TILINGS DEFINED BY AFFINE WEYL GROUPS

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Let $W$ be a Weyl group, presented as a reflection group on a Euclidean vector space $V$, and $C \subset V$ an open Weyl chamber. In a recent paper, Waldspurger proved that the images $(\text{id} - w)(C)$ for $w \in W$ are all disjoint, with union the closed cone spanned by the positive roots. We prove that similarly, the images $(\text{id} - w)(A)$ of the open Weyl alcove $A$, for $w \in W^{\text{aff}}$ in the affine Weyl group, are disjoint and their union is $V$.

1. Introduction

Let $W$ be the Weyl group of a simple Lie algebra, presented as a crystallographic reflection group in a finite-dimensional Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$. Choose a fundamental Weyl chamber $C \subset V$, and let $D$ be its dual cone, that is, the open cone spanned by the corresponding positive roots. Waldspurger [2007] proved the following remarkable result. Consider the linear transformations $(\text{id} - w): V \rightarrow V$ defined by elements $w \in W$.

**Theorem 1.1** (Waldspurger). The images $D_w := (\text{id} - w)(C)$ for $w \in W$ are all disjoint, and their union is the closed cone spanned by the positive roots:

$$D = \bigcup_{w \in W} D_w.$$

For instance, the identity transformation $w = \text{id}$ corresponds to $D_{\text{id}} = \{0\}$ in this decomposition, while the reflection $s_\alpha$ defined by a positive root $\alpha$ corresponds to the open half-line $D_{s_\alpha} = \mathbb{R}_{>0} \cdot \alpha$.

The aim of this note is to prove a similar result for the affine Weyl group $W^{\text{aff}}$. Recall that $W^{\text{aff}} = \Lambda \rtimes W$, where the coroot lattice $\Lambda \subset V$ acts by translations. Let $A \subset C$ be the Weyl alcove, with $0 \in \overline{A}$.

**Theorem 1.2.** The images $V_w = (\text{id} - w)(A)$ for $w \in W^{\text{aff}}$ are all disjoint, and their union is $V$:

$$V = \bigcup_{w \in W^{\text{aff}}} V_w.$$

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The figure above is a picture of the resulting tiling of $V$ for the root system $G_2$. Up to translation by elements of the lattice $\Lambda$, there are five 2-dimensional tiles, corresponding to the five Weyl group elements with trivial fixed point set. With $s_1$ and $s_2$ denoting the simple reflections, the lightly shaded polytopes are labeled by the Coxeter elements $s_1s_2$ and $s_2s_1$, the medium shaded polytopes by $(s_1s_2)^2$ and $(s_2s_1)^2$, and the darkly shaded polytope by the longest Weyl group element $w_0 = (s_1s_2)^3$.

One also has the following related statement.

**Theorem 1.3.** Suppose $S \in \text{End}(V)$ with $\|S\| < 1$. Then the sets $V^{(S)}_w = (S - w)(A)$ for $w \in W^{\text{aff}}$ are all disjoint, and their closures cover $V$:

$$V = \bigcup_{w \in W^{\text{aff}}} V^{(S)}_w.$$ 

Note that for $S = 0$ the resulting decomposition of $V$ is just the Stiefel diagram, while for $S = \tau$ id with $\tau \to 1$ one recovers the decomposition from Theorem 1.2.

The proof of Theorem 1.2 is in large parts parallel to Waldspurger’s [2007] proof of Theorem 1.1. We will nevertheless give full details so the paper is self-contained.

### 2. Notation

With no loss of generality we will take $W$ to be irreducible. Let $\mathcal{R} \subset V$ be the set of roots, $\{\alpha_1, \ldots, \alpha_l\} \subset \mathcal{R}$ a set of simple roots, and

$$C = \{x \mid \langle \alpha_i, x \rangle > 0, \ i = 1, \ldots, l\}$$

the corresponding Weyl chamber. We denote by $\alpha_{\text{max}} \in \mathcal{R}$ the highest root, and $\alpha_0 = -\alpha_{\text{max}}$ the lowest root. The open Weyl alcove is the $l$-dimensional simplex...
defined as

\[ A = \{ x \mid \langle \alpha_i, x \rangle + \delta_{i,0} > 0, \; i = 0, \ldots, l \}. \]

