*Pacific Journal of Mathematics*

# TORUS ACTIONS ON SMALL BLOWUPS OF $\mathbb{CP}^2$

LIAT KESSLER

Volume 244 No. 1 January 2010

PACIFIC JOURNAL OF MATHEMATICS
Vol. 244, No. 1, 2010

# TORUS ACTIONS ON SMALL BLOWUPS OF $\mathbb{CP}^2$

LIAT KESSLER

**A manifold obtained by $k$ simultaneous symplectic blowups of $\mathbb{CP}^2$ of equal sizes $\epsilon$ (where the size of $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is one) admits an effective two dimensional torus action if $k \leq 3$. We show that it does not admit such an action if $k \geq 4$ and $\epsilon \leq 1/(3k2^{2k})$. For the proof, we show a correspondence between the geometry of a symplectic toric four-manifold and the combinatorics of its moment map image. We also use techniques from the theory of J-holomorphic curves.**

## 1. Introduction

Let a torus $\mathbb{T}^\ell = (S^1)^\ell$ act effectively on a symplectic $2n$-dimensional manifold $(M, \omega)$. The action is called *Hamiltonian* if there exists a *moment map*, that is, a map

$$\Phi : M \to (\mathfrak{t}^\ell)^* = \mathbb{R}^\ell$$

that satisfies

$$d\Phi_i = -\iota(\xi_i)\omega$$

for $i = 1, \dots, \ell$, where $\xi_1, \dots, \xi_\ell$ are the vector fields that generate the $\mathbb{T}^\ell$-action. Unless said otherwise, we assume that $M$ is compact and connected. The image of the moment map,

$$\Delta := \Phi(M),$$

is then a convex polytope [Guillemin and Sternberg 1982].

If $\dim \mathbb{T}^\ell = \frac{1}{2} \dim M$, the triple $(M, \omega, \Phi)$ is a *symplectic toric manifold*, and the torus action is called *toric*. The moment map image is a *Delzant polytope*; this means that the edges emanating from each vertex are generated by vectors $v_1, \dots, v_n$ that span the lattice $\mathbb{Z}^n$. By the Delzant theorem, $(M, \omega, \Phi)$ is determined by $\Delta$ up to an equivariant symplectomorphism. Conversely, given a Delzant polytope $\Delta$ in $\mathbb{R}^n$, Delzant [1988] constructs a symplectic toric manifold $(M_\Delta, \omega_\Delta, \Phi_\Delta)$ whose moment map image is $\Delta$.

*MSC2000:* primary 53D20, 53D35, 53D45; secondary 57S15.
*Keywords:* torus action, symplectic manifold, dimension four, symplectic blowup, J-holomorphic curve, moment map, Delzant polygon.

As a result of Delzant's theorem and a combinatorial analysis of Delzant polygons, any symplectic toric four-manifold is obtained from either a standard $\mathbb{CP}^2$ or a Hirzebruch surface by a sequence of equivariant symplectic blowups. (See Lemma 2.9.) However, it may be difficult to determine whether a given symplectic four-dimensional manifold is symplectomorphic to such a manifold.

For instance, let $(M_k, \omega_\epsilon)$ be a symplectic manifold obtained from $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ by $k$ simultaneous symplectic blowups of equal sizes $\epsilon > 0$. (Our normalization convention for the Fubini–Study form $\omega_{\mathrm{FS}}$ is that the size of $\mathbb{CP}^1 \subset \mathbb{CP}^n$, $(1/2\pi)\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}}$, is equal to one.) If $k \geq 4$, this manifold does not admit a toric action that is *consistent with the blowups*, that is, the blowups cannot be performed equivariantly. (See Lemma 2.8.) Does it admit any other toric action?

In [Karshon and Kessler 2007] we show that the answer is "no" when $\epsilon$ is $1/n$ for an integer $n$. In this paper we show that the answer is "no" for $\epsilon \leq 1/(3k2^{2k})$, as a corollary of the following theorem.

**Theorem 1.1.** *If $(M_k, \omega_\epsilon)$ is symplectomorphic to $(M_\Delta, \omega_\Delta)$, for a Delzant polygon $\Delta$, and*

$$\epsilon \leq \frac{1}{3k2^{2k}},$$

*then $(M_\Delta, \omega_\Delta, \Phi_\Delta)$ can be obtained from $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ by $k$ equivariant symplectic blowups of equal size $\epsilon$.*

The theorem becomes false if we do not restrict $\epsilon$; for $\epsilon > \frac{1}{2}$, there is a toric action on $(M_1, \omega_\epsilon)$ that is not consistent with the $\epsilon$-blowup; see Remark 5.5. Theorem 1.1 can be strengthen to the case $\epsilon \leq \frac{1}{3}$; see [Pinsonnault 2008, Corollary 3.14; Kessler 2004, Theorem 3]. However, here we use different methods in the proof; in particular, our arguments illustrate explicitly the behavior of $J_T$-holomorphic curves and their moment map images. ($J_T$ denotes a $\mathbb{T}^2$-invariant complex structure on the manifold that is compatible with the symplectic form.) These novel arguments might be useful in other studies of torus actions on symplectic manifolds.

In proving Theorem 1.1, we apply Gromov's compactness theorem for J-holomorphic curves to show the existence of $J_T$-curves in the homology classes of exceptional divisors obtained by the symplectic $\epsilon$-blowups. In the case presented here, (as opposed to the case $\epsilon = \frac{1}{n}$ for an integer $n$), a priori these might be nonsmooth cusp curves. We claim that in one of these homology classes there is a smooth $J_T$-holomorphic sphere. To prove this claim, we represent $J_T$-holomorphic spheres and cusp curves on the boundary of the moment map image, and reduce the claim to a combinatorial claim on the moment map polygon. A key ingredient is Lemma 4.3, saying that a $J_T$-holomorphic sphere whose moment map image avoids a neighbourhood of a vertex in the moment map polygon $\Delta$ can be pushed, by a gradient flow, to a connected union of preimages of a chain of edges of $\Delta$.

The geometry-combinatorics correspondence is established in Section 2 and Section 4. The relevant results from Gromov's theory of J-holomorphic curves are recalled in Section 3.

To complete the proof of Theorem 1.1 by recursion, we need uniqueness of symplectic blowdowns: symplectic blowdowns along homologous curves result in symplectomorphic manifolds. This is shown in the appendix.

## 2. Reading geometric data from the moment map polygon

**2.1.** An important model for a Hamiltonian action is $\mathbb{C}^n$ with the standard symplectic form, the standard $\mathbb{T}^n$-action given by rotations of the coordinates, and the moment map

$$(z_1, \dots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \dots, |z_n|^2).$$

The image of this moment map is the positive orthant,

$$\mathbb{R}^n_+ = \{(s_1, \dots, s_n) \mid s_j \geq 0 \text{ for all } j\}.$$

A Delzant polytope can be obtained by gluing open subsets of $\mathbb{R}^n_+$ by means of elements of $\mathrm{AGL}(n, \mathbb{Z})$. ($\mathrm{AGL}(n, \mathbb{Z})$ is the group of affine transformations of $\mathbb{R}^n$ that have the form $x \mapsto Ax + \alpha$ with $A \in \mathrm{GL}(n, \mathbb{Z})$ and $\alpha \in \mathbb{R}^n$.) Similarly, a symplectic toric manifold can be obtained by gluing open $\mathbb{T}^n$-invariant subsets of $\mathbb{C}^n$ by means of equivariant symplectomorphisms and reparametrizations of $\mathbb{T}^n$.

**2.2.** The *rational length* of an interval $d$ of rational slope in $\mathbb{R}^n$ is the unique number $\ell = |d|$ such that the interval is $\mathrm{AGL}(n, \mathbb{Z})$-congruent to an interval of length $\ell$ on a coordinate axis. In what follows, intervals are always measured by rational length.

**2.3.** An almost complex structure on a $2n$-dimensional manifold $M$ is an automorphism of the tangent bundle, $J : TM \to TM$, such that $J^2 = -\mathrm{Id}$. It is *compatible* with a symplectic form $\omega$ if $\langle u, v\rangle = \omega(u, Jv)$ is symmetric and positive definite. The *first Chern class* of the symplectic manifold $(M, \omega)$ is defined to be the first Chern class of the complex vector bundle $(TM, J)$ and is denoted $c_1(TM)$. This class is independent of the choice of compatible almost complex structure $J$ [McDuff and Salamon 1998, Section 2.6].

**Lemma 2.4.** *Let $(M, \omega)$ be a compact connected symplectic four-manifold. Let $\Phi : M \to \mathbb{R}^2$ be a moment map for a toric action, and let $\Delta$ be its image.*

(1) *The moment map preimage of a vertex of $\Delta$ is a fixed point for the torus action, and the moment map image of a fixed point is a vertex of $\Delta$.*

(2) *Let $d$ be an edge of $\Delta$ of rational length $\ell$. Then its preimage, $\Phi^{-1}(d)$, is a symplectically embedded 2-sphere in $M$ of symplectic area*

$$\int_{\Phi^{-1}(d)} \omega = 2\pi\ell.$$

(3) *The* (*rational*) *perimeter of $\Delta$ is*

$$\text{perimeter } \Delta = \frac{1}{2\pi}\int_M \omega \wedge c_1(TM).$$

(4) *The area of $\Delta$ is*

$$\frac{1}{(2\pi)^2}\int_M \frac{1}{2!}\omega \wedge \omega.$$

For proof, see [Karshon et al. 2007, Lemma 2.2 and Lemma 2.10].

