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TOM KLEIN AND ANDREW NICAS

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We find the horofunction boundary of the $(2n+1)$-dimensional Heisenberg group with the Korányi metric and show that it is homeomorphic to a $2n$-dimensional disk. We also show that the Busemann points correspond to the $(2n-1)$-sphere boundary of this disk. We show that the compactified Heisenberg group is homeomorphic to a $(2n+1)$-dimensional sphere.

1. Introduction

Gromov [1981, 1.2] defines a boundary for a metric space (X, ρ) as follows. Let $C(X)$ be the space of continuous real-valued functions with the topology of uniform convergence on compact sets, and let $C_*(X)$ be the quotient space of $C(X)$ modulo the constant functions. If (X, ρ) is *proper*, that is, closed balls in X are compact, then the map

$$X \to C_*(X), \quad x \mapsto \text{equivalence class of } \rho(x, \cdot)$$

is an embedding and the closure of X in $C_*(X)$ is compact. The topological boundary of X in $C_*(X)$, denoted $\partial_h X$, is called the *horofunction boundary* of X (with respect to the metric ρ) and its elements are called *horofunctions*. The union of (the image of) X with its horofunction boundary is called the *horofunction compactification* of X (with respect to the metric ρ); see [Bridson and Haefliger 1999, page 267] and [Ballmann et al. 1985, Section 1.3].

The horofunction boundary of a metric space is known explicitly in only a very few cases. When (X, ρ) is a proper CAT(0) space, the horofunction boundary of X coincides with the "visual" boundary defined using equivalence classes of geodesic rays [Bridson and Haefliger 1999, Theorem 8.13]. The notion of an *almost geodesic ray*, as in [Rieffel 2002, Definition 4.3] and Definition 3.1, is a generalization of geodesic ray that serves as a substitute for geodesic ray in a general metric space. A point on the horofunction boundary of a metric space is called a *Busemann point*

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if it is the limit of an almost geodesic ray. In the case of a proper CAT(0) space, all points in the horofunction boundary are Busemann points. Non-Busemann points may exist for other metric spaces; for example, Walsh [2007, Theorem 1.2] has determined precisely when such points exist in the case X is a finite-dimensional normed linear space. We show that non-Busemann points exist for Heisenberg groups equipped with the Korányi metric, a non-CAT(0) metric that is important for geometric analysis on these Lie groups.

The $(2n+1)$-dimensional *Heisenberg group*, denoted $\mathbb{H}^n$, is the simply connected nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with multiplication given by $(w, s)(z, t) = (w+z, s+t+2\,\mathrm{Im}\langle w, z\rangle)$, where $w, z \in \mathbb{C}^n$ and $s, t \in \mathbb{R}$ and $\langle\cdot\,,\cdot\rangle$ is the standard Hermitian inner product on $\mathbb{C}^n$. The identity element is $0 = (0, 0)$ and $(w, s)^{-1} = (-w, -s)$.

The *Korányi gauge* [1985, 1.4] is the real-valued function $\|\cdot\| : \mathbb{H}^n \to \mathbb{R}$ defined by

$$\|(z, t)\| = (|z|^4 + t^2)^{1/4}, \quad \text{where } |z| = \langle z, z\rangle^{1/2}.$$

Since $\|\cdot\|$ is known to be subadditive, that is, $\|ab\| \le \|a\| + \|b\|$ for $a, b \in \mathbb{H}^n$, it readily follows that the function $d(a, b) = \|a^{-1}b\|$ is a metric on $\mathbb{H}^n$, called the *Korányi metric*, which is left invariant with respect to the left action of $\mathbb{H}^n$ on itself; see also [Capogna et al. 2007, page 18], where the gauge has been defined for an isomorphic product on $\mathbb{H}^n$ with $-1/2$ replacing the coefficient 2 in our definition above.[1]

Although a horofunction was defined above as an equivalence class of functions, we will henceforth identify a horofunction on $\mathbb{H}^n$ with its representative that vanishes at $0 = (0, 0) \in \mathbb{H}^n$.

The *Korányi sphere* in $\mathbb{H}^n$, denoted by S^{2n}_K, is the unit sphere for the Korányi gauge:

$$S^{2n}_K = \big\{(w, \mu) \in \mathbb{H}^n \mid \|(w, \mu)\| = 1\big\} = \big\{(w, \mu) \in \mathbb{H}^n \mid |w|^4 + \mu^2 = 1\big\}.$$

Given $(w, \mu) \in S^{2n}_K$, define $h_{(w,\mu)} : \mathbb{H}^n \to \mathbb{R}$ to be the real linear function defined by

$$h_{(w,\mu)}(z, s) = -\,\mathrm{Re}\langle(|w|^2 + i\mu)w, z\rangle \quad \text{for } (z, s) \in \mathbb{H}^n, \tag{1}$$

where $i = \sqrt{-1}$.

We give the following explicit characterization of the horofunctions on $\mathbb{H}^n$.

