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We obtain a characterization of property (T) for von Neumann algebras in terms of 1-cohomology, similar to the Delorme–Guichardet theorem for groups.

0. Introduction

The analogue of group representations in von Neumann algebras is the notion of *correspondences* which is due to Connes [Connes 1982; 1980; Popa 1986], and has been a very useful in defining notions such as property (T) and amenability for von Neumann algebras. It is often useful to view group representations as positive definite functions that we obtain through a GNS construction. Correspondences of a von Neumann algebra N can also be viewed in two separate ways, as *Hilbert N-N bimodules $\mathcal{H}$*, or as *completely positive maps $\phi : N \to N$*, and the equivalence of these two descriptions is also realized via a GNS construction. This allows one to characterize property (T) for von Neumann algebras in terms of completely positive maps.

For a countable group G there is also a notion of conditionally negative definite functions $\psi : G \to \mathbb{C}$, which satisfy $\psi(g^{-1}) = \overline{\psi(g)}$ and the condition that for all $n \in \mathbb{N}$, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $g_1, g_2, \ldots, g_n \in G$, if $\sum_{i=1}^n \alpha_i = 0$ then $\sum_{i,j=1}^n \overline{\alpha_j}\alpha_i \psi(g_j^{-1} g_i) \leq 0$. Real-valued conditionally negative definite functions can be viewed as cocycles $b \in B^1(G, \pi)$, where $\pi : G \to \mathbb{O}(\mathcal{H})$ is an orthogonal representation of G; see [Bekka et al. 2008]. Real-valued conditionally negative definite functions can also be viewed as generators of semigroups of positive definite functions by Schoenberg's theorem. These equivalences then make possible certain connections between 1-cohomology, conditionally negative definite functions, and positive definite deformations, for example the Delorme–Guichardet theorem [Delorme 1977; Guichardet 1977], which states that a group has property (T) of Kazhdan [1967] if and only if the first cohomology vanishes for any unitary representation.

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It was Evans [1977] who introduced the notion of bounded conditionally completely positive/negative maps and related them to the study the infinitesimal generators of norm continuous semigroups of completely positive maps. He noted that this definition gives an analogue to conditionally positive/negative definite functions on groups. We will extend the notion of conditionally completely negative maps to unbounded maps and use a GNS type construction to alternately view them as closable derivations into a Hilbert N-N bimodule. This is done in the same spirit as [Sauvageot 1989; 1990], where Sauvageot makes a connection between quantum Dirichlet forms and differential calculus. Indeed, it is shown in Theorem 1.1 that conditionally completely negative maps are in fact extensions of generators associated to completely Dirichlet forms; however we are coming from a different perspective here and so we will present the correspondence between conditionally completely negative maps and closable derivations in a way more closely related to group theory.

In studying various properties of groups such as property (T) or the Haagerup property, one can give a characterization of these properties in terms of boundedness conditions on conditionally negative definite functions (as, for example in [Akemann and Walter 1981]); hence one would hope that this is possible for von Neumann algebras as well.

We will show that one can indeed obtain a characterization of property (T) in this way. The main result is that a separable finite factor has property (T) if and only if the 1-cohomology spaces of closable derivations vanish whenever the domain contains a non-Γ set (see Section 3 for the definition of a non-Γ set).

Theorem 0.1. *Suppose that N is a separable finite factor. Then the following conditions are equivalent*:

(1) *N has property* (*T*).

(2) *N does not have property Γ*, *and given any weakly dense $*$-subalgebra $N_0 \subset N$ with $1 \in N_0$ such that N_0 contains a non-Γ set*, *every densely defined closable derivation on N_0 into a Hilbert N-N bimodule is inner.*

(3) *There exists a weakly dense $*$-subalgebra $N_0 \subset N$ such that N_0 is countably generated as a vector space and every closable derivation into a Hilbert N-N bimodule whose domain contains N_0 is inner.*

This is the analogue to the Delorme–Guichardet theorem for groups. As a corollary we obtain that if $X_1, \dots, X_n$ generate a finite factor with property (T), and if at least one of the X_j has diffuse spectrum, then the derivations ∂_{X_i} from [Voiculescu 1998] cannot all be closable, and hence the conjugate variables cannot all exist in $L^2(N, \tau)$.

Corollary 0.2. *Suppose that N is a finite factor with property (T). Let $X_1, \dots, X_n$ generate N as a von Neumann algebra such that $\mathbb{C}[X_1, \dots, X_n]$ contains a non-Γ set in the sense of Definition 3.1. Then $\Phi^*(X_1, \dots, X_n) = \infty$.*

We also give an application showing that man amalgamated, free products of finite von Neumann algebras do not have property (T).

Theorem 0.3. *Let N_1 and N_2 be finite von Neumann algebras with normal faithful tracial states τ_1 and τ_2 respectively, and suppose that B is a common von Neumann subalgebra such that $\tau_1|_B = \tau_2|_B$. If there are unitaries $u_i \in \mathcal{U}(N_i)$ such that $E_B(u_i) = 0$ for $i = 1, 2$, then $M = N_1 *_B N_2$ does not have property (T).*

Other than the introduction there are four sections. Section 1 establishes the definitions and notations and gives the connection between closable derivations, conditionally completely negative maps, and semigroups of completely positive maps. In Section 2, we characterize when a closable derivation is inner, in terms of the conditionally completely negative map and the semigroup. In Section 3, we state and prove the main theorem, Theorem 3.2, and in Section 4, we give the application to amalgamated free products (Corollary 4.2).

1. A GNS-type construction

1.1. ***Conditionally completely negative maps.*** Let N be a finite von Neumann algebra with normal faithful trace τ.

Definition. Suppose $\Psi : N \to L^1(N, \tau)$ is a $*$-preserving linear map whose domain is a weakly dense $*$-subalgebra D_Ψ of N such that $1 \in D_\Psi$. Then Ψ is a conditionally completely negative (c.c.n.) map on N if,

(1.1.1) for all $n \in \mathbb{N}$, $x_j, y_j \in D_\Psi$, and $j \leq n$,

$$\sum_{j=1}^{n} x_j y_j = 0 \text{ implies } \sum_{i,j=1}^{n} y_j^* \Psi(x_j^* x_i) y_i \leq 0.$$

It is not hard to see that condition (1.1.1) can be replaced with the condition that

(1.1.1′) for all $n \in \mathbb{N}$, $x_j, y_j \in D_\Psi$, and $j \leq n$,

$$\sum_{j=1}^{n} x_j y_j = 0 \text{ implies } \sum_{i,j=1}^{n} \tau(\Psi(x_j^* x_i) y_i y_j^*) \leq 0.$$

If $\phi : N \to N$ is completely positive and $k \in N$, then $\Psi(x) = k^* x + xk - \phi(x)$ gives a map that is c.c.n. and bounded. If $\delta : N \to L^2(N, \tau)$ is a derivation, then δ is c.c.n. Also if Ψ is a c.c.n. map and $\alpha : N \to N$ is a τ-preserving automorphism, then $\Psi' = \alpha \circ \Psi \circ \alpha^{-1}$ is another c.c.n. map.

One can check that if Ψ_1 and Ψ_2 are c.c.n. such that $D_{\Psi_1} \cap D_{\Psi_2}$ is weakly dense in N, and if $s, t \geq 0$, then $\Psi = s\Psi_1 + t\Psi_2$ is c.c.n. Also if $\{\Psi_t\}_t$ is a family of c.c.n. maps on the same domain and Ψ is the pointwise $\| \cdot \|_1$-limit of $\{\Psi_t\}_t$, then Ψ is c.c.n.

