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OUTER ACTIONS OF A DISCRETE AMENABLE GROUP ON APPROXIMATELY FINITE-DIMENSIONAL FACTORS III: THE TYPE III_λ CASE, $0 < \lambda < 1$, ASYMMETRIZATION AND EXAMPLES

YOSHIKAZU KATAYAMA AND MASAMICHI TAKESAKI

Dedicated to the memory of Masahiro Nakamura

In this last article of the series on outer actions of a countable discrete amenable group on approximately finite-dimensional factors, we analyze outer actions of a countable discrete free abelian group on an approximately finite-dimensional factor of type III_λ with $0 < \lambda < 1$ and compute outer conjugacy invariants. As a byproduct, we discover the asymmetrization technique for coboundary condition on a $\mathbb{T}$-valued cocycle of a torsion-free abelian group, which might have been known by group cohomologists. As the asymmetrization technique gives us a very handy criterion for coboundaries, we present it here in detail.

Introduction

This article concludes the series [Katayama and Takesaki 2003; 2004; 2007] on the outer conjugacy classification of outer actions of a countable discrete amenable group on an approximately finite-dimensional (AFD) factor, by examining outer actions of a countable discrete abelian group G on an AFD factor $\mathcal{R}_\lambda$ of type III_λ

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with $0 < \lambda < 1$. Prior to the outer conjugacy classification theory, the cocycle conjugacy classification theory of actions of a countable discrete amenable group on an AFD factor had been completed through the work of many mathematicians over three decades; see [Connes 1977; 1976b; 1975; Jones 1980; Jones and Takesaki 1984; Ocneanu 1985; Katayama et al. 1998; 1997; Kawahigashi et al. 1992; Sutherland and Takesaki 1985; 1989; 1998].

Unlike the general classification program in operator algebras, the outer conjugacy classification of a countable discrete amenable group on $\mathcal{R}_\lambda$ is almost smooth, as shown in our series of previous work; see [Katayama and Takesaki 2007]. The only nonsmooth part of the classification theory stems from the classification of subgroups N of G; for instance, the classification of subgroups of a torsion-free abelian group of higher rank is nonsmooth. See [Sutherland 1985] for the Borel parameterization of polish groups. When the modular automorphism part $N = \dot{\alpha}^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$ of the outer action $\dot{\alpha}$ of G on $\mathcal{R}_\lambda$ is fixed, the set of invariants becomes a compact abelian group. This is a rare case in the theory of operator algebras. So we are encouraged to make a concrete analysis of outer conjugacy classes of a countable discrete amenable group. Of course, without having a concrete data on the group G involved, we cannot make a fine analysis. So we take a countable discrete free abelian group G and study its outer actions on $\mathcal{R}_\lambda$ and identify the invariants completely. The justification of this restriction rests on the fact that all outer actions of a countable discrete abelian group A can be viewed as outer actions of G by pulling back the outer action via the quotient map $G \to A$. Thanks to all hard analytic work on the cocycle conjugacy classification in the past, as cited in the references, our work is very algebraic and indeed done by cohomological computations.

We will begin by relating the discrete core of $\mathcal{R}_\lambda$ and the core of an AFD factor $\mathcal{R}_1$ of type III_1. This analysis will give us a simple model construction with given invariants, which is presented here in Section 1. We first study single automorphisms and a pair of commuting automorphisms of $\mathcal{R}_\lambda$. Then we will work on the asymmetrization of a cocycle of a countable discrete abelian group; this will provide a powerful tool for analysis of the third cohomology group $\mathrm{H}^3(G, \mathbb{T})$. The general theory of group cohomology is available to us today; for example see [Brown 1994]. But, since we will need to work with individual cocycles to analyze outer actions, we will need a tool to work with a cocycle directly beyond the computation of the cohomology group. For example, we have to identify which data of a given cocycle contributes to the modular automorphism part of the action in question. Thus we will work on the cohomology group based on a very primitive method of chasing cocycles, through which we discover the asymmetrization technique that provides us a handy criterion for the coboundary condition on a cocycle of a torsion-free abelian group. In [Katayama and Takesaki 2003; 2004; 2007], we studied the outer conjugacy classification of countable discrete amenable group outer actions by a resolution of the relevant third cocycle. In the abelian case, we showed that there is a universal resolution group that takes care of all third cocycles at once, which simplifies greatly the investigation of outer actions of a countable discrete abelian group. The reduced modified HJR-sequence will provide us a tool to chase the cocycles, along with the asymmetrization technique. The first step of studying outer actions of a countable discrete abelian group G on a factor $\mathcal{M}$ of type III_λ with $0<\lambda<1$ is to find a countable discrete amenable group H and a surjective homomorphism $\pi_G: H\to G$ such that the pull back $\pi_G^*(c)$ is a coboundary; this process is called the resolution of a cocycle $c\in \mathrm{Z}^3(G,\mathbb{T})$. Then the outer action $\dot\alpha$ is identified with a lifting $\mathfrak{s}_H^*(\alpha)$ of an action α of H through a cross-section $\mathfrak{s}_H: G\to H$ of the homomorphism π_G. Luckily, a countable discrete abelian group G admits a universal resolution $\{H, \pi_G\}$, a group H and a surjective homomorphism $\pi_G: H\to G$ such that $\pi_G^*(\mathrm{H}^3(H,\mathbb{T}))=\{1\}$. We construct the group H via a relatively simple process from a countable discrete free abelian group G. This makes it possible to reduce the study of an outer action $\dot\alpha$ of G to that of an action α of H. Now, the action α of H does not lift to the discrete core $\widetilde{\mathcal{M}}_d$ if $\mathrm{mod}(\alpha)\neq 1$. So we construct a central extension H_m of H by

$$0\to\mathbb{Z}\xrightarrow{\ n\to z_0^n\ } H_\mathrm{m}\longrightarrow H\to 1$$

and work with the characteristic cohomology group $\Lambda(H_\mathrm{m}, L, M, \mathbb{T})$, where the normal subgroup L is the inverse image $L=\pi_G^{-1}(N)$ with $N=\dot\alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$. Thus we are going to investigate the reduced modified HJR-sequence

$$\begin{array}{ccccccc}
\mathrm{H}^2(H,\mathbb{T}) & \xrightarrow{\mathrm{Res}} & \Lambda(H_\mathrm{m}, L, M, \mathbb{T}) & \xrightarrow{\delta} & \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T}) & \xrightarrow{\mathrm{Inf}} & \mathrm{H}^3(H,\mathbb{T}) \\
\| & & \downarrow{\scriptstyle \pi_\mathrm{m}^*} & & \downarrow{\scriptstyle \partial_{G_\mathrm{m}}} & & \| \\
\mathrm{H}^2(H,\mathbb{T}) & \xrightarrow{\mathrm{res}} & \Lambda(H, M, \mathbb{T}) & \xrightarrow{\delta_{\mathrm{HJR}}} & \mathrm{H}^3(G,\mathbb{T}) & \xrightarrow{\pi_G^*} & \mathrm{H}^3(H,\mathbb{T}).
\end{array}$$

Here $\mathfrak{s}$ is a fixed cross-section of the quotient map $G\to Q=G/N$. The groups appearing on the exact sequences above are all compact abelian groups and are indeed computable as shown in this paper.

We cite [Brown 1994; Eilenberg and Mac Lane 1947; Mac Lane and Whitehead 1950; Huebschmann 1981; Jones 1980; Ratcliffe 1980] for the general cohomology theory of abstract groups and [Sutherland 1980] for the cohomology theory related to von Neumann algebras. See [Takesaki 1979; 2003a; 2003b] for the general theory of von Neumann algebras. For information about the discrete core of a factor of type III_λ, see [Connes 1973; 1974; Connes and Takesaki 1977; Falcone and Takesaki 1999; 2001].

1. Simple examples and model construction

Factors of type III_λ and type III_1, and their cores. We begin with the following folk theorem in the structure theory of factors of type III.

Theorem 1.1. *Let $\{\mathcal{M}_{0,1}, \tau, \theta\}$ be a factor of type II_∞ equipped with faithful semifinite normal trace τ and trace scaling automorphism θ by λ with $0 < \lambda < 1$, that is, $\tau \circ \theta = \lambda\tau$. Let $\mathcal{M} = \mathcal{M}_{0,1}^\theta$ be the fixed point subalgebra of $\mathcal{M}_{0,1}$ by θ.*

(i) *The von Neumann algebra $\mathcal{M}$ is a factor of type III_λ.*

(ii) *The triplet $\{\mathcal{M}_{0,1}, \tau, \theta\}$ is conjugate to the discrete core of $\mathcal{M}$.*

(iii) *For an automorphism $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1})$,*

$$\alpha(\mathcal{M}) = \mathcal{M} \quad \textit{is equivalent to} \quad \alpha \circ \theta = \theta \circ \alpha.$$

(iv) *Let $\mathrm{Aut}(\mathcal{M}_{0,1}, \mathcal{M})$ be the group of automorphisms of $\mathrm{Aut}(\mathcal{M}_{0,1})$ leaving $\mathcal{M}$ globally invariant. Then we have the exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{n \to \theta^n} \mathrm{Aut}(\mathcal{M}_{0,1}, \mathcal{M}) \xrightarrow{\alpha \to \alpha|_{\mathcal{M}}} \mathrm{Aut}(\mathcal{M}) \to 1.$$

(v) *The subgroup $\{\theta^n : n \in \mathbb{Z}\}$ is the Galois group of the pair $\{\mathcal{M}_{0,1}, \mathcal{M}\}$ in the sense that $\{\theta^n : n \in \mathbb{Z}\} = \{a \in \mathrm{Aut}(\mathcal{M}_{0,1}) : \alpha(x) = x, x \in \mathcal{M}\}$.*

(vi) *If $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1}, \mathcal{M})$, then the modulus $\mathrm{mod}_{\mathcal{M}_{0,1}}(\alpha)$ as a member of $\mathrm{Aut}(\mathcal{M}_{0,1})$ gives the modulus $\mathrm{mod}_{\mathcal{M}}(\alpha)$ of the restriction $\alpha|_{\mathcal{M}} \in \mathrm{Aut}(\mathcal{M})$ as*

$$\mathrm{mod}_{\mathcal{M}}(\alpha) = \pi_{T'}(\mathrm{mod}_{\mathcal{M}_{0,1}}(\alpha)) \in \mathbb{R}/T'\mathbb{Z},$$

where $T' = -\log\lambda$, $T = 2\pi/T'$ and $\pi_{T'} : s \in \mathbb{R} \mapsto \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}$.

Proof. Statements (i) and (ii) are known in the general structure theory of a factor of type III; see [Takesaki 2003a, Chapter XII, Sections 2 and 6].

We prove statement (v) first. Let ψ be a generalized trace of $\mathcal{M}$, that is, a faithful semifinite normal weight on $\mathcal{M}$ such that $\psi(1) = +\infty$ and $\sigma_T^\psi = \mathrm{id}$. Then the covariant system $\{\mathcal{M}_{0,1}, \theta\}$ is conjugate to the dual system $\{\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}/T\mathbb{Z}, \mathbb{Z}, \widehat{\sigma^\psi}\}$. Thus we may identify them, so that $\mathcal{M}_{0,1}$ admits a periodic one parameter unitary group $\{u^\psi(s) : s \in \mathbb{R}\}$ with

$$u^\psi(T) = 1, \quad \theta(u^\psi(s)) = \lambda^{is}u^\psi(s), \quad \text{and} \quad \sigma_s^\psi = \mathrm{Ad}(u^\psi(s))|_{\mathcal{M}} \quad \text{for } s \in \mathbb{R}.$$

Furthermore, the one parameter unitary group $\{u^\psi(s) : s \in \mathbb{R}\}$ together with $\mathcal{U}(\mathcal{M})$ generates the normalizer $\widetilde{\mathcal{U}}_0(\mathcal{M}) = \{v \in \mathcal{U}(\mathcal{M}_{0,1}) : v\mathcal{M}v^* = \mathcal{M}\}$, giving the semidirect product decomposition $\widetilde{\mathcal{U}}_0 = \mathcal{U}(\mathcal{M}) \rtimes_{\sigma^\psi} \mathbb{R}/T\mathbb{Z}$. Suppose that $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1})$ leaves $\mathcal{M}$ pointwise fixed. We then show that $x \in \mathcal{M}$ and $u^\psi(s)^*\alpha(u^\psi(s))$ for $s \in \mathbb{R}$ commute by calculating

$$\begin{aligned} xu^\psi(s)^*\alpha(u^\psi(s)) &= u^\psi(s)^*u^\psi(s)xu^\psi(s)^*\alpha(u^\psi(s)) = u^\psi(s)^*\sigma_s^\psi(x)\alpha(u^\psi(s)) \\ &= u^\psi(s)^*\alpha(\sigma_s^\psi(x)u^\psi(s)) = u^\psi(s)^*\alpha(u^\psi(s)x) = u^\psi(s)^*\alpha(u^\psi(s))x, \end{aligned}$$

so that $u^\psi(s)^*\alpha(u^\psi(s)) \in \mathcal{M}_{0,1} \cap \mathcal{M}' = \mathbb{C}$. Hence there exists a scalar $\mu(s) \in \mathbb{T}$ such that $\alpha(u^\psi(s)) = \mu(s)u^\psi(s)$ for $s \in \mathbb{R}$. Since $u^\psi(T) = 1$, we have $\mu(T) = 1$. Since $\mu(s+t) = \mu(s)\mu(t)$ for $s, t \in \mathbb{R}$, we have $\mu(s) = \lambda^{ins}$ for $s \in \mathbb{R}$ and some $n \in \mathbb{Z}$. Since $\mathcal{M}$ together with $\{u^\psi(s) : s \in \mathbb{R}\}$ generate the whole algebra $\mathcal{M}_{0,1}$, we conclude that $\alpha = \theta^n$. This shows (v).

We next show (iii). Suppose that $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1})$ leaves $\mathcal{M}$ globally invariant. Let $\beta = \alpha_{\mathcal{M}} = \alpha|_{\mathcal{M}}$ be the automorphism of $\mathcal{M}$ obtained as the restriction of α to $\mathcal{M}$. Then the uniqueness of a generalized trace on $\mathcal{M}$ gives a scalar $s \in \mathbb{R}$ and a unitary $v \in \mathcal{U}(\mathcal{M})$ such that $e^{-s}\psi = \psi \circ (\mathrm{Ad}(v) \circ \beta)$. This means that

$$\mathrm{mod}(\beta) = \mathrm{mod}(\mathrm{Ad}(v) \circ \beta) = \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z},$$

and that σ^ψ and $\mathrm{Ad}(v) \circ \beta$ commute. Hence it is possible to extend $\mathrm{Ad}(v) \circ \beta$ to the automorphism $\gamma \in \mathrm{Aut}\{\mathcal{M}_{0,1}\}$ such that

$$\gamma(u^\psi(t)) = u^\psi(t) \quad \text{for } t \in \mathbb{R} \quad \text{and} \quad \gamma(x) = \mathrm{Ad}(v) \circ \beta(x) \quad \text{for } x \in \mathcal{M}.$$

Now comparing α and γ on $\mathcal{M}$, we find $\gamma(x) = \mathrm{Ad}(v) \circ \beta(x) = \mathrm{Ad}(v) \circ \alpha(x)$ for $x \in \mathcal{M}$. From (v) it follows that α is of the form $\alpha = \theta^n \circ \mathrm{Ad}(v^*) \circ \gamma$ for some $n \in \mathbb{Z}$. Since θ commutes with both γ and $\mathrm{Ad}(v)$, α and θ commute. Hence $\alpha(\mathcal{M}) = \mathcal{M}$ implies $\alpha \circ \theta = \theta \circ \alpha$. The reverse implication is trivial. This proves part (iii).

Part (iv) follows from (iii) and (v).

Let $\{\widetilde{\mathcal{M}}, \mathbb{R}, \tau, \theta\}$ be the noncommutative flow of weights on $\mathcal{M}$, so that the covariant system $\{\mathcal{M}_{0,1}, \mathbb{Z}, \theta\}$ is identified with $\{\mathcal{M} \vee \{\psi^{\mathrm{i}s}\rho(-s) : s \in \mathbb{R}\},\ \theta_{T'}\}$, where $\rho(s)$ for $s \in \mathbb{R}$ is the one-parameter unitary group generating the center $\mathcal{C}$ of $\widetilde{\mathcal{M}}$ such that $\rho(T) = \psi^{\mathrm{i}T}$.

To prove (vi), fix a member $\alpha \in \mathrm{Aut}(\mathcal{M}_{0,1}, \mathcal{M})$ and let $\mathrm{m}(\alpha) = \mathrm{mod}(\alpha) \in \mathbb{R}$ so that $\tau \circ \alpha = e^{-\mathrm{m}(\alpha)}\tau$. Consider the crossed product $\widetilde{\mathcal{M}} = \mathcal{M}_{0,1} \rtimes_\theta \mathbb{Z} \cong \mathcal{M} \overline{\otimes} \mathcal{L}(\ell^2(\mathbb{Z}))$ and the generalized trace $\varphi = \tau \circ \mathcal{E}$ on $\widetilde{\mathcal{M}}$ given by

$$\varphi(x) = \tau \circ \mathcal{E}(x) = \tau\Big(\int_{\mathbb{R}/T\mathbb{Z}} \hat{\theta}_s(x)\, \mathrm{d}s\Big) \quad \text{for } x \in \widetilde{\mathcal{M}}_+.$$

With $U \in \mathcal{U}(\widetilde{\mathcal{M}})$ the unitary corresponding to the crossed product $\mathcal{M}_{0,1} \rtimes_\theta \mathbb{Z}$, we extend α to $\tilde{\alpha} \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ by $\tilde{\alpha}(x) = \alpha(x)$ for $x \in \mathcal{M}_{0,1}$ and $\tilde{\alpha}(U) = U$. Then we have for each $x \in \widetilde{\mathcal{M}}_+$

$$\begin{aligned}\varphi \circ \tilde{\alpha}(x) = \tau\Big(\int_{\mathbb{R}/T\mathbb{Z}} \hat{\theta}_s(\tilde{\alpha}(x))\, \mathrm{d}s\Big) &= \tau\Big(\alpha\Big(\int_{\mathbb{R}/T\mathbb{Z}} \hat{\theta}_s(x)\, \mathrm{d}s\Big)\Big) \\ &= e^{-\mathrm{m}(\alpha)}\tau\Big(\int_{\mathbb{R}/T\mathbb{Z}} \hat{\theta}_s(x)\mathrm{d}s\Big) = e^{-\mathrm{m}(\alpha)}\varphi(x).\end{aligned}$$

Hence we get

$$\text{mod}(\tilde{\alpha}) = [\text{m}(\alpha)]_{T'} = \text{m}(\alpha) + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}. \tag{1-1}$$

Since the covariant systems $\{\mathcal{M}, \alpha\}$ and $\{\widetilde{\mathcal{M}}, \tilde{\alpha}\}$ are cocycle conjugate, we have $\text{mod}(\tilde{\alpha}) = \text{mod}(\alpha)$. This completes the proof. ♡

From now on we denote by $\mathcal{R}_0$ an approximately finite-dimensional factor of type II_1.

A factor $\mathcal{M}_1$ of type III_1 generates a one-parameter family $\{\mathcal{M}_\lambda : 0 < \lambda \leq 1\}$ of factors of type III_λ, who share the same discrete core $\mathcal{M}_{0,1}$. So let $\mathcal{M}_1$ be a factor of type III_1, and let $\{\mathcal{M}_{0,1}, \theta_s, s \in \mathbb{R}\}$ be the noncommutative flow of weights on $\mathcal{M}_1$, that is, $\mathcal{M}_{0,1}$ is a factor of type II_∞ equipped with a trace-scaling one-parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ and a faithful semifinite normal trace τ such that $\mathcal{M}_1 = \mathcal{M}_{0,1}^\theta$ and $\tau \circ \theta_s = e^{-s}\tau$ for $s \in \mathbb{R}$. The following is a folk theorem in the structure theory of type III.

Theorem 1.2. *In the above context, fixing* $T' > 0$, *set* $\lambda = e^{-T'}$ *and* $T = 2\pi/T'$. *Let* $\mathcal{M}_\lambda$ *be the fixed point subalgebra* $\mathcal{M}_{0,1}^{\theta_{T'}}$ *of* $\mathcal{M}_{0,1}$ *under the automorphism* $\theta_{T'}$.

(i) *The subalgebra* $\mathcal{M}_\lambda \subset \mathcal{M}_{0,1}$ *is a factor of type* III_λ, *whose discrete core is conjugate to the pair* $\{\mathcal{M}_{0,1}, \theta_{T'}\}$.

(ii) *The triplet* $\{\mathcal{M}_{0,1}, \mathcal{M}_\lambda, \theta_{T'}\}$ *is a Galois triplet in that*

$$\text{Gal}(\mathcal{M}_{0,1}/\mathcal{M}_\lambda) = \{\theta_{T'}^n : n \in \mathbb{Z}\},$$

where $\text{Gal}(\mathcal{M}/\mathcal{N}) = \{\alpha \in \text{Aut}(\mathcal{M}) : \alpha|_\mathcal{N} = \text{id}\}$ *for any pair* $\mathcal{N} \subset \mathcal{M}$ *of von Neumann algebras. We have the exact sequence*

$$1 \to \{\theta_{T'}^n : n \in \mathbb{Z}\} \longrightarrow \text{Aut}(\mathcal{M}_\lambda)_\text{m} \longrightarrow \text{Aut}(\mathcal{M}_\lambda) \to 1,$$

and

$$\begin{aligned}\text{Aut}(\mathcal{M}_\lambda)_\text{m} &= \{\tilde{\alpha} \in \text{Aut}(\mathcal{M}_{0,1}) : \tilde{\alpha}(\mathcal{M}_\lambda) = \mathcal{M}_\lambda\}\\ &= \{\tilde{\alpha} \in \text{Aut}(\mathcal{M}_{0,1}) : \tilde{\alpha} \circ \theta_{T'} = \theta_{T'} \circ \tilde{\alpha}\}.\end{aligned}$$

(iii) *Another pair* $\{\mathcal{M}_\lambda, \mathcal{M}_1\}$ *forms a Galois pair*

$$\text{Gal}(\mathcal{M}_\lambda/\mathcal{M}_1) = \{\theta_{\dot{s}_{T'}} : \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z},\ s \in \mathbb{R}\},$$

that is, an $\alpha \in \text{Aut}(\mathcal{M}_\lambda)$ *is of the form* $\alpha = \theta_{\dot{s}_{T'}}$ *for some* $\dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}$ *if and only if* $\alpha(x) = x$ *for* $x \in \mathcal{M}_1$.

(iv) *The modulus of* $\theta_{\dot{s}_{T'}} \in \text{Aut}(\mathcal{M}_\lambda)$ *is precisely* $-\dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}$ *itself, that is,*

$$\text{mod}(\theta_{\dot{s}_{T'}}) = -\dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}.$$

If any of $\mathcal{M}_\lambda$, $\mathcal{M}_1$ *and* $\mathcal{M}_{0,1}$ *is approximately finite-dimensional, then all others are approximately finite-dimensional, and the following statements hold:*

(v) *If* $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ *has aperiodic modulus* $\mathrm{m} = \mathrm{mod}(\alpha)$, *that is, if* $k\mathrm{m} \neq 0$ *for every nonzero integer* $k \in \mathbb{Z}$ *or equivalently if* $\{\mathrm{mod}(\alpha)\}_{T'}/T' \notin \mathbb{Q}$, *then* α *is cocycle conjugate to* $\theta_{-\mathrm{m}}$.

(vi) *If an automorphism* $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ *has trivial asymptotic outer period, that is,* $p_a(\alpha) = 0$, *then its cocycle conjugacy class is determined by its modulus* $\mathrm{m} = \mathrm{mod}(\alpha) \in \mathbb{R}/T'\mathbb{Z}$. *In fact, the automorphism* α *is cocycle conjugate to the automorphism* $\theta_{-\mathrm{m}} \otimes \sigma_0$ *on* $\mathcal{M}_\lambda \cong \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0$, *where* $\sigma_0 \in \mathrm{Aut}(\mathcal{R}_0)$ *is any aperiodic automorphism of the approximately finite-dimensional factor* $\mathcal{R}_0$. *If* $\mathrm{m} \neq 0$, *then* $\theta_\mathrm{m} \sim \theta_\mathrm{m} \otimes \sigma_0$.

Proof. We prove statements (v) and (vi). Choose an automorphism $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ such that $\mathrm{m} = \mathrm{mod}(\alpha)$ is aperiodic. Let $\mathcal{R}_0$ be an AFD factor of type II_1 realized as the infinite tensor product of two by two matrix algebras

$$\mathcal{R}_0 = \prod_{n\in\mathbb{Z}}^{\otimes} \{M_n, \tau_n\}$$

relative to the normalized traces $\tau_n = \mathrm{Tr}/2$ on $M_n = \mathrm{M}(2,\mathbb{C})$. Let σ_0 be the Bernoulli shift automorphism of $\mathcal{R}_0$, that is, the automorphism determined by

$$\sigma_0\Big(\prod_{n\in\mathbb{Z}}^{\otimes} x_n\Big) = \prod_{n\in\mathbb{Z}}^{\otimes} x_{n+1}.$$

Then by the grand theorem of Connes [1975] (also [Takesaki 2003b, page 267]) α and $\alpha \otimes \sigma_0$ are cocycle conjugate under the identification of $\mathcal{M}_\lambda$ and $\mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0$ because the asymptotic outer period $p_a(\alpha)$ of α is zero, that is, $p_a(\alpha) = 0$. The same is true for θ_m, that is, $\theta_\mathrm{m} \sim_c \theta_\mathrm{m} \otimes \sigma_0$, where $\sim_c$ means the outer conjugacy. Since $\mathrm{mod}(\alpha_1 \otimes \alpha_2) = \mathrm{mod}(\alpha_1) + \mathrm{mod}(\alpha_2)$ on $\mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \cong \mathcal{M}_\lambda$, we have $\alpha \sim_c \alpha \otimes \sigma_0 \sim_c \alpha \otimes \theta_\mathrm{m} \otimes \theta_{-\mathrm{m}} \sim_c \sigma_0 \otimes \theta_{-\mathrm{m}} \sim_c \theta_{-\mathrm{m}}$. This proves statement (v).

To prove (vi), suppose that $p \in \mathbb{N}$ is the period of $\mathrm{m} \in \mathbb{R}/T'\mathbb{Z}$, that is, the smallest nonnegative integer with $p\mathrm{m} = 0$. We assume that $p \neq 0$. Let $\{e_{j,k} : 1 \leq j, k \leq p\}$ be the standard matrix units of the $p \times p$ matrix algebra $\mathrm{M}(p;\mathbb{C})$, and for each $n \in \mathbb{N}$ set $M_n = \mathrm{M}(p;\mathbb{C})$. Also consider the diagonal unitary

$$u_n = \sum_{i=1}^{p} \exp(2\pi \mathrm{i}((i-1)/p)) e_{i,i} \in \mathrm{U}(p;\mathbb{C}) \subset M_n$$

of order p, that is, $u_n^p = 1$. Now we identify the AFD factor $\mathcal{R}_0$ with the infinite tensor product

$$\mathcal{R}_0 = \prod_{n\in\mathbb{N}}^{\otimes} \{M_n, \tau_n\}, \quad \text{where } \tau_n = \tfrac{1}{n}\mathrm{Tr},$$

and let

$$\sigma_p = \prod_{n\in\mathbb{N}}^{\otimes} \mathrm{Ad}(u_n) \in \mathrm{Aut}(\mathcal{R}_0) \in \mathrm{Aut}(\mathcal{R}_0).$$

Then the automorphism σ_p has the properties

$$\begin{aligned} &\sigma_p^k \notin \mathrm{Int}(\mathcal{R}_0) && \text{for } k = 1, \dots, p-1, \text{ and } \sigma_p^p = \mathrm{id}, \\ &\theta_{\mathrm{m}} \sim_c \theta_{\mathrm{m}} \otimes \sigma_p && \text{on } \mathcal{M}_\lambda \cong \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0, \\ &\theta_{\mathrm{m}} \otimes \theta_{-\mathrm{m}} \sim_c \mathrm{id} \otimes \mathrm{id} \otimes \sigma_p && \text{on } \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \cong \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0, \\ &\sigma_0 \otimes \sigma_p \sim_c \sigma_0 && \text{on } \mathcal{R}_0 \mathbin{\overline{\otimes}} \mathcal{R}_0 \cong \mathcal{R}_0. \end{aligned}$$

If $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$ has the trivial asymptotic outer period $p_a(\alpha) = 0$, then the automorphism α has the properties

$$\begin{aligned} &\alpha \sim_c \sigma_p \otimes \alpha && \text{on } \mathcal{M}_\lambda \cong \mathcal{R}_0 \mathbin{\overline{\otimes}} \mathcal{M}_\lambda, \\ &\theta_{\mathrm{m}} \otimes \alpha \sim_c \mathrm{id} \otimes \sigma_0 && \text{on } \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \cong \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0, \\ &\theta_{-\mathrm{m}} \otimes \sigma_0 \sim_c \theta_{-\mathrm{m}} \otimes \theta_{\mathrm{m}} \otimes \alpha \sim_c \sigma_p \otimes \alpha \sim_c \alpha \end{aligned}$$

under the isomorphisms $\mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{R}_0 \cong \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \cong \mathcal{R}_0 \mathbin{\overline{\otimes}} \mathcal{M}_\lambda \cong \mathcal{M}_\lambda$. This completes the proof. ♡

Thus if $\mathrm{mod}(\alpha)$ is aperiodic, or $p_a(\alpha) = 0$, then the grand theorem of Connes [Connes 1975], or see [Takesaki 2003b, page 270], identifies the cocycle conjugacy class of $\alpha \in \mathrm{Aut}(\mathcal{M}_\lambda)$. But if $\mathrm{mod}(\alpha)$ has nontrivial period, and $p_1 = p_a(\alpha) \neq 0$, then the cocycle conjugacy class of α involves algebraic invariants. For example, one has to consider the extension of α to the discrete core $\widetilde{\mathcal{M}}_{\lambda,d}$ on which α alone cannot act. In fact, one has to consider a larger group $\mathbb{Z}^2$ than the integer group $\mathbb{Z}$.

Invariants for single automorphisms. We consider a single automorphism of a factor $\mathcal{M}$ of type III_λ, which can be viewed as an action of the integer additive group $\mathbb{Z}$. As the integer group $\mathbb{Z}$ appears in many different roles, we denote it here by $G = \mathbb{Z}$. Let a_1 be the generator of the group G, so that $G = \mathbb{Z}a_1$. Sometimes we view G as a multiplicative group, in which case G becomes $G = \{a_1^k : k \in \mathbb{Z}\}$. Since $\mathrm{H}^2(G, \mathbb{T}) = \mathrm{H}^3(G, \mathbb{T}) = \{1\}$, that is, the integer group is cohomologically trivial, there is no distinction between the cocycle conjugacy problem and the outer conjugacy problem of actions of G. Namely, an outer action $\dot{\alpha}$ of G always comes from an action α of G, and outer conjugacy of the outer action $\dot{\alpha}$ of G is the same as the cocycle conjugacy of the action α of G. Hence the obstruction $\mathrm{Ob}(\dot{\alpha})$ of $\dot{\alpha}$ and the characteristic invariant $\chi(\alpha)$ of α are handily identified. The same is true for the modular obstruction $\mathrm{Ob_m}(\dot{\alpha})$ and the modular characteristic invariant $\chi_{\mathrm{m}}(\alpha)$.

Since the single automorphism cocycle conjugacy classification wasn't handled properly in [Katayama et al. 1998; 1997], and more importantly because the presentation of a single automorphism on a factor of type III_λ in [Takesaki 2003b] contains a minor mistake, we present it here in some detail.

In the case that the modulus $\mathrm{m} = \mathrm{mod}(\alpha)$ is aperiodic, the last theorem takes care of the cocycle conjugacy of α, that is, it must be cocycle conjugate to $\theta_{-\mathrm{m}}$. So we handle only the case that $\{\mathrm{mod}(\alpha)\}_{T'}$ is rational multiple of T'.

Suppose $\alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M})) = \mathbb{Z}b_1$ and $b_1 = p_1 a_1$, with $p_1 \in \mathbb{N}$.

Choose a pair $p_1, q_1 \in \mathbb{N}$ of positive integers with $q_1 < p_1$ such that

$$\mathrm{m} = (q_1/p_1)T' + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z} \quad \text{for } 0 \le q_1 < p_1.$$

Form a group extension

$$\begin{gathered} G_{\mathrm{m}} = \{(g, s) \in G \times \mathbb{R} : g\mathrm{m} = \dot{s}_{T'} = s + T'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}\} \\ 0 \to \mathbb{Z} \xrightarrow{k \to (0, kT')} G_{\mathrm{m}} \xrightarrow{\mathrm{pr}_1} G \to 0. \end{gathered} \tag{1-2}$$

Set

$$\begin{aligned} z_0 &= (0, T'), & z_1 &= (a_1, \{\mathrm{m}\}_{T'}), \\ b_1 &= p_1 z_1 - q_1 z_0, & N &= \mathbb{Z}b_1, \quad Q_{\mathrm{m}} = G_{\mathrm{m}}/N. \end{aligned} \tag{1-3}$$

The group G_{m} is equipped with a distinguished homomorphism $\mathrm{k_m} = \mathrm{pr}_2$ to $\mathbb{R}$:

$$\mathrm{k_m}(g, s) = s \in \mathbb{R} \quad \text{for } (g, s) \in G_{\mathrm{m}}. \tag{1-4}$$

Let $\pi_Q : g \in G_{\mathrm{m}} \mapsto \dot{g} \in Q_{\mathrm{m}}$ be the quotient map and further set

$$D_1 = \gcd(p_1, q_1), \quad \text{and} \quad r_1 = p_1/D_1, \quad s_1 = q_1/D_1, \tag{1-5}$$

and find a pair $u_1, v_1 \in \mathbb{Z}$ of integers such that

$$1 = r_1 u_1 - s_1 v_1, \quad \text{or equivalently} \quad D_1 = p_1 u_1 - q_1 v_1,$$

which can be done through the Euclid algorithm. In the event that $q_1 = 0$, the modulus m is trivial, that is, $\mathrm{m} = 0$ and $G_{\mathrm{m}} = G \oplus \mathbb{Z}$.

