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# A TOPOLOGICAL SPHERE THEOREM FOR ARBITRARY-DIMENSIONAL MANIFOLDS

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**We study manifolds with bounded volume, sectional curvature, and injectivity radius. We obtain a topological sphere theorem.**

Sphere theorems are common in differential geometry; one often asks whether a manifold is homeomorphic to a sphere under certain topological or geometric restrictions; see for instance [Grove and Shiohama 1977; Perelman 1995; Shen 1989; Shiohama 1983; Suyama 1991; Wu 1989]. Coghlan and Itokawa [1991] proved a sphere theorem that says that if an even-dimensional, simply connected Riemannian manifold $\mathcal{M}$ has sectional curvature $K_{\mathcal{M}} \in (0, 1]$, volume $V_{\mathcal{M}} \le \frac{3}{2} V_{S^n}$ with $V_{S^n}$ the volume of the standard $n$-dimensional unit sphere $S^n$ in $\mathbb{R}^{n+1}$, then $\mathcal{M}$ must be homeomorphic to $S^n$. In [Wen 2004], we improved this result by relaxing the upper bound on $V_{\mathcal{M}}$ to a bound larger than $\frac{3}{2} V_{S^n}$. In both of these papers, the hypotheses of simple connectivity and even dimension were merely used to deduce that the injectivity radius $i_{\mathcal{M}}$ is no less than $\pi$. Here we find that we can weaken the assumptions on $K_{\mathcal{M}}$ and $i_{\mathcal{M}}$. If the simple connectivity condition is removed, the conclusion holds in *any dimension*.

Before stating our result, we introduce some notation. Let $(\mathcal{M}, g)$ be a compact, connected $n$-dimensional Riemannian manifold with metric $g$. We denote by $K_{\mathcal{M}}$ the sectional curvature of $\mathcal{M}$, by $i_{\mathcal{M}}$ its injectivity radius, and by $V_{\mathcal{M}}$ its volume. For any points $P, Q \in \mathcal{M}$, we denote by $\gamma_{P,Q}$ the shortest geodesic on $\mathcal{M}$ from $P$ to $Q$.

**Theorem 1.** *Given $k > 0$, there exists an $\varepsilon_0 > 0$ such that if a compact connected $n$-dimensional Riemannian manifold $(\mathcal{M}, g)$ satisfies*

$$-k^2 \le K_{\mathcal{M}} \le 1, \quad i_{\mathcal{M}} \ge \pi - \varepsilon_0, \quad V_{\mathcal{M}} \le \tfrac{3}{2} V_{S^n} + \varepsilon_0,$$

*then $\mathcal{M}$ is homeomorphic to $S^n$.*

The examples of real projective spaces $\mathbb{R}P^n$ for $n \ge 2$ and product manifolds $S^n \times S^m$ for $m, n \ge 1$ show that the hypotheses on the lower bound on $i_{\mathcal{M}}$ or the upper bound on $V_{\mathcal{M}}$ cannot be removed.

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In what follows, we denote by $\mathbb{B}(P, r)$ the open geodesic ball in $\mathcal{M}$ with center $P$ and radius $r$, and by $\overline{\mathbb{B}}(P, r)$ its closure. Also, we denote by $\mathcal{B}_r$ the open geodesic ball in $S^n$ with radius $r$. Instead of proving Theorem 1 directly, we will prove a more precise version.

**Proposition 1.** *Let $k > 0$. There exist $\delta, \sigma > 0$ satisfying $\sigma + \delta < \pi$ such that if a compact connected $n$-dimensional Riemannian manifold $(\mathcal{M}, g)$ satisfies*

$$-k^2 \le K_{\mathcal{M}} \le 1, \quad i_{\mathcal{M}} \ge \pi - \sigma, \quad V_{\mathcal{M}} \le 3V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\delta/2}), \tag{1}$$

*then $\mathcal{M}$ is homeomorphic to $S^n$.*

**Remark 1.** The choice of $\sigma$ or $\delta$ here is of course not optimal. We conjecture that $\sigma < \pi/2$ is optimal.

