A TOPOLOGICAL SPHERE THEOREM FOR ARBITRARY-DIMENSIONAL MANIFOLDS

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We study manifolds with bounded volume, sectional curvature, and injectivity radius. We obtain a topological sphere theorem.

Sphere theorems are common in differential geometry; one often asks whether a manifold is homeomorphic to a sphere under certain topological or geometric restrictions; see for instance [Grove and Shiohama 1977; Perelman 1995; Shen 1989; Shiohama 1983; Suyama 1991; Wu 1989]. Coghlan and Itokawa [1991] proved a sphere theorem that says that if an even-dimensional, simply connected Riemannian manifold \( \mathcal{M} \) has sectional curvature \( -k^2 \leq K_{\mathcal{M}} \leq 1 \), volume \( V_{\mathcal{M}} \leq \frac{3}{2} V_{S^n} \) with \( V_{S^n} \) the volume of the standard \( n \)-dimensional unit sphere \( S^n \) in \( \mathbb{R}^{n+1} \), then \( \mathcal{M} \) must be homeomorphic to \( S^n \). In [Wen 2004], we improved this result by relaxing the upper bound on \( V_{\mathcal{M}} \) to a bound larger than \( \frac{3}{2} V_{S^n} \). In both of these papers, the hypotheses of simple connectivity and even dimension were merely used to deduce that the injectivity radius \( i_{\mathcal{M}} \) is no less than \( \pi \). Here we find that we can weaken the assumptions on \( K_{\mathcal{M}} \) and \( i_{\mathcal{M}} \). If the simple connectivity condition is removed, the conclusion holds in any dimension.

Before stating our result, we introduce some notation. Let \((\mathcal{M}, g)\) be a compact, connected \( n \)-dimensional Riemannian manifold with metric \( g \). We denote by \( K_{\mathcal{M}} \) the sectional curvature of \( \mathcal{M} \), by \( i_{\mathcal{M}} \) its injectivity radius, and by \( V_{\mathcal{M}} \) its volume. For any points \( P, Q \in \mathcal{M} \), we denote by \( \gamma_{P,Q} \) the shortest geodesic on \( \mathcal{M} \) from \( P \) to \( Q \).

**Theorem 1.** Given \( k > 0 \), there exists an \( \varepsilon_0 > 0 \) such that if a compact connected \( n \)-dimensional Riemannian manifold \( (\mathcal{M}, g) \) satisfies

\[
-k^2 \leq K_{\mathcal{M}} \leq 1, \quad i_{\mathcal{M}} \geq \pi - \varepsilon_0, \quad V_{\mathcal{M}} \leq \frac{3}{2} V_{S^n} + \varepsilon_0,
\]

then \( \mathcal{M} \) is homeomorphic to \( S^n \).

The examples of real projective spaces \( \mathbb{R}P^n \) for \( n \geq 2 \) and product manifolds \( S^n \times S^m \) for \( m, n \geq 1 \) show that the hypotheses on the lower bound on \( i_{\mathcal{M}} \) or the upper bound on \( V_{\mathcal{M}} \) cannot be removed.

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In what follows, we denote by $\mathbb{B}(P, r)$ the open geodesic ball in $\mathcal{M}$ with center $P$ and radius $r$, and by $\overline{\mathbb{B}}(P, r)$ its closure. Also, we denote by $\mathcal{B}_r$ the open geodesic ball in $S^n$ with radius $r$. Instead of proving Theorem 1 directly, we will prove a more precise version.

**Proposition 1.** Let $k > 0$. There exist $\delta, \sigma > 0$ satisfying $\sigma + \delta < \pi$ such that if a compact connected $n$-dimensional Riemannian manifold $(\mathcal{M}, g)$ satisfies

\( -k^2 \leq K_{\mathcal{M}} \leq 1, \quad i_{\mathcal{M}} \geq \pi - \sigma, \quad V_{\mathcal{M}} \leq 3V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\delta/2}), \)

then $\mathcal{M}$ is homeomorphic to $S^n$.

**Remark 1.** The choice of $\sigma$ or $\delta$ here is of course not optimal. We conjecture that $\sigma < \pi/2$ is optimal.