Proposition 2.8. *Every horofunction on $\mathbb{H}^n$ is of the form h_u for some $u \in S^{2n}_K$.*

The map $\Theta : S^{2n}_K \to S^{2n}_K$, $(w, \mu) \mapsto ((|w|^2+i\mu)^2 w, -\mu)$ is an involution whose fixed point set is the "round" equator $\{(w, 0) \in S^{2n}_K \mid |w| = 1\}$. We will show in

[1] Throughout this paper an unadorned d will always refer to the Korányi metric, while ρ will be used to indicate an arbitrary metric.

Theorem 2.10 that the map $S_K^{2n} \to \partial_h \mathbb{H}^n$, $u \mapsto h_u$, induces a homeomorphism $S_K^{2n}/\langle\Theta\rangle \to \partial_h \mathbb{H}^n$. Theorem 3.3 says the Busemann points in $\partial_h \mathbb{H}^n$ are precisely the horofunctions of the form $h_{(w,0)}$. The space $S_K^{2n}/\langle\Theta\rangle$ is homeomorphic to a $2n$-disk and its $(2n-1)$-sphere boundary corresponds to the set of Busemann points. In summary, we prove the following theorem.

Theorem 4.1. *The horofunction boundary $\partial_h \mathbb{H}^n$ of the Heisenberg group $\mathbb{H}^n$ with the Korányi metric is homeomorphic to the $2n$-disk, with the Busemann points corresponding to the $(2n-1)$-sphere boundary of this disk. The horofunction compactification $\mathbb{H}^n \cup \partial_h \mathbb{H}^n$ is homeomorphic to the $(2n+1)$-sphere.*

2. The horofunction boundary of a metric space

In the notation of Section 1, write $f \mapsto \bar{f}$ for the quotient map $C(X) \to C_*(X)$. If X has a distinguished basepoint $p \in X$, then $C_*(X)$ can be identified with $C(X)_p$, the ideal of functions that vanish at p, via the homeomorphism $C(X)_p \to C_*(X)$, $f \mapsto \bar{f}$ with inverse $\bar{f} \mapsto f - f(p)$. Under this identification the horofunctions are precisely those functions in $C(X)_p$ that arise as limits of the form $\rho_{x_n} - \rho(x_n, p)$, where $\{x_n\}$ is a sequence such that $x_n \to \infty$, and for any $x \in X$, ρ_x denotes the function $\rho_x(y) = \rho(x, y)$.

Remark 2.1. Rieffel [2002, Examples 5.1 and 5.2] shows that even for the hyperbolic group $\mathbb{Z}$, two word metrics coming from different generating sets can give horofunction boundaries that are not homotopy equivalent, even though the metrics are Lipschitz equivalent.

In the case of the Heisenberg group $\mathbb{H}^n$ with the Korányi metric, we show that each point $u \in S_K^{2n}$ gives rise to a horofunction.

For $\lambda \geq 0$, the map $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $(z, s) \mapsto (\lambda z, \lambda^2 s)$ is called *nonisotropic dilation* by λ. It is a homomorphism of $\mathbb{H}^n$ and satisfies $\delta_{\lambda_1} \circ \delta_{\lambda_2} = \delta_{\lambda_1 \lambda_2}$ for $\lambda_1, \lambda_2 \geq 0$. By direct computation,

$$d(g, 0) = \|g^{-1}\| = \|g\| \quad \text{and} \quad d(\delta_\lambda g_1, \delta_\lambda g_2) = \lambda d(g_1, g_2).$$

Definition 2.2. Let $u \in S_K^{2n}$. The *dilation ray* associated to u is the map

$$D_u : [0, \infty) \to \mathbb{H}^n, \quad t \mapsto \delta_t u.$$

We will show that the limit of each dilation ray (considered in $C(\mathbb{H}^n)_0$) determines a horofunction.

Proposition 2.3. *Let $(w, \mu) \in S_K^{2n}$ and let $(z, s) \in \mathbb{H}^n$. Then*

$$\lim_{t \to \infty} d(D_{(w,\mu)}(t), (z, s)) - d(D_{(w,\mu)}(t), 0) = -\operatorname{Re}\langle (|w|^2 + i\mu) w, z\rangle,$$

with convergence uniform on compact subsets with respect to the variable (z, s).

The proof requires the following lemma (stated in sufficient generality for later use in Lemma 4.2).

Lemma 2.4. *For* $0 \le i \le 3$, *let* $\{a_{i,n}\}_{n\in\mathbb{Z}}$ *be sequences of real numbers such that* $a_{i,n} \to a_i$. *Let* $\{t_n\}$ *be a sequence of numbers diverging to infinity, and let*

$$p(t_n) = t_n^4 + a_{3,n}t_n^3 + a_{2,n}t_n^2 + a_{1,n}t_n + a_{0,n}.$$

Then $\lim_{n\to\infty}(p(t_n))^{1/4} - t_n = a_3/4$.