Its faces are indexed by the proper subsets \( I \subset \{0, \ldots, l\} \), where \( A_I \) is given by inequalities \( \langle \alpha_i, x \rangle + \delta_{i,0} > 0 \) for \( i \not\in I \) and equalities \( \langle \alpha_i, x \rangle + \delta_{i,0} = 0 \) for \( i \in I \). Each \( A_I \) has codimension \( |I| \). In particular, \( A_I = A_{\{i\}} \) are the codimension 1 faces, with \( \alpha_i \) as inward-pointing normal vectors. Let \( s_i \) be the affine reflections across the affine hyperplanes supporting \( A_I \), that is,

\[ s_i : x \mapsto x - (\langle \alpha_i, x \rangle + \delta_{i,0})\alpha_i^{\vee} \quad \text{for} \; i = 0, \ldots, l, \]

where \( \alpha_i^{\vee} = 2\alpha_i/\langle \alpha_i, \alpha_i \rangle \) is the simple coroot corresponding to \( \alpha_i \). The Weyl group \( W \) is generated by the reflections \( s_1, \ldots, s_l \), while the affine Weyl group \( W^{\text{aff}} \) is generated by the affine reflections \( s_0, \ldots, s_l \). The affine Weyl group is a semidirect product

\[ W^{\text{aff}} = \Lambda \rtimes W, \]

where the coroot lattice \( \Lambda = \mathbb{Z}[\alpha_1^{\vee}, \ldots, \alpha_l^{\vee}] \subset V \) acts on \( V \) by translations. For any \( w \in W^{\text{aff}} \), we will denote by \( \tilde{w} \in W \) its image under the quotient map \( W^{\text{aff}} \to W \), that is, \( \tilde{w}(x) = w(x) - w(0) \), and by \( \lambda_{\tilde{w}} = w(0) \in \Lambda \) the corresponding lattice vector.

The stabilizer of any element of \( A_I \) is the subgroup \( W^{\text{aff}}_I \) generated by \( s_i \) for \( i \in I \). It is a finite subgroup of \( W^{\text{aff}} \), and the map \( w \mapsto \tilde{w} \) induces an isomorphism onto the subgroup \( W_I \) generated by \( \tilde{s}_i \) for \( i \in I \). Recall that \( W_I \) is itself a Weyl group (not necessarily irreducible): Its Dynkin diagram is obtained from the extended Dynkin diagram of the root system \( \mathfrak{R} \) by removing all vertices that are in \( I \).

3. **The top-dimensional polytopes**

For any \( w \in W^{\text{aff}} \), the subset \( V_w = (\text{id} - w)(A) \) is the relative interior of a convex polytope in the affine subspace \( \text{ran}(\text{id} - w) \). Let

\[ W^{\text{aff}}_{\text{reg}} = \{ w \in W^{\text{aff}} \mid (\text{id} - w) \text{ is invertible} \} \]

and \( W_{\text{reg}} = W \cap W^{\text{aff}}_{\text{reg}} \), so that \( w \in W^{\text{aff}}_{\text{reg}} \) if and only if \( \tilde{w} \in W_{\text{reg}} \). The top dimensional polytopes \( V_w \) are those indexed by \( w \in W^{\text{aff}}_{\text{reg}} \), and the faces of these polytopes are \( V_{w, I} := (\text{id} - w)(A_I) \). For \( w \in W_{\text{reg}} \) and \( i = 0, \ldots, l \), let

\[ n_{w,i} := (\text{id} - \tilde{w}^{-1})^{-1}(\alpha_i). \]

**Lemma 3.1.** For all \( w \in W^{\text{aff}}_{\text{reg}} \), the open polytope \( V_w \) is given by the inequalities

\[ \langle n_{w,i}, \xi + \lambda_w \rangle + \delta_{i,0} > 0 \quad \text{for} \; i = 0, \ldots, l. \]

The face \( V_{w,I} = (\text{id} - w)(A_I) \) is obtained by replacing the inequalities for \( i \in I \) by equalities.
Proof. For any $\xi = (\id - w)x \in V$, we have
\[
\langle a_i, x \rangle = \langle (\id - \tilde{w}^{-1})^{-1}a_i, (\id - \tilde{w})x \rangle = \langle n_{w,i}, (\id - \tilde{w})x \rangle = \langle n_{w,i}, \xi + \tilde{\lambda}_w \rangle,
\]
since $\tilde{w}^{-1}$ is the transpose of $\tilde{w}$ under the inner product $\langle \cdot, \cdot \rangle$. This gives the description of $V_w$ and of its faces $V_{w,i}$.

\[\square\]

Lemma 3.2. Suppose $w \in W^{\text{aff}}_{\text{reg}}$ for $i \in \{0, \ldots, l\}$. Then $V_{w,i} = V_{\sigma,i} \subset \text{ran}(\id - \sigma)$ with $\sigma = ws_i$. In particular, $\sigma$ is an affine reflection, and $n_{w,i}$ is a vector normal to the affine hyperplane $\text{ran}(\id - \sigma)$. One has $\langle n_{w,i}, a_i^\vee \rangle = 1$.