*Proof of Proposition 1.* We proceed by way of contradiction. Suppose there exists a manifold $\mathcal{M}$ satisfying (1) that is not homeomorphic to $S^n$. Take points $p, q$ in $\mathcal{M}$ such that $d(p, q) = d_{\mathcal{M}}$, the diameter $d_{\mathcal{M}}$ of $\mathcal{M}$. Then by a well-known topological fact (see for instance [Brown 1960]), there is a point $x_0 \in \mathcal{M} - \mathbb{B}(p, i_{\mathcal{M}}) \cup \mathbb{B}(q, i_{\mathcal{M}})$. Without loss of generality, let $d(q, x_0) \ge d(p, x_0) = l_0$. Therefore $l_0 \ge i_{\mathcal{M}} \ge \pi - \sigma$. First we show an explicit upper bound on $d_{\mathcal{M}}$.

**Lemma 1.** *$d_{\mathcal{M}} \le \pi - \sigma + \delta$.*

*Proof.* We argue by contradiction. If $d_{\mathcal{M}} > \pi - \sigma + \delta$, then we consider the balls $\mathbb{B}(p, \pi/2 - \sigma/2 + \delta/2)$, $\mathbb{B}(q, \pi/2 - \sigma/2 + \delta/2)$ and $\mathbb{B}(x_0, l_0 - \pi/2 + \sigma/2 - \delta/2)$. They are obviously pairwise disjoint. Therefore since $K_{\mathcal{M}} \le 1$, Günther's volume comparison theorem gives

$$V_{\mathcal{M}} \ge 2V(\mathcal{B}_{\pi/2-\sigma/2+\delta/2}) + V(\mathcal{B}_{l_0-\pi/2+\sigma/2-\delta/2}). \tag{2}$$

In what follows, we check that

$$2V(\mathcal{B}_{\pi/2-\sigma/2+\delta/2}) + V(\mathcal{B}_{l_0-\pi/2+\sigma/2-\delta/2}) > 3V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\delta/2}). \tag{3}$$

Noting that $l_0 - \pi/2 + \sigma/2 - \delta/2 \ge \pi/2 - \sigma/2 - \delta/2 > 0$, we have

$$V(\mathcal{B}_{l_0-\pi/2+\sigma/2-\delta/2}) \ge V(\mathcal{B}_{\pi/2-\sigma/2-\delta/2}).$$

By the definition of $S^n$, we have $V(\mathcal{B}_r) = \omega_{n-1} \int_0^r (\sin t)^{n-1} \mathrm{d}t$ for any $r > 0$, where $\omega_{n-1}$ is the volume of the standard unit $(n-1)$-sphere $S^{n-1}$. Since $\sin t$ is increasing in $(0, \pi/2)$, we have

$$\begin{aligned}\frac{1}{\omega_{n-1}}\big[2V(\mathcal{B}_{\pi/2-\sigma/2+\delta/2}) + V(\mathcal{B}_{l_0-\pi/2+\sigma/2-\delta/2}) - 3V(\mathcal{B}_{\pi/2-\sigma/2}) - V(\mathcal{B}_{\delta/2})\big] \\ \geq 2\int_0^{\pi/2-\sigma/2+\delta/2}(\sin t)^{n-1}\mathrm{d}t + \int_0^{\pi/2-\sigma/2-\delta/2}(\sin t)^{n-1}\mathrm{d}t \\ - 3\int_0^{\pi/2-\sigma/2}(\sin t)^{n-1}\mathrm{d}t - \int_0^{\delta/2}(\sin t)^{n-1}\mathrm{d}t \\ = \int_{\pi/2-\sigma/2}^{\pi/2-\sigma/2+\delta/2}(\sin t)^{n-1}\mathrm{d}t - \int_{\pi/2-\sigma/2-\delta/2}^{\pi/2-\sigma/2}(\sin t)^{n-1}\mathrm{d}t \\ + \int_{\pi/2-\sigma/2}^{\pi/2-\sigma/2+\delta/2}(\sin t)^{n-1}\mathrm{d}t - \int_0^{\delta/2}(\sin t)^{n-1}\mathrm{d}t \\ > \int_{\pi/2-\sigma/2}^{\pi/2-\sigma/2+\delta/2}(\sin t)^{n-1}\mathrm{d}t - \int_{\pi/2-\sigma/2-\delta/2}^{\pi/2-\sigma/2}(\sin t)^{n-1}\mathrm{d}t > 0.\end{aligned}$$