Proof.
$$\begin{aligned}(p(t_n))^{1/4} - t_n &= \frac{p(t_n) - t_n^4}{(p(t_n)^{1/4} + t_n)(p(t_n)^{1/2} + t_n^2)} \\ &= \frac{(p(t_n) - t_n^4)/t_n^3}{[(p(t_n)^{1/4} + t_n)/t_n][(p(t_n)^{1/2} + t_n^2)/t_n^2]} \\ &= \frac{a_{3,n} + a_{2,n}/t_n + a_{1,n}/t_n^2 + a_{0,n}/t_n^3}{[(p(t_n)/t_n^4)^{1/4} + 1][(p(t_n)/t_n^4)^{1/2} + 1]}.\end{aligned}$$

Since $\lim_{n\to\infty} p(t_n)/t_n^4 = 1$ and $\lim_{n\to\infty} a_{i,n}/t_n^{3-i} = 0$ for $0 \le i \le 2$, the result follows. □

Remark 2.5. If a sequence of α-Lipschitz functions between metric spaces converges pointwise, then the sequence converges uniformly on compact subsets to an α-Lipschitz function. Also for any metric space (X, ρ) with basepoint p, and for any $a \in X$, the function $\hat{\rho}_a = \rho_a - \rho(a, p)$ is 1-Lipschitz. Therefore if $\{a_n\}$ is a sequence in X such that the sequence $\{\hat{\rho}_{a_n}\}$ converges pointwise, then it converges uniformly on compact subsets.

Proof of Proposition 2.3. Note that Remark 2.5 applies, and that

$$d(D_{(w,\mu)}(t), 0) = \|\delta_t(w, \mu)\| = t\|(w, \mu)\| = t \quad \text{for } t \ge 0.$$

Thus $d(D_{(w,\mu)}(t), (z, s)) - d(D_{(w,\mu)}(t), 0) = d(D_{(w,\mu)}(t), (z, s)) - t$. Now

$$d(D_{(w,\mu)}(t), (z, s)) = \|(z - tw, s - t^2\mu - 2t\,\mathrm{Im}\langle w, z\rangle)\| = p(t)^{1/4},$$

where

$$\begin{aligned}p(t) &= |z - tw|^4 + (s - t^2\mu - 2t\,\mathrm{Im}\langle w, z\rangle)^2 \\ &= (|w|^4 + \mu^2)t^4 + 4\big(-(\mathrm{Re}\langle w, z\rangle)|w|^2 + (\mathrm{Im}\langle w, z\rangle)\mu\big)t^3 \\ &\qquad\qquad + \text{terms of lower order in } t.\end{aligned}$$

Since $(w, \mu) \in S_K^{2n}$, we have $|w|^4 + \mu^2 = 1$, and so by Lemma 2.4

$$\lim_{t\to\infty} p(t)^{1/4} - t = -(\mathrm{Re}\langle w, z\rangle)|w|^2 + (\mathrm{Im}\langle w, z\rangle)\mu = -\mathrm{Re}\langle(|w|^2 + i\mu)w, z\rangle. \quad □$$

Let $u = (w, \mu) \in S^{2n}_K$. By Proposition 2.3,

$$\lim_{t\to\infty} d_{D_u(t)} - d(D_u(t), 0) = \lim_{t\to\infty} d_{D_u(t)} - t = h_u$$

where h_u is as defined in (1). Hence we have a map $\Phi : S^{2n}_K \to \partial_h \mathbb{H}^n,\ u \mapsto h_u$, and Φ is continuous.

To show that Φ is surjective, that is, every horofunction is of the form h_u for some $u \in S^{2n}_K$, we use the existence of a "conformal inversion" of $\mathbb{H}^n$ in the Korányi sphere S^{2n}_K.

Definition 2.6 [Capogna et al. 2007, page 19]. The *conformal inversion* of $\mathbb{H}^n$ in the sphere S^{2n}_K is the map $j : \mathbb{H}^n - \{0\} \to \mathbb{H}^n - \{0\}$ given by

$$j(z, s) = \left(\frac{-z}{|z|^2 - is}, \frac{-s}{\|(z, s)\|^4}\right).$$

Note that [Capogna et al. 2007] uses a different, but isomorphic, product on $\mathbb{H}^n$, which leads to different constants in the Korányi metric and conformal inversion formulas.

Conformal inversion satisfies the properties [Capogna et al. 2007, page 19]

(1) $j^2 = \mathrm{id}$;

(2) $\delta_\lambda \circ j = j \circ \delta_{1/\lambda}$ for $\lambda > 0$;

(3) $\|j(p)\| = 1/\|p\|$;

(4) $d(j(p), j(q)) = d(p, q)/(\|p\|\|q\|)$.

To show that Φ is surjective, we will need the following relation between the Korányi norm, conformal inversion, and dilation rays.