Proof. For any orthogonal transformation $g \in O(V)$ and any reflection $s \in O(V)$, the dimension of the fixed point set of the orthogonal transformations $g$ and $gs$ differ by $\pm 1$. Since $\tilde{w}$ fixes only the origin, it follows that $\tilde{\sigma}$ has a 1-dimensional fixed point set. Hence $\text{ran}(\id - \sigma)$ is an affine hyperplane, and $\sigma$ is the affine reflection across that hyperplane. Since $s_t$ fixes $A_t$, we have
\[
V_{w,i} = (\id - w)(A_t) = (\id - ws_t)(A_t) = V_{\sigma,i} \subset \text{ran}(\id - \sigma).
\]
By definition $n_{w,i} - \tilde{w}^{-1}n_{w,i} = a_i$. Hence
\[
-2\langle n_{w,i}, a_i \rangle + \langle a_i, a_i \rangle = \|n_{w,i} - a_i\|^2 - \|n_{w,i}\|^2 = \|\tilde{w}^{-1}n_{w,i}\|^2 - \|n_{w,i}\|^2 = 0. \quad \square
\]

The following proposition indicates how the top-dimensional polytopes $V_{w,i}$ are glued along the polytopes of codimension 1.

Proposition 3.3. Let $\sigma \in W^{\text{aff}}$ be an affine reflection, that is, $\text{ran}(\id - \sigma)$ is an affine hyperplane. Consider
\[
(1) \quad \xi \in V_{\sigma} \setminus \bigcup_{|l| \geq 2} V_{\sigma,l}.
\]
Then there are two distinct indices $i, i' \in \{0, \ldots, l\}$ such that $\xi \in V_{\sigma,i} \cap V_{\sigma,i'}$. Furthermore, $w = \sigma s_i$ and $w' = \sigma s_{i'}$ are both in $W^{\text{aff}}_{\text{reg}}$, so that $V_{w,i} = V_{\sigma,i}$ and $V_{w',i'} = V_{\sigma,i'}$, and the polytopes $V_w, V_{w'}$ are on opposite sides of the affine hyperplane $\text{ran}(\id - \sigma)$.

Proof. Let $n$ be a generator of the 1-dimensional subspace $\ker(\id - \tilde{\sigma})$. Then $n$ is a vector normal to $\text{ran}(\id - \sigma)$. The preimage $(\id - \sigma)^{-1}(\xi) \subset V$ is an affine line in the direction of $n$. Since $\xi \in V_{\sigma}$, this line intersects $A_i$; hence it intersects the boundary $\partial \tilde{A}$ in exactly two points $x$ and $x'$. By (1), $x$ and $x'$ are contained in two distinct codimension 1 boundary faces $A_i$ and $A_{i'}$. Since $n$ is inward-pointing at one of the boundary faces, and outward-pointing at the other, the inner products $\langle n, a_i \rangle$ and $\langle n, a_{i'} \rangle$ are both nonzero, with opposite signs. Let $w = \sigma s_i$ and let $w' = \sigma s_{i'}$. We will show that $w \in W^{\text{aff}}_{\text{reg}}$, that is, $\tilde{w} \in W_{\text{reg}}$ (the proof for $w'$ is
similar). Let \( z \in V \) with \( \bar{w}z = z \). Then \( \tilde{\sigma}^{-1}z = \tilde{s}_i z \), so
\[
(id - \tilde{\sigma}^{-1})(z) = (id - \tilde{s}_i)(z) = \langle a_i, z \rangle a_i^\vee.
\]
The left side lies in \( \text{ran}(id - \tilde{\sigma}) \), which is orthogonal to \( n \), while the right side is proportional to \( a_i \). Since \( \langle n, a_i \rangle \neq 0 \), this is only possible if both sides are 0. Thus \( z \) is fixed under \( \tilde{\sigma} \), and hence a multiple of \( n \). On the other hand we have \( \langle a_i, z \rangle = 0 \); hence using again that \( \langle n, a_i \rangle \neq 0 \) we obtain \( z = 0 \). This shows \( \ker(id - \bar{w}) = 0 \).