Theorem 1.3 (invariants for a single automorphism with periodic modulus). *Set* $D_1 = \gcd(p_1, q_1)$. *If p_1 and q_1 are both nonzero, we have the following statements*:

(i) *The pair $\{z_0, z_1\}$ is a free basis of G_{m}, so that every element $g \in G_{\mathrm{m}}$ is written uniquely in the form $g = e_0(g)z_0 + e_1(g)z_1$.*

(ii) *The group G_{m} admits another free basis $\{w_0, w_1\}$ such that $b_1 = D_1 w_1$. Therefore $N = D_1\mathbb{Z}w_1$ and*

$$\begin{gathered} Q_{\mathrm{m}} = \mathbb{Z}\dot{w}_0 \oplus \mathbb{Z}\dot{w}_1, \\ D_1\dot{w}_1 = 0 \quad \text{in } Q_{\mathrm{m}} \cong \mathbb{Z} \oplus \mathbb{Z}_{D_1}, \end{gathered}$$

where the dotted notations indicate their images in the quotient group Q_{m}.

(iii) *The character group* $\widehat{Q}_{\mathrm{m}}$ *of* Q_{m} *and the characteristic cohomology group* $\Lambda(G_{\mathrm{m}}, N, \mathbb{T})$ *are identified under the correspondence*

$$\lambda_\chi(nb_1; g) = \chi(\pi_Q(g))^n \quad \textit{for } g \in G_{\mathrm{m}} \textit{ and } \chi \in \widehat{Q}_{\mathrm{m}}. \tag{1-6}$$

(iv) *The character group* $\widehat{Q}_{\mathrm{m}}$ *is given by the exact sequence*

$$0 \to \mathbb{Z}^2 \longrightarrow \mathbb{R} \oplus (\tfrac{1}{D_1}\mathbb{Z}) \xrightarrow{\exp(2\pi \mathrm{i}\,\cdot)} \mathbb{T} \oplus \mathbb{Z}_{D_1} = \widehat{Q}_{\mathrm{m}} \to 0,$$

which describes the characteristic cohomology group $\Lambda(G_{\mathrm{m}}, N, \mathbb{T})$ *as*

$$\Lambda(G_{\mathrm{m}}, N, \mathbb{T}) \cong \mathbb{T} \oplus \mathbb{Z}_{D_1}. \tag{1-7}$$

If $\chi(z_0)$ *is a root of unity, then the outer period* $p_{\mathrm{o}}(\alpha)$ *of* α *is given as the product* $p_1 s_{\mathrm{o}}$ *with* $s_{\mathrm{o}} \in \mathbb{Z}_+$ *the smallest nonnegative integer* $s \in \mathbb{Z}_+$ *such that* $1 = \chi(z_0)^s$. *If* $\chi(z_0)$ *is not a root of unity, then the corresponding automorphism* α *is aperiodic, that is,* $p_{\mathrm{o}}(\alpha) = 0$.

Proof. (i) Since $\mathrm{pr}_1(z_1) = a_1$ and G is a free abelian group, the exact sequence (1-2) splits along with the cross-section: $m \in G \mapsto mz_1 \in G_{\mathrm{m}}$.

(ii) We set $w_0 = u_1 z_0 - v_1 z_1$ and $w_1 = -s_1 z_0 + r_1 z_1$. Since $z_0 = r_1 w_0 + v_1 w_1$ and $z_1 = s_1 w_0 + u_1 w_1$, the pair $\{w_0, w_1\}$ is a free basis of G_{m} such that

$$G_{\mathrm{m}} = \mathbb{Z}w_0 + \mathbb{Z}w_1, \quad b_1 = D_1 w_1, \quad N = D_1 \mathbb{Z} w_1, \quad Q_{\mathrm{m}} = G_{\mathrm{m}}/N = \mathbb{Z}\dot{w}_0 \oplus \mathbb{Z}\dot{w}_1,$$

as we wanted.

(iii) Since $\mathrm{H}^2(N, \mathbb{T}) = \{1\}$, the second cocycle part of a characteristic cocycle in $\mathrm{Z}(G_{\mathrm{m}}, N, \mathbb{T})$ is taken to be trivial, so that the λ-part vanishes on N and therefore it is a character of G_{m} that vanishes on N and factors through the quotient map $\pi_Q : G_{\mathrm{m}} \to Q_{\mathrm{m}}$. Thus it is of the form $\lambda(b_1; g) = \chi(\pi_Q(g))$ for $g \in G_{\mathrm{m}}$ and $\chi \in \widehat{Q}_{\mathrm{m}}$.

(iv) It follows from (ii) that the character group $\widehat{Q}_{\mathrm{m}}$ is parameterized by $\mathbb{R} \oplus (\frac{1}{D_1}\mathbb{Z})$:

$$\chi_{x,y}(g) = \exp(2\pi \mathrm{i}(x f_0(g) + y f_1(g))) \quad \text{for } g = f_0(g) w_0 + f_1(g) w_1 \in G_{\mathrm{m}},$$

with $(x, y) \in \mathbb{R} \oplus (\frac{1}{D_1}\mathbb{Z})$. This gives the exact sequence

$$0 \to \mathbb{Z}^2 \longrightarrow \mathbb{R} \oplus (\tfrac{1}{D_1}\mathbb{Z}) \xrightarrow{(x,y) \to \chi_{x,y}} \widehat{Q}_{\mathrm{m}} = \mathbb{T} \oplus \mathbb{Z}_{D_1} \to 0. \qquad \heartsuit$$

Model construction. Suppose G is a fixed countable discrete amenable group and let $\{H, \pi_G\}$ be a universal resolution group of the third cocycles of G, that is, $\pi_G : H \to G$ is a surjective homomorphism such that $\pi_G^*(\mathrm{Z}^3(G, \mathbb{T})) \subset \mathrm{B}^3(H, \mathbb{T})$. We require H to be a countable discrete amenable group. Let $M = \mathrm{Ker}(\pi_G)$. Fix a normal subgroup N of G, and set $L = \pi_G^{-1}(N)$. With a fixed invariant homomorphism $\mathrm{m} \in \mathrm{Hom}_G(N, \mathbb{R}/T'\mathbb{Z})$ such that $\mathrm{Ker}(\mathrm{m}) \supset N$, we use the abbreviated notation m for $\mathrm{m}\circ\pi_G$ and form a group extension H_m via

$$0\to\mathbb{Z}\longrightarrow H_\mathrm{m}\xrightarrow{\pi_\mathrm{m}} H\to 1,$$

where $H_\mathrm{m}=\{(g,s)\in H\times\mathbb{R}:\mathrm{m}(g)=\dot{s}_{T'}=s+T'\mathbb{Z}\in\mathbb{R}/T'\mathbb{Z}\}$, with $\pi_\mathrm{m}(g,s)=g\in H$ and $\mathrm{k}(g,s)=s\in\mathbb{R}$ for $(g,s)\in H_\mathrm{m}$. We get the reduced modified HJR-sequence

$$\cdots\longrightarrow \mathrm{H}^2(H,\mathbb{T})\xrightarrow{\mathrm{Res}}\Lambda(H_\mathrm{m},L,M,\mathbb{T})\xrightarrow{\delta}\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T})\to 1.$$

Thus every modular obstruction cocycle $(c,\nu)\in\mathrm{Z}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T})$ is of the form

$$(c,\nu)\equiv\delta(\lambda,\mu)\ \bmod\ \mathrm{B}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T}).$$

Consequently the construction of an outer action $\dot\alpha$ of G on an AFD factor $\mathcal{M}_\lambda$ of type III_λ with $\mathrm{Ob}_\mathrm{m}(\dot\alpha)=([c],\nu)\in\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G,N,\mathbb{T})$ is reduced to the construction of an action $\alpha^{\lambda,\mu}$ of H_m such that

$$(\alpha^{\lambda,\mu})^{-1}(\mathrm{Int}(\mathcal{M}_\lambda))\supset M,\qquad \chi(\alpha^{\lambda,\mu})=[\lambda,\mu]\in\Lambda(H_\mathrm{m},L,M,\mathbb{T}),$$
$$(\alpha^{\lambda,\mu})^{-1}(\mathrm{Cnt}(\mathcal{M}_\lambda))=L,\qquad \mathrm{mod}\,(\alpha^{\lambda,\mu}_g)=\mathrm{m}(\pi_G(g))\quad\text{for } g\in H_\mathrm{m}.$$

So fix a set of invariants $(\lambda,\mu)\in\mathrm{Z}(H_\mathrm{m},L,M,\mathbb{T})$ and $\mathrm{m}\in\mathrm{Hom}_G(G,\mathbb{R}/T'\mathbb{Z})$ such that $\mathrm{Ker}(\mathrm{m})\supset N$. We are going to construct the model action $\alpha^{\lambda,\mu}$ of H_m:

Step I. Let X be a countable but infinite set on which H_m acts freely from the left. In the case that H_m is an infinite group, we take X to be H_m itself and let H_m act on it by left multiplication. So the infinite set X is only needed when H_m is a finite group, in which case X can be taken to be the product set $X=H_\mathrm{m}\times\mathbb{N}$ and H_m acts on the first component by left multiplication. Let $\{M_x, x\in X\}$ be the set of 2×2 matrix algebras $\mathrm{M}(2,\mathbb{C})$ indexed by elements $x\in X$.

Step II. We form the infinite tensor product $\mathcal{R}_0=\prod^{\otimes}_{x\in X}\{M_x,\tau_x\}$ relative to the normalized trace

$$\tau_x\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}=\tfrac12(a_{11}+a_{22}).$$

Then we let σ^0 be the Bernoulli action of H_m on $\mathcal{R}_0$ that is determined by

$$\sigma^0_g\Big(\prod_{x\in X}^{\otimes}a_x\Big)=\prod_{x\in X}^{\otimes}a_{gx}.$$

Step III. Form the twisted partial crossed product of $\mathcal{R}_0$ by N relative to the second cocycle $\mu\in\mathrm{Z}^2(N,\mathbb{T})$ and the action σ^0, that is, $\mathcal{M}_0=\mathcal{R}_0\rtimes_{\sigma^0,\mu}N$. Then let $\{U(m):m\in N\}$ be the projective unitary representation of N to $\mathcal{M}_0$ corresponding to the twisted crossed product, so that

$$U(g)U(h)=\mu(g;h)U(gh)\quad\text{for } g,h\in N,$$
$$U(g)aU(g)^*=\sigma^0_g(a)\qquad\text{for } a\in\mathcal{R}_0,\ g\in N.$$

Let $\sigma^{\lambda,\mu}$ be the action of H_{m} on $\mathcal{M}_0$ determined by

$$\begin{aligned} \sigma_g^{\lambda,\mu}(U(m)) &= \lambda(gmg^{-1}; g)U(gmg^{-1}) \quad &&\text{for } m \in N \text{ and } g \in H_{\mathrm{m}}, \\ \sigma_g^{\lambda,\mu}(a) &= \sigma_g^0(a) &&\text{for } a \in \mathcal{R}_0 \text{ and } g \in H_{\mathrm{m}}. \end{aligned}$$

Step IV. Let $\mathcal{M}_{0,1}$ be the AFD factor of type II_∞ equipped with trace-scaling one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ and set $\mathcal{R}_{0,1} = \mathcal{M}_{0,1} \mathbin{\overline{\otimes}} \mathcal{M}_0$. We then set the action $\tilde{\alpha}^{\lambda,\mu}$ by $\tilde{\alpha}_g^{\lambda,\mu} = \theta_{\mathrm{m}(g)} \otimes \sigma_g^{\lambda,\mu}$ on $\mathcal{R}_{0,1}$ for $g \in H_{\mathrm{m}}$. Set $\mathcal{R} = (\mathcal{R}_{0,1})^{\tilde{\alpha}_{z_0}}$. The automorphism $\tilde{\alpha}_{z_0} = \theta_{T'} \otimes \sigma_{z_0}^{\lambda,\mu}$ scales the trace τ by $\lambda = e^{-T'}$, so the von Neumann algebra $\mathcal{R}$ is an AFD factor of type III_λ. Finally we define the action $\alpha^{\lambda,\mu}$ by $\alpha_g^{\lambda,\mu} = \tilde{\alpha}_g^{\lambda,\mu}|_{\mathcal{R}}$ for $g \in H$; this makes sense because $\tilde{\alpha}_{z_0}$ acts trivially on $\mathcal{R}$.

Theorem 1.4 (model action).

(i) *The action $\alpha = \alpha^{\lambda,\mu}$ constructed above has the invariants*

$$\begin{aligned} N &= \alpha^{-1}(\mathrm{Cnt}(\mathcal{R}_\lambda)), \\ \mathrm{mod}\,(\alpha_g) &= \mathrm{m}(g) \quad \textit{for } g \in H, \\ \chi(\alpha) &= [\lambda, \mu] \in \Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}), \\ \nu_\alpha(g) &= [T\,\mathrm{Log}(\lambda(g; z_0))/2\pi]_T \in \mathbb{R}/T\mathbb{Z} \quad \textit{for } g \in N. \end{aligned}$$

(ii) *Let $\mathfrak{s}_H : G \to H$ be a cross-section of the homomorphism $\pi_G : H \to G$. Then the outer action $\alpha_{\mathfrak{s}_H}^{\lambda,\mu}$ of G has associated modular obstruction given by $\delta([\lambda, \mu]) = [c^{\lambda,\mu}, \nu^\lambda] \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$.*

The construction of (i) *and* (ii) *exhausts all outer actions of G on the approximately finite-dimensional factor $\mathcal{R}$ of type* III_λ, *up to outer conjugacy.*

Proof. (i) Let $\tilde{\alpha}$ denote the action $\tilde{\alpha}^{\lambda,\mu}$ of H_{m} on $\mathcal{R}_{0,1}$. Since $\mathcal{R}$ is the fixed point subalgebra of $\mathcal{R}_{0,1}$ under the automorphism $\tilde{\alpha}_{z_0}$, the restriction $\alpha = \tilde{\alpha}|_{\mathcal{R}}$ of $\tilde{\alpha}$ to $\mathcal{R}$ factors through the quotient group $H = H_{\mathrm{m}}/(\mathbb{Z}z_0)$. Hence α is indeed an action of H. Since $\mathcal{R}_{0,1}$ is a factor of type II_∞ and

$$\tau \circ \tilde{\alpha}_{z_0} = \tau \circ \theta_{\mathrm{m}(z_0)} = e^{-\mathrm{m}(z_0)}\tau = e^{-T'}\tau = \lambda\tau,$$

the fixed point subalgebra $\mathcal{R}$ is a factor of type III_λ and the pair $\{\mathcal{R}_{0,1}, \tilde{\alpha}_{z_0}\}$ is the discrete core of the factor $\mathcal{R}$. Since $\mathcal{R}_{0,1}$ is AFD, $\mathcal{R}$ is as well by the grand theorem of Connes [1976a]. As z_0 is a central element of H_{m}, $\tilde{\alpha}(H_{\mathrm{m}})$ leaves $\mathcal{R}$ globally invariant and hence its restriction to $\mathcal{R}$ makes sense. The inner part $\tilde{\alpha}(N)$, which is given by the projective representation $\{U(g) : g \in N\}$, leaves $\mathcal{R}$ globally invariant, that is, $U(g)$ for $g \in N$ normalizes $\mathcal{R}$; thus we have the inclusion $U(N) \subset \widetilde{\mathcal{U}}_0(\mathcal{R})$. Hence $N = \alpha^{-1}(\mathrm{Cnt}(\mathcal{R}))$. As in (1-1), we have $\mathrm{mod}(\alpha_h) = \mathrm{m}(h)$ for $h \in H$. If $g, g_1, g_2 \in N$ and $h \in H$, then

$$\lambda(g; h) = U^*(g)\tilde{\alpha}_h(U(h^{-1}gh)), \qquad U(g_1)U(g_2) = \mu(g_1; g_2)U(g_1 g_2),$$
$$\nu_\alpha(g) = \partial_{\tilde{\alpha}_{z_0}}(U(g)) = U(g)^*\tilde{\alpha}_{z_0}(U(g)) = \lambda(g; z_0).$$

Hence $\chi(\tilde{\alpha}) = [\lambda, \mu] \in \Lambda(H_{\mathrm{m}}, L, M, \mathbb{T})$ as required. Finally viewing ν_α as a homomorphism of N into $\mathbb{R}/T\mathbb{Z}$, we get $\nu_\alpha \in \mathrm{Hom}_G(N, \mathbb{R}/T\mathbb{Z})$ as stated.

(ii) The assertion follows from the construction of $\alpha^{\lambda,\mu}$. ♡

Actions and outer actions of two commuting automorphisms on an AFD factor $\mathfrak{R}$ of type $\mathbf{III}_\lambda$. In this case, we have to consider the free abelian group $G = \mathbb{Z}^2$ of rank two and its extension $G_{\mathrm{m}} \cong \mathbb{Z}^3$ relative to a homomorphism $\mathrm{m}: G \to \mathbb{R}/T'\mathbb{Z}$. We fix a subgroup N of G, which is going to represent the inverse image $\alpha^{-1}(\mathrm{Cnt}(\mathcal{M}_\lambda))$ of the extended modular automorphism group. We assume that N is in the diagonal form, that is, with a free basis $\{a_1, a_2\}$ of G, the subgroup N is of the form $N = p_1\mathbb{Z}a_1 + p_2\mathbb{Z}a_2$. Of course, one can choose p_1 and p_2 so that $0 \le p_1 \le p_2$ and p_1 divides p_2, but to go beyond the finite rank case, we don't assume that p_1 is a divisor of p_2, which might make the matter slightly more involved. In the case that $G = \mathbb{Z}^2$, we have $\mathrm{H}^3(G, \mathbb{T}) = \{1\}$, so every outer action of G comes from an action of G. Since $\mathrm{H}^2(G, \mathbb{T}) \cong \mathbb{T} \ne \{1\}$, the outer conjugacy class of an action is bigger than the cocycle conjugacy class. To go further, we recall the reduced modified HJR-exact sequence from [Katayama and Takesaki 2007, Theorem 3.11 page 116]:

$$\mathrm{H}^2(G, \mathbb{T}) \xrightarrow{\mathrm{Res}_{Q_{\mathrm{m}}}} \Lambda(G_{\mathrm{m}}, N, \mathbb{T}) \xrightarrow{\delta_{Q_{\mathrm{m}}}} \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G, N, \mathbb{T}) \xrightarrow{\mathrm{Inf}_{Q_{\mathrm{m}}}} \mathrm{H}^3(G, \mathbb{T}) = \{1\},$$

where $Q_{\mathrm{m}} = G_{\mathrm{m}}/N$. Here since $\mathrm{H}^3(G, \mathbb{T}) = \{1\}$, we don't have to consider the resolution group H and its subgroup M. To identify the subgroup $N \subset G$ as a subgroup of G_{m}, we need a little care. First, set

$$\begin{aligned}
z_0 &= (0, T') \in G_{\mathrm{m}},\\
z_1 &= (a_1, q_1T'/p_1) \in G_{\mathrm{m}}, \quad z_2 = (a_2, q_2T'/p_2),\\
b_1 &= (p_1a_1, 0) = p_1z_1 - q_1z_0 \in G_{\mathrm{m}},\\
b_2 &= (p_2a_2, 0) = p_2z_2 - q_2z_0 \in G_{\mathrm{m}},\\
N &= \mathbb{Z}b_1 + \mathbb{Z}b_2 \subset G_{\mathrm{m}} = \mathbb{Z}z_0 + \mathbb{Z}z_1 + \mathbb{Z}z_2,\\
Q_{\mathrm{m}} &= G_{\mathrm{m}}/N.
\end{aligned} \tag{1-8}$$

This gives the following coordinate system in G_{m} and N:

$$\begin{aligned}
g &= e_{1,N}(g)b_1 + e_{2,N}(g)b_2 \in N, \quad \text{that is, } e_{i,N}(g) = \frac{e_i(g)}{p_i} \quad \text{for } i = 1, 2,\\
h &= \tilde{e}_0(h)z_0 + \tilde{e}_1(h)z_1 + \tilde{e}_2(h)z_2 \in G_{\mathrm{m}}.
\end{aligned} \tag{1-9}$$

Theorem 1.5 (invariant). *Define* Z *and* B *by*

$$\mathrm{Z} = \bigl\{b = \{b(i, j) : i = 1, 2,\ j = 0, 1, 2\} \in \mathbb{R}^6 : p_jb(i, j) - q_jb(i, 0) \in \mathbb{Z},\ i = 1, 2,\ j = 1, 2\bigr\}, \tag{1-10}$$

$$\mathrm{B} = \bigl\{b \in \mathrm{Z} : b(i, 0), b(i, i) \in \mathbb{Z},\ i = 1, 2,\ p_2b(1, 2) + p_1b(2, 1) \in \gcd(p_1, p_2)\mathbb{Z}\bigr\},$$

and to each $b \in \mathrm{Z}$ associate a cochain $(\lambda_b, \mu_b) \in \mathrm{Z}(G_{\mathrm{m}}, N, \mathbb{T})$ by

$$\lambda_b(g; h) = \exp\Bigl(2\pi\mathrm{i}\Bigl(\sum_{i=1,2;\, j=0,1,2} b(i, j) e_{i,N}(g)\tilde{e}_j(h)\Bigr)\Bigr), \qquad (1\text{-}11)$$

$$\mu_b(g_1; g_2) = 1 \quad \text{for } g, g_1, g_2 \in N \text{ and } h \in G_{\mathrm{m}}.$$

Then the cochain (λ_b, μ_b) is a characteristic cocycle $(\lambda_b, 1) \in \mathrm{Z}(G_{\mathrm{m}}, N, \mathbb{T})$. The modular obstruction cocycle $(c_b, \nu_b) = \delta(\lambda_b, 1) \in \mathrm{Z}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ corresponding to $(\lambda_b, 1)$ takes the form

$$\begin{aligned} c_b(\dot{g}_1; \dot{g}_2; \dot{g}_3) &= \lambda_b(\mathfrak{n}_N(\dot{g}_2; \dot{g}_3); \mathfrak{s}(\dot{g}_3)) \quad (\text{where } \dot{g}_1, \dot{g}_2, \dot{g}_3 \in Q_{\mathrm{m}}) \\ &= \exp\Bigl(2\pi\mathrm{i}\Bigl(\sum_{\substack{i=1,2\\ j=0,1,2}} \frac{b(i, j)\eta_{p_i}([e_i(\dot{g}_2)]_{p_i}; [e_i(\dot{g}_3)]_{p_i})\{\tilde{e}_j(\dot{g}_1)\}_{p_j}}{p_i}\Bigr)\Bigr), \end{aligned} \qquad (1\text{-}12)$$

$$\nu_b(g) = \Bigl[T \sum_{i=1,2} b(i, 0) e_{i,N}(g)\Bigr]_T \in \mathbb{R}/T\mathbb{Z} \quad \text{for } g \in N,$$

where for the notations η_{p_i} and $\mathfrak{n}_N$ we refer to definitions (3-8) and (3-14), and furthermore $\{\tilde{e}_0(\dot{g}_1)\}_{p_0} = \tilde{e}_0(\dot{g}_1) \in \mathbb{Z}$ for $\dot{g}_1 \in Q_{\mathrm{m}}$. The (i, j)-components $\mathrm{Z}(i, j)$ and $\mathrm{B}(i, j)$ of Z and B give more precise information about the cocycles:

(i) *For $i = 1, 2$, we have*

$$\begin{aligned} \mathrm{Z}_b(i, i) &= \{z = (x, u) \in \mathbb{R}^2 : p_i x - q_i u \in \mathbb{Z}\}, \\ \mathrm{B}_b(i, i) &= \mathbb{Z} \oplus \mathbb{Z}. \end{aligned} \qquad (1\text{-}13)$$

The bicharacter $\lambda^{i,i}_z$ on $N \times G_{\mathrm{m}}$ determined by

$$\lambda^{i,i}_z(g; h) = \exp(2\pi\mathrm{i}(x e_{i,N}(g)\tilde{e}_i(h) + u e_{i,N}(g)\tilde{e}_0(h))) \qquad (1\text{-}14)$$

for each pair $g \in N$ and $h \in G_{\mathrm{m}}$ gives a characteristic cocycle of $\mathrm{Z}(G_{\mathrm{m}}, N, \mathbb{T})$. It is a coboundary if and only if z is in $\mathrm{B}_b(i, i)$. The corresponding cohomology class $[\lambda^{i,i}_z] \in \Lambda_b(i, i)$ has the parameterization

$$\begin{aligned} [\lambda^{i,i}_z] \in \Lambda(i, i) &\sim ([p_i x - q_i u]_{\gcd(p_i, q_i)}, [-v_i x + u_i u]_{\mathbb{Z}}) \\ &\in \mathbb{Z}_{\gcd(p_i, q_i)} \oplus (\mathbb{R}/\mathbb{Z}), \end{aligned} \qquad (1\text{-}15)$$

where the integers u_i and v_i are determined by $p_i u_i - q_i v_i = \gcd(p_i, q_i)$ for $i = 1, 2$ through the Euclid algorithm. For the same i, the associated modular obstruction cohomology class $([c^{i,i}_z, \nu^{i,i}_z]) \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(i, i)$ corresponds to the class:

$$\begin{aligned} ([p_i x - q_i u]_{\gcd(p_i, q_i)}, [-v_i x + u_i u]_{\mathbb{Z}}) &\in \mathbb{Z}_{\gcd(p_i, q_i)} \oplus (\mathbb{R}/\mathbb{Z}), \\ \nu^{i,i}_z(g) = [T u e_{i,N}(g)]_T &\in \mathbb{R}/T\mathbb{Z}. \end{aligned}$$

(ii) *With* $(i, j) = (1, 2)$,

$$\begin{aligned} \mathrm{Z}_b(i, j) &= \big\{(x, u, y, v) \in \mathbb{R}^4 : p_j x - q_j u \in \mathbb{Z},\ p_i y - q_i v \in \mathbb{Z}\big\}; \\ \mathrm{B}_b(i, j) &= \big\{(x, u, y, v) \in \mathrm{Z}_b(i, j) : p_j x + p_i y \in \gcd(p_i, p_j)\mathbb{Z}, \\ &\qquad\qquad u, v \in \mathbb{Z}\big\}. \end{aligned} \tag{1-16}$$

For each element $z = (x, u, y, v) \in \mathrm{Z}_b(i, j)$, *the corresponding bicharacter* λ_z *on* $N \times G_{\mathrm{m}}$ *determined by*

$$\begin{aligned} \lambda_z^{i,j}(g; h) &= \exp\big(2\pi \mathrm{i}(x e_{i,N}(g)\tilde{e}_j(h) + y e_{j,N}(g)\tilde{e}_i(h))\big) \\ &\qquad \times \exp\big(2\pi \mathrm{i}(u e_{i,N}(g)\tilde{e}_0(h) + v e_{j,N}(g)\tilde{e}_0(h))\big), \end{aligned} \tag{1-17}$$

for each pair $g \in N$ *and* $h \in G_{\mathrm{m}}$ *is a characteristic cocycle in* $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. *It is a coboundary if and only if* z *belongs to* $\mathrm{B}_b(i, j)$. *The cohomology class* $[\lambda_z^{i,j}] \in \Lambda_b(i, j)$ *of* λ_z *corresponds to the parameter class*

$$[z] = \begin{pmatrix} [m_{i,j}(x r_{j,i} + y r_{i,j}) - n_{i,j}(u s_{j,i} + v s_{i,j})]_{\mathbb{Z}} \\ [y_{i,j}(x r_{j,i} + y r_{i,j}) + x_{i,j}(u s_{j,i} + v s_{i,j})]_{\mathbb{Z}} \\ [-u w_{i,j} + v w_{j,i}]_{\mathbb{Z}} \end{pmatrix} \in \begin{pmatrix} \Big(\dfrac{1}{D(i, j)}\mathbb{Z}\Big)\Big/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \end{pmatrix}, \tag{1-18}$$

where the integers $D(i, j), \dots, w_{i,j}$ *are those such that*

$$\begin{aligned} D(i, j) &= \gcd(p_i, p_j, q_i, q_j), \\ D_{i,j} &= \gcd(p_i, p_j), \qquad E_{i,j} = \gcd(q_i, q_j), \\ r_{i,j} &= p_i / D_{i,j}, \qquad r_{j,i} = p_j / D_{i,j} \\ s_{i,j} &= q_i / E_{i,j}, \qquad s_{j,i} = q_j / E_{i,j}, \\ m_{i,j} &= D_{i,j} / D(i, j), \qquad n_{i,j} = E_{i,j} / D(i, j), \\ q_i w_{i,j} + q_j w_{j,i} &= E_{i,j}, \qquad x_{i,j} D_{i,j} + y_{i,j} E_{i,j} = D(i, j). \end{aligned} \tag{1-19}$$

The associated modular obstruction class $([c_z^{i,j}], \nu_z^{i,j}) \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(i, j)$ *corresponds to the pair of classes*

$$[z] \in \begin{pmatrix} \Big(\dfrac{1}{D(i, j)}\mathbb{Z}\Big)\Big/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \end{pmatrix},$$

$$\nu_z^{i,j}(g) = \Big[T\Big(u\frac{e_1(g)}{p_1} + v\frac{e_2(g)}{p_2}\Big)\Big]_T \in \mathbb{R}/T\mathbb{Z} \quad \text{for } g \in N.$$

The proof of this special case is not much simpler than the general case, so we will discuss later in the general free abelian group case; see Theorem 4.2.

2. Asymmetrization

Set the notations $X = \mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$ and $X_1 = X\backslash\{1\}$. The signature of a permutation σ is the sign of the product: $\operatorname{sign}(\sigma) = \operatorname{sign}\big\{\prod_{i<j}(\sigma(j)-\sigma(i))\big\}$. Let S be the cyclic permutation

$$S = (2, 3, \dots, n, n+1, 1) \in \Pi(X), \tag{2-1}$$

whose signature is given by

$$\operatorname{sign}(S) = (-1)^n. \tag{2-2}$$

Each element $\sigma \in \Pi(X_1)$ is identified with an element of $\Pi(X)$ so that

$$\sigma = (1, \sigma(2), \sigma(3), \dots, \sigma(n), \sigma(n+1)) \in \Pi(X).$$

This identification of an element of $\Pi(X_1)$ with the corresponding element of $\Pi(X)$ preserves the signature of σ. Then the total permutation group $\Pi(X)$ is the disjoint union of the translations $\{S^k\Pi(X_1) : 0 \le k \le n\}$, that is,

$$\Pi(X) = \bigsqcup_{k=0}^{n} S^k\Pi(X_1). \tag{2-3}$$

Definition 2.1. The *asymmetrization* $\mathrm{AS}\,\xi$ of $\xi \in \mathrm{C}^n(G, A)$ is defined by

$$(\mathrm{AS}\,\xi)(g_1, g_2, \dots, g_n) = \sum_{\sigma\in\Pi(\mathbb{Z}_n)} \operatorname{sign}(\sigma)\xi(g_{\sigma(1)}, g_{\sigma(2)}, \dots, g_{\sigma(n)}). \tag{2-4}$$

Define $\pi_k : G^{n+1} \to G^n$ by

$$\pi_k(g_1, g_2, \dots, g_n, g_{n+1}) = \begin{cases} (g_2, g_3, \dots, g_n, g_{n+1}) & \text{for } k = 0, \\ (g_1, \dots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \dots, g_{n+1}) & \text{for } 1 \le k \le n, \\ (g_1, g_2, \dots, g_n) & \text{for } k = n+1. \end{cases} \tag{2-5}$$

The boundary operation $d \in \operatorname{Hom}(\mathbb{Z}(G^{n+1}), \mathbb{Z}(G^n))$ is then given by

$$d = \sum_{k=0}^{n+1}(-1)^k \circ \pi_k, \tag{2-6}$$

$\partial\xi = d^*\xi$ for $\xi \in \mathrm{C}^{n+1}(G, \mathbb{T})$. We view the asymmetrization AS also as an element of $\operatorname{End}(\mathbb{Z}(G^n))$ determined by

$$\mathrm{AS}(g_1, g_2, \dots, g_n) = \sum_{\sigma\in\Pi(\mathbb{Z}_n)} \operatorname{sign}(\sigma)(g_{\sigma(1)}, g_{\sigma(2)}, \dots, g_{\sigma(n)}).$$

Lemma 2.2. *The asymmetrization and the boundary operation are related by*

$$\mathrm{AS}\circ d = 0 \quad \textit{in } \mathrm{Hom}(\mathbb{Z}(G^{n+1}), \mathbb{Z}(G^n)).$$

Proof. Define $Q \in \mathrm{Hom}(\mathbb{Z}(G^{n+1}), \mathbb{Z}(G^n))$ and $R \in \mathrm{Hom}(\mathbb{Z}(G^{n+1}), \mathbb{Z}(G^n))$ by

$$Q = \sum_{\sigma\in\Pi(X_1)} \sum_{j=1}^{n+1} \big(\mathrm{sign}(S^{j-1}\sigma)\pi_0 S^{j-1}\sigma + (-1)^{n+1}\,\mathrm{sign}(S^j\sigma)\pi_{n+1}S^j\sigma\big),$$

$$Rg = \sum_{\sigma\in\Pi(X_1)} \sum_{j=1}^{n+1} \mathrm{sign}(S^j\sigma)\sum_{k=1}^{n}(-1)^k \pi_k S^j\sigma g \quad \text{for } g\in G^{n+1}.$$

So we have $\mathrm{AS}\circ d = Q + R$. We know that

$$\begin{aligned}
&\pi_0 S^{j-1}\sigma = \pi_{n+1}S^j\sigma \quad \text{for } 1\le j\le n,\\
&\mathrm{sign}(S^{j-1}\sigma)\pi_0 S^{j-1}\sigma + (-1)^{n+1}\,\mathrm{sign}(S^j\sigma)\pi_{n+1}S^j\sigma\\
&\qquad = (-1)^{n(j-1)}\,\mathrm{sign}(\sigma)\pi_0 S^{j-1}\sigma + (-1)^{n+1}(-1)^{nj}\,\mathrm{sign}(\sigma)\pi_{n+1}S^j\sigma = 0.
\end{aligned}$$

Thus we get $Q = 0$.

