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Motivated by the recent concept of a pseudosymmetric braided monoidal category, we define the pseudosymmetric group $PS_n$ to be the quotient of the braid group $B_n$ by the relations

\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]

with $1 \leq i \leq n-2$. It turns out that $PS_n$ is isomorphic to the quotient of $B_n$ by the commutator subgroup $[P_n, P_n]$ of the pure braid group $P_n$ (which amounts to saying that $[P_n, P_n]$ coincides with the normal subgroup of $B_n$ generated by the elements $[\sigma_i^2, \sigma_{i+1}^2]$ with $1 \leq i \leq n-2$), and that $PS_n$ is a linear group.

Introduction

A symmetric category consists of a monoidal category $\mathcal{C}$ equipped with a family of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying natural “bilinearity” conditions together with the symmetry relation $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for all $X, Y \in \mathcal{C}$. This concept was generalized by Joyal and Street [1993] by dropping this symmetry relation from the axioms and arriving thus at the concept of braided category, of central importance in quantum group theory; see [Kassel 1995; Majid 1995].

Inspired by recently introduced categorical concepts of pure-braided structures [Staic 2004] and twines [Bruguières 2006], Panaite, Staic and Van Oystaeyen [Panaite et al. 2009] defined the concept of pseudosymmetric braiding to generalize symmetric braidings. A braiding $c$ on a strict monoidal category $\mathcal{C}$ is pseudosymmetric if it satisfies the modified braid relation

\[
(c_{Y,Z} \otimes \text{id}_X) \circ (\text{id}_Y \otimes c_{Z,X}^{-1}) \circ (c_{X,Y} \otimes \text{id}_Z) = \text{id}_Z \otimes c_{X,Y} \circ (c_{Z,X}^{-1} \otimes \text{id}_Y) \circ (\text{id}_X \otimes c_{Y,Z})
\]

for all $X, Y, Z \in \mathcal{C}$. The main result in [Panaite et al. 2009] asserts that, if $H$ is a Hopf algebra with bijective antipode, then the canonical braiding of the Yetter–Drinfeld category $H \mathcal{YD} H$ is pseudosymmetric if and only if $H$ is commutative and cocommutative.

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It is well known that, at several levels, braided categories correspond to the braid groups \( B_n \), while symmetric categories correspond to the symmetric groups \( S_n \). It is natural to expect that there exist some groups corresponding, in the same way, to pseudosymmetric braided categories. Indeed, it is clear that these groups, denoted by \( \text{PS}_n \) and called (naturally) the pseudosymmetric groups, should be the quotients of the braid groups \( B_n \) by the relations \( \sigma_i \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \). Our aim is to study and find more explicitly the structure of these groups. We prove first that the kernel of the canonical group morphism \( \text{PS}_n \to S_n \) is abelian, and consequently \( \text{PS}_n \) is isomorphic to the quotient of \( B_n \) by the commutator subgroup \( [P_n, P_n] \) of the pure braid group \( P_n \). (This amounts to saying that \([P_n, P_n] \) coincides with the normal subgroup of \( B_n \) generated by the elements \([\sigma_i^2, \sigma_i^2 \sigma_{i+1}] \) with \( 1 \leq i \leq n - 2 \).)

There exist similarities, but also differences, between braid groups and pseudosymmetric groups. Bigelow [2001] and Krammer [2002] proved that braid groups are linear, and we show that so are pseudosymmetric groups. More precisely, we prove that the Lawrence–Krammer representation of \( B_n \) induces a representation of \( \text{PS}_n \) if the parameter \( q \) is chosen to be 1, and that this representation of \( \text{PS}_n \) is faithful over \( \mathbb{R}[t^\pm 1] \). On the other hand, although \( \text{PS}_n \) is an infinite group, like \( B_n \), it does have nontrivial elements of finite order, unlike \( B_n \).

1. Preliminaries

**Definition 1.1** [Panaite et al. 2007]. Let \( \mathcal{C} \) be a strict monoidal category and let \( T_{X,Y} : X \otimes Y \to X \otimes Y \) be a family of natural isomorphisms in \( \mathcal{C} \). We call \( T \) a **strong twine** if, for all \( X, Y, Z \in \mathcal{C} \),

\[
T_{1,1} = \text{id}_I, \quad (T_{X,Y} \otimes \text{id}_Z) \circ T_{X \otimes Y, Z} = (\text{id}_X \otimes T_{Y,Z}) \circ T_{X,Y \otimes Z}, \quad (T_{X,Y} \otimes \text{id}_Z) \circ (\text{id}_X \otimes T_{Y,Z}) = (\text{id}_X \otimes T_{Y,Z}) \circ (T_{X,Y} \otimes \text{id}_Z).
\]

**Definition 1.2** [Panaite et al. 2009]. Let \( \mathcal{C} \) be a strict monoidal category and \( c \) a braiding on \( \mathcal{C} \). We say that \( c \) is **pseudosymmetric** if, for all \( X, Y, Z \in \mathcal{C} \),

\[
(1) \quad (c_{Y,Z} \otimes \text{id}_X) \circ (\text{id}_Y \otimes c_{Z,X}^{-1}) \circ (c_{X,Y} \otimes \text{id}_Z)
\]

\[
= (\text{id}_Z \otimes c_{X,Y}) \circ (c_{Z,X}^{-1} \otimes \text{id}_Y) \circ (\text{id}_X \otimes c_{Y,Z}).
\]

In this case we say that \( \mathcal{C} \) is a **pseudosymmetric braided category**.

