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We prove a dynamical wave trace formula for asymptotically hyperbolic $(n+1)$-dimensional manifolds with negative (but not necessarily constant) sectional curvatures; the formula equates the renormalized wave trace to the lengths of closed geodesics. This result generalizes the classical theorem of Duistermaat and Guillemin for compact manifolds and the results of Guillopé and Zworski, Perry, and Guillarmou and Naud for hyperbolic manifolds with infinite volume. A corollary of this dynamical trace formula is a dynamical resonance-wave trace formula for compact perturbations of convex cocompact hyperbolic manifolds. We define a dynamical zeta function and prove its analyticity in a half plane. In our main result, we produce a prime orbit theorem for the geodesic flow. This is the first such result for manifolds that have neither constant curvature nor finite volume. As a corollary to the prime orbit theorem, using our dynamical resonance-wave trace formula, we show that the existence of pure point spectrum for the Laplacian on negatively curved compact perturbations of convex cocompact hyperbolic manifolds is related to the dynamics of the geodesic flow.

1. Introduction

Mathematicians have been interested for many years in the spectral theory and dynamics of hyperbolic manifolds. Motivated by recent developments in theoretical physics, they have begun to focus on *asymptotically hyperbolic* manifolds; see [Albin 2007; Mazzeo and Melrose 1987; Borthwick and Perry 2002; Joshi and Sá Barreto 2000; Graham and Zworski 2003; Fefferman and Graham 2002]. These manifolds arise in connection with the correspondence between conformal field theory and anti-de Sitter space [Graham 2000; Sá Barreto and Zworski 1997], and are a class of manifolds on which geometric scattering theory can be developed; see [Joshi and Sá Barreto 2000; Guillarmou 2005; 2007].

Recall that on a compact manifold, the spectrum of the Laplacian Δ is a discrete subset $\sigma(\Delta) = \{\lambda_k^2\}_{k=1}^{\infty}$ of $\mathbb{R}^+$, and the wave trace is formally the distribution

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$\sum_{k\geq 1} e^{i\lambda_k t}$. The singularities of the wave trace occur at $t=0$ and at the lengths of closed geodesics. Duistermaat and Guillemin [1975] computed the principal part of the singularities for $t>0$ to be

$$\frac{l(\gamma)\delta(|t|-kl(\gamma))}{\sqrt{|\det(I-\mathcal{P}_\gamma^k)|}};$$

they also computed the asymptotics of the "big singularity" at $t=0$. One then has, at least formally,

$$\sum_{k\geq 1} e^{i\lambda_k|t|} = \sum_{\{\gamma\}}\sum_{k=1}^{\infty} \frac{l(\gamma)\delta(|t|-kl(\gamma))}{\sqrt{|\det(I-\mathcal{P}_\gamma^k)|}} + A(t),$$

where $\{\gamma\}$ are the primitive closed geodesics with length $l(\gamma)$, $\mathcal{P}_\gamma^k$ is the k-times Poincaré map in the cotangent bundle about γ, and the remainder $A(t)$ is exponentially singular at 0 and smooth for $t>0$. This formula shows a beautiful connection via the wave trace between the Laplace and length spectra.

On compact hyperbolic surfaces this dynamical formula is well known. The remainder term $A(t)$ is explicitly computable as a ratio of hyperbolic trigonometric functions, and the result is known as a Selberg trace formula [Selberg 1956]. Hejhal [1975; 1976; 1983] proved Selberg trace formulae for cofinite surfaces, congruence subgroups, and $\mathrm{PSL}(2,\mathbb{R})$. Gangolli and Warner [1975; 1980] proved Selberg trace formulae in higher dimensions. Algebraic group averaging methods have been used to prove trace formulae in [Guillopé and Zworski 1999; Guillarmou and Naud 2006; Perry 2003] on manifolds with constant negative curvature and infinite volume. Further results include [Arthur 1989; Borthwick et al. 2005; Patterson and Perry 2001; Juhl 2001; Müller 1983]. For manifolds with infinite volume, a renormalized wave trace replaces the standard wave trace in the dynamical formula. This renormalized wave trace is known as the 0-trace and was introduced in [Guillopé and Zworski 1995]. Using the 0-trace, Joshi and Sá Barreto [2001] generalized the results of [Duistermaat and Guillemin 1975] to the asymptotically hyperbolic setting. Our dynamical wave trace formula is a refinement of results of [Joshi and Sá Barreto 2001] using techniques in [Jakobson et al. 2008] to provide long time asymptotics with respect to Ehrenfest time $T(\lambda)\sim c\ln(\lambda)$; see [Bérard 1977] and [Zelditch 1994].

Theorem 1.1. *Suppose (X,g) is an asymptotically hyperbolic $(n+1)$-dimensional manifold with negative sectional curvatures. Let $0\text{-tr}\cos\big(t\sqrt{\Delta-n^2/4}\big)$ denote the regularized trace of the wave group, and let $t_0>0$. Let $\mathcal{L}_p$ denote the set of primitive closed geodesics of (X,g), and for $\gamma\in\mathcal{L}_p$, let $l(\gamma)$ denote the length* *of γ. Then*

$$0\text{-tr}\cos\big(t\sqrt{\Delta-n^2/4}\big)=\sum_{\gamma\in\mathcal{L}_p}\sum_{k\in\mathbb{N}}\frac{l(\gamma)\delta(|t|-kl(\gamma))}{\sqrt{|\det(I-\mathcal{P}^k_\gamma)|}}+A(t) \tag{1-1}$$

as a distributional equality in $\mathcal{D}'([t_0,\infty))$,[1] *where $\mathcal{P}^k_\gamma$ is the k-times Poincaré map around γ in the cotangent bundle. The long time asymptotics of the trace formula are as follows. Let $\mathscr{C}^\infty_0([t_0,\infty))\ni\phi(t)=\cos(\lambda t)\rho(t)$ such that* $\operatorname{supp}(\phi)\subset[t_0,T]$, *and $T=T(\lambda)\sim\epsilon\ln(\lambda)$ for a suitably small constant $\epsilon>0$. Then, as $\lambda\to\infty$,*

$$\int_{\mathbb{R}}\phi(t)\,0\text{-tr}\cos\big(t\sqrt{\Delta-n^2/4}\big)dt=\sum_{\gamma\in\mathcal{L}_p}\sum_{k\in\mathbb{N}}\frac{l(\gamma)\phi(kl(\gamma))}{\sqrt{|\det(I-\mathcal{P}^k_\gamma)|}}+O(1). \tag{1-2}$$

Recall the results of [Mazzeo and Melrose 1987]: the Laplacian on an asymptotically hyperbolic manifold of dimension $n+1$ has absolutely continuous spectrum, $\sigma_{\text{ac}}(\Delta)=[n^2/4,\infty)$, and a finite pure point spectrum, $\sigma_{\text{pp}}(\Delta)\subset(0,n^2/4)$. It is natural in this setting to use the spectral parameter $\Lambda=s(n-s)$. Theorem 1.1 and the Poisson formula of [Borthwick 2008] produce a dynamical resonance-wave trace formula in the following corollary.

