A TOPOLOGICAL SPHERE THEOREM FOR ARBITRARY-DIMENSIONAL MANIFOLDS

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We study manifolds with bounded volume, sectional curvature, and injectivity radius. We obtain a topological sphere theorem.

Sphere theorems are common in differential geometry; one often asks whether a manifold is homeomorphic to a sphere under certain topological or geometric restrictions; see for instance [Grove and Shiohama 1977; Perelman 1995; Shen 1989; Shiohama 1983; Suyama 1991; Wu 1989]. Coghlan and Itokawa [1991] proved a sphere theorem that says that if an even-dimensional, simply connected Riemannian manifold \( \mathcal{M} \) has sectional curvature \( K_{\mathcal{M}} \in (0, 1] \), volume \( V_{\mathcal{M}} \leq \frac{3}{2} V_{S^n} \) with \( V_{S^n} \) the volume of the standard \( n \)-dimensional unit sphere \( S^n \) in \( \mathbb{R}^{n+1} \), then \( \mathcal{M} \) must be homeomorphic to \( S^n \). In [Wen 2004], we improved this result by relaxing the upper bound on \( V_{\mathcal{M}} \) to a bound larger than \( \frac{3}{2} V_{S^n} \). In both of these papers, the hypotheses of simple connectivity and even dimension were merely used to deduce that the injectivity radius \( i_{\mathcal{M}} \) is no less than \( \pi \). Here we find that we can weaken the assumptions on \( K_{\mathcal{M}} \) and \( i_{\mathcal{M}} \). If the simple connectivity condition is removed, the conclusion holds in any dimension.

Before stating our result, we introduce some notation. Let \( (\mathcal{M}, g) \) be a compact, connected \( n \)-dimensional Riemannian manifold with metric \( g \). We denote by \( K_{\mathcal{M}} \) the sectional curvature of \( \mathcal{M} \), by \( i_{\mathcal{M}} \) its injectivity radius, and by \( V_{\mathcal{M}} \) its volume. For any points \( P, Q \in \mathcal{M} \), we denote by \( \gamma_{P, Q} \) the shortest geodesic on \( \mathcal{M} \) from \( P \) to \( Q \).

**Theorem 1.** Given \( k > 0 \), there exists an \( \varepsilon_0 > 0 \) such that if a compact connected \( n \)-dimensional Riemannian manifold \( \mathcal{M}, g \) satisfies

\[
-k^2 \leq K_{\mathcal{M}} \leq 1, \quad i_{\mathcal{M}} \geq \pi - \varepsilon_0, \quad V_{\mathcal{M}} \leq \frac{3}{2} V_{S^n} + \varepsilon_0,
\]

then \( \mathcal{M} \) is homeomorphic to \( S^n \).

The examples of real projective spaces \( \mathbb{R}P^n \) for \( n \geq 2 \) and product manifolds \( S^n \times S^m \) for \( m, n \geq 1 \) show that the hypotheses on the lower bound on \( i_{\mathcal{M}} \) or the upper bound on \( V_{\mathcal{M}} \) cannot be removed.

**MSC2000:** 53C20.

**Keywords:** sphere theorem.

Supported in part by National Science Foundation of China (number 10871069) and Shanghai Leading Academic Discipline Project (project number B407).
In what follows, we denote by \( \mathbb{B}(P, r) \) the open geodesic ball in \( \mathcal{M} \) with center \( P \) and radius \( r \), and by \( \overline{\mathbb{B}}(P, r) \) its closure. Also, we denote by \( \mathbb{B}_r \) the open geodesic ball in \( S^n \) with radius \( r \). Instead of proving Theorem 1 directly, we will prove a more precise version.

**Proposition 1.** Let \( k > 0 \). There exist \( \delta, \sigma > 0 \) satisfying \( \sigma + \delta < \pi \) such that if a compact connected \( n \)-dimensional Riemannian manifold \( (\mathcal{M}, g) \) satisfies

\[
-k^2 \leq K_{\mathcal{M}} \leq 1, \quad i_{\mathcal{M}} \geq \pi - \sigma, \quad V_{\mathcal{M}} \leq 3V(\mathbb{B}_{\pi/2-\sigma/2}) + V(\mathbb{B}_{\delta/2}),
\]

then \( \mathcal{M} \) is homeomorphic to \( S^n \).