**Example 2.5.** Figure 1 shows examples of Delzant polygons with three and four edges. On the left there is a *Delzant triangle*,

$$\Delta_\lambda = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq \lambda\}.$$

This is the moment map image of the standard toric action $(a, b) \cdot [z_0 : z_1 : z_2] = [z_0 : az_1 : bz_2]$ on $\mathbb{CP}^2$, with the Fubini–Study symplectic form normalized so that the symplectic area of $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is $2\pi\lambda$. The rational lengths of all its edges is $\lambda$.

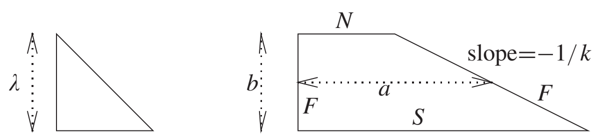


**Figure 1.** A Delzant triangle, $\Delta_\lambda$, and a Hirzebruch trapezoid, $\text{Hirz}_{a,b,k}$.

On the right there is a *Hirzebruch trapezoid*,

$$\text{Hirz}_{a,b,k} = \left\{(x_1, x_2) \,\middle|\, -\frac{b}{2} \leq x_2 \leq \frac{b}{2}, 0 \leq x_1 \leq a - kx_2\right\},$$

where $b$ is the height of the trapezoid, $a$ is its average width, and $k$ is a nonnegative integer such that the east edge has slope $-1/k$ or is vertical if $k = 0$. We assume that $a \geq b$ and that $a - k\frac{b}{2} > 0$. This trapezoid is a moment map image of a standard toric action on a Hirzebruch surface. The rational lengths of its west and east edges are $b$; the rational lengths of its north and south edges are $a \pm kb/2$.

**2.6.** Let $\Delta$ be a Delzant polytope in $\mathbb{R}^n$, let $v$ be a vertex of $\Delta$, and let $\delta > 0$ be smaller than the rational lengths of the edges emanating from $v$. The edges of $\Delta$ emanating from $v$ have the form $\{v + s\alpha_j \mid 0 \leq s \leq \ell_j\}$ where the vectors $\alpha_1, \ldots, \alpha_n$

generate the lattice $\mathbb{Z}^n$ and $\delta < \ell_j$ for all $j$. The *corner chopping of size* $\delta$ of $\Delta$ at $v$ is the polytope $\widetilde{\Delta}$ obtained from $\Delta$ by intersecting with the half-space

$$\{ v + s_1\alpha_1 + \cdots + s_n\alpha_n \mid s_1 + \cdots + s_n \geq \delta\}.$$

See, for example, the chopping of the top right corner in Figure 2. The resulting polytope $\widetilde{\Delta}$ is again a Delzant polytope. The corner chopping operation commutes with $\mathrm{AGL}(n, \mathbb{Z})$-congruence: if $\widetilde{\Delta}$ is obtained from $\Delta$ by a corner chopping of size $\delta > 0$ at a vertex $v \in \Delta$ then, for any $g \in \mathrm{AGL}(n, \mathbb{Z})$, the polytope $g(\widetilde{\Delta})$ is obtained from the polytope $g(\Delta)$ by a corner chopping of size $\delta$ at the vertex $g(v)$.

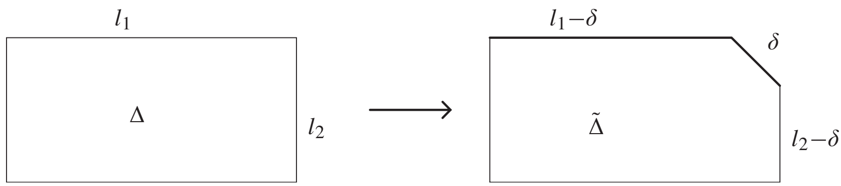


**Figure 2.** Corner chopping.

**2.7.** Recall that a blowup of *size* $\epsilon = r^2/2$ of a $2n$-dimensional symplectic manifold $(M, \omega)$ is a new symplectic manifold $(\widetilde{M}, \tilde{\omega})$ that is constructed in the following way. Let $\Omega \subset \mathbb{C}^n$ be an open subset that contains a ball about the origin of radius greater than $r$, and let $i : \Omega \to M$ be a symplectomorphism onto an open subset of $M$. (We consider $\mathbb{C}^n$ with the standard symplectic form.) The standard symplectic blowup of $\Omega$ of size $r^2/2$ is obtained by removing the open ball $B^{2n}(r)$ of radius $r$ about the origin and collapsing its boundary along the Hopf fibration $\partial B^{2n}(r) \to \mathbb{CP}^{n-1}$; the resulting space is naturally a smooth symplectic manifold [McDuff and Salamon 1998, Section 7.1]. This blowup transports to $M$ through $i$. The resulting copy of $(\mathbb{CP}^{n-1}, \epsilon\omega_{\mathrm{FS}})$ in $\widetilde{M}$ is called the *exceptional divisor*.

If $M$ admits an action of a torus $\mathbb{T}^\ell$, and $i : \Omega \to M$ is $\mathbb{T}^\ell$-equivariant, where $\mathbb{T}^\ell$ acts on $\Omega$ through some homomorphism $\mathbb{T}^\ell \to U(n)$, then the torus action naturally extends to the symplectic blowup of $M$ obtained from $i$, and the blowup is *equivariant*. If the action on $M$ is Hamiltonian, its moment map naturally extends to the blowup; in the case $\ell = n$ we call this a *toric blowup*.

The moment map image of the standard symplectic blowup of $\mathbb{C}^n$ of size $\epsilon$ is obtained from the moment map image $\mathbb{R}^n_+$ of $\mathbb{C}^n$ by corner chopping of size $\epsilon$. See Figure 3 for $n = 2$.

A toric blowup of size $\epsilon$ of a symplectic toric manifold $(M, \omega, \Phi)$ at a fixed point $p$ amounts to chopping off a corner of size $\epsilon$ of its moment map image $\Delta$ at the vertex $v = \Phi(p)$ to get a new polytope $\widetilde{\Delta}$. The preimage of the resulting new facet in $\widetilde{\Delta}$ is the exceptional divisor in $\widetilde{M}$.

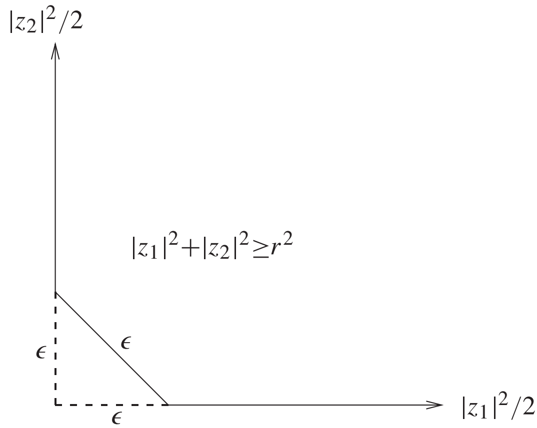


**Figure 3.** Blowup of $\mathbb{C}^2$ of size $\epsilon = r^2/2$.

We restrict our attention to symplectic toric manifolds of dimension 4. Chopping off a corner of size $\epsilon$ of a polygon $\Delta$ can be done if and only if there exist two adjacent edges in $\Delta$ whose rational lengths are both strictly greater than $\epsilon$. As a result, starting from a Delzant triangle of size 1 we can perform one corner chopping of size $\epsilon > 0$ if and only if $\epsilon < 1$, two or three corner choppings of size $\epsilon > 0$ if and only if $\epsilon < \frac{1}{2}$, and no more than three corner choppings of the same size. Therefore:

**Lemma 2.8.** $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ *admits a toric blowup of size* $\epsilon > 0$ *if and only if* $\epsilon < 1$. $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ *admits two or three toric blowups of size* $\epsilon > 0$ *if and only if* $\epsilon < \frac{1}{2}$. $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ *does not admit four or more toric blowups of equal sizes.*

For a detailed proof, see [Karshon and Kessler 2007, Lemma 3.1].

In $\mathbb{R}^2$, all Delzant polygons can be obtained by a simple recursive recipe:

**Lemma 2.9.** (1) *Let* $\Delta$ *be a Delzant polygon with three edges. Then there exists a unique* $\lambda > 0$ *such that* $\Delta$ *is* $\mathrm{AGL}(2,\mathbb{Z})$*-congruent to the Delzant triangle* $\Delta_\lambda$. (See Example 2.5.)

(2) *Let* $\Delta$ *be a Delzant polygon with four or more edges. Let* $s$ *be the nonnegative integer such that the number of edges is* $4+s$. *Then there exist positive numbers* $a \geq b > 0$, *an integer* $0 \leq k \leq 2a/b$, *and positive numbers* $\delta_1, \ldots, \delta_s$, *such that* $\Delta$ *is* $\mathrm{AGL}(2,\mathbb{Z})$*-congruent to a Delzant polygon that is obtained from the Hirzebruch trapezoid* $\mathrm{Hirz}_{a,b,k}$ (see Example 2.5) *by a sequence of corner choppings of sizes* $\delta_1, \ldots, \delta_s$.