The next proposition, a key result in [Panaite et al. 2009], led to the introduction of the concept of pseudosymmetric braiding. Here, it will serve as a source of inspiration for a certain key result for braids, Proposition 2.1.

**Proposition 1.3** [Panaite et al. 2009]. Let \( \mathcal{C} \) be a strict monoidal category and \( c \) a braiding on \( \mathcal{C} \). Then the double braiding \( T_{X,Y} := c_{Y,X} \circ c_{X,Y} \) is a strong twine if and only if \( c \) is pseudosymmetric.
2. Defining relations for $\text{PS}_n$

Let $n \geq 3$ be a natural number. We denote by $B_n$ the braid group on $n$ strands, with its usual presentation by generators $\sigma_i$ with $1 \leq i \leq n - 1$ and relations

\begin{align*}
(2) \quad & \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2, \\
(3) \quad & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{if } 1 \leq i \leq n - 2.
\end{align*}

We begin with the analogue for braids of Proposition 1.3:

**Proposition 2.1.** For all $1 \leq i \leq n - 2$, the relations

\begin{align*}
(4) \quad & \sigma_i \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}, \\
(5) \quad & \sigma_i^2 \sigma_{i+1}^2 = \sigma_{i+1}^2 \sigma_i^2
\end{align*}

are equivalent in $B_n$.

**Proof.** We show first that (4) implies (5):

\[
\sigma_i^2 \sigma_{i+1}^2 = \sigma_i \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i \\
\overset{(3)(4)}{=} \sigma_i \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i \\
\overset{(3)}{=} \sigma_i \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1}^2 \sigma_i^2.
\]

Conversely, we prove that (5) implies (4):

\[
\sigma_i \sigma_{i+1}^{-1} \sigma_i = \sigma_i \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i \\
\overset{(3)(5)}{=} \sigma_i \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i \\
\overset{(3)}{=} \sigma_i^{-1} \sigma_{i+1} \sigma_i^2 \\
\overset{(3)}{=} \sigma_{i+1}^{-1} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \sigma_i^2 \\
\overset{(3)}{=} \sigma_{i+1}^{-1} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \sigma_i^2 \\
\overset{(3)}{=} \sigma_{i+1}^{-1} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \sigma_i^2.
\]

\(\square\)

**Definition 2.2.** For a natural number $n \geq 3$, we define the *pseudosymmetric group* $\text{PS}_n$ as the group with generators $\sigma_i$ for $1 \leq i \leq n - 1$, and relations (2), (3) and (4), or equivalently (2), (3) and (5).

**Proposition 2.3.** For $1 \leq i \leq n - 2$, consider the elements

\begin{align*}
(6) \quad & p_i := \sigma_i \sigma_{i+1}^{-1} \quad \text{and} \quad q_i := \sigma_i^{-1} \sigma_{i+1}
\end{align*}

in $\text{PS}_n$. Then, in $\text{PS}_n$, we have

\begin{align*}
(7) \quad & p_i^3 = q_i^3 = (p_i q_i)^3 = 1 \quad \text{for all } 1 \leq i \leq n - 2.
\end{align*}
Proof. The relations \( p_i^3 = 1 \) and \( q_i^3 = 1 \) follow immediately from (4); actually each of them is equivalent to (4). Now we compute
\[
(p_i q_i)^2 = (\sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1})^2
\]
\[
= \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}
\]
\[
= \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}
\]
\[
= \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}
\]
\[
= \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}
\]
and so \((p_i q_i)^3 = 1\). \qed