As we had seen above, \( n_{w,i} \) is a vector normal to \( \text{ran}(id - \sigma) \) and hence is a multiple of \( n \). By Lemma 3.2, it is a positive multiple if and only if \( \langle n, a_i \rangle > 0 \). But then \( \langle n, a_i' \rangle < 0 \), and so \( n_{w',i'} \) is a negative multiple of \( n \). This shows that \( V_w \) and \( V_{w'} \) are on opposite sides of the hyperplane \( \text{ran}(id - \sigma) \).

Consider the union
\[
X := \bigcup_{w \in W} V_w.
\]
over \( W \subset W_{\text{aff}} \). Thus \( \bigcup_{w \in W} V_w = \bigcup_{i \in \Lambda} (\lambda + X) \). The statement of Theorem 1.2 means in particular that \( X \) is a fundamental domain for the action of \( \Lambda \). Figure 1 and Figure 2 give pictures of \( X \) for the root systems \( B_2 \) and \( G_2 \). The shaded regions are the top-dimensional polytopes (that is, the sets \( V_w \) for \( id - w \) invertible), the dark lines are the 1-dimensional polytopes (corresponding to reflections), and the origin corresponds to \( w = id \).

**Proposition 3.4.** (a) The sets \( \lambda + \text{int}(X) \) for \( \lambda \in \Lambda \) are disjoint, and
\[
\bigcup_{\lambda \in \Lambda} \lambda + \text{int}(X) = \text{V}.
\]
The set $X$ for the root system $G_2$.

(b) The open polytopes $V_w$ for $w \in \mathcal{W}_{\text{aff}}$ are disjoint, and

$$\bigcup_{w \in \mathcal{W}_{\text{aff}}} V_w = V.$$ 

Proof. Since the collection of closed polytopes $\overline{V}_w$ for $w \in \mathcal{W}_{\text{reg}}$ is locally finite, the union $V' := \bigcup_{w \in \mathcal{W}_{\text{aff}}} \overline{V}_w$ is a closed polyhedral subset of $V$. Proposition 3.3 shows that a point $\xi \in V'_w$ cannot contribute to the boundary of this subset unless it lies in $\bigcup_{\sigma \in \mathcal{W}} \bigcup_{|I| \geq 2} V_{w,I}$. We therefore see that the boundary has codimension at least 2, and hence is empty since $V'$ is a closed polyhedron. This proves $V' = V$, and also $\bigcup_{\lambda \in A} (\lambda + \mathbb{X}) = V$ with $X$ as defined in (2). Hence the volume $\text{vol}(X)$ (for the Riemannian measure on $V$ defined by the inner product) must be at least the volume of a fundamental domain for the action of $A$:

$$\text{vol}(X) \geq |W| \text{vol}(A).$$

On the other hand, $\text{vol}(V_w) = \text{vol}((\text{id} - w)(A)) = \det(\text{id} - w) \text{vol}(A)$, so

$$\text{vol}(X) \leq \sum_{w \in W} \text{vol}(V_w) = \text{vol}(A) \sum_{w \in W} \det(\text{id} - w) = |W| \text{vol}(A),$$

where we used that $\sum_{w \in W} \det(\text{id} - w) = |W|$ from [Bourbaki 1975, page 134]. This confirms $\text{vol}(X) = |W| \text{vol}(A)$. It follows that the sets $\lambda + \text{int}(\mathbb{X})$ are pairwise disjoint, or else the inequality (3) would be strict. Similarly that the sets $V_w$ for $w \in \mathcal{W}_{\text{reg}}$ are disjoint, or else the inequality (4) would be strict. (Of course, this also follows from Waldspurger’s Theorem 1.1 since $C_w \subset D_w$.) Hence all $V_w$ for $w \in \mathcal{W}_{\text{aff}}$ are disjoint. 