*Proof.* See [Fulton 1993, Section 2.5 and Notes to Chapter 2]. □

**2.10.** For any Delzant polygon $\Delta$, consider the free Abelian group generated by its edges:

$$\mathbb{Z}[\text{edges of } \Delta].$$

The "length functional"

$$\mathbb{Z}[\text{edges of } \Delta] \to \mathbb{R}$$

is the homomorphism that associates to each basis element its rational length. If $\Delta_{i+1}$ is obtained from $\Delta_i$ by a corner chopping, we consider the injective homomorphism

$$\mathbb{Z}[\text{edges of } \Delta_i] \hookrightarrow \mathbb{Z}[\text{edges of } \Delta_{i+1}] \tag{1}$$

whose restriction to the generators is defined in the following way. If $d$ is an edge of $\Delta$ that does not touch the corner that was chopped, then $d$ is mapped to the edge of $\Delta_{i+1}$ with the same outward normal vector. If $d$ is an edge of $\Delta_i$ that touches the corner that was chopped, then $d$ is mapped to $d+e$ where $e$ is the new edge of $\Delta_{i+1}$, created in the chopping.

The definition of corner chopping in 2.6 implies that the homomorphism (1) respects the length homomorphisms.

By induction and the definition of corner chopping we get the following lemma.

**Lemma 2.11.** *Let*

$$\Delta_0, \Delta_1, \ldots, \Delta_s$$

*be a sequence of Delzant polygons such that, for each $i$, the polygon $\Delta_i$ is obtained from the polygon $\Delta_{i-1}$ by a corner chopping of size $\delta_i$.*

(1) *The image of an edge $d$ of $\Delta_j$ by $s-j$ iterations of homomorphism (1) is a linear combination $\sum_{i=0}^{\ell} m_i c_i$, such that $c_0, \ldots, c_\ell$ are edges of $\Delta_s$ whose union $U_d$ is connected, $\ell \leq (s-j)$, and for $0 \leq i \leq \ell$, the coefficient $m_i$ is a nonnegative integer that is less than or equal to $2^{s-j}$; we say that $d$ is* given *by the chain $U_d$ with multiplicities $m_0, \ldots, m_\ell$.*

(2) area $\Delta_s =$ area $\Delta_0 - \frac{1}{2}\delta_1^2 - \cdots - \frac{1}{2}\delta_s^2$.

(3) perimeter $\Delta_s =$ perimeter $\Delta_0 - \delta_1 - \cdots - \delta_s$.

**Lemma 2.12.** *Let $(M, \omega, \Phi)$ be a four-dimensional symplectic toric manifold, with moment-map polygon $\Delta$ of $n$ edges. Then there are $n-2$ edges of $\Delta$ whose union is connected, such that the classes of their $\Phi$-preimages form a basis to $H_2(M;\mathbb{Z})$. Moreover, for any $n-2$ edges of $\Delta$ whose union is connected, the classes of their preimages form a basis to $H_2(M;\mathbb{Z})$.*

*Proof.* By Lemma 2.9, we can prove this by induction. In the induction step, suppose that $(\widetilde{M}, \tilde{\omega}, \tilde{\Phi})$ with moment map polygon $\widetilde{\Delta}$ of $n+1$ edges is obtained by a toric blowup of $(M, \omega, \Phi)$ with moment map polygon $\Delta$. Let $B_\Delta$ be a set of $n-2$ edges of $\Delta$ whose union is connected, such that the classes of their $\Phi$-preimages form a basis to $H_2(M; \mathbb{Z})$. If $B_\Delta$ consists of an edge that touches the corner that was chopped, set $B_{\widetilde{\Delta}}$ to be the edges of $\widetilde{\Delta}$ with the same outward normal vector as the edges in $B_\Delta$ plus the new edge $e$ of $\widetilde{\Delta}$, created in the chopping. If none of the edges in $B_\Delta$ touches the corner that was chopped, set $B_{\widetilde{\Delta}}$ to be the edges of $\widetilde{\Delta}$ with the same outward normal vector as the edges in $B_\Delta$ plus one of the edges adjacent to $e$ in $\widetilde{\Delta}$. □

**Corollary 2.13.** *Let $(M, \omega, \Phi)$ be a four-dimensional symplectic toric manifold, with moment-map polygon $\Delta$. The number of edges of $\Delta$ is equal to the second Betti number* $\dim H_2(M)$ *plus two.*

By the Delzant theorem, every toric action on $\mathbb{CP}^2$ is obtained from a symplectomorphism of $\mathbb{CP}^2$ with a symplectic toric manifold $M_\Delta$ that is associated to a Delzant polygon $\Delta$. By Corollary 2.13, $\Delta$ must be a triangle. By part (1) of Lemma 2.9, $\Delta$ is $\mathrm{AGL}(2, \mathbb{Z})$-congruent to a Delzant triangle $\Delta_\lambda$. (See Example 2.5.) By our normalization convention for the Fubini–Study form, $\lambda = 1$. It follows that:

**Lemma 2.14.** *Every toric $\mathbb{T}^2$-action on $\mathbb{CP}^2$ is equivariantly symplectomorphic to the standard action.*

## 3. $J$-holomorphic spheres in symplectic 4-manifolds

In this section we will highlight results from the theory of J-holomorphic curves that we will use for the proof of Lemma 4.3, and to show uniqueness of symplectic blowdowns in the appendix.

Let $(M, \omega)$ be a compact symplectic manifold. Let $\mathcal{J} = \mathcal{J}(M, \omega)$ be the space of almost complex structures on $M$ that are compatible with $\omega$. The space $\mathcal{J}$ is contractible [McDuff and Salamon 1998]. Given $J \in \mathcal{J}$, a *parametrized $J$-holomorphic sphere* is a map $u : \mathbb{CP}^1 \to M$, such that $du : T\mathbb{CP}^1 \to TM$ satisfies the Cauchy–Riemann equation $du \circ i = J \circ du$. Such a $u$ represents a homology class in $H_2(M; \mathbb{Z})$ that we denote $[u]$. A $J$-holomorphic sphere is called *simple* if it cannot be factored through a branched covering of $\mathbb{CP}^1$. One similarly defines a holomorphic curve in $(M, J)$ whose domain is a Riemann surface other than $\mathbb{CP}^1$.

For any class $A \in H_2(M; \mathbb{Z})$, consider the universal moduli space of simple parametrized holomorphic spheres in the class $A$,

$$\mathcal{M}(A, \mathcal{J}) = \{(u, J) \mid J \in \mathcal{J}, u : \mathbb{CP}^1 \to M \text{ is simple } J\text{-holomorphic, and } [u] = A\},$$

and the projection map

$$p_A : \mathcal{M}(A, \mathcal{J}) \to \mathcal{J}.$$

For a fixed $J \in \mathcal{J}$, we denote by $\mathcal{M}(A, J)$ the space $p_A^{-1}(J)$.

The automorphism group $\mathrm{PSL}(2, \mathbb{C})$ of $\mathbb{CP}^1$ acts on $\mathcal{M}(A, \mathcal{J})$ by reparametrizations. The quotient $\mathcal{M}(A, \mathcal{J})/\mathrm{PSL}(2, \mathbb{C})$ is the space of unparametrized J-holomorphic spheres representing $A \in H_2(M)$.

**Lemma 3.1.** *Let* $0 \neq A \in H_2(M; \mathbb{Z})$. *The action of* $G = \mathrm{PSL}(2, \mathbb{C})$ *on* $\mathcal{M}(A, \mathcal{J})$ *is free and proper.*

*Proof.* For any sphere $u \in \mathcal{M}(A, \mathcal{J})$, the stabilizer $G_u = \{\psi \in G \mid u \circ \psi = u\}$ is trivial, since $u$ is simple; this proves that the action is free.

We now need to show that the action map $(u, \psi) \mapsto (u, u \circ \psi)$ is proper. Let $K \subset \mathcal{M}(A, \mathcal{J}) \times \mathcal{M}(A, \mathcal{J})$ be a compact subset. Without loss of generality $K = K_1 \times K_2$, where $K_1$ and $K_2$ are compact in $\mathcal{M}(A, \mathcal{J})$. Because $\mathcal{M}(A, \mathcal{J})$ is Hausdorff and first countable, it is enough to show that for every sequence $\{(u_n, \psi_n)\}$ in the preimage of $K_1 \times K_2$ there exists a subsequence such that $\{\psi_n\}$ converges uniformly and $\{u_n\}$ converges in the $C^\infty$ topology. Take such a sequence $\{u_n, \psi_n\}$. Because $u_n \in K_1$ and $K_1$ is compact, after passing to a subsequence we may assume that $\{u_n\}$ $C^\infty$-converges.

By [McDuff and Salamon 2004, Lemma D.1.2], if the sequence $\psi_n$ does not have a uniformly convergent subsequence, then there exist points $x, y \in \mathbb{CP}^1$ and a subsequence $\psi_\mu$ which converges to the point $y$ uniformly in compact subsets of $\mathbb{CP}^1 \setminus \{x\}$. In particular $\psi_\mu$ converges to a point on a half sphere, hence $u_\mu \circ \psi_\mu$, restricted to a half sphere, converge to a constant map. However, the sequence of holomorphic spheres $\{u_n \circ \psi_n\}$, (as a sequence in the compact subset $K_2$ of $\mathcal{M}(A, \mathcal{J})$), has a $C^\infty$-convergent (hence u.c.s.-convergent) subsequence whose limit is in the nontrivial homology class $A$, and we get a contradiction. □

Gromov [1985] introduced a notion of weak convergence of a sequence of holomorphic curves. This notion is preserved under reparametrization of the curve, and it implies convergence in homology. *Gromov's compactness theorem* guarantees that, given a converging sequence of almost complex structures, a corresponding sequence of holomorphic curves with bounded symplectic area has a weakly converging subsequence. The limit under weak convergence might not be a curve; it might be a cusp curve, that is, a connected union of holomorphic curves. As a result of Gromov's compactness, we have the following lemma.

