabla')$ coincide. Let $\mathfrak{p}'$ and κ' respectively be the Poincaré and Katz ranks of ∇'. From relation (11) we get

$$[\Lambda : \Lambda_L] = [\Lambda : \Lambda_L(\nabla')] = [\Lambda : \Lambda_{\kappa'}(\nabla')] + [\Lambda_{\kappa'}(\nabla') : \Lambda_L(\nabla')].$$

According to [Corel 2004, proof of Proposition 5.1, page 1390], we have

$$0 \leqslant [\Lambda_{\kappa'}(\nabla') : \Lambda_L(\nabla')] \leqslant \tfrac{1}{2} n(n-1)\kappa' - \tfrac{1}{2} i(\mathrm{End}\,\nabla'),$$

while Proposition 1 yields

$$(\mathfrak{p}' - \kappa') \sum_{j \geqslant 1} \rho^j \leqslant [\Lambda : \Lambda_{\kappa'}(\nabla')] \leqslant \tfrac{1}{2} n(n-1)(\mathfrak{p}' - \kappa') - \tfrac{1}{2} \sum_{j \geqslant 0} \Delta^j (\Delta^j - 1),$$

where $\rho^j = \mathrm{rk}(\delta_\Lambda - \lambda \mathrm{id}_V)^j$ and $\Delta^j = \rho^j - \rho^{j+1}$. Clearly $i(\mathrm{End}\,\nabla') = i(\mathrm{End}\,\nabla)$ and $\mathfrak{p}'$ and κ' respectively are equal to the Poincaré and Katz ranks of $\mathrm{End}\,\nabla$. Adding these two inequalities, we get

$$(\mathfrak{p}' - \kappa') \sum_{j \geqslant 1} \rho^j \leqslant [\Lambda : \Lambda_L] \leqslant \tfrac{1}{2} n(n-1)\mathfrak{p}' - \tfrac{1}{2} i(\mathrm{End}\,\nabla) - \tfrac{1}{2} \sum_{j \geqslant 0} \Delta^j (\Delta^j - 1).$$

for the case where the map δ_Λ has only one eigenvalue. If we have several eigenvalues $\lambda_1, \ldots, \lambda_t$, then $\mathfrak{p}' = \mathfrak{p}$. Relation (11) implies

$$\begin{aligned} \sum_{i=1}^t (\mathfrak{p}_i - \kappa_i) \sum_{j \geqslant 1} \rho_i^j &\leqslant [\Lambda : \Lambda_L] \\ &\leqslant \sum_{i=1}^t \tfrac{1}{2} n_i(n_i - 1)\mathfrak{p}_i - \sum_{i=1}^t \tfrac{1}{2} i(\mathrm{End}\,\nabla_i) - \tfrac{1}{2} \sum_{i=1}^t \sum_{j \geqslant 0} \Delta_i^j (\Delta_i^j - 1), \end{aligned}$$

where $n_i = \dim_{\mathbb{C}} E_i$ and $\mathfrak{p}_i$ is the Poincaré rank of the subconnection $\mathrm{End}\,\nabla_i$ over the lattice $\Lambda \cap W_i$. However, we have $\mathfrak{p}_i \leqslant \mathfrak{p}$ and $\mathfrak{p}_i < \mathfrak{p}$ if and only if $J(\delta_i) = 0$;

therefore we can replace everywhere $\mathfrak{p}_i$ with $\mathfrak{p}$. By definition,

$$K(\delta) = K(\delta_\Lambda) = \frac{1}{2}\sum_{i=1}^{t}\sum_{j\geqslant 0}\Delta_i^j(\Delta_i^j - 1) \quad \text{and} \quad J(\delta_i) = \sum_{j\geqslant 1}\rho_i^j.$$

The eigenvalues of the irregular part (determinant factors) of $\operatorname{End}\nabla_i$ are among those of $\operatorname{End}\nabla$; therefore we have $\kappa_i \leqslant \kappa$ and hence

$$\sum_{i=1}^{t}(\mathfrak{p}-\kappa_i)J(\delta_i) \geqslant (\mathfrak{p}-\kappa)\sum_{i=1}^{t}J(\delta_i) = (\mathfrak{p}-\kappa)J(\delta),$$

since $\delta = \bigoplus_{i=1}^{t}\delta_i$ and distinct δ_i have no common eigenvalue. Finally, by the argument used in [Corel 2004, top of page 1391], we get

$$\sum_{i=1}^{t}\tfrac{1}{2}i(\operatorname{End}\nabla_i) + \sum_{i<j}n_i n_j\mathfrak{p} = \tfrac{1}{2}i(\operatorname{End}\nabla) = \tfrac{1}{2}i_x(\mathscr{E}nd\,\nabla). \qquad \square$$

R. R. Gontsov [2004] gives an analogous relation, with the weaker bounds $\operatorname{rk}\delta$ instead of $J(\delta)$ and $\sum_{i=1}^{t}\Delta_0^i(\Delta_0^i - 1)$ instead of $K(\delta)$.

3.2. *Fuchs inequalities for a holomorphic vector bundle.*

Definition 6. With the previous notation, we define

$$J_x(\nabla) = J(\delta_x), \qquad K_x(\nabla) = K(\delta_x), \qquad L_x(\nabla) = (\mathfrak{p}_x(\mathscr{E}nd\,\nabla) - \kappa_x(\mathscr{E}nd\,\nabla))J(\delta_x),$$

where δ_x is a representative of $(PM\,\nabla)_x$ if $\mathfrak{p}_x(\nabla) > 0$, and $\delta_x = 0$ otherwise.