**Remark 1.** The choice of \( \sigma \) or \( \delta \) here is of course not optimal. We conjecture that \( \sigma < \pi/2 \) is optimal.

**Proof of Proposition 1.** We proceed by way of contradiction. Suppose there exists a manifold \( \mathcal{M} \) satisfying (1) that is not homeomorphic to \( S^n \). Take points \( p, q \) in \( \mathcal{M} \) such that \( d(p, q) = d_{\mathcal{M}} \), the diameter \( d_{\mathcal{M}} \) of \( \mathcal{M} \). Then by a well-known topological fact (see for instance [Brown 1960]), there is a point \( x_0 \in \mathcal{M} - \mathbb{B}(p, i_{\mathcal{M}}) \cup \overline{\mathbb{B}}(q, i_{\mathcal{M}}) \).

Without loss of generality, let \( d(q, x_0) = d(p, x_0) = l_0 \). Therefore \( l_0 \geq i_{\mathcal{M}} \geq \pi - \sigma \).

First we show an explicit upper bound on \( d_{\mathcal{M}} \).

**Lemma 1.** \( d_{\mathcal{M}} \leq \pi - \sigma + \delta \).

**Proof.** We argue by contradiction. If \( d_{\mathcal{M}} > \pi - \sigma + \delta \), then we consider the balls \( \mathbb{B}(p, \pi/2 - \sigma/2 + \delta/2), \mathbb{B}(q, \pi/2 - \sigma/2 + \delta/2) \) and \( \mathbb{B}(x_0, l_0 - \pi/2 + \sigma/2 - \delta/2) \).

They are obviously pairwise disjoint. Therefore since \( K_{\mathcal{M}} \leq 1 \), Günther’s volume comparison theorem gives

\[
V_{\mathcal{M}} \geq 2V(\mathbb{B}_{\pi/2-\sigma/2+\delta/2}) + V(\mathbb{B}_{l_0-\pi/2+\sigma/2-\delta/2}).
\]

In what follows, we check that

\[
2V(\mathbb{B}_{\pi/2-\sigma/2+\delta/2}) + V(\mathbb{B}_{l_0-\pi/2+\sigma/2-\delta/2}) > 3V(\mathbb{B}_{\pi/2-\sigma/2}) + V(\mathbb{B}_{\delta/2}).
\]

Noting that \( l_0 - \pi/2 + \sigma/2 - \delta/2 \geq \pi/2 - \sigma/2 - \delta/2 > 0 \), we have

\[
V(\mathbb{B}_{l_0-\pi/2+\sigma/2-\delta/2}) \geq V(\mathbb{B}_{\pi/2-\sigma/2-\delta/2}).
\]

By the definition of \( S^n \), we have \( V(\mathbb{B}_r) = \omega_{n-1} \int_0^r (\sin t)^{n-1} \, dt \) for any \( r > 0 \), where \( \omega_{n-1} \) is the volume of the standard unit \( (n - 1) \)-sphere \( S^{n-1} \). Since \( \sin t \) is
increasing in \((0, \pi/2)\), we have

\[
\frac{1}{\omega_{n-1}} \left[ 2V(B_{\pi/2-\sigma/2+\delta/2}) + V(B_{0-\pi/2+\sigma/2-\delta/2}) - 3V(B_{\pi/2-\sigma/2}) - V(B_{\delta/2}) \right] \\
\geq 2 \int_0^{\pi/2-\sigma/2+\delta/2} (\sin t)^{n-1} dt + \int_0^{\pi/2-\sigma/2-\delta/2} (\sin t)^{n-1} dt \\
- 3 \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt - \int_0^{\delta/2} (\sin t)^{n-1} dt \\
= \int_{\pi/2-\sigma/2}^{\pi/2-\sigma/2+\delta/2} (\sin t)^{n-1} dt - \int_{\pi/2-\sigma/2}^{\pi/2-\sigma/2+\delta/2} (\sin t)^{n-1} dt \\
+ \int_{\pi/2-\sigma/2}^{\pi/2-\sigma/2+\delta/2} (\sin t)^{n-1} dt - \int_0^{\delta/2} (\sin t)^{n-1} dt \\
> \int_{\pi/2-\sigma/2}^{\pi/2-\sigma/2+\delta/2} (\sin t)^{n-1} dt - \int_{\pi/2-\sigma/2}^{\pi/2-\sigma/2+\delta/2} (\sin t)^{n-1} dt > 0.
\]