To proceed, we quote the following result from Waldspurger’s paper, where it is stated in greater generality.
Proposition 3.5 [Waldspurger 2007, Lemme]. Given $w \in W$ and a proper subset $I \subset \{0, \ldots, l\}$, there exists a unique $q \in W_I$ such that

$$\ker(id - w q) \cap \{x \in V \mid \langle a_i, x \rangle > 0 \text{ for all } i \in I\} \neq \emptyset.$$

Following [Waldspurger 2007] we use this to prove,

**Proposition 3.6.** Every element of $V$ is contained in some $V_\sigma$ for $\sigma \in W_{\text{aff}}$:

$$\bigcup_{w \in W_{\text{aff}}} V_\sigma = V. \tag{5}$$

**Proof.** Let $\xi \in V$ be given. Pick $w \in W_{\text{aff}}$ with $\xi \in V_w$, and let $I \subset \{0, \ldots, l\}$ with $\xi \in V_\sigma$. Then $x := (id - w)^{-1}(\xi) \in A_I$ is fixed under $W_I$. By Proposition 3.5 we may choose $\tilde{w}q \in W_I$ and $n \in V$ such that

(a) $\tilde{w}q(n) = n$,

(b) $\langle a_i, n \rangle > 0$ for all $i \in I$.

Taking $\|n\|$ sufficiently small, we have $x + n \in A$, and

$$(id - w q)(x + n) = (id - w q)(x) + (id - \tilde{w}q)n = (id - w)(x) = \xi.$$ 

This shows $\xi \in V_w$.

\[\square\]

### 4. Disjointness of the sets $\lambda + X$

To finish the proof of Theorem 1.2, we have to show that the union (5) is disjoint. Waldspurger’s Theorem 1.1 shows that all $D_\sigma = (id - w)(C)$ for $w \in W$ are disjoint. (We refer to his paper for a very simple proof of this fact.) Hence the same is true of $V_\sigma \subset D_\sigma$ for $w \in W$. It remains to show that the sets $\lambda + X$ for $\lambda \in \Lambda$, with $X$ given by (2), are disjoint.

The closure $\overline{X} = \bigcup_{w \in W} V_\sigma$ only involves the top-dimensional polytopes:

**Lemma 4.1.** The closure of the set $X$ is the union $\overline{X} = \bigcup_{w \in W_{\text{reg}}} V_\sigma$. Furthermore, $\text{int}(\overline{X}) = \text{int}(X)$.

**Proof.** We must show that for any $\xi \in V_\sigma$ with $\sigma \in W \setminus W_{\text{reg}}$, there exists a $w \in W_{\text{reg}}$ such that $\xi \in V_w$. Using induction, it suffices to find $\sigma' \in W$ such that $\xi \in V_{\sigma'}$ and $\dim(\ker(id - \sigma')) = \dim(\ker(id - \sigma)) - 1$. Let $\pi : V \rightarrow \ker(id - \sigma)^\perp = \text{ran}(id - \sigma)$ denote the orthogonal projection. Then $id - \sigma$ restricts to an invertible transformation of $\pi(V)$, and $V_{\sigma'}$ is the image of $\pi(A)$ under this transformation. We have

$$\pi(A) = \pi(\tilde{A}) = \bigcup_{i=0}^{l} \pi(\tilde{A}_i),$$

...
and this continues to hold if we remove the index \( i = 0 \) from the right side, as well as all indices \( i \) for which \( \dim \pi(A_i) < \dim \pi(V) \). That is, for each point \( x \in \pi(\bar{A}) \) there exists an index \( i \neq 0 \) such that \( x \in \pi(A_i) \), with \( \dim \pi(A_i) = \dim \pi(V) \). Taking \( x \) to be the preimage of \( \zeta \) under \( (\id - \sigma)|_x(V) \), we have \( \zeta \in V_{\sigma,i} \) with \( i \neq 0 \) and \( \dim V_{\sigma,i} = \dim \ran(\id - \sigma) \). Let \( \sigma' = \sigma s_j \in W \). Then \( V_{\sigma,i} = V_{\sigma',i} \); hence \( \dim(\ran(\id - \sigma')) \geq \dim V_{\sigma,i} = \dim(\ran(\id - \sigma)) \), which shows \( \dim \ker(\id - \sigma') \leq \dim \ker(\id - \sigma) \). By elementary properties of reflection groups, the dimensions of the fixed point sets of \( \sigma \) and \( \sigma' \) differ by either +1 or −1. Hence \( \dim(\ker(\id - \sigma')) = \dim(\ker(\id - \sigma)) - 1 \), proving the first assertion of the lemma.