Corollary 1.2. *Suppose (X,g) is a negatively curved*[2] *compact perturbation of a convex cocompact $(n+1)$-dimensional hyperbolic manifold (see Definition 3.4). Then we have the distributional equality*

$$\tfrac{1}{2}\sum_{s\in\mathcal{R}^{sc}}e^{(s-n/2)|t|}=\sum_{\gamma\in\mathcal{L}_p}\sum_{k\in\mathbb{N}}\frac{l(\gamma)\delta(|t|-kl(\gamma))}{\sqrt{|\det(I-\mathcal{P}^k_\gamma)|}}+C(t)$$

as an element of $\mathcal{D}'([t_0,\infty))$ for any $t_0>0$. The resonances $\mathcal{R}^{sc}$ are summed with multiplicity, *and the long time asymptotics of the trace formula are given by* (1-2).

The next result is analyticity of the dynamical zeta function in a half-plane determined by the topological entropy of the geodesic flow.

Theorem 1.3. *Suppose (X,g) is an asymptotically hyperbolic negatively curved $(n+1)$-dimensional manifold. Let L_p be the set of primitive closed orbits of the geodesic flow on X, and for $\gamma\in L_p$, let $l_p(\gamma)$ be the length of the primitive period of γ. Then the dynamical zeta function*

$$Z(s)=\exp\bigg(\sum_{\gamma\in L_p}\sum_{k\in\mathbb{N}}\frac{e^{-ksl_p(\gamma)}}{k}\bigg)$$

[1] $\mathcal{D}'(X)$ is the dual of $\mathscr{C}^\infty_0(X)$.

[2] We will use "negatively curved" to mean all sectional curvatures are negative.

converges absolutely for $\mathrm{Re}(s) > h$, *where h is the topological entropy of the geodesic flow* (4-5). *The weighted dynamical zeta function*

$$\tilde{Z}(s) = \exp\biggl(\sum_{\gamma \in L_p} \sum_{k \in \mathbb{N}} \frac{e^{-ksl_p(\gamma)}}{k\sqrt{|\det(I - \mathcal{P}_\gamma^k)|}}\biggr)$$

converges absolutely for $\mathrm{Re}(s) > \mathfrak{p}(-H/2)$, *where* $\mathfrak{p}$ *is the topological pressure of* (4-4) *and H is the Sinai–Ruelle–Bowen potential of* (4-3).

Our main result, which is primarily an application of [Parry and Pollicott 1983], produces a prime orbit theorem for the geodesic flow on negatively curved $(n+1)$-dimensional asymptotically hyperbolic manifolds with positive topological entropy. The prime orbit theorem for infinite-volume hyperbolic manifolds is due to Perry [2001] and Guillarmou and Naud [2006]; see also earlier work of Guillopé [1986] and Lalley [1988].

Theorem 1.4. *Suppose* (X, g) *is an asymptotically hyperbolic* $(n+1)$*-dimensional manifold with negative sectional curvatures. Let* L_p *be the set of primitive closed orbits of the geodesic flow, and for* $\gamma \in L_p$, *let* $l_p(\gamma)$ *be the length of the primitive period of* γ. *Let h be the topological entropy of the geodesic flow* (4-5), *and assume* $h > 0$. *The dynamical zeta function*

$$Z(s) = \exp\biggl(\sum_{\gamma \in L_p} \sum_{k \in \mathbb{N}} \frac{e^{-ksl_p(\gamma)}}{k}\biggr)$$

has a nowhere vanishing analytic extension to an open neighborhood of $\mathrm{Re}(s) \geq h$ *except for a simple pole at* $s = h$. *Moreover, the length spectrum counting function*

(1-3) $$N(T) := \#\{\gamma \in \mathcal{L} : l(\gamma) \leq T\} \quad \textit{satisfies} \ \lim_{T \to \infty} \frac{TN(T)}{e^{hT}} = 1.$$

Finally, we use the prime orbit theorem and the trace formula to prove a result that shows that the existence of pure point spectrum is related to the topological entropy of the geodesic flow and the curvature bounds for negatively curved compact perturbations of convex cocompact hyperbolic manifolds. In the constant curvature case, this result is due to [Patterson 1987].

Corollary 1.5. *Let* (X, g) *be a negatively curved compact perturbation of a convex cocompact* $(n+1)$*-dimensional hyperbolic manifold with topological entropy h for the geodesic flow. Then there exist* $0 < k_2 \leq 1 \leq k_1$ *such that the sectional curvatures* κ *satisfy* $-k_1^2 \leq \kappa \leq -k_2^2$. *If* $h > nk_1/2$, *then* $\sigma_{\mathrm{pp}}(\Delta) \neq \varnothing$, *and moreover, there is a* $\Lambda_0 = s_0(n - s_0) \in \sigma_{\mathrm{pp}}(\Delta)$ *with* $s_0 \geq h + n(1 - k_1)/2$. *If* $h \leq nk_2/2$, *then* $\sigma_{\mathrm{pp}}(\Delta) = \varnothing$.

The paper is organized as follows. In Section 2, we recall some basic spectral and geometric properties of asymptotically hyperbolic manifolds, including the 0-renormalization and key results of [Joshi and Sá Barreto 2001] for the wave group. In Section 3, we prove the dynamical trace formula and its corollary. In Section 4, after defining the dynamical zeta function and recalling some basic definitions and results from dynamics, we prove our preliminary result for the dynamical zeta function. We then prove the prime orbit theorem and its corollary relating the pure point spectrum to the dynamics of the geodesic flow in Section 5. Concluding remarks comprise Section 6. In the appendix, we provide additional technical details concerning the long time asymptotics of the trace formula.