Clearly, the estimates (2) and (3) contradict the assumptions (1). \(\Box\)

**Lemma 2.** If \(\delta > 0\) and \(\sigma = 2/3 \int_0^{\delta/2} (\sin t)^{n-1} dt\) satisfy \(\sigma + \delta < \pi\), then

\[
V(B_{\delta/2}) + V(B_{\pi/2-\sigma/2}) > \frac{3}{2} V_{S^n}.
\]

**Proof.** In fact, since \(|\sin t| \leq 1\),

\[
V(B_{\delta/2}) = \omega_{n-1} \int_0^{\delta/2} (\sin t)^{n-1} dt = \frac{3}{2} \omega_{n-1} \pi,
\]

\[
> 3 \omega_{n-1} \int_{\pi/2-\sigma/2}^{\pi/2} (\sin t)^{n-1} dt
\]

\[
= 3V(B_{\pi/2}) - V(B_{\pi/2-\sigma/2}) = \frac{3}{2} V_{S^n} - V(B_{\pi/2-\sigma/2}). \Box
\]

**Lemma 3.** There exists a point \(E\) on \(\partial B(p, \pi/2 - \sigma/2)\), that is, the boundary of \(\overline{B}(p, \pi/2 - \sigma/2)\), such that

\[
d(E, q) \leq \pi/2 - \sigma/2 + \delta \quad \text{and} \quad d(E, x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta.
\]

**Proof.** Since \(i_{\pi} \geq \pi - \sigma\), the boundary \(\partial B(p, \pi/2 - \sigma/2)\) is arc-connected in \(M\). Let \(W = \gamma_{p, q} \cap \partial B(p, \pi/2 - \sigma/2)\) and \(T = \gamma_{p, q} \cap \partial B(p, \pi/2 - \sigma/2)\). Take a continuous curve \(f(t) (t \in [0, 1])\) on \(\partial B(p, \pi/2 - \sigma/2)\) such that \(W = f(0)\) and \(T = f(1)\). Let \(\Gamma\) be the image curve of \(f\), and let

\[
\Gamma_1 = \{x \in \Gamma \mid d(x, q) \leq \pi/2 - \sigma/2 + \delta\},
\]

\[
\Gamma_2 = \{x \in \Gamma \mid d(x, x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta\}.
\]

It is clear that \(\Gamma_1\) and \(\Gamma_2\) both are nonempty closed since \(T \in \Gamma_1\) and \(W \in \Gamma_2\). We will prove that there exists a point \(E\) on \(\Gamma\) satisfying (5). For this, we need only to
verify that $\Gamma_1 \cap \Gamma_2 \neq \emptyset$. First we shall exclude the case that there exists a point $E$ in $\Gamma$ such that

\[(6) \quad d(E, q) > \pi/2 - \sigma/2 + \delta \quad \text{and} \quad d(E, x_0) > l_0 - \pi/2 + \sigma/2 + \delta.\]

In fact, if (6) occurs, there must exist a point $F$ in the shortest geodesic $\gamma_p$ issuing from $p$ and passing through $E$, such that $d(F, p) = \pi/2 - \sigma/2 + \delta/2$. By the triangle inequality, we have

\[(7) \quad d(F, q) \geq d(E, q) - d(E, F) > \pi/2 - \sigma/2 + \delta/2,\]

\[d(F, x_0) \geq d(E, x_0) - d(E, F) > l_0 - \pi/2 + \sigma/2 + \delta/2.\]

Therefore the four balls $B(p, \pi/2 - \sigma/2)$, $B(q, \pi/2 - \sigma/2)$, $B(x_0, l_0 - \pi/2 + \sigma/2)$ and $B(F, \delta/2)$ are pairwise disjoint. Applying again Günther’s volume comparison theorem, we get

\[V_{\kappa} > V(B(p, \pi/2 - \sigma/2)) + V(B(q, \pi/2 - \sigma/2))\]

\[\quad + V(B(x_0, l_0 - \pi/2 + \sigma/2)) + V(B(F, \delta/2))\]

\[\geq 2V(B_{\pi/2-\sigma/2}) + V(B_{\pi/2-\sigma/2}) + V(B_{\delta/2})\]

\[= 3V(B_{\pi/2-\sigma/2}) + V(B_{\delta/2}),\]

which contradicts the assumption on $V_{\kappa}$. Thus (6) cannot hold, which means $\Gamma = \Gamma_1 \cup \Gamma_2$. Since $\Gamma$ is connected, we get a point $E \in \Gamma_1 \cap \Gamma_2 \neq \emptyset$; this point clearly satisfies (5). \hfill \square