It follows in particular that the closure of \( \text{int}(\bar{X}) \) equals that of \( X \). Suppose \( \zeta \in \text{int}(\bar{X}) \). By Proposition 3.6 there exists \( \lambda \in \Lambda \) with \( \zeta \in \lambda + X \). It follows that \( \text{int}(\bar{X}) \) meets \( \lambda + X \), and hence also meets \( \lambda + \text{int}(\bar{X}) \). Since the \( \Lambda \)-translates of \( \text{int}(\bar{X}) \) are pairwise disjoint (see Proposition 3.4), it follows that \( \lambda = 0 \), that is, \( \zeta \in X \). This shows \( \zeta \in X \cap \text{int}(\bar{X}) = \text{int}(X) \); hence \( \text{int}(\bar{X}) \subset \text{int}(X) \). The opposite inclusion is obvious.

Since we already know that the sets \( \lambda + \text{int}(X) \) are disjoint, we are interested in \( X \setminus \text{int}(X) \subset \partial X = \bar{X} \setminus \text{int}(X) \). Let us call a closed codimension 1 boundary face of the polyhedron \( \bar{X} \) horizontal if its supporting hyperplane contains \( V_{w,0} \) for some \( w \in W_{\text{reg}} \), and vertical if its supporting hyperplane contains \( V_{w,i} \) for some \( w \in W_{\text{reg}} \) and \( i \neq 0 \). These two cases are exclusive:

**Lemma 4.2.** Let \( n \) be the inward-pointing normal vector to a codimension 1 face of \( \bar{X} \). Then \( \langle n, \alpha_{\text{max}} \rangle \neq 0 \). In fact, \( \langle n, \alpha_{\text{max}} \rangle < 0 \) for the horizontal faces and \( \langle n, \alpha_{\text{max}} \rangle > 0 \) for the vertical faces.

**Proof.** Given a codimension 1 boundary face of \( \bar{X} \), pick any point \( \zeta \) in that boundary face not lying in \( \bigcup_{w \in W_{\text{reg}}} \bigcup_{|I| \geq 2} V_{w,I} \). Let \( w \in W_{\text{reg}} \) and \( i \in \{0, \ldots, l\} \) such that \( \zeta \in V_{w,i} \) and \( n_{w,i} \) is an inward-pointing normal vector. By Proposition 3.3, there is a unique \( i' \neq i \) such that \( \zeta \in V_{w',i'} \), where \( w' = ws_{j_l}s_{j_l}' \). Since \( V_{w'} \) and \( V_{w'} \) lie on opposite sides of the affine hyperplane spanned by \( V_{w,i} \), and \( \zeta \) is a boundary point of \( \bar{X} \), we have \( w' \notin W \). Thus one of \( i \) and \( i' \) must be zero. If \( i = 0 \) (so that the given boundary face is horizontal) we obtain \( \langle n_{w,0}, \alpha_{\text{max}} \rangle = -\langle n_{w,0}, \alpha_0 \rangle < 0 \). If \( i' = 0 \) we similarly obtain \( \langle n_{w',0}, \alpha_{\text{max}} \rangle < 0 \); hence \( \langle n_{w,i}, \alpha_{\text{max}} \rangle > 0 \).

**Lemma 4.3.** Let \( \zeta \in X \setminus \text{int}(X) \). Then there exists a vertical boundary face of \( \bar{X} \) containing \( \zeta \). Equivalently, the complement \( \partial \bar{X} \setminus (X \setminus \text{int}(X)) \) is contained in the union of horizontal boundary faces.

**Proof.** The alcove \( A \) is invariant under multiplication by any scalar in \( (0, 1) \). Hence, the same is true for the sets \( V_w \) for \( w \in W \), as well as for \( X \) and \( \text{int}(X) \). Hence, if \( \zeta \in X \setminus \text{int}(X) \) there exists \( t_0 > 1 \) such that \( t_0 \zeta \in X \setminus \text{int}(X) \) for \( 1 \leq t < t_0 \). The closed codimension 1 boundary face containing this line segment is necessarily vertical,
since a line through the origin intersects the affine hyperplane \{ x \mid \langle a_{w,0}, x - \xi \rangle = 0 \}
in at most one point.

**Proposition 4.4.** For any \( \xi \in X \), there exists \( \epsilon > 0 \) such that \( \xi + s\alpha_{\text{max}} \in \text{int}(X) \) for \( 0 < s < \epsilon \).