We need the notation $\sigma_{k,k+1}$ for the flip of k and $k+1$:

$$\sigma_{k,k+1} = (1, 2, \dots, k-2, k-1, k+1, k, k+2, k+3, \dots, n+1) \in \Pi(X).$$

Then we get

$$\mathrm{sign}(\sigma_{k,k+1}\rho)\pi_k\sigma_{k,k+1}\rho g + \mathrm{sign}(\rho)\pi_k\rho g = 0 \quad \text{for } \rho\in\Pi(X) \text{ and } 1\le k\le n.$$

Hence we come to

$$\begin{aligned}
R &= \sum_{\sigma\in\Pi(X_1)} \sum_{j=1}^{n+1} \mathrm{sign}(S^j\sigma)\sum_{k=1}^{n}(-1)^k\pi_k S^j\sigma\\
&= \sum_{k=1}^{n}(-1)^k \sum_{j=1}^{n+1}\sum_{\sigma\in\Pi(X_1)} \mathrm{sign}(S^j\sigma)\pi_k S^j\sigma = \sum_{k=1}^{n}(-1)^k \sum_{\rho\in\Pi(X)} \mathrm{sign}(\rho)\pi_k\rho\\
&= \sum_{k=1}^{n}(-1)^k \sum_{\rho\in\Pi_0(X)} \big(\mathrm{sign}(\rho)\pi_k\rho + \mathrm{sign}(\sigma_{k,k+1}\rho)\pi_k\sigma_{k,k+1}\rho\big) = 0,
\end{aligned}$$

where $\Pi_0(X)$ is the group of even permutations of X, that is, the alternating group. Therefore we conclude $\mathrm{AS}\circ d = Q + R = 0$. ♡

Let $\mathcal{A}$ be a G-module with action α. We recall the dimension shifting theorem and the dimension shift map ∂. First, a new G-module $\widetilde{\mathcal{A}}$ is defined through the following:

(i) $\mathrm{Map}(G, \mathcal{A})$ is the module $\mathcal{A}^G$ of all $\mathcal{A}$-valued functions on G with pointwise addition.

(ii) The group $\mathcal{A}$ is the submodule of $\mathrm{Map}(G, \mathcal{A})$ of constant $\mathcal{A}$-valued functions.

(iii) The action α of G on $\mathcal{A}$ extends to the enlarged additive group $\mathrm{Map}(G, \mathcal{A})$ by

$$(\alpha_h f)(g) = \alpha_h(f(gh)) \quad \text{for } f \in \mathrm{Map}(G, \mathcal{A}) \text{ and } g, h \in G.$$

(iv) Finally $\widetilde{\mathcal{A}}$ is the quotient G-module $\widetilde{\mathcal{A}} = \mathrm{Map}(G, \mathcal{A})/\mathcal{A}$.

Thus we obtain the equivariant short exact sequence

$$0 \to \mathcal{A} \longrightarrow \mathrm{Map}(G, \mathcal{A}) \longrightarrow \widetilde{\mathcal{A}} \to 0. \tag{2-7}$$

The short exact sequence (2-7) splits as follows:

(i) First, set $j(f)(g) = f(g) - f(e)$ for $f \in \mathrm{Map}(G, \mathcal{A})$ and $g \in G$, where $e \in G$ is the neutral element of G. Then the map j is a homomorphism of $\mathrm{Map}(G, \mathcal{A})$ onto the subgroup $\mathrm{Map}_0(G, \mathcal{A})$ of all $\mathcal{A}$-valued functions on G vanishing at e. Then we get $\mathrm{Ker}(j) = \mathcal{A} \subset \mathrm{Map}(G, \mathcal{A})$, so that the map j is viewed as a bijection from $\widetilde{\mathcal{A}}$ onto $\mathrm{Map}_0(G, \mathcal{A})$.

(ii) The map j transforms the action $\tilde{\alpha}$ of G on $\widetilde{\mathcal{A}}$ to the action, denoted by $\tilde{\alpha}$ again, on $\mathrm{Map}_0(G, \mathcal{A})$ defined by $(\tilde{\alpha}_h f)(g) = \alpha_h(f(gh)) - \alpha_h(f(h))$ for $g, h \in G$ and $f \in \mathrm{Map}_0(G, \mathcal{A})$.

With the map j, we will identify $\widetilde{\mathcal{A}}$ and $\mathrm{Map}_0(G, \mathcal{A})$. Thus we have a short exact sequence

$$0 \to \mathcal{A} \xrightarrow{\ i\ } \mathrm{Map}(G, \mathcal{A}) \underset{\mathfrak{s}}{\overset{j}{\rightleftarrows}} \widetilde{\mathcal{A}} = \mathrm{Map}_0(G, \mathcal{A}) \to 0.$$

Let $\mathfrak{s}$ denote the embedding of $\widetilde{\mathcal{A}} = \mathrm{Map}_0(G, \mathcal{A}) \hookrightarrow \mathrm{Map}(G, \mathcal{A})$, which is a right inverse of the map j. If $\tilde{u} \in \mathrm{Z}^{n-1}_\alpha(G, \widetilde{\mathcal{A}})$, then

$$0 = \partial_G \tilde{u} = j\ (\tilde{\partial}_G \mathfrak{s}(\tilde{u})),$$

where $\tilde{\partial}_G$ means the coboundary operator in $\mathrm{C}^n_{\tilde{\alpha}}(G, \mathrm{Map}(G, \mathcal{A}))$, so that we have $\partial_G \mathfrak{s}(\tilde{u}) \in \mathrm{Z}^n_\alpha(G, \mathcal{A})$. We denote the cohomology class $[\tilde{\partial}_G \mathfrak{s}(\tilde{u})] \in \mathrm{H}^n_\alpha(G, \mathcal{A})$ by $\partial[\tilde{u}]$ for each $[\tilde{u}] \in \mathrm{H}^{n-1}_{\tilde{\alpha}}(G, \widetilde{\mathcal{A}})$. It is known as the dimension shift theorem that the map ∂ is an isomorphism of $\mathrm{H}^{n-1}_{\tilde{\alpha}}(G, \widetilde{\mathcal{A}})$ onto $\mathrm{H}^n_\alpha(G, \mathcal{A})$.

Definition 2.3. Suppose that the group G admits a torsion-free central element $z_0 \in G$. A cocycle $c \in \mathrm{Z}^n_\alpha(G, \mathcal{A})$ is said to be of the *standard form* (relative to the central element z_0) if

(i) for each $k_1, \dots, k_n \in \mathbb{Z}$ and $g_1, g_2, \dots, g_n \in G$,

$$c(z_0^{k_1} g_1, \dots, z_0^{k_n} g_n) = \alpha_{g_1}(d_c(k_1; g_2, \dots, g_n)) + c(g_1, g_2, \dots, g_n); \tag{2-8}$$

(ii) the map $k \in \mathbb{Z} \mapsto d_c(k; g_2, g_3, \dots, g_n) \in \mathcal{A}$ belongs to $\mathrm{Z}^1_{\alpha_{z_0}}(\mathbb{Z}, \mathcal{A})$ for each $g_2, g_3, \dots, g_n \in G$, that is,

$$d_c(k+\ell; g_2, g_3, \dots, g_n) = d_c(k; g_2, g_3, \dots, g_n) + \alpha^k_{z_0}(d_c(\ell; g_2, g_3, \dots, g_n)). \tag{2-9}$$

(iii) For each $k \in \mathbb{Z}$ and $g_1, g_2, \dots, g_n \in G$, we have

$$(\partial_G d_c)(k; g_1, g_2, \dots, g_n) = \alpha^k_{z_0}(c(g_1, g_2, \dots, g_n)) - c(g_1, g_2, \dots, g_n). \tag{2-10}$$

Remark 2.4. If we choose d_c so that

$$c(z_0 g_1, z_0^{k_2} g_2, \dots, z_0^{k_n} g_n) = \alpha_{g_1}(d_c(g_2, g_3, \dots, g_n)) + c(g_1, g_2, \dots, g_n),$$
$$(\partial_G d_c)(g_1, g_2, \dots, g_n) = \alpha_{z_0}(c(g_1, g_2, \dots, g_n)) - c(g_1, g_2, \dots, g_n)$$

and we define $d_c(k; g_2, g_3, \dots, g_n)$ inductively by

$$d_c(k; g_2, g_3, \dots, g_n) = d_c(g_2, g_3, \dots, g_n) + \alpha_{z_0}(d_c(k-1; g_2, g_3, \dots, g_n)), \tag{2-11}$$

Then the cocycle identity (2-8) for $c(g_2, g_3, \dots, g_n)$ and $d_c(k; g_2, g_3, \dots, g_n)$ can be fulfilled automatically.

In the sequel, we often write $d_c(g_2, g_3, \dots, g_n)$ for the d-part of a standard cocycle c without referring to the first variable k in $d_c(k; g_2, g_3, \dots, g_n)$.

Lemma 2.5. *In the above context, every cocycle $c \in \mathrm{Z}^n_\alpha(G, \mathcal{A})$ is cohomologous to a cocycle c_{s} of the standard form.*

Proof. For $n = 1$, the cocycle identity $c(z_0^k g) = \alpha_g(c(z_0^k)) + c(g)$ for $k \in \mathbb{Z}$ and $g \in G$ shows that with $d_c(k) = c(z_0^k)$ the cochains d_c and c satisfy Definition 2.3(i). Now we have

$$\alpha^k_{z_0}(c(g)) - c(g) = c(z_0^k g) - c(z_0^k) - c(g) = c(g) + \alpha_g(c(z_0^k)) - c(z_0^k) - c(g)$$
$$= \alpha_g(d_c(k)) - d_c(k) = (\partial_G d_c)(k; g),$$

which shows the property of Definition 2.3(iii) for c and d_c.

Now assume our claim is valid for $1, \dots, n-1$ and for any G-module $\{\mathcal{A}, \alpha\}$. Choose an equivariant short exact sequence

$$0 \to \mathcal{A} \xrightarrow{\ i\ } M \underset{\mathfrak{s}}{\overset{j}{\rightleftarrows}} \widetilde{\mathcal{A}} \to 0$$

such that $\mathrm{H}^n_\alpha(G, M) = \{0\}$ for $n \geq 1$, and the cross-section $\mathfrak{s} : \widetilde{\mathcal{A}} \to M$ is a homomorphism of $\widetilde{\mathcal{A}}$ into M, but is not equivariant. Then $\partial_G \mathfrak{s} : \mathrm{Z}^{n-1}_\alpha(G, \widetilde{\mathcal{A}}) \to \mathrm{Z}^n_\alpha(G, \mathcal{A})$ gives rise to an isomorphism $\partial : \mathrm{H}^{n-1}_\alpha(G, \widetilde{\mathcal{A}}) \to \mathrm{H}^n_\alpha(G, \mathcal{A})$. For a standard cocycle

$\tilde{c} \in \mathrm{Z}_\alpha^{n-1}(G, \widetilde{\mathcal{A}})$, we set, for each $z_0^{k_1} g_1, \dots, z_0^{k_{n-1}} g_{n-1} \in G$,

$$\bar{c}(z_0^{k_1} g_1, \dots, z_0^{k_{n-1}} g_{n-1}) = \alpha_{g_1}(\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, g_3, \dots, g_{n-1}))) + \mathfrak{s}(\tilde{c}(g_1, g_2, g_3, \dots, g_{n-1})).$$

Since $j(\bar{c}) = \tilde{c}$, we have $c = \partial_G \bar{c} \in \mathrm{Z}_\alpha^n(G, \mathcal{A})$. We then compute

$$\begin{aligned}
c(z_0^{k_1} g_1, \dots, z_0^{k_n} g_n) &= (\partial_G \bar{c})(z_0^{k_1} g_1, \dots, z_0^{k_n} g_n) \\
&= \alpha_{z_0^{k_1} g_1}\big(\bar{c}(z_0^{k_2} g_2, z_0^{k_3} g_3, \dots, z_0^{k_n} g_n)\big) \\
&\qquad + \sum_{j=1}^{n-1} (-1)^j \bar{c}(z_0^{k_1} g_1, \dots, z_0^{k_j} g_j z_0^{k_{j+1}} g_{j+1}, \dots, g_n) \\
&\qquad + (-1)^n \bar{c}(z_0^{k_1} g_1, \dots, z_0^{k_{n-1}} g_{n-1}) \\
&= \alpha_{z_0^{k_1} g_1}\big(\alpha_{g_2}(\mathfrak{s}(d_{\tilde{c}}(k_2; g_3, \dots, g_n))) + \bar{c}(g_2, g_3, \dots, g_n)\big) \\
&\quad - \big(\alpha_{g_1 g_2}(\mathfrak{s}(d_{\tilde{c}}(k_1 + k_2; g_3, \dots, g_n))) + \bar{c}(g_1 g_2, g_3, \dots, g_n)\big) \\
&\quad + \sum_{j=2}^{n-1} (-1)^j \big(\alpha_{g_1}(\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, \dots, g_j g_{j+1}, \dots, g_n))) \\
&\qquad + \bar{c}(g_1, \dots, g_j g_{j+1}, \dots, g_n)\big) \\
&\quad + (-1)^n \alpha_{g_1}\big(\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, g_3, \dots, g_{n-1}))\big) + (-1)^n \bar{c}(g_1, g_2, g_3, \dots, g_{n-1}) \\
&= (\partial_G \bar{c})(g_1, g_2, \dots, g_n) + \alpha_{z_0^{k_1} g_1}(\alpha_{g_2}(\mathfrak{s}(d_{\tilde{c}}(k_2; g_3, \dots, g_n)))) \\
&\quad + \alpha_{g_1}\big(\alpha_{z_0}^{k_1}(\bar{c}(g_2, g_3, \dots, g_n)) - \bar{c}(g_2, g_3, \dots, g_n)\big) \\
&\quad - \alpha_{g_1 g_2}\big(\mathfrak{s}(d_{\tilde{c}}(k_1; g_3, \dots, g_n)) + \alpha_{z_0}^{k_1}(\mathfrak{s}(d_{\tilde{c}}(k_2; g_3, \dots, g_n)))\big) \\
&\quad + \sum_{j=2}^{n-1} (-1)^j \alpha_{g_1}(\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, \dots, g_j g_{j+1}, \dots, g_n))) \\
&\quad + (-1)^n \big(\alpha_{g_1}(\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, g_3, \dots, g_{n-1})))\big) \\
&= (\partial_G \bar{c})(g_1, g_2, \dots, g_n) \\
&\quad + \alpha_{g_1}\Big(\alpha_{z_0}^{k_1}(\bar{c}(g_2, g_3, \dots, g_n)) - \bar{c}(g_2, g_3, \dots, g_n) - \alpha_{g_2}\big(\mathfrak{s}(d_{\tilde{c}}(k_1; g_3, \dots, g_n))\big) \\
&\qquad + \sum_{j=2}^{n-1} (-1)^j \mathfrak{s}(d_{\tilde{c}}(k_1; g_2, \dots, g_j g_{j+1}, \dots, g_n)) \\
&\qquad + (-1)^n (\mathfrak{s}(d_{\tilde{c}}(k_1; g_2, g_3, \dots, g_{n-1})))\Big) \\
&= (\partial_G \bar{c})(g_1, g_2, \dots, g_n) \\
&\quad + \alpha_{g_1}\Big(\alpha_{z_0}^{k_1}\big(\bar{c}(g_2, g_3, \dots, g_n) - \bar{c}(g_2, g_3, \dots, g_n)\big) \\
&\qquad - \partial_G(\mathfrak{s} \circ d_{\tilde{c}})(k_1; g_2, g_3, \dots, g_n)\Big).
\end{aligned}$$

Consequently, we get

$$c(z_0^{k_1}g_1, \dots, z_0^{k_n}g_n) = \alpha_{g_1}(d_c(k_1; g_2, g_3, \dots, g_n)) + c(g_1, g_2, \dots, g_n)$$

with

$$\begin{aligned} c(g_1, g_2, \dots, g_n) &= (\partial_G \bar{c})(g_1, g_2, \dots, g_n), \\ d_c(g_2, g_3, \dots, g_n) &= \alpha_{z_0}(\bar{c}(g_2, g_3, \dots, g_n)) - \bar{c}(g_2, g_3, \dots, g_n) \\ &\qquad - \partial_G(\mathfrak{s} \circ d_{\tilde{c}})(g_2, g_3, \dots, g_n). \end{aligned}$$

We now check the requirement (2-10) for d_c and c:

$$\begin{aligned} \alpha_{z_0}(c(g_1, g_2, \dots, g_n)) - c(g_1, g_2, \dots, g_n) \\ &= \alpha_{z_0}(\partial_G \bar{c}(g_1, g_2, \dots, g_n)) - \partial_G \bar{c}(g_1, g_2, \dots, g_n) \\ &= \partial_G\big(\alpha_{z_0}(\bar{c}(g_1, g_2, \dots, g_n)) - \bar{c}(g_1, g_2, \dots, g_n)\big) \\ &= \partial_G\big(d_c(g_2, g_3, \dots, g_n) + \partial_G \mathfrak{s} \circ d_{\tilde{c}}(g_2, g_3, \dots, g_n)\big) \\ &= \partial_G d_c(g_2, g_3, \dots, g_n). \end{aligned}$$

Thus the cocycle c is standard. ♡

We now state the main result on the asymmetrization, which extends the work of Olesen, Pedersen, and Takesaki [Olesen et al. 1980]:

Theorem 2.6. *Let Q be a countable torsion-free abelian group.*

(i) *The asymmetrization* AS *maps the group $\mathrm{Z}^n(Q, \mathbb{T})$ of $\mathbb{T}$-valued n-th cocycles onto the compact group $X^n(Q, \mathbb{T})$ of all asymmetric multicharacters on n variables of Q.*

(ii) *The following sequence is exact for each $n \in \mathbb{N}$:*

$$1 \to \mathrm{B}^n(Q, \mathbb{T}) \longrightarrow \mathrm{Z}^n(Q, \mathbb{T}) \xrightarrow{\mathrm{AS}} X^n(Q, \mathbb{T}) \to 1.$$

Consequently,

$$\mathrm{H}^n(\mathbb{Z}^m, \mathbb{T}) \cong X^n(\mathbb{Z}^m, \mathbb{T}) \cong \begin{cases} \mathbb{T}^{m!/(n!(m-n)!)} & \text{if } m \geq n \\ 0 & \text{if } m < n. \end{cases}$$

More generally, if Q is a countable torsion-free abelian group, then the cohomology group $\mathrm{H}^n(Q, \mathbb{T})$ is naturally isomorphic to the Pontrjagin–Kampen dual of the n-th exterior power $Q \wedge Q \wedge \cdots \wedge Q$ of Q.

(iii) *The group $X^n(Q, \mathbb{T})$ is a subgroup of $\mathrm{Z}^n(Q, \mathbb{T})$ such that*

$$\mathrm{Z}^n(Q, \mathbb{T}) = X^n(Q, \mathbb{T})\mathrm{B}^n(Q, \mathbb{T}), \quad X^n(Q, \mathbb{T}) \cap \mathrm{B}^n(Q, \mathbb{T}) = \mathrm{Ker}(\mathrm{Power}\, n!),$$

and $\mathrm{AS}\, c = c^{n!}$ for $c \in X^n(Q, \mathbb{T})$.

Remark 2.7. If the group Q has torsion, then the theorem fails as seen in the case that $Q=\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}$ for $p\geq 2$, $\mathrm{H}^3(Q,\mathbb{T})\cong\mathbb{Z}_p$ and $X^3(Q,\mathbb{T})=\{0\}$.

For the proof, we need some preparation. First, if $n=1$, then the claim is trivially true for any abelian group Q with no assumption on torsion. We then assume the claim is true for cocycle dimension $1,\dots,n-1$ with $n\in\mathbb{N}$ fixed and for any torsion-free abelian group Q. With this induction hypothesis, we prepare a couple of lemmas for cocycle dimension n.

Lemma 2.8. (i) *If M is an abelian group such that a cocycle $c\in\mathrm{Z}^n(M,\mathbb{T})$ is a coboundary if and only if* $\mathrm{AS}\,c=1$, *then the same is true for the product group $Q=M\times\mathbb{Z}$.*

(ii) *If M is an abelian group such that the asymmetrization* $\mathrm{AS}\,c$ *of each cocycle $c\in\mathrm{Z}^n(M,\mathbb{T})$ is a multicharacter, then the same is true for the product group $Q=M\times\mathbb{Z}$.*

Proof. Let z_0 denote the element of Q corresponding to the product decomposition $Q=M\times\mathbb{Z}$, so that every element $q\in Q$ is written uniquely in the form $q=mz_0^k$ for $m\in M$ and $k\in\mathbb{Z}$.

(i) In Lemma 2.2, we proved the triviality of the asymmetrization of a coboundary. Thus we prove the converse. Suppose $\mathrm{AS}\,c=1$ for $c\in\mathrm{Z}^n(Q,\mathbb{T})$. By Lemma 2.5 the cocycle c is cohomologous to a cocycle c_{s} of standard form, and $\mathrm{AS}\,c_{\mathrm{s}}=\mathrm{AS}\,c=1$ by Lemma 2.2. So we may and do assume that c is standard:

$$c(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)=d_c(p_2,p_3,\dots,p_n)^{\ell_1}c_M(p_1,p_2,\dots,p_n),$$

where $\tilde{p}_i=p_iz_0^{\ell_i}\in Q=M\times\mathbb{Z}$. As Q does not act on $\mathbb{T}$, the d-part d_c is a cocycle in $\mathrm{Z}^{n-1}(Q,\mathbb{T})$.

We look at the asymmetrization of c:

$$\begin{aligned}(\mathrm{AS}\,c)(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)&=\prod_{\sigma\in S_n}\Big(d_c(p_{\sigma(2)},p_{\sigma(3)},\dots,p_{\sigma(n)})^{\ell_{\sigma(1)}}\\&\qquad\qquad\times c_M(p_{\sigma(1)},p_{\sigma(2)},\dots,p_{\sigma(n)})\Big)^{\operatorname{sign}\sigma}\\&=\prod_{\sigma\in S_n}d_c(p_{\sigma(2)},p_{\sigma(3)},\dots,p_{\sigma(n)})^{\ell_{\sigma(1)}\operatorname{sign}\sigma}\\&\qquad\qquad\times\prod_{\sigma\in S_n}c_M(p_{\sigma(1)},p_{\sigma(2)},\dots,p_{\sigma(n)})^{\operatorname{sign}\sigma},\end{aligned}$$

that is,

$$(\mathrm{AS}\,c)(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)=\prod_{\sigma\in S_n}d_c(p_{\sigma(2)},p_{\sigma(3)},\dots,p_{\sigma(n)})^{\ell_{\sigma(1)}\operatorname{sign}\sigma}\times(\mathrm{AS}\,c_M)(p_1,p_2,\dots,p_n).\tag{2-12}$$

To compute the first term of the above expression, we take a closer look at the permutation group S_n. In particular, we have to pay attention to the fact that the first term in the variables of d_c is missing. To this end, we fix k with $1 \le k \le n$, which represents the missing term in d_c, and consider the cyclic permutation

$$S_{n-1}(k) = (1, 2, \dots, k-1, k+1, \dots, n) \in \Pi(\{1, 2, \dots, k-1, k+1, \dots, n\}).$$

For $\sigma = (k, \sigma(2), \sigma(3), \dots, \sigma(n)) \in S_n$, define ρ, $\tilde{\rho}$ and $\tilde{\sigma}$ through

$$\begin{aligned}
\rho &= S^{(n-k+1)}\sigma \\
&= \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n \\ \sigma(n-k+2) & \sigma(n-k+3) & \cdots & \sigma(n) & k & \sigma(2) & \cdots & \sigma(n-k+1) \end{pmatrix}, \\
\tilde{\rho} &= \begin{pmatrix} 1 & 2 & \cdots & k-1 & k+1 & \cdots & n \\ \sigma(n-k+2) & \sigma(n-k+3) & \cdots & \sigma(n) & \sigma(2) & \cdots & \sigma(n-k+1) \end{pmatrix}, \\
\tilde{\sigma} &= S_{n-1}(k)^{k-1}\tilde{\rho} \\
&= \begin{pmatrix} 1 & 2 & \cdots & k-1 & k+1 & \cdots & n \\ \sigma(2) & \sigma(3) & \cdots & \sigma(k) & \sigma(k+1) & \cdots & \sigma(n) \end{pmatrix} = (\sigma(2), \sigma(3), \dots, \sigma(n)).
\end{aligned}$$

Then observing $\operatorname{sign}\tilde{\rho} = \operatorname{sign}\rho$, we compute

$$\begin{aligned}
\operatorname{sign}\sigma &= \operatorname{sign} S^{k-1} \operatorname{sign}\rho = (-1)^{(n-1)(k-1)} \operatorname{sign}\tilde{\rho} \\
&= (-1)^{(n-1)(k-1)} \operatorname{sign}(S_{n-1}(k)^{n-k}) \operatorname{sign}\tilde{\sigma} \\
&= (-1)^{(n-1)(k-1)+(n-2)(n-k)} \operatorname{sign}\tilde{\sigma} = (-1)^{k-1} \operatorname{sign}\tilde{\sigma}.
\end{aligned}$$

Hence the first term of (2-12) becomes

$$\begin{aligned}
&\prod_{\sigma \in S_n} \big(d_c(p_{\sigma(2)}, p_{\sigma(3)}, \dots, p_{\sigma(n)})\big)^{\ell_{\sigma(1)} \operatorname{sign}\sigma} \\
&\qquad = \prod_{k=1}^{n} \Bigg(\prod_{\tilde{\sigma} \in S_{n-1}(k)} \big(d_c(p_{\tilde{\sigma}(1)}, p_{\tilde{\sigma}(2)}, \dots, p_{\tilde{\sigma}(n-1)})\big)^{\operatorname{sign}\tilde{\sigma}} \Bigg)^{\ell_k (-1)^{k-1}} \\
&\qquad = \prod_{k=1}^{n} \big((\mathrm{AS}\, d_c)(p_1, p_2, \dots, \breve{p}_k, \dots, p_n)\big)^{\ell_k (-1)^{k-1}}
\end{aligned}$$

where the notation $\smile$ stands for removing the corresponding variable. Thus (2-12) is replaced by

$$\begin{aligned}
\text{(2-12}'\text{)} \quad (\mathrm{AS}\, c)(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) &= \prod_{k=1}^{n} \big((\mathrm{AS}\, d_c)(p_1, p_2, \dots, \breve{p}_k, \dots, p_n)\big)^{\ell_k (-1)^{k-1}} \\
&\qquad \times (\mathrm{AS}\, c_M)(p_1, p_2, \dots, p_n).
\end{aligned}$$

The condition $\mathrm{AS}\,c = 1$ yields that $\mathrm{AS}\,c_M = 1$ with $\ell_k = 0$ for $k = 1, \dots, n$ and $\mathrm{AS}\,d_c = 1$ with $\ell_1 = 1$ and $\ell_k = 0$ for $k = 2, \dots, n$ and $p_1 = e$. Hence c_M and d_c are both coboundaries by the induction hypothesis. Choose $b \in \mathrm{C}^{n-1}(M, \mathbb{T})$ and $a \in \mathrm{C}^{n-2}(M, \mathbb{T})$ such that $c_M = \partial_M b$ and $d_c = \partial_M a$. Then the cocycle c has the form

$$\begin{aligned} c(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) &= d_c(p_2, p_3, \dots, p_n)^{\ell_1} c(p_1, p_2, \dots, p_n) \\ &= ((\partial_M a)(p_2, p_3, \dots, p_n))^{\ell_1} (\partial_M b)(p_1, p_2, \dots, p_n). \end{aligned}$$

Setting $f(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{n-1}) = a(p_2, p_3, \dots, p_{n-1})^{-\ell_1} b(p_1, p_2, \dots, p_{n-1})$ where $\tilde{p}_i = z_0^{\ell_i} p_i \in Q$ for $i = 1, \dots, n-1$, we compute

$$\begin{aligned} &(\partial_Q f)(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \\ &\quad = f(\tilde{p}_2, \tilde{p}_3, \dots, \tilde{p}_n) \times \prod_{k=1}^{n-1} f(\tilde{p}_1, \dots, \tilde{p}_k \tilde{p}_{k+1}, \dots, \tilde{p}_n)^{(-1)^k} \\ &\qquad\qquad \times f(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{n-1})^{(-1)^n} \\ &\quad = a(p_3, \dots, p_n)^{-\ell_2} a(p_3, \dots, p_n)^{\ell_1+\ell_2} \\ &\qquad \times \prod_{k=2}^{n-1} a(p_2, \dots, p_k p_{k+1}, \dots, p_n)^{-\ell_1(-1)^k} \\ &\qquad\qquad \times a(p_2, p_3, \dots, p_{n-1})^{-\ell_1(-1)^n} \\ &\qquad \times b(p_2, p_3, \dots, p_n) \times \prod_{k=1}^{n-1} b(p_1, \dots, p_k p_{k+1}, \dots, p_n)^{(-1)^k} \\ &\qquad\qquad \times b(p_1, p_3, \dots, p_n)^{(-1)^n} \\ &\quad = a(p_3, \dots, p_n)^{\ell_1} \prod_{k=2}^{n-1} a(p_2, \dots, p_k p_{k+1}, \dots, p_n)^{-\ell_1(-1)^k} \\ &\qquad\qquad \times a(p_2, p_3, \dots, p_{n-1})^{-\ell_1(-1)^n} \times (\partial_M b)(p_1, p_2, \dots, p_n) \\ &\quad = ((\partial_M a)(p_2, p_3, \dots, p_n))^{\ell_1} (\partial_M b)(p_1, p_2, \dots, p_n) \\ &\quad = c(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n). \end{aligned}$$

Therefore c is a coboundary. This completes the proof of part (i).

(ii) Fix a standard cocycle $c \in \mathrm{Z}^n(Q, \mathbb{T})$ by

$$c(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = d_c(p_2, p_3, \dots, p_n)^{\ell_1} c(p_1, p_2, \dots, p_n)$$

with $d_c \in \mathrm{Z}^{n-1}(M, \mathbb{T})$ and $c_M \in \mathrm{Z}^n(M, \mathbb{T})$. Observing that $\mathrm{AS}\,c_M$ and $\mathrm{AS}\,d_c$ are both multicharacters by the assumptions, we compute with (2-12′), for $\tilde{q}_1 = q_1 z_0^{k_1}$,

$$
\begin{aligned}
&(\mathrm{AS}\, c)(\tilde p_1\tilde q_1, \tilde p_2, \dots, \tilde p_n)\\
&\quad = (\mathrm{AS}\, d_c)(p_2, \dots, p_n)^{\ell_1+k_1}\\
&\qquad \times \prod_{j=2}^{n}\big((\mathrm{AS}\, d_c)(p_1q_1, p_2, \dots, \breve p_j, \dots, p_n)\big)^{\ell_j(-1)^{j-1}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (\mathrm{AS}\, c_M)(p_1q_1, p_2, \dots, p_n)\\
&\quad = (\mathrm{AS}\, d_c)(p_2, \dots, p_n)^{\ell_1}\\
&\qquad \times \prod_{j=2}^{n}\big((\mathrm{AS}\, d_c)(p_1, p_2, \dots, \breve p_j, \dots, p_n)\big)^{\ell_j(-1)^{j-1}}\\
&\qquad \times (\mathrm{AS}\, d_c)(p_2, \dots, p_n)^{k_1}\\
&\qquad\quad \times \prod_{j=2}^{n}\big((\mathrm{AS}\, d_c)(q_1, p_2, \dots, \breve p_j, \dots, p_n)\big)^{\ell_j(-1)^{j-1}}\\
&\qquad\qquad\qquad \times (\mathrm{AS}\, c_M)(p_1, p_2, \dots, p_n)(\mathrm{AS}\, c_M)(q_1, p_2, \dots, p_n)\\
&\quad = (\mathrm{AS}\, c)(\tilde p_1, \tilde p_2, \dots, \tilde p_n)(\mathrm{AS}\, c)(\tilde q_1, \tilde p_2, \dots, \tilde p_n).
\end{aligned}
$$

Thus AS c is indeed multiplicative on the first variable, so that it is an asymmetric multicharacter of $Q = M \times \mathbb{Z}$. ♡

Lemma 2.9. *Suppose that* $c \in \mathrm{Z}^n(Q, \mathbb{T})$ *has a trivial asymmetrization, that is,* $\mathrm{AS}\, c = 1$. *Assume the following*:

(a) *M is a finitely generated subgroup of Q*;

(b) *a_0 is in Q but not M*;

(c) *$f \in \mathrm{C}^{n-1}(M, \mathbb{T})$ cobounds the restriction c_M of c to M, that is, $\partial_M f = c_M$.*

Then the cochain f has an extension to the subgroup $N = \langle M, a_0\rangle$ generated by M and a_0 such that $\partial_N f = c_N$, where c_N is the restriction of c to the subgroup N.