**Proof of Proposition 1.** We proceed by way of contradiction. Suppose there exists a manifold $\mathcal{M}$ satisfying (1) that is not homeomorphic to $S^n$. Take points $p, q$ in $\mathcal{M}$ such that $d(p, q) = d_{\mathcal{M}}$, the diameter $d_{\mathcal{M}}$ of $\mathcal{M}$. Then by a well-known topological fact (see for instance [Brown 1960]), there is a point $x_0 \in \mathcal{M} - \mathbb{B}(p, i_{\mathcal{M}}) \cup \mathbb{B}(q, i_{\mathcal{M}})$.

Without loss of generality, let $d(q, x_0) \geq d(p, x_0) = l_0$. Therefore $l_0 \geq i_{\mathcal{M}} \geq \pi - \sigma$.

First we show an explicit upper bound on $d_{\mathcal{M}}$.

**Lemma 1.** $d_{\mathcal{M}} \leq \pi - \sigma + \delta$.

**Proof:** We argue by contradiction. If $d_{\mathcal{M}} > \pi - \sigma + \delta$, then we consider the balls $\mathbb{B}(p, \pi/2 - \sigma/2 + \delta/2)$, $\mathbb{B}(q, \pi/2 - \sigma/2 + \delta/2)$ and $\mathbb{B}(x_0, l_0 - \pi/2 + \sigma/2 - \delta/2)$. They are obviously pairwise disjoint. Therefore since $K_{\mathcal{M}} \leq 1$, Günther’s volume comparison theorem gives

\( V_{\mathcal{M}} \geq 2V(\mathcal{B}_{\pi/2-\sigma/2+\delta/2}) + V(\mathcal{B}_{l_0-\pi/2+\sigma/2-\delta/2}). \)

In what follows, we check that

\( 2V(\mathcal{B}_{\pi/2-\sigma/2+\delta/2}) + V(\mathcal{B}_{l_0-\pi/2+\sigma/2-\delta/2}) > 3V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\delta/2}). \)

Noting that $l_0 - \pi/2 + \sigma/2 - \delta/2 \geq \pi/2 - \sigma/2 - \delta/2 > 0$, we have

\( V(\mathcal{B}_{l_0-\pi/2+\sigma/2-\delta/2}) \geq V(\mathcal{B}_{\pi/2-\sigma/2-\delta/2}). \)

By the definition of $S^n$, we have $V(\mathcal{B}_r) = \omega_{n-1} \int_0^r (\sin t)^{n-1} \, dt$ for any $r > 0$, where $\omega_{n-1}$ is the volume of the standard unit $(n-1)$-sphere $S^{n-1}$. Since $\sin t$ is
increasing in \((0, \pi/2)\), we have
\[
\frac{1}{\omega_{n-1}} \left[ 2V(\mathcal{B}_{\pi/2-\sigma/2}^+\mathcal{B}/2) + V(\mathcal{B}_{\pi/2-\sigma/2}^+\mathcal{B}/2) - 3V(\mathcal{B}_{\pi/2-\sigma/2}^+\mathcal{B}/2) - V(\mathcal{B}_{\pi/2-\sigma/2}^+\mathcal{B}/2) \right]
\geq 2 \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt + \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt
- 3 \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt - \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt
\]
\[
= \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt - \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt
+ \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt - \int_0^{\pi/2-\sigma/2} (\sin t)^{n-1} dt
\geq 0.
\]
Clearly, the estimates (2) and (3) contradict the assumptions (1). \(\square\)

**Lemma 2.** If \(\delta > 0\) and \(\sigma = 2/3\) \(\int_0^{\pi/2} (\sin t)^{n-1} dt\) satisfy \(\sigma + \delta < \pi\), then
\[
V(\mathcal{B}_{\delta/2}) + V(\mathcal{B}_{\pi/2-\sigma/2}) > \frac{3}{2} V S^n.
\]

**Proof.** In fact, since \(|\sin t| \leq 1\),
\[
V(\mathcal{B}_{\delta/2}) = \omega_{n-1} \int_0^{\delta/2} (\sin t)^{n-1} dt = \frac{3}{2} \omega_{n-1} \sigma
\]
\[
> 3 \omega_{n-1} \int_0^{\pi/2} (\sin t)^{n-1} dt
\]
\[
= 3V(\mathcal{B}_{\pi/2}) - V(\mathcal{B}_{\pi/2-\sigma/2}) = \frac{3}{2} V S^n - V(\mathcal{B}_{\pi/2-\sigma/2}).\quad \square
\]