Consider now the symmetric group \( S_n \) with its usual presentation by generators \( s_i \) with \( 1 \leq i \leq n - 1 \) and relations (2), (3) and \( s_i^2 = 1 \) for all \( 1 \leq i \leq n - 1 \). We denote by \( \pi : B_n \to S_n \), \( \beta : B_n \to \text{PS}_n \) and \( \alpha : \text{PS}_n \to S_n \) the canonical surjective group homomorphisms given by \( \pi(s_i) = s_i \), \( \alpha(s_i) = s_i \) and \( \beta(s_i) = s_i \) for all \( 1 \leq i \leq n - 1 \). Obviously we have \( \pi = \alpha \circ \beta \); hence in particular we obtain \( \text{Ker}(\alpha) = \beta(\text{Ker}(\pi)) \). We denote as usual \( \text{Ker}(\pi) = P_n \), the pure braid group on \( n \) strands. It is well known (see [Kassel and Turaev 2008, page 21]) that \( P_n \) is generated by the elements
\[
a_{ij} := \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1}^{-1} \sigma_{j-1}^{-1} \quad \text{for } 1 \leq i < j \leq n
\]
that satisfy certain relations, of which we will use only one, namely, that for \( 1 \leq i < j \leq n \) and \( 1 \leq r < s \leq n \),
\[
a_{ij} a_{rs} = a_{rs} a_{ij} \quad \text{if } s < i \text{ or } i < r < s < j.
\]
Alternatively, \( P_n \) is generated by the elements
\[
b_{ij} := \sigma_{j-1}^{-1} \sigma_{j-2}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^2 \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1}^{-1} \sigma_{j-1}^{-1} \quad \text{for } 1 \leq i < j \leq n.
\]
It is easy to see that in \( B_n \) we have
\[
\sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} = \sigma_i^{-1} \sigma_{i+1}^2 \sigma_i \quad \text{and} \quad \sigma_{i+1}^{-1} \sigma_i^2 \sigma_{i+1} = \sigma_i \sigma_{i+1}^2 \sigma_i^{-1},
\]
and by using repeatedly these relations we obtain an equivalent description of the elements \( a_{ij} \) and \( b_{ij} \):
\[
a_{ij} = \sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i \quad \text{for } 1 \leq i < j \leq n,
\]
\[
b_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1} \quad \text{for } 1 \leq i < j \leq n.
\]
Now, for all $1 \leq i < j \leq n$, we define $A_{i,j}$ and $B_{i,j}$ as the elements in $PS_n$ given by $A_{i,j} := \beta(a_{ij})$ and $B_{i,j} := \beta(b_{ij})$. From the discussion above it follows that Ker$(\alpha)$ is generated by \{A_{i,j}\}_{1 \leq i < j \leq n}$ and also by \{B_{i,j}\}_{1 \leq i < j \leq n}.

**Lemma 2.4.** The following relations hold in $PS_n$ for $1 \leq i < j < n$:

\begin{align}
A_{i,j+1} &= \sigma_j A_{i,j} \sigma_j^{-1}, \\
B_{i,j+1} &= \sigma_j^{-1} B_{i,j} \sigma_j.
\end{align}

**Proof.** These relations are consequences of corresponding relations in $B_n$ for the $a_{ij}$ and $b_{ij}$, which in turn follow immediately from (8) and (10). \hfill \Box

**Lemma 2.5.** For all $i, j \in \{1, 2, \ldots, n\}$ with $i + 1 < j$, we have in $PS_n$

\begin{align}
A_{i,j} &= \sigma_i A_{i+1,j} \sigma_i^{-1} , \\
B_{i,j} &= \sigma_i^{-1} B_{i+1,j} \sigma_i.
\end{align}

**Proof.** We prove (16), while (17) is similar and left to the reader. Note that in $PS_n$ we have $\sigma_i^{-1} \sigma_i^2 \sigma_i = \sigma_i + 1 \sigma_i^2 \sigma_i^{-1}$, which together with the second of (11) implies $\sigma_i \sigma_i^2 \sigma_i^{-1} = \sigma_i + 1 \sigma_i^2 \sigma_i^{-1}$; hence

\[
A_{i,j} = \sigma_j \sigma_{j-1} \cdots (\sigma_i + 1 \sigma_i^2 \sigma_i^{-1}) \cdots \sigma_{j-2} \sigma_{j-1}^{-1} \\
= \sigma_j \sigma_{j-1} \cdots (\sigma_i \sigma_i^2 \sigma_i^{-1}) \cdots \sigma_{j-2} \sigma_{j-1}^{-1} \\
= \sigma_i \sigma_{j-1} \sigma_{j-2} \cdots \sigma_i^2 \cdots \sigma_{j-2} \sigma_{j-1} \sigma_i^{-1} = \sigma_i A_{i+1,j} \sigma_i^{-1} .
\]

\hfill \Box

**Proposition 2.6.** For all $1 \leq i < j \leq n$, we have $A_{i,j} = B_{i,j}$ in $PS_n$.

**Proof.** We use (16) repeatedly:

\[
A_{i,j} = \sigma_i A_{i+1,j} \sigma_i^{-1} = \sigma_i \sigma_{i+1} A_{i+2,j} \sigma_{i+1}^{-1} \sigma_i^{-1} \\
\ldots \\
= \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} A_{j-1,j} \sigma_{i+1}^{-1} \sigma_i^{-1} \\
= \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^{-1} = (13) B_{i,j} .
\]

\hfill \Box

**Lemma 2.7.** For all $1 \leq i < j \leq n$ and $1 \leq h \leq k < n$, we have in $PS_n$

\begin{align}
A_{i,j} \sigma_i^2 &= \sigma_i^2 A_{i,j} , \\
A_{h,k+1} \sigma_k^2 &= \sigma_k^2 A_{h,k+1}.
\end{align}

