Pacific Journal of Mathematics

KASHIWARA AND ZELEVINSKY INVOLUTIONS IN AFFINE TYPE *A*

NICOLAS JACON AND CÉDRIC LECOUVEY

Volume 243 No. 2 December 2009

PACIFIC JOURNAL OF MATHEMATICS
Vol. 243, No. 2, 2009

KASHIWARA AND ZELEVINSKY INVOLUTIONS IN AFFINE TYPE A

NICOLAS JACON AND CÉDRIC LECOUVEY

We describe how the Kashiwara involution $*$ on crystals of affine type A is encoded by the combinatorics of aperiodic multisegments. This affords an elementary proof that $*$ coincides with the Zelevinsky involution τ on the set of simple modules for the affine Hecke algebras. We then give efficient procedures for computing $*$ and τ. Remarkably, these procedures do not use the underlying crystal structure. They also allow one to explicitly match to each other the Ginzburg and Ariki parametrizations of simple modules associated to affine and cyclotomic Hecke algebras, respectively.

1. Introduction

The Kashiwara involution $*$ in affine type A is a fundamental anti-isomorphism of the quantum group $\mathcal{U}_v$ associated to the affine root system $A_{e-1}^{(1)}$. It induces a subtle involution on $B_e(\infty)$, the Kashiwara crystal corresponding to the negative part $\mathcal{U}_v^-$ of $\mathcal{U}_v$. The Zelevinsky involution yields an involution τ of the affine Hecke algebra of type A. When q is specialized to an e-th root of 1, τ also induces an involution on $B_e(\infty)$. In this paper, we show by using the combinatorics of aperiodic multisegments that the Kashiwara and Zelevinsky involutions coincide on $B(\infty)$. We also provide efficient procedures for computing these involutions. In addition, our results yield an explicit matching of the Ginzburg and Ariki parametrizations of the simple modules associated to affine and cyclotomic Hecke algebras respectively.

All our computations can be made independently of the crystal structure on $B_e(\infty)$. Moreover, they do not require the determination of i-induction or i-restriction operations on simple modules.

We now describe the context and the results of the paper more precisely. The Zelevinsky involution [1980] was introduced in connection with the representation theory of the linear group $\mathrm{GL}(n, \mathbb{F}_\mathfrak{p})$ over the $\mathfrak{p}$-adic field $\mathbb{F}_\mathfrak{p}$. Results of Mœglin and Waldspurger [1986] then allow one to link it with a natural involution τ of the affine type A Hecke algebra $\mathcal{H}_n^a(q)$ over the field $\mathbb{F}$ with generic parameter q. For $e \geq 2$ an integer and q specialized at ξ, a primitive e-root of 1, it was conjectured in

MSC2000: 20C08, 20C20, 81R50.
Keywords: affine Hecke algebra, Zelevinsky involution, crystals, Kashiwara involution.

[Vignéras 1997] that this involution should be related to the modular representation theory of $\mathrm{GL}(n, \mathbb{F}_{\mathfrak{p}})$. In the sequel we will refer to τ as the Zelevinsky involution of $\mathcal{H}_n^a(\xi)$ (see Section 3 for a complete definition).

The involution τ induces an involution on the set of simple $\mathcal{H}_n^a(\xi)$-modules. There exist essentially two different parametrizations of these modules in the literature. In the geometric construction of Chriss and Ginzburg [1997], under the assumption $\mathbb{F} = \mathbb{C}$, simple $\mathcal{H}_n^a(\xi)$-modules are labeled by aperiodic multisegments. These simple modules can also be regarded as simple modules associated to Ariki–Koike algebras $\mathcal{H}_n^{\mathbf{v}}(\xi)$. The Specht module theory developed by Dipper, James and Mathas then provides a labeling of the simple $\mathcal{H}_n^a(\xi)$-modules by Kleshchev multipartitions. Both constructions allow one to endow the set of simple $\mathcal{H}_n^a(\xi)$-modules with the structure of a crystal isomorphic to $B_e(\infty)$. The Kashiwara crystal operators then yield the modular branching rules for the Ariki–Koike algebras and affine Hecke algebras of type A [Ariki 2006; Ariki et al. 2008].

Grojnowski [1999] uses i-induction and i-restrictions operators to define an abstract crystal structure on the set of simple $\mathcal{H}_n^a(\xi)$-modules. He then proves that this crystal is in fact isomorphic to $B_e(\infty)$. This approach is valid over an arbitrary field $\mathbb{F}$ and does not require the Specht module theory of Dipper, James and Mathas. This notably allows the extension Grojnowski's methods to the representation theory of the cyclotomic Hecke–Clifford superalgebras [Brundan and Kleshchev 2001]. Nevertheless, this approach does not match up the abstract crystal obtained with the labelings of the simple modules by aperiodic multisegments or Kleshchev multipartitions. Since the i-induction operation on simple modules is difficult to obtain in general, it is also not really suited to explicit computations.

The identification of $\mathcal{U}_v^-$ with the composition subalgebra of the Hall algebra associated to the cyclic quiver of type $A_e^{(1)}$ yields two different structures of crystal on the set of aperiodic multisegments. They both come from two different parametrizations of the canonical basis of $\mathcal{U}_v^-$ which correspond under the anti-isomorphism ρ on $\mathcal{U}_v^-$ switching the generators f_i and f_{-i}. In particular ρ provides an involution on the crystal $B_e(\infty)$ which can be easily computed. The use of the composition algebra also allows an explicit description of the structure of Kashiwara crystal on the set of aperiodic multisegments. This was obtained in [Leclerc et al. 1999] by Leclerc, Thibon and Vasserot. In addition, these authors prove that the involution τ on $B_e(\infty)$ satisfies the identity $\tau = \sharp \circ \rho$ where $\sharp$ is the twofold symmetry on $B_e(\infty)$ which switches the sign of each arrow.

In this paper, we first establish that the two crystal structures on aperiodic multisegments obtained by identifying $\mathcal{U}_v^-$ with the composition algebra correspond up to conjugation by the Kashiwara involution $*$. This implies that $* = \tau$ on $B_e(\infty)$. An equivalent identity can also be established by using [Grojnowski 1999] but, as mentioned above, this approach requires subtle considerations on the representation theory of $\mathcal{H}_n^a(\xi)$ and does not allow one to compute $* = \tau$ efficiently. Our proof, in contrast, uses only elementary properties of crystal graphs and yields efficient procedures for computing the involution $* = \tau$. This notably allows us to generalize an algorithm of Mœglin and Waldspurger that gives the Zelevinsky involution when $e = \infty$.

As a consequence, extending [Vazirani 2002], we completely solve the following natural problem. Given a simple $\mathcal{H}_n^a(\xi)$-module L_ψ (with ψ an aperiodic multisegment ψ), we find all the Ariki–Koike algebras $\mathcal{H}_n^{\boldsymbol{v}}(q)$ and the simple $\mathcal{H}_n^{\boldsymbol{v}}(q)$-modules $D^{\boldsymbol{\lambda}}$ (with $\boldsymbol{\lambda}$ a Kleshchev multipartition) such that $D^{\boldsymbol{\lambda}} \simeq L_\psi$ as $\mathcal{H}_n^a(\xi)$-modules. The procedure that yields the Kashiwara involution also allows the computation of the commutor of $A_e^{(1)}$-crystals introduced in [Kamnitzer and Tingley 2009].

The paper is organized as follows. In Section 2, we review the identification of $\mathcal{U}_v^-$ with the composition algebra and the two structures of crystal it gives on the set of aperiodic multisegments. We also recall basic facts on the Kashiwara involution. Section 3 is devoted to the definition of the Zelevinsky involution on the set of simple $\mathcal{H}_n^a(\xi)$-modules and to results from [Leclerc et al. 1999]. In Section 4, we prove the identity $* = \tau$. The problem of determining the algebras $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$ and the simple $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$-modules isomorphic to a given simple $\mathcal{H}_n^a(\xi)$-module is studied in Section 5. In Sections 6 and 7, we give a simple combinatorial procedure for computing the involutions τ, ρ and $\sharp$ on $B_e(\infty)$. We prove in fact that all these computations can essentially be obtained from the Mullineux involution on e-regular partitions and the crystal isomorphisms described in [Jacon and Lecouvey 2009a]. We also investigate several consequences of our results.

2. Quantum groups and crystals in affine type A

The quantum group $\mathcal{U}_v$. Let v be an indeterminate and $e \geq 2$ an integer. Write $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ for the quantum group of type $A_{e-1}^{(1)}$. This is an associative $\mathbb{Q}(v)$-algebra with generators $e_i, f_i, t_i, t_i^{-1}, i \in \mathbb{Z}/e\mathbb{Z}$ and ∂; a description of the relations satisfied by these generators can be found in [Uglov 2000, §2.1]. Write $\{\Lambda_0, \dots, \Lambda_{e-1}, \delta\}$ and $\{\alpha_0, \dots, \alpha_{e-1}\}$ respectively for the set of fundamental weights and the set of simple roots associated to $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$. Let P be the weight lattice of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$. We denote by $\mathcal{U}_v = \mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$ the subalgebra generated by $e_i, f_i, t_i, t_i^{-1}, i \in \mathbb{Z}/e\mathbb{Z}$. Then $\overline{P} = P/\mathbb{Z}\delta$ is the set of classical weights of $\mathcal{U}_v$. For any $i \in \mathbb{Z}/e\mathbb{Z}$, we also denote by Λ_i and α_i the restriction of Λ_i and $\alpha_i \in P$ to $\overline{P}^\wedge$. Let $\mathcal{U}_v^-$ be the subalgebra of $\mathcal{U}_v$ generated by the f_i's with $i \in \mathbb{Z}/e\mathbb{Z}$.

Aperiodic multisegments. We now turn to the set of aperiodic multisegments and define its crystal structures.

Definition 2.1. Let $l \in \mathbb{Z}_{>0}$ and $i \in \mathbb{Z}/e\mathbb{Z}$. The *segment of length l and head i* is the sequence of consecutive residues $[i, i+1, \dots, i+l-1]$. We denote it by $[i; l)$. Similarly, the *segment of length l and tail i* is the sequence of consecutive residues $[i-l+1, \dots, i-1, i]$. We denote it by $(l; i]$.

Definition 2.2. A collection of segments is called a *multisegment*. If the collection is the empty set, we call it the empty multisegment and denote it by $\varnothing$.

It is convenient to write a multisegment ψ in the form

$$\psi = \sum_{\substack{i \in \mathbb{Z}/e\mathbb{Z} \\ l \in \mathbb{N}_{>0}}} m_{[i;l)}[i; l).$$

Definition 2.3. A multisegment ψ is *aperiodic* if, for every $l \in \mathbb{Z}_{>0}$, there exists some $i \in \mathbb{Z}/e\mathbb{Z}$ such that $(l; i]$ does not appear in ψ. Equivalently, a multisegment ψ is aperiodic if, for each $l \in \mathbb{Z}_{>0}$, there exists some $i \in \mathbb{Z}/e\mathbb{Z}$ such that $[i; l)$ does not appear in ψ. We denote by Ψ_e the set of aperiodic multisegments.

Let $B_e(\infty)$ be the (abstract) crystal basis of $\mathcal{U}_v^-$. By results of Ringel and Lusztig, the algebra $\mathcal{U}_v^-$ is isomorphic to the composition algebra of the Hall algebra associated to the cyclic quiver Γ_e of length e. This yields in particular a natural parametrization of the vertices of $B_e(\infty)$ by Ψ_e. We can thus regard the vertices of $B_e(\infty)$ as aperiodic multisegments. The corresponding crystal structure was described in [Leclerc et al. 1999, Theorem 4.1]. In fact we shall need in the sequel two different structures of crystal on Ψ_e. They are linked by the involution ρ which negates all the segments of a given multisegment, that is, such that

$$\psi^\rho = \sum_{\substack{i \in \mathbb{Z}/e\mathbb{Z} \\ l \in \mathbb{N}_{>0}}} m_{[i;l)}(l; -i] \tag{2-1}$$

for any multisegment $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z},\, l \in \mathbb{N}_{>0}} m_{[i;l)}[i; l)$. The involution ρ has a natural algebraic interpretation since it also yields a linear automorphism of the Hall algebra associated to Γ_e. Since we do not use Hall algebras in this paper, we only recall below the two crystal structures relevant for our purpose.

Let ψ be a multisegment and let $\psi_{\geq l}$ be the multisegment obtained from ψ by deleting the segments of length less than l, for $l \in \mathbb{Z}_{>0}$. Denote by $m_{[i;l)}$ the multiplicity of $[i; l)$ in ψ. For any $i \in \mathbb{Z}/e\mathbb{Z}$, set

$$\widehat{S}_{l,i} = \sum_{k \geq l} (m_{[i+1;k)} - m_{[i;k)}).$$

Let $\widehat{l_0}$ be the minimal value of l that attains $\min_{l>0} \widehat{S}_{l,i}$.