Clearly, the estimates (2) and (3) contradict the assumptions (1). □

**Lemma 2.** *If $\delta > 0$ and $\sigma = 2/3\int_0^{\delta/2}(\sin t)^{n-1}\mathrm{d}t$ satisfy $\sigma + \delta < \pi$, then*

$$V(\mathcal{B}_{\delta/2}) + V(\mathcal{B}_{\pi/2-\sigma/2}) > \tfrac{3}{2}V_{S^n}. \tag{4}$$

*Proof.* In fact, since $|\sin t| \leq 1$,

$$\begin{aligned}V(\mathcal{B}_{\delta/2}) &= \omega_{n-1}\int_0^{\delta/2}(\sin t)^{n-1}\mathrm{d}t = \tfrac{3}{2}\omega_{n-1}\sigma \\ &> 3\omega_{n-1}\int_{\pi/2-\sigma/2}^{\pi/2}(\sin t)^{n-1}\mathrm{d}t \\ &= 3V(\mathcal{B}_{\pi/2}) - V(\mathcal{B}_{\pi/2-\sigma/2}) = \tfrac{3}{2}V_{S^n} - V(\mathcal{B}_{\pi/2-\sigma/2}).\end{aligned}$$ □

**Lemma 3.** *There exists a point $E$ on $\partial\mathbb{B}(p, \pi/2 - \sigma/2)$, that is, the boundary of $\overline{\mathbb{B}}(P, \pi/2-\sigma/2)$, such that*

$$d(E, q) \leq \pi/2 - \sigma/2 + \delta \quad \text{and} \quad d(E, x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta. \tag{5}$$

*Proof.* Since $i_{\mathcal{M}} \geq \pi - \sigma$, the boundary $\partial\mathbb{B}(p, \pi/2 - \sigma/2)$ is arc-connected in $\mathcal{M}$. Let $W = \gamma_{p,x_0} \cap \partial\mathbb{B}(p, \pi/2 - \sigma/2)$ and $T = \gamma_{p,q} \cap \partial\mathbb{B}(p, \pi/2 - \sigma/2)$. Take a continuous curve $f(t)$ $(t \in 0, 1])$ on $\partial\mathbb{B}(p, \pi/2 - \sigma/2)$ such that $W = f(0)$ and $T = f(1)$. Let $\Gamma$ be the image curve of $f$, and let

$$\begin{aligned}\Gamma_1 &= \{x \in \Gamma \mid d(x, q) \leq \pi/2 - \sigma/2 + \delta\}, \\ \Gamma_2 &= \{x \in \Gamma \mid d(x, x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta\}.\end{aligned}$$

It is clear that $\Gamma_1$ and $\Gamma_2$ both are nonempty closed since $T \in \Gamma_1$ and $W \in \Gamma_2$. We will prove that there exists a point $E$ on $\Gamma$ satisfying (5). For this, we need only to verify that $\Gamma_1 \cap \Gamma_2 \neq \varnothing$. First we shall exclude the case that there exists a point $E$ in $\Gamma$ such that

$$d(E,q) > \pi/2 - \sigma/2 + \delta \quad \text{and} \quad d(E,x_0) > l_0 - \pi/2 + \sigma/2 + \delta. \tag{6}$$

In fact, if (6) occurs, there must exist a point $F$ in the shortest geodesic $\overline{\gamma}_p$ issuing from $p$ and passing through $E$, such that $d(F,p) = \pi/2 - \sigma/2 + \delta/2$. By the triangle inequality, we have

$$\begin{aligned} d(F,q) &\geq d(E,q) - d(E,F) > \pi/2 - \sigma/2 + \delta/2, \\ d(F,x_0) &\geq d(E,x_0) - d(E,F) > l_0 - \pi/2 + \sigma/2 + \delta/2. \end{aligned} \tag{7}$$

Therefore the four balls $\mathbb{B}(p, \pi/2 - \sigma/2)$, $\mathbb{B}(q, \pi/2 - \sigma/2)$, $\mathbb{B}(x_0, l_0 - \pi/2 + \sigma/2)$ and $\mathbb{B}(F, \delta/2)$ are pairwise disjoint. Applying again Günther's volume comparison theorem, we get

$$\begin{aligned} V_{\mathcal{M}} &> V(\mathbb{B}(p, \pi/2 - \sigma/2)) + V(\mathbb{B}(q, \pi/2 - \sigma/2)) \\ &\qquad + V(\mathbb{B}(x_0, l_0 - \pi/2 + \sigma/2)) + V(\mathbb{B}(F, \delta/2)) \\ &\geq 2V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\delta/2}) \\ &= 3V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\delta/2}), \end{aligned}$$

which contradicts the assumption on $V_{\mathcal{M}}$. Thus (6) cannot hold, which means $\Gamma = \Gamma_1 \cup \Gamma_2$. Since $\Gamma$ is connected, we get a point $E \in \Gamma_1 \cap \Gamma_2 \neq \varnothing$; this point clearly satisfies (5). □