**Lemma 3.** There exists a point \(E\) on \(\partial \mathbb{B}(p, \pi/2 - \sigma/2)\), that is, the boundary of \(\overline{\mathbb{B}}(P, \pi/2 - \sigma/2)\), such that
\[
d(E, q) \leq \pi/2 - \sigma/2 + \delta \quad \text{and} \quad d(E, x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta.
\]

**Proof.** Since \(i_{\mathcal{M}} \geq \pi - \sigma\), the boundary \(\partial \mathbb{B}(p, \pi/2 - \sigma/2)\) is arc-connected in \(\mathcal{M}\). Let \(W = \gamma_{p, x_0} \cap \partial \mathbb{B}(p, \pi/2 - \sigma/2)\) and \(T = \gamma_{p, q} \cap \partial \mathbb{B}(p, \pi/2 - \sigma/2)\). Take a continuous curve \(f(t) (t \in [0, 1])\) on \(\partial \mathbb{B}(p, \pi/2 - \sigma/2)\) such that \(W = f(0)\) and \(T = f(1)\). Let \(\Gamma\) be the image curve of \(f\), and let
\[
\Gamma_1 = \{x \in \Gamma \mid d(x, q) \leq \pi/2 - \sigma/2 + \delta\},
\]
\[
\Gamma_2 = \{x \in \Gamma \mid d(x, x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta\}.
\]
It is clear that \(\Gamma_1\) and \(\Gamma_2\) both are nonempty closed since \(T \in \Gamma_1\) and \(W \in \Gamma_2\). We will prove that there exists a point \(E\) on \(\Gamma\) satisfying (5). For this, we need only to
verify that \( \Gamma_1 \cap \Gamma_2 \neq \emptyset \). First we shall exclude the case that there exists a point \( E \) in \( \Gamma \) such that

\begin{equation}
(6) \quad d(E, q) > \pi/2 - \sigma/2 + \delta \quad \text{and} \quad d(E, x_0) > l_0 - \pi/2 + \sigma/2 + \delta.
\end{equation}

In fact, if (6) occurs, there must exist a point \( F \) in the shortest geodesic \( \gamma_p \) issuing from \( p \) and passing through \( E \), such that \( d(F, p) = \pi/2 - \sigma/2 + \delta/2 \). By the triangle inequality, we have

\begin{equation}
(7) \quad d(F, q) \geq d(E, q) - d(E, F) > \pi/2 - \sigma/2 + \delta/2, \\
       d(F, x_0) \geq d(E, x_0) - d(E, F) > l_0 - \pi/2 + \sigma/2 + \delta/2.
\end{equation}

Therefore the four balls \( \mathbb{B}(p, \pi/2 - \sigma/2), \mathbb{B}(q, \pi/2 - \sigma/2), \mathbb{B}(x_0, l_0 - \pi/2 + \sigma/2) \) and \( \mathbb{B}(F, \delta/2) \) are pairwise disjoint. Applying again Günther’s volume comparison theorem, we get

\[
V_{\mathcal{M}} > V(\mathbb{B}(p, \pi/2 - \sigma/2)) + V(\mathbb{B}(q, \pi/2 - \sigma/2)) \\
\quad \quad \quad \quad \quad + V(\mathbb{B}(x_0, l_0 - \pi/2 + \sigma/2)) + V(\mathbb{B}(F, \delta/2)) \\
\quad \quad \quad \quad \quad \geq 2V(\mathbb{B}_{\pi/2-\sigma/2}) + V(\mathbb{B}_{\pi/2-\sigma/2}) + V(\mathbb{B}_{\delta/2}) \\
\quad \quad \quad \quad \quad = 3V(\mathbb{B}_{\pi/2-\sigma/2}) + V(\mathbb{B}_{\delta/2}),
\]

which contradicts the assumption on \( V_{\mathcal{M}} \). Thus (6) cannot hold, which means \( \Gamma = \Gamma_1 \cup \Gamma_2 \). Since \( \Gamma \) is connected, we get a point \( E \in \Gamma_1 \cap \Gamma_2 \neq \emptyset \); this point clearly satisfies (5). \( \square \)