Theorem 2.4. *Let ψ be a multisegment, $i \in \mathbb{Z}/e\mathbb{Z}$ and let $\widehat{l_0}$ be as above. Then we have*

$$\widehat{f_i}\psi = \psi_{\widehat{l_0},i},$$

where the multisegment $\psi_{\widehat{l_0},i}$ *is defined by*

$$\psi_{\widehat{l_0},i} = \begin{cases} \psi + [i;1) & \text{if } \widehat{l_0} = 1, \\ \psi + [i;\widehat{l_0}) - [i+1;\widehat{l_0}-1) & \text{if } \widehat{l_0} > 1. \end{cases}$$

The crystal structure on Ψ_e obtained from the action of the operators $\widehat{f_i}$, for $i \in \mathbb{Z}/e\mathbb{Z}$, does not coincide with that initially described by Leclerc, Thibon and Vasserot. The LTV crystal structure stated in [Leclerc et al. 1999] is obtained by using the crystal operators

$$\widetilde{f_i} = \rho \circ \widehat{f_{-i}} \circ \rho, \qquad i \in \mathbb{Z}/e\mathbb{Z} \tag{2-2}$$

rather than the operators $\widehat{f_i}$. More precisely, set $S_{l,i} = \sum_{k \geq l} (m_{(k;i-1]} - m_{(k;i]})$. Let l_0 be the minimal l that attains $\min_{l>0} S_{l,i}$. Then, the crystal structure corresponding to the $\widetilde{f_i}$'s is given as follows.

Theorem 2.5. *Let* ψ *be a multisegment and let* $i \in \mathbb{Z}/e\mathbb{Z}$ *and* l_0 *be as above. Then we have*

$$\widetilde{f_i}\,\psi = \psi_{l_0,i},$$

where the multisegment $\psi_{l_0,i}$ *is defined by*

$$\psi_{l_0,i} = \begin{cases} \psi + (1;i] & \text{if } l_0 = 1, \\ \psi + (l_0;i] - (l_0-1;i-1] & \text{if } l_0 > 1. \end{cases}$$

Let ψ be a multisegment. Then to compute $\widetilde{e_i}\,\psi$, we proceed as follows. If $\min_{l>0} S_{l,i} = 0$, then $\widetilde{e_i}\,\psi = 0$. Otherwise, let l_0 be the maximal l that attains $\min_{l>0} S_{l,i}$. Then, $\widetilde{e_i}\,\psi$ is obtained from ψ by replacing $(l_0;i]$ with $(l_0-1;i-1]$.

In the sequel, we identify $B_e(\infty)$ with the crystal structure obtained on Ψ_e by considering the operators $\widetilde{f_i}$, $i \in \mathbb{Z}/e\mathbb{Z}$ (see also Remark 2.7). Then ρ induces an involution on $B_e(\infty)$ and the crystal operators $\widetilde{f_i}$ and $\widehat{f_i}$ are related by (2-2). We denote by $\mathrm{wt}(\psi)$ the weight of the aperiodic multisegment ψ considered as a vertex of the crystal $B_e(\infty)$. Set

$$\mathrm{wt}(\psi) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \mathrm{wt}_i(\psi)\, \Lambda_i. \tag{2-3}$$

For any $i \in \mathbb{Z}/e\mathbb{Z}$, define

$$\varepsilon_i(\psi) = \max\{k \in \mathbb{N} \mid \widetilde{e}_i^{\,k}(u) \neq 0\} \quad \text{and} \quad \varphi_i(\psi) = \mathrm{wt}_i(\psi) + \varepsilon_i(\psi).$$

The Kashiwara involution. The Kashiwara involution $*$ is the $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-antiautomorphism such that $q^* = q$ and defined on the generators by

$$e_i^* = e_i, \quad f_i^* = f_i, \quad t_i^* = t_i^{-1}. \tag{2-4}$$

Since $*$ stabilizes $\mathcal{U}_v^-$, it induces an involution (also denoted $*$) on $B_e(\infty)$ the crystal graph of $\mathcal{U}_v^-$. By setting for any vertex $b \in B_e(\infty)$ and any $i \in \mathbb{Z}/e\mathbb{Z}$

$$\widetilde{e}_i^*(b) = \widetilde{e}_i(b^*)^*, \quad \widetilde{f}_i^*(b) = \widetilde{f}_i(b^*)^*, \quad \varepsilon_i^*(b) = \varepsilon_i(b^*) \quad \text{and } \varphi_i^*(b) = b^* \tag{2-5}$$

we obtain another crystal structure on $B_e(\infty)$ [Kashiwara 1995].

Let $i \in \mathbb{Z}/e\mathbb{Z}$ and write B_i for the crystal with set of vertices $\{b_i(k) \mid k \in \mathbb{Z}\}$ and such that

$$\mathrm{wt}(b_i(k)) = k\alpha_i, \quad \varepsilon_j(b_i(k)) = \begin{cases} -k & \text{if } i = j, \\ -\infty & \text{if } i \neq j, \end{cases} \quad \varphi_j(b_i(k)) = \begin{cases} k & \text{if } i = j, \\ -\infty & \text{if } i \neq j, \end{cases}$$

$$\widetilde{e}_j b_i(k) = \begin{cases} b_i(k+1) & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad \widetilde{f}_j b_i(k) = \begin{cases} b_i(k-1) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Set $b_i = b_i(0)$.

Recall that on the tensor product $B \otimes B' = \{b \otimes b' \mid b \in B, b' \in B'\}$ of the crystals B and B', the action of $\widetilde{e}_i$ and $\widetilde{f}_i$ is given by

$$\widetilde{f}_i(u \otimes v) = \begin{cases} \widetilde{f}_i(u) \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v), \\ u \otimes \widetilde{f}_i(v) & \text{if } \varphi_i(u) \leq \varepsilon_i(v), \end{cases} \tag{2-6}$$

$$\widetilde{e}_i(u \otimes v) = \begin{cases} u \otimes \widetilde{e}_i(v) & \text{if } \varphi_i(u) < \varepsilon_i(v), \\ \widetilde{e}_i(u) \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v). \end{cases} \tag{2-7}$$

The embedding of crystals $\theta_i : B_e(\infty) \hookrightarrow B_e(\infty) \otimes B_i$ which sends the highest-weight vertex $b_\varnothing$ of $B_e(\infty)$ on $b_\varnothing \otimes B_i$ allows one, at least theoretically, to compute the action of the operators $\widetilde{e}_i^*$ and $\widetilde{f}_i^*$.

Proposition 2.6 [Kashiwara 1995, Proposition 8.1]. *If $b \in B_e(\infty)$ and $\varepsilon_i^*(b) = m$, then*

(1) $\theta_i(b) = (\widetilde{e}_i^*)^m b \otimes \widetilde{f}_i^m b_i$,

(2) $\theta_i(\widetilde{f}_i^* b) = (\widetilde{e}_i^*)^m b \otimes \widetilde{f}_i^{m+1} b_i$ *and*

(3) $\theta_i(\widetilde{e}_i^* b) = (\widetilde{e}_i^*)^m b \otimes \widetilde{f}_i^{m-1} b_i$ *if* $m > 0$ *and* $\theta_i(\widetilde{e}_i^* b) = 0$ *if* $m = 0$.

Remark 2.7. (1) As already seen on pages 290–291, Ψ_e is equipped with two crystal structures. One is obtained from the action of the crystal operators $\widehat{f}_i$ and the other is related to the operators $\widetilde{f}_i$ and yields the Kashiwara crystal graph structure $B_e(\infty)$ on Ψ_e. We shall see in Section 4 that the actions of $\widehat{f}_i$ and $\widetilde{f}_i^*$ coincide, for all $i \in \mathbb{Z}/e\mathbb{Z}$.

(2) Proposition 2.6 does not provide an efficient procedure for computing the involution $*$. Indeed, to obtain $\theta_i(b)$, we have first to determine a path from b to the highest-weight vertex of $B_e(\infty)$. Moreover, computing a section of the embedding θ_i is difficult in general.

Crystals of $\mathcal{U}_{\boldsymbol{v}}$-modules of highest weight. Let $l \in \mathbb{N}$. A tuple $\boldsymbol{v} = (v_0, \dots, v_{l-1}) \in \mathbb{Z}^l$ is called a *multicharge* and l is by definition its level. One can associate to $\boldsymbol{v}$ the abstract $\mathcal{U}_v$-irreducible module $V_e(\Lambda_{\boldsymbol{v}})$ with highest weight $\Lambda_{\boldsymbol{v}} = \Lambda_{v_0 \pmod e} + \cdots + \Lambda_{v_{l-1} \pmod e}$. There exist distinct realizations of $V_e(\Lambda_{\boldsymbol{v}})$ as an irreducible component of a Fock space $\mathfrak{F}_e^{\boldsymbol{v}}$ whose structure depends on $\boldsymbol{v}$. As a $\mathbb{C}(v)$-vector space, the Fock space $\mathfrak{F}_e^{\boldsymbol{v}}$ of level l admits the set of all l-partitions as a natural basis. Namely the underlying vector space is

$$\mathfrak{F}_e = \bigoplus_{n \geq 0} \bigoplus_{\boldsymbol{\lambda} \in \Pi_{l,n}} \mathbb{C}(v)\boldsymbol{\lambda},$$

where $\Pi_{l,n}$ is the set of l-partitions with rank n. Consider $\mathfrak{v} = (\mathfrak{v}_0, \dots, \mathfrak{v}_{l-1}) \in (\mathbb{Z}/e\mathbb{Z})^l$. We write $\boldsymbol{v} \in \mathfrak{v}$ when $v_c \in \mathfrak{v}_c$ for any $c = 0, \dots, l-1$. As $\mathcal{U}_v$-modules, the Fock spaces $\mathfrak{F}_e^{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathfrak{v}$ are all isomorphic but with distinct actions for $\mathcal{U}_v$. For each of these actions, the empty l-partition $\boldsymbol{\varnothing} = (\varnothing, \dots, \varnothing)$ is a highest-weight vector of highest weight $\Lambda_{\boldsymbol{v}}$. We denote by $V_e(\boldsymbol{v})$ the irreducible component with highest-weight vector $\boldsymbol{\varnothing}$ in $\mathfrak{F}_e^{\boldsymbol{v}}$. The modules $V_e(\boldsymbol{v})$ when $\boldsymbol{v}$ runs over $\mathfrak{v}$ are all isomorphic to the abstract module $V_e(\Lambda_{\boldsymbol{v}})$. However, the actions of the Chevalley operators on these modules do not coincide in general.

The module $\mathfrak{F}_e^{\boldsymbol{v}}$ admits a crystal graph $B_e^{\boldsymbol{v}}$ labeled by l-partitions. We now recall the crystal structures on $B_e^{\boldsymbol{v}}$ and $B_e(\boldsymbol{v})$, the crystal associated to $V_e(\boldsymbol{v})$. We omit the description of the $\mathcal{U}_v$-module structures on $\mathfrak{F}_e^{\boldsymbol{v}}$ and $V_e(\boldsymbol{v})$, which are not needed in our proofs; see [Jimbo et al. 1991] for a full account.

Let $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{l-1})$ be an l-partition, which we identify with its Young diagram. The nodes of $\boldsymbol{\lambda}$ are the triplets $\gamma = (a, b, c)$ where $c \in \{0, \dots, l-1\}$ and a, b are respectively the row and column indices of the node γ in λ^c. The content of γ is the integer $c(\gamma) = b - a + v_c$ and the residue $\operatorname{res}(\gamma)$ of γ is the element of $\mathbb{Z}/e\mathbb{Z}$ such that

$$\operatorname{res}(\gamma) \equiv c(\gamma) \pmod e. \tag{2-8}$$

We say that γ is an i-node of $\boldsymbol{\lambda}$ when $\operatorname{res}(\gamma) \equiv i \pmod e$. This node is removable when $\gamma = (a, b, c) \in \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \backslash \{\gamma\}$ is an l-partition. Similarly γ is addable when $\gamma = (a, b, c) \notin \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \cup \{\gamma\}$ is an l-partition.

The crystal structure on $B_e^{\boldsymbol{v}}$ (and in fact, the $\mathcal{U}_v$-module structure on $\mathfrak{F}_e^{\boldsymbol{v}}$ itself) is conditioned by the total order $\prec_{\boldsymbol{v}}$ on the set of addable and removable i-nodes of the multipartitions. Consider $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$, two i-nodes in $\boldsymbol{\lambda}$. We define the order $\prec_{\boldsymbol{v}}$ by setting

$$\gamma_1 \prec_{\boldsymbol{v}} \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2) \text{ or} \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 > c_2. \end{cases}$$

Starting from any l-partition $\boldsymbol{\lambda}$, consider its set of addable and removable i-nodes. Let w_i be the word obtained first by writing the addable and removable i-nodes of $\boldsymbol{\lambda}$ in increasing order with respect to $\prec_{\boldsymbol{v}}$, encoding each addable i-node by the letter A and each removable i-node by the letter R. Write $\widetilde{w}_i = A^r R^s$ for the word derived from w_i by deleting as many factors RA as possible. If $r > 0$, let γ be the rightmost addable i-node in $\widetilde{w}_i$. When $\widetilde{w}_i \neq \varnothing$, the node γ is called the good i-node.