Lemma 2.7. *For $p, q \in \mathbb{H}^n - \{0\}$, we have $d(p, q) = \|q\| d(j(\delta_{1/\|p\|} p), \delta_{\|p\|} j(q))$.*

Proof. $d(p, q) = \|p\|\|q\| d(j(p), j(q))$

$$= \|q\| d(\delta_{\|p\|} j(p), \delta_{\|p\|} j(q)) = \|q\| d(j(\delta_{1/\|p\|} p), \delta_{\|p\|} j(q)). \qquad \square$$

Proposition 2.8. *$\Phi : S^{2n}_K \to \partial_h \mathbb{H}^n$ is surjective. Thus every horofunction of $\mathbb{H}^n$ with the Korányi metric is of the form $h_u : \mathbb{H}^n \to \mathbb{R},\ (z, s) \mapsto -\mathrm{Re}\langle(|w|^2 + i\mu)w, z\rangle$, where $u \in S^{2n}_K$.*

Proof. Let $h : \mathbb{H}^n \to \mathbb{R}$ be a horofunction. Then there is a sequence $\{p_n\}$ in $\mathbb{H}^n$ such that $p_n \to \infty$ and for all $q \in \mathbb{H}^n$

$$\lim_{n\to\infty} d(p_n, q) - \|p_n\| = h(q).$$

The points $\delta_{1/\|p_n\|} p_n$ lie on S^{2n}_K because $\|\delta_{1/\|p_n\|} p_n\| = \|p_n\|/\|p_n\| = 1$. Since S^{2n}_K is compact, $\{\delta_{1/\|p_n\|} p_n\}$ has a convergent subsequence, so by passing to the subsequence, we may assume that $\lim_{n\to\infty} \delta_{1/\|p_n\|} p_n = u \in S^{2n}_K$.

The triangle inequality gives

$$\text{(2)}\quad \left|d(j(\delta_{1/\|p_n\|}p_n), \delta_{\|p_n\|}j(q)) - d(j(u), \delta_{\|p_n\|}j(q))\right| \le d(j(\delta_{1/\|p_n\|}p_n), j(u)).$$

Since $\lim_{n\to\infty}\delta_{1/\|p_n\|}p_n = u$ and $v \mapsto d(j(v), j(u))$ is continuous, the right side of (2) tends to 0 as $n\to\infty$.

By Lemma 2.7, for $q \neq 0$,

$$\text{(3)}\quad \lim_{n\to\infty} d(p_n, q) - \|p_n\| = \lim_{n\to\infty}(\|q\| d(j(\delta_{1/\|p_n\|}p_n), \delta_{\|p_n\|}j(q)) - \|p_n\|).$$

By (2), the right side of (3) is the same as

$$\text{(4)}\quad \lim_{n\to\infty}\|q\| d(j(u), \delta_{\|p_n\|}j(q)) - \|p_n\|.$$

Using properties of the metric and inversion, we have

$$\begin{aligned} d(j(u), \delta_{\|p_n\|}j(q)) = d(j(u), j(\delta_{1/\|p_n\|}q)) &= \frac{d(u, \delta_{1/\|p_n\|}q)}{\|u\|\|\delta_{1/\|p_n\|}q\|} \\ &= \frac{\|p_n\|}{\|q\|} d(u, \delta_{1/\|p_n\|}q) \ = \frac{1}{\|q\|} d(\delta_{\|p_n\|}u, q). \end{aligned}$$

Hence (4) is the same as $\lim_{n\to\infty} d(\delta_{\|p_n\|}u, q) - \|p_n\|$, which by Proposition 2.3 is $h_u(q)$. □

The map $\Phi : S^{2n}_K \to \partial_h \mathbb{H}^n$ is not injective, but we can state explicitly which points get identified under Φ.

Lemma 2.9. *For $(w, \mu), (w', \mu') \in S^{2n}_K$, we have $\Phi(w, \mu) = \Phi(w', \mu')$ if and only if $(w', \mu') = (w, \mu)$ or $(w', \mu') = ((|w|^2 + i\mu)^2 w, -\mu)$.*

Proof. Since $(w, \mu), (w', \mu') \in S^{2n}_K$, we have

$$\text{(5)}\quad |w|^4 + \mu^2 = 1 = |w'|^4 + (\mu')^2.$$

The condition $\Phi(w', \mu') = \Phi(w, \mu)$ is equivalent to

$$-\operatorname{Re}\langle(|w'|^2 + i\mu')w', z\rangle = -\operatorname{Re}\langle(|w|^2 + i\mu)w, z\rangle \quad \text{for all } z \in \mathbb{C}^n,$$

which is in turn equivalent to

$$\text{(6)}\quad (|w'|^2 + i\mu')w' = (|w|^2 + i\mu)w$$

since $\operatorname{Re}\langle\cdot\,,\cdot\rangle$ is a Euclidean inner product on $\mathbb{C}^n$.