**Proof.** If \( \xi \in \text{int}(X) \) there is nothing to show; hence suppose \( \xi \in X \setminus \text{int}(X) \). Suppose first that \( \xi \) is not in the union of horizontal boundary faces of \( \bar{X} \). Then there exists an open neighborhood \( U \) of \( \xi \) such that \( U \cap X = U \cap \bar{X} \). All boundary faces of \( \bar{X} \) meeting \( \xi \) are vertical, and their inward-pointing normal vectors \( n \) all satisfy \( \langle n, \alpha_{\text{max}} \rangle > 0 \). Hence, \( \xi + s\alpha_{\text{max}} \in \text{int}(U \cap \bar{X}) = \text{int}(U \cap X) \subset X \) for \( s > 0 \) sufficiently small.

For the general case, suppose by way of contradiction that for all \( \epsilon > 0 \), there is \( s \in (0, \epsilon) \) with \( \xi + s\alpha_{\text{max}} \notin \text{int}(X) \). Since \( \xi \) is contained in some vertical boundary face, one can choose \( t > 1 \) so that \( \xi' := t\xi \in X \setminus \text{int}(X) \), but \( \xi' \) is not in the closure of the union of horizontal boundary faces. Given \( \epsilon > 0 \), pick \( s \in (0, \epsilon) \) such that \( \xi' + (s/t)\alpha_{\text{max}} \notin \text{int}(X) \). Since \( \text{int}(X) \) is invariant under multiplication by scalars in \( (0, 1) \), the complement \( V \setminus \text{int}(X) \) is invariant under multiplication by scalars in \( (1, \infty) \); hence we obtain \( \xi' + s\alpha_{\text{max}} \notin \text{int}(X) \). This contradicts what we have shown above, and completes the proof. \( \square \)

**Proposition 4.5.** The sets \( \lambda + X \) for \( \lambda \in \Lambda \) are disjoint.

**Proof.** Suppose \( \xi \in (\lambda + X) \cap (\lambda' + X) \). By Proposition 4.4, we can choose \( s > 0 \) so that \( \xi + s\alpha_{\text{max}} \in (\lambda + \text{int}(X)) \cap (\lambda' + \text{int}(X)) \). Since the \( \Lambda \)-translates of \( \text{int}(X) \) are disjoint, it follows that \( \lambda = \lambda' \). \( \square \)

This completes the proof of Theorem 1.2. We conclude with some remarks on the properties of the decomposition \( V = \bigcup_{w \in W_{\text{aff}}} V_w \).

**Remarks 4.6.** (a) The group of symmetries \( \tau \) of the extended Dynkin diagram (that is, the outer automorphisms of the corresponding affine Lie algebra) acts by symmetries of the decomposition \( V = \bigcup_{w \in W_{\text{aff}}} V_w \), as follows. Identify the nodes of the extended Dynkin diagram with the simple affine reflections \( s_0, \ldots, s_l \). Then \( \tau \) extends to a group automorphism of \( W_{\text{aff}} \), taking \( s_i \) to \( \tau(s_i) \). This automorphism is implemented by a unique Euclidean transformation \( g : V \to V \), that is,

\[ g \varphi g^{-1} = \tau(w) \text{ for all } w \in W_{\text{aff}}. \]

Then \( g \) preserves \( A \), and consequently

\[ g V_w = g(\text{id} - w)(A) = (\text{id} - \tau(w))(A) = V_{\tau(w)} \text{ for } w \in W_{\text{aff}}. \]

(b) It is immediate from the definition that \(-w : V \to V, x \mapsto -wx \) takes \( V_{w^{-1}} \) into \( V_w \):