**Proof.** Note first that (18) is obvious for $j = i + 1$. Assume that $i + 1 < j$; using the fact that $A_{r,s} = B_{r,s}$ for all $r, s$, we compute

\[
A_{i,j} \sigma_i^2 = \sigma_i A_{i+1,j} \sigma_i = \sigma_i B_{i+1,j} \sigma_i = \sigma_i^2 B_{i,j} = \sigma_i^2 A_{i,j} .
\]
Note also that (19) is obvious for \( h = k \). Assume that \( h < k \); using again \( A_{r,s} = B_{r,s} \) for all \( r, s \), we compute

\[
A_{h,k+1} \sigma_k^2 = \sigma_k A_{h,k} \sigma_k = \sigma_k B_{h,k} \sigma_k = \sigma_k^2 B_{h,k+1} = \sigma_k^2 A_{h,k+1}. \qedhere
\]

### 3. The structure of \( \text{PS}_n \)

We denote by \( \mathcal{P}_n \) the kernel of the morphism \( \alpha : \text{PS}_n \to S_n \) defined above.

**Proposition 3.1.** \( \mathcal{P}_n \) is an abelian group.

**Proof.** It is enough to prove that any two elements \( A_{i,j} \) and \( A_{k,l} \) commute in \( \text{PS}_n \).

We only have to analyze the following seven cases for the numbers \( i, j, k, l \):

(i) \( i < j < k < l \). This is an obvious consequence of (9).

(ii) \( i < j = k < l \). We write

\[
A_{i,j} = \sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i, \\
A_{j,l} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{j+1} \sigma_{j+1}^{-1} \cdots \sigma_{l-2}^{-1} \sigma_{l-1}^{-1},
\]

and we obtain \( A_{i,j} A_{j,l} = A_{j,l} A_{i,j} \) by using (2) and the fact that \( \sigma_{j-1}^2 \) and \( \sigma_j^2 \) commute in \( \text{PS}_n \).

(iii) \( i < k < j < l \). This follows since \( A_{k,l} = B_{k,l} \) in \( \text{PS}_n \) (Proposition 2.6), and \( a_{ij} \) and \( b_{kl} \) commute in \( P_n \) if \( i < k < j < l \), which is easily seen geometrically.

(iv) \( i = k < j = l \). This is trivial.

(v) \( i < k < l < j \). This is an obvious consequence of (9).

(vi) \( i = k < j < l \). In case \( j = i + 1 \), we have \( A_{i,j} = \sigma_i^2 \) and so we obtain \( A_{i,j} A_{i,l} = A_{i,l} A_{i,j} \) by using (18); assuming now \( i + 1 < j \), by using repeatedly (16) we can compute

\[
A_{i,j} A_{i,l} = \sigma_i A_{i+1,j} A_{i+1,l} \sigma_i^{-1} \\
= \sigma_i \sigma_{i+1} A_{i+2,j} A_{i+2,l} \sigma_{i+1}^{-1} \sigma_i^{-1} \\
\cdots \\
= \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} A_{j-1,j} A_{j-1,l} \sigma_{j-2}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1},
\]

and similarly

\[
A_{i,l} A_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} A_{j-1,l} A_{j-1,j} \sigma_{j-2}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1};
\]

these are equal since \( A_{j-1,j} = \sigma_{j-1}^2 \) and by (18), \( \sigma_{j-1}^2 A_{j-1,l} = A_{j-1,l} \sigma_{j-1}^2. \)
(vii) \( i < k < j = l \). In case \( j = k + 1 \), we have \( A_{k,j} = \sigma_k^2 \) and so we obtain

\[ A_{i,j}A_{k,j} = A_{k,j}A_{i,j} \]

by using (19); assuming now \( k + 1 < j \), by repeatedly using (14) we can compute

\[
A_{i,j}A_{k,j} = \sigma_{j-1}A_{i,j-1}A_{k,j-1}\sigma_{j-1}
\]

\[
= \sigma_{j-1}\sigma_{j-2}A_{i,j-2}A_{k,j-2}\sigma_{j-2}\sigma_{j-1}
\]

\[ \vdots \]

\[
= \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{k+1}A_{i,k+1}A_{k,k+1}\sigma_{k+1}^{-1}\cdots\sigma_{j-2}\sigma_{j-1},
\]

and similarly

\[
A_{k,j}A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{k+1}A_{i,k+1}A_{k,k+1}\sigma_{k+1}^{-1}\cdots\sigma_{j-2}\sigma_{j-1},
\]

these are equal since \( A_{k,k+1} = \sigma_k^2 \) and by (19), \( A_{i,k+1}\sigma_k^2 = \sigma_k^2 A_{i,k+1} \). \( \square \)

Let \( G \) be a group. If \( x, y \in G \) we denote by \([x, y] := x^{-1}y^{-1}xy\) the commutator of \( x \) and \( y \), and by \( G' \) the commutator subgroup of \( G \) (the subgroup of \( G \) generated by all commutators \([x, y]\)), which is the smallest normal subgroup \( N \) of \( G \) with the property that \( G/N \) is abelian. Moreover, \( G' \) is a characteristic subgroup of \( G \), that is, \( \theta(G') = G' \) for all \( \theta \in \text{Aut}(G) \).