We say that Ψ is symmetric if $\tau(\Psi(x)y) = \tau(x\Psi(y))$ for all $x, y \in D_\Psi$. We say that Ψ is conservative if $\tau \circ \Psi = 0$. We also say that Ψ is closable if the quadratic form q on $L^2(N, \tau)$ given by $D(q) = D_\Psi$ and $q(x) = \tau(\Psi(x)x^*)$ is closable. Note that we will see in Section 1.3 that if $\Psi : D_\Psi \to L^2(N, \tau) \subset L^1(N, \tau)$ is a conservative symmetric c.c.n. map, then Ψ is automatically closable.

If Ψ is a conservative symmetric c.c.n. map, then $\tau(\Psi(1)x) = \tau(\Psi(x)) = 0$ for all $x \in D_\Psi$; hence $\Psi(1) = 0$. Also, if Ψ is symmetric and $\Psi(1) \geq 0$, then given any $x \in D_\Psi$, if we let $x_1 = x,\ x_2 = 1,\ y_1 = -1$, and $y_2 = x$, then the above condition implies that $\tau(\Psi(x)x^*) \geq 0$, so that we actually have positivity instead of just the symmetry condition.

1.2. ***Closable derivations.*** Let $\mathcal{H}$ be a Hilbert N-N bimodule. A derivation of N is a (possibly unbounded) map $\delta : N \to \mathcal{H}$ defined on a weakly dense $*$-subalgebra D_δ of N such that $1 \in D_\delta$, and such that $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in D_\delta$. The map δ is closable if it is closable as an operator from $L^2(N, \tau)$ to $\mathcal{H}$.

The map δ is inner if $\delta(x) = x\xi - \xi x$ for some $\xi \in \mathcal{H}$. It is spanning if $\overline{\mathrm{sp}} D_\delta \delta(D_\delta) = \mathcal{H}$, and it is real if

$$\langle x\delta(y), \delta(z)\rangle_{\mathcal{H}} = \langle \delta(z^*), \delta(y^*)x^*\rangle_{\mathcal{H}} \quad \text{for all } x, y, z \in D_\delta.$$

If $\delta' : D_\delta \to \mathcal{H}'$ is another derivation, then we say that δ and δ' are equivalent if there exists a unitary map $U : \mathcal{H} \to \mathcal{H}'$ such that $U(x\delta(y)z) = xU(\delta(y))z = x\delta'(y)z$ for all $x, y, z \in D_\delta$.

Recall that if $\mathcal{H}$ is a Hilbert N-N bimodule, then we can define the adjoint bimodule $\mathcal{H}^\circ$, where $\mathcal{H}^\circ$ is the conjugate Hilbert space of $\mathcal{H}$ and the bimodule structure is given by $x\xi^\circ y = (y^*\xi x^*)^\circ$. If $\delta : D_\delta \to \mathcal{H}$ is a closable derivation, then we may define the adjoint derivation $\delta^\circ : D_\delta \to \mathcal{H}^\circ$ by setting $\delta^\circ(x) = \delta(x^*)^\circ$; then δ° is a closable derivation and the derivations $\frac{1}{2}(\delta + \delta^\circ)$ and $\frac{1}{2}(\delta - \delta^\circ)$ are real derivations from D_δ to $\mathcal{H} \oplus \mathcal{H}^\circ$.

1.3. ***From conditionally completely negative maps to closable derivations.*** Let Ψ be a conservative symmetric c.c.n. map on N with domain D_Ψ. We associate to Ψ a derivation in the following way (compare with [Sauvageot 1989]):

Let

$$\mathcal{H}_0 = \Big\{ \sum_{i=1}^{n} x_i \otimes y_i \in D_\Psi \otimes D_\Psi \ \Big|\ \sum_{i=1}^{n} x_i y_i = 0 \Big\}.$$

Define a sesquilinear form on $\mathcal{H}_0$ by

$$\Big\langle \sum_{i=1}^{n} x_i' \otimes y_i', \sum_{j=1}^{m} x_j \otimes y_j \Big\rangle_\Psi = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \tau(\Psi(x_j^* x_i') y_i' y_j^*).$$

The positivity of $\langle \cdot, \cdot \rangle_\Psi$ is equivalent to the c.c.n. condition on Ψ. Let $\mathcal{H}$ be the closure of $\mathcal{H}_0$ after we mod out by the kernel of $\langle \cdot, \cdot \rangle_\Psi$. If $p = \sum_{k=1}^n x_k \otimes y_k$ such that $\sum_{k=1}^n x_k y_k = 0$, then

$$x \mapsto -\frac{1}{2} \sum_{i,j=1}^{n} \tau(x_j^* x x_i \Psi(y_i y_j^*)) \quad \text{and} \quad y \mapsto -\frac{1}{2} \sum_{i,j=1}^{n} \tau(\Psi(x_j^* x_i) y_i y y_j^*)$$

are both positive normal functionals on N with norm $\langle p, p \rangle_\Psi$. We also have left and right commuting actions of D_Ψ on $\mathcal{H}_0$ given by

$$xpy = x\Big(\sum_{k=1}^{n} x_k \otimes y_k\Big)y = \sum_{k=1}^{n} (x x_k) \otimes (y_k y),$$

and by the preceding remarks we have, for all $x, y \in D_\Psi$,

$$\begin{aligned} &\langle xp, xp \rangle_\Psi = \langle x^* x p, p \rangle_\Psi \le \|x^* x\| \langle p, p \rangle_\Psi = \|x\|^2 \langle p, p \rangle_\Psi, \\ &\langle py, py \rangle_\Psi \le \|y\|^2 \langle p, p \rangle_\Psi. \end{aligned}$$

Hence the above actions of D_Ψ pass to commuting left and right actions on $\mathcal{H}$, and they extend to left and right actions of N on $\mathcal{H}$ given by the formulas

$$\begin{aligned} \Big\langle x\big[\textstyle\sum_{i=1}^n x_i' \otimes y_i'\big], \big[\sum_{j=1}^m x_j \otimes y_j\big] \Big\rangle_{\mathcal{H}} &= \sum_{i=1}^{n} \sum_{j=1}^{m} \tau(x_j^* x x_i' \Psi(y_i' y_j^*)), \\ \Big\langle \big[\textstyle\sum_{i=1}^n x_i' \otimes y_i'\big] y, \big[\sum_{j=1}^m x_j \otimes y_j\big] \Big\rangle_{\mathcal{H}} &= \sum_{i=1}^{n} \sum_{j=1}^{m} \tau(\Psi(x_j^* x_i') y_i' y y_j^*). \end{aligned}$$

Since the above forms are normal, the left and right actions commute and are normal, thus making $\mathcal{H}$ into a Hilbert N-N bimodule.

Define $\delta_\Psi : D_\Psi \to \mathcal{H}$ by $x \mapsto [x \otimes 1 - 1 \otimes x]$. Then δ_Ψ is a derivation such that, for all $x, y \in D_\Psi$,

$$\begin{aligned} \langle \delta_\Psi(x), \delta_\Psi(y) \rangle_{\mathcal{H}} &= \langle x \otimes 1 - 1 \otimes x, y \otimes 1 - 1 \otimes y \rangle_\Psi \\ &= -\tfrac{1}{2}\tau(\Psi(y^* x)) + \tfrac{1}{2}\tau(\Psi(x) y^*) + \tfrac{1}{2}\tau(\Psi(y^*) x) - \tfrac{1}{2}\tau(\Psi(1) x y^*) \\ &= \tau(\Psi(x) y^*). \end{aligned}$$

Also δ_Ψ is real since, for all $x, y, z \in D_\Psi$,

$$\begin{aligned}\langle x\delta_\Psi(y), \delta_\Psi(z)\rangle_{\mathcal{H}} &= \langle xy\otimes 1 - x\otimes y, z\otimes 1 - 1\otimes z\rangle_\Psi \\ &= -\tfrac{1}{2}\tau(\Psi(z^*xy)) + \tfrac{1}{2}\tau(\Psi(xy)z^*) + \tfrac{1}{2}\tau(\Psi(z^*x)y) - \tfrac{1}{2}\tau(\Psi(x)yz^*) \\ &= -\tfrac{1}{2}\tau(\Psi(1)z^*xy) + \tfrac{1}{2}\tau(\Psi(z^*)xy) + \tfrac{1}{2}\tau(\Psi(y)z^*x) - \tfrac{1}{2}\tau(\Psi(yz^*)x) \\ &= \langle 1\otimes z^* - z^*\otimes 1, 1\otimes y^*x^* - y^*\otimes x^*\rangle_\Psi = \langle \delta_\Psi(z^*), \delta_\Psi(y^*)x^*\rangle_{\mathcal{H}}.\end{aligned}$$

It follows that δ_Ψ is closable if Ψ is. Also if $\Psi : D_\Psi \to L^2(N, \tau) \subset L^1(N, \tau)$, then we would have $D_\Psi = D(\delta_\Psi^*\delta_\Psi)$, which would show that δ (and hence also Ψ) is closable.