**Lemma 3.2.** *Let* $\{J_n\} \subset \mathcal{J}$ *be a sequence of almost complex structures that converges in the* $C^\infty$ *topology to an almost complex structure* $J_\infty \in \mathcal{J}$. *For each* $n$, *let* $f_n : \mathbb{CP}^1 \to M$ *be a parametrized* $J_n$*-holomorphic sphere. Suppose that the set of areas* $\omega([f_n])$ *is bounded. Then one of the following two possibilities occurs.*

(1) *There exist a* $J_\infty$*-holomorphic sphere* $u : \mathbb{CP}^1 \to M$ *and elements* $A_n \in \mathrm{PSL}(2, \mathbb{C})$ *such that a subsequence of the* $f_n \circ A_n$*'s converges to* $u$ *in the* $C^\infty$ *topology. In particular, there exist infinitely many* $n$*'s for which* $[f_n] = [u]$.

(2) *There exist two or more $J_\infty$-holomorphic spheres $u_\ell : \mathbb{CP}^1 \to M$ that are nonconstant and simple and positive integers $m_\ell$, for $\ell = 1, \ldots, L$, and infinitely many n's for which*

$$[f_n] = \sum_{\ell=1}^{L} m_\ell [u_\ell] \quad \text{in } H_2(M).$$

For details, see [Karshon et al. 2007, Lemma A.3].

In the proof of Lemma 4.3, we will use the following Lemma.

**Lemma 3.3.** *Let $(M, \omega)$ be a closed symplectic four-manifold. Let $E \in H_2(M; \mathbb{Z})$ be a homology class that can be represented by an embedded symplectic sphere and such that $c_1(TM)(E) = 1$. Then for every $J \in \mathcal{J}$ there exists a $J$-holomorphic cusp curve in the class $E$.*

To deduce the lemma from Gromov's compactness we need the existence of a dense set $U \subset \mathcal{J}$ such that for any $J \in U$, the class $E$ is represented by an embedded $J$-holomorphic sphere.

For any positive number $K$, let

$$\mathcal{N}_K = \{A \in H_2(M; \mathbb{Z}) \mid A \neq 0, c_1(TM)(A) \leq 0, \text{ and } \omega(A) < K\}.$$

The importance of this set lies in the fact that if a homology class $E$ with $\omega(E) \leq K$ and $c_1(TM)(E) \leq 1$ is represented by a J-holomorphic cusp curve with two or more components, then at least one of these components must lie in a homology class in $\mathcal{N}_K$; see Lemma A.5 in [Karshon et al. 2007]. Let

$$U_K = \mathcal{J} \setminus \bigcup_{A \in \mathcal{N}_K} \text{image } p_A.$$

Let $(M, \omega)$ be a compact symplectic four-manifold. Then the subset $U_K \subset \mathcal{J}$ is open, dense, and path connected. This is proved in [McDuff 1990, Lemma 3.1; 1991, Section 3] and presented in [Karshon et al. 2007, Lemma A.8 and Lemma A.10]. The following is also shown in [Karshon et al. 2007, Lemma A.12].

**Lemma 3.4.** *Let $(M, \omega)$ be a compact symplectic four-manifold. Let $E \in H_2(M)$ be a homology class that can be represented by an embedded symplectic sphere and such that $c_1(TM)(E) = 1$.*

(1) *The projection map $p_E : \mathcal{M}(E, \mathcal{J}) \to \mathcal{J}$ is open.*

(2) *Let $K \geq \omega(E)$. Then, for any $J \in U_K$, the class $E$ is represented by an embedded $J$-holomorphic sphere.*

Lemma 3.3 now follows.

For the proof of Theorem 1.1, we also need the following lemmas.

**Lemma 3.5.** *Let $(M, \omega)$ be a compact symplectic four-manifold. Let $A \in H_2(M;\mathbb{Z})$ be a homology class which is represented by an embedded symplectic sphere $C$.*

(1) *There exists an almost complex structure $J_0 \in \mathcal{J}$ for which $C$ is a $J_0$-holomorphic sphere.*

(2) *For any $J \in \mathcal{J}$ and any simple parametrized $J$-holomorphic sphere $f : \mathbb{CP}^1 \to M$ in the class $A$, the map $f$ is an embedding.*

plus 1pt plus 1pt The lemma is a consequence of the adjunction formula. For details and references see, for example, [Karshon and Kessler 2007, Lemma 5.3].

**Lemma 3.6.** *Let $(M, \omega)$ be a compact symplectic four-manifold. Let $A \in H_2(M;\mathbb{Z})$ be a homology class that is represented by an embedded symplectic sphere, and such that $c_1(TM)(A) = 1$. Let $J \in \operatorname{image} p_A$, and $(u, J) \in \mathcal{M}(A, J)$.*

*If $A = \sum_{i=1}^{n} m_i[u_i]$, where each component $u_i$ is a simple parametrized $J$-holomorphic sphere and $m_i \in \mathbb{N}$, then all the components but one must be constants, and the nonconstant component differs from $u$ by reparametrization of $\mathbb{CP}^1$.*

*Proof.* By Lemma 3.5, $u$ is an embedding, so the adjunction equality

$$0 = 2 + A \cdot A - c_1(TM)(A)$$

holds; since $c_1(TM)(A) = 1$ this implies $A \cdot A = -1$. If $n > 1$ and there is more than one nonconstant component, then for $1 \le i \le n$, $\omega([u]) > \omega([u_i])$ so $u \neq u_i$, hence by positivity of intersections of J-holomorphic spheres in a four-manifold [McDuff and Salamon 2004, Theorem 2.6.3], $[u_i] \cdot [u] \ge 0$. Thus $0 \le \sum_{i=1}^{n} m_i([u_i] \cdot [u]) = A \cdot A$, in contradiction to $A \cdot A = -1$.

Thus, all the components but one must be constants. By a similar argument, the nonconstant component differs from $u$ at most by reparametrization of $\mathbb{CP}^1$. □

**Lemma 3.7.** *Let $(M, \omega)$ be a closed symplectic four-manifold. Let $E \in H_2(M;\mathbb{Z})$ be a homology class that can be represented by an embedded symplectic sphere and such that $c_1(TM)(E) = 1$. Let*

$$U_E = \operatorname{image} p_E.$$

(1) *$U_E \subset \mathcal{J}$ is open, dense, and path connected. Between any two elements in $U_E$ there is a path in $U_E$ that is transversal to $p_E$.*

(2) *The map*

$$\widetilde{p_E} : \mathcal{M}(E, \mathcal{J}) / \operatorname{PSL}(2, \mathbb{C}) \to U_E$$

*induced from the projection map $p_E$ is proper.*

(3) *For $J_0, J_1 \in U_E$, the sets $\mathcal{M}(E, J_0)/\operatorname{PSL}(2, \mathbb{C})$ and $\mathcal{M}(E, J_1)/\operatorname{PSL}(2, \mathbb{C})$ consist each of a single point, and there exists a path $\{J_t\}_{0 \le t \le 1}$ such that*

$$\mathcal{W}(E; \{J_t\}) = \{(u_t, J_t) \mid u_t \in \mathcal{M}(E, J_t)\} / \operatorname{PSL}(2, \mathbb{C})$$

*is a compact one-dimensional manifold, and each $u_t$ is an embedding.*

*Proof.* (1) Since $p_E$ is an open map by Lemma 3.4(1), its image $U_E$ is an open set in $\mathcal{J}$. Set $K = \omega(E)$. By part (2) of Lemma 3.4, $U_K \subseteq U_E$. Since $U_K$ is dense in $\mathcal{J}$, so is $U_E$. Since $U_E$ is open, $\mathcal{J}$ locally path connected, and $U_K$ is dense in $U_E$ and path connected, we get that $U_E$ is path connected.

By the regularity criterion of Hofer–Lizan–Sikorav [1997], any element in $U_E$ is a regular value for $p_E$. A path between regular values for $p_E$ can be perturbed to a path with the same endpoints that is transversal to $p_E$; see [McDuff and Salamon 2004, Theorem 3.1.7(ii); Karshon et al. 2007, Lemma A.9(d)].

(2) This follows from Gromov's compactness in the following way. Let $D \subset U_E$ be a compact subset. We need to show that $p_E^{-1}(D)/\operatorname{PSL}(2, \mathbb{C})$ is compact. Because $\mathcal{M}(E, \mathcal{J})$ is Hausdorff and first countable, it is enough to show that for every sequence $\{(f_n, J_n)\}$ in $p_E^{-1}(D)$ there exists a subsequence that, after reparametrization, has a limit in $p_E^{-1}(D)$ in the $C^\infty$ topology.