**Lemma 1** and the triangle inequalities easily imply another result:

**Corollary 1.** The point \( E \) obtained in **Lemma 3** satisfies the inequalities

\[
\pi/2 - \delta/6 < d(E, p) = \pi/2 - \sigma/2, \\
\pi/2 - \delta/6 \leq d(E, q) \leq \pi/2 - \sigma/2 + \delta, \\
\pi/2 - \delta/6 \leq d(E, x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta.
\]

On the other hand,

\begin{equation}
(9) \quad d(p, q) \leq \pi - \sigma + \delta \quad \text{and} \quad \pi - \sigma \leq l_0 = d(p, x_0) \leq \pi - \sigma + \delta.
\end{equation}

Take \( E \in \partial \mathcal{B}(p, \pi/2 - \sigma/2) \) satisfying (5). We consider a geodesic triangle \( (\gamma_{E, p}, \gamma_{E,x_0}, \gamma_{p,x_0}) \) in \( \mathcal{M} \). Since \( K_{\mathcal{M}} \geq -k^2 \), Toponogov’s comparison theorem gives

\[
(10) \quad \cosh[kd(p, x_0)] \\
\quad \quad \leq \cosh[kd(E, p)] \cosh[kd(E, x_0)] - \sinh[kd(E, p)] \sinh[kd(E, x_0)] \cos \alpha \\
\quad \quad = \cosh[k(d(E, p) + d(E, x_0)) - \sinh[kd(E, p)] \sinh[kd(E, x_0)](1+\cos \alpha),
\]
where the angle \( \alpha \) is defined by \( \alpha = \angle(\hat{\gamma}_{E,p}, \hat{\gamma}_{E,x_0}) \). By Corollary 1, we have

\[
1 + \cos \alpha \leq \frac{\cosh(k(d(E, p) + d(E, x_0))) - \cosh(kd(p, x_0))}{\sinh(kd(E, p)) \sinh(kd(E, x_0))} \leq \frac{\cosh(k(l_0 + \delta)) - \cosh(kl_0)}{\sinh^2(k(\pi/2 - \delta/6))}.
\]

(11)

Clearly \( t \mapsto \cosh(k(t + c)) - \cosh(k t) \) is increasing in \([0, \infty)\) for \( c > 0 \), so we get

\[
1 + \cos \alpha < \frac{\cosh(k(\pi + 2\delta)) - \cosh(k(\pi + \delta))}{\sinh^2(k(\pi/2 - \delta/6))} < \frac{\cosh(k(\pi + 3\delta)) - \cosh(k(\pi + \delta))}{\sinh^2(k(\pi/2 - \delta/6))}.
\]

(12)

Similarly, if we consider the geodesic triangle \((\gamma_{E,p}, \gamma_{E,q}, \gamma_{p,q})\) and the angle \( \beta = \angle(\hat{\gamma}_{E,p}, \hat{\gamma}_{E,q}) \), we have

\[
1 + \cos \beta \leq \frac{\cosh(k(d(E, p) + d(E, q))) - \cosh(kd(p, q))}{\sinh(kd(E, p)) \sinh(kd(E, q))} \leq \frac{\cosh(k(\pi - \sigma + \delta)) - \cosh(kl_0)}{\sinh^2(k(\pi/2 - \delta/6))} \leq \frac{\cosh(k(\pi - \sigma + 2\delta)) - \cosh(k(\pi - \sigma))}{\sinh^2(k(\pi/2 - \delta/6))} < \frac{\cosh(k(\pi + 3\delta)) - \cosh(k(\pi + \delta))}{\sinh^2(k(\pi/2 - \delta/6))}.
\]

(13)

Likewise, if we think of the geodesic triangle \((\gamma_{E,q}, \gamma_{E,x_0}, \gamma_{q,x_0})\) and the angle \( \gamma = \angle(\hat{\gamma}_{E,q}, \hat{\gamma}_{E,x_0}) \) then, noting that \( d(q, x_0) \geq l_0 \geq \pi - \varepsilon_0 \), we have

\[
1 + \cos \gamma \leq \frac{\cosh(k(d(E, q) + d(E, x_0))) - \cosh(kd(q, x_0))}{\sinh(kd(E, q)) \sinh(kd(E, x_0))} < \frac{\cosh(k(\pi + 3\delta)) - \cosh(k(\pi + \delta))}{\sinh^2(k(\pi/2 - \delta/6))}.
\]

(14)

Now we will conclude the proof of Proposition 1 using the following lemma, whose proof will be postponed.