We will also assume that δ_Ψ is spanning by restricting to $\overline{\mathrm{sp}} D_\Psi \delta(D_\Psi) \subset \mathcal{H}$.

It is not really much of a restriction that $\Psi(1) = 0$, since if Ψ is any symmetric c.c.n. map with $\Psi(1) \in L^2(N, \tau)$, then $\Psi'(x) = \Psi(x) - \frac{1}{2}\Psi(1)x - \frac{1}{2}x\Psi(1)$ defines a symmetric c.c.n. map with $\Psi'(1) = 0$.

1.4. ***From closable derivations to conditionally completely negative maps.*** Let $\mathcal{H}$ be a Hilbert N-N bimodule, and suppose that $\delta : N \to \mathcal{H}$ is a closable real derivation defined on a weakly dense $*$-subalgebra D_δ of N with $1 \in D_\delta$.

Define

$$D_\Psi = \bigl\{x \in D(\overline{\delta}) \cap N \mid y \mapsto \langle \overline{\delta}(x), \delta(y^*)\rangle \text{ gives a normal linear functional on } N\bigr\}.$$

Then by [Sauvageot 1990; Davies and Lindsay 1992], $D(\overline{\delta}) \cap N$ is a $*$-subalgebra, and hence one can show that D_Ψ is a dense $*$-subalgebra of N. We define the map $\Psi_\delta : D_\Psi \to L^1(N, \tau)$ by letting $\Psi_\delta(x)$ be the Radon–Nikodym derivative of the normal linear functional $y \mapsto \langle \overline{\delta}(x), \delta(y^*)\rangle$. Since δ is closable, Ψ_δ is also.

Since δ is real, Ψ_δ is a symmetric $*$-preserving map such that $\tau \circ \Psi = 0$, and if $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n, y_1, y_x, \dots, y_n \in D_\Psi$ such that $\sum_{i=1}^n x_i y_i = 0$, then

$$\begin{aligned}\sum_{i,j=1}^n \tau(\Psi(x_j^*x_i)y_iy_j^*) &= \sum_{i,j=1}^n \langle \delta(x_j^*x_i), \delta(y_jy_i^*)\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^n \langle x_j^*\delta(x_i), y_j\delta(y_i^*) + \delta(y_j)y_i^*\rangle_{\mathcal{H}} + \langle \delta(x_j^*)x_i, y_j\delta(y_i^*) + \delta(y_j)y_i^*\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^n \langle \delta(x_i)y_i, x_j\delta(y_j)\rangle_{\mathcal{H}} + \langle x_i\delta(y_i), \delta(x_j)y_j\rangle_{\mathcal{H}} = -2\Bigl\|\sum_{i=1}^n \delta(x_i)y_i\Bigr\|_{\mathcal{H}}^2 \le 0.\end{aligned}$$

Hence Ψ_δ is a conservative symmetric c.c.n. map on D_Ψ.

Note that if we restrict ourselves to closable derivations that are spanning, then an easy calculation shows that the constructions above are inverses of each other in the sense that $\Psi_{\delta_\Psi}|_{D_\Psi} = \Psi$ and $\overline{\delta_{\Psi_\delta}} \cong \overline{\delta}$.

1.5. ***Closable derivations and c.c.n. maps from groups.*** Let Γ be a discrete group, (C, τ_0) a finite von Neumann algebra with a normal faithful trace, and σ a cocycle action of Γ on (C, τ_0) by τ_0-preserving automorphisms. Denote by $N = C \rtimes_\sigma \Gamma$ the corresponding cross-product algebra with trace τ given by $\tau(\Sigma c_g u_g) = \tau_0(c_e)$, where $c_g \in C$ and $\{u_g\}_g \subset N$ denote the canonical unitaries implementing the action σ on C.

Let $(\pi_0, \mathcal{H}_0)$ be a unitary or orthogonal representation of Γ, and let $b : \Gamma \to \mathcal{H}_0$ be an (additive) cocycle of Γ, that is, $b(gh) = \pi_0(g)b(h) + b(g)$ for all $g, h \in \Gamma$. Set $\mathcal{H}_{\pi_0}$ to be the Hilbert space $\mathcal{H}_0 \overline{\otimes}_{\mathbb{R}} L^2(N, \tau)$ if π_0 is an orthogonal representation and $\mathcal{H}_0 \overline{\otimes}_{\mathbb{C}} L^2(N, \tau)$ if π_0 is a unitary representation. We let N act on the right of $\mathcal{H}_{\pi_0}$ by $(\xi \otimes \hat{x})y = \xi \otimes (\widehat{xy})$ for $x, y \in N$ and $\xi \in \mathcal{H}_0$ and on the left by $c(\xi \otimes \hat{x}) = \xi \otimes (\widehat{cx})$ and $u_g(\xi \otimes \hat{x}) = (\pi_0(g)\xi) \otimes (\widehat{u_g x})$ for $c \in C,\ x \in N,\ g \in \Gamma$, and $\xi \in \mathcal{H}_0$. Let D_Γ be the $*$-subalgebra generated by C and $\{u_g\}_g$. We define δ_b by $\delta_b(c_g u_g) = c_g \delta_b(u_g) = b(g) \otimes \widehat{c_g u_g}$ for $c_g \in C$ and $g \in \Gamma$; then we can extend δ_b linearly so that δ_b is a derivation on D_Γ. If $(\pi_0, \mathcal{H}_0)$ is an orthogonal representation and 1_g denotes the Dirac delta function at g, then

$$\begin{aligned}
\langle c u_g \delta_b(u_h), \delta_b(u_k)\rangle &= \langle \pi_0(g)b(h), b(k)\rangle \langle \widehat{c u_g u_h}, \hat{u}_k\rangle \\
&= \langle -\pi_0(g)\pi_0(h)b(h^{-1}), -\pi_0(k)b(k^{-1})\rangle \langle \widehat{c u_g u_h}, \hat{u}_k\rangle 1_k(gh) \\
&= \langle b(k^{-1}), b(h^{-1})\rangle \langle \hat{u}_k^*, \widehat{u_h^* u_g^* c^*}\rangle \\
&= \langle \delta_b(u_k^*), \delta_b(u_h^*) u_g^* c^*\rangle,
\end{aligned}$$

for all $g, h, k \in \Gamma$ and $c \in C$, thus showing that δ_b is real.