**Proposition 3.2.** \( \Psi_n \cong P_n/P_n' \cong \mathbb{Z}^{n(n-1)/2} \).

*Proof.* For \( 1 \leq i \leq n-2 \) we define \( t_i \in P_n \) by

\[
t_i := [\sigma_i^2, \sigma_{i+1}^2] = [a_{i,i+1}, a_{i+1,i+2}].
\]

These elements are the relators added to the ones of \( B_n \) in order to obtain \( \text{PS}_n \); therefore, as a particular case of a general fact about groups given by generators and relations (see for instance [Coxeter and Moser 1972, page 2]), the kernel of the map \( \beta : B_n \to \text{PS}_n \) defined above coincides with the normal subgroup of \( B_n \) generated by \( \{t_i\}_{1 \leq i \leq n-2} \), which will be denoted by \( L_n \). We obviously have \( L_n \subseteq P_n \), and if we consider the map \( \beta \) restricted to \( P_n \), we have a surjective morphism \( P_n \to \Psi_n \) with kernel \( L_n \), so \( \Psi_n \cong P_n/L_n \). By Proposition 3.1 we know that \( \Psi_n \) is abelian, so we obtain \( P_n' \subseteq L_n \). On the other hand, since \( P_n' \) is characteristic in \( P_n \) and \( P_n \) is normal in \( B_n \), it follows (see [Suzuki 1982, Proposition 6.14]) that \( P_n' \) is normal in \( B_n \), and since \( t_1, \ldots, t_{n-2} \in P_n' \) and \( L_n \) is the normal subgroup of \( B_n \) generated by \( \{t_i\}_{1 \leq i \leq n-2} \), we obtain \( L_n \subseteq P_n' \). Thus, we have obtained \( L_n = P_n' \) and so \( \Psi_n \cong P_n'/P_n' \). On the other hand, it is well known that \( P_n/P_n' \cong \mathbb{Z}^{n(n-1)/2} \); see for instance [Kassel and Turaev 2008, Corollary 1.20]. \( \square \)

As a consequence of the equality \( L_n = P_n' \), we obtain \( B_n/P_n' \).

**Corollary 3.3.** \( \text{PS}_n \cong B_n/P_n' \).

The extension with abelian kernel \( 1 \to \Psi_n \to \text{PS}_n \to S_n \to 1 \) induces an action of \( S_n \) on \( \Psi_n \), given by \( \sigma \cdot a = \tilde{\sigma} a \tilde{\sigma}^{-1} \) for \( \sigma \in S_n \) and \( a \in \Psi_n \), where \( \tilde{\sigma} \) is an element of \( \text{PS}_n \) with \( \alpha(\tilde{\sigma}) = \sigma \). In particular, on generators we have

\[
s_k \cdot A_{i,j} = \sigma_k A_{i,j} \sigma_k^{-1},
\]
for 1 \leq k \leq n - 1 and 1 \leq i < j \leq n. By using some of the formulas given above, one can describe explicitly this action as

\begin{align*}
(20a) \quad s_k \cdot A_{i,j} &= A_{i,j} & \text{if } k < i - 1, \\
(20b) \quad s_{i-1} \cdot A_{i,j} &= A_{i-1,j}, \\
(20c) \quad s_i \cdot A_{i,j} &= A_{i+1,j} & \text{if } j - i > 1 \text{ and } s_i \cdot A_{i,i+1} = A_{i,i+1}, \\
(20d) \quad s_k \cdot A_{i,j} &= A_{i,j} & \text{if } i < k < j - 1, \\
(20e) \quad s_{j-1} \cdot A_{i,j} &= A_{i,j-1} & \text{if } j - i > 1 \text{ and } s_{j-1} \cdot A_{j-1,j} = A_{j-1,j}, \\
(20f) \quad s_j \cdot A_{ij} &= A_{i,j+1} & \text{for } 1 \leq i < j < n, \\
(20g) \quad s_k \cdot A_{i,j} &= A_{i,j} & \text{if } j < k.
\end{align*}

Note that the first equality in (20c) follows by using (17) together with the fact that \( A_{i,j} = B_{i,j} \) (Proposition 2.6), and the first equality in (20e) follows by an easy computation using also the fact that \( A_{i,j} = B_{i,j} \). Also, one can easily see that these formulas may be expressed more compactly as follows: If \( \sigma \in \{s_1, \ldots, s_{n-1}\} \) and \( 1 \leq i < j \leq n \), then \( \sigma \cdot A_{i,j} = A_{\sigma(i),\sigma(j)} \), where we made the convention \( A_{r,t} := A_{t,r} \) for \( t < r \). Since \( s_1, \ldots, s_{n-1} \) generate \( S_n \), we have found the action of \( S_n \) on \( A_{i,j} \):

**Proposition 3.4.** For any \( \sigma \in S_n \) and \( 1 \leq i < j \leq n \), the action of \( \sigma \) on \( A_{i,j} \) is given by \( \sigma \cdot A_{i,j} = A_{\sigma(i),\sigma(j)} \), with the convention \( A_{r,t} := A_{t,r} \) for \( t < r \).