2. Asymptotically hyperbolic manifolds

A manifold with boundary $(X^{n+1}, \partial X)$ is *asymptotically hyperbolic* if there exists a boundary-defining function x such that a neighborhood of ∂X admits a product decomposition $(0, \epsilon)_x \times \partial X$, with respect to which the metric takes the form

$$g = \frac{dx^2 + h(x, y, dx, dy)}{x^2}, \tag{2-1}$$

where $h|_{\{x=0\}}$ is independent of dx. There is no one canonical metric on ∂X but rather a conformal class of metrics induced by $h|_{\{x=0\}} = h_0$, and $(\partial X, [h_0])$ is called the conformal infinity; see [Graham and Zworski 2003; Guillarmou 2007]. Mazzeo and Melrose [1987] observed that X is a complete Riemannian manifold; moreover, along any smooth curve in $X - \partial X$ approaching a point $p \in \partial X$, the sectional curvatures of g approach $-|dx|^2_{x^2 g}$. For each $h \in [h_0]$ there exists a unique (near the boundary) boundary-defining function x such that $|dx|_{x^2 g} = 1$, near $Y := \partial X$. With this normalization, the sectional curvatures approach -1 at ∂X, hence the name "asymptotically hyperbolic." A large class of interesting asymptotically hyperbolic metrics are the conformally compact metrics. Four-dimensional conformally compact Einstein metrics have received recent attention in both geometric analysis and mathematical physics due to their relation to quantum field theory and quantum gravity; see [Albin 2007; Fefferman and Graham 2002].

The wave group and renormalized wave trace. The (even) wave kernel is the Schwartz kernel of the fundamental solution to

$$(\partial_t^2 + \Delta - \tfrac{1}{4}n^2)U(t, w, w') = 0, \qquad U(0, w, w') = \delta(w - w'), \qquad \tfrac{\partial}{\partial t}U(0, w, w') = 0.$$

Due to the semigroup property with respect to time, the wave kernel is also referred to as the wave group and written $\cos(t\sqrt{\Delta - n^2/4})$. In [2001], Joshi and Sà Barreto constructed the wave group as an element of an operator calculus on a manifold with corners obtained by blowing up $\mathbb{R}^+ \times X \times X$. This construction was heavily influenced by Melrose's work [1993] with b-manifolds.

Since asymptotically hyperbolic spaces have infinite volume, one must introduce an integral renormalization to take the trace of the wave group. Recall that the finite part $\text{f.p.}_{\epsilon=0} f(\epsilon)$ is defined as f_0 when $f(\epsilon) = f_0 + \sum_k f_k \epsilon^{-\lambda_k} (\log \epsilon)^{m_k} + o(1)$, with $\mathrm{Re}(\lambda_k) \geq 0$ and $m_k \in \mathbb{N} \cup \{0\}$. Then f_0 is unique, as shown for example in [Hörmander 1983].

Definition 2.1 [Guillopé and Zworski 1995]. The 0-*regularized integral* ${}^0\!\int \omega$ of a smooth function (or density) f on X is defined, if it exists, as the finite part

$$ {}^0\!\!\int_X f := \text{f.p.}_{\epsilon=0} \int_{x(p)>\epsilon} f(p)\, \mathrm{d}\mathrm{vol}_g(p), $$

where x is a boundary-defining function.

For an operator A with smooth Schwartz kernel $A(z, y)$ on $X \times X$, we may then define the 0-trace of A to be

$$ \text{0-tr}(A) := {}^0\!\!\int_X A(z, z)\, \mathrm{d}\mathrm{vol}_g(z). $$

Joshi and Sà Barreto showed that the wave group for an asymptotically hyperbolic manifold has a well-defined 0-trace. In [2001, Theorems 4.1 and 4.2], they prove that the singular support of $\text{0-tr}\cos(t\sqrt{\Delta - n^2/4})$ is contained in the set of lengths of closed geodesics of (X, g), and that there exists a compact subset $X_\epsilon \subset X$ that contains all the closed geodesics of X. These results allow us to generalize local dynamical arguments and results for compact manifolds to the asymptotically hyperbolic setting.

3. Dynamical trace formula

The local arguments of [Duistermaat and Guillemin 1975] together with [Joshi and Sá Barreto 2001, Theorem 4.2] provide the leading terms in the renormalized wave trace; however, to bound the remainder term we adapt the local techniques of [Jakobson et al. 2008], and this requires the following two lemmas.

Lemma 3.1. *Let (X, g) be a smooth, complete, $(n+1)$-dimensional Riemannian manifold whose sectional curvatures κ satisfy $-k_1^2 \leq \kappa \leq -k_2^2$ for some $0 < k_2 \leq k_1$. Then the Poincaré map about a closed orbit γ of the geodesic flow has eigenvalues λ_i for $i = 1, \dots, 2n$ such that*

$$ \begin{aligned} e^{k_2 l(\gamma)} &\leq |\lambda_i| \leq e^{k_1 l(\gamma)} && \text{for } i = 1, \dots, n, \\ e^{-k_1 l(\gamma)} &\leq |\lambda_i| \leq e^{-k_2 l(\gamma)} && \text{for } i = n+1, \dots, 2n, \end{aligned} $$

where $l(\gamma)$ is the period (or length) of γ.

Proof. Let P_γ be the Poincaré map about the closed orbit γ of the geodesic flow. Since the flow is Anosov [Anosov 1967], P_γ has n expanding eigenvalues $\{\lambda_i\}_{i=1}^n$ and n contracting eigenvalues $\{\lambda_i\}_{i=n+1}^{2n}$. We proceed to estimate the expanding eigenvalues using Rauch's comparison theorem. Let M_i be complete manifolds of dimension $n+1$ with constant negative curvature $-k_i^2$ for $i = 1, 2$. Consider Jacobi fields J and J_i along any geodesic γ_0 on X and γ_i on M_i such that

$$J(0) = J_i(0) = 0, \quad \langle J'(0), \gamma_0'(0)\rangle = \langle J_i'(0), \gamma_i'(0)\rangle, \quad |J'(0)| = |J_i'(0)|.$$

Assume that γ_0 and γ_i do not have conjugate points on $(0, a]$ for some $a > 0$. Then, by Rauch's comparison theorem [do Carmo 1992]

$$|J_2(t)| \le |J(t)| \le |J_1(t)| \quad \text{for } t \in (0, a]. \tag{3-1}$$

By definition of the Lyapunov exponents [Barreira and Pesin 2002] for the Poincaré map on the geodesic flow, the expanding eigenvalues λ_i of the Poincaré map about a closed geodesic $\tilde{\gamma}$ satisfy

$$|\lambda_i| = e^{k_i l(\tilde{\gamma})} \tag{3-2}$$

on the constant curvature manifolds M_i. Since the eigenvalues of the Poincaré map are determined by the Jacobi fields along the closed geodesics, it follows from (3-1) and (3-2) that the expanding eigenvalues for P_γ satisfy

$$e^{k_2 l(\gamma)} \le |\lambda_i| \le e^{k_1 l(\gamma)} \quad \text{for } i = 1, \dots, n.$$

For each contracting eigenvalue λ_i with $i \in \{n+1, \dots, 2n\}$, there is an expanding eigenvalue $\lambda_{j(i)}$ with $j(i) \in \{1, \dots, n\}$ such that $|\lambda_i|^{-1} = |\lambda_{j(i)}|$. The inequality for the contracting eigenvalues follows immediately. □

The next result allows us to estimate the remainder in the trace formula by separating the periodic orbits and applying the stationary phase method of [Jakobson et al. 2008]. This separation lemma is a generalization of [Jakobson et al. 2008, Lemma 2.3] to our $(n+1)$-dimensional asymptotically hyperbolic variable negative curvature setting.