Proposition 2.8. *The crystal graph $B_e^{\boldsymbol{v}}$ of $\mathfrak{F}_e^{\boldsymbol{v}}$ is the graph with*

(1) *as vertices, the l-partitions*;

(2) *as edges, $\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ is obtained by adding to $\boldsymbol{\lambda}$ its good i-node.*

For each $i \in \mathbb{Z}/e\mathbb{Z}$, we set $\varepsilon_i(\boldsymbol{\lambda}) = s$ and $\varphi_i(\boldsymbol{\lambda}) = r$.

Since $V_e(\boldsymbol{v})$ is the irreducible module with highest-weight vector $\boldsymbol{\varnothing}$ in $\mathfrak{F}_e^{\boldsymbol{v}}$, its crystal graph $B_e(\boldsymbol{v})$ can be realized as the connected component of the highest-weight vertex $\boldsymbol{\varnothing}$ in $B_e^{\mathbf{v}}$. The vertices of $B_e(\boldsymbol{v})$ are labeled by the so-called Uglov l-partitions associated to $\boldsymbol{v}$.

Set

(2-9) $$\mathcal{V}_l = \{\boldsymbol{v} = (v_0, \dots, v_{l-1}) \in \mathbb{Z}^l \mid v_0 \leq \cdots \leq v_{l-1} \text{ and } v_{l-1} - v_0 < e\}.$$

Definition 2.9. Assume that $\boldsymbol{v} \in \mathcal{V}_l$. The l-partition $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{l-1})$ is called a FLOTW l-partition associated to $\boldsymbol{v}$ if, for all $i = 1, 2, \dots$, we have

(2-10) $$\lambda_i^j \geq \lambda_{i+v_{j+1}-v_j}^{j+1} \text{ for all } j = 0, \dots, l-2 \text{ and } \lambda_i^{l-1} \geq \lambda_{i+e+v_0-v_{l-1}}^0,$$

and for all $k > 0$, among the residues appearing in $\boldsymbol{\lambda}$ at the right ends of rows of length k, at least one element of $\{0, 1, \dots, e-1\}$ does not occur.

The set of FLOTW l-partitions associated to $\boldsymbol{v}$ is denoted by $\Phi_e(\boldsymbol{v})$. (The acronym comes from the names of the authors of [Foda et al. 1999].)

The following result was obtained by Jimbo, Misra, Miwa and Okado, but the presentation we adopt here comes from [Foda et al. 1999].

Proposition 2.10 [Jimbo et al. 1991]. *When $\boldsymbol{v} \in \mathcal{V}_l$, the set of vertices of $B_e(\boldsymbol{v})$ coincides with the set of FLOTW l-partitions associated to $\boldsymbol{v}$.*

Consider $\boldsymbol{v} \in \mathcal{V}_l$ and $\boldsymbol{\lambda} \in \Phi_e(\boldsymbol{v})$. We associate to each nonzero part λ_i^c of $\boldsymbol{\lambda}$ the segment

(2-11) $$[(1-i+v_c) \pmod e, (2-i+v_c) \pmod e, \dots, (\lambda_i^c - i + v_c) \pmod e].$$

The multisegment $f_{\boldsymbol{v}}(\boldsymbol{\lambda})$ is then the formal sum of all the segments associated to the parts λ_i^c of $\boldsymbol{\lambda}$. Since $f_{\boldsymbol{v}}(\boldsymbol{\lambda})$ is aperiodic by Definition 2.9, we get a well defined map

(2-12) $$f_{\boldsymbol{v}} : B_e(\boldsymbol{v}) \to \Psi_e.$$

Example 2.11. Let $e = 4$ and consider the FLOTW bipartition (2.1, 1) associated to $\boldsymbol{v} = (0, 1)$. Then

$$f_{\boldsymbol{v}}(2.1, 1) = [0, 1] + [3] + [1].$$

Let $\boldsymbol{v} = (0, 1, 3)$ and consider the FLOTW 3-partition (2, 1, 1). We have

$$f_{\boldsymbol{v}}(2, 1, 1) = [0, 1] + [1] + [3].$$

Let $T_\Lambda = \{t_\Lambda\}$ be the crystal defined by $\mathrm{wt}(t_{\Lambda_{\boldsymbol{v}}}) = \Lambda$, $\epsilon_i(t_{\Lambda_{\boldsymbol{v}}}) = \varphi_i(t_{\Lambda_{\boldsymbol{v}}}) = -\infty$ and $\widetilde{e}_i t_{\Lambda_{\boldsymbol{v}}} = \widetilde{f}_i t_{\Lambda_{\boldsymbol{v}}} = 0$. We have a unique crystal embedding $B_e(\boldsymbol{v}) \hookrightarrow B_e(\infty) \otimes T_\Lambda$.

Theorem 2.12 [Ariki et al. 2008]. *For any $\boldsymbol{v} \in \mathcal{V}_l$, the map $f_{\boldsymbol{v}}$ coincides with the unique crystal embedding $B_e(\boldsymbol{v}) \hookrightarrow B_e(\infty) \otimes T_\Lambda$.*

According to Proposition 8.2 in [Kashiwara 1995], we have

$$f_{\boldsymbol{v}}(\Phi_e(\boldsymbol{v})) = \{\psi \in \Psi_e \mid \varepsilon_i(\psi^*) \le r_i \text{ for any } i \in \mathbb{Z}/e\mathbb{Z},$$

where r_i is the number of entries in $\boldsymbol{v}$ equal to i and ψ^* is the image of ψ under the Kashiwara involution of the crystal $B_e(\infty)$.

Given any $\psi \in \Psi_e$, write $\boldsymbol{v}(\psi)$ for the element of $\mathcal{V}_l$ defined by the conditions

$$r_i = \varepsilon_i(\psi^*) = \varepsilon_i^*(\psi) \quad \text{for any } i \in \mathbb{Z}/e\mathbb{Z}. \tag{2-13}$$

Then, by the previous considerations, there exists a unique l-partition $\underline{\boldsymbol{\lambda}}(\psi) = (\lambda^0, \dots, \lambda^{l-1}) \in \Phi_e(\boldsymbol{v}(\psi))$ such that

$$f_{\boldsymbol{v}(\psi)}(\underline{\boldsymbol{\lambda}}(\psi)) = \psi. \tag{2-14}$$

3. The Zelevinsky involution of $\mathcal{H}_n^a(q)$

Three natural involutions on $\mathcal{H}_n^a(q)$. Denote by $\mathcal{H}_n(q)$ the Hecke algebra of type A with parameter q over the field $\mathbb{F}$. This is the unital associative $\mathbb{F}$-algebra generated by $T_1, \dots, T_{n-1}$ and the relations

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad (i = 1, \dots, n-2), \\ T_iT_j &= T_jT_i \quad (|j-i| > 1), \\ (T_i - q)(T_i + 1) &= 0 \quad (i = 1, \dots, n-1). \end{aligned}$$

The affine Hecke algebra $\mathcal{H}_n^a(q)$ is the $\mathbb{F}$-algebra which as an $\mathbb{F}$-module is isomorphic to

$$\mathcal{H}_n(q) \otimes_R \mathbb{F}[X_1^{\pm 1}, \dots, X_n^{\pm 1}].$$

The algebra structure is obtained by requiring that $\mathcal{H}_n(q)$ and $\mathbb{F}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ are both subalgebras and for any $i = 1, \dots, n$

$$T_iX_iT_i = qX_{i+1}, \quad T_iX_j = X_jT_i \text{ if } i \ne j.$$

In the sequel, we assume that $q = \xi$ is a primitive e-th root of the unity and write $\mathcal{H}_n^a(\xi)$ for the affine Hecke algebra with parameter ξ. We have three involutive automorphisms τ, $\flat$ and $\sharp$ on $\mathcal{H}_n^a(\xi)$. There are defined on the generators by

$$\begin{aligned} T_i^\tau &= -\xi T_{n-i}^{-1}, \ X_j^\tau = X_{n+1-j}. \\ T_i^\flat &= T_{n-i}, \qquad X_j^\flat = X_{n+1-j}^{-1}. \\ T_i^\sharp &= -\xi T_i^{-1}, \ X_j^\sharp = X_j^{-1}. \end{aligned} \tag{3-1}$$

The involution $\sharp$ was considered in [Iwahori and Matsumoto 1965] and the involution τ, called the Zelevinsky involution, in [Mœglin and Waldspurger 1986]. One can easily check that they are connected as follows:

$$x^\tau = (x^\flat)^\sharp = (x^\sharp)^\flat \quad \text{for all } x \in \mathcal{H}_n^a(\xi). \tag{3-2}$$

The involutions τ, $\sharp$ and $\flat$ on $B_e(\infty)$. We denote by Mod_n^a the category of finite-dimensional $\mathcal{H}_n^a(\xi)$-modules such that for $j = 1, \dots, n$ the eigenvalues of X_j are powers of ξ.

For any multisegment $\psi = \sum_{i \in \mathbb{Z}/e\mathbb{Z}, l \in \mathbb{N}_{>0}} m_{(l;i]}(l; i]$, we write

$$|\psi| = \sum_{\substack{i \in \mathbb{Z}/e\mathbb{Z} \\ l \in \mathbb{N}_{>0}}} l m_{(l;i]}.$$

The geometric realization of $\mathcal{H}_n^a(\xi)$ due to Ginzburg allows one to label the simple $\mathcal{H}_n^a(\xi)$-modules in Mod_n^a by the aperiodic multisegments such that $|\psi| = n$. We do not use Ginzburg's construction in the sequel and just refer to [Chriss and Ginzburg 1997] for a complete exposition or a short review; see also [Ariki et al. 2008; Leclerc et al. 1999]. Let L_ψ be the simple $\mathcal{H}_n^a(\xi)$-module corresponding to ψ under this parametrization.

The three involutions $\tau, \flat, \sharp$ on $\mathcal{H}_n^a(\xi)$ induce involutions on the set of simple $\mathcal{H}_n^a(\xi)$-modules that we will denote in the same way. This yields involutions on the set of aperiodic multisegments (also denoted by τ, $\flat$ and $\sharp$) satisfying

$$L_\psi^\tau = L_{\psi^\tau}, \quad L_\psi^\flat = L_{\psi^\flat}, \quad L_\psi^\sharp = L_{\psi^\sharp}$$

for each aperiodic multisegment ψ. Thus we have three involutions on the vertices of $B_e(\infty)$. By (3-2), they satisfy the relation

$$\tau = \sharp \circ \flat = \flat \circ \sharp. \tag{3-3}$$

By [Leclerc et al. 1999, §2.4], the involution $\flat$ is given by

$$\psi = \sum_{\substack{i \in \mathbb{Z}/e\mathbb{Z} \\ l \in \mathbb{N}_{>0}}} m_{(l;i]}[i; l) \quad \Rightarrow \quad \psi^\flat = \sum_{\substack{i \in \mathbb{Z}/e\mathbb{Z} \\ l \in \mathbb{N}_{>0}}} m_{(l;i]}(l; -i]. \tag{3-4}$$

By comparing with the action (2-1) of ρ on $B_e(\infty)$, this immediately gives:

Lemma 3.1. *The involutions ρ and $\flat$ coincide on $B_e(\infty)$, and their action is given by* (3-4).

Since the action of $\rho = \flat$ on $B_e(\infty)$ is immediate, it is indifferent to describe τ or $\sharp$ on $B_e(\infty)$. The following proposition makes explicit the involution $\sharp$ on $B_e(\infty)$.

Theorem 3.2 [Leclerc et al. 1999]. *Let ψ be an aperiodic multisegment. Then $\psi^\sharp$ is the aperiodic multisegment obtained from ψ by the twofold symmetry $i \leftrightarrow -i$ in the graph $B_e(\infty)$.*

4. The equality $* = \tau$ on $B_e(\infty)$

The aim of this section is to prove that $*$ and τ coincide on $B_e(\infty)$.

Theorem 4.1. $\widetilde{f}_j^*(\psi) = \widehat{f}_j(\psi)$ *for any multisegment $\psi \in \Psi_e$ and any $j \in \mathbb{Z}/e\mathbb{Z}$.*

Before proving the theorem, we see how it implies the desired equality $* = \tau$:

Corollary 4.2. *The involutions $*$ and τ coincide on $B_e(\infty)$.*

Proof. Let $\psi \in \Psi_e$. There exist $i_1, \dots, i_n$ in $\mathbb{Z}/e\mathbb{Z}$ such that $\psi = \widetilde{f}_{i_1} \dots \widetilde{f}_{i_n} \varnothing$. Hence, we obtain

$$\psi^* = \widetilde{f}_{i_1}^* \dots \widetilde{f}_{i_n}^* \varnothing = \widehat{f}_{i_1} \dots \widehat{f}_{i_n} \varnothing.$$

Using (2-2), this gives

$$\psi^* = \rho(\widetilde{f}_{-i_1} \dots \widetilde{f}_{-i_n} \varnothing) = (\psi^\sharp)^\rho = (\psi^\rho)^\sharp = \psi^\tau,$$

where the two last equalities follow from Lemma 3.1 and (3-3). □

To prove Theorem 4.1, we proceed by induction on $|\psi|$. Using Proposition 2.6, we shall see that it suffices to establish the equivalence

$$\widetilde{f}_i \widehat{f}_j(\psi) = \widehat{f}_j \widetilde{f}_i(\psi) \quad \Longleftrightarrow \quad \widetilde{f}_i \widetilde{f}_j^*(\psi) = \widetilde{f}_j^* \widetilde{f}_i(\psi)$$

for any $i, j \in \mathbb{Z}/e\mathbb{Z}$. Now, by definition, the operator $\widetilde{f}_i$ adds an entry i on the right end, and $\widehat{f}_j$ an entry j at the left end, of one of the segments of ψ. This will imply:

$(*)$ $\widetilde{f}_i \widehat{f}_j(\psi) = \widehat{f}_j \widetilde{f}_i(\psi)$ *except possibly when $i = j$ and $\widetilde{f}_i(\psi) = \psi + [i]$ or $\widehat{f}_i(\psi) = \psi + [i]$.*

On the other hand, it is easy to derive from Proposition 2.6 that

$\widetilde{f}_i \widetilde{f}_j^*(\psi) = \widetilde{f}_j^* \widetilde{f}_i(\psi)$ *except possibly when $i = j$ and $\widetilde{f}_i(\psi) = \psi + [i]$ or $\widetilde{f}_i^*(\psi) = \psi + [i]$.*

The case where the operators do not commute being easily tractable, this will imply Theorem 4.1.