Also we have

$$\begin{aligned}
\left|\langle \delta_b(c_g u_g), \delta_b(\textstyle\sum_{h\in\Gamma} d_h u_h)\rangle\right| &= \left|\textstyle\sum_{h\in\Gamma} \langle b(g), b(h)\rangle \langle \widehat{c_g u_g}, \widehat{d_h u_h}\rangle\right| \\
&= \|b(g)\|^2 \left|\langle \widehat{c_g u_g}, \textstyle\sum_{h\in\Gamma} \widehat{d_h u_h}\rangle\right| \\
&\le \|b(g)\|^2 \|c_g\| \left\|\textstyle\sum_{h\in\Gamma} d_h u_h\right\|_1,
\end{aligned}$$

for all $g \in \Gamma,\ c_g \in C$ and $\sum_{h\in\Gamma} d_h u_h \in D_\Gamma$. Hence if $x = \sum_{g\in\Gamma} c_g u_g \in D_\Gamma$ and $y \in D_\Gamma$, then $|\langle \delta_b(x), \delta_b(y)\rangle| \le (\sum_{g\in\Gamma} \|b(g)\|^2 \|c_g\|) \|y\|_1$. In particular this shows that δ_b is closable.

Now suppose that $\psi : \Gamma \to \mathbb{C}$ is a real-valued conditionally negative definite function on Γ such that $\psi(e) = 0$, and let (π_ψ, b_π) be the representation and cocycle that correspond to ψ through the GNS construction [Bekka et al. 2008]. Let $(\mathcal{H}, \delta)$ denote the Hilbert N-N bimodule and closable derivation constructed out of (π_ψ, b_π) as above, and let Ψ be the symmetric c.c.n. map associated to $(\mathcal{H}, \delta)$ as in Section 1.4. Then a calculation shows that $\Psi(\sum_g c_g u_g) = \sum_g \psi(g) c_g u_g$, and in fact it is easy to show that this equation still describes a c.c.n. map even if ψ is not real valued.

Conversely, if $(\mathcal{H}, \delta)$ is a Hilbert N-N bimodule and a closable derivation such that δ is defined on the $*$-subalgebra generated by C and $\{u_g\}_g$, then we can associate to it a representation π_0 on $\mathcal{H}_0 = \overline{\text{sp}}\{\delta(u_g)u_g^* \mid g \in \Gamma\}$ by $\pi_0(g)\xi' = u_g \xi' u_g^*$ for $\xi' \in \mathcal{H}_0$. Also we may associate to δ a group cocycle b on Γ by $b(g) = \delta(u_g)u_g^*$ for $g \in \Gamma$. If Ψ is a c.c.n. map that is also defined on the $*$-subalgebra generated by C and $\{u_g\}_g$, then we can associate to it a conditionally negative definite function ψ by $\psi(g) = \tau(\Psi(u_g)u_g^*)$. Furthermore if δ is real, then by taking only the real span above, we see that $\mathcal{H}_0$ is a real Hilbert space and π_0 is an orthogonal representation; also ψ is real valued if and only if Ψ is symmetric, and if $(\mathcal{H}, \delta)$ and Ψ correspond to each other as in Sections 1.3 and 1.4, then (π_0, b) and ψ correspond to each other via the GNS construction.

1.6. ***Examples from free probability.*** We now have two main examples of closable derivations, those that are inner, and those that come from cocycles on groups. Voiculescu [1998; 1999] uses certain derivations in a key role for his nonmicrostates approach to free entropy and mutual free information. These derivations will give us more examples of closable derivations under certain circumstances.

1.6.1. *The derivation ∂_X from* [Voiculescu 1998]. Let $B \subset N$ be a $*$-subalgebra with $1 \in B$ and $X = X^* \in N$. If we denote by $B[X]$ the subalgebra generated by B and X, and if X and B are algebraically free (that is, they do not satisfy any nontrivial algebraic relations), then there is a well-defined unique derivation

$$\partial_X : B[X] \to B[X] \otimes B[X] \subset L^2(N, \tau) \otimes L^2(N, \tau)$$

such that $\partial_X(X) = 1 \otimes 1$ and $\partial_X(b) = 0$ for all $b \in B$.

We note that if ∂_X is inner, then identifying $L^2(N, \tau) \otimes L^2(N, \tau)$ with the Hilbert–Schmidt operators gives the existence of a Hilbert–Schmidt operator that commutes with B. Therefore if B contains a diffuse element (that is, one generating a von Neumann algebra without minimal projections), then ∂_X is not inner.

From [Voiculescu 1998], the conjugate variable $J(X:B)$ of X with respect to B is an element in $L^1(W^*(B[X]), \tau)$ such that $\tau(J(X:B)m) = \tau \otimes \tau(\partial_X(m))$ for all $m \in B[X]$, that is, $J(X:B) = \partial_X^*(1 \otimes 1)$.

If $J(X:B)$ exists and is in $L^2(N, \tau)$ (as in the case when we perturb a set of generators by free semicircular elements), then ∂_X is a closable derivation by [Voiculescu 1998, Corollary 4.2]

1.6.2. *The derivation $\delta_{A:B}$ from* [Voiculescu 1999]. Suppose $A, B \subset N$ are two $*$-subalgebras with $1 \in A, B$. If we denote by $A \vee B$ the subalgebra generated by A and B, and if A and B are algebraically free, then we may define a unique derivation

$$\delta_{A:B} : A \vee B \to (A \vee B) \otimes (A \vee B) \subset L^2(N, \tau) \otimes L^2(N, \tau)$$

by $\delta_{A:B}(a) = a \otimes 1 - 1 \otimes a$ for all $a \in A$, and $\delta_{A:B}(b) = 0$ for all $b \in B$.

For the same reason as above, if B contains a diffuse element and $A \neq \mathbb{C}$, then the derivation is not inner.

Recall from [Voiculescu 1999] that the liberation gradient $j(A:B)$ of (A, B) is an element in $L^1(W^*(A \cup B), \tau)$ such that $\tau(j(A:B)m) = \tau \otimes \tau(\delta_{A:B}(m))$ for all $m \in A \vee B$, that is, $j(A:B) = \delta^*_{A:B}(1 \otimes 1)$.

If $j(A:B)$ exists and is in $L^2(N, \tau)$, then $\delta_{A:B}$ is a closable derivation by [Voiculescu 1999, Corollary 6.3]

1.7. ***Generators of completely positive semigroups.*** Suppose N is a finite von Neumann algebra with normal faithful trace τ. A weak*-continuous semigroup $\{\phi_t\}_{t\geq 0}$ on N is said to be symmetric if $\tau(x\phi_t(y)) = \tau(\phi_t(x)y)$ for all $x, y \in N$, and completely Markovian if each ϕ_t is a unital c.p. map on N. We denote by Δ the generator of a symmetric completely Markovian semigroup $\{\phi_t\}_{t\geq 0}$ on N, that is, Δ is the densely defined operator on N described by

$$D(\Delta) = \{x \in N : (x - \phi_t(x))/t \text{ has a weak limit as } t \to 0\},$$

and $\Delta(x) = \lim_{t\to 0}(x - \phi_t(x))/t$. We also let Δ denote the generator of the corresponding semigroup on $L^2(N, \tau)$. Then Δ describes a completely Dirichlet form [Davies and Lindsay 1992] on $L^2(N, \tau)$ by

$$D(\mathscr{E}) = D(\Delta^{1/2}) \quad \text{and} \quad \mathscr{E}(x) = \|\Delta^{1/2}(x)\|_2^2.$$

From [Davies and Lindsay 1992], $D(\mathscr{E}) \cap N$ is a weakly dense $*$-subalgebra, and hence it follows from [Sauvageot 1989] that there exists a Hilbert N-N bimodule $\mathscr{H}$ and a closable derivation $\delta : D(\mathscr{E}) \cap N \to \mathscr{H}$ such that $\mathscr{E}(x) = \|\delta(x)\|^2$ for all $x \in D(\mathscr{E}) \cap N$. Conversely it follows from [Sauvageot 1990] that if $D(\delta)$ is a weakly dense $*$-subalgebra with $1 \in D(\delta)$ and $\delta : D(\delta) \to \mathscr{H}$ is a closable derivation, then the closure of the quadratic form given by $\|\delta(x)\|^2$ is completely Dirichlet on $L^2(N, \tau)$ and hence generates a symmetric completely Markovian semigroup as above (see also [Cipriani and Sauvageot 2003]).