Proof. To apply the structure theory of abelian groups, we use the additive group operation in the group Q. From the general theory of abelian groups, it follows that M and N are both free abelian groups and there exists a free basis $\{z_1, z_2, \dots, z_m\}$ of N and nonnegative integers $\{p_1, p_2, \dots, p_r\} \subset \mathbb{Z}_+$ for $1 \le r \le m$, such that $N = \langle z_1, z_2, \dots, z_m\rangle$ and $M = \langle p_1 z_1, \dots, p_r z_r\rangle$. With the assumption for $n-1$, every $(n-1)$-cocycle $\mu \in \mathrm{Z}^{n-1}(M, \mathbb{T})$ is cohomologous to an asymmetric multicharacter μ_{a}, that is, there exist $a_{i_1, i_2, \dots, i_r} \in \mathbb{R}$ such that

$$
\mu_{\mathrm{a}}(g_1, g_2, \dots, g_{n-1}) = \exp\Big(2\pi \mathrm{i} \sum_{\substack{i_j \in \{1,2,\dots,r\}\\ 1 \le i_1 < i_2 < \cdots < i_{n-1} \le n-1}} a_{i_1, i_2, \dots, i_{n-1}} \big(e_{i_1,M} \wedge e_{i_2,M} \wedge \cdots \wedge e_{i_{n-1},M}\big)(g_1, g_2, \dots, g_{n-1})\Big),
$$

where $\{e_{i,M} : 1 \le i \le r\}$ is the coordinate system of M relative to the basis $\{p_1 z_1, \dots, p_r z_r\}$. Setting

$$\nu_a(g_1, g_2, \dots, g_{n-1}) = \exp\Big(2\pi \mathrm{i} \sum_{\substack{i_j \in \{1,2,\dots,r\} \\ 1 \le i_1 < i_2 < \cdots < i_{n-1} \le n-1}} \frac{a_{i_1,i_2,\dots,i_{n-1}}}{p_{i_1} p_{i_2} \cdots p_{i_{n-1}}} \Big(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{n-1}}\Big)(g_1, g_2, \dots, g_{n-1})\Big),$$

where $\{e_i : 1 \le i \le m\}$ is the coordinate system of N in the basis $\{z_1, \dots, z_m\}$, we obtain an extension ν of μ_{a}. Choose $\xi \in \mathrm{C}^{n-2}(M, \mathbb{T})$ so that $\mu = (\partial_M \xi)\mu_{\mathrm{a}}$, and extend ξ to a cochain $\xi \in \mathrm{C}^{n-2}(N, \mathbb{T})$. Then the second cocycle $(\partial_N \xi)\nu$ gives an extension of the original $(n-1)$-cocycle $\mu \in \mathrm{Z}^{n-1}(M, \mathbb{T})$. Thus we obtain the surjectivity of the restriction map $\mathrm{res} : \mu \in \mathrm{Z}^{n-1}(N, \mathbb{T}) \mapsto \mu_M \in \mathrm{Z}^{n-1}(M, \mathbb{T})$, that is, the exactness of the sequence

$$\mathrm{Z}^{n-1}(N, \mathbb{T}) \xrightarrow{\mathrm{res}} \mathrm{Z}^{n-1}(M, \mathbb{T}) \to 1.$$

By induction on generators, Lemma 2.9 yields that the restriction c_N of c to N is a coboundary. Hence there exists $\xi \in \mathrm{C}^{n-1}(N, \mathbb{T})$ such that $c_N = \partial_N \xi$. Then we have $\partial_M f = c_M = \partial_M \xi_M$, so $\mu_M = \xi_M^{-1} f \in \mathrm{Z}^{n-1}(M, \mathbb{T})$. By the first arguments, we can extend μ_M to an element $\nu \in \mathrm{Z}^{n-1}(N, \mathbb{T})$. Set $f = \nu\xi \in \mathrm{C}^{n-1}(N, \mathbb{T})$. The newly defined cochain f on N extends the original $f \in \mathrm{C}^{n-1}(M, \mathbb{T})$ and cobounds the cocycle c_N, that is, $\partial_N f = (\partial_N \nu)(\partial_N \xi) = \partial_N \xi = c_N$. ♡

We may now complete the proof of Theorem 2.6 by proceeding from cocycle dimension $1, \dots, n-1$ to the cocycle dimension n.

Proof of Theorem 2.6. Suppose that $c \in \mathrm{Z}^n(Q, \mathbb{T})$ and $\mathrm{AS}\, c = 1$. Let $\{z_k : k \in \mathbb{N}\}$ be a sequence of generators of Q and let $M_m = \langle z_1, z_2, \dots, z_m \rangle$ for $m \in \mathbb{N}$. The sequence $\{M_m\}$ is then increasing and $Q = \bigcup M_m$. The triviality assumption $\mathrm{AS}\, c = 1$ and Lemma 2.8(i) yield that the restriction c_m of the cocycle c to each M_m is a coboundary, so that there exists $f_m \in \mathrm{C}^{n-1}(M_m, \mathbb{T})$ such that $c_m = \partial_{M_m} f_m$. The last lemma however allows us to choose the sequence $\{f_m\}$ so that each f_m is an extension of the previous f_{m-1}. Hence the sequence $\{f_m\}$ gives a cochain $f \in \mathrm{C}^{n-1}(Q, \mathbb{T})$ such that $f|_{M_m} = f_m$ for $m \in \mathbb{N}$, and therefore $\partial_Q f = c$. Thus we conclude that $\mathrm{Ker(AS)} \subset \mathrm{B}^n(Q, \mathbb{T})$. The inclusion $\mathrm{Ker(AS)} \supset \mathrm{B}^n(Q, \mathbb{T})$ was proved in Lemma 2.2. Hence $\mathrm{Ker(AS)} = \mathrm{B}^n(Q, \mathbb{T})$.

Lemma 2.8(ii) for $\{M_m\}_{m \in \mathbb{N}}$ yields that the asymmetrization $\mathrm{AS}\, c$ is a multi-character for any $c \in \mathrm{Z}^n(Q, \mathbb{T})$.

Set $c_{\mathrm{a}} = \mathrm{AS}\, c$ for an arbitrary cocycle $c \in \mathrm{Z}^n(Q, \mathbb{T})$. Then $c_{\mathrm{a}} \in X^n(Q, \mathbb{T})$. Since Q is torsion free, the group $X^n(Q, \mathbb{T})$ is indefinitely divisible. So the $n!$-fold power mapping $\xi \in X^n(Q, \mathbb{T}) \mapsto \xi^{n!} \in X^n(Q, \mathbb{T})$ is surjective. But the asymmetrization AS on $X^n(Q, \mathbb{T})$ is precisely the $n!$-fold power. Hence there exists $\xi \in X^n(Q, \mathbb{T})$

such that $\mathrm{AS}\,\xi = \xi^{n!} = c_{\mathrm{a}}$. Now we have $\mathrm{AS}(\xi^{-1}c) = \xi^{-n!}c_{\mathrm{a}} = 1$. Therefore $\xi^{-1}c \in \mathrm{B}^n(Q, \mathbb{T})$. Consequently, we conclude

$$\mathrm{Z}^n(Q, \mathbb{T}) = X^n(Q, \mathbb{T})\mathrm{B}^n(Q, \mathbb{T}),$$
$$X^n(Q, \mathbb{T}) \cap \mathrm{B}^n(Q, \mathbb{T}) = X^n(Q, \mathbb{T}) \cap \mathrm{Ker}(\mathrm{AS}) = \{c \in X^n(Q, \mathbb{T}) : c^{n!} = 1\}. \quad \heartsuit$$

Corollary 2.10. *If G is a discrete abelian group, then the asymmetrization of every n-cocycle $c \in \mathrm{Z}^n(G, \mathbb{T})$ is a multicharacter, that is,* $\mathrm{AS}\,c \in X^n(G, \mathbb{T})$.

Proof. Let F be a free abelian group large enough so that there exists a surjective homomorphism $\pi : F \to G$. Consider the pull back $\pi^*(c)$ and its asymmetrization, $\mathrm{AS}\,\pi^*(c) = \pi^*(\mathrm{AS}\,c)$. It follows from Theorem 2.6 that the pull back $\pi^*(\mathrm{AS}\,c)$ is a multicharacter of F; consequently the original asymmetrization $\mathrm{AS}\,c$ is a multicharacter of G. ♡

3. Universal resolution for a countable discrete abelian group

We discuss a universal resolution group for a countable discrete abelian group. We consider only the case that the abelian group under consideration has infinitely many generators since the finitely generated case can be covered by the infinite generator case. Let $G = \mathbb{Z}^{<\mathbb{N}}$ be the free abelian group of a finite sequences of integers, that is, every element $g \in G$ is of the form

$$g = (g_1, g_2, \dots, g_i, \dots, g_\ell, 0, 0, \dots) \quad \text{for } g_i \in \mathbb{Z},$$

with $\ell = \ell(g) \in \mathbb{N}$, the index of the last nonzero term of $g \in \mathbb{Z}^{<\mathbb{N}}$. With

$$a_i = (0, 0, \dots, 0, 1, 0, 0, \dots), \tag{3-1}$$

where the 1 is in the i-th slot, every element $g \in \mathbb{Z}^{<\mathbb{N}}$ is written uniquely

$$g = \sum_{i \in \mathbb{N}} e_i(g) a_i. \tag{3-2}$$

We call $\{a_i : i \in \mathbb{N}\}$ the *standard basis* of $\mathbb{Z}^{<\mathbb{N}}$. We also fix a subgroup N of G that is generated by a sequence $\{p_i a_i : i \in \mathbb{N}\}$ with $p_i \in \mathbb{Z}_+$ and $i \in \mathbb{N}$. We will use the matrix

$$P = \begin{pmatrix} p_1 & 0 & 0 & \cdots \\ 0 & p_2 & 0 & \cdots \\ 0 & 0 & p_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{so that } N = P\mathbb{Z}^{<\mathbb{N}}.$$

Let M be the additive group of upper triangular matrices with integer coefficients, that is,

$$M = \left\{ m = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & \cdots \\ 0 & 0 & m_{23} & m_{24} & \cdots \\ 0 & 0 & 0 & \ddots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix} : m_{i,j} \in \mathbb{Z} \right\},$$

and set $e_{j,k}(m) = m_{jk}$ for $j < k$ and $m \in M$. For $i < j$, let $a_i \wedge a_j$ be the element of M such that $e_{k,\ell}(a_i \wedge a_j) = \delta_{ik}\delta_{j\ell}$, that is, the matrix with only (i, j)-component 1 and all others 0; equivalently $a_i \wedge a_j$ with $i < j$ is the (i, j)-matrix unit of M. Let $\mathfrak{n}_M$ be the M-valued second cocycle of G defined by

$$e_{j,k}(\mathfrak{n}_M(g; h)) = e_j(g)e_k(h) \quad \text{for } g, h \in G \text{ and } 1 \le j < k,$$

(3-3)
$$\mathfrak{n}_M(g; h) = \begin{pmatrix} 0 & e_1(g)e_2(h) & e_1(g)e_3(h) & e_1(g)e_4(h) & \cdots \\ 0 & 0 & e_2(g)e_3(h) & e_2(g)e_4(h) & \cdots \\ 0 & 0 & \ddots & e_3(g)e_4(h) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \end{pmatrix}.$$

Let H be the group extension of G associated with $\mathfrak{n}_M \in \mathrm{Z}^2(G, M)$:

$$H = M \times_{\mathfrak{n}_M} G \quad \text{and} \quad L = M \times_{\mathfrak{n}_M} N.$$

The group operation in H is given by $(m, g)(n, h) = (m+n+\mathfrak{n}_M(g; h), g+h)$ for $(m, g), (n, h) \in H$. The inverse $(m, g)^{-1}$ is given by

$$(m, g)^{-1} = (-m + \mathfrak{n}_M(g, -g), -g)$$

because $(0, 0) = (m, g)(m', g') = (m + m' + \mathfrak{n}_M(g; g'), g + g')$, $g' = -g$ and $m' = -m + \mathfrak{n}_M(g; g)$. To determine the commutator subgroup $[H, H]$, we take $(m, g), (n, h) \in H$ and compute

$$\begin{aligned}
&(m, g)(n, h)(m, g)^{-1}(n, h)^{-1} \\
&\quad = (m, g)(n, h)(-m + \mathfrak{n}_M(g; g), -g)(-n + \mathfrak{n}_M(h; h); -h) \\
&\quad = (m + n + \mathfrak{n}_M(g, h), g + h) \\
&\qquad\qquad \times (-m - n + \mathfrak{n}_M(g; g) + \mathfrak{n}_M(h; h) + \mathfrak{n}_M(g; h), -g - h) \\
&\quad = (\mathfrak{n}_M(g; h) + \mathfrak{n}_M(g; g) + \mathfrak{n}_M(h; h) + \mathfrak{n}_M(g; h) + \mathfrak{n}_M(g + h; -(g + h)), 0) \\
&\quad = (\mathfrak{n}_M(g; h) - \mathfrak{n}_M(h; g), 0) \\
&\quad = \Big(\sum_{j<k} (e_j(g)e_k(h) - e_j(h)e_k(g))(a_j \wedge a_k), 0\Big).
\end{aligned}$$

Lemma 3.1. *The commutator subgroup $[H, H]$ of H is the center M of H.*

Proof. From the computation above, it follows that for each pair $j < k$

$$\mathfrak{s}_H(a_j)\mathfrak{s}_H(a_k)\mathfrak{s}_H(a_j)^{-1}\mathfrak{s}_H(a_k)^{-1} = a_j \wedge a_k,$$

with $\mathfrak{s}_H$ the cross-section of $\pi_0 : (m, g) \in H \mapsto g \in G$ given by $\mathfrak{s}_H(g) = (0, g) \in H$ for $g \in G$. Thus $[H, H]$ contains the generators $a_j \wedge a_k$ for $j < k$ of M. ♡

Theorem 3.2. *The pair $\{H, \pi_0\}$ is a universal resolution of the third cocycle group $\mathrm{Z}^3(G, \mathbb{T})$ of G. If K is a countable discrete abelian group, then for any surjective homomorphism $\pi : \mathbb{Z}^{<\mathbb{N}} \to K$, the composed map $\pi_K = \pi \circ \pi_0 : H \to K$ makes the pair $\{H, \pi_K\}$ a universal resolution of the third cocycle group $\mathrm{Z}^3(K, \mathbb{T})$.*

Proof. Since $\mathbb{Z}^{<\mathbb{N}}$ is a free abelian group on countably infinite generators, there exists a surjective homomorphism from G to any countable abelian group K. So it is sufficient to prove that

$$\pi_0^*(\mathrm{Z}^3(G, \mathbb{T})) \subset \mathrm{B}^3(H, \mathbb{T}).$$

For each triplet $\xi, \eta, \zeta \in \mathrm{Hom}(G, \mathbb{R})$, we define a multihomomorphism, called the *tensor product* and denoted by $\xi \otimes \eta \otimes \zeta \in \mathrm{C}^3(G, \mathbb{R})$, as follows:

$$(\xi \otimes \eta \otimes \zeta)(g; h; k) = \xi(g)\eta(h)\zeta(k) \quad \text{for } g, h, k \in G.$$

Then the tensor product $\xi \otimes \eta \otimes \zeta$ generates the third cocycle group $\mathrm{Z}^3(G, \mathbb{R})$ up to coboundary, that is,

$$\big\langle \big\{\xi \otimes \eta \otimes \zeta : \xi, \eta, \zeta \in \mathrm{Hom}(G, \mathbb{R})\big\}\big\rangle + \mathrm{B}^3(G, \mathbb{R}) = \mathrm{Z}^3(G, \mathbb{R}).$$

Now for each pair $\eta, \zeta \in \mathrm{Hom}(G, \mathbb{R})$, we define a cochain $B_{\eta,\zeta} \in \mathrm{C}^1(H, \mathbb{R})$ by

$$B_{\eta,\zeta}(g) = \sum_{j<k} \eta(a_j)\zeta(a_k)e_{j,k}(m_0(g)) \quad \text{for } g = (m_0(g), \pi_0(g)) \in H. \tag{3-4}$$

Then we have

$$\begin{aligned}
&\big(\partial_H(\pi_0^*\xi \otimes B_{\eta,\zeta})\big)(g_1; g_2; g_3)\\
&\quad = \xi(\pi_0(g_2))B_{\eta,\zeta}(g_3) - \xi(\pi_0(g_1) + \pi_0(g_2))B_{\eta,\zeta}(g_3)\\
&\qquad\qquad + \xi(\pi_0(g_1))B_{\eta,\zeta}(g_2g_3) - \xi(\pi_0(g_1))B_{\eta,\zeta}(g_2)\\
&\quad = -\xi(\pi_0(g_1))B_{\eta,\zeta}(g_3) + \xi(\pi_0(g_1))\big(\textstyle\sum_{j<k} \eta(a_j)\zeta(a_k)e_{j,k}(m_0(g_2g_3))\big)\\
&\qquad\qquad - \xi(\pi_0(g_1))B_{\eta,\zeta}(g_2)\\
&\quad = -\xi(\pi_0(g_1))B_{\eta,\zeta}(g_3)\\
&\qquad + \xi(\pi_0(g_1))\big(\textstyle\sum_{j<k} \eta(a_j)\zeta(a_k)(e_{j,k}(m_0(g_2) + m_0(g_3) + \pi_0^*\mathfrak{n}_M(g_2; g_3))\big)\\
&\qquad\qquad - \xi(\pi_0(g_1)B_{\eta,\zeta}(g_2)\\
&\quad = \xi(\pi_0(g_1))\big(\textstyle\sum_{j<k} \eta(a_j)\zeta(a_k)e_j(\pi_0(g_2))e_k(\pi_0(g_3))\big).
\end{aligned}$$

Choosing $\xi, \eta, \zeta \in \mathrm{Hom}(G, \mathbb{T})$ to be $\xi = e_i,\ \eta = e_j$ and $\zeta = e_k$ for $i < j < k$, we obtain

$$\pi_0^*(e_i \otimes e_j \otimes e_k) = \partial_H(\pi_0^* e_i \otimes B_{e_j, e_k}).$$

Every third cocycle in $\mathrm{Z}^3(G, \mathbb{T})$ is cohomologous to a cocycle $c_a \in \mathrm{Z}^3(G, \mathbb{T})$ of the form

$$c_a(g_1; g_2; g_3) = \exp\Big(2\pi \mathrm{i}\Big(\sum_{i<j<k} a(i, j, k) e_i(g_1) e_j(g_2) e_k(g_3)\Big)\Big). \tag{3-5}$$

So with $b_a \in \mathrm{C}^2(H, \mathbb{T})$ defined by

$$b_a(g_1; g_2) = \exp\Big(2\pi \mathrm{i}\Big(\sum_{i<j<k} a(i, j, k) e_i(\pi_0(g_1)) B_{e_j, e_k}(g_3)\Big)\Big), \tag{3-6}$$

we have

$$\pi_0^* c_a = \partial_H b_a. \tag{3-7}$$

Hence we get $\pi_0^*(\mathrm{Z}^3(G, \mathbb{T})) \subset \mathrm{B}^3(H, \mathbb{T})$, from which we conclude that the pair $\{H, \pi_0\}$ is a universal resolution of $\mathrm{Z}^3(G, \mathbb{T})$. ♡

Remark 3.3. The μ-part of every characteristic cocycle $(\lambda, \mu) \in \mathrm{Z}(H, M, \mathbb{T})$ is trivial.

Proof. Since $M \triangleleft H$ is central, λ is a bicharacter of $M \times H$; in particular $\lambda(m, \cdot)$ is a character of H for every $m \in M$. Hence it must vanish on the commutator subgroup, that is, $\lambda(m, n) = 1$ for all $m, n \in M$. Thus $\mu \in \mathrm{Z}^2(M, \mathbb{T})$ is a coboundary. ♡

Consider $(\lambda, \mu) \in \mathrm{Z}(H, L, \mathbb{T})$ with $L = M \times_{\mathfrak{n}_M} N$. We may and do assume the triviality $\mu_M = 1$ of the restriction of μ to M. We then have the corresponding crossed extension

$$1 \to \mathbb{T} \longrightarrow E \underset{u}{\overset{j}{\rightleftarrows}} L \to 1$$

The triviality of μ_M means that the cross-section u is multiplicative on M, that is, $u(mn) = u(m)u(n)$ for $m, n \in M$. Here we use the multiplicative group operation since M sits in the noncommutative group H.

Lemma 3.4. *If $\mathfrak{s}_H$ is a cross-section of the quotient map $\pi_0 : H \to \mathbb{Z}^{<\mathbb{N}} = H/M$ with $\mathfrak{n}_M = \partial \mathfrak{s}_H \in \mathrm{Z}^2(\mathbb{Z}^{<\mathbb{N}}, M)$, then each characteristic cocycle in $\mathrm{Z}(H, L, M, \mathbb{T})$ is cohomologous to the one $(\lambda, \mu) \in \mathrm{Z}(H, L, M, \mathbb{T})$ such that*

$$\lambda(m; n\mathfrak{s}_H(h)) = \lambda(m; \mathfrak{s}_H(h)) \qquad \textit{for } m, n \in M,\ h \in \mathbb{Z}^{<\mathbb{N}},$$

$$\mu(m\mathfrak{s}_H(g); n\mathfrak{s}_H(h)) = \lambda(n; \mathfrak{s}_H(g))\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h)) \quad \textit{for } m, n \in M,\ g, h \in N.$$

Proof. In the crossed extension $E \in \mathrm{Xext}(H_{\mathrm{m}}, L, M, \mathbb{T})$ associated with $(\lambda, \mu) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$ given by $1 \to \mathbb{T} \to E \to L \to 1$, we redefine the cross-section u for $m \in M$ and $g \in N$ as $u(m\mathfrak{s}_H(g)) = u(m)u(\mathfrak{s}_H(g))$, so that $\mu(m; \mathfrak{s}_H(g)) = 1$. We now compute, for $m, n \in M$ and $h \in \mathbb{Z}^{<\mathbb{N}}$,

$$\begin{aligned}\lambda(m; n\mathfrak{s}_H(h))u(m) &= \alpha_{n\mathfrak{s}_H(h)}(u(m)) = u(n)\alpha_{\mathfrak{s}_H(h)}(u(m))u(n)^{-1}\\ &= \lambda(m; \mathfrak{s}_H(h))u(mn)u(n)^{-1}\\ &= \lambda(m; \mathfrak{s}_H(h))u(m);\end{aligned}$$

for $g, h \in N$, we complete the proof with the computation

$$\begin{aligned}&\mu(m\mathfrak{s}_H(g); n\mathfrak{s}_H(h))u(m\mathfrak{s}_H(g)n\mathfrak{s}_H(h))\\ &\qquad = u(m\mathfrak{s}_H(g))u(n\mathfrak{s}_H(h))\\ &\qquad = u(m)u(\mathfrak{s}_H(g))u(n)u(\mathfrak{s}_H(h)))\\ &\qquad = u(m)\alpha_{\mathfrak{s}_H(g)}(u(n))u(\mathfrak{s}_H(g))u(\mathfrak{s}_H(h))\\ &\qquad = \lambda(n; \mathfrak{s}_H(g))u(m)u(n)\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h))u(\mathfrak{s}_H(g)\mathfrak{s}_H(h))\\ &\qquad = \lambda(n; \mathfrak{s}_H(g))u(mn)\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h))u(\mathfrak{s}_H(g)\mathfrak{s}_H(h))\\ &\qquad = \lambda(n; \mathfrak{s}_H(g))\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h))u(m\mathfrak{s}_H(g)n\mathfrak{s}_H(h)).\end{aligned}$$ ♡

Groups G, H_{m}, G_{m} and Q_{m}. First, we fix notations. To work on the quotient group $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$ with $p \in \mathbb{N}$ and $p \geq 2$, we set

$$\begin{aligned}&[i]_p = i + p\mathbb{Z} \in \mathbb{Z}_p, \quad \text{where } i = np + \{i\}_p,\ 0 \leq \{i\}_p < p,\\ &\eta_p([i]_p, [j]_p) = \{i\}_p + \{j\}_p - \{i+j\}_p = \begin{cases}0 & \text{if } \{i\}_p + \{j\}_p < p,\\ p & \text{if } \{i\}_p + \{j\}_p \geq p.\end{cases}\end{aligned} \tag{3-8}$$

We shall call the $p\mathbb{Z}$-valued cocycle $\eta_p \in \mathrm{Z}^2(\mathbb{Z}_p, p\mathbb{Z})$ the *Gauss cocycle*, which can be written

$$\eta_p([i]_p, [j]_p) = p\left(\left[\frac{i+j}{p}\right] - \left[\frac{i}{p}\right] - \left[\frac{j}{p}\right]\right), \tag{3-8$'$}$$

where $[x]$ for $x \in \mathbb{R}$ is the largest integer less than or equal to x.

Given a homomorphism m of the group G to $\mathbb{R}/T'\mathbb{Z}$ such that $\mathrm{Ker}(\mathrm{m}) \supset N$, we consider the group extension

$$G_{\mathrm{m}} = \{(g, s) \in G \times \mathbb{R} : \dot{s}_{T'} = s + T'\mathbb{Z} = \mathrm{m}(g) \in \mathbb{R}/T'\mathbb{Z}\},$$

$$0 \to \mathbb{Z} \xrightarrow{n \to z_0^n} G_{\mathrm{m}} \xrightarrow{\pi_{\mathrm{m}}} G \to 1,$$

where $z_0 = (0, T') \in G_{\mathrm{m}}$. Identifying m with $\mathrm{m} \circ \pi_0 \in \mathrm{Hom}(H, \mathbb{R}/T'\mathbb{Z})$, we also form a group extension

$$\begin{aligned} H_{\mathrm{m}} &= \{(h, s) \in H \times \mathbb{R} : \mathrm{m}(h) = \dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}\} \\ &= \{(m, g, s) \in M \times G \times \mathbb{R} : \mathrm{m}(g) = \dot{s}_{T'} \in \mathbb{R}/T'\mathbb{Z}\}, \end{aligned}$$

$$0 \to \mathbb{Z} \xrightarrow{n \to z_0^n} H_{\mathrm{m}} \longrightarrow H \to 1,$$

where the central element $z_0 = (1, T') \in H_{\mathrm{m}}$ appears in both G_{m} and H_{m}. We hope that this abuse of notation for two distinct elements in the different groups will not cause a headache later; it is just like the zero elements in ring theory.

By the assumption $N \subset \mathrm{Ker}(\mathrm{m})$, the homomorphism m factors through the quotient group $Q = G/N$, so that it is also viewed as a homomorphism of $Q \to \mathbb{R}/T'\mathbb{Z}$; therefore we can form the group extension Q_{m} as before, which sits on the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathfrak{s}_{\mathrm{m}}(N) & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & G_{\mathrm{m}} & \underset{\mathfrak{s}_{\mathrm{m}}}{\overset{\pi_{\mathrm{m}}}{\rightleftarrows}} & G & \longrightarrow & 1 \\
 & & \| & & \pi_Q \downarrow \uparrow \mathfrak{s} & & \pi_Q \downarrow \uparrow \mathfrak{s} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & Q_{\mathrm{m}} & \underset{\dot{\mathfrak{s}}_{\mathrm{m}}}{\overset{\dot{\pi}_{\mathrm{m}}}{\rightleftarrows}} & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 1 & & 1 & &
\end{array}$$

From the assumption $\mathrm{Ker}(\mathrm{m}) \supset N$, it follows that $\mathrm{m}(p_i a_i) = 0$, so that there exists an integer $q_i \in \mathbb{Z}$ with $0 \le q_i < p_i$ such that

$$\begin{aligned} \mathrm{m}_i &= \{\mathrm{m}(a_i)\}_{T'} = q_i T'/p_i \in ((T'/p_i)\mathbb{Z}), \\ \mathrm{m}(a_i) &= \dot{\mathrm{m}}_i = \mathrm{m}_i + \mathbb{T}'\mathbb{Z} \in \mathbb{R}/T'\mathbb{Z}. \end{aligned} \tag{3-9}$$

For $g \in G$, we set

$$\begin{aligned} G_{\mathrm{m}} \ni z_i &= \begin{cases} (a_i, \mathrm{m}_i) & \text{if } i \neq 0, \\ (0, T') & \text{if } i = 0, \end{cases} \\ \mathfrak{s}_{\mathrm{m}}(g) &= \sum_{i \in \mathbb{N}} e_i(g) z_i = \Big(g, \sum_{i \in \mathbb{N}} e_i(g)\mathrm{m}_i\Big) = (g, \mathrm{n}(g)), \\ \mathrm{n}(g) &= \sum_{i \in \mathbb{N}} e_i(g)\mathrm{m}_i. \end{aligned} \tag{3-10}$$

Then G_{m} decomposes as

$$
\begin{aligned}
G_{\mathrm{m}} &= \mathbb{Z}z_0 \oplus \mathfrak{s}_{\mathrm{m}}(G) = \sum_{i\in\mathbb{N}_0}^{\oplus} \mathbb{Z}z_i, \text{ where } \mathbb{N}_0 = \mathbb{N}\cup\{0\},\\
\tilde{g} &= \tilde{e}_0(\tilde{g})z_0 + \sum_{i\in\mathbb{N}} \tilde{e}_i(\tilde{g})z_i \in G_{\mathrm{m}};\\
\tilde{g} &= (g,s) = (0, \tilde{e}_0(g,s)T') + \Big(\sum_{i\in\mathbb{N}} \tilde{e}_i(\tilde{g})a_i, \sum_{i\in\mathbb{N}} \tilde{e}_i(\tilde{g})\mathrm{m}_i\Big)\\
&= (0, \tilde{e}_0(g,s)T') + \sum_{i\in\mathbb{N}} \tilde{e}_i(\tilde{g})z_i;\\
\tilde{e}_0(g,s) &= (s - \mathrm{n}(g))/T' \in \mathbb{Z},\\
\tilde{e}_i(g,s) &= e_i(g) \quad \text{for } i \in \mathbb{N}.
\end{aligned}
\tag{3-11}
$$

In particular, if $g \in N$, we have $g = (g,0) = -(\mathrm{n}(g)/T')z_0 + \sum_{i\in\mathbb{N}} e_i(g)z_i$, so that

$$\tilde{e}_0(g) = -\mathrm{n}(g)/T' \neq 0 \quad \text{unless } \mathrm{n}(g) = \sum_{i\in\mathbb{N}} e_i(g)\mathrm{m}_i = 0.$$

We then have $\mathrm{m}(g) = [\mathrm{n}(g)]_{T'} \in \mathbb{R}/T'\mathbb{Z}$. Setting $b_j = p_j a_j$ for $j \in \mathbb{N}$, we write every $g \in N$ uniquely in the form

$$g = \sum_{j\in\mathbb{N}} \frac{e_j(g)}{p_j} b_j = \sum_{j\in\mathbb{N}} e_{j,N}(g) b_j, \tag{3-12}$$

where $e_{j,N}(g) = e_j(g)/p_j$; also in H_{m} we have

$$b_j = p_j z_j - p_j m_j z_0 = p_j z_j - q_j z_0. \tag{3-13}$$

Remark. The element $(a_i, 0)$ is *not* a member of G_{m}.