Lemma 3.2. *Let (X, g) be an asymptotically hyperbolic $(n+1)$-dimensional manifold with negative sectional curvatures. Let $\mathcal{N}(\gamma, \epsilon)$ denote the ϵ-neighborhood of a geodesic γ in the unit tangent bundle SX with respect to the Sasaki metric. Then there exist positive constants T_0, B, and δ (depending only on the injectivity radius* $\mathrm{inj}(X)$ *and the curvature bounds) such that for any $T > T_0$ the sets $N(\gamma, e^{-BT})$ are disjoint for all pairs of closed geodesics γ on X with length $l_\gamma \in [T-\delta, T]$.*

Proof. Since the sectional curvatures of any asymptotically hyperbolic manifold approach -1 at ∂X, there exist $0 < k_2 \le 1 \le k_1$ such that

$$-k_1^2 \le \kappa \le -k_2^2, \tag{3-3}$$

for all sectional curvatures κ on X. Let $B > 2k_1$ and choose $0 < \delta < \mathrm{inj}(X)/3$, and let T_0 be such that $2e^{-k_1 T_0} < \delta$. We proceed by contradiction. For a given geodesic

γ_1, assume there is a second geodesic γ_2 with $T-\delta < l(\gamma_1) \le l(\gamma_2) \le T$ such that the corresponding neighborhoods intersect. Assume the geodesics are not inverses of each other; by the choice of δ, they cannot be integer multiples of each other unless they are inverses. Let $\gamma_j(t)$ for $0 \le t \le l(\gamma_j)$ denote the geodesic on X, and let its corresponding lift to SX be denoted by $\tilde{\gamma}_j(t) = (\gamma_j(t), \gamma_j'(t))$. We may assume without any loss of generality that $d_{SX}(\tilde{\gamma_2}(0), \tilde{\gamma_1}(0)) \le 2e^{-2k_1T}$. For any $0 \le t \le l(\gamma_2)$, by Lemma 3.1,

$$d_{SX}(\tilde{\gamma_2}(t), \tilde{\gamma_1}(t)) = d_{SX}(G^t\tilde{\gamma_2}(0), G^t(\tilde{\gamma_1}(0)) \le 2e^{-2k_1T}e^{k_1t} \le 2e^{-k_1T}, \tag{3-4}$$

where G^t is the geodesic flow. This implies

$$d_X(\gamma_2(t), \gamma_1(t)) \le 2e^{-2k_1T}. \tag{3-5}$$

Consequently, the entire geodesics γ_i lie in the $2e^{-k_1T}$ neighborhood of each other. Now, reparametrize γ_1 by defining

$$\beta_1(s) := \gamma_1(l_1s/l_2) \quad \text{for } 0 \le s \le l_2,$$

where $\gamma_i : [0, l_i] \to X$. By the triangle inequality,

$$\begin{aligned} d(\gamma_2(t), \beta_1(t)) &\le d(\gamma_2(t), \gamma_1(t)) + d(\gamma_1(t), \beta_1(t)) \\ &\le 2e^{-k_1t} + t(1-l_1/l_2) \le 2e^{-k_1T} + \delta < \tfrac{2}{3}\operatorname{inj}(X). \end{aligned}$$

For any $0 \le t \le l_2$, there exists a unique shortest geodesic $\alpha_t(s)$ in X connecting $\gamma_2(t)$ and $\beta_1(t)$. Let the parameter $s \in [0, 1]$ so that $\alpha_t(0) = \gamma_2(t)$ and $\alpha_t(1) = \beta_1(t)$. Define the mapping

$$\Phi(t, s) : [0, l_2] \times [0, 1] \to X, \quad (t, s) \mapsto \alpha_t(s).$$

We will derive a contradiction by showing that Φ defines a homotopy between $\gamma_2(t)$ and $\beta_1(t)$. First, $\Phi(t, 0) = \gamma_2(t)$, and $\Phi(t, 1) = \beta_1(t)$. Moreover, since both γ_2 and β_1 have period l_2, we have $\alpha_0(s) = \alpha_{l_2}(s)$ for all $s \in [0, 1]$, so that $\Phi(\,\cdot\,, s)$ is a closed curve in X. Finally, $\Phi(t, s)$ is continuous since the function $d(\gamma_2(t), \beta_1(t))$ is a continuous function of t. This shows that Φ is indeed a homotopy between $\gamma_2(t)$ and $\beta_1(t)$. Since β_1 is just a reparametrization of γ_1, this shows that the γ_i lie in the same free homotopy class, which contradicts the fact that on a complete manifold with pinched negative curvature there is at most one closed geodesic in each free homotopy class [Eberlein et al. 1993]. □

With these preliminary lemmas, we may now prove the dynamical wave trace formula.

Proof of Theorem 1.1. Note that dependence of the renormalized trace of the wave group on the choice of boundary defining function is absorbed by the remainder term in the right side of the formula. Fix $t_0 > 0$, and let $\phi \in \mathscr{C}_0^\infty([t_0, \infty))$. The singular support of $0\text{-tr}\cos(t\sqrt{\Delta - n^2/4})$ lies in $\{kl(\gamma) : \gamma \in \mathscr{L}_p\}$ by [Joshi and Sá Barreto 2001, Theorem 4.2]. The arguments of [Duistermaat and Guillemin 1975, Theorem 4.5] are local, so $0\text{-tr}\cos(t\sqrt{\Delta - n^2/4})$ has an expansion at each singularity $T \in \{kl(\gamma) : \gamma \in \mathscr{L}_p\}$ with leading term

$$\frac{l(\gamma)\delta(|t| - kl(\gamma))}{\sqrt{|\det(I - \mathcal{P}_\gamma^k)|}}, \tag{3-6}$$

and

$$0\text{-tr}\cos(t\sqrt{\Delta - n^2/4}) - \left(\sum_{\gamma\in\mathscr{L}_p}\sum_{k\in\mathbb{N}} \frac{l(\gamma)\delta(|t| - kl(\gamma))}{\sqrt{|\det(I - \mathcal{P}_\gamma^k)|}}\right) = A(t).$$