Take such a sequence, $\{(f_n, J_n)\}$. Because $J_n \in D$ and $D$ is compact and contained in $U_E$, after passing to a subsequence we may assume that $\{J_n\}$ converges to $J_\infty \in U_E$. Each $f_n$ is a $J_n$-holomorphic sphere in the class $E$. Suppose that there exists a subsequence that, after reparametrization, converges to some $u : \mathbb{CP}^1 \to M$ in the $C^\infty$ topology. Then $u$ must be in the class $E$ and it must be $J_\infty$-holomorphic. If $u$ is not simple, we get a contradiction to Lemma 3.6. Then the pair $(u, J_\infty)$ is in the moduli space $\mathcal{M}(E, \mathcal{J})$, and since $J_\infty \in D$, this pair is in $p_E^{-1}(D)$.

Now suppose that there does not exist such a subsequence. Then there exist two or more nonconstant simple $J_\infty$-holomorphic spheres $u_\ell : \mathbb{CP}^1 \to M$ and positive integers $m_\ell$ such that $\sum m_\ell[u_\ell] = E$, by Lemma 3.2. This contradicts Lemma 3.6.

(3) For $J \in U_E = \operatorname{image} p_E$, the set $\mathcal{M}(E, J) = p_E^{-1}(J) \neq \varnothing$. Hence, by Lemma 3.6, the set $\mathcal{M}(E, J)/\operatorname{PSL}(2, \mathbb{C})$ consists of a single point. For $J_0, J_1 \in U_E$, by part (1), there is a path $\{J_t\}$ in $U_E$ from $J_0$ to $J_1$, that is transversal to $p_E$. Hence, by [McDuff and Salamon 2004, Theorem 3.1.7], $\mathcal{W}^*(E; \{J_t\}) = \{(u_t, J_t) \mid u_t \in \mathcal{M}(E, J_t)\}$ is a manifold of dimension $1 + 6 = 1 + \operatorname{index} p_E$. By Lemma 3.1, the action of $\operatorname{PSL}(2, \mathbb{C})$ on $\mathcal{W}^*(E; \{J_t\})$ is free and proper, thus

$$\mathcal{W}(E; \{J_t\}) = \{(u_t, J_t) \mid u_t \in \mathcal{M}(E, J_t)\}/\operatorname{PSL}(2, \mathbb{C})$$

is a manifold of dimension one. $\mathcal{W}(E; \{J_t\})$ is the inverse image of the path $\{J_t\}$ under the map $\widetilde{p_E}$, hence, by part (2), it is compact.

By Lemma 3.5, each $u_t$ is an embedding. □

## 4. Representing $J_T$-holomorphic curves on the moment map polygon

**Notation.** For $(M, \omega, \Phi)$, let $J_T$ denote a $\mathbb{T}^n$-invariant complex structure on $M$ that is compatible with $\omega$. By Delzant's construction [1988], such a structure exists.

**Claim 4.1.** *Let $(M, \omega, \Phi)$ be a four-dimensional symplectic toric manifold, with moment-map polygon $\Delta$. The preimage under $\Phi$ of an edge $d$ of $\Delta$ is an embedded $J_T$-holomorphic sphere.*

*Proof.* By part (2) of Lemma 2.4, $Y = \Phi^{-1}(d)$ is a symplectically embedded 2-sphere in $M$. Being a connected component of a fixed point set of a holomorphic $S^1$-action, $TY = J_T TY$. As an almost complex manifold of real dimension two, $(Y, J_T|_{TY})$ is a complex manifold. Thus the embedded sphere $Y$ is an embedded holomorphic sphere in the complex manifold $(M, J_T)$. □

**Lemma 4.2.** *Let $(M, \omega, \Phi)$ be a four-dimensional symplectic toric manifold, with moment-map polygon $\Delta$.*

- *Any $J_T$-holomorphic sphere is homologous in $H_2(M; \mathbb{Z})$ to a linear combination with coefficients in $\mathbb{N}$ of inverse images under $\Phi$ of edges of $\Delta$.*
- *For any set $S$ of $n-2$ edges whose union is connected, any simple $J_T$-holomorphic sphere $C$ that is not the preimage of an edge of $\Delta$ is homologous to a linear combination with coefficients in $\mathbb{N}$ of preimages of edges of $\Delta$ whose union is connected and that are contained in $S$; if the intersection of $C$ with each of the two edges of $\Delta$ that are not in $S$ is positive, then all the $n-2$ edges of $S$ appear with positive coefficients in this linear combination.*

*Proof.*

- Let $\Psi$ be an $S^1$-moment map obtained by composing $\Phi$ with projection in a rational direction along which there is not any edge of $\Delta$. Denote by $v_{\min}$ ($v_{\max}$) the vertex of minimal (maximal) value of that projection. Let $D_1, \ldots, D_m$ be a chain of $\Phi$-preimages of edges between $v_{\min}$ and $v_{\max}$. Let $D'_1, \ldots, D'_{m'}$ be the other chain of $\Phi$-preimages of edges between $v_{\min}$ and $v_{\max}$.

  Without loss of generality we assume that $C$ is a simple $J_T$-holomorphic sphere that is not the $\Phi$-preimage of an edge of $\Delta$. By Lemma 2.12, in $H_2(M; \mathbb{Z})$
  $$[C] = \sum_{i=1}^{m} a_i D_i + \sum_{j=1}^{m'} b_j D'_j, \quad \text{with } a_1 = b_1 = 0.$$
  Adapting the proof of Lemma C.6 in [Karshon 1999] we get that
  $$a_{i+1}/k_{i+1} \geq a_i/k_i \geq 0, \quad \text{for } 1 \leq i < m \quad (1 \leq i < m'), \tag{2}$$
  where $k_i$ is the order of the stabilizer of the $i$-th sphere in a chain.

  Notice that (2) implies that
  $$a_\ell > 0 \Rightarrow a_i > 0 \text{ for all } \ell \leq i \leq m,$$
  $$b_\ell > 0 \Rightarrow b_j > 0 \text{ for all } \ell \leq j \leq m'.$$

Hence, $C$ is homologous in $H_2(M;\mathbb{Z})$ to a linear combination with coefficients in $\mathbb{N}$ of inverse images under $\Phi$ of, at most $n-2$, edges of $\Delta$ whose union is connected.

- It is enough to observe that for any set $S$ of $n-2$ edges whose union is connected, there is an $S^1$-moment map $\Psi$, obtained by composing $\Phi$ with projection in a rational direction along which there is not any edge of $\Delta$, such that the vertex $v_{\min}$ is the vertex between the two edges of $\Delta$ that are not in $S$. Then the previous proof gives the required. □

**Lemma 4.3.** *Let $(M,\omega,\Phi)$ be a four-dimensional symplectic toric manifold with moment-map polygon $\Delta$. Let $J_T$ be a $\mathbb{T}^2$-invariant $\omega$-compatible complex structure on $M$, and $g_T$ be the Riemannian metric defined by $(\omega, J_T)$. Let $\imath^*$ be a projection in a rational direction along which there is not any edge of $\Delta$. Let $v_{\min}$ be the vertex of $\Delta$ of minimal value of that projection.*

*Let $C$ be a $J_T$-holomorphic sphere such that $\Phi(C)$ avoids the vertex $v_{\min}$. Let $\alpha$ and $\beta$ be the points of $\Phi(C)$ on the boundary of $\Delta$, that are closest to $v_{\min}$ from left and right. Let $v_\alpha$ and $v_\beta$ be the vertices following $\alpha$ and $\beta$. Then the gradient flow of $\Psi=\imath^*\circ\Phi$ with respect to $g_T$ carries $C$ to a family of $J_T$-holomorphic spheres; this family weakly converges to a connected union of preimages of edges of $\Delta$ (maybe with multiplicities). These edges form a chain that we denote $L_C$. The vertices of $L_C$ closest to $v_{\min}$ from left and right are $v_\alpha$ and $v_\beta$.*

*Proof.* The function $\Psi=\imath^*\circ\Phi : M\to\mathbb{R}$ is a moment map associated with a Hamiltonian action on $(M,\omega)$ of $S^1$ embedded in $\mathbb{T}^2$ by $\imath : S^1\hookrightarrow\mathbb{T}^2$.

Let $\xi_M$ be the vector field generating the $S^1$-action. The gradient flow $\eta_t$ of $\Psi$ with respect to the invariant metric $g_T$ is generated by $-J_T\xi_M$. This flow is equivariant with respect to the action, that is, for each $t$, the diffeomorphism $\eta_t : M\to M$ is $\mathbb{T}^2$-equivariant. Consequently, it sends a set that is a $\Phi$-preimage of a vertex or a $\Phi$-preimage of an edge to itself.