If $(w', \mu') = (w, \mu)$ then obviously (6) holds. If $w' = (|w|^2 + i\mu)^2 w$ and $\mu' = -\mu$ then (5) implies $|w| = |w'|$ and $(|w'|^2 + i\mu')w' = (|w|^2 - i\mu)(|w|^2 + i\mu)^2 w = (|w|^4 + \mu^2)(|w|^2 + i\mu)w = (|w|^2 + i\mu)w$, so (6) holds.

Conversely, suppose (6) is valid. Taking the Euclidean norm of both sides yields $|w'| = |w|$, and thus by (5) we have $\mu^2 = (\mu')^2$. Hence $\mu' = \mu$ or $\mu' = -\mu$. If $\mu' = \mu$, then dividing both sides of (6) by $|w'|^2 + i\mu' = |w|^2 + i\mu$ yields $w = w'$.

If $\mu' = -\mu$, then multiplying both sides of (6) by $(|w|^2 + i\mu)$ and observing that $(|w|^2 + i\mu)(|w|^2 - i\mu) = |w|^4 + \mu^2 = 1$ yields $w' = (|w|^2 + i\mu)^2 w$. $\square$

We define an involution $\Theta : S_K^{2n} \to S_K^{2n}$ of the Korányi sphere by

$$\Theta(w, \mu) = ((|w|^2 + i\mu)^2 w, -\mu) \quad \text{for } (w, \mu) \in S_K^{2n}. \tag{7}$$

It is straightforward to verify that $\Theta^2 = \mathrm{id}$ and that the fixed point set of Θ is the round $(2n-1)$-sphere $\{(z, 0) \in \mathbb{H}^n \mid |z| = 1\}$.

By Lemma 2.9, $\Phi(u) = \Phi(v)$ for $u, v \in S_K^{2n}$ if and only if $u = v$ or $\Theta(u) = v$. Hence Θ induces an injection $\overline{\Phi} : S_K^{2n}/\langle\Theta\rangle \to \partial_h \mathbb{H}^n$.

Theorem 2.10. *$\overline{\Phi} : S_K^{2n}/\langle\Theta\rangle \to \partial_h \mathbb{H}^n$ is a homeomorphism.*

Proof. We have already shown $\overline{\Phi}$ is a continuous injection. By Proposition 2.8, it is also surjective. Since $S_K^{2n}/\langle\Theta\rangle$ is compact, $\overline{\Phi}$ is a homeomorphism. $\square$

3. Busemann points

Definition 3.1 [Rieffel 2002, Definition 4.3]. Let (X, ρ) be a metric space and let $T \subseteq [0, \infty)$ be an unbounded subset with $0 \in T$. Let $\gamma : T \to X$ be a function.

(a) γ is a *geodesic ray* if $\rho(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in T$.

(b) γ is an *almost geodesic ray* if for every $\epsilon > 0$ there is an integer N such that $t, s \in T$ and $t \geq s \geq N$ implies

$$|\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t| < \epsilon.$$

(c) γ is a *weakly geodesic ray* if for every $y \in X$ and every $\epsilon > 0$ there is an integer N such that $s, t \geq N$ implies

$$|\rho(\gamma(t), \gamma(0)) - t| < \epsilon \quad \text{and} \quad |\rho(\gamma(t), y) - \rho(\gamma(s), y) - (t - s)| < \epsilon.$$

Rieffel proves that geodesic implies almost geodesic implies weakly geodesic [2002, Lemma 4.5], and that every horofunction is the limit of a weak geodesic [2002, Theorem 4.7] (when X is proper and has a countable basis).

Remark 3.2. It follows from Proposition 2.3 that for the Korányi metric on the Heisenberg group and for a unit vector u, the dilation ray D_u is a weak geodesic converging to the horofunction h_u.

A horofunction $h \in \partial_h X$ is a *Busemann point* [Rieffel 2002, Definition 4.8] if it is the limit of an almost geodesic ray, that is, there is an almost geodesic ray $\gamma : T \to X$ such that $\lim_{t\to\infty} \rho(\gamma(t), \cdot) - \rho(\gamma(t), p) = h - h(p)$, where p is a basepoint.

Let $(w, 0) \in S_K^{2n}$ with $|w| = 1$. Then by direct computation, the dilation ray $D_{(w,0)} : [0, \infty) \to \mathbb{H}^n$ is the Euclidean straight line $D_{(w,0)}(t) = (tw, 0)$, and

$d(D_{(w,0)}(t), D_{(w,0)}(s)) = |t-s|$. Hence $D_{(w,0)}$ is a geodesic ray, and it follows from Proposition 2.3 that $h_{(w,0)}$ is the limit of this ray. In particular, $h_{(w,0)}$ is a Busemann point. For the Korányi metric on the Heisenberg group, these are the only Busemann points.

Theorem 3.3. *A horofunction $h_{(w,\mu)} \in \partial_h \mathbb{H}^n$ with $(w,\mu) \in S_K^{2n}$ is a Busemann point if and only if $\mu = 0$.*

Proof. We showed above that $h_{(w,0)}$ with $|w| = 1$ is a Busemann point.