By the assumption of negative sectional curvatures together with the calculation of [Mazzeo and Melrose 1987] that shows that all sectional curvatures approach -1 at ∂X, there exist $0 < k_2 \le 1 \le k_1$ such that $-k_1^2 \le \kappa \le -k_2^2$, for all sectional curvatures κ. Therefore, the "clean intersection" condition of [Duistermaat and Guillemin 1975] is satisfied. Since the arguments therein are localized to small neighborhoods around each $\gamma \in \mathscr{L}$, by (3-6), [Duistermaat and Guillemin 1975], and [Joshi and Sá Barreto 2001, Theorem 4.1], we have

$$\int \phi(t)\, 0\text{-tr}(\cos(t\sqrt{\Delta - n^2/4}))\, dt = \sum_{\substack{\gamma\in\mathscr{L}_p,\, k\in\mathbb{N} \\ kl(\gamma)\in\operatorname{supp}(\phi)}} \frac{l(\gamma)\phi(kl(\gamma))}{\sqrt{|\det(I - \mathcal{P}_\gamma^k)|}} + A(\phi),$$

where $\operatorname{supp}(\phi)$ is the support of ϕ and $A(\phi) = \int A(t)\phi(t)dt$. By Lemma 3.2, the periodic orbits are separated, and by [Joshi and Sá Barreto 2001, Theorem 4.1] the closed geodesics lie in a compact subset of X, so we may apply the local estimates from the proof of [Jakobson et al. 2008, Theorem 1.3]. Assuming ϕ satisfies the hypotheses in our theorem, the dynamical estimates in [ibidem, Section 3] give the long time asymptotics for our trace formula. Further details of the technical modifications necessary to adapt their dynamical arguments to our setting may be found in the appendix. □

As a corollary to this theorem, we combine the dynamical trace formula with Borthwick's Poisson formula [2008] to produce a dynamical resonance-wave trace formula. To state this result we recall a few definitions. The Poisson formula relates the renormalized wave trace to the poles, called *resonances*, of the meromorphically continued resolvent. Closely related to the resolvent is the scattering operator whose poles essentially coincide with those of the resolvent; it is more convenient to state the trace formula in terms of scattering resonances. Recall that the multiplicities of the resonances are given by

$$m(\zeta) = \operatorname{rank} \operatorname{Res}_\zeta(\Delta - s(n-s))^{-1},$$

where ζ is a pole of the resolvent, $(\Delta - s(n-s))^{-1}$.

Definition 3.3. Let (X, g) be an asymptotically hyperbolic $(n+1)$-dimensional manifold with boundary defining function x. For $\operatorname{Re} s = n/2$ with $s \neq n/2$, a function $f_1 \in \mathscr{C}^\infty(\partial X)$ determines a unique solution u of

$$(\Delta - s(n-s))u = 0, \quad u \sim x^{n-s} f_1 + x^s f_2 \text{ as } x \to 0,$$

where $f_2 \in \mathscr{C}^\infty(\partial X)$. This defines the *scattering operator* $S(s) : f_1 \mapsto f_2$.

Heuristically, the scattering operator, which is classically a scattering matrix, acts as a Dirichlet to Neumann map, and physically it describes the scattering behavior of particles. The scattering operator extends meromorphically to $s \in \mathbb{C}$ as a family of pseudodifferential operators of order $2s - n$. Renormalizing the scattering operator as

$$\tilde{S}(s) := \frac{\Gamma(s - n/2)}{\Gamma(n/2 - s)} \Lambda^{n/2-s} S(s) \Lambda^{n/2-s}, \quad \text{where } \Lambda := \tfrac{1}{2}(\Delta_h + 1)^{1/2},$$

gives a meromorphic family of Fredholm operators with poles of finite rank. Here Δ_h is the Laplacian on ∂X for the metric $h(x)|_{x=0}$. Note that this definition depends on the boundary-defining function. The multiplicity of a pole or zero of $S(s)$ is defined to be

$$\nu(\zeta) = -\operatorname{tr}\big(\operatorname{Res}_\zeta \tilde{S}'(s) \tilde{S}(s)^{-1}\big).$$

The scattering multiplicities are related to the resonance multiplicities in [Borthwick and Perry 2002; Guillarmou 2005; Guillopé and Zworski 1999] by

$$\nu(\zeta) = m(\zeta) - m(n - \zeta) + \sum_{k \in \mathbb{N}} \big(\chi_{n/2-k}(s) - \chi_{n/2+k}(s)\big) d_k,$$

where $d_k = \dim\ker \tilde{S}(\frac{1}{2}n + k)$ and χ_p denotes the characteristic function of the set $\{p\}$. The numbers d_k are determined by natural conformal operators acting on the conformal infinity; see [Graham et al. 1992; Guillopé and Zworski 1999; Guillarmou and Naud 2006].

The set of resolvent resonances will be denoted $\mathscr{R}$, while the set of scattering resonances are

$$\mathscr{R}^{\mathrm{sc}} := \mathscr{R} \cup \bigcup_{k=1}^{\infty} \{\tfrac{1}{2}n - k \text{ with multiplicity } d_k\}.$$

Finally, we recall that for a discrete torsion-free group Γ of isometries of $\mathbb{H}^{n+1}$, the quotient $\mathbb{H}^{n+1}/\Gamma$ is said to be *convex cocompact* when its convex core is compact; a compact perturbation is defined as follows.

Definition 3.4. We call an asymptotically hyperbolic manifold (X, g) a *compact perturbation* of a convex cocompact hyperbolic manifold if there exists a convex cocompact manifold (X_0, g_0) (possibly disconnected) such that $(X - K, g) \cong (X_0 - K_0, g_0)$ for some compact sets $K \subset X$ and $K_0 \subset X_0$.