\[ -w(V_{w^{-1}}) = V_w. \]
(c) For any positive root $\alpha$, let $s_\alpha$ be the corresponding reflection. Then
\[(\text{id} - s_\alpha)(\xi) = \langle \alpha, \xi \rangle \alpha^\vee,\]
where $\alpha^\vee$ is the coroot corresponding to $\alpha$. Hence $D_{s_\alpha}$ is the relative interior of the line segment from 0 to $\lambda \alpha^\vee$, where $\lambda$ is the maximum value of the linear functional $\zeta \mapsto \langle \alpha, \zeta \rangle$ on the closed alcove $\overline{A}$. This maximum is achieved at one of the vertices. Let $\sigma_1^\vee, \ldots, \sigma_l^\vee$ be the fundamental coweights, defined by $\langle \alpha, \sigma_j^\vee \rangle = \delta_{ij}$ for $i, j = 1, \ldots, l$. Let $c_i \in \mathbb{N}$ be the coefficients of $\alpha_{\text{max}}$ relative to the simple roots: $\alpha_{\text{max}} = \sum_{i=1}^l c_i \alpha_i$. Then the nonzero vertices of $A$ are $\sigma_i^\vee / c_i$. Similarly let $a_i \in \mathbb{Z}_{\geq 0}$ be the coefficients of $\alpha_i$, so that $\alpha_i = \sum_{i=1}^l a_i \alpha_i$. Then the value of $\alpha$ at the $i$-th vertex of $\overline{A}$ is $a_i / c_i$, and $\lambda$ is the maximum of those values. There are two interesting cases: First, if $\alpha = \alpha_{\text{max}}$, then all $a_i / c_i = 1$, and $\alpha^\vee = \alpha$. That is, the open line segment from the origin to the highest root always appears in the decomposition. Second, if $\alpha = \alpha_i$, then $a_i = 1$ while all other $a_j$ vanish. In this case, one obtains the open line segment from the origin to $(1 / c_i) \alpha^\vee_i$.

(d) Every $V_w$ contains a distinguished ‘base point’. Indeed, let $\rho \in V$ be the half-sum of positive roots, and $h^\vee = 1 + \langle \alpha_{\text{max}}, \rho \rangle$ the dual Coxeter number. Then $\rho / h^\vee \in A$, and consequently $\rho / h^\vee - w(\rho / h^\vee) \in V_w$.

5. Proof of Theorem 1.3

The proof is very similar to the proof of Proposition 3.4; hence we will be brief. Each $V_w^{(S)} = (S - w)(A)$ is the interior of a simplex in $V$, with codimension 1 faces $V_{w,i}^{(S)} = (S - w)(A_i)$. As in the proof of Lemma 3.1, we see that
\[n_{w,i}^{(S)} = (S - i\omega^{-1})^{-1} a_i\]
is an inward-pointing normal vector to the face $V_{w,i}^{(S)}$. For $S = 0$ this simplifies to $n_{w,i}^{(0)} = - w a_i$. If $w' = w_s i$, then $V_{w,i}^{(S)} = V_{w',i}^{(S)}$, so that $n_{w,i}^{(S)}$ and $n_{w',i}^{(S)}$ are proportional. Since $n_{w,i}^{(0)} = - n_{w',i}^{(0)}$, continuity implies that $n_{w,i}^{(S)}$ is a negative multiple of $n_{w',i}^{(S)}$. As a consequence, we see that $V_w^{(S)}$ and $V_{w'}^{(S)}$ are on opposite sides of affine hyperplane supporting $V_{w,i}^{(S)} = V_{w',i}^{(S)}$. Arguing as in the proof of Proposition 3.4, this shows that
\[\bigcup_{w \in W^{\text{af}}} V_w^{(S)} = V.\]
Letting $X^{(S)} = \bigcup_{w \in W} V_w^{(S)}$, it follows that $V = \bigcup_{i \in A} (S + X^{(S)})$. Therefore we have $\text{vol}(X^{(S)}) \geq |W| \text{vol}(A)$. But

\[
\text{vol}(X^{(S)}) \leq \sum_{w \in W} \text{vol}((S - w)(A)) = \text{vol}(A) \sum_{w \in W} |\det(S - w)| = \text{vol}(A) \sum_{w \in W} \det(id - Sw^{-1}) = |W| \text{vol}(A),
\]

using [Bourbaki 1975, page 134]. It follows that $\text{vol}(X^{(S)}) = |W| \text{vol}(A)$, which implies (as in the proof of Proposition 3.4) that all $\text{int}(\overline{V}_w^{(S)}) = V_w^{(S)}$ are disjoint. This completes the proof.

**Remark 5.1.** Theorem 1.3 and its proof go through for any $S$ in the component of $0$ in the set $\{S \in \text{End}(V) \mid \det(S - w) \neq 0 \text{ for all } w \in W\}$. For instance, the fact that $\det(id - Sw^{-1}) > 0$ follows by continuity from $S = 0$. On the other hand, the result becomes false if, for example, $S$ is a positive matrix with $S > 2 \text{id}$, since then $\sum_{w \in W} |\det(S - w)| = \sum_{w \in W} \det(S - w) = \det(S)|W|$; see [Bourbaki 1975, page 134].

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