**Lemma 3.5.** Let \( F \) be a free \( \mathbb{Z} \)-module of rank \( m \), and let \( \{X_1, \ldots, X_m\} \) be a generating system for \( F \) over \( \mathbb{Z} \). Then \( \{X_1, \ldots, X_m\} \) is a basis of \( F \) over \( \mathbb{Z} \).

**Proof.** Assume \( X_1, \ldots, X_m \) are linearly dependent over \( \mathbb{Z} \) and take \( \sum_{i=1}^m a_i X_i = 0 \) a nontrivial linear combination over \( \mathbb{Z} \). Choose a prime number \( p \) such that \( |a_i| < p \) for all \( 1 \leq i \leq m \), and consider \( \overline{F} := F / pF \), a linear space over the field \( \mathbb{Z}_p = \mathbb{Z} / p\mathbb{Z} \), and \( \overline{X_i} \), the images of the elements \( X_i \) in \( \overline{F} \). These elements generate \( \overline{F} \) over \( \mathbb{Z}_p \), and since the dimension of \( \overline{F} \) over \( \mathbb{Z}_p \) is \( m \), it follows that \( \{\overline{X}_1, \ldots, \overline{X}_m\} \) is a basis of \( \overline{F} \) over \( \mathbb{Z}_p \). Thus, it follows that \( a_i \equiv 0 \pmod{p} \) for all \( 1 \leq i \leq m \), which is a contradiction because we have chosen \( p \) so that \( |a_i| < p \) for all \( 1 \leq i \leq m \). \( \square \)

**Proposition 3.6.** In \( \text{PS}_n \), there is no element of order 2 whose image in \( S_n \) is the transposition \( s_1 = (1, 2) \). Consequently, the extension \( 1 \to \mathbb{Q}_n \to \text{PS}_n \to S_n \to 1 \) is not split.

**Proof.** Take \( x \in \text{PS}_n \) such that \( \alpha(x) = s_1 \). Since \( \alpha(\sigma_1) = s_1 \), we obtain that \( x\alpha_1^{-1} \in \text{Ker}(\alpha) = \mathbb{Q}_n \). By Proposition 3.2 and Lemma 3.5, it follows that the abelian group \( \mathbb{Q}_n \) is freely generated by \( \{A_{i,j}\}_{1 \leq i < j \leq n} \), so we can write uniquely
\[ x = \prod_{1 \leq i < j \leq n} A_{i,j}^{m_{ij}} \sigma_1, \text{ with } m_{ij} \in \mathbb{Z}. \] We compute
\[
x^2 = \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{m_{ij}} \sigma_1 \right) \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{m_{ij}} \sigma_1 \right)
= \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{m_{ij}} \right) \left( \prod_{1 \leq i < j \leq n} \sigma_1 A_{i,j}^{m_{ij}} \sigma_1^{-1} \right)^2
= \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{m_{ij}} \right) \left( \prod_{1 \leq i < j \leq n} A_{i,j}^{m_{ij}} \right) A_{1,2}
= A_{1,2}^{2m_{12}+1} \left( \prod_{3 \leq j \leq n} A_{1,j}^{m_{12}+m_{2j}} A_{2,j}^{m_{12}+m_{2j}} \right) \left( \prod_{3 \leq i < j \leq n} A_{i,j}^{2m_{ij}} \right),
\]
and this element cannot be trivial because \(2m_{12} + 1\) cannot be 0. Note that for the last equality we used the commutation relations
\[
\sigma_1 A_{1,2} \sigma_1^{-1} = A_{1,2},
\]
\[
\sigma_1 A_{1,j} \sigma_1^{-1} = A_{2,j} \quad \text{for all } j \geq 3,
\]
\[
\sigma_1 A_{2,j} \sigma_1^{-1} = A_{1,j} \quad \text{for all } j \geq 3,
\]
\[
\sigma_1 A_{i,j} \sigma_1^{-1} = A_{i,j} \quad \text{for all } 3 \leq i < j,
\]
which can be easily proved by using some of the formulas given above. □

**Remark 3.7.** As is well known [Brown 1982], any extension with abelian kernel corresponds to a 2-cocycle. Specifically, the extension \(1 \to \mathcal{P}_n \to \text{PS}_n \to S_n \to 1\) corresponds to an element in \(H^2(S_n, \mathbb{Z}^{n(n-1)/2})\). We illustrate this by computing explicitly the corresponding 2-cocycle for \(n = 3\). We consider the set-theoretical section \(f : S_3 \to \text{PS}_3\) defined by \(f(1) = 1, f(s_2) = \sigma_2, f(s_1) = \sigma_1, f(s_1s_2) = \sigma_1 \sigma_2, f(s_2s_1) = \sigma_2 \sigma_1, f(s_3s_1s_2) = \sigma_3 \sigma_1 \sigma_2\). The 2-cocycle afforded by this section is defined by \(u : S_3 \times S_3 \to \mathcal{P}_3, (x, y) \mapsto f(x) f(y) f(xy)^{-1}\), and a direct computation gives its explicit formula as in Table 1, where we have chosen an additive notation for the abelian group \(\mathcal{P}_3 \cong \mathbb{Z}^3\).