Proof of Corollary 1.2. This corollary is an immediate consequence of our dynamical trace formula and of [Borthwick 2008, Theorem 1.2]. We note that in terms of the resolvent resonances the spectral side of the trace formula becomes

$$\tfrac{1}{2}\sum_{s\in\mathcal{R}} m(s)e^{(s-n/2)|t|} + \tfrac{1}{2}\sum_{k\in\mathbb{N}} d_k e^{-k|t|}. \tag{3-7}$$ □

Remark. A nice application of our trace formula would be to count resonances lying in strips in the complex plane. See for example [Perry 2003, Theorem 1.3] and [Guillopé and Zworski 1999, Theorem 2]. Those results use the existence of only *one* closed geodesic; by incorporating more of the length spectrum, one expects these estimates to be improved to give a fractal Weyl law with exponent determined by the entropy of the geodesic flow. This remains an interesting open problem.

4. Dynamical zeta function

The dynamical zeta function is to the geodesic length spectrum as the Riemann zeta function is to the prime numbers. Let

$$Z(s) = \exp\left(\sum_{\gamma\in L_p}\sum_{k\in\mathbb{N}} \frac{e^{-ksl_p(\gamma)}}{k}\right), \tag{4-1}$$

where L_p consists of primitive closed orbits of the geodesic flow and $l_p(\gamma)$ is the primitive period (or length) of $\gamma \in L_p$. This definition is the same as Parry and Pollicott's [1983] dynamical zeta function for Axiom A flows. We also consider the weighted dynamical zeta function

$$\tilde{Z}(s) = \exp\left(\sum_{\gamma\in L_p}\sum_{k\in\mathbb{N}} \frac{e^{-ksl_p(\gamma)}}{k\sqrt{|\det(I - \mathcal{P}_\gamma^k)|}}\right),$$

where $\mathcal{P}_\gamma^k$ is the k-times Poincaré map of the geodesic flow around the closed orbit γ. As observed in [Patterson and Perry 2001], the weighted zeta function is particularly interesting for its connections to the resonances of the resolvent; Perry [2003] and Guillarmou and Naud [2006] used the Hadamard factorization of this zeta function to prove Selberg trace formulae for convex cocompact hyperbolic manifolds.

We recall some definitions from dynamics. Let SX denote the unit tangent bundle, and let G^t be the geodesic flow on SX. Since X is complete and has negative sectional curvatures, the geodesic flow is Anosov [Anosov 1967]; see for example [Bolton 1979; Eberlein 1972; 1973b; 1973c; Klingenberg 1974]. Consequently, $T(SX)_\xi$ splits for each $\xi \in SX$ into a direct sum

$$T(SX)_\xi = E^s_\xi \oplus E^u_\xi \oplus E_\xi, \tag{4-2}$$

where E^s_ξ is exponentially contracting, E^u_ξ is exponentially expanding, and E_ξ is the one-dimensional subspace tangent to the flow. The *Sinai–Ruelle–Bowen potential* is a Hölder continuous function defined by

$$H(\xi) := \frac{d}{dt}\Big|_{t=0} \ln \det dG^t|_{E^u_\xi}. \tag{4-3}$$

This potential is the instantaneous rate of expansion at ξ. The topological pressure $\mathfrak{p}$ of a function $f : SX \to \mathbb{R}$ is defined as follows. For large T and small $\delta > 0$, a finite set $Y \subset SX$ is (T, δ) separated if, given $\xi, \xi' \in Y$ with $\xi \neq \xi'$, there is a $t \in [0, T]$ with $d(G^t\xi, G^t\xi') \geq \delta$. Here the distance on SX is given by the Sasaki metric. Then

$$\mathfrak{p}(f) = \lim_{\delta\to 0} \limsup_{T\to\infty} T^{-1} \log \sup\left\{\sum_{\xi\in Y} \exp \int_0^T f(G^t\xi)dt : Y \text{ is } (T, \delta) \text{ separated}\right\}. \tag{4-4}$$

In the compact setting, the topological pressure of a function $f : SX \to \mathbb{R}$ may be equivalently defined by $\mathfrak{p}(f) = \sup_\mu (h_\mu + \int f\,d\mu)$, where the supremum is taken over all G^t invariant measures μ, and h_μ denotes the *measure theoretical entropy* of the geodesic flow; see [Bowen 1975; Katok and Hasselblatt 1995].

The pressure of a function is a concept in dynamical systems from statistical mechanics; it measures the growth rate of the number of separated orbits weighted according to the values of f [Walters 1975]. In particular, $\mathfrak{p}(0)$ is equal to the *topological entropy* h of the geodesic flow, given by

$$h = \lim_{\delta\to 0} \limsup_{T\to\infty} T^{-1} \log \sup \#\{Y \subset SX : Y \text{ is } (T, \delta) \text{ separated}\}. \tag{4-5}$$

When X has no conjugate points, the topological entropy of the geodesic flow is equivalent to the volume growth rate

$$\lambda(X) = \limsup_{r\to\infty} \frac{1}{r} \log \mathrm{Vol}(B_r(x)),$$

where $B_r(x)$ is the ball of radius r and center x in the universal covering of X, and $\mathrm{Vol}(B_r(x))$ is its volume; see [Manning 1979; Freire and Mañé 1982].

For convex cocompact hyperbolic manifolds $\mathbb{H}^{n+1}/\Gamma$, the topological entropy is equal to δ, the exponent of convergence for the Poincaré series for Γ, which is also equal to the dimension of the limit set of Γ. For the Sinai–Ruelle–Bowen potential, $\mathfrak{p}(-H)$ is equal to zero, and the corresponding equilibrium measure that attains the supremum is the Liouville measure μ_L on the unit tangent bundle; thus

$$h_{\mu_L} = \int_{SX} H \, d\mu_L.$$

Proof of Theorem 1.3. Note that the sum is

$$\sum_{\gamma \in L} k(\gamma)^{-1} \exp(-\textstyle\int_\gamma s\,dt) = \sum_{r\in\mathbb{N}} a_r,$$

where L consists of all closed orbits of the geodesic flow, $k(\gamma)$ is the multiplicity of γ, and

$$a_r = \sum_{\epsilon(r-1/2)\le l(\gamma)<\epsilon(r+1/2)} k(\gamma)^{-1} \exp(-\textstyle\int_\gamma s\,dt).$$

The arguments proving [Franco 1977, Lemma 2.8] are entirely local and show that $a_r \le \exp(-(r\epsilon\mathfrak{p}(-s)))/r\epsilon$. Moreover, $\mathfrak{p}(-s) = \mathfrak{p}(0) - s$ by [Walters 1975], so that the series converges absolutely when $\mathrm{Re}(s) > \mathfrak{p}(0)$. For the weighted dynamical zeta function, note that $\mathcal{P}_\gamma$ has expanding eigenvalues $\lambda_1, \ldots, \lambda_n$ and contracting eigenvalues $\lambda_{n+1}, \ldots, \lambda_{2n}$, and

$$|\det(I - \mathcal{P}_\gamma)| = \prod_1^{2n} |1-\lambda_i| = \prod_{i=1}^{n} |\lambda_i| \prod_{j=1}^{n} \left|1 - \frac{1}{|\lambda_j|}\right| \prod_{k=n+1}^{2n} |1-\lambda_k|.$$