Set $L$ to be the chain of edges of $\Delta$ that do not touch $v_{\min}$. Let

$$B=\{p\in M : i^*\circ\Phi(p)>r\}$$

for some $i^*(v_{\min})<r<\min\{i^*(v'),i^*(v'')\}$, where $v'$ ($v''$) is the vertex following $v_{\min}$ immediately from the left (right). Then $\bigcap_{t>0}(\eta_t(B))\supseteq\Phi^{-1}L$. On the other hand, a point $p\in B$ that is not in $\Phi^{-1}(L)$, is sent to $v_{\min}$ by the gradient flow $\eta_t$ as $t\to-\infty$, that is, for $t'$ big enough, $q=\eta_{-t'}(p)$ is not in $B$. Since $\eta_{t'}$ is a diffeomorphism, there cannot be $b\in B$ such that $\eta_{t'}(b)=\eta_{t'}(q)=\eta_0(p)=p$, in particular, $p$ is not in the intersection $\bigcap_{t>0}(\eta_t(B))$. So

$$\bigcap_{t>0}(\eta_t(B))=\Phi^{-1}(L).$$

We choose $B$ big enough such that, for some complex coordinates, the complexified toric action on $M-B$ is the standard action of the complex torus on an open subset of $\mathbb{C}^2$. In particular, for $t_1, t_2$ close to 0, if $t_1 > t_2 > 0$, $\eta_{t_1}(M-B) \supset \eta_{t_2}(M-B)$, hence $\eta_{t_1}(B) \subset \eta_{t_2}(B)$. Since $\eta_t$ is a flow, (that is, a homomorphism from $(\mathbb{R}, +)$ to $(\mathrm{Diff}, \circ)$), this implies that for any $t_1 > t_2 > 0$, $\eta_{t_1}(B) \subset \eta_{t_2}(B)$, that is, $\eta_t$ is monotonic on $B$.

Now, choose $B$ such that, in addition to the above, its image contains $\Phi(C)$. Consider a sequence $\{C_i\}$, where $C_i = \eta_i(C)$, with discrete $i \to \infty$. Each $C_i$ is a $J_T$-holomorphic sphere in the homology class $[C]$. By Gromov's compactness theorem, there is a subsequence $\{C_\mu\}$ that weakly converges to a $J_T$-holomorphic (maybe nonsmooth) cusp curve $C'$ in $[C]$. In particular, each point in the limit $C'$ is the limit of a sequence of points in $\{C_\mu\}$, hence, since $C_\mu = \eta_\mu(C) \subset \eta_\mu(B)$, and $\eta_t$ is monotonic on $B$, we get that $C' \subset \cap_\mu(\eta_\mu(B)) \subset \Phi^{-1}(L)$. Thus, since each edge preimage is an irreducible $J_T$-holomorphic sphere in the complex manifold $(M, J_T)$ (by Claim 4.1), the irreducible components of $C'$ are preimages of edges in $L$. We conclude that the cusp curve $C'$ is a connected union of preimages of the edges of a subchain $L_C$ of $L$, with positive multiplicities.

Let $p_\alpha$ ($p_\beta$) be the preimage of $v_\alpha$ ($v_\beta$) in $M$. The chain $L_C$ includes $v_\alpha$ and $v_\beta$, as the limits of $\eta_\mu(p_\alpha)$ and $\eta_\mu(p_\beta)$. Assume a vertex $v$ on $L_C$ is closer to $v_{\min}$ from the left than $v_\alpha$. Let $e_v$ be the edge that touches $v$ from below. Then $L_C$ intersects $e_v$ at $v$, hence $\Phi^{-1}(L_C)$ intersects $\Phi^{-1}(e_v)$ at the point $\Phi^{-1}(v)$, maybe with multiplicities. However $C \cap \Phi^{-1}(e_v) = \varnothing$, in contradiction to $[\Phi^{-1}(L_C)] = [C]$. Similarly, the vertex on $L_C$ closest to $v_{\min}$ from the right is $v_\beta$. □

**Claim 4.4.** *Let $(M, \omega, \Phi)$ be a four-dimensional symplectic toric manifold with moment-map polygon $\Delta$.*

*Every $J_T$-cusp curve $C$ is homologous in $H_2(M;\mathbb{Z})$ to a linear combination with coefficients in $\mathbb{N}$ of preimages of edges of $\Delta$ whose union is connected. In particular, $C$ is homologous to a $\mathbb{T}^2$-invariant $J_T$-cusp curve.*

We already know that a $J_T$-cusp curve $C$ is homologous to a linear combination with coefficients in $\mathbb{N}$ of preimages of edges of $\Delta$ (by applying the first part of Lemma 4.2 to the components of the cusp curve). However, the union of these edges might not be connected. The "connected" part that we add here plays an important role in the proof of Theorem 1.1.

*Proof.* Let $\mathrm{i}^*$ be a projection in a rational direction along which there is not any edge of $\Delta$. Let $v_{\min}$ be the vertex of $\Delta$ of minimal value of $\mathrm{i}^*$. If for any component of $C$ that is not a $\Phi$-preimage of an edge of $\Delta$, the moment map image avoids a neighbourhood of $v_{\min}$, then the claim follows from Lemma 4.3 (and the fact that $C$ is connected). Otherwise, there is such a component $D$; by positivity of intersections, the intersection number of $D$ with the preimage of each of the two edges adjacent to $v_{\min}$ is positive. Thus, by the second part of Lemma 4.2, $D$ is homologous to a linear combination with coefficients in $\mathbb{N}$ of $\Phi$-preimages of all the edges of $\Delta$ but the two adjacent to $v_{\min}$. By the first part of Lemma 4.2, each component of $C$ is homologous to a linear combination of $\Phi$-preimages of edges of $\Delta$ with coefficients in $\mathbb{N}$. Combining such representatives of $D$ and the other components of $C$ gives the claim. □

**Lemma 4.5.** *Let $(M, \omega, \Phi)$ be a symplectic toric four-manifold with moment map polygon $\Delta$. Let $C$ be an embedded symplectic sphere in $(M, \omega)$ which satisfies $c_1(TM)(C) = 1$.*

*Then $C$ is homologous in $H_2(M; \mathbb{Z})$ to a linear combination with coefficients in $\mathbb{N}$ of preimages of edges of $\Delta$ whose union is connected.*

*Proof.* By Lemma 3.3 there exists a $J_T$-holomorphic cusp curve in the class $[C]$. Now apply Claim 4.4. □

## 5. No toric action on $(M_k, \omega_\epsilon)$ for $k > 3$ and small $\epsilon$

For $\epsilon > 0$, denote by

$$(M_k, \omega_\epsilon)$$

a symplectic manifold that is obtained from $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ by $k$ simultaneous symplectic blowups of equal sizes $\epsilon$. For description of symplectic blowup, see 2.7. The $k$ *simultaneous blowups* are obtained from embeddings $i_1 : \Omega_1 \to M, \ldots, i_k : \Omega_k \to M$ whose images are disjoint. We denote by $E_1, \ldots, E_k$ the homology classes in $H_2(M_k; \mathbb{Z})$ of the exceptional divisors obtained by the blowups, and by $L$ the homology class of a line $\mathbb{CP}^1 \subset M_k$.

**5.1.** By McDuff and Polterovich [1994], for $k \leq 8$ there exists a symplectic blowup of $\mathbb{CP}^2$ $k$ times by size $\epsilon$ if and only if $\epsilon$ satisfies the following conditions. If $k = 2, 3, 4$: $\epsilon < \frac{1}{2}$. If $k = 5, 6$: $\epsilon < \frac{2}{5}$. If $k = 7$: $\epsilon < \frac{3}{8}$. If $k = 8$: $\epsilon < \frac{6}{17}$. According to Biran [1997], for $k \geq 9$, there exist $k$ symplectic blowups of equal sizes $\epsilon$ if and only if $\epsilon$ satisfies the volume constraint, that is, $\epsilon < 1/\sqrt{k}$.

Assume that $(M_k, \omega_\epsilon)$ admits a toric action with moment map polygon $\Delta$. By Lemma 4.5, each $E_i$ can be represented by a linear combination with coefficients in $\mathbb{N}$ of preimages of edges of $\Delta$. We call the union of these edges, with the $\mathbb{N}$-multiplicities, a $\Delta$-*representative* of $E_i$. If this union is connected, we call it a *connected $\Delta$-representative*. We observe the following properties of $\Delta$-representatives of $E_1, \ldots, E_k$.

**Claim 5.2.** *Assume that $(M_k, \omega_\epsilon)$ admits a toric action with moment map image $\Delta$. Choose $\Delta$-representatives for $E_1, \ldots, E_k$. For $m \leq k$, the number of edges in the union of the $\Delta$-representatives of $m$ different $E_i$'s is $> m$, unless each of these $\Delta$-representatives is a single edge with multiplicity one.*

*Proof.* Assume that the union of the chosen $\Delta$-representatives of $E_1, \dots, E_m$ (without loss of generality) is a subset of the set of edges $C_1, \dots, C_m$, that is, in $H_2(M_k;\mathbb{Z})$, for $1 \le i \le m$,

$$E_i = \sum_{j=1}^{m} a^i_j [\Phi^{-1} C_j], \quad a^i_j \in 0 \cup \mathbb{N}. \tag{3}$$

Denote by $A$ the $m \times m$ matrix of the coefficients $a^i_j$. Since the homology classes $E_1, \dots, E_m$ are independent, the matrix $A$ is invertible (over $\mathbb{R}$). We get that

$$([\Phi^{-1}C_1], \dots, [\Phi^{-1}C_m])^t = A^{-1}(E_1, \dots, E_m)^t. \tag{4}$$