Next we define a cross-section $\dot{\mathfrak{s}}_{\mathrm{m}} : Q \to Q_{\mathrm{m}}$ so that the diagram

$$
\begin{array}{ccc}
G_{\mathrm{m}} & \xleftarrow[\mathfrak{s}_{\mathrm{m}}]{} & G\\
{\scriptstyle \mathfrak{s}}\uparrow & & \uparrow{\scriptstyle \mathfrak{s}}\\
Q_{\mathrm{m}} & \xleftarrow[\dot{\mathfrak{s}}_{\mathrm{m}}]{} & Q
\end{array}
$$

commutes. First, we set

$$
\begin{aligned}
&\dot{g} = g + N \in Q = G/N \quad \text{for } g \in G, \quad \mathfrak{s}(q) = \sum_{i\in\mathbb{N}} \{e_i(q)\}_{p_i} a_i \quad \text{for } q \in Q,\\
&\dot{a}_i = \pi_{Q_{\mathrm{m}}}(a_i), \quad \dot{z}_i = (\dot{a}_i, \mathrm{m}_i),\\
&\dot{\mathfrak{s}}_{\mathrm{m}}(q) = \sum_{i\in\mathbb{N}} \{e_i(q)\}_{p_i} \dot{z}_i = \sum_{i\in\mathbb{N}} \{e_i(q)\}_{p_i} (\dot{a}_i, \mathrm{m}_i) = \Big(q, \sum_{i\in\mathbb{N}} \{e_i(q)\}_{p_i} \mathrm{m}_i\Big),\\
&\mathfrak{s}(q,s) = (\mathfrak{s}(q), s) \in G_{\mathrm{m}} \quad \text{for } (q,s) \in Q_{\mathrm{m}}.
\end{aligned}
$$

The cross-section $\mathfrak{s}: Q_{\mathrm{m}} \to G_{\mathrm{m}}$ gives rise to an N-valued cocycle

$$\mathfrak{n}_N = \partial_Q \mathfrak{s} \in \mathrm{Z}^2(Q_{\mathrm{m}}, N), \tag{3-14}$$

which is given by

$$\begin{aligned}
\mathfrak{n}_N(\tilde{q}_1; \tilde{q}_2) &= \mathfrak{s}(q_1, s_1) + \mathfrak{s}(q_2, s_2) - \mathfrak{s}(q_1 + q_2, s_1 + s_2) \\
&= (\mathfrak{s}(q_1) + \mathfrak{s}(q_2) - \mathfrak{s}(q_1 + q_2), 0) \\
&= \Big(\sum_{i \in \mathbb{N}} \eta_{p_i}([e_i(q_1)]_{p_i}; [e_i(q_2)]_{p_i}) a_i, 0\Big) \\
&= \sum_{i \in \mathbb{N}} (\eta_{p_i}([e_i(q_1)]_{p_i}; [e_i(q_2)]_{p_i}) a_i, 0) \in N = N \times \{0\}
\end{aligned}$$

for each pair $\tilde{q}_1 = (q_1, s_1), \tilde{q}_2 = (q_2, s_2) \in Q_{\mathrm{m}}$.

For each element $h = (m, g) \in H$ with $m \in M$ and $g \in G$, we write $m = m_0(h)$ and $g = \pi_G(h)$. Then we have $L = \pi_G^{-1}(N)$ and

$$m_0(gh) = m_0(g) + m_0(h) + \mathfrak{n}_M(\pi_G(g); \pi_G(h)) \quad \text{for } g, h \in H.$$

For short, we write $e_{i,j}(\tilde{g}) = e_{i,j}(m_0(g))$ for $\tilde{g} = (m_0(g), g, s) \in H_{\mathrm{m}}$ and $i, j \in \mathbb{N}$. With $\mathfrak{s}_H(g) = (0, g) \in H$ for each $g \in G$, we have

$$\mathfrak{n}_M(g; h) = \mathfrak{s}_H(g) + \mathfrak{s}_H(h) - \mathfrak{s}_H(g + h) = \partial_G \mathfrak{s}_H(g; h) \quad \text{for } g, h \in G.$$

With $\dot{\mathfrak{s}} = \mathfrak{s}_H \circ \mathfrak{s}$, we obtain a cross-section $\dot{\mathfrak{s}}$ of $\pi_Q \circ \pi_G : H \to Q = H/L$, which gives rise to an L-valued second cocycle $\mathfrak{n}_L \in \mathrm{Z}^2(Q, L)$; for $q_1, q_2 \in Q$, it is

$$\begin{aligned}
\mathfrak{n}_L(q_1; q_2) &= \dot{\mathfrak{s}}(q_1)\dot{\mathfrak{s}}(q_2)\dot{\mathfrak{s}}(q_1 + q_2)^{-1} \\
&= \mathfrak{s}_H\big(\mathfrak{s}(q_1)\big)\mathfrak{s}_H\big(\mathfrak{s}(q_2)\big)\mathfrak{s}_H\big(\mathfrak{s}(q_1 + q_2)\big)^{-1} \\
&= \mathfrak{n}_M\big(\mathfrak{s}(q_1); \mathfrak{s}(q_2)\big)\mathfrak{s}_H\big(\mathfrak{s}(q_1) + \mathfrak{s}(q_2)\big)\mathfrak{s}_H\big(\mathfrak{s}(q_1 + q_2)\big)^{-1} \\
&= \mathfrak{n}_M\big(\mathfrak{s}(q_1); \mathfrak{s}(q_2)\big)\mathfrak{s}_H\big(\mathfrak{n}_N(q_1; q_2) + \mathfrak{s}(q_1 + q_2)\big)\mathfrak{s}_H\big(\mathfrak{s}(q_1 + q_2)\big)^{-1} \\
&= \mathfrak{n}_M\big(\mathfrak{s}(q_1); \mathfrak{s}(q_2)\big)\mathfrak{n}_M\big(\mathfrak{n}_N(q_1; q_2); \mathfrak{s}(q_1 + q_2)\big)^{-1}\mathfrak{s}_H\big(\mathfrak{n}_N(q_1; q_2)\big).
\end{aligned} \tag{3-15}$$

We further compute the (j, k)- and k-components as

$$\begin{aligned}
e_{j,k}\big(\mathfrak{n}_M(\mathfrak{s}(q_1); \mathfrak{s}(q_2))\big) &= e_j(\mathfrak{s}(q_1))e_k(\mathfrak{s}(q_2)) \\
&= \{e_j(q_1)\}_{p_j}\{e_k(q_2)\}_{p_k}, \\
e_{j,k}\big(\mathfrak{n}_M(\mathfrak{n}_N(q_1; q_2); \mathfrak{s}(q_1 + q_2))\big) &= e_j(\mathfrak{n}_N(q_1; q_2))e_k(\mathfrak{s}(q_1 + q_2)) \\
&= \eta_{p_j}([e_j(q_1)]_{p_j}; [e_j(q_2)]_{p_j})\{e_k(q_1 + q_2)\}_{p_k}, \\
e_k\big(\mathfrak{s}_H(\mathfrak{n}_N(q_1; q_2))\big) &= \eta_{p_k}([e_k(q_1)]_{p_k}; [e_k(q_2)]_{p_k}).
\end{aligned} \tag{3-16}$$

Since

$$H_{\mathrm{m}} = M \times_{\pi_{\mathrm{m}}^*(\mathfrak{n}_M)} \Big(\sum_{i\in\mathbb{N}}^{\oplus} \mathbb{Z} z_i \oplus \mathbb{Z} z_0\Big),$$

for each $h = (m, g) \in H$, we set

$$\mathfrak{s}_{\mathrm{m}}(h) = (m, \mathfrak{s}_{\mathrm{m}}(g)) = \Big(m, \sum_{i\in\mathbb{N}} e_i(g) z_i\Big) = \Big(m, g, \sum_{i\in\mathbb{N}} e_i(g)\mathrm{m}_i\Big), \tag{3-17}$$

and we identify $\ell = (m, Pg) \in L$ with $(m, Pg, 0) \in H_{\mathrm{m}}$, so that L is a subgroup of H_{m}, while H is not.

4. The characteristic cohomology group $\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T})$

Since H is a universal resolution group for $G = \mathbb{Z}^{<\mathbb{N}}$, every third cohomology class $[c] \in \mathrm{H}^3(G, \mathbb{T})$ is of the form $[c] = \delta_{\mathrm{HJR}}[\lambda, \mu]$ for some $[\lambda, \mu] \in \Lambda(H, M, \mathbb{T})$. So every outer action $\dot{\alpha}$ of G on a factor $\mathcal{M}$ of type III_λ comes from an action α of H, that is, the outer action $\dot{\alpha}$ is given by

$$\dot{\alpha}_g = \alpha_{\mathfrak{s}_H(g)} \quad \text{for } g \in G. \tag{4-1}$$

But the action α of H does not give rise to an action of H on the reduced (discrete) core $\widetilde{\mathcal{M}}_d$. Instead, the action α of H on $\mathcal{M}$ gives rise naturally to an action, denoted by the same notation α, of H_{m} on $\widetilde{\mathcal{M}}_d$, where

$$\mathrm{m}(h) = \ \mathrm{mod}\ (\alpha_h) \in \mathbb{R}/T'\mathbb{Z} \quad \text{for } h \in H.$$

If $N = \dot{\alpha}^{-1}(\mathrm{Cnt_r}(\mathcal{M}))) \subset G$, then $L = \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$. We make a basic assumption on the subgroup N that

$$N = PG = P\mathbb{Z}^{<\mathbb{N}}.$$

In the case that G is finitely generated free abelian group, the fundamental structure theorem for finitely generated abelian groups guarantees that every subgroup of G is of this form.

We study first the characteristic cohomology group $\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T})$ and modified HJR-map $\delta : \Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) \to \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$.

We introduce a series of notations first:

$$\begin{aligned} &\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \mathbb{Z}_+,\\ &\Delta_0 = \{(i, j, k) \in \mathbb{N}_0^3 : i < j < k\} \cup \{(i, i, k) \in \mathbb{N}_0^3; i < k\}\\ &\qquad\qquad \cup \{(k, i, k) \in \mathbb{N}_0^3 : i < k\},\\ &\Delta = \Delta_0 \cap \mathbb{N}^3. \end{aligned} \tag{4-2}$$

For each $g \in H_{\mathrm{m}}$, let $m_0(g)$ be the M-component of g, that is,

$$m_0(g) = g\mathfrak{s}_H(\pi_G(g))^{-1} \in M \quad \text{for } g \in H_{\mathrm{m}}. \tag{4-3}$$

We regard e_i and $e_{j,k}$ as functions defined on H_{m} by fixing the coordinate system

$$\tilde{g} = \Big(\sum_{1 \le j < k} e_{j,k}(g)(a_j \wedge a_k), \sum_{i \in \mathbb{N}_0} \tilde{e}_i(\tilde{g}) z_i \Big) \in H_{\mathrm{m}}, \text{ with } g = \pi_{\mathrm{m}}(\tilde{g}) \in H. \tag{4-4}$$

We then introduce a cochain $B_{jk} \in \mathrm{C}^1(H_{\mathrm{m}}, \mathbb{R})$ defined for $h \in H_{\mathrm{m}}$ by

$$B_{jk}(h) = \begin{cases} -e_{j,k}(m_0(h)) & \text{if } j < k, \\ -\frac{1}{2}(e_j e_j)(h) & \text{if } j = k, \\ e_{k,j}(m_0(h)) - (e_j e_k)(h) & \text{if } j > k, \end{cases} \tag{4-5}$$

The cochain enjoys the property

$$\partial_H B_{jk} = \pi_0^*(e_j \otimes e_k) \quad \text{for } j, k \in \mathbb{N}. \tag{4-6}$$

We continue to define the following cochains for each $a \in \mathbb{R}^{\mathbb{N}_0^3}$:

$$\begin{aligned}
X_a(i, j, k) &= a(i, j, k) e_{j,k} \otimes e_i + a(j, i, k) e_{i,k} \otimes e_j + a(k, i, j) e_{i,j} \otimes e_k, \\
X_a(i, k) &= a(i, i, k) e_{i,k} \otimes e_i + a(k, i, k) e_{i,k} \otimes e_k, \\
Y_a(i, j, k) &= a(i, j, k)\big(B_{ij} \otimes e_k + e_k \otimes B_{ji} - B_{ik} \otimes e_j - e_j \otimes B_{ki}\big) \\
&\quad + a(j, i, k)\big(B_{ji} \otimes e_k + e_k \otimes B_{ij} - B_{jk} \otimes e_i - e_i \otimes B_{kj}\big) \\
&\quad + a(k, i, j)\big(B_{ki} \otimes e_j + e_j \otimes B_{ik} - B_{kj} \otimes e_i - e_i \otimes B_{jk}\big), \\
Y_a(i, k) &= a(i, i, k)(B_{ii} \otimes e_k + e_k \otimes B_{ii} - B_{ik} \otimes e_i - e_i \otimes B_{ki}) \\
&\quad + a(k, i, k)(B_{ki} \otimes e_k + e_k \otimes B_{ik} - B_{kk} \otimes e_i - e_i \otimes B_{kk}), \\
Z(\cdots)(g; h) &= Y(\cdots)(m_0(h); g), \\
Z_a(i, j, k) &= a(i, j, k)\big(e_j \otimes e_{i,k} - e_k \otimes e_{i,j}\big) \\
&\quad + a(j, i, k)\big(e_k \otimes e_{i,j} + e_i \otimes e_{j,k}\big) + a(k, i, j)\big(e_j \otimes e_{i,k} - e_i \otimes e_{j,k}\big), \\
Z_a(i, k) &= a(i, i, k) e_i \otimes e_{i,k} + a(k, i, k) e_k \otimes e_{i,k}; \\
f_{i,j,k} &= 2(e_i e_j) \otimes e_k - 3 e_i \otimes (e_j e_k) + e_j \otimes (e_i e_k) \\
&\qquad - 2(e_i e_k) \otimes e_j - e_k \otimes (e_i e_j), \\
U_a(i, j, k) &= \tfrac{1}{6}\big(a(i, j, k) f_{i,j,k} + a(j, i, k) f_{j,i,k} + a(k, i, j) f_{k,i,j} \\
&\qquad - (\mathrm{AS}\, a)(i, j, k) f_{i,j,k}\big), \\
U_a(i, k) &= -a(i, i, k) B_{ii} \otimes e_k + a(k, i, k)(B_{kk} \otimes e_i - e_k \otimes (e_i e_k)), \\
V_a(i, j, k) &= Z_a(i, j, k) + \pi_G^* U_a(i, j, k), \\
V_a(i, k) &= Z_a(i, k) + \pi_G^* U_a(i, k).
\end{aligned}$$

The infinite summations

$$
\begin{aligned}
X_a &= \sum_{i<j<k} X_a(i,j,k) + \sum_{i<k} X_a(i,k),\\
Y_a &= \sum_{i<j<k} Y_a(i,j,k) + \sum_{i<k} Y_a(i,k),\\
U_a &= \sum_{i<j<k} U_a(i,j,k) + \sum_{i<k} U_a(i,k),\\
V_a &= \sum_{i<j<k} V_a(i,j,k) + \sum_{i<k} V_a(i,k),\\
Z_a &= \sum_{i<j<k} Z_a(i,j,k) + \sum_{i<k} Z_a(i,k)
\end{aligned}
\tag{4-7}
$$

will become all finite sums as soon as variables from M or H_{m} are fed in. So no divergence problem in the infinite sums will occur.

The cochain $f_{i,j,k}$ relates basic cocycles $e_i \otimes e_j \otimes e_k$ and the asymmetric tricharacter

$$
\begin{aligned}
\det_{ijk} &= (e_i \otimes e_j \otimes e_k + e_j \otimes e_k \otimes e_i + e_k \otimes e_i \otimes e_j)\\
&\qquad - (e_j \otimes e_i \otimes e_k + e_i \otimes e_k \otimes e_j + e_k \otimes e_j \otimes e_i) = e_i \wedge e_j \wedge e_k
\end{aligned}
$$

as

$$
\det_{ijk} = \partial_L f_{i,j,k} + 6 e_i \otimes e_j \otimes e_k \quad \text{for } i < j < k,
\tag{4-8}
$$

which can be confirmed by a direct computation.

Let Z be the set of all pairs (a, b) of functions $a : (i, j, k) \in \mathbb{N}^3 \mapsto a(i, j, k) \in \mathbb{R}$ and $b : (i, j) \in \mathbb{N}_0^2 \mapsto b(i, j) \in \mathbb{R}$ such that a satisfies

$$
\begin{aligned}
a(i,j,k) &= 0 \quad \text{for } j,k \in \mathbb{N}_0 \text{ with } j \geq k,\\
a(0,j,k) &= 0 \quad \text{for every } j,k \in \mathbb{N}_0,\\
(\mathrm{AS}\, a)(i,j,k) &= a(i,j,k) - a(j,i,k) + a(k,i,j)\\
&\in \Big(\frac{1}{\gcd(p_i, p_j, p_k)}\mathbb{Z}\Big).
\end{aligned}
\tag{4-9Z-a}
$$

and b satisfies

$$
\begin{aligned}
b(i,j)p_j - b(i,0)q_j &\in \mathbb{Z} \quad \text{for } i,j \in \mathbb{N} \text{ with } i < j,\\
b(0,j) &= 0 \quad \text{for } j \in \mathbb{N}_0.
\end{aligned}
\tag{4-9Z-b}
$$

Let Z_a be the set of $a \in \mathbb{R}^{\mathbb{N}^3}$ satisfying (4-9Z-a), and let Z_b be the set of all $b \in \mathbb{R}^{\mathbb{N}_0^2}$ satisfying (4-9Z-b). So we have $\mathrm{Z} = \mathrm{Z}_a \oplus \mathrm{Z}_b$.

Let B be the subgroup of Z consisting of all those $(a, b) \in \mathrm{Z}$ such that a satisfies the coboundary condition

$$\text{(4-9B-a)} \qquad \begin{aligned} a(i,j,k), a(k,i,j), a(j,i,k) &\in \mathbb{Z} &&\text{if } i<j<k,\\ a(i,i,k) &\in 2\mathbb{Z} &&\text{if } i<k,\\ a(k,i,k) &\in 2\mathbb{Z} &&\text{if } i<k, \end{aligned}$$

and b satisfies the coboundary condition

$$\text{(4-9B-b)} \qquad \begin{aligned} &\frac{b(i,j)}{p_i}+\frac{b(j,i)}{p_j} \in \Big(\frac{1}{p_i}\mathbb{Z}\Big)+\Big(\frac{1}{p_j}\mathbb{Z}\Big)=\Big(\frac{1}{\operatorname{lcm}(p_i,p_j)}\mathbb{Z}\Big) &&\text{if } i<j,\\ &b(i,0)\in\mathbb{Z} \text{ and } b(i,i)\in\mathbb{Z} &&\text{if } i\in\mathbb{N}. \end{aligned}$$

Let B_a (respectively B_b) be the set of all $b \in \mathbb{R}^{\mathbb{N}_0^2}$ satisfying (4-9B-a) (respectively (4-9B-b)). Thus we have $\mathrm{B} = \mathrm{B}_a \oplus \mathrm{B}_b$. Set $\mathrm{H}_a = \mathrm{Z}_a/\mathrm{B}_a$ and $\mathrm{H}_b = \mathrm{Z}_b/\mathrm{B}_b$. With $D(i, j, k) = \gcd(p_i, p_j, p_k)$ for each triplet $i < j < k$ with $i, j, k \in \mathbb{N}$, we set

$$\begin{aligned} \mathrm{Z}_a(i,j,k) &= \{(u,v,w)\in\mathbb{R}^3 : u-v+w \in ((1/D(i,j,k))\mathbb{Z})\},\\ \mathrm{B}_a(i,j,k) &= \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}, \end{aligned}$$

where $u = a(i, j, k)$, $v = a(j, i, k)$ and $w = a(k, i, j)$. For a pair $i, k \in \mathbb{N}$ with $i < k$, we set

$$\mathrm{Z}_a(i,k) = \{(x,y)\in\mathbb{R}^2\} = \mathbb{R}\oplus\mathbb{R} \quad\text{and}\quad \mathrm{B}_a(i,k) = (2\mathbb{Z})\oplus(2\mathbb{Z}),$$

where $x = a(i, i, k)$ and $y = a(k, i, k)$. We then naturally define

$$\begin{aligned} \Lambda_a(i,j,k) &= \mathrm{Z}_a(i,j,k)/\mathrm{B}_a(i,j,k)\\ &\cong \Big(\Big(\frac{1}{D(i,j,k)}\mathbb{Z}\Big)\Big/\mathbb{Z}\Big)\oplus\mathbb{R}/\mathbb{Z}\oplus\mathbb{R}/\mathbb{Z} &&\text{for } i<j<k,\\ \Lambda_a(i,k) &= \mathrm{Z}_a(i,k)/\mathrm{B}_a(i,k) = \mathbb{R}/(2\mathbb{Z})\oplus\mathbb{R}/(2\mathbb{Z}) &&\text{for } i<k. \end{aligned}$$

Here the second isomorphism above can be seen easily by considering the matrix

$$A = \begin{pmatrix} 1 & -1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3,\mathbb{Z}).$$

For each ordered pair $i, j \in \mathbb{N}$ with $i < j$, we put $D_{i,j} = \gcd(p_i, p_j)$ and define

$$\text{(4-10)} \qquad \begin{aligned} &\mathrm{Z}_b(i,j) = \{(x,u,y,v)\in\mathbb{R}^4 : p_j x - q_j u \in\mathbb{Z},\ p_i y - q_i v\in\mathbb{Z}\},\\ &\mathrm{B}_b(i,j) = \{(x,u,y,v)\in\mathrm{Z}_b(i,j) : p_j x + p_i y \in D_{i,j}\mathbb{Z},\ u,v\in\mathbb{Z}\},\\ &\mathrm{Z}_b(i,i) = \{z=(x,u)\in\mathbb{R}^2 : p_i x - q_i u\in\mathbb{Z}\}, \quad \mathrm{B}_b(i,i)=\mathbb{Z}\oplus\mathbb{Z},\\ &\Lambda_b(i,j) = \mathrm{Z}_b(i,j)/\mathrm{B}_b(i,j), \quad \Lambda_b(i,i) = \mathrm{Z}_b(i,i)/\mathrm{B}_b(i,i). \end{aligned}$$

Definition 4.1. To each $(a, b) \in \mathrm{Z}$ we associate a cochain $(\lambda_{a,b}, \mu_a)$ defined by

$$
\begin{aligned}
\lambda_{a,b}(g; h) &= \exp(2\pi \mathrm{i}((Y_a + X_{\mathrm{AS}\,a})(g; h))) \\
&\qquad \times \exp\bigl(2\pi \mathrm{i}\bigl(\textstyle\sum_{i\in\mathbb{N}, j\in\mathbb{N}_0} b(i, j) e_{i,N}(g)\tilde{e}_j(h)\bigr)\bigr), \\
\eta_a(g; h) &= \exp(2\pi \mathrm{i}(Y_a(g; h))), \qquad\qquad (4\text{-}11) \\
\mu_a(g; h) &= \exp(2\pi \mathrm{i} V_a(g; h)) \\
&= \lambda_{a,b}(m_0(h); g) \exp(2\pi \mathrm{i} U_a(\pi_G(g); \pi_G(h)))
\end{aligned}
$$

for each $(g, h) \in L \times H_{\mathrm{m}}$. In the case that $b = 0$ (respectively $a = 0$) we denote the corresponding cochains by (λ_a, μ_a) (respectively λ_b). Let Z_a (respectively B_a) be the set of $\{(\lambda_a, \mu_a) : a \in \mathrm{Z}_a\}$ (respectively $\{(\lambda_b, 1) : b \in \mathrm{Z}_b\}$), and let $\Lambda = \Lambda_a \oplus \Lambda_b$, $\Lambda_a = Z_a/B_a$ and $\Lambda_b = Z_b/B_b$.

Theorem 4.2. (a) *The cochain (λ_a, μ_a) is a characteristic cocycle belonging to $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$ and the correspondence $a \in \mathrm{Z}_a \mapsto (\lambda_a, \mu_a) \in Z_a$ gives the following commutative diagram of exact sequences*:

$$
(4\text{-}12a)\qquad
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 \to & \mathrm{B}_a & \longrightarrow & a \in \mathrm{Z}_a & \longrightarrow & [a] \in \mathrm{H}_a & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & B_a & \longrightarrow & (\lambda_a, \mu_a) \in Z_a & \longrightarrow & [\lambda_a, \mu_a] \in \Lambda_a & \to 1 \\
 & & & & & \downarrow & \\
 & & & & & 0 &
\end{array}
$$

(b) *The correspondence $b \in \mathrm{Z}_b \mapsto (\lambda_b, 1) \in Z_b$ gives the following commutative diagram of exact sequences*:

$$
(4\text{-}12b)\qquad
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 \to & \mathrm{B}_b & \longrightarrow & b \in \mathrm{Z}_b & \longrightarrow & [b] \in \mathrm{H}_b & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & B_b & \longrightarrow & (\lambda_b, 1) \in Z_b & \longrightarrow & [\lambda_b] \in \Lambda_b & \to 1 \\
 & & & & & \downarrow & \\
 & & & & & 0 &
\end{array}
$$

(c) *The characteristic cohomology group $\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) = \Lambda_a \oplus \Lambda_b$ has further fine structure*:

(i) *The group Λ_a has the Cartesian product decomposition*

$$\Lambda_a = \prod_{i<j<k} \Lambda_a(i,j,k) \oplus \prod_{i<j} \Lambda_a(i,j), \tag{4-13a}$$

$$\begin{aligned} \textit{where} \quad \Lambda_a(i,j,k) &\cong \mathbb{Z}_{D(i,j,k)} \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}, \\ D(i,j,k) &= \gcd(p_i, p_j, p_k), \\ \Lambda_a(i,j) &\cong \mathbb{R}/(2\mathbb{Z}) \oplus \mathbb{R}/(2\mathbb{Z}). \end{aligned}$$

(ii) *The fiber product decomposition of Λ_b is the family $\{\Lambda_b(i,j) : i, j \in \mathbb{N}\}$ and each group $\Lambda_b(i,j)$ is described by*

$$\begin{aligned} &\Lambda_b(i,j) \cong \mathbb{Z}/(\gcd(p_i, p_j, q_i, q_j)\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}) \quad \textit{for } i < j, \\ &\Lambda_b(i,i) \cong \mathbb{Z}/(\gcd(p_i, q_i)\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}). \end{aligned} \tag{4-13a}$$

The group $\Lambda_b(i,j)$ is equipped with three homomorphisms, and $\Lambda_b(i,i)$ has two:

$$\begin{aligned} &\pi_{ij} : \Lambda_b(i,j) \to \Big(\frac{1}{D(i,j)}\mathbb{Z}\Big)\Big/\mathbb{Z}, \\ &\pi^i_{i,j} : \Lambda_b(i,j) \to \mathbb{R}/\mathbb{Z}, \qquad \pi^j_{i,j} : \Lambda_b(i,j) \to \mathbb{R}/\mathbb{Z}, \\ &\pi_{ii} : \Lambda_b(i,i) \to (1/p_i)\mathbb{Z}/\mathbb{Z}, \qquad \pi^i_i : \Lambda_b(i,i) \to \mathbb{R}/\mathbb{Z}, \end{aligned} \tag{4-14}$$

These are such that for each $z = (x, u, y, v) \in \mathrm{Z}_b(i,j)$

$$\begin{aligned} &\pi_{ij}([\lambda_z]) = [m_{i,j}(x r_{j,i} + y r_{i,j}) - n_{i,j}(u s_{j,i} + v s_{i,j})]_\mathbb{Z}, \\ &\pi^i_{i,j}([\lambda_z]) = [u]_\mathbb{Z}, \qquad \pi^j_{i,j}([\lambda_z]) = [v]_\mathbb{Z}, \\ &\pi_{ii}([\lambda_z]) = [p_i x - q_i u]_\mathbb{Z}, \qquad \pi^i_i([\lambda_z]) = [u]_\mathbb{Z}, \end{aligned} \tag{4-15}$$

where

$$\begin{aligned} &D(i,j) = \gcd(p_i, p_j, q_i, q_j), \\ &D_{i,j} = \gcd(p_i, p_j), \quad E_{i,j} = \gcd(q_i, q_j), \\ &r_{i,j} = p_i/D_{i,j}, \quad r_{j,i} = p_j/D_{i,j} \quad s_{i,j} = q_i/E_{i,j}, \quad s_{j,i} = q_j/E_{i,j}, \\ &m_{i,j} = D_{i,j}/D(i,j), \quad n_{i,j} = E_{i,j}/D(i,j), \\ &q_i w_{i,j} + q_j w_{j,i} = E_{i,j}, \quad x_{i,j} D_{i,j} + y_{i,j} E_{i,j} = D(i,j). \end{aligned} \tag{4-16}$$

The group Λ_b is the fiber product of $\{\Lambda_b(i,j) : i, j \in \mathbb{N}\}$ relative to the maps $\{\pi^i_{i,j}, \pi^j_{i,j}, \pi^i_i : i, j \in \mathbb{N}\}$ in the sense that Λ_b is the group of all those $\lambda_b \in \prod_{(i,j)\in\mathbb{N}^2} \Lambda_b(i,j)$ such that

$$\pi^i_{i,j}[\lambda_b(i,j)] = \pi^i_i[\lambda_b(i,i)] = \pi^i_{ki}[\lambda_b(k,i)] \quad \textit{for } i, j, k \in \mathbb{N}. \tag{4-17}$$

We will prove the theorem in several steps.

First, we observe that the asymmetrization of $f_{i,j,k}$ is given by

$$\begin{aligned} \mathrm{AS}\, f_{i,j,k} &= 2(e_i e_j) \wedge e_k - 3e_i \wedge (e_j e_k) + e_j \wedge (e_i e_k) \\ &\qquad\qquad\qquad\qquad - 2(e_i e_k) \wedge e_j - e_k \wedge (e_i e_j) \\ &= 3\big((e_j e_k) \wedge e_i - (e_i e_k) \wedge e_j + (e_i e_j) \wedge e_k\big). \end{aligned} \tag{4-18}$$

Lemma 4.3. (i) *The difference $X_a - Y_a$ is equal to $X_{\mathrm{AS}\,a}$ on $M \times H_{\mathrm{m}}$. In particular, if the integers*

$$e_{i,j}(m)e_k(g),\ e_{j,k}(m)e_i(g),\ e_{i,k}(m)e_j(g),\ e_{j,k}(m)e_i(g)$$

are all divisible by $\gcd(p_i, p_j, p_k)$, *then for each $a \in Z$*

$$Y_a(i,j,k)(m;g) \equiv X_a(i,j,k)(m;g) \mod \mathbb{Z} \quad \text{for } m \in M \text{ and } g \in H_{\mathrm{m}}.$$

Therefore, if either $g \in L$ or $m \in L \wedge H_{\mathrm{m}}$, then

$$\begin{aligned} X_a(i,j,k)(m;g) &\equiv Y_a(i,j,k)(m;g) \mod \mathbb{Z}, \\ X_a(i,j,k)(h_1 \wedge g; h_2) &\equiv Y_a(i,j,k)(h_1 \wedge g; h_2) \mod \mathbb{Z} \end{aligned} \tag{4-19}$$

for each $h_1, h_2 \in H_{\mathrm{m}}$.

(ii) *For every $m \in M$ and $g \in H_{\mathrm{m}}$ and $i < k$ we have*

$$X_a(i,k)(m;g) = Y_a(i,k)(m;g). \tag{4-20}$$

Proof. (i) We simply compute for $i < j < k$:

$$\begin{aligned} &(X_a(i,j,k) - Y_a(i,j,k))(m;g) \\ &\quad = a(i,j,k)e_{j,k}(m)e_i(g) + a(j,i,k)e_{i,k}(m)e_j(g) \\ &\qquad\qquad + a(k,i,j)e_{i,j}(m)e_k(g) \\ &\qquad - a(i,j,k)(e_{i,k}(m)e_j(g) - e_{i,j}(m)e_k(g)) \\ &\qquad - a(j,i,k)(e_{i,j}(m)e_k(g) + e_{j,k}(m)e_i(g)) \\ &\qquad - a(k,i,j)(e_{i,k}(m)e_j(g) - e_{j,k}(m)e_i(g)) \\ &\quad = (a(i,j,k) - a(j,i,k) + a(k,i,j)) \\ &\qquad\qquad \times (e_{j,k}(m)e_i(g) - e_{i,k}(m)e_j(g) + e_{i,j}(m)e_k(g)). \end{aligned}$$

Thus we conclude $(X_a - Y_a)(m;g) = X_{\mathrm{AS}\,a}(m;g)$ for $m \in M$ and $g \in H_{\mathrm{m}}$.

(ii) The assertion follows from an easy direct computation. ♡

Lemma 4.4. *If $a \in \mathbb{R}^\Delta$ is asymmetric modulo $((1/(p_i p_j p_k))\mathbb{Z})$ in that*

$$(\mathrm{AS}\, a)(i, j, k) = a(i, j, k) - a(j, i, k) + a(k, i, j) \in ((1/(p_i p_j p_k))\mathbb{Z}) \tag{4-21}$$

for each triplet $i < j < k$, then the cochain μ_a of (4-11), that is,

$$\mu_a(g; h) = \exp(2\pi \mathrm{i}(V_a(g; h))) \quad \textit{for } g, h \in L,$$

is a second cocycle $\mu_a \in \mathrm{Z}^2(L, \mathbb{T})$.