By Lemma 3.1,

$$|\lambda_i|^{-1} \le e^{-k_2 l(\gamma)} \quad \text{for } i = 1, \ldots, n,$$
$$|\lambda_i| \le e^{-k_2 l(\gamma)} \quad \text{for } i = n+1, \ldots 2n.$$

Therefore,

$$\lim_{|l(\gamma)|\to\infty} \frac{\prod_{i=1}^n |\lambda_i|}{|\det(I - \mathcal{P}_\gamma)|} = 1,$$

so we may replace $|\det(I - \mathcal{P}_\gamma)|$ by this product of expanding eigenvalues. Since H is the rate of expansion of volume in E^u, the summand for γ is

$$k(\gamma)^{-1} \exp\Bigl(\int_\gamma \tfrac{1}{2}H - s\Bigr).$$

Then, after we similarly define a_r, [Franco 1977, Lemma 2.8] shows that

$$a_r \le \exp(r\epsilon\mathfrak{p}(\tfrac{1}{2}H - s))/(r\epsilon),$$

so that the series converges absolutely when $\mathfrak{p}(\frac{1}{2}H - s) < 0$. Thus, the series converges absolutely when $\mathrm{Re}(s) > \mathfrak{p}(\frac{1}{2}H)$, since $\mathfrak{p}(\frac{1}{2}H - s) = \mathfrak{p}(\frac{1}{2}H) - s$. □

Remark. As observed in [Chen and Manning 1981], if the exponent $1/2$ in the denominator of the weighted dynamical zeta function is replaced by $t \in \mathbb{R}$, then by the preceding arguments, $\tilde{Z}(s)$ converges absolutely for $\mathrm{Re}(s) > \mathfrak{p}(-tH)$.

5. Prime orbit theorem

To prove the prime orbit theorem we require further definitions to describe the geodesic flow. The following definitions are from [Eberlein 1972; 1973a]; see also [Bishop and O'Neill 1969] and [Eberlein et al. 1993]. For $\xi \in SX$, the *positive prolongational limit set* is

$$P^+(\xi) = \{y \in SX : \text{for any neighborhoods } O, U \text{ of } \xi, y, \text{ respectively, there is a sequence } t_n \subset \mathbb{R} \text{ with } t_n \to \infty \text{ such that } G^{t_n}(O) \cap U \neq \varnothing\}.$$

Then ξ is nonwandering if $\xi \in P^+(\xi)$. The flow is *topologically transitive* on $\Omega \subset SM$ if for any open sets $O, U \subset \Omega$, there exists $t \in \mathbb{R}$ such that $G^t(U) \cap O \neq \varnothing$. The flow is *topologically mixing* if there exists $A > 0$ such that $G^t(U) \cap O \neq \varnothing$ for all $|t| > A$.

A closed invariant set $\Omega \subset SX$ without fixed points is *hyperbolic* if the tangent bundle restricted to Ω is a Whitney sum $T_\Omega SX = E + E^s + E^u$ of three TG^t invariant subbundles, where E is the one-dimensional bundle tangent to the flow, and E^s and E^u are exponentially contracting and expanding, respectively:

$$\|TG^t(v)\| \leq Ke^{-\lambda t}\|v\| \quad \text{for } v \in E^s \text{ and } t \geq 0,$$
$$\|TG^{-t}(v)\| \leq Ke^{-\lambda t}\|v\| \quad \text{for } v \in E^u \text{ and } t \geq 0.$$

Parry and Pollicott [1983] defined a *basic set* to be a topologically transitive hyperbolic set U with no fixed points for which periodic orbits are dense and which admits an open set $O \supset U$ such that $U = \bigcap_{t \in \mathbb{R}} G^t O$.

Proof of Theorem 1.4. With the work of Bishop and O'Neill [1969], Eberlein [1973b; 1972; 1973a], and Eberlein and O'Neill [1973] on "visibility manifolds" (complete manifolds with nonpositive curvature), we are able to give a quick proof of the prime orbit theorem.

We first use results from [Eberlein 1972]: For X asymptotically hyperbolic and negatively curved, the nonwandering set $\Omega \subset SX$ is closed and invariant under the flow; see [page 502]. By [Theorems 3.9 and 3.10], the periodic vectors are dense in Ω since $h > 0$. By [Theorem 3.13], Ω is connected, and by [Theorem 3.11], the geodesic flow restricted to Ω is topologically transitive. Since the flow is Anosov, Ω is a hyperbolic set; see (4-2). Since the periodic orbits are dense in Ω, and since Ω is closed, Ω is a compact subset of SX by [Joshi and Sá Barreto 2001, Theorem 4.1]. Clearly Ω cannot have fixed points for the geodesic flow, and by definition of the nonwandering set, we have $\bigcap_{t \in \mathbb{R}} G^t O = \Omega$ for any open neighborhood $O \supset \Omega$.

Therefore Ω is a basic set. By [Parry and Pollicott 1983, Proposition 1], the flow restricted to Ω is topologically mixing. Because all Anosov flows are a priori Axiom A flows, the geodesic flow restricted to Ω is a topologically mixing Axiom A flow restricted to a basic set and it satisfies the hypotheses of [Parry and Pollicott 1983, Theorems 1 and 2]; applying these results completes the proof. □

Proof of Corollary 1.5. The proof consists of applying the prime orbit theorem and analyzing the dominant terms on the spectral and dynamical sides of the trace formula. First, assume the topological entropy satisfies $h > nk_1/2$. Let ϕ be a test function satisfying the hypotheses in the statement of Theorem 1.1. By the Dirichlet box principle (see [Jakobson et al. 2008, Section 4]; also [Karnaukh 1996; Phillips and Rudnick 1994; Rubinstein and Sarnak 1994]), there exist infinitely many pairs $(\lambda, T(\lambda) = \epsilon \ln(\lambda))$ with $\lambda \to \infty$ such that for $\gamma \in \mathscr{L}$ with

$$t_0 \leq l(\gamma) \leq T(\lambda),$$

there exists $m \in \mathbb{Z}$ such that $|\lambda l(\gamma) - 2\pi m| \leq 1/2$. This implies $\cos(\lambda l(\gamma)) \geq 1/2$. Consequently, after integrating against the test function ϕ, the dynamical side of the trace formula is bounded below by

$$\frac{N(T(\lambda))T(\lambda)}{2e^{nk_1 T(\lambda)/2}} + O(1).$$

By the prime orbit theorem, this is bounded below by $(1/2)e^{(h-nk_1/2)T(\lambda)} + O(1)$. The hypothesis on h implies that the dynamical side of the trace formula grows exponentially, so this must also hold for the spectral side of the trace formula. By [Borthwick 2008, Theorem 1.1], this is only possible if there exists $\Lambda_0 \in \sigma_{\mathrm{pp}}(\Delta)$ with corresponding $s_0 \geq h + n(1-k_1)/2$.