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**Table 1.** The 2-cocycle for \(n = 3\) associated to the section \(f\).
4. $\text{PS}_n$ is linear

Bigelow [2001] and Krammer [2002] proved that the braid group $B_n$ is linear. More precisely, let $R$ be a commutative ring, let $q$ and $t$ be two invertible elements in $R$, and let $V$ be a free $R$-module of rank $n(n-1)/2$ with a basis $\{x_{i,j}\}_{1 \leq i < j \leq n}$. Then the map $\rho : B_n \to \text{GL}(V)$, defined by

$$
\sigma_k x_{k,k+1} = t q^2 x_{k,k+1}, \\
\sigma_k x_{i,k} = (1 - q) x_{i,k} + q x_{i,k+1} \quad \text{for } i < k, \\
\sigma_k x_{i,k+1} = x_{i,k} + t q^{k-i+1} (q - 1) x_{k,k+1} \quad \text{for } i < k, \\
\sigma_k x_{k,j} = t q (q - 1) x_{k,k+1} + q x_{k+1,j} \quad \text{for } k + 1 < j, \\
\sigma_k x_{k+1,j} = x_{k,j} + (1 - q) x_{k+1,j} \quad \text{for } k + 1 < j, \\
\sigma_k x_{i,j} = x_{i,j} \quad \text{for } i < j < k \text{ or } k + 1 < i < j, \\
\sigma_k x_{i,j} = x_{i,j} + t q^{k-i} (q - 1)^2 x_{k,k+1} \quad \text{for } i < k < k + 1 < j,
$$

and $\rho(x)(v) = x v$ for $x \in B_n$ and $v \in V$, gives a representation of $B_n$, and if also $R = \mathbb{R}[t^\pm 1]$ and $q \in \mathbb{R} \subseteq R$ with $0 < q < 1$, then the representation is faithful; see [Krammer 2002].

We consider now the general formula for $\rho$, in which we take $q = 1$:

$$
\sigma_k x_{k,k+1} = t x_{k,k+1}, \\
\sigma_k x_{i,k} = x_{i,k+1} \quad \text{for } i < k, \\
\sigma_k x_{i,k+1} = x_{i,k} \quad \text{for } i < k, \\
\sigma_k x_{k,j} = x_{k+1,j} \quad \text{for } k + 1 < j, \\
\sigma_k x_{k+1,j} = x_{k,j} \quad \text{for } k + 1 < j, \\
\sigma_k x_{i,j} = x_{i,j} \quad \text{for } i < j < k \text{ or } k + 1 < i < j, \\
\sigma_k x_{i,j} = x_{i,j} \quad \text{for } i < k < k + 1 < j.
$$

One can easily see that these formulas imply

$$
\sigma_k^2 x_{k,k+1} = t^2 x_{k,k+1} \quad \text{and} \quad \sigma_k^2 x_{i,j} = x_{i,j} \quad \text{if } (i, j) \neq (k, k + 1).
$$

One can then check that $\rho(\sigma_k^2)$ commutes with $\rho(\sigma_{k+1}^2)$ for all $1 \leq k \leq n - 2$, and so for $q = 1$ it turns out that $\rho$ is a representation of $\text{PS}_n$.

**Theorem 4.1.** This representation of $\text{PS}_n$ is faithful if $R = \mathbb{R}[t^\pm 1]$. Therefore, $\text{PS}_n$ is linear.

**Proof.** We first prove that $A_{i,j} x_{i,j} = t^2 x_{i,j}$, and $A_{i,j} x_{k,l} = x_{k,l}$ if $(i, j) \neq (k, l)$. We do it by induction over $|j - i|$. If $|j - i| = 1$, the relations follow from the fact that $A_{i,i+1} = \sigma_i^2$. Assume the relations hold for $|j - i| = s - 1$. We want to prove
them for $|j - i| = s$. We recall that $A_{i,j} = \sigma_{j-1}A_{i,j-1}\sigma_{j-1}^{-1}$; see (14). We compute

$$A_{i,j}x_{i,j} = \sigma_{j-1}A_{i,j-1}\sigma_{j-1}^{-1}x_{i,j} = \sigma_{j-1}A_{i,j-1}x_{i,j-1} = \sigma_{j-1}t^2x_{i,j-1} \quad \text{(by induction)}
$$

$$= t^2x_{i,j}.$$  

On the other hand, if $(i, j) \neq (k, l)$ then $\sigma_{j-1}^{-1}x_{k,l} = x_{u,v}$ with $(i, j - 1) \neq (u, v)$, and so

$$A_{i,j}x_{k,l} = \sigma_{j-1}A_{i,j-1}\sigma_{j-1}^{-1}x_{k,l} = \sigma_{j-1}A_{i,j-1}x_{u,v} = \sigma_{j-1}x_{u,v} \quad \text{(by induction)}$$

$$= \sigma_{j-1}\sigma_{j-1}^{-1}x_{k,l} = x_{k,l},$$

as desired.