More on the crystal operators $\widetilde{f}_i$ and $\widehat{f}_i$. We begin with refinements of the actions of the operators $\widetilde{f}_i$ and $\widehat{f}_i$. In [Ariki et al. 2008], we have obtained an alternative description of the action of the crystal operators on Ψ_e. Consider $\psi \in \Psi_e$ and $i \in \mathbb{Z}/e\mathbb{Z}$. We encode segments in ψ with *tail i* by the symbol R, and those with tail $i-1$ by A. For any nonnegative integer l, write $w_{i,l} = R^{m_{(l;i]}} A^{m_{(l;i-1]}}$, where $m_{(l;i]}$ and $m_{(l;i-1]}$ are the number of segments $(l;i]$ and $(l;i-1]$ in ψ. Set $w_i = \prod_{l\geq 1} w_{i,l}$. Write $\widetilde{w}_i = A^{a_i(\psi)} R^{r_i(\psi)}$ for the word derived from w_i by deleting as many factors RA as possible. If $a_i(\psi) > 0$, denote by $l_{0,i}(\psi) > 0$ the length of the rightmost segment A in $\widetilde{w}_i$. If $a_i(\psi) = 0$, set $l_{0,i}(\psi) = 0$. When there is no risk of confusion, we simply write l_0 instead of $l_{0,i}(\psi)$.

Lemma 4.3 [Ariki et al. 2008]. *With the notation above we have $\varepsilon_i(\psi) = r_i(\psi)$ and*

$$\widetilde{f}_i \psi = \begin{cases} \psi + (l_0; i] - (l_0 - 1; i-1] & \text{if } a_i(\psi) > 0, \\ \psi + (1; i] & \text{if } a_i(\psi) = 0. \end{cases}$$

We can compute similarly the action of the crystal operators $\widehat{f}_i$ (with $i \in \mathbb{Z}/e\mathbb{Z}$) on ψ. We encode the segments in ψ with *head i* by the symbol $\widehat{R}$ and those with head $i+1$ by $\widehat{A}$. For any nonnegative integer l, write $\widehat{w}_{i,l} = \widehat{R}^{m_{[i;l)}} \widehat{A}^{m_{[i+1;l)}}$ where $m_{[i;l)}$ and $m_{[i+1;l)}$ are the number of segments $[i;l)$ and $[i+1;l)$ in ψ. Set $\widehat{w}_i = \prod_{l\geq 1} \widehat{w}_{i,l}$. Write $\overline{w}_i = \widehat{A}^{\widehat{a}_i(\psi)} \widehat{R}^{\widehat{r}_i(\psi)}$ for the word derived from $\widehat{w}_i$ by deleting as many factors $\widehat{R}\widehat{A}$ as possible. If $\widehat{a}_i(\psi) > 0$, let $\widehat{l_{0,i}}(\psi) > 0$ be length of the rightmost segment $\widehat{A}$ in $\overline{w}_i$. If $\widehat{a}_i(\psi) = 0$, set $\widehat{l_{0,i}}(\psi) = 0$. When there is no risk of confusion, we also simply write $\widehat{l_0}$ instead of $\widehat{l_{0,i}}(\psi)$.

Lemma 4.4. *With the notation above, plus $\widehat{\varepsilon}_i(\psi) = \max\{p \mid \widehat{e}_i^{\,p}(\psi) \neq 0\}$, we have $\widehat{\varepsilon}_i(\psi) = \widehat{r}_i(\psi)$ and*

$$\widehat{f}_i \psi = \begin{cases} \psi + [i; \widehat{l_0}) - [i+1; \widehat{l_0} - 1) & \text{if } \widehat{a}_i(\psi) > 0, \\ \psi + [i; 1) & \text{if } \widehat{a}_i(\psi) = 0. \end{cases} \tag{4-1}$$

Remark 4.5. By [Grojnowski 1999, Theorem 9.13], for any $i \in \mathbb{Z}/e\mathbb{Z}$, the integers $\varepsilon_i(\psi) = r_i(\psi)$ and $\widehat{\varepsilon}_i(\psi) = \widehat{r}_i(\psi)$ give the maximal size of a Jordan block with eigenvalue ξ^i corresponding to the action of the generator X_n and X_1, respectively, on the simple $\mathcal{H}_n^a(\xi)$-module L_ψ.

Equality of the crystal operators $\widetilde{f}_i^*$ and $\widehat{f}_i$. Our purpose is now to establish the equality

$$\widetilde{f}_i^*(\psi) = \widehat{f}_i(\psi) \text{ for any } \psi \in \Psi_e. \tag{4-2}$$

This is achieved by showing that the relations $\widetilde{f}_i^* \widetilde{f}_j \psi = \widetilde{f}_j \widetilde{f}_i^* \psi$ and $\widehat{f}_i \widetilde{f}_j \psi = \widetilde{f}_j \widehat{f}_i \psi$ are both equivalent to a very simple condition on ψ.

Lemma 4.6. *Fix $i \in \mathbb{Z}/e\mathbb{Z}$.*

(1) *Consider $\psi, \chi \in \Psi_e$ such that $\psi = \widehat{f}_i \chi$ and fix $j \in \mathbb{Z}/e\mathbb{Z}$. Then*

$$l_{0,j}(\chi) \neq l_{0,j}(\psi) \iff i = j,\ \widehat{a}_i(\chi) = 0 \text{ and } a_i(\chi) = 1.$$

(2) *Consider $\psi, \chi \in \Psi_e$ such that $\psi = \widetilde{f}_i \chi$ and fix $j \in \mathbb{Z}/e\mathbb{Z}$. Then*

$$\widehat{l}_{0,j}(\chi) \neq \widehat{l}_{0,j}(\psi) \iff i = j,\ a_i(\chi) = 0 \text{ and } \widehat{a}_i(\chi) = 1.$$

Proof. We prove (1); the arguments for (2) are similar. Assume first that $\widehat{l}_{0,i}(\chi) = \widehat{l}_0 > 1$. Hence $\widehat{a}_i(\chi) > 0$ and $\psi = \chi - [i+1, \dots, i+\widehat{l}_0-1] + [i, \dots, i+\widehat{l}_0-1]$. If j is distinct (mod e) from $i+\widehat{l}_0-1$ and $i+\widehat{l}_0$, then neither $[i+1, \dots, i+\widehat{l}_0-1]$ nor $[i, \dots, i+\widehat{l}_0-1]$ are segments A or R for j. We have $w_j(\psi) = w_j(\chi)$ and then $l_{0,j}(\chi) = l_{0,j}(\psi)$. Thus we can restrict ourselves to the cases $j \equiv i+\widehat{l}_0-1 \pmod e$ and $j \equiv i+\widehat{l}_0 \pmod e$. We write

$$\widehat{w}_i(\chi) = \cdots [i, \dots, i+\widehat{l}_0-2]^{m_{[i,\dots,i+\widehat{l}_0-2]}} [i+1, \dots, i+\widehat{l}_0-1]^{m_{[i+1,\dots,i+\widehat{l}_0-1]}} \\ [i, \dots, i+\widehat{l}_0-1]^{m_{[i,\dots,i+\widehat{l}_0-1]}} [i+1, \dots, i+\widehat{l}_0]^{m_{[i+1,\dots,i+\widehat{l}_0]}} \dots$$

where we have only shown the segments of length $\widehat{l}_0-1$ and $\widehat{l}_0$ of $\widehat{w}_i(\chi)$. Since $\psi = \widehat{f}_i \chi$, we have

$$\widehat{w}_i(\psi) = \cdots [i, \dots, i+\widehat{l}_0-2]^{m_{[i,\dots,i+\widehat{l}_0-2]}} [i+1, \dots, i+\widehat{l}_0-1]^{m_{[i+1,\dots,i+\widehat{l}_0-1]}-1} \\ [i, \dots, i+\widehat{l}_0-1]^{m_{[i,\dots,i+\widehat{l}_0-1]}+1} [i+1, \dots, i+\widehat{l}_0]^{m_{[i+1,\dots,i+\widehat{l}_0]}} \dots$$

In particular, by (4-1), we must have

$$m_{[i,\dots,i+\widehat{l}_0-2]} < m_{[i+1,\dots,i+\widehat{l}_0-1]} \quad \text{and} \quad m_{[i,\dots,i+\widehat{l}_0-1]} \geq m_{[i+1,\dots,i+\widehat{l}_0]}.$$

When $j = (i+\widehat{l}_0-1) \pmod e$, $[i+1, \dots, i+\widehat{l}_0-1]$ and $[i, \dots, i+\widehat{l}_0-1]$ are of type R for j. Hence, by considering only the segments of lengths $\widehat{l}_0-1$ and $\widehat{l}_0$, we can write

$$w_j(\chi) = \cdots [i+1, \dots, i+\widehat{l}_0-1]^{m_{[i+1,\dots,i+\widehat{l}_0-1]}} [i, \dots, i+\widehat{l}_0-2]^{m_{[i,\dots,i+\widehat{l}_0-2]}} \\ [i, \dots, i+\widehat{l}_0-1]^{m_{[i,\dots,i+\widehat{l}_0-1]}} [i-1, \dots, i+\widehat{l}_0-2]^{m_{[i-1,\dots,i+\widehat{l}_0-2]}} \cdots,$$

$$w_j(\psi) = \cdots [i+1, \dots, i+\widehat{l}_0-1]^{m_{[i+1,\dots,i+\widehat{l}_0-1]}-1} [i, \dots, i+\widehat{l}_0-2]^{m_{[i,\dots,i+\widehat{l}_0-2]}} \\ [i, \dots, i+\widehat{l}_0-1]^{m_{[i,\dots,i+\widehat{l}_0-1]+1}} [i-1, \dots, i+\widehat{l}_0-2]^{m_{[i-1,\dots,i+\widehat{l}_0-2]}} \cdots.$$

Since $m_{[i,\dots,i+\widehat{l}_0-2]} < m_{[i+1,\dots,i+\widehat{l}_0-1]}$, the cancellation procedures of the factors RA in $w_j(\chi)$ and $w_j(\psi)$ yield the same final word. Hence $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$ and we have also $l_{0,j}(\chi) = l_{0,j}(\psi) = 1$.

When $j = (i+\widehat{l}_0) \pmod e$, $[i+1, \dots, i+\widehat{l}_0-1]$ and $[i, \dots, i+\widehat{l}_0-1]$ are of type A for j. We obtain also $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$ by considering the segments of lengths $\widehat{l}_0-1$ and $\widehat{l}_0$. Thus $l_{0,j}(\chi) = l_{0,j}(\psi)$.

Observe that we have always $\widetilde{w}_j(\psi) = \widetilde{w}_j(\chi)$ for any $j \in \mathbb{Z}/e\mathbb{Z}$ when $\widehat{l_0} > 1$. In particular,

$$\widehat{a}_i(\chi) > 0 \implies a_j(\chi) = a_j(\widehat{f}_i\chi) \text{ for any } j \in \mathbb{Z}/e\mathbb{Z}. \tag{4-3}$$

Now assume $\widehat{l_0} = 1$, that is, $\psi = \chi + [i]$. Write

$$\widehat{w}_i(\chi) = [i]^{m_{[i]}}[i+1]^{m_{[i+1]}}\cdots \quad \text{and} \quad \widehat{w}_i(\psi) = [i]^{m_{[i]}+1}[i+1]^{m_{[i+1]}}\cdots,$$

with $m_{[i]} \geq m_{[i+1]}$.

When $j = i+1 \pmod e$, $[i]$ is of type A for j and $[i+1]$ is of type R. Thus we can write

$$w_j(\chi) = [i+1]^{m_{[i+1]}}[i]^{m_{[i]}}\cdots \quad \text{and} \quad w_j(\psi) = [i+1]^{m_{[i+1]}}[i]^{m_{[i]}+1}\cdots.$$

Since $m_{[i]} \geq m_{[i+1]}$, the rightmost segments A in $\widetilde{w}_j(\chi)$ and $\widetilde{w}_j(\psi)$ are the same and we still have $l_{0,j}(\chi) = l_{0,j}(\psi)$.