From Sections 1.3 and 1.4 and from the remarks above, we obtain the following.

Theorem 1.1. *Let $N_0 \subset N$ be a weakly dense $*$-subalgebra with $1 \in N_0$, and suppose $\Psi : N_0 \to L^1(N, \tau)$ is a closable, conservative, symmetric c.c.n. map such that $\Psi^{-1}(L^2(N, \tau))$ is weakly dense in N. Then $\Delta = \Psi|_{\Psi^{-1}(L^2(N,\tau))}$ is closable as a densely defined operator on $L^2(N, \tau)$, and $\overline{\Delta}$ is the generator of a symmetric completely Markovian semigroup on N. Conversely, if Δ is the generator of a symmetric completely Markovian semigroup on N, then Δ extends to a conservative, symmetric c.c.n. map $\Psi : N_0 \to L^1(N, \tau)$, where N_0 is the $*$-subalgebra generated by $D(\Delta)$.*

2. A characterization of inner derivations

Let N be a finite von Neumann algebra with normal faithful trace τ. Given a symmetric c.c.n. map Ψ on N, we will now give a characterization of when Ψ is norm bounded.

Theorem 2.1. *Let $\Psi : D_\Psi \to L^1(N, \tau)$ be a closable, conservative, symmetric c.c.n. map with weakly dense domain D_Ψ. Let $\delta : D_\Psi \to \mathcal{H}$ be the closable derivation associated with Ψ. Then the following conditions are equivalent:*

(a) *δ extends to an everywhere-defined derivation δ' that is inner and such that for any $x \in N$ there exists a constant $C_x > 0$ such that $|\langle \delta'(x), \delta'(y)\rangle| \leq C_x \|y\|_1$ for all $y \in N$.*

(b) *There exists a constant $C > 0$ such that $|\langle \delta(x), \delta(y)\rangle| \leq C\|x\|\|y\|_1$ for all $x, y \in D_\Psi$.*

(c) *Ψ is norm bounded on $(D_\Psi)_1$.*

(d) *The image of Ψ is contained in $N \subset L^1(N, \tau)$, and $-\Psi$ extends to a mapping that generates a norm continuous semigroup of normal c.p. maps.*

(e) *There exists a $k \in N$ and a normal c.p. map $\phi : N \to N$ with the property that $\Psi(x) = k^*x + xk - \phi(x)$ for all $x \in D_\Psi$.*

Proof. (a) implies (c): Let δ' be the everywhere-defined extension of δ, and let Ψ' be the c.c.n. map associated with δ'. Since for any $x \in N$ there exists a constant $C_x > 0$ such that $|\langle \delta'(x), \delta'(y)\rangle| \leq C_x\|y\|_1$ for all $y \in N$, the image of Ψ' is contained in N. Also since $\Psi'(1) = 0$ we have $\Psi'(x^*x) - x^*\Psi'(x) - \Psi'(x^*)x \leq 0$ for all $x \in D_{\Psi'}$, and so $-\Psi'$ is a dissipation [Lindblad 1976; Kishimoto 1976]. Since $-\Psi'$ is also everywhere-defined, it is bounded by [Kishimoto 1976, Theorem 1].

(b) is equivalent to (c): If (b) holds, then for all $x, y \in D_\Psi$,

$$|\tau(\Psi(x)y^*)| = |\langle \delta(x), \delta(y)\rangle| \leq C\|x\|\|y\|_1.$$

So by taking the supremum over all $y \in D_\Psi$ such that $\|y\|_1 \leq 1$, we find that $\|\Psi(x)\| \leq C\|x\|$ for all $x \in D_\Psi$.

Suppose now Ψ is bounded by $C > 0$. Then for all $x, y \in D_\Psi$,

$$|\langle \delta(x), \delta(y)\rangle| = |\tau(\Psi(x)y^*)| \leq \|\Psi(x)\|\|y\|_1 \leq C\|x\|\|y\|_1.$$

(c) implies (d): This follows from [Evans 1977, Proposition 2.10].

(d) implies (e): This is [Christensen and Evans 1979, Theorem 3.1].

(e) implies (a): Suppose that for $k \in N$ and ϕ c.p., we have $\Psi(x) = k^*x + xk - \phi(x)$ for all $x \in N$. Let $\phi' = \tau(\phi(1))^{-1}\phi$, and let $(\mathcal{H}, \xi)$ be the pointed Hilbert N-bimodule associated with ϕ'. Hence if we set $\delta'(x) = (\tau(\phi(1))/2)^{1/2}[x, \xi]$, then we have $\delta' \cong \delta$. By replacing k with $\frac{1}{2}(k+k^*)$ and ϕ with $\frac{1}{2}(\phi+\phi^*)$, we may assume that ϕ is symmetric; it is then easy to verify that there exists a constant $C > 0$ such that $|\langle \delta'(x), \delta'(y)\rangle| \le C\|x\|\|y\|_1$ for all $x, y \in N$. Hence δ' gives an everywhere-defined extension of δ satisfying the required properties. $\square$

Our next result is in the same spirit as Theorem 2.1. It provides several equivalent conditions for a closable derivation to be inner.

Theorem 2.2. *Let $\Psi : D_\Psi \to L^1(N,\tau)$ be a closable, conservative, symmetric c.c.n. map with weakly dense domain D_Ψ. Let $\delta : D_\Psi \to \mathcal{H}$ be the closable derivation associated with Ψ. Then the following conditions are equivalent*:

(α) *δ is inner.*

(β) *δ is bounded on $(D_\Psi)_1$.*

(γ) *Ψ is $\|\cdot\|_1$-bounded on $(D_\Psi)_1$.*

(δ) *Ψ can be approximated uniformly by c.p. maps in the sense that, for all $\varepsilon > 0$, there exists a $k \in N$ and a normal c.p. map ϕ such that*

$$\|\Psi(x) - k^*x - xk + \phi(x)\|_1 \le \varepsilon\|x\| \quad \textit{for all } x \in D_\Psi.$$

Proof. (α) implies (δ): Suppose there is a $\xi \in \mathcal{H}$ such that $\delta(x) = x\xi - \xi x$ for all $x \in D_\Psi$. Let $\varepsilon > 0$. Since the subspace of "left and right bounded" vectors is dense in $\mathcal{H}$, we may choose $\xi_0 \in \mathcal{H}$ so that there exists a constant $C > 0$ such that $\|x\xi_0\| \le C\|x\|_2$ for all $x \in N$, $\|\xi_0\| \le \|\xi\|$, and also $\|\xi - \xi_0\| < \varepsilon/8\|\xi\|$. Since ξ_0 is "bounded", we may let ϕ_{ξ_0} be the normal c.p. map associated with $\xi_0/\|\xi_0\|$. Let $\phi = 2\|\xi_0\|^2\phi_{\xi_0}$, and let $k = \phi(1)/2$.

Note that since δ is real, ξ_0 is real also, that is, $\langle x\xi_0, \xi_0 y\rangle = \langle y^*\xi_0, \xi_0 x^*\rangle$ for all $x, y \in N$.