Lemma 1 and the triangle inequalities easily imply another result:

**Corollary 1.** The point $E$ obtained in Lemma 3 satisfies the inequalities

\[(8) \quad \pi/2 - \delta/6 \leq d(E, p) = \pi/2 - \sigma/2,\]

\[\pi/2 - \delta/6 \leq d(E, q) \leq \pi/2 - \sigma/2 + \delta,\]

\[\pi/2 - \delta/6 \leq d(E, x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta.\]

On the other hand,

\[(9) \quad d(p, q) \leq \pi - \sigma + \delta \quad \text{and} \quad \pi - \sigma \leq l_0 = d(p, x_0) \leq \pi - \sigma + \delta.\]

Take $E \in \partial B(p, \pi/2 - \sigma/2)$ satisfying (5). We consider a geodesic triangle $(\gamma_{E, p}, \gamma_{E, x_0}, \gamma_{p, x_0})$ in $\mathcal{M}$. Since $K_{\mathcal{M}} \geq -k^2$, Toponogov’s comparison theorem gives

\[(10) \quad \cosh kd(p, x_0)\]

\[\leq \cosh kd(E, p) \cosh kd(E, x_0) - \sinh kd(E, p) \sinh kd(E, x_0) \cos \alpha\]

\[= \cosh [kd(E, p) + d(E, x_0)] - \sinh kd(E, p) \sinh kd(E, x_0)](1 + \cos \alpha),\]
where the angle $\alpha$ is defined by $\alpha = \angle (\hat{\gamma}_{E,p}, \hat{\gamma}_{E,x_0})_E$. By Corollary 1, we have

$$1 + \cos \alpha \leq \frac{\cosh(k(d(E, p) + d(E, x_0))) - \cosh(kd(p, x_0))}{\sinh(kd(E, p)) \sinh(kd(E, x_0))}$$

$$\leq \frac{\cosh(k(l_0 + \delta)) - \cosh(kl_0)}{\sinh^2(k(\pi/2 - \delta/6))}.$$  

(11)

Clearly $t \mapsto \cosh(k(t+c)) - \cosh(kt)$ is increasing in $[0, \infty)$ for $c > 0$, so we get

$$1 + \cos \alpha \leq \frac{\cosh(k(\pi + 2\delta)) - \cosh(k(\pi + \delta))}{\sinh^2(k(\pi/2 - \delta/6))} \leq \frac{\cosh(k(\pi + 3\delta)) - \cosh(k(\pi + \delta))}{\sinh^2(k(\pi/2 - \delta/6))}.$$  

(12)

Similarly, if we consider the geodesic triangle $(\gamma_{E,p}, \gamma_{E,q}, \gamma_{p,q})$ and the angle $\beta = \angle (\hat{\gamma}_{E,p}, \hat{\gamma}_{E,q})_E$, we have

$$1 + \cos \beta \leq \frac{\cosh(k(d(E, p) + d(E, q))) - \cosh(kd(p, q))}{\sinh(kd(E, p)) \sinh(kd(E, q))}$$

$$\leq \frac{\cosh(k(\pi - \sigma + \delta)) - \cosh(kl_0)}{\sinh^2(k(\pi/2 - \delta/6))} \leq \frac{\cosh(k(\pi - \sigma + 2\delta)) - \cosh(k(\pi - \sigma))}{\sinh^2(k(\pi/2 - \delta/6))} \leq \frac{\cosh(k(\pi + 3\delta)) - \cosh(k(\pi + \delta))}{\sinh^2(k(\pi/2 - \delta/6))}.$$  

(13)

Likewise, if we think of the geodesic triangle $(\gamma_{E,q}, \gamma_{E,x_0}, \gamma_{q,x_0})$ and the angle $\gamma = \angle (\hat{\gamma}_{E,q}, \hat{\gamma}_{E,x_0})_E$, then, noting that $d(q, x_0) \geq l_0 \geq \pi - c_0$, we have

$$1 + \cos \gamma \leq \frac{\cosh(k(d(E, q) + d(E, x_0))) - \cosh(kd(q, x_0))}{\sinh(kd(E, q)) \sinh(kd(E, x_0))}$$

$$\leq \frac{\cosh(k(\pi + 3\delta)) - \cosh(k(\pi + \delta))}{\sinh^2(k(\pi/2 - \delta/6))}.$$  

(14)

Now we will conclude the proof of Proposition 1 using the following lemma, whose proof will be postponed.