When $j = i \pmod e$, $[i]$ is of type R for j. Observe that $\widehat{a}_i(\chi) = 0$. Set $\widetilde{w}_i(\chi) = A^{a_i(\chi)}R^{r_i(\chi)}$. Then $\widetilde{w}_i(\psi)$ is obtained by applying the cancellation procedure of the factors RA to the word $w = RA^{a_i(\chi)}R^{r_i(\chi)}$. Clearly, $l_{0,j}(\chi) \neq l_{0,j}(\psi)$ if and only if $a_i(\chi) = 1$, for in this case $l_{0,j}(\chi) > 1$ and $l_{0,j}(\psi) = 1$. □

Proposition 4.7. *For any $\chi \in \Psi_e$ and $i, j \in \mathbb{Z}/e\mathbb{Z}$, we have*

$$\widetilde{f}_i\widehat{f}_j\chi \neq \widehat{f}_j\widetilde{f}_i\chi \iff i = j \text{ and } a_i(\chi) + \widehat{a}_i(\chi) = 1.$$

Proof. Assume $i \neq j$, or $i = j$ and $a_i(\chi) + \widehat{a}_i(\chi) > 1$. By Lemma 4.6, we have $\widehat{l}_{0,j}(\chi) = \widehat{l}_{0,j}(\widetilde{f}_i\chi) = \widehat{l_0}$ and $l_{0,i}(\chi) = l_{0,i}(\widehat{f}_j\chi) = l_0$. Hence

$$\widetilde{f}_i\widehat{f}_j\chi = \chi + [j;\widehat{l_0}) + (l_0;i] - [j+1;\widehat{l_0}-1) - (l_0-1;i-1] = \widehat{f}_j\widetilde{f}_i\chi$$

with $[j+1;\widehat{l_0}-1) = \varnothing$ if $\widehat{l_0} = 1$ and $(l_0-1;i-1] = \varnothing$ if $l_0 = 1$.

Now, assume $i = j$, $a_i(\chi) = 1$ and $\widehat{a}_i(\chi) = 0$. We have

$$\widetilde{f}_i\widehat{f}_i\chi = \chi + 2[i] \text{ and } \widehat{f}_i\widetilde{f}_i\chi = \chi + [i] + [i-l_0+1,\dots,i] - [i-l_0+1,\dots,i-1]$$

with $l_0 = l_{0,i}(\chi) > 1$. Similarly, if we assume $i = j$, $a_i(\chi) = 1$ and $\widehat{a}_i(\chi) = 0$, we obtain

$$\widehat{f}_i\widetilde{f}_i\chi = \chi + 2[i] \text{ and } \widetilde{f}_i\widehat{f}_i\chi = \chi + [i] + [i+1,\dots,i+\widehat{l_0}-1] - [i,\dots,i+\widehat{l_0}-1]$$

with $\widehat{l_0} = \widehat{l}_{0,i}(\chi) > 1$. In both cases, $\widetilde{f}_i\widehat{f}_j\chi \neq \widehat{f}_i\widetilde{f}_i\chi$, which completes the proof. Observe that we then have

$$\widetilde{f}_i\widehat{f}_i\chi = (\widehat{f}_i)^2\chi \quad \text{and} \quad \widehat{f}_i\widetilde{f}_i\chi = (\widetilde{f}_i)^2\chi. \tag{4-4}$$

□

Proposition 4.8. *Consider $\psi \in \Psi_e$ and $i, j \in \mathbb{Z}/e\mathbb{Z}$.*

(1) *If $i \neq j$, we have $\widetilde{f}_i \widetilde{f}_j^* \psi = \widetilde{f}_j^* \widetilde{f}_i \psi$.*

(2) *If $i = j$, set $m = \varepsilon_i^*(\psi)$. Then*

$$\widetilde{f}_i \widetilde{f}_i^* \psi \neq \widetilde{f}_i^* \widetilde{f}_i \psi \quad \Longleftrightarrow \quad \varphi_i((\widetilde{e}_i^*)^m \psi) = \varepsilon_i^*(\psi) + 1.$$

Proof. (1) This is a classical property of crystals. Write $\theta_j(\psi) = (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^m b_j$ where $m = \varepsilon_j^*(\psi)$. Then by (2-6), we have $\theta_j(\widetilde{f}_i \psi) = \widetilde{f}_i (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^m b_j$ for $i \neq j$. By Proposition 2.6, we obtain $\theta_j(\widetilde{f}_j^* \widetilde{f}_i \psi) = \widetilde{f}_i (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^{m+1} b_j$. We have also $\theta_j(\widetilde{f}_j^* \psi) = (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^{m+1} b_j$ and since $i \neq j$, this yields $\theta_j(\widetilde{f}_i \widetilde{f}_j^* \psi) = \widetilde{f}_i (\widetilde{e}_j^*)^m \psi \otimes \widetilde{f}_j^{m+1} b_j$. Hence $\theta_j(\widetilde{f}_i \widetilde{f}_j^* \psi) = \theta_j(\widetilde{f}_j^* \widetilde{f}_i \psi)$ and we have $\widetilde{f}_i \widetilde{f}_j^* \psi = \widetilde{f}_j^* \widetilde{f}_i \psi$ because θ_j is an embedding of crystals.

(2) Using the same arguments, we derive

$$\theta_i(\widetilde{f}_i \widetilde{f}_i^* \psi) = \begin{cases} \widetilde{f}_i (\widetilde{e}_i^*)^m \psi \otimes \widetilde{f}_i^{m+1} b_i & \text{if } \varphi_i((\widetilde{e}_i^*)^m \psi) > m+1, \\ (\widetilde{e}_i^*)^m \psi \otimes \widetilde{f}_i^{m+2} b_i & \text{if } \varphi_i((\widetilde{e}_i^*)^m \psi) \leq m+1, \end{cases}$$

$$\theta_i(\widetilde{f}_i^* \widetilde{f}_i \psi) = \begin{cases} \widetilde{f}_i (\widetilde{e}_i^*)^m \psi \otimes \widetilde{f}_i^{m+1} b_i & \text{if } \varphi_i((\widetilde{e}_i^*)^m \psi) > m, \\ (\widetilde{e}_i^*)^m \psi \otimes \widetilde{f}_i^{m+2} b_i & \text{if } \varphi_i((\widetilde{e}_i^*)^m \psi) \leq m. \end{cases}$$

Thus we obtain $\theta_i(\widetilde{f}_i \widetilde{f}_i^* \psi) = \theta_i(\widetilde{f}_i^* \widetilde{f}_i \psi)$ except when $\varphi_i((\widetilde{e}_i^*)^m \psi) = m+1$. Observe that we have in this case

$$\widetilde{f}_i \widetilde{f}_i^* \psi = (\widetilde{f}_i^*)^2 \psi \neq (\widetilde{f}_i)^2 \psi = \widetilde{f}_i^* \widetilde{f}_i \psi. \tag{4-5}$$ □

Lemma 4.9. *Consider $\psi \in \Psi_e$ and $i \in \mathbb{Z}/e\mathbb{Z}$. Set $\mathrm{wt}(\psi) = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} \mathrm{wt}_i(\psi) \Lambda_i$. Then*

$$\mathrm{wt}_i(\psi) = a_i(\psi) - r_i(\psi) + \widehat{a}_i(\psi) - \widehat{r}_i(\psi), \tag{4-6}$$

$$\varphi_i(\psi) = a_i(\psi) + \widehat{a}_i(\psi) - \widehat{r}_i(\psi). \tag{4-7}$$

Proof. For the first equality, set $\psi = \sum_{l \geq 1} m_{(l;i]}(l; i] = \sum_{l \geq 1} m_{[i;l)}[i; l)$. During the cancellation procedures described just before and just after Lemma 4.3, pairs of segments (R, A) or $(\widehat{R}, \widehat{A})$ are deleted. Thus assertion 1 is equivalent to the equality $\mathrm{wt}_i(\psi) = \Delta_i(\psi)$, where

$$\Delta_i(\psi) = \sum_{l \geq 1} m_{(l;i-1]} - m_{(l;i]} + \sum_{l \geq 1} m_{[i+1;l)} - m_{[i;l)}. \tag{4-8}$$

We proceed by induction on $|\psi|$. For $\psi = \varnothing$, (4-8) is satisfied. Now assume the equalities (4-8) hold for any $i \in \mathbb{Z}/e\mathbb{Z}$ with $|\psi| = n$. Set $\psi' = \widetilde{f}_j \psi$. We have $\mathrm{wt}(\psi') = \mathrm{wt}(\psi) - \alpha_j$. Since $\alpha_j = 2\Lambda_j - \Lambda_{j+1} - \Lambda_{j-1}$, this gives

$$\mathrm{wt}_i(\psi') = \begin{cases} \mathrm{wt}_i(\psi) & \text{if } i \notin \{j-1, j, j+1\}, \\ \mathrm{wt}_i(\psi) - 2 & \text{if } i = j, \\ \mathrm{wt}_i(\psi) + 1 & \text{if } i \in \{j-1, j+1\}. \end{cases} \tag{4-9}$$

The multisegment ψ' is obtained by adding the segments $[j]$ to ψ or by replacing a segment $(l-1; j-1]$ in ψ by the segment $(l, j]$. This shows that the relations (4-9) are also satisfied by the $\Delta_i(\psi')$'s. Hence $\Delta_i(\psi') = \mathrm{wt}_i(\psi')$ for any $i \in \mathbb{Z}/e\mathbb{Z}$.

Proof of (4-7): By (2-3), we have $\mathrm{wt}_i(\psi) = \varphi_i(\psi) - \varepsilon_i(\psi)$. Lemma 4.3 then gives $\mathrm{wt}_i(\psi) = \varphi_i(\psi) - r_i(\psi)$. Thus $\varphi_i(\psi) = a_i(\psi) + \widehat{a}_i(\psi) - \widehat{r}_i(\psi)$ by (4-6). □

To prove (4-2), we proceed by induction on $n = |\psi|$. We easily check that $\widetilde{f}_i^*(\varnothing) = \widehat{f}_i(\varnothing) = [i]$ for the empty multisegment. Now assume that $\widetilde{f}_i^*(\psi) = \widehat{f}_i(\psi)$ for any multisegment $\psi \in \Psi_e$ such that $|\psi| \le n$.

Proposition 4.10. *Under the induction hypothesis we have, for any $\chi \in \Psi_e$ such that $|\chi| \le n$,*

(4-10) $\widetilde{f}_i \widehat{f}_j \chi \neq \widehat{f}_j \widetilde{f}_i \chi \iff \widetilde{f}_i \widetilde{f}_j^* \chi \neq \widetilde{f}_j^* \widetilde{f}_i \chi \iff i = j$ *and* $a_i(\chi) + \widehat{a}_i(\chi) = 1$.

Proof. Note first that the proposition does not follow directly from the induction hypothesis for $\left|\widetilde{f}_j \chi\right| = n+1$. By this induction hypothesis, we have $(\widetilde{e}_i^*)^{\varepsilon_i^*(\chi)} \chi = (\widehat{e}_i)^{\widehat{r}_i(\chi)} \chi$. Set $\chi' = (\widehat{e}_i)^{\widehat{r}_j(\chi)} \chi$. Formula (4-7) gives

$$\varphi_i(\chi') = a_i(\chi') + \widehat{a}_i(\chi') - \widehat{r}_i(\chi') = a_i(\chi') + \widehat{a}_i(\chi) + \widehat{r}_i(\chi),$$

for we have $\widehat{r}_i(\chi') = 0$ and $\widehat{a}_i(\chi') = \widehat{a}_i(\chi) + \widehat{r}_i(\chi)$. Observe that $\varepsilon_i^*(\chi) = \widehat{r}_i(\chi)$ be the induction hypothesis. Moreover, we have $a_i(\chi) = a_i(\chi')$ by (4-3), since $\widehat{a}_i(\varphi) > 0$ for any $\varphi = (\widehat{e}_i)^a \chi$ with $a \in \{1, \dots, \widehat{r}_j(\chi)\}$. This gives the equivalences

$$\varphi_i(\chi') = \varepsilon_i^*(\chi) + 1 \iff a_i(\chi') + \widehat{a}_i(\chi) = 1 \iff a_i(\chi) + \widehat{a}_i(\chi) = 1.$$

Now Propositions 4.7 and 4.8 yield (4-10). □

We are now able to prove the main result of this section.

Theorem 4.11. *For any multisegment $\psi \in \Psi_e$ and any $j \in \mathbb{Z}/e\mathbb{Z}$, we have $\widetilde{f}_j^*(\psi) = \widehat{f}_j(\psi)$.*

Proof. We argue by induction on $n = |\psi|$. We already know that for all $j \in \mathbb{Z}/e\mathbb{Z}$, we have $\widetilde{f}_j^*(\varnothing) = \widehat{f}_j(\varnothing) = [j]$. Now assume $\widetilde{f}_j^* \chi = \widehat{f}_j \chi$ for any $j \in \mathbb{Z}/e\mathbb{Z}$ and any $\chi \in \Psi_e$ such that $|\chi| \le n$. Consider $\psi \in \Psi_e$ such that $|\psi| = n+1$.There exists $i \in \mathbb{Z}/e\mathbb{Z}$ and $\chi \in \Psi_e$ such that $\psi = \widetilde{f}_i \chi$ and $|\chi| = n$.