Suppose now that $h_{(w,\mu)} \in \partial_h \mathbb{H}^n$ is a Busemann point. Then by definition there exists an almost geodesic ray $\gamma : T \to \mathbb{H}^n$ (where $T \subseteq [0,\infty)$ is an unbounded set with $0 \in T$) such that

$$\lim_{t\to\infty} d(\gamma(t), \cdot\,) - d(\gamma(t), 0) = h_{(w,\mu)}. \tag{8}$$

Let $t_n \in T$, $n = 1, 2, \ldots$, be such that $t_n \to \infty$, and let $p_n = \gamma(t_n)$. Since γ is an almost geodesic ray, for every $\epsilon > 0$ there exists an integer N such that $n \geq m \geq N$ implies

$$|d(p_n, p_m) + d(p_m, \gamma(0)) - t_n| < \epsilon. \tag{9}$$

By setting $n = m$ in (9), we see that $n \geq N$ implies

$$|d(p_n, \gamma(0)) - t_n| < \epsilon. \tag{10}$$

By (8), given $\epsilon > 0$ there exists an integer N' such that $n \geq N'$ implies

$$|d(p_n, \gamma(0)) - \|p_n\| - h_{(w,\mu)}(\gamma(0))| < \epsilon. \tag{11}$$

Combining (9), (10), and (11) yields that

$$|d(p_n, p_m) - (\|p_n\| - \|p_m\|)| < 4\epsilon \quad \text{for } n \geq m \geq N'' = \max(N, N'). \tag{12}$$

Keeping m fixed in (12), and letting $n \to \infty$ shows that $m \geq N''$ implies $|h_{(w,\mu)}(p_m) + \|p_m\|| \leq 4\epsilon$ and so

$$\lim_{m\to\infty} h_{(w,\mu)}(p_m) + \|p_m\| = 0. \tag{13}$$

Let $p_n = (z_n, \lambda_n)$. Then by passing to a subsequence we can assume $\{\delta_{1/\|p_n\|} p_n\}$ converges to $u \in S_K^{2n}$. From the proof of Proposition 2.8, $h_{(w,\mu)} = h_u$, so by Lemma 2.9, $u = (w,\mu)$ or $u = ((|w|^2 + i\mu)^2 w, -\mu)$. By replacing (w,μ) with $((|w|^2 + i\mu)^2 w, -\mu)$ if necessary, we can assume $\delta_{1/\|p_n\|} p_n$ converges to (w,μ).

In particular, $\{z_m/\|p_m\|\}$ converges to w. By Equation (13),

$$\lim_{m\to\infty} -\operatorname{Re}\langle(|w|^2 + i\mu, z_m)\rangle + \|p_m\| = 0$$

and so

$$\lim_{m\to\infty}[-\operatorname{Re}\langle(|w|^2+i\mu)w, z_m/\|p_m\|\rangle+1]\|p_m\|=0.$$

Since $\|p_m\|\to\infty$, it follows that

$$\lim_{m\to\infty}\operatorname{Re}\langle(|w|^2+i\mu)w, z_m/\|p_m\|\rangle=1.$$

The left side is $\operatorname{Re}\langle(|w|^2+i\mu)w, w\rangle=|w|^4$, so $|w|^4=1$. Since $|w|^4+\mu^2=1$, it follows that $\mu=0$. □

Remark 3.4. Walsh has studied the horofunction boundary of the *discrete* Heisenberg group, $H_3=\langle a, b \mid [[a, b], a]=[[a, b], b]=1\rangle$, where $[x, y]$ denotes the commutator of x and y, with the word length metric coming from the set of generators $\{a, b, a^{-1}, b^{-1}\}$. In this case he was able to explicitly determine the Busemann points in the horofunction boundary [Walsh 2008, Theorems 3.3 and 3.4]. See also [Webster and Winchester 2006, Example 3.5].

Theorem 3.3 answers in the affirmative the following question, in the special case of the Heisenberg group with the Korányi metric, which concerns the global geometry of simply connected nilpotent Lie groups.

Question 3.5. Given a left invariant metric on a nonabelian simply connected nilpotent Lie group N, must the horofunction boundary of N necessarily contain non-Busemann points?

4. The topological types of the horofunction boundary and compactification

Recall from Section 2 the involution $\Theta\colon S^{2n}_K\to S^{2n}_K$ of the Korányi sphere, given by Equation (7), which fixes the equator of S^{2n}_K and interchanges its two hemispheres. By Theorem 2.10, $S^{2n}_K/\langle\Theta\rangle$ is homeomorphic to the horofunction boundary of $\mathbb{H}^n$ with the Korányi metric.