Proof. Observing

$$(\partial_L \mu_a)(g_1; g_2; g_3) = \exp(2\pi \mathrm{i}(\partial_L V_a(g_1; g_2; g_3))) \quad \text{for } g_1, g_2, g_3 \in L,$$

we compute the coboundary of V_a:

$$\begin{aligned}
\partial_L V_a(i, j, k) &= \partial_L Z_a(i, j, k) + \partial_L U_a(i, j, k)\\
&= a(i, j, k)(e_j \otimes e_i \otimes e_k - e_k \otimes e_i \otimes e_j)\\
&\quad + a(j, i, k)(e_k \otimes e_i \otimes e_j + e_i \otimes e_j \otimes e_k)\\
&\quad + a(k, i, j)(e_j \otimes e_i \otimes e_k - e_i \otimes e_j \otimes e_k)\\
&\quad + \tfrac{1}{6}\partial_L\big(a(i, j, k) f_{i,j,k} + a(j, i, k) f_{j,i,k} + a(k, i, j) f_{k,i,j}\\
&\qquad\qquad - (\mathrm{AS}\, a)(i, j, k) f_{i,j,k}\big)\\
&= a(i, j, k)(e_j \otimes e_i \otimes e_k - e_k \otimes e_i \otimes e_j)\\
&\quad + a(j, i, k)(e_k \otimes e_i \otimes e_j + e_i \otimes e_j \otimes e_k)\\
&\quad + a(k, i, j)(e_j \otimes e_i \otimes e_k - e_i \otimes e_j \otimes e_k)\\
&\quad + \tfrac{1}{6}\big(a(i, j, k)(\det_{ijk} - 6 e_i \otimes e_j \otimes e_k)\\
&\qquad + a(j, i, k)(\det_{jik} - 6 e_j \otimes e_i \otimes e_k)\\
&\qquad\quad + a(k, i, j)(\det_{kij} - 6 e_k \otimes e_i \otimes e_j)\\
&\qquad\qquad - (\mathrm{AS}\, a)(i, j, k)(\det_{ijk} - 6 e_i \otimes e_j \otimes e_k)\big)\\
&\equiv -(\mathrm{AS}\, a)(i, j, k)(e_i \otimes e_j \otimes e_k - e_j \otimes e_i \otimes e_k + e_k \otimes e_i \otimes e_j)\\
&\equiv 0 \quad \operatorname{mod} \mathbb{Z} \quad \text{on } L \times L \times L,
\end{aligned}$$

since $e_i \otimes e_j \otimes e_k$ takes values in $p_i p_j p_k \mathbb{Z}$ on $L \times L \times L$. Also we have

$$\begin{aligned}
\partial_L V_a(i, k) &= \partial_L Z_a(i, k) + \partial_L U_a(i, k)\\
&= a(i, i, k) e_i \otimes e_i \otimes e_k + a(k, i, k) e_k \otimes e_i \otimes e_k - a(i, i, k) e_i \otimes e_i \otimes e_k\\
&\qquad + a(k, i, k)(e_k \otimes e_k \otimes e_i - e_k \otimes (e_i \otimes e_k + e_k \otimes e_i))\\
&= 0.
\end{aligned}$$

Hence μ_a is a second cocycle on L. ♡

Lemma 4.5. (i) *For every* $(a, b) \in \mathrm{Z}$, *the pair* $\{\lambda_{a,b}, \mu_a\}$ *is a characteristic cocycle in* $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$.

(ii) *Every characteristic cocycle* $(\lambda, \mu) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$ *is cohomologous to some* $(\lambda_{a,b}, \mu_a)$.

(iii) *The characteristic cocycle* $\{\lambda_{a,b}, \mu_a\} \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$ *is a coboundary if and only if* $(a, b) \in \mathrm{B}$.

Proof. (i) We first check the cocycle identities for $g, g_1, g_2 \in L$ and $h, h_1, h_2 \in H_{\mathrm{m}}$:

(a) $$((\partial_L \otimes \mathrm{id})\lambda_{a,b})(g_1; g_2; h) = \mu_a(h^{-1}g_1h; h^{-1}g_2h)/\mu_a(g_1; g_2) = \lambda_{a,b}(g_2 \wedge h; g_1),$$

(b) $$((\mathrm{id} \otimes \partial_{H_{\mathrm{m}}})\lambda_{a,b})(g; h_1; h_2) = 1/\lambda_{a,b}(g \wedge h_1; h_2) = \lambda_{a,b}(h_1 \wedge g; h_2),$$

(c) $$\lambda_{a,b}(g; h) = \mu_a(h; h^{-1}gh)/\mu_a(g; h) \quad \text{for } g, h \in L.$$

Second, we compute for $g_1, g_2 \in L$ and $h \in H_{\mathrm{m}}$ that

$$\begin{aligned} &X_a(i, j, k)(g_2 \wedge h; g_1) \\ &\quad = a(i, j, k)e_{j,k}(g_2 \wedge h)e_i(g_1) + a(j, i, k)e_{i,k}(g_2 \wedge h)e_j(g_1) + a(k, i, j)e_{i,j}(g_2 \wedge h)e_k(g_1) \\ &\quad = a(i, j, k)e_i(g_1)(e_j(g_2)e_k(h) - e_k(g_2)e_j(h)) \\ &\qquad + a(j, i, k)e_j(g_1)(e_i(g_2)e_i(h) - e_k(g_2)e_i(h)) \\ &\qquad + a(k, i, j)e_k(g_1)(e_i(g_2)e_j(h) - e_j(g_2)e_i(h)) \\ &\quad = \big(a(i, j, k)e_i \otimes (e_j \otimes e_k - e_k \otimes e_j) + a(j, i, k)e_j \otimes (e_i \otimes e_i - e_k \otimes e_i) \\ &\qquad + a(k, i, j)e_k \otimes \big(e_i \otimes e_j - e_j \otimes e_i\big)\big)(g_1; g_2; h). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (\partial_L \otimes \mathrm{id})Y_a(i, j, k) = {} & a(i, j, k)(e_i \otimes e_j \otimes e_k - e_i \otimes e_k \otimes e_j) \\ & + a(j, i, k)(e_j \otimes e_i \otimes e_k - e_j \otimes e_k \otimes e_i) \\ & + a(k, i, j)(e_k \otimes e_i \otimes e_j - e_k \otimes e_j \otimes e_i). \end{aligned} \tag{4-22}$$

Since $X_{\mathrm{AS}\,a}(i, j, k)(g_2 \wedge h; g_1) \equiv 0 \mod \mathbb{Z}$, Lemma 4.3 yields, for each $g_1, g_2 \in L$ and $h \in H_{\mathrm{m}}$,

$$\begin{aligned} \big((\partial_L \otimes \mathrm{id})Y_a(i, j, k)\big)(g_1; g_2; h) &= X_a(i, j, k)(g_2 \wedge h; g_1) \\ &\equiv Y_a(i, j, k)(g_2 \wedge h; g_1) \mod \mathbb{Z}. \end{aligned}$$

Similarly, we have

$$((\partial_L \otimes \mathrm{id})Y_a(i, k))(g_1, g_2; h) \equiv Y_a(i, k)(g_2 \wedge h; g_1) \mod \mathbb{Z}.$$

Next, we have

$$\begin{aligned} X_a(i,j,k)&(h_1\wedge g;h_2)\\ &=a(i,j,k)e_{j,k}(h_1\wedge g)e_i(h_2)+a(j,i,k)e_{i,k}(h_1\wedge g)e_j(h_2)\\ &\qquad\qquad+a(k,i,j)e_{i,j}(h_1\wedge g)e_k(h_2)\\ &=\big(a(i,j,k)(e_k\otimes e_j-e_j\otimes e_k)\otimes e_i\\ &\qquad+a(j,i,k)(e_k\otimes e_i-e_i\otimes e_k)\otimes e_j\\ &\qquad\quad+a(k,i,j)(e_j\otimes e_i-e_i\otimes e_j)\otimes e_k\big)(g;h_1;h_2), \end{aligned} \tag{4-23}$$

$$\begin{aligned} (\mathrm{id}\otimes\partial_{H_\mathrm{m}})&Y_a(i,j,k)(g;h_1;h_2)\\ &=\big(a(i,j,k)(e_k\otimes e_j\otimes e_i-e_j\otimes e_k\otimes e_i)\\ &\qquad+a(j,i,k)(e_k\otimes e_i\otimes e_j-e_i\otimes e_k\otimes e_j)\\ &\qquad\quad+a(k,i,j)(e_j\otimes e_i\otimes e_k-e_i\otimes e_j\otimes e_k)\big)(g;h_1;h_2)\\ &=X_a(i,j,k)(h_1\wedge g;h_2) \end{aligned}$$

and $(\mathrm{id}\otimes\partial_{H_\mathrm{m}})X_{\mathrm{AS}\,a}(i,j,k)=0$. Hence Lemma 4.3 again yields, for each $g\in L$ and $h_1,h_2\in H_\mathrm{m}$,

$$\begin{aligned} (\mathrm{id}\otimes\partial_{H_\mathrm{m}})&(Y_a(i,j,k)+X_{\mathrm{AS}\,a}(i,j,k))(g;h_1;h_2)\\ &\equiv(Y_a(i,j,k)+X_{\mathrm{AS}\,a}(i,j,k))(h_1\wedge g;h_2)\mod\mathbb{Z}. \end{aligned}$$

Similarly, we get $((\mathrm{id}\otimes\partial_{H_\mathrm{m}})Y_a(i,k))(g,h_1;h_2)=Y_a(i,k)(h_1\wedge g;h_2)$ for $g\in L$ and $h_1,h_2\in H_\mathrm{m}$, and $X_{\mathrm{AS}\,a}(i,k)=0$. Thus so far we have established formulas (a) and (b).

Now we work on (c). Fixing $g,h\in L$, we compute its right hand side as

$$\begin{aligned} \frac{\mu_a(h;h^{-1}gh)}{\mu_a(g;h)}&=\frac{\mu_a(h;(g\wedge h)g)}{\mu_a(g;h)}=\lambda_a(g\wedge h;h)\frac{\mu_a(h;g)}{\mu_a(g;h)}\\ &=\lambda_a(g\wedge h;h)\frac{\mu_a(m_0(h)\mathfrak{s}_H(h);m_0(g)\mathfrak{s}_H(g))}{\mu_a(m_0(g)\mathfrak{s}_H(g);m_0(h)\mathfrak{s}_H(h))}\\ &=\exp(2\pi\mathrm{i}(X_a(g\wedge h;h)))(\exp(2\pi\mathrm{i}(\mathrm{AS}\,V_a(h;g))))\\ &=\exp(2\pi\mathrm{i}((Y_a+X_{\mathrm{AS}\,a})(g\wedge h;h)))(\exp(2\pi\mathrm{i}(\mathrm{AS}\,V_a(h;g))). \end{aligned}$$

Next we prove that

$$\lambda_a(\mathfrak{s}_H(g);\mathfrak{s}_H(h))=\lambda_a(g\wedge h;h)(\mathrm{AS}\,\mu_a)(\mathfrak{s}_H(h);\mathfrak{s}_H(g))\quad\text{for } g,h\in N.$$

First we observe that

$$X_{\mathrm{AS}\,a}(g;h)\equiv 0\bmod\mathbb{Z}\quad\text{for } g,h\in L.$$

To prove (c), we ignore the term $X_{\mathrm{AS}\,a}$ and compute

$$\begin{aligned} &X_a(i,j,k)(g\wedge h;h)\\ &\qquad= a(i,j,k)e_{j,k}(g\wedge h)e_i(h)+a(j,i,k)e_{i,k}(g\wedge h)e_j(h)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+a(k,i,j)e_{i,j}(g\wedge h)e_k(h)\\ &\qquad=\big(a(i,j,k)(e_j\otimes(e_ke_i)-e_k\otimes(e_je_i))\\ &\qquad\qquad\qquad+a(j,i,k)(e_i\otimes(e_ke_j)-e_k\otimes(e_ie_j))\\ &\qquad\qquad\qquad\qquad\qquad+a(k,i,j)(e_i\otimes(e_je_k)-e_j\otimes(e_ie_k))\big)(g;h), \end{aligned}$$

and also

$$\begin{aligned} X_a(i,k)(g\wedge h;h)&=a(i,i,k)(e_i(g)e_k(h)-e_k(g)e_i(h))e_i(h)\\ &\qquad\qquad+a(k,i,k)(e_i(g)e_k(h)-e_k(g)e_i(h))e_k(h)\\ &=a(i,i,k)(e_i\otimes(e_ie_k)-e_k\otimes e_i^2)(g;h)\\ &\qquad\qquad+a(k,i,k)(e_i\otimes e_k^2-e_k\otimes(e_ie_k))(g;h). \end{aligned}$$

Next we determine the asymmetrization of $U_a(i,j,k)$ based on (4-18):

$$\begin{aligned} \mathrm{AS}\,U_a(i,j,k)&=\tfrac{1}{6}\big(a(i,j,k)\,\mathrm{AS}\,f_{i,j,k}+a(j,i,k)\,\mathrm{AS}\,f_{j,i,k}+a(k,i,j)\,\mathrm{AS}\,f_{k,i,j}\\ &\qquad\qquad\qquad\qquad\qquad-(\mathrm{AS}\,a)(i,j,k)\,\mathrm{AS}\,f_{i,j,k}\big)\\ &=\tfrac{1}{2}\big(a(i,j,k)((e_je_k)\wedge e_i-(e_ie_k)\wedge e_j+(e_ie_j)\wedge e_k)\\ &\qquad+a(j,i,k)((e_ie_k)\wedge e_j-(e_je_k)\wedge e_i+(e_ie_j)\wedge e_k)\\ &\qquad+a(k,i,j)((e_ie_j)\wedge e_k-(e_je_k)\wedge e_i+(e_ie_k)\wedge e_j)\\ &\qquad-(a(i,j,k)-a(j,i,k)+a(k,i,j))\\ &\qquad\qquad\times((e_je_k)\wedge e_i-(e_ie_k)\wedge e_j+(e_ie_j)\wedge e_k)\big)\\ &=\tfrac{1}{2}\big(a(j,i,k)((e_ie_k)\wedge e_j-(e_je_k)\wedge e_i+(e_ie_j)\wedge e_k)\\ &\qquad+a(k,i,j)((e_ie_j)\wedge e_k-(e_je_k)\wedge e_i+(e_ie_k)\wedge e_j)\\ &\qquad+(a(j,i,k)-a(k,i,j))\\ &\qquad\qquad\times((e_je_k)\wedge e_i-(e_ie_k)\wedge e_j+(e_ie_j)\wedge e_k)\big)\\ &=-a(k,i,j)(e_je_k)\wedge e_i+a(k,i,j)(e_ie_k)\wedge e_j\\ &\qquad\qquad\qquad\qquad+a(j,i,k)(e_ie_j)\wedge e_k. \end{aligned}$$

Hence we get

$$\begin{aligned} \text{(4-24)}\quad \mathrm{AS}\,U_a(i,j,k)&=-a(k,i,j)((e_je_k)\otimes e_i-e_i\otimes(e_je_k))\\ &\qquad+a(k,i,j)((e_ie_k)\otimes e_j-e_j\otimes(e_ie_k))\\ &\qquad\qquad+a(j,i,k)((e_ie_j)\otimes e_k-e_k\otimes(e_ie_j)). \end{aligned}$$

We also check the asymmetrization of $U_a(i,k)$:

$$\begin{aligned}\mathrm{AS}\,U_a(i,k) &= a(i,i,k)e_k\wedge B_{ii} + a(k,i,k)(B_{kk}\wedge e_i - e_k\wedge(e_ie_k))\\ &= \tfrac12 a(i,i,k)(e_i^2\otimes e_k - e_k\otimes e_i^2) + \tfrac12 a(k,i,k)(e_i\otimes e_k^2 - e_k^2\otimes e_i)\\ &\qquad + a(k,i,k)((e_ie_k)\otimes e_k - e_k\otimes(e_ie_k)).\end{aligned}$$

We then combine these with the above computations for $X_a(i,j,k)$, paying attention to the order of variables in the first and second term:[1]

$$\begin{aligned}&X_a(i,j,k)(g\wedge h;h) + \mathrm{AS}\,U_a(i,j,k)(\mathfrak{s}_H(h);\mathfrak{s}_H(g))\\ &\quad= a(i,j,k)(e_j\otimes(e_ke_i) - e_k\otimes(e_je_i))\\ &\qquad + a(j,i,k)(e_i\otimes(e_ke_j) - e_k\otimes(e_ie_j))\\ &\qquad + a(k,i,j)(e_i\otimes(e_je_k) - e_j\otimes(e_ie_k))\\ &\qquad + \big(a(k,i,j)(e_je_k)\wedge e_i - a(k,i,j)(e_ie_k)\wedge e_j\\ &\qquad\qquad - a(j,i,k)(e_ie_j)\wedge e_k\big)\\ &\quad= a(i,j,k)(e_j\otimes(e_ke_i) - e_k\otimes(e_je_i))\\ &\qquad + a(j,i,k)\big(e_i\otimes(e_ke_j) - e_k\otimes(e_ie_j) - (e_ie_j)\wedge e_k\big)\\ &\qquad + a(k,i,j)\big(e_i\otimes(e_je_k) - e_j\otimes(e_ie_k) + (e_je_k)\wedge e_i - (e_ie_k)\wedge e_j\big)\\ &\quad= a(i,j,k)(e_j\otimes(e_ke_i) - e_k\otimes(e_je_i))\\ &\qquad + a(j,i,k)\big(e_i\otimes(e_ke_j) - e_k\otimes(e_ie_j) - (e_ie_j)\otimes e_k + e_k\otimes(e_ie_j)\big)\\ &\qquad + a(k,i,j)\big(e_i\otimes(e_je_k) - e_j\otimes(e_ie_k) + (e_je_k)\otimes e_i\\ &\qquad\qquad - e_i\otimes(e_je_k) - (e_ie_k)\otimes e_j + e_j\otimes(e_ie_k)\big)\\ &\quad= a(i,j,k)(e_j\otimes(e_ke_i) - e_k\otimes(e_je_i))\\ &\qquad + a(j,i,k)(e_i\otimes(e_ke_j) - (e_ie_j)\otimes e_k)\\ &\qquad + a(k,i,j)((e_je_k)\otimes e_i - (e_ie_k)\otimes e_j).\end{aligned}$$

and

$$\begin{aligned}&X_a(i,k)(g\wedge h;h) + \mathrm{AS}\,U_a(i,k)(\mathfrak{s}_H(h);\mathfrak{s}_H(g))\\ &\quad= a(i,i,k)(e_i\otimes(e_ie_k) - e_k\otimes e_i^2)\\ &\qquad + a(k,i,k)(e_i\otimes e_k^2 - e_k\otimes(e_ie_k))\\ &\qquad + \tfrac12 a(i,i,k)(e_k\otimes e_i^2 - e_i^2\otimes e_k) + \tfrac12 a(k,i,k)(e_k^2\otimes e_i - e_i\otimes e_k^2)\\ &\qquad + a(k,i,k)(e_k\otimes(e_ie_k) - (e_ie_k)\otimes e_k)\\ &\quad= a(i,i,k)\big(e_i\otimes(e_ie_k) - \tfrac12(e_k\otimes e_i^2 + e_i^2\otimes e_k)\big)\\ &\qquad + a(k,i,k)\big(\tfrac12(e_i\otimes e_k^2 + e_k^2\otimes e_i) - (e_ie_k)\otimes e_k\big).\end{aligned}$$

[1] In the first term, the variables g and h appear in this order, but in the second they appear in the opposite order.

We now compare these with $Y_a(i,j,k)$:

$$\begin{aligned}
&Y_a(i,j,k)(\mathfrak{s}_H(g);\mathfrak{s}_H(h))\\
&\quad= a(i,j,k)(e_j\otimes(e_ie_k)-e_k\otimes(e_ie_j))\\
&\qquad + a(j,i,k)(e_i\otimes(e_je_k)-(e_ie_j)\otimes e_k)\\
&\qquad + a(k,i,j)((e_je_k)\otimes e_i-(e_ke_i)\otimes e_j)\\
&\quad= X_a(i,j,k)(g\wedge h;h)+\mathrm{AS}\,U_a(i,j,k)(\mathfrak{s}_H(h);\mathfrak{s}_H(g))\\
&\quad\equiv Y_a(i,j,k)(g\wedge h;h)+\mathrm{AS}\,U_a(i,j,k)(\mathfrak{s}_H(h);\mathfrak{s}_H(g)),
\end{aligned}$$

and

$$\begin{aligned}
&Y_a(i,k)(\mathfrak{s}_H(g);\mathfrak{s}_H(h))\\
&\quad= \big(a(i,i,k)(B_{ii}\otimes e_k+e_k\otimes B_{ii}-B_{ik}\otimes e_i-e_i\otimes B_{ki})\\
&\qquad + a(k,i,k)(B_{ki}\otimes e_k+e_k\otimes B_{ik}-B_{kk}\otimes e_i-e_i\otimes B_{kk})\big)\\
&\quad= a(i,i,k)\big(e_i\otimes(e_ie_k)-\tfrac{1}{2}(e_k\otimes e_i^2+e_i^2\otimes e_k)\big)\\
&\qquad + a(k,i,k)\big(\tfrac{1}{2}(e_i\otimes e_k^2+e_k^2\otimes e_i)-(e_ie_k)\otimes e_k\big)\\
&\quad= X_a(i,k)(g\wedge h;h)+\mathrm{AS}\,U_a(i,k)(\mathfrak{s}_H(h);\mathfrak{s}_H(g))\\
&\quad= Y_a(i,k)(g\wedge h;h)+\mathrm{AS}\,U_a(i,k)(\mathfrak{s}_H(h);\mathfrak{s}_H(g)).
\end{aligned}$$

Therefore, we have

$$\lambda_{a,b}(\mathfrak{s}_H(g);\mathfrak{s}_H(h))=\lambda_{a,b}(g\wedge h;h)\frac{\mu_a(\mathfrak{s}_H(h);\mathfrak{s}_H(g))}{\mu_a(\mathfrak{s}_H(g);\mathfrak{s}_H(h))}.$$

Since we have $Y_a(mg;nh)=Y_a(m;h)+Y_a(g;n)+Y_a(g;h)$ for every $m,n\in M$ and $g,h\in H_\mathrm{m}$, we get, for each $m,n\in M$ and $g,h\in N$,

$$\begin{aligned}
&\lambda_{a,b}(m\mathfrak{s}_H(g);n\mathfrak{s}_H(h))\\
&\quad=\lambda_{a,b}(m;\mathfrak{s}_H(h))\lambda_{a,b}(\mathfrak{s}_H(g);n)\lambda_{a,b}(\mathfrak{s}_H(g);\mathfrak{s}_H(h))\\
&\quad=\frac{\lambda_{a,b}(m;\mathfrak{s}_H(h))}{\lambda_{a,b}(n;\mathfrak{s}_H(g))}\lambda_{a,b}(g\wedge h;h)\frac{\mu_a(\mathfrak{s}_H(h);\mathfrak{s}_H(g))}{\mu_a(\mathfrak{s}_H(g);\mathfrak{s}_H(h))}\\
&\quad=\frac{\mu_a(n\mathfrak{s}_H(h);(n\mathfrak{s}_H(h))^{-1}m\mathfrak{s}_H(g)n\mathfrak{s}_H(h))}{\mu_a(m\mathfrak{s}_H(g);n\mathfrak{s}_H(h))}.
\end{aligned}$$

This proves the cocycle identity (c). Consequently $\{\lambda_{a,b},\mu_a\}$ is a characteristic cocycle in $\mathrm{Z}(H_\mathrm{m},L,M,\mathbb{T})$.

(ii) Suppose that $(\lambda,\mu)\in\mathrm{Z}(H_\mathrm{m},L,M,\mathbb{T})$. Since M is central in H_m, the λ-part is a bicharacter on $M\times H_\mathrm{m}$, so there exists an $a=\{a(i,j,k)\}\in\mathbb{R}^\Delta$ such that

$$\lambda(m;h)=\exp\Big(2\pi\mathrm{i}\Big(\sum_{i,j<k}a(i,j,k)e_{j,k}(m)e_i(h)\Big)\Big)\quad\text{for } m\in M \text{ and } h\in H_\mathrm{m}.$$

As $[H_{\mathrm{m}}, H_{\mathrm{m}}] = M$, for each fixed $m \in M$ the character $\lambda(m; \cdot)$ on H_{m} must vanish on M, that is, $\lambda(m; n) = 1$ for $m, n \in M$. Thus the restriction μ_M of the second cocycle μ to M is a coboundary. Hence, replacing μ by a cohomologous cocycle if necessary, we may and do assume that $\mu_M = 1$. Now consider the corresponding $E \in \mathrm{Xext}(H_{\mathrm{m}}, L, M, \mathbb{T})$, with diagram

$$1 \to \mathbb{T} \longrightarrow E \underset{\mathfrak{s}_j}{\overset{j}{\rightleftarrows}} L \to 1.$$

Redefining the cross-section $\mathfrak{s}_j$ as $\mathfrak{s}_j(m\mathfrak{s}_H(g)) = \mathfrak{s}_j(m)\mathfrak{s}_j(\mathfrak{s}_H(g))$ for $m \in M$ and $g \in N$, we may and do assume that $\mu(m; g) = 1$ for $m \in M$ and $g \in L$. Now we compute the second cocycle μ with $m, n \in M$ and $g, h \in L$:

$$\begin{aligned}\mu(mg; nh)\mathfrak{s}_j(mgnh) &= \mathfrak{s}_j(mg)\mathfrak{s}_j(ng) = \mathfrak{s}_j(m)\mathfrak{s}_j(g)\mathfrak{s}_j(n)\mathfrak{s}_j(h)\\ &= \mathfrak{s}_j(m)\lambda(n; g)\mathfrak{s}_j(n)\mathfrak{s}_j(g)\mathfrak{s}_j(h)\\ &= \lambda(n; g)\mu(g; h)\mathfrak{s}_j(m)\mathfrak{s}_j(n)\mathfrak{s}_j(gh)\\ &= \lambda(n; g)\mu(g; h)\mathfrak{s}_j(mngh) = \lambda(n; g)\mu(g; h)\mathfrak{s}_j(mgnh),\end{aligned}$$

which gives $\mu(mg; nh) = \lambda(n; g)\mu(g; h)$ for $m, n \in M$ and $g, h \in L$. In particular, we have

$$\mu(g; h) = \lambda(m_0(h); g)\mu(\mathfrak{s}_H(\pi_G(g)); \mathfrak{s}_H(\pi_G(h)) \quad \text{for } g, h \in L,$$

where $m_0(h) = h\mathfrak{s}_H(\pi_G(h))^{-1} \in M$. Now with $g_1, g_2, g_3 \in N$, we compute the coboundary:

$$\begin{aligned}1 &= (\partial_L\mu)(\mathfrak{s}_H(g_1); \mathfrak{s}_H(g_2); \mathfrak{s}_H(g_3))\\ &= \frac{\mu(\mathfrak{s}_H(g_2); \mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1); \mathfrak{s}_H(g_2)\mathfrak{s}_H(g_3))}{\mu(\mathfrak{s}_H(g_1)\mathfrak{s}_H(g_2); \mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1); \mathfrak{s}_H(g_2))}\\ &= \frac{\mu(\mathfrak{s}_H(g_2); \mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1); \mathfrak{n}_M(g_2; g_3)\mathfrak{s}_H(g_2+g_3))}{\mu(\mathfrak{n}_M(g_1; g_2)\mathfrak{s}_H(g_1+g_2); \mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1); \mathfrak{s}_H(g_2))}\\ &= \lambda(\mathfrak{n}_M(g_2; g_3); \mathfrak{s}_H(g_1))\frac{\mu(\mathfrak{s}_H(g_2); \mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1); \mathfrak{s}_H(g_2+g_3))}{\mu(\mathfrak{s}_H(g_1+g_2); \mathfrak{s}_H(g_3))\mu(\mathfrak{s}_H(g_1); \mathfrak{s}_H(g_2))}.\end{aligned} \tag{4-25}$$

Thus the cocycle $c_a \in \mathrm{Z}^3(N, \mathbb{T})$ given by

$$\begin{aligned}c_a(g_1; g_2; g_3) &= \lambda(\mathfrak{n}_M(g_2; g_3); g_1)\\ &= \exp\Big(2\pi\mathrm{i}\Big(\sum_{i,j<k} a(i, j, k)e_{j,k}(\mathfrak{n}_M(g_2; g_3))e_i(g_1)\Big)\Big)\\ &= \exp\Big(2\pi\mathrm{i}\Big(\sum_{i,j<k} a(i, j, k)e_i(g_1)e_j(g_2)e_k(g_3)\Big)\Big)\end{aligned}$$

is a coboundary in $\mathrm{Z}^3(N, \mathbb{T})$. Thus we get, for every $g_1, g_2, g_3 \in N$,

$$\begin{aligned}
1 &= (\mathrm{AS}\, c_a)(g_1, g_2, g_3) \\
&= \exp\Bigl(2\pi\mathrm{i}\Bigl(\sum_{i,j<k} a(i,j,k) \sum_{\sigma\in\Pi(i,j,k)} \operatorname{sign}(\sigma) e_i(g_{\sigma(i)}) e_j(g_{\sigma(j)}) e_k(g_{\sigma(k)})\Bigr)\Bigr) \\
&= \exp\Bigl(2\pi\mathrm{i}\Bigl(\sum_{i,j<k} a(i,j,k) \det_{ijk}(g_1; g_2; g_3)\Bigr)\Bigr) \\
&= \exp\Bigl(2\pi\mathrm{i}\Bigl(\sum_{(i,j,k)\in\Delta} (\mathrm{AS}\, a)(i,j,k) \det_{ijk}(g_1, g_2, g_3)\Bigr)\Bigr).
\end{aligned}$$

Thus the coefficient $a = \{a(i,j,k)\} \in \mathbb{R}^\Delta$ is asymmetric in the sense of Lemma 4.4, so that it gives the second cocycle $\mu_a = \exp(2\pi\mathrm{i} V_a) \in \mathrm{Z}^2(L, \mathbb{T})$. Then the cocycle $\mu\mu_a^{-1} \in \mathrm{Z}^2(L, \mathbb{T})$ falls in the subgroup $\pi_G^*(\mathrm{Z}^2(N, \mathbb{T})) \subset \mathrm{B}^2(L, \mathbb{T})$ because

$$\begin{aligned}
\mu(m\mathfrak{s}_H(g); n\mathfrak{s}_H(h)) &= \lambda(n; \mathfrak{s}_H(g))\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h)) \\
&= \frac{\mu_a(m\mathfrak{s}_H(g); n\mathfrak{s}_H(h))}{\mu_a(\mathfrak{s}_H(g); \mathfrak{s}_H(h))}\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h)) \\
&= \frac{\mu(\mathfrak{s}_H(g); \mathfrak{s}_H(h))}{\mu_a(\mathfrak{s}_H(g); \mathfrak{s}_H(h))}\mu_a(m\mathfrak{s}_H(g); n\mathfrak{s}_H(h)), \\
\mu_a^{-1}\mu &= \pi_G^* \circ \mathfrak{s}_H^*(\mu\mu_a^{-1}) \in \pi_G^*(\mathrm{Z}^2(N, \mathbb{T})).
\end{aligned}$$

Thus there exists a cochain $f \in \mathrm{C}^1(L, \mathbb{T})$ such that

$$\mu_a(g; h) = \mu(g; h)\frac{f(g)f(h)}{f(gh)} \quad \text{for } g, h \in L.$$

Since $1 = \mu(m; h) = \mu_a(m; h)$ for $m \in M$ and $h \in L$, we have $f(mh) = f(m)f(h)$. Since $(\partial_1 f)(m; h) = 1$ for $m \in M$ and $h \in H_{\mathrm{m}}$, we have $\partial f(\lambda, \mu) = (\lambda, \mu_a)$.