When $i \neq j$ or $a_i(\chi) + \widehat{a}_i(\chi) > 1$, we have by Proposition 4.10 $\widetilde{f}_j^* \psi = \widetilde{f}_j^* \widetilde{f}_i \chi = \widetilde{f}_i \widetilde{f}_j^* \chi$. By our induction hypothesis, we can thus write $\widetilde{f}_j^* \psi = \widetilde{f}_i \widehat{f}_j \chi$. Since $a_i(\chi) + \widehat{a}_i(\chi) > 1$, this finally gives $\widetilde{f}_j^* \psi = \widehat{f}_j \widetilde{f}_i \chi = \widehat{f}_j \psi$.

When $i = j$ and $a_i(\chi) + \widehat{a}_i(\chi) = 1$, we obtain $\widetilde{f}_i^* \psi = \widetilde{f}_i^* \widetilde{f}_i \chi = \widetilde{f}_i^2 \psi$ by (4-5). Similarly, we have $\widehat{f}_i \psi = \widehat{f}_i \widetilde{f}_i \chi = \widetilde{f}_i^2 \psi$ by (4-4). Thus $\widetilde{f}_i^* \psi = \widehat{f}_i \psi$. □

Remark 4.12. Theorem 4.1 and Proposition 4.10 imply, for any $\psi \in \Psi_e$, that

$$\widetilde{f}_i \widetilde{f}_j^* \psi = \widetilde{f}_j^* \widetilde{f}_i \psi \quad \iff \quad i \neq j \text{ or } a_i(\psi) + \widehat{a}_i(\psi) > 1.$$

5. Affine Hecke algebra of type A and Ariki–Koike algebras

Identification of simple modules. Let $\boldsymbol{v} = (v_0, \dots, v_{l-1}) \in \mathcal{V}_l$. The Ariki–Koike algebra $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$ is the quotient $\mathcal{H}_n^a(\xi)/I_{\boldsymbol{v}}$ where $I_{\boldsymbol{v}} = \langle P_{\boldsymbol{v}} = \prod_{i=0}^{l-1}(X_1 - \xi^{v_i})\rangle$. Then each simple $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$-module is isomorphic to a simple $\mathcal{H}_n^a(\xi)$-module of Mod_n^a. By the Specht module theory developed by Dipper, James and Mathas [Dipper et al. 1998], the simple $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$-modules are parametrized by certain l-partitions of n called Kleshchev multipartitions. Let $\Phi_e^K(\boldsymbol{v})$ be the set of Kleshchev l-partitions. Given μ in $\Phi_e^K(\boldsymbol{v})$, write D^μ for the simple $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$-module associated to μ under this parametrization. In fact, we shall need in the sequel the parametrization of the simple $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$-modules by FLOTW l-partitions. The correspondence between the parametrizations by Kleshchev and FLOTW l-partitions has been detailed in [Jacon and Lecouvey 2009a]. In particular, the bijection

$$\Gamma : \Phi_e(\boldsymbol{v}) \to \Phi_e^K(\boldsymbol{v}) \tag{5-1}$$

is an isomorphism of $\mathcal{U}_v$-crystals which can easily be made explicit. This means that, given any $\boldsymbol{\lambda}$ in $\Phi_e(\boldsymbol{v})$, we can compute $\Gamma(\boldsymbol{\lambda})$ directly from $\boldsymbol{\lambda}$ without using the crystal structures on $\Phi_e(\boldsymbol{v})$ and $\Phi_e^K(\boldsymbol{v})$. We then set $\widetilde{D}^{\boldsymbol{\lambda}} = D^{\Gamma(\boldsymbol{\lambda})}$. This gives the natural labeling

$$\mathrm{Irr}(\mathcal{H}_n^{\boldsymbol{v}}(\xi)) = \{\widetilde{D}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Phi_e(\boldsymbol{v})\},$$

which coincides with the parametrization of the simple $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$-modules in terms of Geck–Rouquier canonical basic set obtained in [Jacon 2004].

The simple $\mathcal{H}_n^a(\xi)$-module L_ψ with $\psi \in \Psi_e$ isomorphic to $\widetilde{D}^{\boldsymbol{\lambda}}$ is given by the following result:

Theorem 5.1 [Ariki et al. 2008, Theorem 6.2]. *For $\boldsymbol{\lambda} \in \Phi_e(\boldsymbol{v})$ we have $\widetilde{D}^{\boldsymbol{\lambda}} \simeq L_{f_v(\boldsymbol{\lambda})}$, where f_v is the crystal embedding of Theorem 2.12.*

Conversely, given any simple $\mathcal{H}_n^a(\xi)$-module L_ψ, it is natural to search for the Ariki–Koike algebras $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$ with $\boldsymbol{v}$ in $\mathcal{V}_l$ and the simple $\mathcal{H}_n^{\boldsymbol{v}}(\xi)$-module $\widetilde{D}^{\boldsymbol{\lambda}}$ such that $\widetilde{D}^{\boldsymbol{\lambda}} \simeq L_\psi$. This problem turns out to be more complicated. Indeed we have first to determine all the multicharges $\boldsymbol{v}$ such that $f_{\boldsymbol{v}}^{-1}(\psi) \neq \varnothing$ and next we need to compute the l-partition $\boldsymbol{\lambda}$ satisfying $f_{\boldsymbol{v}}(\boldsymbol{\lambda}) = \psi$. Note that $\boldsymbol{\lambda}$ is necessarily unique for a given $\boldsymbol{v}$ since $f_{\boldsymbol{v}}$ is injective. We will then say that $\boldsymbol{v}$ is an *admissible multicharge* with respect to ψ when $f_{\boldsymbol{v}}^{-1}(\psi) \neq \varnothing$. Then $\boldsymbol{\lambda} = f_{\boldsymbol{v}}^{-1}(\psi)$ is its corresponding *admissible multipartition*. In the next paragraphs, we shall completely solve the problem of determining all the admissible multicharges and FLOTW multipartitions associated to an aperiodic multisegment ψ. To obtain the corresponding Kleshchev multipartition, it then suffices to apply Γ.

Admissible multicharges. Let $\psi \in \Psi_e$. To find a multicharge $\boldsymbol{v}$ such that $f_{\boldsymbol{v}}^{-1}(\psi) \neq \varnothing$, we compute $\varepsilon_i^*(\psi)$ for all $i \in \mathbb{Z}/e\mathbb{Z}$ by using the equality $\varepsilon_i^*(\psi) = \widehat{r_i}(\psi)$ established in Theorem 4.1. For a multicharge $\boldsymbol{v}$ in $\mathcal{V}_l$ and $i \in \mathbb{Z}/e\mathbb{Z}$, let $\kappa_i(\boldsymbol{v})$ be the nonnegative integers such that

$$\boldsymbol{v} = (\underbrace{0, \dots, 0}_{\kappa_0(\boldsymbol{v})}, \underbrace{1, \dots, 1}_{\kappa_1(\boldsymbol{v})}, \dots, \underbrace{e-1, \dots, e-1}_{\kappa_{e-1}(\boldsymbol{v})}).$$

Then

$$f_{\boldsymbol{v}}^{-1}(\psi) \neq \varnothing \quad \Longleftrightarrow \quad \kappa_i(\boldsymbol{v}) \geq \varepsilon_i^*(\psi) \text{ for all } i \in \mathbb{Z}/e\mathbb{Z}.$$

Observe that the multicharge $\boldsymbol{v}(\psi)$ with $\kappa_i(\boldsymbol{v}(\psi)) = \varepsilon_i^*(\psi)$ (defined at the end of Section 2, page 295) is the multicharge of minimal level among all the admissible multicharges. It is of particular interest for the computation of the involution $\sharp$ as we shall see.

Admissible multipartitions. Consider $\psi \in \Psi_e$, $l \in \mathbb{N}$ and an admissible multicharge $\boldsymbol{v} \in \mathcal{V}_l$ with respect to ψ. The aim of this section is to give a simple procedure for computing the admissible l-partition $\boldsymbol{\lambda} \in \Phi_e(\boldsymbol{v})$ associated to ψ (i.e., such that $f_{\boldsymbol{v}}(\boldsymbol{\lambda}) = \psi$).

We begin with a general lemma on FLOTW l-partitions. Consider $\boldsymbol{v} \in \mathcal{V}_l$ and $\boldsymbol{\lambda} \in \Phi_e(\boldsymbol{v})$ a nonempty l-partition. Let m be the length of the minimal nonzero part of $\boldsymbol{\lambda}$. Let $\boldsymbol{\mu}$ be the l-partition obtained by deleting in $\boldsymbol{\lambda}$ the parts of length m.

Lemma 5.2. *The l-partition $\boldsymbol{\mu}$ belongs to $\Phi_e(\boldsymbol{v})$.*

Proof. Assume that $\boldsymbol{\mu} \notin \Phi_e(\boldsymbol{v})$. Then one of the following situations happens.

(i) There exists $c \in \{0, 1, \dots, l-1\}$ and $i \in \mathbb{N}$ such that $\mu_i^c < \mu_{i+v_{c+1}-v_c}^{c+1}$. This implies in particular that $\mu_{i+v_{c+1}-v_c}^{c+1} \neq 0$. Since $\boldsymbol{\lambda}$ belongs to $\Phi_e(\boldsymbol{v})$, we have $\lambda_i^c \geq \lambda_{i+v_{c+1}-v_c}^{c+1}$. Thus $\lambda_{i+v_{c+1}-v_c}^{c+1} = \mu_{i+v_{c+1}-v_c}^{c+1}$, $\lambda_i^c = m$ and $\mu_i^c = 0$. We have $\lambda_{i+v_{c+1}-v_c}^{c+1} \neq 0$ and $\lambda_{i+v_{c+1}-v_c}^{c+1} \leq m$. This contradicts the fact that $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by deleting the minimal nonzero parts.

(ii) There exists $i \in \mathbb{N}$ such that $\mu_i^{l-1} < \mu_{i+v_0-v_{l-1}+e}^0$. We obtain a contradiction similarly. □

For $\psi \in \Psi_e$, define $l_1 > \cdots > l_r > 0$ as the decreasing sequence of (distinct) lengths of the segments appearing in ψ. For any $t = 1, \dots, r$, write a_t for the number of segments in ψ with length l_t. Set $\psi_0 = \varnothing$ the empty multisegment and $\psi_r = \psi$. For any $1 \leq t \leq r-1$ let ψ_t be the multisegment obtained from ψ by deleting successively the segments of length $l_{t+1}, \dots, l_r$. Clearly ψ_t is aperiodic.

Assume $\boldsymbol{\lambda} \in \Phi_e(\boldsymbol{v})$ is associated to ψ. Since $f_{\boldsymbol{v}}(\boldsymbol{\lambda}) = \psi$, the sequence $l_1 > \cdots > l_r$ is also the decreasing sequence of distinct parts appearing in $\boldsymbol{\lambda}$. Moreover, for ant $t = 1, \dots, r$, $\boldsymbol{\lambda}$ contains a_t parts equal to l_t. Set $\boldsymbol{\lambda}[r] = \boldsymbol{\lambda}$. Let $\boldsymbol{\lambda}[t]$, $t = 0, \dots, r-1$ be the l-partitions obtained by deleting successively the parts of lengths $l_r, \dots, l_{t+1}$

in $\boldsymbol{\lambda}$. By Lemma 5.2, the l-partitions $\boldsymbol{\lambda}[t]$, $t = 0, \dots, r-1$, all belong to $\Phi_e(\boldsymbol{v})$. Since $f_{\boldsymbol{v}}(\boldsymbol{\lambda}) = \psi$ we must also have $f_{\boldsymbol{v}}(\boldsymbol{\lambda}[t]) = \psi_t$ for any $t = 0, \dots, r$, by definition of the map $f_{\boldsymbol{v}}$ (see (2-11)).

We are going to compute $\boldsymbol{\lambda}$ from ψ by induction on the lengths $l_1 > \cdots > l_r > 0$ of the segments of ψ. To do this, we have to determine the sequence of l-partitions $\boldsymbol{\lambda}[t]$, $t = 0, \dots, r$ associated to the segments ψ_t, $t = 0, \dots, r$. We have $\boldsymbol{\lambda}[0] = \boldsymbol{\varnothing}$. Thus, it suffices to explain how $\boldsymbol{\lambda}[t+1] \in \Phi_e(\boldsymbol{v})$ can be obtained from $\boldsymbol{\lambda}[t] \in \Phi_e(\boldsymbol{v})$.

The l-partition $\boldsymbol{\lambda}[t+1]$ is constructed by adding a_{t+1} parts of lengths l_{t+1} to $\boldsymbol{\lambda}[t]$ such that the parts added give segments $[k_{t+1}; l_{t+1})$ in the correspondence (2-11). Since the nonzero parts $\boldsymbol{\lambda}[t]$ are greater to l_{t+1}, these new parts can only appear at the bottom of the partitions composing $\boldsymbol{\lambda}[t]$. The procedure for computing $\boldsymbol{\lambda}[t+1]$ from $\boldsymbol{\lambda}[t]$ consists of three steps:

(a) For $c = 0, \dots, l-1$, consider the integers

$$i_c = \min\{a \in \mathbb{N} \mid \boldsymbol{\lambda}[t]^c_a = 0\},$$

that is, the sequence of depths of the partitions appearing in $\boldsymbol{\lambda}$.