Theorem 4 (Fuchs inequalities). *Let ∇ be a meromorphic connection over a vector bundle $\mathscr{E}$ of rank n. The sum of exponents satisfies the inequalities*

$$\deg\mathscr{E} - \tfrac{1}{2}n(n-1)h(\mathscr{E}nd\,\mathscr{E}) + \tfrac{1}{2}i(\mathscr{E}nd\,\nabla) + K(\mathscr{E}) \leqslant \sum_{x\in X}\sum_{i=1}^{n}\mu_i^{\mathscr{E},x} \leqslant \deg\mathscr{E} - L(\mathscr{E}),$$

where

$$L(\mathscr{E}) = \sum_{x\in X}L_x(\nabla), \quad i(\mathscr{E}nd\,\nabla) = \sum_{x\in X}i_x(\mathscr{E}nd\,\nabla), \quad K(\mathscr{E}) = \sum_{x\in X}\frac{1}{\rho_x(\nabla)}K_x(\nabla).$$

Proof. Assume ∇ is unramified. Proposition 4 and relation (10) imply

$$L(\mathscr{E}) \leqslant \deg\mathscr{E} - \deg\mathscr{E}_L \leqslant \tfrac{1}{2}n(n-1)h(\mathscr{E}nd\,\mathscr{E}) - \tfrac{1}{2}i(\mathscr{E}nd\,\nabla) - K(\mathscr{E})$$

when ∇ has no ramification, whence the result by Theorem 3. Let now $\tilde{X}\xrightarrow{\pi}X$ be an admissible ramified cover. The connection $\pi^*\nabla$ is unramified; hence we have

$$\begin{aligned} L(\pi^*\mathscr{E}) &\leqslant \deg\pi^*\mathscr{E} - \deg\mathscr{E}_L(\pi) \\ &\leqslant \tfrac{1}{2}n(n-1)h(\mathscr{E}nd\,\pi^*\mathscr{E}) - \tfrac{1}{2}i(\mathscr{E}nd\,\pi^*\nabla) - K(\pi^*\mathscr{E}). \end{aligned} \tag{12}$$

Let $\tilde{x} \in \tilde{X}$ and put for simplicity $e = e_{\tilde{x}}(\pi)$ and $x = \pi(\tilde{x})$. If the map $(PM\nabla)_x$ has matrix M for a given local basis of sections of $\mathscr{E}$ and a local coordinate at x, then there exist a local basis of sections of $\pi^*\mathscr{E}$ and a local coordinate at $\tilde{x}$ such that $(PM\pi^*\nabla)_{\tilde{x}}$ has matrix eM. Therefore $K_{\tilde{x}}(\pi^*\mathscr{E}) = K_x(\mathscr{E})$, and $K(\pi^*\mathscr{E}) = \sum_{x\in X}|\pi^{-1}(x)|K_x(\mathscr{E})$. If π is the minimal covering with apparent ramification at $x_0 \notin \mathscr{S}$, we get $|\pi^{-1}(x)| = \rho_x(\nabla)\deg\pi$. Since $\mathscr{E}nd\,\pi^*\nabla = \pi^*\mathscr{E}nd\,\nabla$ and $\mathfrak{p}_{\tilde{x}}(\pi^*\nabla) = e\mathfrak{p}_x(\nabla)$, we get $\kappa_{\tilde{x}}(\mathscr{E}nd\,\pi^*\nabla) = e\,\kappa_x(\mathscr{E}nd\,\nabla)$ and $i_{\tilde{x}}(\mathscr{E}nd\,\pi^*\nabla) = e\,i_x(\mathscr{E}nd\,\nabla)$. In particular, $J_{\tilde{x}}(\pi^*\nabla) = eJ_x(\nabla)$. Therefore we get

$$J(\pi^*\mathscr{E}) = \sum_{\tilde{x}\in\tilde{X}} e_{\tilde{x}}(\pi)J_{\pi(\tilde{x})}(\nabla) = \deg\pi \sum_{x\in X} J_x(\nabla) = \deg\pi\, J(\nabla).$$

The same holds for any subconnection; therefore $L(\pi^*\mathscr{E}) = (\deg\pi)L(\mathscr{E})$. In the same way, $h(\mathscr{E}nd\,\pi^*\mathscr{E}) = (\deg\pi)h(\mathscr{E}nd\,\mathscr{E})$ and $i(\mathscr{E}nd\,\pi^*\nabla) = (\deg\pi)i(\mathscr{E}nd\,\nabla)$. Since $\deg\pi^*\mathscr{E} = \deg\pi\,\deg\mathscr{E}$, dividing out (12) by $\deg\pi$ gives the claim. □

Corollary 6. *The ∇-defect of $\mathscr{E}$ satisfies*

$$L(\mathscr{E}) \leqslant d_\nabla(\mathscr{E}) \leqslant \tfrac{1}{2}n(n-1)h(\mathscr{E}nd\,\mathscr{E}) - \tfrac{1}{2}i(\mathscr{E}nd\,\nabla) - K(\mathscr{E}).$$

Note 3. We can refine all these estimates by replacing the quantity $L(\mathscr{E})$ with $\ell(\mathscr{E}) = \sum_{x\in X}\sum_{i=1}^{t}(\mathfrak{p}_x(\mathrm{End}\,\nabla_i) - \kappa_x(\mathrm{End}\,\nabla_i))J_x(\nabla_i)$, where $(\nabla_1, \dots, \nabla_t)$ are the local formal subconnections given by the Sibuya splitting of $\mathscr{E}$ at x.

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