Then if $x, y \in D_\Psi$, we have

$$\begin{aligned}
&\tau((\Psi(x) - k^*x - xk + \phi(x))y^*) \\
&\qquad = \tau(\Psi(x)y^*) - \tfrac{1}{2}\tau(\phi(1)xy^*) - \tfrac{1}{2}\tau(x\phi(1)y^*) + \tau(\phi(x)y^*) \\
&\qquad = \langle \delta(x), \delta(y)\rangle - \langle xy^*\xi_0, \xi_0\rangle - \langle y^*x\xi_0, \xi_0\rangle + 2\langle x\xi_0 y^*, \xi_0\rangle \\
&\qquad = \langle x\xi - \xi x, y\xi - \xi y\rangle - \langle x\xi_0 - \xi_0 x, y\xi_0 - \xi_0 y\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|\langle \Psi(x) - k^*x - xk + \phi(x), y\rangle| \\
&\quad \le \|x\xi - \xi x\|\|y\xi - \xi y - y\xi_0 + \xi_0 y\| + \|y\xi_0 - \xi_0 y\|\|x\xi - \xi x - x\xi_0 + \xi_0 x\| \\
&\quad \le 4\|x\|\|\xi\|\|y\|\|\xi - \xi_0\| + 4\|y\|\|\xi_0\|\|x\|\|\xi - \xi_0\| \\
&\quad \le \varepsilon\|x\|\|y\|.
\end{aligned}$$

The desired result follows by taking the supremum over all $y \in (D_\Psi)_1$.

(δ) implies (γ): Let $k \in N$ and ϕ c.p. such that $\|\Psi(x) - k^*x - xk + \phi(x)\|_1 \leq \|x\|$. By [Popa 2006, 1.1.2], $\|\phi(x)\|_2 \leq \|\phi(1)\|_2 \|x\|$ for all $x \in N$. Hence, for all $x \in D_\Psi$,

$$\begin{aligned}\|\Psi(x)\|_1 &\leq \|\Psi(x) - k^*x - xk + \phi(x)\|_1 + \|k^*x - xk + \phi(x)\|_2 \\ &\leq (1 + 2\|k\|_2 + \|\phi(1)\|_2)\|x\|.\end{aligned}$$

Thus Ψ is bounded in $\|\cdot\|_1$ on $(D_\Psi)_1$.

(γ) implies (β): If $\|\Psi(x)\|_1 \leq C\|x\|$ for all $x \in D_\Psi$, then

$$\|\delta(x)\|^2 = \tau(\Psi(x)x^*) \leq \|\Psi(x)\|_1 \|x\| \leq C\|x\|^2 \quad \text{for all } x \in D_\Psi.$$

(β) implies (α): Since δ is bounded on $(D_\Psi)_1$, we may extend δ to a derivation on the C^*-algebra A that is generated by D_Ψ. Let $X = \{\delta(u)u^* \mid u \in \mathcal{U}(A)\}$. For each $v \in \mathcal{U}(A)$, we let v act on $\mathcal{H}$ by $v \cdot \xi = v\xi + \delta(v)$. Let ξ_0 be the center of the set X. Then since $\|v \cdot \xi - v \cdot \eta\| = \|\xi - \eta\| = \|\xi v - \eta v\|$ for all $\xi, \eta \in \mathcal{H}$, the center of the set $v \cdot X$ is $v \cdot \xi_0$, and the center of the set Xv is $\xi_0 v$. Further we have $v \cdot X = Xv$, and thus $v \cdot \xi_0 = \xi_0 v$. Since v was arbitrary and every $x \in A$ is a linear combination of unitaries, we have $\delta(x) = \xi_0 x - x\xi_0$ for all $x \in A$. □

In general, Ψ may be unbounded in $\|\cdot\|_1$ even if $\|\phi_t(x) - x\|_2$ converges to 0 uniformly on N_1. However, the next section shows that this cannot happen if N has property (T) and the domain of Ψ contains a “critical set” as in [Connes and Jones 1985, Proposition 1].

3. Property (T) in terms of closable derivations

Suppose M is a finite von Neumann algebra with countable decomposable center. We will say that M has property (T) if the inclusion $(M \subset M)$ is rigid in the sense of [Popa 2006]; that is, M has property (T) if and only if there exists a normal faithful tracial state τ' on M such that one of these equivalent conditions hold:

- For all $\varepsilon > 0$, there exists a finite $F' \subset M$ and a $\delta' > 0$ such that if $\mathcal{H}$ is a Hilbert M-M bimodule with a vector $\xi \in \mathcal{H}$ satisfying the conditions

$$\|\langle \cdot \xi, \xi\rangle - \tau'\| \leq \delta', \quad \|\langle \xi \cdot, \xi\rangle - \tau'\| \leq \delta', \quad \|y\xi - \xi y\| \leq \delta' \quad \text{for all } y \in F',$$

  then there exists a $\xi_0 \in \mathcal{H}$ such that $\|\xi_0 - \xi\| \leq \varepsilon$ and $x\xi_0 = \xi_0 x$ for all $x \in M$.
- For all $\varepsilon > 0$, there exists a finite $F \subset M$ and a $\delta > 0$ such that if $\phi : M \to M$ is a normal, completely positive map with

$$\tau' \circ \phi \leq \tau', \quad \phi(1) \leq 1, \quad \|\phi(y) - y\|_2 \leq \delta \quad \text{for all } y \in F,$$

  then $\|\phi(x) - x\|_2 \leq \varepsilon$ for all $x \in M$ with $\|x\| \leq 1$.

Furthermore, Popa [2006] showed that the above definition is independent of the trace τ', and in the case when N is a factor, this agrees with the original definition in [Connes and Jones 1985].

In this section we will obtain a characterization of property (T) in terms of certain boundedness conditions on c.c.n. maps. Since we are dealing with unbounded maps, the domain of a map will be of crucial importance. We will thus want to consider c.c.n. maps whose domain contains a "critical set", which by [Popa 1986, Remark 4.1.6] motivates the following.

Definition 3.1. Suppose that N is a II_1 factor, and let $N_0 \subset N$ be a weakly dense $*$-subalgebra of N with $1 \in N_0$. Then N_0 contains a non-Γ set if there is a finite $F \subset N_0$ and a $K > 0$ such that for all $\xi \in L^2(N, \tau)$, if $\langle \xi, 1\rangle = 0$ then we have $\|\xi\|_2^2 \leq K \sum_{x \in F} \|x\xi - \xi x\|_2^2$.

Note that by [Connes 1976, Lemma 2.4] one can check that N_0 has a non-Γ set if and only if there exists a finitely generated subgroup $\mathcal{G} \subset \operatorname{Int} C^*(N_0)$ such that there is no nonnormal $\mathcal{G}$-invariant state on N. Also it follows from the definition that $N \subset N$ contains a non-Γ set if and only if N does not have property Γ of [Murray and von Neumann 1943]. Also, if Λ is a countable ICC group, then by [Effros 1975] Λ is not inner amenable if and only if $\mathbb{C}\Lambda$ contains a non-Γ set.

We now come to the main result, which is to give several equivalent characterizations of property (T); in particular we obtain a 1-cohomology characterization of property (T), which is the analogue of the Delorme–Guichardet theorem from group theory.

Theorem 3.2. *Suppose that N is a separable finite factor with normal faithful trace τ. Let $N_0 \subset N$ be a weakly dense $*$-subalgebra such that $1 \in N_0$ and N_0 is countably generated as a vector space. Consider the following conditions*:

(a) *N has property* (*T*).

(b) *There exists a finite $F \subset N_0$ and a $K > 0$ such that if $\mathcal{H}$ is a Hilbert N-N bimodule with $\xi \in \mathcal{H}$ and if $\delta_\xi = \max_{x \in F}\{\|x\xi - \xi x\|\}$, then there exists a $\xi_0 \in \mathcal{H}$ such that $x\xi_0 = \xi_0 x$ for all $x \in N$ and $\|\xi_0 - \xi\| \leq \delta_\xi K$.*

(c) *Every densely defined closable derivation on N_0 is inner.*

(d) *Every closable*, *conservative*, *symmetric c.c.n. map on N_0 is bounded in $\|\cdot\|_1$ on $(N_0)_1$.*

(e) *There exists a finite $F' \subset N_0$ and a $K' > 0$ such that if $\phi : N \to N$ is a c.p. map with $\phi(1) \leq 1$, $\tau \circ \phi \leq \tau$, and $\phi = \phi^*$, and if $\delta'_\phi = \max_{x \in F'}\{\|x - \phi(x)\|_2\}$, then $\tau((y - \phi(y))y^*) \leq K'\delta'_\phi$ for all $y \in (N)_1$.*

Then (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a). *If moreover N_0 contains a non-Γ set*, *then also* (a) $\Rightarrow$ (b).