(b) Let $c_1, c_2, \dots, c_p \in \{0, \dots, l-1\}$ be such that

$$k_{t+1} \equiv 1 - i_{c_1} + v_{c_1} \equiv \cdots \equiv 1 - i_{c_p} + v_{c_p} \pmod{e},$$

with $p \geq a_{t+1}$. These integers must exist by (2-11), since $f^{-1}_{\mathbf{v}(\psi)}(\psi_{t+1}) \neq \varnothing$. Without loss of generality, we can assume $c_1 \blacktriangleleft \cdots \blacktriangleleft c_p$, where $\blacktriangleleft$ is the total order on $\{0, \dots, l-1\}$ such that

$$c \blacktriangleleft c' \iff \begin{cases} v_c - i_c < v_{c'} - i_{c'} \text{ or} \\ v_c - i_c = v_{c'} - i_{c'} \text{ and } c < c' \text{ as integers.} \end{cases} \tag{5-2}$$

(c) The problem reduces to determining a_{t+1} partitions among the partitions $\lambda[t]^{c_f}$, $f = 1, \dots, p$, which, once completed with a part l_{t+1}, yield an l-partition of $\Phi_e(\boldsymbol{v})$. Set $S[t+1] = \{(c_1, i_{c_1}), \dots, (c_{a_{t+1}}, i_{c_{a_{t+1}}})\}$. Let $\widehat{\boldsymbol{\lambda}}[t+1]$ be the l-partition defined by

$$\widehat{\boldsymbol{\lambda}}[t+1]^c_i = \begin{cases} \boldsymbol{\lambda}[t]^c_i & \text{if } (c, i) \notin S[t+1], \\ l_{t+1} & \text{if } (c, i) \in S[t+1]. \end{cases}$$

So $\widehat{\boldsymbol{\lambda}}[t+1]$ is obtained by adding a_{t+1} parts l_{t+1} at the bottom of the partitions $\boldsymbol{\lambda}[t]^c$ with $c \in \{c_1, \dots, c_{a_{t+1}}\}$. This means that the new parts are added at the bottom of the a_{t+1} first partitions considered following (5-2).

Lemma 5.3. *With the notation above,* $\widehat{\boldsymbol{\lambda}}[t+1] = \boldsymbol{\lambda}[t+1]$.

Proof. It suffices to prove that $\widehat{\boldsymbol{\lambda}}[t+1]$ belongs to $\Phi_e(\boldsymbol{v})$. Indeed, this will give $f_{\boldsymbol{v}}(\widehat{\boldsymbol{\lambda}}[t+1]) = f_{\boldsymbol{v}}(\boldsymbol{\lambda}[t+1]) = \psi_{t+1}$, and thus $\widehat{\boldsymbol{\lambda}}[t+1] = \boldsymbol{\lambda}[t+1]$, since $f_{\boldsymbol{v}}$ is an embedding. The second condition in Definition 2.9 of a FLOTW l-partition is clearly satisfied, since the multisegment ψ_{t+1} is aperiodic. We have to check the first condition. So assume for a contradiction that $\widehat{\boldsymbol{\lambda}}[t{+}1]$ does not satisfy (2-10).

(a) Suppose first that $\widehat{\boldsymbol{\lambda}}[t{+}1]^s_i < \widehat{\boldsymbol{\lambda}}[t{+}1]^{s+1}_{i+v_{s+1}-v_s}$, where $s \in \{1, \dots, l-1\}$ and i is a nonnegative integer. Since $\boldsymbol{\lambda}[t] \in \Phi_e(\boldsymbol{v})$, we have $\widehat{\boldsymbol{\lambda}}[t{+}1]^s_i = \boldsymbol{\lambda}[t]^s_i = 0$ and $\boldsymbol{\lambda}[t]^{s+1}_{i+v_{s+1}-v_s} = 0$, $\widehat{\boldsymbol{\lambda}}[t{+}1]^{s+1}_{i+v_{s+1}-v_s} = l_{t+1}$. Thus $(s{+}1, i+v_{s+1}-v_s) \in S[t{+}1]$. We have two cases to consider.

- Assume $i + v_{s+1} - v_s > 1$ and
$$\widehat{\boldsymbol{\lambda}}[t{+}1]^{s+1}_{i+v_{s+1}-v_s-1} = \boldsymbol{\lambda}[t]^{s+1}_{i+v_{s+1}-v_s-1} > 0.$$
Then $i = 1$ or $\widehat{\boldsymbol{\lambda}}[t{+}1]^s_{i-1} \neq 0$. Indeed we must have $\widehat{\boldsymbol{\lambda}}[t{+}1]^s_{i-1} = \widehat{\boldsymbol{\lambda}}[t]^s_{i-1} \geq \boldsymbol{\lambda}[t]^{s+1}_{i+v_{s+1}-v_s-1}$ because $\boldsymbol{\lambda}[t]$ belongs to $\Phi_e(\boldsymbol{v})$. We have $(s{+}1, i+v_{s+1}-v_s) \in S[t{+}1]$. In particular , $k_{t+1} \equiv v_{s+1} - (i + v_{s+1} - v_s) + 1 \equiv v_s - i + 1 \pmod e$. Since $\boldsymbol{\lambda}[t]^s_i = 0$, this means that $(s, i) \in S[t{+}1]$. But this is a contradiction, because by the second choice in (5-2), we should have $\widehat{\boldsymbol{\lambda}}[t{+}1]^s_i = l_{t+1} \neq 0$.
- Assume $i + v_{s+1} - v_s = 1$; then $i = 1$ and we have $v_{s+1} = v_s$. Thus $(s, i) \in S[t{+}1]$ and we derive a contradiction similarly.

(b) Now suppose we have $\widehat{\boldsymbol{\lambda}}[t{+}1]^{l-1}_i > \widehat{\boldsymbol{\lambda}}[t{+}1]^0_{i+v_0-v_{l-1}+e}$. The proof is analogous. We obtain that $(l-1, i) \in S[t{+}1]$ and $(0, i+v_0-v_{l-1}+e) \in S[t{+}1]$. This contradicts the first choice in (5-2). □

By using the procedure above, we are now able to compute the l-partitions $\boldsymbol{\lambda}[t]$, $t = 1, \dots, r$, from ψ and from its associated admissible multicharge $\boldsymbol{v}$. This gives a recursive algorithm for computing the admissible l-partition $\boldsymbol{\lambda}$ from ψ.

Example 5.4. Let $e = 4$. We consider the aperiodic multisegment

$$\psi = [0; 6) + [0; 5) + [3; 5) + [1; 4) + 2[3; 3) + [0; 3) + [2; 2) + [2; 1).$$

We have $\widehat{w}_0(\psi) = \widehat{R}\widehat{A}\widehat{R}\widehat{R}$, $\widehat{w}_1(\psi) = \widehat{A}\widehat{A}\widehat{R}$, $\widehat{w}_2(\psi) = \widehat{R}\widehat{R}\widehat{A}\widehat{A}\widehat{A}$ and $\widehat{w}_3(\psi) = \widehat{R}\widehat{R}\widehat{A}\widehat{R}\widehat{R}\widehat{A}$. This gives

$$\varepsilon^*_0(\psi) = 2, \quad \varepsilon^*_1(\psi) = 1, \quad \varepsilon^*_2(\psi) = 0, \quad \varepsilon^*_3(\psi) = 0.$$

Thus the multicharge $(0, 0, 1)$ is an admissible multicharge. Actually this is the one with minimal level. We now use the algorithm to compute the associated admissible l-partition $\boldsymbol{\lambda}$. Using the same notation, we successively obtain

$$\boldsymbol{\lambda}[0] = (\varnothing, \varnothing), \quad \boldsymbol{\lambda}[1] = (6, \varnothing, \varnothing), \quad \boldsymbol{\lambda}[2] = (6.5, 5, \varnothing), \quad \boldsymbol{\lambda}[3] = (6.5, 5, 4),$$
$$\boldsymbol{\lambda}[4] = (6.5, 5.3, 4.3.3), \quad \boldsymbol{\lambda}[5] = (6.5.2, 5.3, 4.3.3), \quad \boldsymbol{\lambda}[6] = (6.5.2, 5.3.1, 4.3.3).$$

This last is the admissible 3-partition associated to the multicharge $(0, 0, 1)$ and ψ. We easily check that

$$f_{(0,0,1)}(6.5.2, 5.3.1, 4.3.3) = \psi.$$

This means that the modules L_ψ and $\widetilde{D}^{\boldsymbol{\lambda}}$ are isomorphic.

The multicharge $(0, 0, 1, 2, 3)$ is another example of an admissible multicharge (with level 5) and its associated admissible multipartition is

$$\boldsymbol{\lambda} = (6.3, 5.3, 4.3, 2, 5.1).$$

6. Computation of the involution ♯

The generalized Mullineux involution. The twofold symmetry $i \leftrightarrow -i$ defines a skew crystal isomorphism from $B_e(\boldsymbol{v})$ to $B_e(\boldsymbol{v}^\sharp)$, where $\boldsymbol{v} = (v_0, \dots, v_{l-1})$ and $\boldsymbol{v}^\sharp = (-v_{l-1}, \dots, -v_0)$ belong to $\mathcal{V}_l$; see (2-9). Given $\boldsymbol{\lambda} \in \Phi_e(\boldsymbol{v})$, write $m_l^{\boldsymbol{v}}(\boldsymbol{\lambda}) \in \Phi_e(\boldsymbol{v}^\sharp)$ for the image of $\boldsymbol{\lambda}$ under this skew isomorphism. Ford and Kleshchev [1997] proved that for $l = 1$, the map $m_l^{\boldsymbol{v}}$ reduces to the Mullineux involution m_1 on e-restricted partitions. Thus we call $m_l^{\boldsymbol{v}}$ the generalized Mullineux involution.

From the start of Section 5 we know that the set $\Phi_e^K(\boldsymbol{v})$ of Kleshchev l-partitions also has the structure of an affine crystal isomorphic to $B_e(\boldsymbol{v})$. In particular the twofold symmetry $i \leftrightarrow -i$ also defines a bijection $m_l^{\boldsymbol{v},K}$ from $\Phi_e^K(\boldsymbol{v})$ to $\Phi_e^K(\boldsymbol{v}^\sharp)$. In [Jacon and Lecouvey 2009b], we gave an explicit procedure for finding $m_l^{\boldsymbol{v},K}$. Given $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{l-1}) \in \Phi_e^K(\boldsymbol{v})$, the l-partition $\boldsymbol{\mu} = m_l^{\boldsymbol{v},K}(\boldsymbol{\lambda})$ is obtained by computing first

$$\boldsymbol{\nu} = (m_1(\lambda^0), \dots, m_1(\lambda^{l-1})),$$

i.e., the l-partition obtained by applying the Mullineux map to each partition of $\boldsymbol{\lambda}$. In general $\boldsymbol{\nu}$ does not belong to $\Phi_e^K(\boldsymbol{v}^\sharp)$ and we have to apply a straightening algorithm, detailed in §4.3 of the cited paper, to obtain $\boldsymbol{\mu}$.

Now recall from (5-1) the bijection (in fact a crystal isomorphism)

$$\Gamma : \Phi_e(\boldsymbol{v}) \to \Phi_e^K(\boldsymbol{v}),$$

which can be made explicit by using the results in [Jacon and Lecouvey 2009a]. This allows one to compute the map $m_l^{\boldsymbol{v}}$, since

$$m_l^{\boldsymbol{v}} = \Gamma^{-1} \circ m_l^{\boldsymbol{v},K} \circ \Gamma. \tag{6-1}$$

Remark 6.1. The previous procedure yielding the generalized Mullineux map $m_l^{\boldsymbol{v}}$ can be optimized. In particular the conjugation by Γ can be avoided. Nevertheless, the pattern of the computation remains essentially the same: it uses the original Mullineux map m_1 and the results in [Jacon and Lecouvey 2009a] on affine crystal isomorphisms. Since it requires technical combinatorial developments that are not essential for our purposes, we omit it here.

Remark 6.2. In the case $e = \infty$, the map Γ is the identity and m_1 is simply the conjugation operation on the partitions. As observed in [Jacon and Lecouvey 2009b, §4.4], the algorithm for computing $m_l^{\boldsymbol{v}} = m_l^{\boldsymbol{v},K}$ then simplifies considerably.