Next we look at one of the cocycle identities, for $g_1, g_2 \in L$ and $h \in H_{\mathrm{m}}$:

$$\begin{aligned}
\lambda(g_1g_2; h) &= \lambda(g_1; h)\lambda(g_2; h)\frac{\mu_a(g_1; g_2)}{\mu_a(h^{-1}g_1h; h^{-1}g_2h)} \\
&= \frac{1}{\lambda(g_2 \wedge h; g_1)}\lambda(g_1; h)\lambda(g_2; h) \\
&= \lambda(h \wedge g_2; g_1)\lambda(g_1; h)\lambda(g_2; h) \\
&= \exp\Bigl(2\pi\mathrm{i}\Bigl(\sum_{i,j<k} a(i,j,k) e_i(g_1) e_{j,k}(h \wedge g_2)\Bigr)\Bigr)\lambda(g_1; h)\lambda(g_2; h),
\end{aligned}$$

which gives the partial coboundary condition

$$(\partial_L \otimes \mathrm{id})\lambda = \exp\Bigl(2\pi\mathrm{i}\Bigl(\sum_{i,j<k} a(i,j,k) e_i \otimes (e_j \otimes e_k - e_k \otimes e_j)\Bigr)\Bigr).$$

Another cocycle identity for $g \in L$ and $h_1, h_2 \in H_{\mathrm{m}}$ is

$$\begin{aligned}\lambda(g; h_1h_2) &= \lambda(g; h_1)\lambda(h_1^{-1}gh_1; h_2),\\ &= \lambda(g \wedge h_1; h_2)\lambda(g; h_1)\lambda(g; h_2)\\ &= \exp\Big(2\pi\mathrm{i}\Big(\sum_{i,j<k} a(i,j,k)e_{j,k}(g \wedge h_1)e_i(h_2)\Big)\Big)\lambda(g; h_1)\lambda(g; h_2);\end{aligned}$$

this gives the second partial coboundary condition

$$(\mathrm{id} \otimes \partial_{H_{\mathrm{m}}})\lambda = \exp\Big(2\pi\mathrm{i}\Big(\sum_{i,j<k} a(i,j,k)(e_k \otimes e_j - e_j \otimes e_k) \otimes e_i\Big)\Big).$$

Setting $\eta_a = \exp(2\pi\mathrm{i}(Y_a))$, we obtain, by (4-22) and (4-23),

$$(\partial_L \otimes \mathrm{id})\lambda = (\partial_L \otimes \mathrm{id})\eta_a \quad \text{and} \quad (\mathrm{id} \otimes \partial_{H_{\mathrm{m}}})\lambda = (\mathrm{id} \otimes \partial_{H_{\mathrm{m}}})\eta_a.$$

Therefore the cochain $\bar{\eta}_a\lambda = \chi$ is a bicharacter on $L \times H_{\mathrm{m}}$. Since $M = [H_{\mathrm{m}}, H_{\mathrm{m}}]$, the bicharacter χ vanishes on $L \times M$, that is, $\lambda(m; g) = \eta_a(m; g)$ for $m \in M$ and $g \in L$. Thus we get

$$\begin{aligned}1 &= \lambda(m; g)\bar{\eta}_a(m; g) = \exp(2\pi\mathrm{i}(X_a(m; g) - Y_a(m; g)))\\ &= \exp(2\pi\mathrm{i}(X_{\mathrm{AS}\,a}(m; g))) = \lambda_{\mathrm{AS}\,a}(m; g),\end{aligned}$$

which is equivalent to the fact that $(\mathrm{AS}\,a)(i,j,k) \in ((1/\gcd(p_i, p_j, p_k))\mathbb{Z})$. Thus we conclude the cocycle condition (4-9Z-a) on the parameter $\{a(i,j,k)\}$. Therefore the coefficient $a \in \mathbb{R}^\Delta$ satisfies the requirement for the element $(a, 0) \in Z$. Consequently, it follows from (i) that $(\lambda_{a,0}, \mu_a) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. Then the cocycle identity (c) for $(\lambda_{a,0}, \mu_a)$ yields that

$$\lambda(g; h) = \frac{\mu_a(h; h^{-1}gh)}{\mu_a(g; h)} = \lambda_{a,0}(g; h) = \eta_a(g; h) \quad \text{for } g, h \in L.$$

Thus the bicharacter $\chi = \bar{\eta}_a\lambda$ on $L \times H_{\mathrm{m}}$ vanishes on $L \times L$. Since Lemma 4.3(i) yields for each $m \in M$ and $h \in H_{\mathrm{m}}$ that

$$\begin{aligned}\chi(m; h) &= \lambda(m; h)\bar{\eta}_a(m; h) = \lambda_a(m; h)\bar{\eta}_a(m; h)\\ &= \lambda_{\mathrm{AS}\,a}(m; h) = \exp\Big(2\pi\mathrm{i}\Big(\sum_{i,j<k} (\mathrm{AS}\,a)(i,j,k)e_{j,k}(m)e_i(h)\Big)\Big),\end{aligned}$$

we conclude that χ is of the form

$$\chi(g; h) = \chi_0(\pi_G(g); \pi_G(h)) \exp\Big(2\pi\mathrm{i}\Big(\sum_{i<j<k} (\mathrm{AS}\,a)(i,j,k)e_{j,k}(g)e_i(h)\Big)\Big)$$

for $g \in L$ and $h \in H_{\mathrm{m}}$, where χ_0 is a bicharacter on $N \times G_{\mathrm{m}}$ and $\pi_G : H_{\mathrm{m}} \to G_{\mathrm{m}}$ the quotient map with $M = \mathrm{Ker}(\pi_G)$. We choose $b(i, j) \in \mathbb{R}$ so that

$$\exp(2\pi \mathrm{i}(b(i, j))) = \chi_0(b_i; z_j) \quad \text{for } i \in \mathbb{N} \text{ and } j \in \mathbb{N}_0.$$

Then we must have

$$1 = \chi_0(b_i; b_j) = \chi_0(b_i; p_j z_j - q_j z_0) = \exp(2\pi \mathrm{i}(b(i, j)p_j - b(i, 0)q_j)),$$

so that $b(i, j) \in \mathbb{R}$ for $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$ satisfies the condition

$$b(i, j)p_j \equiv b(i, 0)q_j \mod \mathbb{Z} \quad \text{for } i, j \in \mathbb{N}.$$

Hence χ_0 is written in the form

$$\chi_0(g; \tilde{h}) = \exp\Bigl(2\pi \mathrm{i}\Bigl(\sum_{i,j \in \mathbb{N}} b(i, j)e_{i,N}(g)e_j(\tilde{h}) + \sum_{i \in \mathbb{N}} b(i, 0)e_{i,N}(g)\tilde{e}_0(\tilde{h})\Bigr)\Bigr)$$

for each pair $g \in N$ and $\tilde{h} \in H_{\mathrm{m}}$, where each coefficient $b(i, j)$ satisfies

$$b(i, j)p_j - b(i, 0)q_j \in \mathbb{Z} \quad \text{for } i, j \in \mathbb{N} \quad \text{and} \quad b(0, i) = 0 \quad \text{for } i \in \mathbb{N}_0.$$

Consequently the pair (a, b) is a member of Z and we conclude that (λ, μ) is cohomologous to the characteristic cocycle $(\lambda_{a,b}, \mu_a) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$.

(iii) Suppose $(\lambda, \mu) = (\lambda_{a,b}, \mu_a) = \partial f$ with $f \in \mathrm{C}^1(L, \mathbb{T})$. Since $\mu_M = 1$ and $\mu_a(m; g) = 1$ for $m \in M$ and $g \in L$, we have $f(mg) = f(m)f(g)$ for $m \in M$ and $g \in L$, so that the restriction of f to M is of the form

$$f_c(m) = \exp\Bigl(2\pi \mathrm{i}\Bigl(\sum_{1 \le i < j} c(i, j)e_{i,j}(m)\Bigr)\Bigr) \quad \text{for } m \in M.$$

Since M is central in H_{m}, we have for every pair $(m, g) \in M \times H_{\mathrm{m}}$

$$1 = \frac{f_c(g^{-1}mg)}{f_c(m)} = \lambda(m; g) = \exp\Bigl(2\pi \mathrm{i}\Bigl(\sum_{i, j<k} a(i, j, k)e_{j,k}(m)e_i(g)\Bigr)\Bigr),$$

which yields the integrality condition $a(i, j, k) \in \mathbb{Z}$ for every $(i, j, k) \in \Delta$. that $\lambda_a(m; h) = 1$ for $m \in M$ and $h \in H_{\mathrm{m}}$. Since $\chi = 1$ on $L \times L$, for every $g, h \in L$ we have

$$\begin{aligned} 1 &= \lambda_{0,b}(g; h) = \lambda_a(m_0(g); h)\lambda_{0,b}(\mathfrak{s}_H(g); h) = \lambda(g; h) \\ &= f(h^{-1}gh)/f(g) = f_c(g \wedge h); \end{aligned}$$

$$c(i, j) \in \Bigl(\frac{1}{p_i p_j}\mathbb{Z}\Bigr) \quad \text{for } i, j \in \mathbb{N}.$$

This computation also shows that

$$\lambda_{0,b}(g; h) = f_c(g \wedge h) \quad \text{for } g \in L \text{ and } h \in H_{\mathrm{m}}.$$

Furthermore, we have for each $m, n \in M$ and $g, h \in L$

$$\mu_a(mg; nh) = \lambda_{a,b}(n; g)\mu_a(g; h) = \mu_a(g; h),$$

so that μ_a is of the form $\mu_a = \pi_G^*(\tilde{\mu})$ with

$$\tilde{\mu}_a(g; h) = \exp(2\pi\mathrm{i}(U_a(g; h))) \quad \text{for } g, h \in N.$$

Since $\lambda_a(\mathfrak{n}_M(g_2; g_3); g_1) = 1$ for $g_1, g_2, g_3 \in H_{\mathrm{m}}$, we have $\tilde{\mu}_a \in \mathrm{Z}^2(N, \mathbb{T})$ by (4-25). We first compute for each $g, h \in L$ that

$$(\mathrm{AS}\,\mu_a)(g; h) = \frac{f(g)f(h)}{f(gh)} \frac{f(hg)}{f(g)f(h)} = \frac{f(hgh^{-1}h)}{f(gh)} = f_c(h \wedge g) = 1.$$

Since $\mathrm{AS}\,U_a(i, j, k)$ is also integer valued, we have

$$\begin{aligned}
\mathrm{AS}\,\mu_a &= \exp\Big(2\pi\mathrm{i}\Big(\sum_{i<k} \mathrm{AS}\,U_a(i, k)\Big)\Big) \\
&= \exp\Big(2\pi\mathrm{i}\Big(\sum_{i<k}\big(\tfrac{1}{2}a(i, i, k)(e_i^2 \otimes e_k - e_k \otimes e_i^2)\big)\Big)\Big) \\
&\qquad \times \exp\Big(2\pi\mathrm{i}\Big(\sum_{i<k} \tfrac{1}{2}a(k, i, k)(e_i \otimes e_k^2 - e_k^2 \otimes e_i)\Big)\Big) \\
&= 1.
\end{aligned}$$

Thus we get

$$a(i, i, k), a(k, i, k) \in 2\mathbb{Z} \quad \text{and} \quad U_a(i, k) \equiv 0 \mod \mathbb{Z}.$$

Consequently, $\tilde{\mu}_a$ is a coboundary as a member of $\mathrm{Z}^2(N, \mathbb{T})$. Hence there exists a cochain $\tilde{f} \in \mathrm{C}^1(N, \mathbb{T})$ such that

$$\frac{f(g)f(h)}{f(gh)} = \mu_a(g; h) = \tilde{\mu}_a(\pi_G(g); \pi_G(h)) = \frac{\tilde{f}(\pi_G(g))\tilde{f}(\pi_G(h))}{\tilde{f}(\pi_G(gh))}.$$

Thus f is of the form

$$\begin{aligned}
f(g) &= f_c(m_0(g))f(\mathfrak{s}_H(g)) = \chi(g)\tilde{f}(\pi_G(g)) \quad \text{for } g \in L, \\
f_c(m) &= \chi(m) \quad \text{for } m \in M.
\end{aligned}$$

where $\chi \in \mathrm{Hom}(L, \mathbb{T})$. Since $L/[L, L] \cong M/PMP \oplus N$, the homomorphism χ is of the form

$$\chi(g) = \exp\Big(2\pi\mathrm{i}\Big(\sum_{j<k} c(j, k)e_{j,k}(g) + \sum_{k\in\mathbb{N}_0} c(k)\tilde{e}_k(g)\Big)\Big) \quad \text{for } g \in L,$$

where

$$c(i, j) \in \Big(\frac{1}{p_i p_j}\mathbb{Z}\Big) \quad \text{for } i < j \quad \text{and} \quad c(k) \in \mathbb{R}.$$

Since Y_a is integer valued, the λ part becomes for $g \in N$ and $h \in H_{\mathrm{m}}$

$$\begin{aligned}\lambda(g;h) &= \exp\Big(2\pi \mathrm{i}\Big(\sum_{j\in\mathbb{N},k\in\mathbb{N}_0} b(j,k)e_{j,N}(g)\tilde{e}_k(h)\Big)\Big),\\ &= \frac{f(h^{-1}gh)}{f(g)} = \frac{f((g\wedge h)g)}{f(g)} = f_c(g\wedge h)\\ &= \exp\Big(2\pi \mathrm{i}\Big(\sum_{1\le j<k} c(j,k)e_{j,k}(g\wedge h)\Big)\Big)\\ &= \exp\Big(2\pi \mathrm{i}\Big(\sum_{1\le j<k} c(j,k)\big(e_j(g)e_k(h)-e_k(g)e_j(h)\big)\Big)\Big)\\ &= \exp\Big(2\pi \mathrm{i}\Big(\sum_{1\le j<k} c(j,k)\big(p_je_{j,N}(g)e_k(h)-p_ke_{k,N}(g)e_j(h)\big)\Big)\Big).\end{aligned}$$

Hence we conclude that for $j<k$ and $i\in\mathbb{N}$

$$\begin{aligned} b(i,0)&\in\mathbb{Z}, \qquad b(j,k)\equiv c(j,k)p_j \mod \mathbb{Z},\\ b(i,i)&\in\mathbb{Z}, \qquad b(k,j)\equiv -c(j,k)p_k \mod \mathbb{Z}.\end{aligned}$$

Thus we have for $i<j$

$$\begin{aligned} b(i,j) &= c(i,j)p_i + m_{i,j} \quad \text{for some } m_{i,j}\in\mathbb{Z},\\ b(j,i) &= -c(i,j)p_j + m_{j,i} \quad \text{for some } m_{j,i}\in\mathbb{Z},\end{aligned}$$

$$\frac{b(i,j)}{p_i}+\frac{b(j,i)}{p_j} = \frac{m_{i,j}}{p_i}+\frac{m_{j,i}}{p_j} \in \Big(\frac{1}{p_i}\mathbb{Z}\Big)+\Big(\frac{1}{p_j}\mathbb{Z}\Big) = \Big(\frac{1}{\mathrm{lcm}(p_i,p_j)}\mathbb{Z}\Big).$$

Conversely suppose $(a,b)\in \mathrm{B}$, that is,

$$a(i,j,k)\in\mathbb{Z} \quad \text{for } i<j<k \quad \text{and} \quad a(i,i,k), a(k,i,k)\in 2\mathbb{Z} \quad \text{for } i<k,$$

and $b(i,j)/p_i + b(j,i)/p_j \in ((1/\mathrm{lcm}(p_i,p_j))\mathbb{Z})$; also $b(i,i)\in\mathbb{Z}$ and $b(i,0)\in\mathbb{Z}$ for $i\in\mathbb{N}$. So we can write

$$\frac{b(i,j)}{p_i}+\frac{b(j,i)}{p_j} = \frac{m_{i,j}}{p_i}+\frac{m_{j,i}}{p_j} \quad \text{for some } m_{i,j}, m_{j,i}\in\mathbb{Z}.$$

Set $c(i,j) = b(i,j)/p_i - m_{i,j}/p_i$ for $i<j$ and $c(i,i) = b(i,i)$, so that

$$\frac{b(j,i)}{p_j} = -c(i,j) + \frac{m_{j,i}}{p_j}.$$

Then we have

$$\begin{aligned}\sum_{i,j\in\mathbb{N}} & b(i,j)e_{i,N}(g)e_j(h)\\ &\equiv \sum_{i<j} c(i,j)\big(p_i e_{i,N}(g)e_j(h) - p_j e_{j,N}(g)e_i(h)\big) \mod \mathbb{Z},\\ &= \sum_{i<j} c(i,j)e_{i,j}(g\wedge h).\end{aligned}$$

Thus with $f_c(g) = \exp\big(2\pi \mathrm{i}\big(\sum_{1\le i<j} c(i,j)e_{i,j}(g)\big)\big)$ for $g\in L$, we have

$$\exp\Big(2\pi\mathrm{i}\Big(\sum_{i,j} b(i,j)e_{i,N}(g)e_j(h)\Big)\Big) = \frac{f_c(h^{-1}gh)}{f_c(g)} = \partial_1 f_c(g;h),$$

where $e_{i,N}(g)$ means $e_{i,N}\circ\pi_G$. We then compute the coboundary of f_c for $g,h\in L$ as

$$\begin{aligned}(\partial_L f_c)(g;h) &= \frac{f_c(g)f_c(h)}{f_c(gh)}\\ &= \exp\Big(2\pi\mathrm{i}\Big(\sum_{i<j} c(i,j)(e_{i,j}(g)+e_{i,j}(h)-e_{i,j}(gh))\Big)\Big)\\ &= \exp\Big(-2\pi\mathrm{i}\Big(\sum_{i<j} c(i,j)e_i(g)e_j(h)\Big)\Big) = 1,\end{aligned}$$

because $e_i(g)\in p_i\mathbb{Z}$ and $e_j(h)\in p_j\mathbb{Z}$ if $g,h\in L$ and

$$p_i c(i,j)p_j = b(i,j)p_j - m_{i,j}p_j \equiv b(i,0)q_j \equiv 0 \mod \mathbb{Z}.$$

As $a(i,j,k)\in\mathbb{Z}$ for every triplet $(i,j,k)\in\Delta$, we get trivially

$$\lambda_{a,0}=1,\quad \tilde\mu_a = \mathfrak{s}_H^*\mu_a\in \mathrm{Z}^2(N,\mathbb{T}),\quad\text{and}\quad \mu_a=\pi_G^*(\tilde\mu_a).$$

Since $\partial_N U_a(i,j,k)$ for $i<j<k$ is integer valued, the cochain

$$\tilde\mu_a^{ijk} = \exp(2\pi\mathrm{i}(U_a(i,j,k)))$$

belongs to $\mathrm{Z}^2(N,\mathbb{T})$. Since AS $U_a(i,j,k)$ is integer valued by (4-24), AS $\tilde\mu_a^{ijk}=1$ and therefore $\tilde\mu_a^{ijk}\in \mathrm{B}^2(N,\mathbb{T})$. Because $\tilde\mu_a^{ik} = \exp(2\pi\mathrm{i}(U_a(i,k)))=1$ for $i<k$, we conclude that $\tilde\mu_a\in\mathrm{B}^2(N,\mathbb{T})$. Thus there exists a cochain $\tilde f\in \mathrm{C}^1(N,\mathbb{T})$ such that $\tilde\mu_a=\partial_N\tilde f$. Define a cochain $f\in\mathrm{C}^1(L,\mathbb{T})$ by $f=(\pi_G^*\tilde f)f_c$. Then we get for each pair $g\in L$ and $h\in H_{\mathrm{m}}$

$$(\partial_1 f)(g;h) = \frac{f(h^{-1}gh)}{f(g)} = \frac{\tilde f(\pi_G(h^{-1}gh))f_c(h^{-1}gh)}{\tilde f(\pi_G(g))f_c(g)} = \frac{f_c(h^{-1}gh)}{f_c(g)} = \lambda_{a,b}(g;h)$$

and for $g, h \in L$

$$(\partial_2 f)(g; h) = \frac{\tilde{f}(\pi_G(g)) f_c(g) \tilde{f}(\pi_G(h)) f_c(h)}{\tilde{f}(\pi_G(gh)) f_c(gh)}$$
$$= \partial_L f_c(g; h)(\partial_N \tilde{f})(\pi_G(g); \pi_G(h)) = \tilde{\mu}_a(\pi_G(g); \pi_G(h)) = \mu_a(g; h).$$

Therefore we conclude $\partial f = \{\lambda_{a,b}, \mu_a\} \in \mathrm{B}(H_{\mathrm{m}}, L, M, \mathbb{T})$. ♡

Lemma 4.6. *The cocycle λ_b corresponding to $b \in \mathrm{Z}_b$ does not depend on the M-component, that is,*

$$\lambda_b(mg; n\tilde{h}) = \lambda_b(g; \tilde{h}) \quad \text{for } m, n \in M,\ g \in L \text{ and } \tilde{h} \in H_{\mathrm{m}}.$$

We will view λ_b as a bicharacter on $N \times G_{\mathrm{m}}$ rather than on $L \times H_{\mathrm{m}}$.

(i) *For $i \in \mathbb{Z}$, set*

$$\mathrm{Z}_b(i, i) = \{z = (x, u) \in \mathbb{R}^2 : p_i x - q_i u \in \mathbb{Z}\} \quad \text{and} \quad \mathrm{B}_b(i, i) = \mathbb{Z} \oplus \mathbb{Z}.$$

The bicharacter $\lambda_z^{i,i}$ on $N \times G_{\mathrm{m}}$ determined by

$$\lambda_z^{i,i}(g; h) = \exp(2\pi \mathrm{i}(x e_{i,N}(g)\tilde{e}_i(h) + u e_{i,N}(g)\tilde{e}_0(h))) \quad \text{for } g \in N \text{ and } h \in G_{\mathrm{m}},$$

gives a characteristic cocycle of $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. It is a coboundary if and only if z is in $\mathrm{B}_b(i, i)$. The corresponding cohomology class $[\lambda_z^{i,i}] \in \Lambda_b(i, i)$ is given by

$$[\lambda_z^{i,i}] = ([p_i x - q_i u]_{\gcd(p_i, q_i)}, [-v_i x + u_i u]_{\mathbb{Z}}) \in \mathbb{Z}_{\gcd(p_i, q_i)} \oplus (\mathbb{R}/\mathbb{Z}),$$

where the integers u_i and v_i are determined by $p_i u_i - q_i v_i = \gcd(p_i, q_i)$ through the Euclid algorithm.

(ii) *Fix a pair $i, j \in \mathbb{N}$ of indices and set*

$$\mathrm{Z}_b(i, j) = \{(x, u, y, v) \in \mathbb{R}^4 : p_j x - q_j u \in \mathbb{Z},\ p_i y - q_i v \in \mathbb{Z}\},$$
$$\mathrm{B}_b(i, j) = \{(x, u, y, v) \in \mathrm{Z}_b(i, j) : p_j x + p_i y \in \gcd(p_i, p_j)\mathbb{Z},\ u, v \in \mathbb{Z}\}.$$

To each element $z = (x, u, y, v) \in \mathrm{Z}_b(i, j)$, there corresponds a bicharacter λ_z on $N \times G_{\mathrm{m}}$ determined by

$$\lambda_z^{i,j}(g; h) = \exp(2\pi \mathrm{i}(x e_{i,N}(g)\tilde{e}_j(h) + y e_{j,N}(g)\tilde{e}_i(h)))$$
$$\times \exp(2\pi \mathrm{i}(u e_{i,N}(g)\tilde{e}_0(h) + v e_{j,N}(g)\tilde{e}_0(h))) \text{ for } g \in N \text{ and } h \in G_{\mathrm{m}},$$

which is a characteristic cocycle in $\mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T})$. It is a coboundary if and only if $z \in \mathrm{B}_b(i, j)$. The cohomology class $[\lambda_z^{i,j}] \in \Lambda_b(i, j)$ of λ_z corresponds *to the parameter class*

$$[z] = \begin{pmatrix} [m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}} \\ [y_{i,j}(xr_{j,i} + yr_{i,j}) + x_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}} \\ [-uw_{i,j} + vw_{j,i}]_{\mathbb{Z}} \end{pmatrix} \in \begin{pmatrix} \left(\dfrac{1}{D(i,j)}\mathbb{Z}\right)\Big/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \end{pmatrix},$$

where $D(i,j), \dots, w_{j,i}$ are given in (4-17) of Theorem 4.2.

Proof. (i) Set $D_i = \gcd(p_i, q_i)$; set $r_i = p_i/D_i$ and $s_i = q_i/D_i$, and choose integers $u_i, v_i \in \mathbb{Z}$ so that $r_i u_i - s_i v_i = 1$, where such a pair $(u_i, v_i) \in \mathbb{Z}^2$ can be determined through the Euclid algorithm. Next we set $e_1 = (1,0)$ and $e_2 = (0,1)$. Set

$$f_1 = u_i e_1 + v_i e_2 \quad \text{and} \quad f_2 = s_i e_1 + r_i e_2,$$

so that $e_1 = r_i f_1 - v_i f_2$ and $e_2 = -s_i f_1 + u_i f_2$. Then $\mathrm{Z}_b(i,i)$ is given by

$$\mathrm{Z}_b(i,i) = \left(\frac{1}{D_i}\mathbb{Z}\right) f_1 + \mathbb{R} f_2,$$

and $\mathrm{B}_b(i,i) = \mathbb{Z}e_1 + \mathbb{Z}e_2 = \mathbb{Z}f_1 + \mathbb{Z}f_2$, so that

$$\Lambda_b(i,i) = \mathrm{Z}_b(i,i)/\mathrm{B}_b(i,i) \cong \left(\frac{1}{D_i}\mathbb{Z}\Big/\mathbb{Z}\right)\dot{f}_1 \oplus (\mathbb{R}/\mathbb{Z})\dot{f}_2,$$

where the dotted elements indicate the corresponding elements in the quotient group $\Lambda_b(i,i)$. Now we chase the parameter:

$$\begin{aligned} z &= xe_1 + ue_2 = x(r_i f_1 - v_i f_2) + u(-s_i f_1 + u_i f_2) \\ &= (r_i x - s_i u) f_1 + (-v_i x + u_i u) f_2; \\ \dot{z} &= [r_i x - s_i u]_{\mathbb{Z}} \dot{f}_1 + [-v_i x + u_i u]_{\mathbb{Z}} \dot{f}_2, \end{aligned}$$

and

$$\lambda_z^{i,i}(g; \tilde{h}) = \exp(2\pi \mathrm{i}((xe_{i,N}(g)e_i(\tilde{h}) + ue_{i,N}(g)e_0(\tilde{h}))))$$

for each pair $g \in N$ and $\tilde{h} \in G_{\mathrm{m}}$.

(ii) First we fix the standard basis $\{e_1, \dots, e_4\}$ of $\mathbb{R}^4$ and set

$$g_0 = r_{i,j}e_1 - r_{j,i}e_3 \quad \text{and} \quad g_1 = u_{j,i}e_1 + u_{i,j}e_3,$$

where we choose $u_{i,j}, u_{j,i} \in \mathbb{Z}$ so that $r_{i,j}u_{i,j} + r_{j,i}u_{j,i} = 1$. Since

$$e_1 = u_{i,j}g_0 + r_{j,i}g_1 \quad \text{and} \quad e_2 = -u_{j,i}g_0 + r_{i,j}g_1,$$

we have $\mathbb{Z}e_1 + \mathbb{Z}e_3 = \mathbb{Z}g_0 + \mathbb{Z}g_1$. Also we have

$$\mathrm{B}_b(i,j) + \mathbb{R}g_0 = \mathbb{R}g_0 + \mathbb{Z}g_1 + \mathbb{Z}e_2 + \mathbb{Z}e_4.$$

Consider an integer 3×4 matrix

$$T = \begin{pmatrix} m_{i,j}r_{j,i} & -n_{i,j}s_{j,i} & m_{i,j}r_{i,j} & -n_{i,j}s_{i,j} \\ y_{i,j}r_{j,i} & x_{i,j}s_{j,i} & y_{i,j}r_{i,j} & x_{i,j}s_{i,j} \\ 0 & -w_{i,j} & 0 & w_{j,i} \end{pmatrix}.$$

We claim that

$$T(\mathrm{Z}_b(i, j) + \mathbb{R}g_0) = \left(\frac{1}{D(i, j)}\mathbb{Z}\right) \oplus \mathbb{R} \oplus \mathbb{R}.$$

To prove the claim, for each vector $z = xe_1 + ue_2 + ye_3 + ve_4 \in \mathbb{R}^4$, we simply compute

$$Tg_0 = 0,$$

$$Tz = \begin{pmatrix} m_{i,j}r_{j,i} & -n_{i,j}s_{j,i} & m_{i,j}r_{i,j} & -n_{i,j}s_{i,j} \\ y_{i,j}r_{j,i} & x_{i,j}s_{j,i} & y_{i,j}r_{i,j} & x_{i,j}s_{i,j} \\ 0 & -w_{i,j} & 0 & w_{j,i} \end{pmatrix} \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix}$$

$$= \begin{pmatrix} m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j}) \\ y_{i,j}(xr_{j,i} + yr_{i,j}) + x_{i,j}(us_{j,i} + vs_{i,j}) \\ -uw_{i,j} + vw_{j,i} \end{pmatrix}.$$

Suppose

$$\frac{k}{D(i, j)} = m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j}) \in \left(\frac{1}{D(i, j)}\right)\mathbb{Z}.$$

Then we have

$$\begin{aligned} k &= (m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j}))D(i, j) \\ &= (xp_j - uq_j) + (yp_i - vq_i) \\ &= ((x + tr_{i,j})p_j - uq_j) + ((y - tr_{j,i})p_i - vq_i). \end{aligned}$$

A choice of $t \in \mathbb{R}$, such that $(x + tr_{i,j})p_j - uq_j$ is an integer, yields the integrality of the other term $(y - tr_{j,i})p_i - vq_i$, so that $z + tg_0 \in \mathrm{Z}_b(i, j)$. Now we prove that

$$T^{-1}\mathbb{Z}^3 = \mathrm{B}_b(i, j) + \mathbb{R}g_0.$$

Since T is a matrix with integer coefficients and the generators g_1, e_2, e_4 are all integer vectors, we have $T(\mathrm{B}_b(i, j)) \subset \mathbb{Z}^3$. Conversely, suppose that $Tz \in \mathbb{Z}^3$. Then we have

$$\begin{aligned} k &= m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j}) \in \mathbb{Z}, \\ \ell &= y_{i,j}(xr_{j,i} + yr_{i,j}) + x_{i,j}(us_{j,i} + vs_{i,j}) \in \mathbb{Z}, \\ m &= -uw_{i,j} + vw_{j,i} \in \mathbb{Z}. \end{aligned}$$

Hence we get

$$xr_{j,i} + yr_{i,j} = x_{i,j}k + n_{i,j}\ell \in \mathbb{Z}, \quad n = us_{j,i} + vs_{i,j} = -y_{i,j}k + m_{i,j}\ell \in \mathbb{Z},$$
$$u = nw_{j,i} - ms_{i,j} \in \mathbb{Z}, \quad v = nw_{i,j} + ms_{j,i} \in \mathbb{Z},$$
$$xp_j + yp_i = (xr_{j,i} + yr_{i,j})D_{i,j} \in D_{i,j}\mathbb{Z}.$$

Therefore $z \in \mathrm{B}_b(i, j) + \mathbb{R}g_0$.

Consequently, we conclude

$$\Lambda_b(i, j) \cong \mathrm{Z}_b(i, j)/\mathrm{B}_b(i, j) \cong \Big(\Big(\frac{1}{D(i, j)}\mathbb{Z}\Big)\Big/\mathbb{Z}\Big) \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}),$$

in the sense that the cohomology class $[\lambda_z^{i,j}] \in \Lambda_b(i, j)$ corresponds to

$$[z] = \begin{pmatrix} [m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}} \\ [y_{i,j}(xr_{j,i} + yr_{i,j}) + x_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}} \\ [-uw_{i,j} + vw_{j,i}]_{\mathbb{Z}} \end{pmatrix} \in \begin{pmatrix} ((1/D(i, j))\mathbb{Z})/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \end{pmatrix}.$$

For each $i, j \in \mathbb{N}$, define maps $\pi_i^i : \Lambda_b(i, i) \to \mathbb{R}/\mathbb{Z}$, $\pi_{i,j}^i : \Lambda_b(i, j) \to \mathbb{R}/\mathbb{Z}$, $\pi_{i,j}^j : \Lambda_b(i, j) \to \mathbb{R}/\mathbb{Z}$ and $\pi_{ij} : \Lambda_b(i, j) \to ((1/D(i, j))\mathbb{Z})/\mathbb{Z}$ by

$$\pi_i^i([\lambda_z^{i,i}]) = [u]_{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z} \quad \text{and} \quad \pi_{ii}([\lambda_z^{i,i}]) = [xr_i - us_i]_{\mathbb{Z}} \in ((1/D_i)\mathbb{Z})/\mathbb{Z}$$

for each $z = (x, u) \in \mathrm{Z}_b(i, i)$, and

$$\pi_{i,j}^i([\lambda_z^{i,j}]) = [u]_{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z}, \qquad \pi_{i,j}^j([\lambda_z^{i,j}]) = [v]_{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z},$$
$$\pi_{ij}([\lambda_z^{i,j}]) = [m_{i,j}(xr_{j,i} + yr_{i,j}) - n_{i,j}(us_{j,i} + vs_{i,j})]_{\mathbb{Z}} \in \Big(\frac{1}{D(i, j)}\mathbb{Z}\Big)\Big/\mathbb{Z}$$

for each $z = (x, u, y, v) \in \mathrm{Z}_b(i, j)$. The maps $\pi_{i,j}^i$ and $\pi_{i,j}^j$ are both well defined because the coboundary condition on z implies the integrality of u and v.