Proof. (b) implies (c). Let $\delta : N_0 \to \mathcal{H}$ be a closable derivation, and note that by Section 1.2, we may assume that δ is real. Let $\phi_t : N \to N$ be the semigroup of normal symmetric c.p. maps associated with δ. Then for all $y \in N_0$, we have $\|\delta(y)\|^2 = \lim_{t\to\infty} \tau(\frac{1}{t}(y - \phi_t(y))y^*)$.

Let $(\mathcal{H}_t, \xi_t)$ be the pointed correspondence obtained from ϕ_t. Then since ϕ_t is unital and symmetric, $\|y\xi_t - \xi_t y\|_2^2 = 2\tau((y-\phi_t(y))y^*)$ for all $y \in N$. Let $F \subset N_0$ and $K > 0$ be as in (b), and let $C = \sup_{0<t\le 1, x\in F} \tau((x - \phi_t(x))x^*)/t$. Then

$$\begin{aligned}\|\delta(y)\|^2 &= \lim_{t\to 0} \tau(\tfrac{1}{t}(y - \phi_t(y))y^*)\\ &= \lim_{t\to 0} \|y\xi_t - \xi_t y\|_2^2/2t\\ &\le 2 \sup_{\substack{0<t\le 1\\ x\in F}} \|x\xi_t - \xi_t x\|_2^2 K^2/t = 4 \sup_{\substack{0<t\le 1\\ x\in F}} \tau((x - \phi_t(x))x^*)K^2/t = 4CK^2\end{aligned}$$

for all $y \in (N_0)_1$. Thus δ is inner by Theorem 2.2.

(c) implies (d). This follows from Theorem 2.2.

(d) implies (e). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $(N_0)_1$ such that $N_0 = \mathrm{sp}\{x_n\}_{n\in\mathbb{N}}$. If (e) does not hold, then for each $k \in \mathbb{N}$ there exists a c.p. map $\phi_k : N \to N$ such that $\phi_k(1) \le 1$, $\tau \circ \phi_k \le \tau$, and $\phi_k = \phi_k^*$, and there exists a $y_k \in (N_0)_1$ such that $\tau((y_k - \phi_k(y_k))y_k^*) > 4^k \delta'_k$, where $\delta'_k = \max_{j\le k}\{\|x_j - \phi_k(x_j)\|_2\}$.

Let $\Psi_k = (\mathrm{id} - \phi_k)/\delta'_k$, and let $\Psi = \sum_{k=1}^{\infty} 2^{-k}\Psi_k$. Then since $N_0 = \mathrm{sp}\{x_n\}_{n\in\mathbb{N}}$, $\Psi : N_0 \to L^2(N, \tau)$ is a well-defined symmetric c.c.n. map with $\Psi(1) \ge 0$. Also since $\phi_k(1) \le 1$, $\tau \circ \phi_k \le \tau$, and $\phi_k = \phi_k^*$, if we let $(\mathcal{H}_k, \xi_k)$ be the pointed Hilbert N-N bimodule corresponding to ϕ_k then, for all $x \in N$,

$$\begin{aligned}2\tau((x - \phi_k(x))x^*) &\ge \tau \circ \phi_k(x^*x) + \tau(x^*x\phi_k(1)) - 2\tau(\phi(x)x^*)\\ &= \|x\xi_k - \xi_k x\|^2 \ge 0.\end{aligned}$$

Thus

$$\|\Psi(y_k)\|_1 \ge \tau(\Psi(y_k)y_k^*) \ge 2^{-k}\tau((y_k - \phi_k(y_k)y_k^*)/\delta'_k > 2^k \quad \text{for all } k \in \mathbb{N}.$$

Hence if we let $\Psi'(x) = \Psi(x) - x\Psi(1)/2 - \Psi(1)x/2$, then Ψ' is a closable, conservative, symmetric c.c.n. map that is unbounded in $\|\cdot\|_1$ on $(N_0)_1$.

(e) implies (a). Let F' and K' be as in (e), and let $\varepsilon > 0$. Suppose $\phi : N \to N$ is a c.p. map such that $\phi(1) \le 1$, $\tau \circ \phi \le \tau$, $\phi = \phi^*$, and $\|x - \phi(x)\|_2 < \varepsilon^2/2K'$ for all $x \in F'$. Let $(\mathcal{H}_\phi, \xi_\phi)$ be the pointed Hilbert N-N bimodule associated with ϕ. Then since $\|\phi(1)\|_2 \le 1$ by [Popa 2006, Lemma 1.1.3], we have

$$\|y - \phi(y)\|_2^2 \le \|y\xi_\phi - \xi_\phi y\|^2 = 2\tau((y - \phi(y))y^*) \le 2K'\delta'_\phi < \varepsilon^2 \quad \text{for all } y \in (N)_1.$$

Hence N has property (T) by [Peterson and Popa 2005, Lemma 3].

(a) implies (b). If N_0 contains a non-Γ set, then [Popa 1986, Remark 4.1.6] shows that [Connes and Jones 1985, Proposition 1] applies to give the desired result. □

The proof in Theorem 3.2 that (d) implies (e) can be suitably adapted to the case of inclusions of σ-compact and locally compact groups, thus showing that an inclusion of groups has relative property (T) if and only if "δ depends linearly on ε", answering a question of Jolissaint — see [2005, Theorem 1.2].

Let $B \subset N$ with $1 \in B$ be a $*$-subalgebra, and suppose $X = X \in N$. Recall from [Voiculescu 1998] that a dual operator to $(X; B)$ in $L^2(N, \tau)$ is an operator $Y \in \mathcal{B}(L^2(N, \tau))$ such that

$$[B, Y] = 0 \quad \text{and} \quad [X, Y] = P_1,$$

where P_1 is the orthogonal projection onto $\mathbb{C}1$.

Corollary 3.3. *Suppose N is a separable finite factor with property* (T), *let $B \subset M$ with $1 \in B$ be a $*$-subalgebra, and let $X = X^* \in N$ such that $B[X]$ generates N as a von Neumann algebra. Suppose that B is diffuse and $B[X]$ contains a non-Γ set. Then the conjugate variable $J(X:B)$ does not exist in $L^2(N, \tau)$, that is, $\Phi^*(X:B) = \infty$. Also $(X; B)$ does not have a dual operator in $L^2(N, \tau)$.*

Proof. If the conjugate variables $J(X:B)$ did exist in $L^2(N, \tau)$, then we would have, as in Section 1.6.1, a closable derivation on $B[X]$ that is not inner. Therefore by Theorem 3.2 this cannot happen.

The fact that $(X; B)$ does not have a dual operator in $L^2(N, \tau)$ then follows directly from [Voiculescu 1998]. □

4. Property (T) and amalgamated free products

We include here an application of the above ideas, showing that a large class of amalgamated free products do not have property (T). We first prove that if N has property (T), then even though a c.c.n. maps may be unbounded on some domains, it must still satisfy a certain condition on its rate of growth.