For the standard sphere $S^{2n}=\{(w,\mu)\in\mathbb{H}^n \mid |w|^2+\mu^2=1\}$, the map

$$S^{2n}_K \xrightarrow[\cong]{f} S^{2n}, \qquad (w,\mu)\mapsto(|w|w,\mu)$$

is a homeomorphism that restricts to a homeomorphism from the northern hemisphere $(S^{2n}_K)^+=\{(w,\mu)\in S^{2n}_K \mid \mu\ge 0\}$ of S^{2n}_K to the northern hemisphere of the standard sphere. The composite map

$$(S^{2n}_K)^+\hookrightarrow S^{2n}_K\to S^{2n}_K/\langle\Theta\rangle$$

is a continuous bijection and hence a homeomorphism, so we find that $\partial_h\mathbb{H}^n\cong S^{2n}_K/\langle\Theta\rangle\cong(S^{2n}_K)^+\cong(S^{2n})^+$ is homeomorphic to a $2n$-disk, and that under this homeomorphism, the Busemann points from Theorem 3.3 correspond to the S^{2n-1} boundary of the disk.

Theorem 4.1. *The horofunction boundary $\partial_h \mathbb{H}^n$ of the Heisenberg group $\mathbb{H}^n$ with the Korányi metric is homeomorphic to the $2n$-disk, with the Busemann points corresponding to the S^{2n-1} boundary of this disk. The horofunction compactification $\mathbb{H}^n \cup \partial_h \mathbb{H}^n$ is homeomorphic to the $(2n+1)$-sphere.*

The fact that the horofunction compactification $\mathbb{H}^n \cup \partial_h \mathbb{H}^n$ is homeomorphic to the $(2n+1)$-sphere will follow from the next two lemmas.

Let $B_K^{2n+1} = \{x \in \mathbb{H}^n \mid \|x\| \le 1\}$ denote the unit ball for the Korányi metric. Define a map $\Psi : B_K^{2n+1} \to \mathbb{H}^n \cup \partial_h \mathbb{H}^n$ by $\Psi(x) = \delta_{1/(1-\|x\|)} x$ if $\|x\| < 1$ and $\Psi(x) = h_x$ if $\|x\| = 1$, where we recall that $h_x(z, s) = -\operatorname{Re}\langle(|w|^2 + i\mu)w, z\rangle$ for $x = (w, \mu)$.

Then we get an induced map $\overline{\Psi}$ through

$$\begin{tikzcd} B_K^{2n+1} \arrow[r, "\Psi"] \arrow[d] & \mathbb{H}^n \cup \partial_h \mathbb{H}^n \\ B_K^{2n+1}/\langle\Theta\rangle. \arrow[ur, "\overline{\Psi}"'] & \end{tikzcd}$$

Note that $\overline{\Psi}|_{S_K^{2n}/\langle\Theta\rangle} = \overline{\Phi}$ from Theorem 2.10.

Lemma 4.2. $\overline{\Psi} : B_K^{2n+1}/\langle\Theta\rangle \to \mathbb{H}^n \cup \partial_h \mathbb{H}^n$ *is a homeomorphism.*

Proof. $\Psi|_{\operatorname{Int} B_K^{2n+1}} : \operatorname{Int} B_K^{2n+1} \to \mathbb{H}^n$ is a homeomorphism with inverse $\delta_{1/(1+\|x\|)}$, and we showed in Theorem 2.10 that $\overline{\Psi}|_{S_K^{2n}/\langle\Theta\rangle} = \overline{\Phi} : S_K^{2n}/\langle\Theta\rangle \to \partial_h \mathbb{H}^n$ is a homeomorphism. Thus we need only show that Ψ is continuous.

Let $\{x_n\}$ be a sequence in $\operatorname{Int} B_K^{2n+1}$ converging to $u \in S_K^{2n}$. Let $q_n = \delta_{1/(1-\|x_n\|)} x_n$. We will prove that $d_{q_n} - \|x_n\|/(1 - \|x_n\|) \to h_u$ uniformly on compact subsets.

Let $x_n = (w_n, \mu_n)$ and let $u = (w, \mu)$, so that $w_n \to w$ and $\mu_n \to \mu$. We have

$$d_{q_n}(z, s) - \frac{\|x_n\|}{1 - \|x_n\|} = \left(\left|z - \frac{w_n}{1-\|x_n\|}\right|^4 + \left(s - \frac{\mu_n}{(1-\|x_n\|)^2} - \frac{2}{1-\|x_n\|} \operatorname{Im}\langle w_n, z\rangle\right)^2\right)^{1/4} - \frac{\|x_n\|}{1 - \|x_n\|}.$$

Let $t_n = \|x_n\|/(1 - \|x_n\|)$, so that $t_n \to \infty$. Then

$$d_{q_n}(z, s) - \|x_n\|/(1 - \|x_n\|) = (p(t_n))^{1/4} - t_n,$$

where

$$\begin{aligned} p(t_n) &= \left|z - t_n \frac{w_n}{\|x_n\|}\right|^4 + \left(s - t_n^2 \frac{\mu_n}{\|x_n\|^2} - 2t_n \frac{\operatorname{Im}\langle w_n, z\rangle}{\|x_n\|}\right)^2 \\ &= \left(\frac{|w_n|^4 + \mu_n^2}{\|x_n\|^4}\right) t_n^4 + 4\left(-\frac{\operatorname{Re}\langle w_n, z\rangle |w_n|^2}{\|x_n\|^3} + \frac{\mu_n \operatorname{Im}\langle w_n, z\rangle}{\|x_n\|^3}\right) t_n^3 \\ &\qquad + \text{terms of lower order in } t_n. \end{aligned}$$