**Lemma 4.** For $k > 0$, there exists a positive number $\delta_0 \in (0, 3\pi/5)$ such that $\delta_0$ is a solution of

$$\cosh(k(\pi + 3t)) - \cosh(k(\pi + t)) - (1 - \sqrt{3}/2) \sinh^2(k(\pi/2 - t/6)) = 0.$$  

(15)

Take $\delta = \delta_0$ in Lemma 4, take the $\sigma$ from Lemma 2, and let $E$ be the point given by Lemma 3. Obviously, $\sigma < \delta/3$, hence $\sigma + \delta < 4\delta/3 < \pi$. Applying (12)–(14),
one immediately deduces
\[ \cos \alpha < -\sqrt{3}/2, \quad \cos \beta < -\sqrt{3}/2, \quad \cos \gamma < -\sqrt{3}/2. \]
That is,
\[ \alpha > 2\pi/3, \quad \beta > 2\pi/3, \quad \gamma > 2\pi/3. \]
However, since \( 0 \leq \gamma \leq 2\pi - (\alpha + \beta) \), we get a contradiction. Thus our hypothesis on \( \mathcal{M} \) was wrong, so \( \mathcal{M} \) must be homeomorphic to \( S^n \).

In Theorem 1 or Proposition 1, we require that the sectional curvature \( K_{\mathcal{M}} \) is in the interval \( [-k^2, 1] \) for some \( k > 0 \). Trivially the result holds if \( K_{\mathcal{M}} \in (0, 1] \).

In the situation \( 0 \leq K_{\mathcal{M}} \leq 1 \), we can simplify our proof by comparing against Euclidean space; however the estimates (12)–(14) would need to be changed for the case \( k = 0 \).

**Theorem 2.** Suppose \((\mathcal{M}, g)\) is a compact connected \( n \)-dimensional Riemannian manifold with sectional curvature \( 0 \leq K_{\mathcal{M}} \leq 1 \). Let \( \delta > 0 \), and let
\[ \sigma = \frac{2}{3} \int_0^{\delta/2} (\sin t)^{n-1} dt \quad \text{such that} \quad (2 - \sqrt{3})(\pi - \sigma)^2 - 16\delta(\pi - \sigma + 2\delta) \geq 0. \]
Assume also that \( i_{\mathcal{M}} \geq \pi - \sigma \) and \( 0 < V_{\mathcal{M}} \leq 3V(\mathbb{B}_{\pi/2 - \sigma/2}) + V(\mathbb{B}_{\delta/2}). \) Then \( \mathcal{M} \) is homeomorphic to \( S^n \).

**Proof.** We prove this result by contradiction. If some manifold \( \mathcal{M} \) satisfies the assumptions of Theorem 2 and is not homeomorphic to \( S^n \), there is a point \( x_0 \in \mathcal{M} \) such that \( x_0 \in \mathcal{M} - \mathbb{B}(p, i_{\mathcal{M}}) \cup \mathbb{B}(q, i_{\mathcal{M}}) \), with \( d(p, q) = d_{\mathcal{M}} \). Assume that \( d(q, x_0) \geq d(p, x_0) = l_0 \geq i_{\mathcal{M}} \). By Lemma 3, there exists a point \( E \in \partial \mathbb{B}(p, \pi/2 - \sigma/2) \) satisfying (5). By triangle inequality, we get because \( K_{\mathcal{M}} \geq 0 \) that
\[ d(E, q) \geq \pi/2 - \sigma/2 \quad \text{and} \quad d(E, x_0) \geq \pi/2 - \sigma/2. \]