Let Λ_b be the set of all

$$\lambda_b = \{\lambda_b(i, i), \lambda_b(i, j)\} \in \prod_{i \in \mathbb{N}} \Lambda_b(i, i) \times \prod_{\substack{i<j \\ i,j \in \mathbb{N}}} \Lambda_b(i, j)$$

such that $\pi_i^i(\lambda_b(i, i)) = \pi_{i,j}^i(\lambda_b(i, j)) = \pi_{ki}^i(\lambda_b(k, i))$ for all $i, j, k \in \mathbb{N}$. Finally we have $\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) = \Lambda_a \oplus \Lambda_b$. This completes the proof. ♡

Remark 4.7. The direct sum homomorphism $\pi_{ij} \oplus \pi_{i,j}^i \oplus \pi_{i,j}^j$ is a homomorphism of $\Lambda_a(i, j)$ onto the direct sum group:

$$\Lambda_b(i, j) \xrightarrow{\pi_{ij} \oplus \pi_{i,j}^i \oplus \pi_{i,j}^j} \Big(\Big(\frac{1}{D(i, j)}\mathbb{Z}\Big)\Big/\mathbb{Z}\Big) \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}).$$

By multiplying $\pi_{ii}(\lambda_z)$ by D_i, we get

$$D_i \pi_{ii}([\lambda_z]) = [x p_i - u q_i]_{D_i \mathbb{Z}} \in \mathbb{Z}/(D_i \mathbb{Z}).$$

Similarly, we have

$$D(i, j)\pi_{ij}(\lambda_z) = [(x p_j + y p_i) - (u q_j + v q_i)]_{D(i,j)} \in \mathbb{Z}/(D(i, j)\mathbb{Z}).$$

The kernel of $\pi_{ij} \oplus \pi^i_{i,j} \oplus \pi^j_{i,j}$ is given by

$$\operatorname{Ker}(\pi_{ij} \oplus \pi^i_{i,j} \oplus \pi^j_{i,j}) = \begin{pmatrix} \{0\} \\ \Bigl(\dfrac{1}{m_{i,j}}\mathbb{Z}\Bigr)\Big/\mathbb{Z} \\ \{0\} \end{pmatrix}.$$

At the parameter level, the kernel is described as follows:

$$[\lambda_z] \in \operatorname{Ker}(\pi_{ij} \oplus \pi^i_{i,j} \oplus \pi^j_{i,j}) \text{ if and only if } x p_j + y p_i \in D(i, j)\mathbb{Z},\ u, v \in \mathbb{Z}.$$

5. The reduced modified HJR-sequence

We are now going to investigate the reduced modified HJR-exact sequence

(5-1)
$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{H}^2(H, \mathbb{T}) & = & \mathrm{H}^2(H, \mathbb{T}) \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) & \xrightarrow{\mathrm{res}} & \Lambda(H, M, \mathbb{T}) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_{\mathrm{HJR}}} \\
\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}) & \xrightarrow{\partial_{Q_{\mathrm{m}}}} & \mathrm{H}^3(G, \mathbb{T}) \\
\downarrow{\scriptstyle \mathrm{Inf}} & & \downarrow{\scriptstyle \mathrm{inf}} \\
\mathrm{H}^3(H, \mathbb{T}) & = & \mathrm{H}^3(H, \mathbb{T})
\end{array}$$

We refer to [Katayama and Takesaki 2007, page 116] for details. We first discuss the second cohomology group $\mathrm{Z}^2(H, \mathbb{T})$ and the restriction map Res. Each second cocycle $\mu \in \mathrm{Z}^2(H, \mathbb{T})$ gives rise to a group extension equipped with a cross-section

$$1 \to \mathbb{T} \longrightarrow E \underset{\mathfrak{s}_j}{\overset{j}{\rightleftarrows}} H \to 1$$

such that $\mathfrak{s}_j(g)\mathfrak{s}_j(h) = \mu(g;h)\mathfrak{s}_j(gh)$ for $g, h \in H$. With

$$\lambda_\mu(g;h) = \mu(h;h^{-1}gh)/\mu(g;h) \quad \text{for } g, h \in H,$$

we obtain a characteristic cocycle $(\lambda_\mu, \mu) \in \mathrm{Z}(H, H, \mathbb{T})$. This corresponds to the case that $P = 1$ in the previous section. So we set

$$\begin{aligned} \mathrm{Z}^2 &= \big\{a \in \mathbb{R}^{\mathbb{N}^3} : a(i,j,k) = 0 \text{ if } j \geq k,\ (\mathrm{AS}\, a)(i,j,k) \in \mathbb{Z}\big\}, \\ \mathrm{B}^2 &= \big\{a \in \mathrm{Z}^2 : a(i,j,k) \in \mathbb{Z},\ a(i,i,k), a(k,i,k) \in 2\mathbb{Z}\big\}. \end{aligned} \tag{5-2}$$

Theorem 5.1. (i) *Each element $a \in \mathrm{Z}^2$ gives rise to a cocycle*

$$\mu_a = \exp(2\pi \mathrm{i} V_a) \in \mathrm{Z}^2(H, \mathbb{T}) \tag{5-3}$$

and the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{B}^2 & \longrightarrow & a \in \mathrm{Z}^2 & \longrightarrow & [a] \in \mathrm{H}^2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{B}^2(H,\mathbb{T}) & \longrightarrow & \mu_a \in \mathrm{Z}^2(H,\mathbb{T}) & \longrightarrow & [\lambda_a] \in \mathrm{H}^2(H,\mathbb{T}) & \longrightarrow & 1 \end{array}$$

describes the second cohomology $\mathrm{H}^2(H, \mathbb{T})$. *More precisely, with*

$$\begin{aligned} &\mathrm{Z}^2(i,j,k) = \{(x,y,z) \in \mathbb{R}^3 : x - y + z \in \mathbb{Z}\}, && \mathrm{Z}^2(i,k) = \mathbb{R}^2, \\ &\mathrm{B}^2(i,j,k) = \mathbb{Z}^3, && \mathrm{B}^2(i,k) = (2\mathbb{Z})^2, \\ &\mathrm{H}^2(i,j,k) = \mathrm{Z}^2(i,j,k)/\mathrm{B}^2(i,j,k), && \mathrm{H}^2(i,k) = \mathrm{Z}^2(i,k)/\mathrm{B}^2(i,k) \end{aligned}$$

for each triplet $i < j < k$ (respectively pair $i < k$) and

$$a(i,j,k) = x, \quad a(j,i,k) = y, \quad a(k,i,j) = z,$$
$$(\text{respectively} \quad a(i,i,k) = x, \quad a(k,i,k) = y),$$

we set

$$\begin{aligned} \mu_a^{ijk} &= \exp(2\pi \mathrm{i}(V_a(i,j,k))) \in \mathrm{Z}^2(H,\mathbb{T}), \\ \mu_a^{ik} &= \exp(2\pi \mathrm{i}(V_a(i,k))) \in \mathrm{Z}^2(H,\mathbb{T}). \end{aligned}$$

Then we have

$$\begin{aligned} \mathrm{Z}^2(H,\mathbb{T}) &= \prod_{i<j<k} \mathrm{Z}^2(i,j,k) \times \prod_{i<k} \mathrm{Z}^2(i,k), \\ \mathrm{B}^2(H,\mathbb{T}) &= \prod_{i<j<k} \mathrm{B}^2(i,j,k) \times \prod_{i<k} \mathrm{B}^2(i,k), \\ \mu_a &= \Big(\prod_{i<j<k} \mu_a^{ijk}\Big)\Big(\prod_{i<k} \mu_a^{ik}\Big) \in \mathrm{Z}^2(H,\mathbb{T}), \end{aligned}$$

$$\mathrm{H}^2(H,\mathbb{T}) \cong \prod_{i<j<k} \mathrm{H}^2(i,j,k) \times \prod_{i<k} \mathrm{H}^2(i,k),$$
$$[\mu_a] = ([\mu_a^{ijk}], [\mu_a^{ik}] : i < j < k \text{ and } i < k) \in \mathrm{H}^2(H,\mathbb{T}).$$

Each $\mathrm{H}^2(i,j,k)$ *for* $i<j<k$, *(respectively* $\mathrm{H}^2(i,k)$ *for* $i<k$*), is given by*

$$\mathrm{H}^2(i,j,k) \cong (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}),$$
$$(\textit{respectively } \mathrm{H}^2(i,k) \cong (\mathbb{R}/2\mathbb{Z}) \oplus (\mathbb{R}/2\mathbb{Z})).$$

Proof. Most of the claims have been proved already except the claim for the structure of $\mathrm{H}^2(i,j,k)$. To prove this, it is convenient to introduce a matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3,\mathbb{Z}), \quad \text{for which } A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then observe that $A\mathrm{Z}^2(i,j,k) = (\mathbb{Z}\oplus\mathbb{R}\oplus\mathbb{R})$ and $A\mathrm{B}^2 = \mathbb{Z}^3$; we conclude

$$\mathrm{H}^2(i,j,k) \cong \{0\} \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}). \qquad \heartsuit$$

Theorem 5.2. (i) *Each second cocycle* $\mu_a \in \mathrm{Z}^2(H,\mathbb{T})$ *for* $a \in \mathrm{Z}^2$ *gives the corresponding characteristic cocycle*

$$\mathrm{Res}(\mu_a) = (\lambda_a, \mu_a) = \pi_{\mathrm{m}}^*(\lambda_a|_{L\times H_{\mathrm{m}}}, \mu_a|_L) \in \mathrm{Z}(H_{\mathrm{m}}, L, M, \mathbb{T}).$$

The image $\mathrm{Res}(\mathrm{Z}^2(H,\mathbb{T}))$ *is therefore given by*

$$\mathrm{Res}(\mathrm{Z}^2(H,\mathbb{T})) = \{(\lambda_a,\mu_a) : a \in \mathrm{Z}_a, (\mathrm{AS}\,a)(i,j,k) \in \mathbb{Z},\ i<j<k\}.$$

The (i,j,k)*-component* $\mathrm{Res}(i,j,k)$ *of the restriction map* Res *gives rise to the following commutative diagram of short exact sequences*:

$$\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
\mathrm{B}^2(i,j,k) = \mathbb{Z}^3 & \xrightarrow{X_a(i,j,k)\longrightarrow X_a(i,j,k)} & \mathrm{B}_a(i,j,k) = \mathbb{Z}^3 \\
\downarrow & & \downarrow \\
\mathrm{Z}^2(i,j,k) = A^{-1}(\mathbb{Z}\oplus\mathbb{R}^2) & \xrightarrow{X_a(i,j,k)\longrightarrow X_a(i,j,k)} & \mathrm{Z}_a(i,j,k) = A^{-1}((1/D)\mathbb{Z}\oplus\mathbb{R}^2) \\
\downarrow & & \downarrow \\
\mathrm{H}^2(i,j,k) = \{0\}\oplus\mathbb{T}^2 & \xrightarrow{\mathrm{Res}(i,j,k)} & \Lambda_a(i,j,k) = \mathbb{Z}_D \oplus \mathbb{T}^2 \\
\downarrow & & \downarrow \\
1 & & 0,
\end{array}$$

where $D = D(i, j, k) = \gcd(p_i, p_j, p_k)$. *Also the restriction map* $\mathrm{Res}_a(i, k) : \mathrm{H}^2(i, k) \to \Lambda_a(i, k)$ *is given by*

$$\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
\mathrm{B}^2(i,k) = (2\mathbb{Z})^2 & \xrightarrow{X_a(i,k) \longrightarrow X_a(i,k)} & \mathrm{B}_a(i,k) = (2\mathbb{Z})^2 \\
\downarrow & & \downarrow \\
\mathrm{Z}^2(i,k) = \mathbb{R}^2 & \xrightarrow{X_a(i,k) \longrightarrow X_a(i,k)} & \mathrm{Z}_a(i,k) = \mathbb{R}^2 \\
\downarrow & & \downarrow \\
\mathrm{H}^2(i,k) = (\mathbb{R}/2\mathbb{Z})^2 & \xrightarrow{\mathrm{Res}(i,k)} & \Lambda_a(i,k) = (\mathbb{R}/2\mathbb{Z})^2 \\
\downarrow & & \downarrow \\
1 & & 1
\end{array}$$

Consequently, we get

$$\Lambda_a(i, j, k)/\,\mathrm{Res}(i, j, k)(\mathrm{H}^2(i, j, k)) \cong \mathbb{Z}/(D\mathbb{Z}),$$
$$\Lambda_a(i, k)/\,\mathrm{Res}(i, k)(\mathrm{H}^2(i, k)) \cong \{0\}.$$

(ii) *The modified* HJR-*map* $\delta : \Lambda(H_{\mathrm{m}}, L, M, \mathbb{T}) \to \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ *enjoys these properties*:

(a) *The* (i, j, k)*-component and* (i, k)*-component of* $\mathrm{Ker}(\delta)$ *are given by*

$$\mathrm{Ker}(\delta)_{ijk} = \{0\} \oplus (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z}),$$
$$\mathrm{Ker}(\delta)_{ik} = (\mathbb{R}/2\mathbb{Z}) \oplus (\mathbb{R}/2\mathbb{Z}) = \Lambda_a(i, k).$$

(b) *The image* $\delta([\lambda_a, \mu_a]) \in \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ *for* $a \in \mathrm{Z}_a$ *depends only on the asymmetrization* $\mathrm{AS}\, a$, *that is*,

$$\delta([\lambda_a, \mu_a]) = \delta([\lambda_{\hat{a}}, 1]),$$

where

$$\hat{a}(i, j, k) = (\mathrm{AS}\, a)(i, j, k) \in ((1/D)\mathbb{Z}) \quad \text{for } i < j < k,$$
(5-4)
$$\hat{a}(j, i, k) = \hat{a}(k, i, j) = \hat{a}(i, i, k) = \hat{a}(i, j, j) = \hat{a}(k, i, k) = 0.$$

(c) *Set* $\mathrm{Z}_{\hat{a}} = \{a \in \mathrm{Z}_a : a$ *satisfies the requirement* (5-4)$\}$. *If* $a \in \mathrm{Z}_{\hat{a}}$, *then the image* $c_a = \delta(\lambda_a, 1) \in \mathrm{Z}^{\mathrm{out}}(G_{\mathrm{m}}, N, \mathbb{T})$ *under the modified* HJR-*map* δ *is in the pull back* $\pi^*_{\mathrm{m}}(\mathrm{H}^3(Q, \mathbb{T}))$ *and given by*

(5-5)
$$c_a(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3) = c_a(q_1, q_2, q_3) = \exp\Big(2\pi \mathrm{i}\Big(\sum_{i<j<k} a(i, j, k)\{e_i(q_1)\}_{p_i}\{e_j(q_2)\}_{p_j}\{e_k(q_3)\}_{p_k}\Big)\Big)$$

for each $\tilde{q}_1 = (q_1, s_1)$, $\tilde{q}_2 = (q_2, s_2)$ *and* $\tilde{q}_3 = (q_3, s_3) \in Q_{\mathrm{m}}$.

(d) *The modified* HJR*-map* δ_{HJR} *is injective on* Λ_b *and* $\mathrm{Ker}(\delta)$ *is precisely the connected component of* $\Lambda(H_{\mathrm{m}}, L, M, \mathbb{T})$*. If* $b \in \mathrm{Z}_b$*, then*

$$[c_b, \nu_b] = \delta(\lambda_b, 1) \in \mathrm{Z}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$$

is given by

$$c_b(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3) = \exp\Bigl(2\pi \mathrm{i}\Bigl(\sum_{i\in\mathbb{N}, j\in\mathbb{N}_0} b(i, j) e_{i,N}(\mathfrak{n}_N(\tilde{q}_2; \tilde{q}_3)) \tilde{e}_j(\mathfrak{s}(\tilde{q}_1))\Bigr)\Bigr) \tag{5-6}$$

where

$$\begin{aligned} e_{i,N}(\mathfrak{n}_N(\tilde{q}_2; \tilde{q}_3)) &= \eta_{p_i}([e_i(q_2)]_{p_i}; [e_i(q_3)]_{p_i})/p_i, \\ \tilde{e}_i(\mathfrak{s}(\tilde{q}_1)) &= \{e_i(q_1)\}_{p_i} \text{ for } i \geq 1, \\ \tilde{e}_0(\mathfrak{s}(\tilde{q}_1)) &= \tilde{e}_0(q_1). \end{aligned} \tag{5-7}$$

The d-part d_{c_b} *of* c_b *is given by* ν_b*:*

$$\begin{aligned} d_{c_b}(q_2; q_3) &= \exp\Bigl(2\pi \mathrm{i}\Bigl(\sum_{j\in\mathbb{N}} b(j, 0) \eta_{p_j}([e_j(q_2)]_{p_j}; [e_j(q_3)]_{p_j})/p_j\Bigr)\Bigr) \\ &= \exp\bigl(2\pi \mathrm{i}\bigl(\{\nu_b(\mathfrak{n}_N(q_2; q_3))\}_T/T\bigr)\bigr), \\ \nu_b(g) &= \pi_T\Bigl(T \sum_{j\in\mathbb{N}} b(j, 0) e_{j,N}(g)\Bigr) \in \mathbb{R}/T\mathbb{Z} \quad \text{for } g \in N, \end{aligned} \tag{5-8}$$

where $\pi_T : s \in \mathbb{R} \mapsto s_T = s + T\mathbb{Z} \in \mathbb{R}/T\mathbb{Z}$ *is the quotient map.*

The modular obstruction group $\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ *looks like*

$$\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}) = \mathrm{H}^{\mathrm{out}}_a \oplus \mathrm{H}^{\mathrm{out}}_b \quad \text{and} \quad \mathrm{H}^{\mathrm{out}}_b \cong \Lambda_b, \tag{5-9}$$

$$\delta([\lambda_a, \mu_a]) = [c_{\mathrm{AS}\,a}] \in \prod_{i<j<k} \Bigl(\Bigl(\frac{1}{\gcd(p_i, p_j, p_k)}\mathbb{Z}\Bigr)\Big/\mathbb{Z}\Bigr) \quad \text{for } a \in \mathrm{Z}_a,$$

$$[c_b, \nu_b] = \delta([\lambda_b, 1]) \quad \text{for } \nu_b \in \mathrm{Hom}(N, \mathbb{R}/T\mathbb{Z}),$$

$$\begin{aligned} [c_b^{i,i}] &= ([p_i b(i, i) - q_i b(i, 0)]_{D_i\mathbb{Z}}, [-v_i b(i, i) + u_i b(i, 0)]_{\mathbb{Z}}) \\ &\in \mathbb{Z}/(D_i\mathbb{Z}) \oplus \mathbb{R}/\mathbb{Z}, \end{aligned}$$

$$[c_b^{i,j}] = \begin{pmatrix} [m_{i,j}(b(i, j) r_{j,i} + b(j, i) r_{i,j}) - n_{i,j}(b(i, 0) s_{j,i} + b(j, 0) s_{i,j})]_{\mathbb{Z}} \\ [y_{i,j}(b(i, j) r_{j,i} + b(j, i) r_{i,j}) + x_{i,j}(b(i, 0) s_{j,i} + b(j, 0) s_{i,j})]_{\mathbb{Z}} \\ [-b(i, 0) w_{i,j} + b(j, 0) w_{j,i}]_{\mathbb{Z}} \end{pmatrix}$$

$$\in \begin{pmatrix} ((1/D(i, j))\mathbb{Z})/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \\ \mathbb{R}/\mathbb{Z} \end{pmatrix}, \quad \text{where } D(i, j) = \gcd(p_i, p_j, q_i, q_j).$$

(iii) *The map* $\partial_{Q_{\rm m}} : {\rm H}^{\rm out}_{{\rm m},\mathfrak{s}}(G, N, \mathbb{T}) \to {\rm H}^3(G, \mathbb{T})$ *in the modified HJR-exact sequence above is given by*

$$\begin{aligned}\partial_{Q_{\rm m}}([c_{\hat a}][c_b\nu_b]) &= [c^G_{\hat a}] \in {\rm H}^3(G, \mathbb{T}) = X^3(G, \mathbb{T}) \quad \text{for } \hat a \in {\rm Z}_{\hat a},\\ &\text{where } c^G_{\hat a} = \exp\Big(2\pi{\rm i}\Big(\sum_{i<j<k} ({\rm AS}\, a)(i, j, k) e_i \otimes e_j \otimes e_k\Big)\Big),\\ \partial_{Q_{\rm m}}({\rm H}^{\rm out}_{{\rm m},\mathfrak{s}}(G, N, \mathbb{T})) &= \pi^*_Q({\rm H}^3(Q, \mathbb{T})).\end{aligned} \tag{5-10}$$

Proof. (i) The assertion has been already proved.

(ii) For each $i < j < k$, let $D(i, j, k) = \gcd(p_i, p_j, p_k) \in \mathbb{Z}$. Fix $a \in {\rm Z}_a$, that is, $a \in \mathbb{R}^\Delta$ such that

$$\begin{aligned}({\rm AS}\, a)(i, j, k) = a(i, j, k) - a(j, i, k) + a(k, i, j) \in ((1/D(i, j, k))\mathbb{Z}),\\ a(i, j, k) = 0 \quad \text{if } j \ge k.\end{aligned}$$

Set

$$z_a(i, j, k) = \begin{pmatrix} a(i, j, k)\\ a(j, i, k)\\ a(k, i, j)\end{pmatrix} \in {\rm Z}_a = A^{-1}\begin{pmatrix}((1/D(i, j, k))\mathbb{Z})\\ \mathbb{R}\\ \mathbb{R}\end{pmatrix}.$$

Then we get

$$\begin{aligned}Az_a(i, j, k) &= \begin{pmatrix}({\rm AS}\, a)(i, j, k)\\ a(j, i, k)\\ a(k, i, j)\end{pmatrix} \in \begin{pmatrix}(1/D(i, j, k))\mathbb{Z}\\ \mathbb{R}\\ \mathbb{R}\end{pmatrix}\\ A{\rm B}_a(i, j, k) &= \mathbb{Z}^3,\end{aligned}$$

so that

$$[\lambda^{i,j,k}_a, \mu^{ijk}_a] \sim \begin{pmatrix}[({\rm AS}\, a)(i, j, k)]_{\mathbb{Z}}\\ [a(j, i, k)]_{\mathbb{Z}}\\ [a(k, i, j)]_{\mathbb{Z}}\end{pmatrix} \in \begin{pmatrix}((1/D(i, j, k))\mathbb{Z})\\ \mathbb{R}/\mathbb{Z}\\ \mathbb{R}/\mathbb{Z}\end{pmatrix}.$$

If $({\rm AS}\, a)(i, j, k) \in \mathbb{Z}$, the second cocycle μ^{ijk}_a extends to a second cocycle on H, which gives $(\lambda^{i,j,k}_a, \mu^{i,j,k}_a) = {\rm Res}(\mu^{i,j,k}_a)$. Since ${\rm Range}({\rm Res}) = {\rm Ker}(\delta)$, the image $\delta(\lambda^{i,j,k}_a, \mu^{i,j,k}_a)$ depends only on the first term $({\rm AS}\, a)(i, j, k)$ of $Az_a(i, j, k)$. Hence we conclude $\delta([\lambda_a, \mu_a]) = \delta([\lambda_{\hat a}], 1)$. We also have $\Lambda_a(i, k) = {\rm Res}(i, k)({\rm H}^2(i, k))$, so that the map δ kills the entire $\Lambda_a(i, k)$. This proves (ii)(a) and (ii)(b).

(ii)(c) Set $c_a = \delta(\lambda_a, \mu_a)$ with $a \in {\rm Z}_{\hat a}$. We then look at the crossed extension $E_{\lambda_a,\mu_a} \in {\rm Xext}(H_{\rm m}, L, M, \mathbb{T})$, given by

$$1 \to \mathbb{T} \longrightarrow E \underset{\mathfrak{s}_j}{\overset{j}{\rightleftarrows}} L \to 1.$$

Since

$$a(i,j,k) \in \Big(\frac{1}{\gcd(p_i,p_j,p_k)}\mathbb{Z}\Big) \quad \text{and} \quad e_i(g) \in p_i\mathbb{Z} \quad \text{for } g \in L,$$

we have $\mu_a = 1$. Hence observing that $\lambda_a(g;\tilde{h}) = 1$ for every $g \in L \wedge H_{\mathrm{m}}$ and $\tilde{h} \in H_{\mathrm{m}}$, we get from (3-15) and (3-16) that

$$\begin{aligned}
c_a(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3) &= \alpha_{\mathfrak{s}(\tilde{q}_1)}(\mathfrak{s}_j(\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3)))\mathfrak{s}_j(\mathfrak{n}_L(\tilde{q}_1;\tilde{q}_2\tilde{q}_3)) \\
&\qquad \times \{\mathfrak{s}_j(\mathfrak{n}_L(\tilde{q}_1;\tilde{q}_2))\mathfrak{s}_j(\mathfrak{n}_L(\tilde{q}_1\tilde{q}_2;\tilde{q}_3))\}^{-1} \\
&= \lambda_a(\mathfrak{s}(\tilde{q}_1)\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3)\mathfrak{s}(\tilde{q}_1)^{-1};\mathfrak{s}(\tilde{q}_1)) \\
&= \lambda_a((\mathfrak{s}(\tilde{q}_1)\wedge\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3))\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3);\mathfrak{s}(\tilde{q}_1)) \\
&= \lambda_a(\mathfrak{s}(\tilde{q}_1)\wedge\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3);\mathfrak{s}(\tilde{q}_1))\lambda_a(\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3);\mathfrak{s}(\tilde{q}_1)) \\
&= \lambda_a(\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3);\mathfrak{s}(\tilde{q}_1)) \\
&= \exp\Big(2\pi \mathrm{i}\Big(\sum_{i<j<k} a(i,j,k)e_{j,k}(\mathfrak{n}_L(\tilde{q}_2;\tilde{q}_3))e_i(\mathfrak{s}(\tilde{q}_1))\Big)\Big) \\
&= \exp\Big(2\pi \mathrm{i}\Big(\sum_{i<j<k} a(i,j,k)\{e_i(\tilde{q}_1)\}_{p_i}\{e_j(\tilde{q}_2)\}_{p_j}\{e_k(\tilde{q}_3)\}_{p_k}\Big)\Big) \\
&= \exp\Big(2\pi \mathrm{i}\Big(\sum_{i<j<k} a(i,j,k)\{e_i(q_1)\}_{p_i}\{e_j(q_2)\}_{p_j}\{e_k(q_3)\}_{p_k}\Big)\Big) \\
&= c_a(q_1;q_2;q_3)
\end{aligned}$$

for each $\tilde{q}_1 = (q_1,s_1)$, $\tilde{q}_2 = (\tilde{q}_2,s_2)$ and $\tilde{q}_3 = (q_3,s_3) \in Q_{\mathrm{m}}$. The assertion (ii)(c) follows.

(ii)(d) Since $\mathrm{Res}(\mathrm{H}^2(H,\mathbb{T})) \cap \Lambda_b = \{0\}$, the modified HJR-map δ is injective on Λ_b. Now fix $b \in \mathrm{Z}_b$. Since $\mu_b = 1$ and $\lambda_b(m;\tilde{h}) = 1$ for every pair $m \in M$ and $\tilde{h} \in H_{\mathrm{m}}$, we have, as in (ii)(c),

$$\begin{aligned}
c_b(\tilde{q}_1;\tilde{q}_2;\tilde{q}_3) &= \lambda_b(\mathfrak{n}_N(q_2;q_3);\mathfrak{s}(\tilde{q}_1)) \\
&= \exp\Big(2\pi \mathrm{i}\Big(\sum_{i\in\mathbb{N},\, j\in\mathbb{N}_0} b(i,j)e_{i,N}(\mathfrak{n}_N(q_2;q_3))\tilde{e}_j(\mathfrak{s}(\tilde{q}_1))\Big)\Big) \\
&= \exp\Big(2\pi \mathrm{i}\Big(\sum_{i,j\in\mathbb{N}} b(i,j)e_{i,N}(\mathfrak{n}_N(q_2;q_3))e_j(\mathfrak{s}(q_1))\Big)\Big) \\
&\qquad \times \exp\Big(2\pi \mathrm{i}\Big(\sum_{i\in\mathbb{N}} b(i,0)e_{i,N}(\mathfrak{n}_N(q_2;q_3))\tilde{e}_0(\tilde{q}_1)\Big)\Big)
\end{aligned}$$

where $e_{i,N}(\mathfrak{n}_N(q_2; q_3))$ is given by (5-7). Also we compute

$$\begin{aligned} d_{c_b}(q_2; q_3) = \lambda_b(\mathfrak{n}_N(q_2; q_3); z_0) &= \exp\Big(2\pi \mathrm{i}\Big(\frac{\nu_b(\mathfrak{n}_N(q_2; q_3))}{T}\Big)\Big) \\ &= \exp\Big(2\pi \mathrm{i}\Big(\sum_{i\in\mathbb{N}} b(i,0)e_{i,N}(\mathfrak{n}_N(q_2; q_3))\Big)\Big), \\ \nu_b(g) = \pi_T\Big(T\sum_{i\in\mathbb{N}} b(i,0)e_{i,N}(g)\Big) &\in \mathbb{R}/T\mathbb{Z} \quad \text{for } g\in N, \end{aligned}$$

with $\pi_T : s \in \mathbb{R} \mapsto s_T = s + T\mathbb{Z} \in \mathbb{R}/T\mathbb{Z}$ the quotient map.

The last assertion, (5-9), on $\mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T})$ follows almost automatically from the above computations and Lemma 4.6 in the last section.

(iii) We now compute the map

$$\partial_{\pi_{\mathrm{m}}} : \mathrm{H}^{\mathrm{out}}_{\mathrm{m},\mathfrak{s}}(G, N, \mathbb{T}) \to \mathrm{H}^3(G, \mathbb{T}).$$

We continue to work on the cocycle $(\lambda_{a,b}, 1)$ for $a \in \mathrm{Z}_{\hat{a}}$ whose restriction to $\{H_{\mathrm{m}}, K\}$ gives rise to the crossed extension $U \in \mathrm{Xext}(H_{\mathrm{m}}, K, \mathbb{T})$, given by

$$1 \to \mathbb{T} \longrightarrow U \underset{\mathfrak{s}_j}{\overset{j}{\rightleftarrows}} K \to 1,$$

where the group K is given by

$$K = \mathrm{Ker}(\nu_b \circ \pi_G) = \big\{g \in L : \textstyle\sum_{i\in\mathbb{N}} b(j,0)e_{j,N}(g) \in \mathbb{Z}\big\}.$$

Then the third cocycle $c_G \in \mathrm{Z}^3(G, \mathbb{T})$,

$$\begin{aligned} c_G(g_1; g_2; g_3) &= \alpha_{\mathfrak{s}_H(g_1)}(\mathfrak{s}_j(\mathfrak{n}_M(g_2; g_3)))\mathfrak{s}_j(\mathfrak{n}_M(g_1; g_2g_3)) \\ &\qquad\qquad \times \big(\mathfrak{s}_j(\mathfrak{n}_M(g_1; g_2))\mathfrak{s}_j(\mathfrak{n}_M(g_1g_2; g_3))\big)^{-1} \\ &= \lambda_{a,b}(\mathfrak{n}_M(g_2; g_3); g_1) = \lambda_a(\mathfrak{n}_M(g_2; g_3); g_1) \\ &= \exp\Big(2\pi \mathrm{i}\Big(\sum_{i<j<k} a(i,j,k)e_i(g_1)e_j(g_2)e_k(g_3)\Big)\Big) \\ &= c_a^G(g_1; g_2; g_3) \quad \text{for } g_1, g_2, g_3 \in G, \end{aligned}$$

is precisely the image $\partial_{\pi_{\mathrm{m}}} \circ \delta(\lambda_{a,b}, 1)$. ♡

6. Concluding remark

The history of cocycle (respectively outer) conjugacy analysis of group actions and group outer actions on an AFD factor goes back to the seminal work of Connes [1977; 1976b]. Steady progress was then made over the course of three decades; see especially the work of V. F. R. Jones [1980] and A. Ocneanu [1985].

We have now computed the invariants, which determine the outer conjugacy class, of an outer action of a countable discrete abelian group on an AFD factor of type III_λ for $0 < \lambda < 1$. The reduction of outer conjugacy analysis of an outer action of a countable discrete amenable group on an AFD factor of type III_λ down to the associated complete invariants was successfully carried out in [Katayama and Takesaki 2003; 2004; 2007]. As we have shown here, the invariants can be computed as soon as the group is specified, except in the case of type III_0.

Toward the one parameter automorphism group. After completing the classification of cocycle (respectively outer) conjugacy of countable discrete amenable group (respectively outer) actions on an AFD factor, it is natural to consider the same problem for a continuous group. The first step is obviously to study the one-parameter automorphism group $\{\alpha_t : t \in \mathbb{R}\}$ of an approximately finite-dimensional factor $\mathcal{R}_0$ of type II_1. Indeed, Y. Kawahigashi [1989; 1990; 1991b; 1991a] has already classified, up to cocycle (or stable) conjugacy, most one parameter automorphism groups of $\mathcal{R}_0$ constructed from concrete data; this was extended to the case of type III by U. K. Hui [2002]. However the general ones with full Connes spectrum are left untouched. One of difficulties is the lack of a technique that would allow us to create a one cocycle $\{u_s : s \in \mathbb{R}\}$ for a projection $p \in \mathrm{Proj}(\mathcal{R}_0)$ such that the perturbed one-parameter automorphism group $\{\mathrm{Ad}(u_t) \circ \alpha_t : t \in \mathbb{R}\}$ leaves the projection p invariant; this would allow us to localize analysis of the action. If a projection $p \in \mathrm{Proj}(\mathcal{R}_0)$ is differentiable relative to α, then the associated derivation δ_α generates a desired cocycle. But we don't know the answer to this:

Question. Does the C*-algebra

$$A = \{x \in \mathcal{R}_0 : \lim_{t \to 0} \|x - \alpha_t(x)\| = 0\}$$

contain a nontrivial projection?

If $p \in \mathrm{Proj}(A)$, then for each smooth function $f \in C_c^\infty(\mathbb{R})$ with compact support, the element

$$p(f) = \alpha_f(p) = \int_{\mathbb{R}} f(t)\alpha_t(p)\mathrm{d}t$$

is smooth, and one can choose f so that $\|p - p(f)\|$ is arbitrarily small, so that $\mathrm{Sp}(p(f))$ is concentrated on a neighborhood of the two points $\{0, 1\}$; this allows us to generate a nontrivial differentiable projection q near p via the contour integral

$$q = \frac{1}{2\pi \mathrm{i}} \oint_{|z-1|=r} (z - p(f))^{-1}\mathrm{d}z.$$

On the other hand, thanks to the exponential functional calculus, one can generate plenty of differentiable unitaries. For example, if $h \in A_{\mathrm{s.a}}$, then for a real-valued smooth function f, we get a differentiable unitary element $\exp(\mathrm{i} f(h))$ of A that can stay near the unitary $\exp(ih)$ in norm. Hence the group of differentiable unitaries is σ^*-strongly dense in the unitary group $\mathcal{U}(\mathcal{R}_0)$.

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