Theorem 4.1. *Suppose N is a finite von Neumann algebra with normal faithful tracial state τ. If N has property* (T) *and $\Psi : D_\Psi \to L^2(N, \tau) \subset L^1(N, \tau)$ is a conservative, symmetric c.c.n. map, then any sequence $\{x_n\}_n$ in $(D_\Psi)_1$ such that $\|\Psi(x_n)\|_2 \to \infty$ satisfies $\|\Psi(x_n)\|_2 / \|\Psi(x_n)\| \to 0$.*

Proof. Let $\{\Phi_t\}_t$ be the semigroup of unital normal symmetric c.p. maps associated with Ψ as in Section 1.7, and for each $\beta > 0$, let $\varepsilon_\beta = \sup_{t \leq \beta, x \in N_1} \|\Phi_t(x) - x\|_2$. Since N has property (T), we know $\varepsilon_\beta \to 0$ as $\beta \to 0$.

For all $\beta > 0$ and $x \in (D_\Psi)_1$, we have

$$\begin{aligned}\int_0^\beta \Phi_t \circ \Psi(x)dt &= \lim_{s\to 0}\int_0^\beta \Phi_t((\Phi_s(x)-x)/s)dt\\ &= \lim_{s\to 0}\frac{1}{s}\Big(\int_0^\beta \Phi_{t+s}(x)dt - \int_0^\beta \Phi_t(x)dt\Big)\\ &= \lim_{s\to 0}\frac{1}{s}\Big(\int_s^{\beta+s} \Phi_t(x)dt - \int_0^\beta \Phi_t(x)dt\Big)\\ &= \lim_{s\to 0}\frac{1}{s}\Big(\int_\beta^{\beta+s} \Phi_t(x)dt - \int_0^s \Phi_t(x)dt\Big) = \Phi_\beta(x)-x.\end{aligned}$$

Hence for all $x \in (D_\Psi)_1$,

$$\begin{aligned}\|\Psi(x)\|_2 &\le \Big\|\frac{1}{\beta}\int_0^\beta \Phi_t\circ\Psi(x)dt\Big\|_2 + \Big\|\frac{1}{\beta}\int_0^\beta (\Phi_t\circ\Psi(x)-\Psi(x))dt\Big\|_2\\ &\le \frac{\varepsilon_\beta}{\beta} + \frac{1}{\beta}\int_0^\beta \big\|(\Phi_t\circ\Psi(x)-\Psi(x))\big\|_2 dt\\ &\le \frac{\varepsilon_\beta}{\beta} + \|\Psi(x)\|\frac{1}{\beta}\int_0^\beta \varepsilon_t dt \le \frac{\varepsilon_\beta}{\beta} + \|\Psi(x)\|\varepsilon_\beta.\end{aligned}$$

Thus $\varepsilon_\beta \ge \|\Psi(x)\|_2\beta/(1+\|\Psi(x)\|\beta)$, and since $\varepsilon_\beta \to 0$ the result follows. □

Corollary 4.2. *Let N_1 and N_2 be finite von Neumann algebras with normal faithful tracial states τ_1 and τ_2, respectively, and suppose B is a common von Neumann subalgebra such that $\tau_1|_B = \tau_2|_B$. Suppose also that there are unitaries $u_i \in \mathcal{U}(N_i)$ such that $E_B(u_i) = 0$ for $i = 1, 2$. Then $M = N_1 *_B N_2$ does not have property* (T).

Proof. Let $\tau = \tau_1 *_B \tau_2$ be the trace for M, and let $\mathcal{H} = L^2(M,\tau)\otimes_B L^2(M,\tau)$. Define δ to be the unique derivation from the algebraic amalgamated free product to $\mathcal{H}$ that satisfies $\delta(a) = a\otimes_B 1 - 1\otimes_B a$ for all $a \in N_1$, and $\delta(b) = 0$ for all $b \in N_2$. By [Nica et al. 2002, Corollary 5.4], $\delta^*(1\otimes_B 1) = 0$. In particular and just as in the nonamalgamated case, δ is a closable derivation. Furthermore if u_1 and u_2 are the unitaries as above and $z \in N_0$, then $\langle \delta((u_1u_2)^n), \delta(z)\rangle$ is equal to

$$\begin{aligned}&\sum_{j=0}^{n-1}\langle (u_1u_2)^j u_1 \otimes_B u_2(u_1u_2)^{n-j-1} - (u_1u_2)^j \otimes_B (u_1u_2)^{n-j}, \delta(z)\rangle\\ &\qquad = \sum_{j=0}^{n-1}\langle 1\otimes_B 1, u_1^*(u_2^*u_1^*)^j\delta(z)(u_2^*u_1^*)^{n-j-1}u_2^* - (u_2^*u_1^*)^j\delta(z)(u_2^*u_1^*)^{n-j}\rangle,\end{aligned}$$

also, for each $0 \leq j < n$, we may use the Leibniz rule for the derivation to rewrite $(u_2^* u_1^*)^j \delta(z)(u_2^* u_1^*)^{n-j}$ as the sum

$$
\begin{aligned}
&\delta((u_2^*u_1^*)^j z(u_2^*u_1^*)^{n-j}) + \\
&-\sum_{k=0}^{j-1}(u_2^*u_1^*)^k u_2^*\delta(u_1^*)(u_2^*u_1^*)^{j-k-1} z(u_2^*u_1^*)^{n-j} + \\
&-\sum_{i=0}^{n-j-1}(u_2^*u_1^*)^j z(u_2^*u_1^*)^i u_2^*\delta(u_1^*)(u_2^*u_1^*)^{n-j-i-1}.
\end{aligned} \tag{3}
$$

However when we take the inner product with $1 \otimes_B 1$, the first term will be 0 as mentioned above, and by freeness (since u_1 and u_2 have expectation 0) the other terms will be 0, except, when $i = n - j - 1$ in the third term. In that case, we have

$$
\begin{aligned}
-\langle 1 \otimes_B 1, (u_2^*u_1^*)^j z(u_2^*u_1^*)^{n-j-1} u_2^*\delta(u_1^*)\rangle &= -\langle 1 \otimes_B 1, (u_2^*u_1^*)^j z(u_2^*u_1^*)^{n-j} \otimes_B 1\rangle \\
&= -\tau(E_B((u_1u_2)^{n-j} z^*(u_1u_2)^j)).
\end{aligned}
$$

Similarly,

$$
\langle 1 \otimes_B 1, u_1^*(u_2^*u_1^*)^j \delta(z)(u_2^*u_1^*)^{n-j-1}u_2^*\rangle = \tau(E_B(u_2(u_1u_2)^{n-j-1}z^*(u_1u_2)^j u_1)).
$$

Hence from the above equalities we get

$$
\begin{aligned}
\langle \delta((u_1u_2)^n), \delta(z)\rangle &= \sum_{j=0}^{n-1} \tau(u_2(u_1u_2)^{n-j-1}z^*(u_1u_2)^j u_1 + (u_1u_2)^{n-j}z^*(u_1u_2)^j) \\
&= 2n\tau((u_1u_2)^n z^*).
\end{aligned}
$$

In particular, the c.c.n. map Ψ associated with δ has $\Psi((u_1u_2)^n) = 2n(u_1u_2)^n$, and so $\|\Psi((u_1u_2)^n)\|_2 \to \infty$ but $\|\Psi((u_1u_2)^n)\|_2/\|\Psi((u_1u_2)^n)\| \not\to 0$; hence M does not have property (T) by Theorem 4.1. $\square$

Note that we only used the fact that u_1 and u_2 are unitaries to insure that $2n\|(u_1u_2)^n\|_2 \to \infty$. Also note that the conditions of Corollary 4.2 are satisfied when M is a free product (with amalgamation over $\mathbb{C}$) as well as when M is a group von Neumann algebra coming from an amalgamated free product of groups. We also mention that from the calculation above we are able to compute explicitly the semigroup of c.p. maps that δ generates — it is the semigroup given by $\phi_t = (e^{-2t}\mathrm{id} + (1 - e^{-2t})E_B) *_B \mathrm{id}$.

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