Now consider the geodesic triangle \((\gamma_{p,E}, \gamma_{x_0,E}, \gamma_{p,x_0})\); let \( \alpha = \angle(\dot{\gamma}_{E,p}, \dot{\gamma}_{E,x_0}) \). By Toponogov’s comparison theorem,
\[ d^2(p, x_0) \leq d^2(E, p) + d^2(E, x_0) - 2d(E, p)d(E, x_0) \cos \alpha, \]
so
\[ 1 + \cos \alpha \leq \frac{(d(E, p) + d(E, x_0))^2 - d^2(p, x_0)}{2d(E, p)d(E, x_0)} \]
\[ \leq \frac{(l_0 + \delta)^2 - l_0^2}{2(\pi/2 - \sigma/2)^2} < \frac{2\delta(\pi - \sigma + 2\delta)}{(\pi/2 - \sigma/2)^2}. \]
Similarly, consider the triangle \( (\gamma_{E,p}, \gamma_{E,q}, \gamma_{p,q}) \), with \( \beta = \angle(\gamma_{E,p}, \gamma_{E,q}) \) and the triangle \( (\gamma_{E,q}, \gamma_{E,x_0}, \gamma_{q,x_0}) \), with \( \gamma = \angle(\gamma_{E,q}, \gamma_{E,x_0}) \). Then

\[
1 + \cos \beta \leq \frac{(d(E, p) + d(E, q))^2 - d^2(p, q)}{2d(E, p)d(E, q)} \\
\leq \frac{(\pi - \sigma + \delta)^2 - (\pi - \sigma)^2}{2(\pi/2 - \sigma/2)^2} < \frac{2\delta(\pi - \sigma + 2\delta)}{(\pi/2 - \sigma/2)^2}.
\]

(19)

\[
1 + \cos \gamma \leq \frac{(d(E, q) + d(E, x_0))^2 - d^2(q, x_0)}{2d(E, q)d(E, x_0)} \\
\leq \frac{2\delta(l_0 + \delta)}{(\pi/2 - \sigma/2)^2} < \frac{2\delta(\pi - \sigma + 2\delta)}{(\pi/2 - \sigma/2)^2}.
\]

Let \( \delta \) and \( \sigma \) satisfy (16). Then from (18) and (19), one can infer again that

\[
\alpha > 2\pi/3, \quad \beta > 2\pi/3, \quad \gamma > 2\pi/3,
\]

which is impossible as above. \( \square \)

**Proof of Lemma 4.** First, we will show that the Equation (15) indeed contains a positive solution \( \delta_0 \). Define

\[
F(t, k) = \cosh(k(\pi + 3t)) - \cosh(k(\pi + t)) - (1 - \sqrt{3}/2) \sinh^2(k(\pi/2 - t/6)).
\]

For fixed \( k > 0 \) and for \( t \in [0, 3\pi] \),

\[
\frac{dF}{dt} = k \left\{ 3 \sinh(k(3t + \pi)) - \sinh(k(t + \pi)) + \frac{2-\sqrt{3}}{12} \sinh(k(\pi - t/3)) \right\} > 0,
\]

which implies that \( F(t, k) \) is increasing with respect to \( t \) in \( [0, 3\pi] \). Moreover, \( F(0, k) < 0 \) and \( F(3\pi, k) > 0 \). So (15) has a unique solution \( \delta_0 \in (0, 3\pi) \) for any \( k > 0 \). Consider the function \( k \mapsto F(3\pi/5, k) \). Then

\[
\frac{dF}{dk} \left( \frac{3\pi}{5}, k \right) = \frac{14\pi}{5} \sinh \left( \frac{14k\pi}{5} \right) - \frac{8\pi}{5} \sinh \left( \frac{8k\pi}{5} \right) - \frac{(2-\sqrt{3})\pi}{5} \sinh \left( \frac{4k\pi}{5} \right).
\]

We can check that

\[
\frac{14\pi}{5} \sinh \left( \frac{14k\pi}{5} \right) - \frac{8\pi}{5} \sinh \left( \frac{8k\pi}{5} \right) > \frac{4\pi}{5} e^{8\pi/5} > \frac{(2-\sqrt{3})\pi}{5} \sinh \left( \frac{4k\pi}{5} \right),
\]

which implies that \( F(3\pi/5, k) \) is increasing in \( [0, \infty) \). Note that \( F(3\pi/5, 0) = 0 \); thus \( F(3\pi/5, k) > 0 \) for \( k > 0 \). This shows there is a solution in \( 0 < \delta_0 < 3\pi/5 \). \( \square \)

**Acknowledgments**

I thank Professor Dong Ye, whose comments simplified many proofs of this paper. Also, I thank the referee for suggestions.
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Received October 5, 2008. Revised April 9, 2009.

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