Next, assume the topological entropy satisfies $h \leq nk_2/2$. We proceed by contradiction. Assume there exists $\Lambda_0 \in \sigma_{\mathrm{pp}}(\Delta)$; then the corresponding $s_0 > n/2$. For $\lambda = 2k\pi$, $\cos(\lambda) = 1$. Integrating against the test function ϕ, the spectral side of the trace formula is bounded below by $(1/2)e^{(s_0-n/2)T(\lambda)} + O(e^{\delta T(\lambda)})$. If $\sigma_{\mathrm{pp}}(\Delta) = 0 < \Lambda_0 < \Lambda_1 < \cdots < n^2/4$, then $\delta = s_1 - n/2$, and if $\sigma_{\mathrm{pp}}(\Delta) = \{\Lambda_0\}$, then by [Borthwick 2008, Theorem 1.1], we can take any $\delta > 0$. So, we see that the spectral side of the trace formula grows exponentially as $(\lambda, T(\lambda)) \to \infty$. Note that the dynamical side of the trace formula is bounded above by

$$\frac{N(T(\lambda))T(\lambda)}{e^{nk_2 T(\lambda)/2}} + O(1) \leq e^{(h-nk_2/2)T(\lambda)} + O(1).$$

By the hypothesis on h, the dynamical side of the trace formula does not grow exponentially, which gives a contradiction and proves the second statement of the corollary. □

6. Concluding remarks and further directions

This paper was motivated by [Guillopé and Zworski 1999; Guillarmou 2007; Guillarmou and Naud 2006]; our aim was to prove trace formulae for asymptotically hyperbolic $(n+1)$-dimensional manifolds and to understand the relationship between the resonances and dynamics of these manifolds. Although our trace formulae do not provide an explicit expression for the remainder terms, they — like classical trace formulae — provide a connection between the Laplace and length spectra. Applications include computing remainder terms for both the length and resonance counting functions; see for example [Guillarmou and Naud 2006; Jakobson et al. 2008]. Another application is to counting resonances in regions of $\mathbb{C}$ corresponding to physical phenomena [Guillopé and Zworski 1999]. It would be interesting to study the remainder terms in our formulae in greater depth; it is almost certain that an explicit formula for the remainder does not exist in this context of variable curvature, but perhaps one may show exponential decay at infinity. Numerical methods indicate that the remainder term $A(t)$ decays as $t \to \infty$ without the use of Ehrenfest time. It would also be interesting to study the behavior as $t \to 0$. Ideally, we would like to generalize our trace formulae to all asymptotically hyperbolic manifolds. The dynamical trace requires hypotheses on the geodesic flow to allow summation of periodic orbits and to control the remainder term; assuming globally negative (but not necessarily constant) curvature guarantees this, but based on [Anosov 1967; Eberlein 1973b; Klingenberg 1974], we expect a weaker hypothesis to suffice. Such a hypothesis may be quite technical. The Poisson formula is more delicate and remains an open problem for asymptotically hyperbolic manifolds. Borthwick's recent work [2008] is progress in this direction, although Guillarmou's careful study [2007] of the scattering phase shows that the Poisson formula for asymptotically hyperbolic manifolds is subtle and elusive.

Appendix to the proof of Theorem 1.1

By [Joshi and Sá Barreto 2001], there exists a compact subset $X' \subset X$ which contains all the closed geodesics. Note that the 0-regularized integral over X of any function f satisfies

$$\text{(6-1)} \qquad {}^{0}\!\!\int f = \int_{X'} f + {}^{0}\!\!\int_{X-X'} f.$$

The contribution to the renormalized wave trace coming from ${}^{0}\!\int_{X-X'}$ is smooth and has exponential decay as $t \to \infty$ due to the iterative construction of [Joshi and Sá Barreto 2001] in which the first order parametrix at ∂X is the hyperbolic wave kernel that has exponential decay as $t \to \infty$. In particular, we have the distributional equality

$$\begin{aligned} 0\text{-tr}\cos\bigl(t\sqrt{\Delta - n^2/4}\bigr) &= \sum_{\gamma\in\mathcal{L}_p}\sum_{k\in\mathbb{N}} \frac{l(\gamma)\delta(|t|-kl(\gamma))}{\sqrt{|\det(I-\mathcal{P}_\gamma^k)|}} + A(t) \\ &= \int_{X'} U(t,w,w)\,\mathrm{d}\mathrm{vol}(w) + E(t), \end{aligned}$$

where $E(t) = O(t^{-\infty})$ as $t\to\infty$. So, to estimate the remainder term coming from the dynamics in the renormalized wave trace, we only need to work over X'.

In the definition of the $\hbar := \lambda^{-1}$ pseudodifferential calculus used to approximate the wave kernel, we define $T = T(\lambda) = \epsilon \ln(\lambda)$, for some small constant ϵ as in the arguments of [Jakobson et al. 2008, Section 3]. Since the estimates in [ibidem, Section 3.4] are on the universal cover M of X, which is not assumed to be compact, these estimates generalize identically to our setting. In [ibidem, (3.11)], we may define the cutoff function η for X' rather than X, so that this equation becomes

$$\int_{X'} U(t,w,w)\mathrm{d}\mathrm{vol}(w) = \int_M E_N(t,w,w;\hbar)\eta(w)\mathrm{d}\mathrm{vol}(w) + O(e^{c|t|}\hbar^N),$$

for some constant $c>0$. The remaining estimates for the error in the microlocal parametrix construction of the wave kernel depend only on the curvature bounds and our dynamical and separation lemmas; our prime orbit theorem is required in estimate [ibidem, (3.39)], but this is no problem since the proof of the prime orbit theorem is completely independent. Finally, we note that although the estimates in [Jakobson et al. 2008] are for surfaces, they generalize naturally to higher dimensions; the higher-dimensional analogue of the main result therein is the subject of ongoing work of D. Jakobson, I. Polterovich, and R. Schubert.

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