Note that $(|w_n|^4+\mu_n^2)/\|x_n\|^4=1$, so $p(t_n)$ is "monic". The coefficient of $4t_n^3$ in the second line above converges to

$$-(\operatorname{Re}\langle w,z\rangle)|w|^2+(\operatorname{Im}\langle w,z\rangle)\mu=-\operatorname{Re}\langle(|w|^2+i\mu)w,z\rangle=h_u(z,s),$$

and the lower power t_n coefficients converge, so by Lemma 2.4 and Remark 2.5 the proof is complete. □

Define the following subsets of B_K^{2n+1}:

$$\begin{aligned}
(S_K^{2n})^+&=\{(w,\mu)\in S_K^{2n}\mid \mu\ge 0\}, & (B_K^{2n+1})^+&=\{(w,\mu)\in B_K^{2n+1}\mid \mu\ge 0\},\\
(S_K^{2n})^-&=\{(w,\mu)\in S_K^{2n}\mid \mu\le 0\}, & (B_K^{2n+1})^-&=\{(w,\mu)\in B_K^{2n+1}\mid \mu\le 0\},\\
S_+&=(S_K^{2n})^+\cup\{(w,\mu)\in B_K^{2n+1}\mid \mu=0\},\\
S_-&=(S_K^{2n})^-\cup\{(w,\mu)\in B_K^{2n+1}\mid \mu=0\}.
\end{aligned}$$

The next lemma will show that $B_K^{2n+1}/\langle\Theta\rangle$ is homeomorphic to S^{2n+1}. Define a homeomorphism $\Theta':S_+\to S_-$ by

$$\Theta'(w,\mu)=\begin{cases}((|w|^2+i\mu)^2w,-\mu) & \text{if } \mu>0,\\ (w,0) & \text{if } \mu=0.\end{cases}$$

Lemma 4.3. *Θ' is isotopic to $\Theta'':S_+\to S_-$, where $\Theta''(w,\mu)=(w,-\mu)$.*

Proof. Define $\theta:(S_K^{2n})^+\to\mathbb{R}$ by $\theta(w,\mu)=\arccos(|w|^4-\mu^2)$, where arccos is the branch with $0\le\arccos(x)\le\pi$. Then $e^{i\theta(w,\mu)}=(|w|^2+i\mu)^2$ for $(w,\mu)\in(S_K^{2n})^+$. Define an isotopy $F:S_+\times I\to S_-$ by $F((w,\mu),t)=(e^{i\theta(w,\mu)(1-t)}w,-\mu)$. Then $F_0=\Theta'$ and $F_1=\Theta''$. □

Completion of the proof of Theorem 4.1. By Lemma 4.2, $\mathbb{H}^n\cup\partial_h\mathbb{H}^n$ is homeomorphic to $B_K^{2n+1}/\langle\Theta\rangle$, which in turn is homeomorphic to $(B_K^{2n+1})^+\cup_{\Theta'}(B_K^{2n+1})^-$, which, since Θ' is isotopic to Θ'' by Lemma 4.3, is in turn homeomorphic to $(B_K^{2n+1})^+\cup_{\Theta''}(B_K^{2n+1})^-$ by a standard theorem in geometric topology. Since the latter is homeomorphic to S^{2n+1} (we are essentially gluing two disks by the identity along their boundaries), the proof of Theorem 4.1 is complete. □

Remark 4.4. Suppose that a group G acts on a metric space X by isometries. Then G acts on the function space $C(X)_p$ by the rule $gf(x)=f(g^{-1}x)-f(g^{-1}p)$ for $g\in G$ and $f\in C(X)_p$. This action is equivariant for the embedding $X\hookrightarrow C(X)_p$, so the action of G on X extends to the horofunction boundary ∂_hX. In the specific case of the Heisenberg group with the Korányi metric, the explicit calculation of the horofunctions (Proposition 2.8) makes it straightforward to show that the action of $\mathbb{H}^n$ on itself extends to the trivial action on the horofunction boundary.

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TOM KLEIN
DEPARTMENT OF MATHEMATICS AND STATISTICS
MCMASTER UNIVERSITY
HAMILTON, ONTARIO L8S 4K1
CANADA

klein@math.mcmaster.ca

ANDREW NICAS
DEPARTMENT OF MATHEMATICS AND STATISTICS
MCMASTER UNIVERSITY
HAMILTON, ONTARIO L8S 4K1
CANADA

nicas@mcmaster.ca