The algorithm. Let $\psi \in \Psi_e$. To compute $\psi^\sharp$, we first determine an admissible multicharge $\boldsymbol{v}$ with respect to ψ and the associated admissible multipartition $\boldsymbol{\lambda}$. Then we apply the above algorithm to compute $m_l^{\boldsymbol{v}}(\boldsymbol{\lambda})$. It turns out that the complexity of this algorithm considerably increases with the level of $\boldsymbol{v}$. Hence, the use of the admissible multicharge $\boldsymbol{v}(\psi)$ with minimal level is preferable. Let us summarize the different steps of the procedure we have to apply to compute $\psi^\sharp$:

(1) For $i \in \mathbb{Z}/e\mathbb{Z}$, we compute $\varepsilon_i^*(\psi)$ using Theorem 4.1, which gives the equalities $\varepsilon_i^*(\psi) = \widehat{r_i}(\psi)$ for all $i \in \mathbb{Z}/e\mathbb{Z}$. We then put

$$\boldsymbol{v}(\psi) = (\underbrace{0, \dots, 0}_{\varepsilon_0^*(\boldsymbol{v})}, \underbrace{1, \dots, 1}_{\varepsilon_1^*(\boldsymbol{v})}, \dots, \underbrace{e-1, \dots, e-1}_{\varepsilon_{e-1}^*(\boldsymbol{v})}).$$

By Theorem 4.1, $\boldsymbol{v}(\psi)$ is an admissible multicharge in $\mathcal{V}_l$.

(2) Using the procedure in the previous section (pages 304–306), we compute the admissible FLOTW multipartition $\underline{\boldsymbol{\lambda}}(\psi)$ with respect to $\boldsymbol{v}(\boldsymbol{\lambda})$ and ψ.

(3) We compute the image $m_l^{\boldsymbol{v}}(\underline{\boldsymbol{\lambda}}(\psi))$ of $\underline{\boldsymbol{\lambda}}(\psi)$ under the generalized Mullineux involution (page 307).

(4) We finally obtain the aperiodic multisegment $\psi^\sharp = f_{\boldsymbol{v}(\psi)}(m_l^{\boldsymbol{v}}(\underline{\boldsymbol{\lambda}}(\psi)))$ by applying the map (2-12) defined on page 294.

Remark 6.3. In the case $e = \infty$, our algorithm for computing the Zelevinsky involution is essentially equivalent to that described in [Mœglin and Waldspurger 1986], except we use multipartitions rather than multisegments.

7. Further remarks

Computation of the Kashiwara involution. We established in Section 4 that the crystal operators $\widetilde{f}_i^*$ and $\widehat{f}_i$ coincide for any $i \in \mathbb{Z}/e\mathbb{Z}$. Given $\psi \in \Psi_e$, we can thus compute ψ^* by determining a path $\psi = \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} \varnothing$ in $B_e(\infty)$ from the empty multisegment to ψ. We then have $\psi^* = \widehat{f}_{i_1} \cdots \widehat{f}_{i_n} \varnothing$.

By combining the algorithm described in Section 6 for computing $\sharp$ with the relation $* = \tau = \rho \circ \sharp$, we obtain another procedure computing $*$ on $B_e(\infty)$, not requiring the determination of a path in the crystal $B_e(\infty)$.

Example 7.1. For $e = 2$, the involution $\sharp$ is just the identity, so the Kashiwara involution coincides with ρ on $B_e(\infty)$.

Crystal commutor for $B_e(\boldsymbol{v})\otimes B_e(\boldsymbol{v}')$. Kamnitzer and Tingley [2009] introduced a crystal commutor for any symmetrizable Kac–Moody algebra. Recall that a crystal commutor for $B_e(\boldsymbol{v}) \otimes B_e(\boldsymbol{v}')$ is a crystal isomorphism

$$\sigma_{\boldsymbol{v},\boldsymbol{v}'} : B_e(\boldsymbol{v}) \otimes B_e(\boldsymbol{v}') \to B_e(\boldsymbol{v}') \otimes B_e(\boldsymbol{v}).$$

This isomorphism is unique if and only if $B_e(\boldsymbol{v}) \otimes B_e(\boldsymbol{v}')$ does not contain two isomorphic connected components, that is, if the decomposition of the corresponding tensor product is without multiplicity. Such a crystal commutor is defined by specifying the images of the highest-weight vertices of $B_e(\boldsymbol{v})\otimes B_e(\boldsymbol{v}')$. It is easy to verify, using (2-7), that the highest-weight vertices of $B_e(\boldsymbol{v})\otimes B_e(\boldsymbol{v}')$ are precisely the vertices of the form $\varnothing \otimes \boldsymbol{\lambda}$, with $\boldsymbol{\lambda} \in B_e(\boldsymbol{v}')$, such that $\varepsilon_i(\boldsymbol{\lambda}) \leq r_i$ for any $i \in \mathbb{Z}/e\mathbb{Z}$ (r_i is the number of entries in $\boldsymbol{v}$ equal to i). Denote by $\mathcal{H}_{\boldsymbol{v},\boldsymbol{v}'}$ the set of highest-weight vertices in $B_e(\boldsymbol{v}) \otimes B_e(\boldsymbol{v}')$.

For any $\boldsymbol{\lambda} \in B_e(\boldsymbol{v})$, write for short $\boldsymbol{\lambda}^* = \underline{\boldsymbol{\lambda}}(f_{\boldsymbol{v}'}(\boldsymbol{\lambda})^*)$, where $\underline{\boldsymbol{\lambda}}$ is defined by (2-14). Since $*$ is an involution, we have $\boldsymbol{\lambda}^* \in B_e(\boldsymbol{w})$, where $\boldsymbol{w} \in \mathcal{V}_l$ is the multicharge having $\varepsilon_i(\boldsymbol{\lambda})$ entries equal to i and level $l=\sum_{i\in\mathbb{Z}/e\mathbb{Z}} \varepsilon_i(\boldsymbol{\lambda})$. The condition $\varepsilon_i(\boldsymbol{\lambda}) \leq r_i$ for any $i \in \mathbb{Z}/e\mathbb{Z}$ then implies that $\boldsymbol{\lambda}^* \in B_e(\boldsymbol{v})$. We have the following theorem which is the main result of [Kamnitzer and Tingley 2009].

Theorem 7.2. (1) *Assume $\varnothing \otimes \boldsymbol{\lambda} \in \mathcal{H}_{\boldsymbol{v},\boldsymbol{v}'}$. Then $\varnothing \otimes \boldsymbol{\lambda}^* \in \mathcal{H}_{\boldsymbol{v}',\boldsymbol{v}}$.*

(2) *The map $\sigma_{\boldsymbol{v},\boldsymbol{v}'} : \mathcal{H}_{\boldsymbol{v},\boldsymbol{v}'} \to \mathcal{H}_{\boldsymbol{v}',\boldsymbol{v}}$ taking $\varnothing\otimes\boldsymbol{\lambda}$ to $\varnothing\otimes\boldsymbol{\lambda}^*$ defines a crystal commutor for $B_e(\boldsymbol{v}) \otimes B_e(\boldsymbol{v}')$.*

The results from Sections 4 and 6 then allow one to compute the crystal commutor of Kamnitzer and Tingley for affine type A crystals.

Example 7.3. Assume $e = 2$. The crystal commutor $\sigma_{\boldsymbol{v},\boldsymbol{v}'}$ satisfies

$$\sigma_{\boldsymbol{v},\boldsymbol{v}'}(\varnothing \otimes \boldsymbol{\lambda}) = (\varnothing \otimes \underline{\boldsymbol{\lambda}}(\rho \circ f_{\boldsymbol{v}'}(\boldsymbol{\lambda})) \quad \text{for any } \varnothing \otimes \boldsymbol{\lambda} \in \mathcal{H}_{\boldsymbol{v},\boldsymbol{v}'}.$$

Acknowledgments

The authors are very grateful to the anonymous referee for a careful reading and for having pointing several inaccuracies in a previous version of this paper.

References

[Ariki 2006] S. Ariki, "Proof of the modular branching rule for cyclotomic Hecke algebras", *J. Algebra* **306**:1 (2006), 290–300. MR 2008m:20010 Zbl 1130.20005

[Ariki et al. 2008] S. Ariki, N. Jacon, and C. Lecouvey, "The modular branching rule for affine Hecke algebras of type A", preprint, 2008. arXiv 0808.3915

[Brundan and Kleshchev 2001] J. Brundan and A. Kleshchev, "Hecke–Clifford superalgebras, crystals of type $A_{2l}^{(2)}$ and modular branching rules for $\hat{S}_n$", *Represent. Theory* **5** (2001), 317–403. MR 2002j:17024 Zbl 1005.17010

[Chriss and Ginzburg 1997] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997. MR 98i:22021 Zbl 0879.22001

[Dipper et al. 1998] R. Dipper, G. James, and A. Mathas, "Cyclotomic q-Schur algebras", *Math. Z.* **229**:3 (1998), 385–416. MR 2000a:20033 Zbl 0934.20014

[Foda et al. 1999] O. Foda, B. Leclerc, M. Okado, J.-Y. Thibon, and T. A. Welsh, "Branching functions of $A_{n-1}^{(1)}$ and Jantzen–Seitz problem for Ariki–Koike algebras", *Adv. Math.* **141**:2 (1999), 322–365. MR 2000f:17036 Zbl 0930.17023

[Ford and Kleshchev 1997] B. Ford and A. S. Kleshchev, "A proof of the Mullineux conjecture", *Math. Z.* **226**:2 (1997), 267–308. MR 98k:20015a Zbl 0958.20018

[Grojnowski 1999] I. Grojnowski, "Affine $\widehat{\mathfrak{sl}}_p$ controls the modular representation theory of the symmetric group and related Hecke algebras", preprint, 1999. arXiv math/9907129

[Iwahori and Matsumoto 1965] N. Iwahori and H. Matsumoto, "On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups", *Inst. Hautes Études Sci. Publ. Math.* **25** (1965), 5–48. MR 32 #2486 Zbl 0228.20015

[Jacon 2004] N. Jacon, "On the parametrization of the simple modules for Ariki–Koike algebras at roots of unity", *J. Math. Kyoto Univ.* **44**:4 (2004), 729–767. MR 2006c:20008 Zbl 1085.20001

[Jacon and Lecouvey 2009a] N. Jacon and C. Lecouvey, "Crystal isomorphisms for irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of higher level", *Algebr. Represent. Theory* (2009).

[Jacon and Lecouvey 2009b] N. Jacon and C. Lecouvey, "On the Mullineux involution for Ariki–Koike algebras", *J. Algebra* **321**:8 (2009), 2156–2170. MR MR2501515 Zbl 05565626

[Jimbo et al. 1991] M. Jimbo, K. C. Misra, T. Miwa, and M. Okado, "Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}(n))$ at $q = 0$", *Comm. Math. Phys.* **136**:3 (1991), 543–566. MR 93a:17015 Zbl 0749.17015

[Kamnitzer and Tingley 2009] J. Kamnitzer and P. Tingley, "A definition of the crystal commutor using Kashiwara's involution", *J. Algebraic Combin.* **29**:2 (2009), 261–268. MR MR2475637 Zbl 05550567

[Kashiwara 1995] M. Kashiwara, "On crystal bases", pp. 155–197 in *Representations of groups (Banff, AB, 1994)*, CMS Conf. Proc. **16**, Amer. Math. Soc., Providence, RI, 1995. MR 97a:17016 Zbl 0851.17014

[Leclerc et al. 1999] B. Leclerc, J.-Y. Thibon, and E. Vasserot, "Zelevinsky's involution at roots of unity", *J. Reine Angew. Math.* **513** (1999), 33–51. MR 2001f:20011 Zbl 0949.17006

[Mœglin and Waldspurger 1986] C. Mœglin and J.-L. Waldspurger, "Sur l'involution de Zelevinski", *J. Reine Angew. Math.* **372** (1986), 136–177. MR 88c:22019 Zbl 0594.22008

[Uglov 2000] D. Uglov, "Canonical bases of higher-level q-deformed Fock spaces and Kazhdan–Lusztig polynomials", pp. 249–299 in *Physical combinatorics (Kyoto, 1999)*, Progr. Math. **191**, Birkhäuser, Boston, 2000. MR 2001k:17030 Zbl 0963.17012

[Vazirani 2002] M. Vazirani, "Parameterizing Hecke algebra modules: Bernstein–Zelevinsky multisegments, Kleshchev multipartitions, and crystal graphs", *Transform. Groups* **7**:3 (2002), 267–303. MR 2003g:20009 Zbl 1061.20007

[Vignéras 1997] M.-F. Vignéras, "À propos d'une conjecture de Langlands modulaire", pp. 415–452 in *Finite reductive groups* (Luminy, 1994), Progr. Math. **141**, Birkhäuser, Boston, 1997. MR 98b:22035

[Zelevinsky 1980] A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of GL(n)", *Ann. Sci. École Norm. Sup.* (4) **13**:2 (1980), 165–210. MR 83g:22012 Zbl 0441.22014

Received December 1, 2008. Revised April 22, 2009.

NICOLAS JACON
UNIVERSITÉ DE FRANCHE-COMTÉ
EQUIPE D'ALGÉBRE ET DE THÉORIE DES NOMBRES
UFR SCIENCES ET TECHNIQUES
16 ROUTE DE GRAY
25030 BESANÇON
FRANCE
njacon@univ-fcomte.fr

CÉDRIC LECOUVEY
UNIVERSITÉ DU LITTORAL — CÔTE D'OPALE
LABORATOIRE DE MATHÉMATIQUES PURES ET APPLIQUÉES JOSEPH LIOUVILLE
CENTRE UNIVERSITAIRE DE LA MI-VOIX
B.P. 699
62228 CALAIS
FRANCE
Cedric.Lecouvey@lmpa.univ-littoral.fr