The homology classes $L, E_1, \dots, E_k$ form a basis of $H_2(M_k;\mathbb{Z})$, therefore each $[\Phi^{-1}C_j] = d_j L + \sum_i b^j_i E_i$, with unique integers as coefficients. The coefficients do not change if we write $[\Phi^{-1}C_j]$ as a linear combination of $L$, $E_i$ in $H_2(M_k;\mathbb{R})$. By this and (4), all the entries of $A^{-1}$ are in $\mathbb{Z}$, so in $H_2(M_k;\mathbb{Z})$,

$$[\Phi^{-1}C_j] = \sum_{i=1}^{m} b^j_i E_i, \quad b^j_i \in \mathbb{Z}.$$

Since the size of each $E_i$ is $\epsilon$ we deduce that the length $|C_j|$ of each $C_j$ is an integer multiple of $\epsilon$. Since $|C_j| > 0$, it must be a multiple of $\epsilon$ by $N_j \in \mathbb{N}$. However, by (3), for $1 \le i \le m$,

$$\epsilon = \sum_{j=1}^{m} a^i_j |C_j|, \quad a^i_j \in 0 \cup \mathbb{N}.$$

Thus

$$\epsilon = \sum_{j=1}^{m} a^i_j N_j \epsilon, \quad a^i_j \in 0 \cup \mathbb{N},\ N_j \in \mathbb{N}.$$

We get that in each line (and each column) of (the invertible matrix) $A$ there is 1 in one entry and 0 in each of the other entries, that is, each of the $\Delta$-representatives is a single edge with multiplicity one. □

**Claim 5.3.** *Assume that $(M_k, \omega_\epsilon)$ admits a toric action with moment map image $\Delta$. Choose connected $\Delta$-representatives for $E_1, \dots, E_k$. Denote their union by $U$. If none of the chosen connected $\Delta$-representatives is a single edge of $\Delta$ with multiplicity one, then $U$ is connected and consists of at least $k+1$ edges.*

*Proof.* By Claim 5.2, $U$ consists of more than $k$ edges. Assume that $U$ is disconnected. Then it consists of at most $k+1$ edges, hence it consists of exactly $k+1$ edges out of the $k+3$ edges of $\Delta$. Since none of the $\Delta$-representatives is a single edge, Claim 5.2 implies that the $m_j$ edges of a connected component $j$ support at most $m_j - 1$ of the $E_i$'s. Thus the nonconnected $k+1$ edges support at most

$\sum_{j=1}^{c}(m_j - 1) = k + 1 - c < k$ of these classes, where $c > 1$ is the number of connected components, and we get a contradiction. □

For a convex polygon $\Delta$ in $\mathbb{R}^2$, we denote by

$$(M_\Delta, \omega_\Delta, \Phi_\Delta)$$

a symplectic toric manifold whose moment map image is $\Delta$.

The main ingredient of the proof of Theorem 1.1 is:

**Claim 5.4.** *If $(M_k, \omega_\epsilon)$ is symplectomorphic to $(M_\Delta, \omega_\Delta)$, and*

$$\epsilon \leq \frac{1}{3k2^{2k}},$$

*then one of the classes $E_1, \dots, E_k$ is realized by an embedded $\mathbb{T}^2$-invariant symplectic exceptional sphere; equivariantly blowing down along it yields $(M_{k-1}, \omega_\epsilon)$ with a toric action.*

*Proof.* If $k \geq 1$, the moment map image $\Delta$ is a Delzant polygon of $k + 3 \geq 4$ edges, so by Lemma 2.9, up to $\mathrm{AGL}(2, \mathbb{Z})$-congruence, it is obtained by $(k - 1)$ corner-choppings of sizes $(\delta_1, \dots, \delta_{k-1})$ from a standard Hirzebruch trapezoid $\Sigma$ with west and east edges $F_w$, $F_e$, south edge $S$, north edge $N$, and slope $-1/d$.

By part (1) of Lemma 2.11,

$$|S| + |N| < 2^k \text{ perimeter } \Delta, \tag{5}$$

and $F_w$ and $F_e$ are given by two disjoint connected unions of edges of $\Delta$ with multiplicities $\leq 2^k$.

For each class $E_i$, we choose a connected $\Delta$-representative, that is a connected union of edges (with multiplicities in $\mathbb{N}$) whose preimage is in $E_i$. Assume that none of these $\Delta$-representatives is a single edge of $\Delta$ with multiplicity one. By Claim 5.3, the union $U$ of these $\Delta$-representatives is connected and consists of at least $k + 1$ edges of the $k + 3$ edges of $\Delta$. Then, (at least) one of the two chains of edges giving $F_w$ and $F_e$ as above is contained in $U$ : the connected at most two edges that are not in $U$ can overlap at most one chain giving $F_w$ or $F_e$, since the two chains are separated at each end by an edge. Thus

$$|F| = |F_w| = |F_e| \leq 2^k k\epsilon. \tag{6}$$

Then

$$\tfrac{1}{2}(1 - k\epsilon^2) = \text{area } \Delta = \tfrac{1}{2}(|S| + |N|)|F| - \sum_{i=1}^{k-1} \tfrac{1}{2}{\delta_i}^2 \leq \tfrac{1}{2} 2^k |F| \text{ perimeter } \Delta - \sum_{i=1}^{k-1} \tfrac{1}{2}{\delta_i}^2$$
$$= \tfrac{1}{2} 2^k (3 - k\epsilon)|F| - \sum_{i=1}^{k-1} \tfrac{1}{2}{\delta_i}^2 \leq \tfrac{1}{2} 2^k (3 - k\epsilon) 2^k k\epsilon - \sum_{i=1}^{k-1} \tfrac{1}{2}{\delta_i}^2.$$

The first (in)equality holds by part (4) of Lemma 2.4, the second holds by part (2) of Lemma 2.11, the third inequality holds by Equation (5), the fourth follows from part (3) of Lemma 2.4 and the fact that the Poincare dual to $c_1(TM_k)$ equals $3L - \sum_{i=1}^{k} E_i$, and the last holds by Equation (6).

We get that

$$1 - k\epsilon^2 \leq 2^{2k}(3 - k\epsilon)(k\epsilon) \leq 2^{2k}(3k\epsilon - k\epsilon^2).$$

So, $1 \leq 2^{2k}3k\epsilon - k\epsilon^2(2^{2k} - 1)$, thus $1 < 3k2^{2k}\epsilon$, in contradiction to the assumption on $\epsilon$. Therefore, (at least) one of the classes $E_1, \dots, E_k$ is represented by the inverse image under the moment map of a single edge of $\Delta$ with multiplicity one. By Claim 4.1, such a representative $C_T$ is an embedded $J_T$-holomorphic sphere. It is $\mathbb{T}^2$-invariant: let $a \in \mathbb{T}^2$; because $\mathbb{T}^2$ is connected, $[aC_T] = [C_T]$; by positivity of intersections and since $E_i \cdot E_i = -1$, $aC_T$ and $C_T$ must coincide. Because $C_T$ is an embedded $J_T$-sphere and $J_T$ is compatible with $\omega_\epsilon$, $C_T$ is symplectic.

Without loss of generality, the class $E_1$ is represented by such a $J_T$-holomorphic sphere $C_T$. Set $J_0$ to be an almost complex structure on $(M_k, \omega_\epsilon)$ for which the exceptional divisors obtained by the symplectic blowups are disjoint embedded $J_0$-holomorphic spheres $S_1, \dots, S_k$ that represent the classes $E_1, \dots, E_k$. (Such a structure exists by Lemma 3.5.) By Lemma A.1 in the appendix, the symplectic manifold resulting from $(M_k, \omega_\epsilon)$ by blowing down along $C_T$ is symplectomorphic to the symplectic manifold obtained by blowing down along $S_1$, which is $(M_{k-1}, \omega_\epsilon)$. □

*Proof of Theorem 1.1.* Assume that $(M_k, \omega_\epsilon)$ is symplectomorphic to $(M_\Delta, \omega_\Delta)$ and $\epsilon \leq 1/(3k2^{2k})$. After $k$ iterations of Claim 5.4, we get $\mathbb{CP}^2$ with a toric action. By Lemma 2.14, this manifold is equivariantly symplectomorphic to $\mathbb{CP}^2$ with its standard toric action. By reversing our steps we get $\mathbb{CP}^2$ blown up equivariantly $k$ times by equal sizes $\epsilon$. □

**Remark 5.5.** Theorem 1.1 becomes false if we do not restrict $\epsilon$. For $\epsilon > \frac{1}{2}$, let $(M_1, \omega_\epsilon, \Phi_1)$ be $\mathbb{CP}^2$ blown up equivariantly by size $\epsilon$. The moment map image is obtained by chopping off a corner of size $\epsilon$ from a Delzant triangle of edge-size 1, to get a trapezoid $\text{Hirz}_{(1+\epsilon)/2,1-\epsilon,1}$, that is, of height $(1-\epsilon)$, average width $(1+\epsilon)/2$, and slope $-1$. Let $(N, \omega_2, \Phi_2)$ be a Hirzebruch surface whose image is a trapezoid $\text{Hirz}_{(1+\epsilon)/2,1-\epsilon,3}$ (Notice that the north edge is then of size $2\epsilon - 1$, which is $> 0$ if and only if $\epsilon > \frac{1}{2}$.) See Figure 4. Since these Hirzebruch trapezoids have the same average width and height and the inverse of their slopes differ by 2, the corresponding manifolds are isomorphic as symplectic manifolds with Hamiltonian $S^1$-action (by [Karshon 2003, Lemma 3]), however they are not isomorphic as symplectic toric manifolds (their moment map polygons are not equivalent).