To show that the representation is faithful, take $b \in \text{PS}_n$ such that $\rho(b) = \text{id}_V$ and consider $\alpha(b)$, the image of $b$ in $S_n$. From the way $\rho$ is defined it follows that

$$bx_{i,j} = t^px_{\alpha(b)(i),\alpha(b)(j)} \quad \text{for all } 1 \leq i < j \leq n,$$

with $p \in \mathbb{Z}$, where we made the convention $x_{r,s} := x_{s,r}$ if $1 \leq s < r \leq n$. Since $x_{i,j}$ is a basis in $V$ and we assumed $\rho(b) = \text{id}_V$, we find that the permutation $\alpha(b) \in S_n$ has the property that if $1 \leq i < j \leq n$, then either $\alpha(b)(i) = i$ and $\alpha(b)(j) = j$ or $\alpha(b)(i) = j$ and $\alpha(b)(j) = i$. Since we assumed $n \geq 3$, the only such permutation is the trivial one. Thus, we have obtained that $b \in \text{Ker}(\alpha) = \mathfrak{P}_n$ and so we can write $b = \prod_{1 \leq i < j \leq n}A_{i,j}^{m_{i,j}}$, with $m_{i,j} \in \mathbb{Z}$. By using the formulas given above for the action of $A_{i,j}$ on $x_{k,l}$ we immediately obtain $bx_{k,l} = t^{2m_{i,j}}x_{k,l}$ for all $1 \leq k < l \leq n$. Using again the assumption $\rho(b) = \text{id}_V$, we obtain $t^{2m_{i,j}} = 1$ and hence $m_{k,l} = 0$ for all $1 \leq k < l \leq n$, that is $b = 1$, finishing the proof. \hfill $\square$

5. Pseudosymmetric groups and pseudosymmetric braidings

We recall from [Kassel 1995, XIII.2] that to braid groups one can associate the so-called braided category $\mathcal{B}$, a universal braided monoidal category. Similarly, we can construct a pseudosymmetric braided category $\mathcal{P}$ associated to pseudosymmetric groups. Namely, the objects of $\mathcal{P}$ are natural numbers $n \in \mathbb{N}$. The set of morphisms from $m$ to $n$ is empty if $m \neq n$ and is $\text{PS}_n$ if $m = n$. The monoidal structure of $\mathcal{P}$ is defined as the one for $\mathcal{B}$, and so is the braiding, namely

$$c_{m,n} : n \otimes m \to m \otimes n,$$

$$c_{0,n} = \text{id}_n = c_{n,0},$$

$$c_{m,n} = (\sigma_m\sigma_{m-1}\cdots\sigma_1)(\sigma_{m+1}\sigma_m\cdots\sigma_2)\cdots(\sigma_{m+n-1}\sigma_{m+n-2}\cdots\sigma_n) \quad \text{if } m, n > 0.$$
We denote by $t_{m,n} = c_{n,m} \circ c_{m,n}$ the double braiding. In view of Proposition 1.3, to prove that $c$ is pseudosymmetric it is enough to check that, for all $m,n,p \in \mathbb{N}$,

$$(21) \quad (t_{m,n} \otimes \text{id}_p) \circ (\text{id}_m \otimes t_{n,p}) = (\text{id}_m \otimes t_{n,p}) \circ (t_{m,n} \otimes \text{id}_p).$$

Note that $t_{m,n} \otimes \text{id}_p$ and $\text{id}_m \otimes t_{n,p}$ are elements in $\mathcal{P}_{m+n+p}$, which is an abelian group, and the composition $\circ$ between $t_{m,n} \otimes \text{id}_p$ and $\text{id}_m \otimes t_{n,p}$ is just the multiplication in the group $\mathcal{P}_{m+n+p}$, so (21) is obviously true.

Let $\mathcal{C}$ be a strict braided monoidal category with braiding $c$, let $n$ be a natural number and let $V \in \mathcal{C}$. Consider the automorphisms $c_1, \ldots, c_{n-1}$ of $V^\otimes n$ defined by $c_i = \text{id}_{V^\otimes (i-1)} \otimes c_{V^\otimes (i-1)} \otimes \text{id}_{V^\otimes (n-i-1)}$. It is well known (see [Kassel 1995, XV.4]) that there exists a unique group morphism $\rho_n^c : B_n \rightarrow \text{Aut}(V^\otimes n)$ such that $\rho_n^c(\sigma_i) = c_i$ for all $1 \leq i \leq n-1$. It is clear that, if $c$ is pseudosymmetric, then $\rho_n^c$ factorizes to a group morphism $\mathbb{P}S_n \rightarrow \text{Aut}(V^\otimes n)$. Thus, pseudosymmetric braided categories provide representations of pseudosymmetric groups.

References


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