Lemma 4. For \( k > 0 \), there exists a positive number \( \delta_0 \in (0, 3\pi/5) \) such that \( \delta_0 \) is a solution of

\[
cosh(k(\pi + 3\delta)) - \cosh(k(\pi + \delta)) - (1 - \sqrt{3}/2) \sinh^2(k(\pi/2 - \delta/6)) = 0.
\]

(15)

Take \( \delta = \delta_0 \) in Lemma 4, take the \( \sigma \) from Lemma 2, and let \( E \) be the point given by Lemma 3. Obviously, \( \sigma < \delta/3 \), hence \( \sigma + \delta < 4\delta/3 < \pi \). Applying (12)–(14),
one immediately deduces
\[ \cos \alpha < -\sqrt{3}/2, \quad \cos \beta < -\sqrt{3}/2, \quad \cos \gamma < -\sqrt{3}/2. \]

That is,
\[ \alpha > 2\pi/3, \quad \beta > 2\pi/3, \quad \gamma > 2\pi/3. \]

However, since \(0 \leq \gamma \leq 2\pi - (\alpha + \beta)\), we get a contradiction. Thus our hypothesis on \(\mathcal{M}\) was wrong, so \(\mathcal{M}\) must be homeomorphic to \(S^n\). \(\square\)

In Theorem 1 or Proposition 1, we require that the sectional curvature \(K_{\mathcal{M}}\) is in the interval \([-k^2, 1]\) for some \(k > 0\). Trivially the result holds if \(K_{\mathcal{M}} \in (0, 1]\). In the situation \(0 \leq K_{\mathcal{M}} \leq 1\), we can simplify our proof by comparing against Euclidean space; however the estimates (12)–(14) would need to be changed for the case \(k = 0\).

**Theorem 2.** Suppose \((\mathcal{M}, g)\) is a compact connected n-dimensional Riemannian manifold with sectional curvature \(0 \leq K_{\mathcal{M}} \leq 1\). Let \(\delta > 0\), and let

\[ \sigma = \frac{2}{3} \int_0^{\delta/2} (\sin t)^{n-1} \, dt \quad \text{such that} \quad (2-\sqrt{3})(\pi-\sigma)^2 - 16\delta(\pi-\sigma+2\delta) \geq 0. \]

Assume also that \(i_{\mathcal{M}} \geq \pi - \sigma\) and \(0 < V_{\mathcal{M}} \leq 3V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\delta/2})\). Then \(\mathcal{M}\) is homeomorphic to \(S^n\).

**Proof.** We prove this result by contradiction. If some manifold \(\mathcal{M}\) satisfies the assumptions of Theorem 2 and is not homeomorphic to \(S^n\), there is a point \(x_0 \in \mathcal{M}\) such that \(x_0 \in \mathcal{M} - \mathcal{B}(p, i_{\mathcal{M}}) \cup \mathcal{B}(q, i_{\mathcal{M}})\), with \(d(p, q) = d_{\mathcal{M}}\). Assume that \(d(q, x_0) \geq d(p, x_0) = l_0 \geq i_{\mathcal{M}}\). By Lemma 3, there exists a point \(E \in \partial \mathcal{B}(p, \pi/2 - \sigma/2)\) satisfying (5). By triangle inequality, we get because \(K_{\mathcal{M}} \geq 0\) that

\[ d(E, q) \geq \pi/2 - \sigma/2 \quad \text{and} \quad d(E, x_0) \geq \pi/2 - \sigma/2. \]