Theorem 1.1 and Lemma 2.8 yield the following corollary.

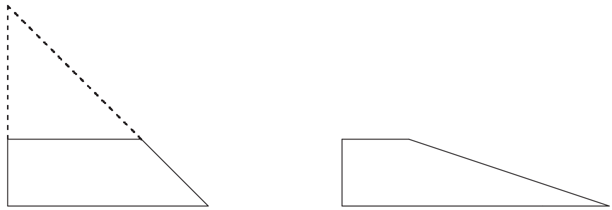

**Figure 4.** Symplectomorphic but not equivariantly symplectomorphic symplectic toric manifolds.

**Corollary 5.6.** *$(M_k, \omega_\epsilon)$ with $\epsilon \leq 1/(3k2^{2k})$ admits a toric action if and only if $k \leq 3$.*

By the sharpness of the constrains listed in 5.1, when $\epsilon \leq 1/(3k2^{2k})$ there exists a symplectic blowup of $\mathbb{CP}^2$ $k$ times by size $\epsilon$.

Since $H^1(M_k, \mathbb{R}) = \{0\}$, any effective $(S^1)^2$-action on $(M_k, \omega_\epsilon)$ is toric.

**Corollary 5.7.** *$(M_k, \omega_\epsilon)$ with $\epsilon \leq 1/(3k2^{2k})$ admits an effective $(S^1)^2$-action if and only if $k \leq 3$.*

## Appendix: Uniqueness of blowdown

**Lemma A.1.** *Let $(M, \omega)$ be a compact four-dimensional symplectic manifold. Let $J_0, J_1 \in \mathcal{J}$. Let $A$ be a class in $H_2(M; \mathbb{Z})$ such that $c_1(TM)(A) = 1$ and $\omega(A) > 0$. Assume that $A$ is represented by an embedded $J_0$-holomorphic sphere $C_0$ and by an embedded $J_1$-holomorphic sphere $C_1$.*

*Then for $i = 0, 1$, there are neighbourhoods $U_i$ of $C_i$, each symplectomorphic to a tubular neighbourhood of $\mathbb{CP}^1$, and a symplectomorphism $\phi$ of $(M, \omega)$, that sends $(U_0, C_0)$ to $(U_1, C_1)$, and induces the identity map on $H_2(M; \mathbb{Z})$.*

*Proof.* By part (3) of Lemma 3.7, there is a smooth family (with parameter $0 \leq t \leq 1$) of $J_t$-holomorphic embeddings $\rho_t$ from $\mathbb{CP}^1$ to the manifold. Their images are all in the homology class $A$. Notice that the pullbacks of $\omega$ to $\mathbb{CP}^1$ by the homotopic maps are all in the same cohomology class. Hence, by Moser, there is a family of diffeomorphisms $\phi_t : \mathbb{CP}^1 \to \mathbb{CP}^1$, starting at the identity map, that satisfy $\phi_t^*(\rho_0^*(\omega)) = \rho_t^*(\omega)$. Hence we may assume that $\rho_0$ is a symplectic embedding of the standard $\mathbb{CP}^1$ and compose the embeddings $\{\rho_t\}$ on the family $\{\phi_t\}$ to get a one-parameter family of symplectic embeddings of the standard $\mathbb{CP}^1$ into $M$. Moreover, using a parametrized version of Weinstein's tubular neighbourhood theorem, this family can be extended to a one-parameter family of symplectic embeddings $\sigma_t$ of a neighbourhood of $\mathbb{CP}^1$ (as the zero-section) in the tautological bundle with a symplectic form, into $M$; denote the image of $\sigma_t$ by $U_t$.

We get a "partial flow" that moves along the neighbourhoods $U_t$. Differentiating it by $t$ gives vector fields $X_t$, defined at $U_t$. The Lie derivative $\mathrm{Lie}_{X_t}\,\omega$ is 0. By Cartan's formula,

$$\mathrm{Lie}_{X_t}\,\omega = d(\iota_{X_t}\omega) + (\iota_{X_t})d\omega = d\iota_{X_t}\omega,$$

where the last equality holds since $\omega$ is closed. Thus the one form $\iota_{X_t}\omega$ on $U_t$ is closed. Therefore, and since $\mathbb{CP}^1$ is simply connected, when we consider $X_t$ as a vector field defined at a neighbourhood of $\mathbb{CP}^1 \times [0, 1] \subseteq M \times [0, 1]$, we get a function $h$ defined on a (maybe smaller) neighbourhood of $\mathbb{CP}^1 \times [0, 1] \subseteq M \times [0, 1]$, such that $\iota_{X_t}\omega = dh_t$. Using partition of unity in $M \times [0, 1]$, we expand $h$ to a smooth function $H : M \times [0, 1] \to \mathbb{R}$, whose restriction to a small neighborhood of image $\rho_t$ coincides with $h_t$.

This gives a Hamiltonian flow on $M$, thus a family of symplectomorphisms $\{\alpha_t\}_{0\le t\le 1}$, starting from the identity map. Take $\alpha_1$ to be $\phi$. □

## Acknowledgments

I am grateful to Yael Karshon and Leonid Polterovich for giving the right answers and asking the right questions, and to Dusa McDuff for helpful discussions.

## References

[Biran 1997] P. Biran, "Symplectic packing in dimension 4", *Geom. Funct. Anal.* **7**:3 (1997), 420–437. MR 98i:57057 Zbl 0892.53022

[Delzant 1988] T. Delzant, "Hamiltoniens périodiques et images convexes de l'application moment", *Bull. Soc. Math. France* **116**:3 (1988), 315–339. MR 90b:58069 Zbl 0676.58029

[Fulton 1993] W. Fulton, *Introduction to toric varieties*, Annals of Math. Studies **131**, Princeton Univ. Press, 1993. The William H. Roever Lectures in Geometry. MR 94g:14028 Zbl 0813.14039

[Gromov 1985] M. Gromov, "Pseudoholomorphic curves in symplectic manifolds", *Invent. Math.* **82**:2 (1985), 307–347. MR 87j:53053 Zbl 0592.53025

[Guillemin and Sternberg 1982] V. Guillemin and S. Sternberg, "Convexity properties of the moment mapping", *Invent. Math.* **67**:3 (1982), 491–513. MR 83m:58037 Zbl 0503.58017

[Hofer et al. 1997] H. Hofer, V. Lizan, and J.-C. Sikorav, "On genericity for holomorphic curves in four-dimensional almost-complex manifolds", *J. Geom. Anal.* **7** (1997), 149–159. MR 2000d:32045 Zbl 0911.53014

[Karshon 1999] Y. Karshon, *Periodic Hamiltonian flows on four-dimensional manifolds*, Mem. Amer. Math. Soc. **141**, Amer. Math. Soc., Providence, R.I., 1999. MR 2000c:53113 ZBL 0982.70011

[Karshon 2003] Y. Karshon, "Maximal tori in the symplectomorphism groups of Hirzebruch surfaces", *Math. Res. Lett.* **10**:1 (2003), 125–132. MR 2004f:53101 Zbl 1036.53063

[Karshon and Kessler 2007] Y. Karshon and L. Kessler, "Circle and torus actions on equal symplectic blow-ups of $\mathbb{CP}^2$", *Math. Res. Lett.* **14**:5 (2007), 807–823. MR 2009c:53121 Zbl 1139.53041

[Karshon et al. 2007] Y. Karshon, L. Kessler, and M. Pinsonnault, "A compact symplectic four-manifold admits only finitely many inequivalent toric actions", *J. Symplectic Geom.* **5**:2 (2007), 139–166. MR 2008m:53199 Zbl 1136.53060

[Kessler 2004] L. Kessler, *Torus actions on small blow ups of* $\mathbb{CP}^2$, thesis, The Hebrew University of Jerusalem, 2004.

[McDuff 1990] D. McDuff, "The structure of rational and ruled symplectic 4-manifolds", *J. Amer. Math. Soc.* **3**:3 (1990), 679–712. MR 91k:58042 Zbl 0723.53019

[McDuff 1991] D. McDuff, "Blow ups and symplectic embeddings in dimension 4", *Topology* **30**:3 (1991), 409–421. MR 92m:57039 Zbl 0731.53035

[McDuff and Polterovich 1994] D. McDuff and L. Polterovich, "Symplectic packings and algebraic geometry", *Invent. Math.* **115**:3 (1994), 405–434. With an appendix by Y. Karshon. MR 95a:58042 Zbl 0833.53028

[McDuff and Salamon 1998] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 2nd ed., Oxford Math. Monogr., The Clarendon Press, Oxford Univ. Press, New York, 1998. MR 2000g:53098 Zbl 1066.53137

[McDuff and Salamon 2004] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, Amer. Math. Soc. Colloquium Publ. **52**, Amer. Math. Soc., Providence, RI, 2004. MR 2004m:53154 Zbl 1064.53051

[Pinsonnault 2008] M. Pinsonnault, "Maximal compact tori in the Hamiltonian group of 4-dimensional symplectic manifolds", *J. Mod. Dyn.* **2**:3 (2008), 431–455. MR MR2417479 Zbl 1154.57023

Received September 8, 2008.

LIAT KESSLER
MASSACHUSSETS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS
CAMBRIDGE, MA 02139
UNITED STATES

kessler@math.mit.edu