Now consider the geodesic triangle \((\gamma_{p,E}, \gamma_{x_0,E}, \gamma_{p,x_0})\); let \(\alpha = \angle(\gamma_{E,p}, \gamma_{E,x_0})\)\(E\). By Toponogov’s comparison theorem,

\[ d^2(p, x_0) \leq d^2(E, p) + d^2(E, x_0) - 2d(E, p)d(E, x_0) \cos \alpha, \]

so

\[ 1 + \cos \alpha \leq \frac{(d(E, p) + d(E, x_0))^2 - d^2(p, x_0)}{2d(E, p)d(E, x_0)} \]

\[ \leq \frac{(l_0 + \delta)^2 - l_0^2}{2(\pi/2 - \sigma/2)^2} \leq \frac{2\delta(\pi - \sigma + 2\delta)}{(\pi/2 - \sigma/2)^2}. \]
Similarly, consider the triangle \((\gamma_{E,p}, \gamma_{E,q}, \gamma_{p,q})\), with \(\beta = \angle(\hat{\gamma}_{E,p}, \hat{\gamma}_{E,q})\) and the triangle \((\gamma_{E,q}, \gamma_{E,x_0}, \gamma_{q,x_0})\), with \(\gamma = \angle(\hat{\gamma}_{E,q}, \hat{\gamma}_{E,x_0})\). Then

\[
1 + \cos \beta \leq \frac{(d(E, p) + d(E, q))^2 - d^2(p, q)}{2d(E, p)d(E, q)} \leq \frac{(\pi - \sigma + \delta)^2 - (\pi - \sigma)^2}{2(\pi/2 - \sigma/2)^2} < \frac{2\delta(\pi - \sigma + 2\delta)}{(\pi/2 - \sigma/2)^2},
\]

(19)

\[
1 + \cos \gamma \leq \frac{(d(E, q) + d(E, x_0))^2 - d^2(q, x_0)}{2d(E, q)d(E, x_0)} \leq \frac{2\delta(l_0 + \delta)}{(\pi/2 - \sigma/2)^2} < \frac{2\delta(\pi - \sigma + 2\delta)}{(\pi/2 - \sigma/2)^2}.
\]

Let \(\delta\) and \(\sigma\) satisfy (16). Then from (18) and (19), one can infer again that

\[
\alpha > 2\pi/3, \quad \beta > 2\pi/3, \quad \gamma > 2\pi/3,
\]

which is impossible as above.

Proof of Lemma 4. First, we will show that the Equation (15) indeed contains a positive solution \(\delta_0\). Define

\[
F(t, k) = \cosh(k(\pi + 3t)) - \cosh(k(\pi + t)) - (1 - \sqrt{3}/2) \sinh^2(k(\pi/2 - t/6)).
\]

For fixed \(k > 0\) and for \(t \in [0, 3\pi]\),

\[
\frac{dF}{dt} = k\left\{3 \sinh(3t + \pi) - \sinh(k(t + \pi)) + \frac{2 - \sqrt{3}}{12} \sinh(k(\pi - t/3))\right\} > 0,
\]

which implies that \(F(t, k)\) is increasing with respect to \(t\) in \([0, 3\pi]\). Moreover, \(F(0, k) < 0\) and \(F(3\pi, k) > 0\). So (15) has a unique solution \(\delta_0 \in (0, 3\pi)\) for any \(k > 0\). Consider the function \(k \mapsto F(3\pi/5, k)\). Then

\[
\frac{dF}{dk}(\frac{3\pi}{5}, k) = \frac{14\pi}{5} \sinh\left(\frac{14k\pi}{5}\right) - \frac{8\pi}{5} \sinh\left(\frac{8k\pi}{5}\right) - \frac{(2 - \sqrt{3})\pi}{5} \sinh\left(\frac{4k\pi}{5}\right).
\]

We can check that

\[
\frac{14\pi}{5} \sinh\left(\frac{14k\pi}{5}\right) - \frac{8\pi}{5} \sinh\left(\frac{8k\pi}{5}\right) > \frac{4\pi}{5} e^{8\pi/5} > \frac{(2 - \sqrt{3})\pi}{5} \sinh\left(\frac{4k\pi}{5}\right),
\]

which implies that \(F(3\pi/5, k)\) is increasing in \([0, \infty)\). Note that \(F(3\pi/5, 0) = 0\); thus \(F(3\pi/5, k) > 0\) for \(k > 0\). This shows there is a solution in \(0 < \delta_0 < 3\pi/5\). □

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References


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