Lemma 1 and the triangle inequalities easily imply another result:

**Corollary 1.** *The point $E$ obtained in Lemma 3 satisfies the inequalities*

$$\begin{aligned} \pi/2 - \delta/6 &< d(E,p) = \pi/2 - \sigma/2, \\ \pi/2 - \delta/6 &\leq d(E,q) \leq \pi/2 - \sigma/2 + \delta, \\ \pi/2 - \delta/6 &\leq d(E,x_0) \leq l_0 - \pi/2 + \sigma/2 + \delta. \end{aligned} \tag{8}$$

*On the other hand,*

$$d(p,q) \leq \pi - \sigma + \delta \quad \textit{and} \quad \pi - \sigma \leq l_0 = d(p,x_0) \leq \pi - \sigma + \delta. \tag{9}$$

Take $E \in \partial\mathbb{B}(p, \pi/2 - \sigma/2)$ satisfying (5). We consider a geodesic triangle $(\gamma_{E,p}, \gamma_{E,x_0}, \gamma_{p,x_0})$ in $\mathcal{M}$. Since $K_{\mathcal{M}} \geq -k^2$, Toponogov's comparison theorem gives

$$\begin{aligned} &\cosh[kd(p,x_0)] \\ &\quad \leq \cosh[kd(E,p)]\cosh[kd(E,x_0)] - \sinh[kd(E,p)]\sinh[kd(E,x_0)]\cos\alpha \\ &\quad = \cosh[k(d(E,p) + d(E,x_0))] - \sinh[kd(E,p)]\sinh[kd(E,x_0)](1 + \cos\alpha), \end{aligned} \tag{10}$$

where the angle $\alpha$ is defined by $\alpha = \angle(\dot\gamma_{E,p}, \dot\gamma_{E,x_0})|_E$. By Corollary 1, we have

$$1+\cos\alpha \le \frac{\cosh(k(d(E,p)+d(E,x_0))) - \cosh(kd(p,x_0))}{\sinh(kd(E,p))\sinh(kd(E,x_0))} \le \frac{\cosh(k(l_0+\delta)) - \cosh(kl_0)}{\sinh^2(k(\pi/2-\delta/6))}. \tag{11}$$

Clearly $t \mapsto \cosh(k(t+c)) - \cosh(kt)$ is increasing in $[0,\infty)$ for $c > 0$, so we get

$$1+\cos\alpha < \frac{\cosh(k(\pi+2\delta)) - \cosh(k(\pi+\delta))}{\sinh^2(k(\pi/2-\delta/6))} < \frac{\cosh(k(\pi+3\delta)) - \cosh(k(\pi+\delta))}{\sinh^2(k(\pi/2-\delta/6))}. \tag{12}$$

Similarly, if we consider the geodesic triangle $(\gamma_{E,p}, \gamma_{E,q}, \gamma_{p,q})$ and the angle $\beta = \angle(\dot\gamma_{E,p}, \dot\gamma_{E,q})|_E$, we have

$$\begin{aligned} 1+\cos\beta &\le \frac{\cosh(k(d(E,p)+d(E,q))) - \cosh(kd(p,q))}{\sinh(kd(E,p))\sinh(kd(E,q))} \\ &\le \frac{\cosh(k(\pi-\sigma+\delta)) - \cosh(kl_0)}{\sinh^2(k(\pi/2-\delta/6))} \\ &\le \frac{\cosh(k(\pi-\sigma+2\delta)) - \cosh(k(\pi-\sigma))}{\sinh^2(k(\pi/2-\delta/6))} \\ &< \frac{\cosh(k(\pi+3\delta)) - \cosh(k(\pi+\delta))}{\sinh^2(k(\pi/2-\delta/6))}. \end{aligned} \tag{13}$$

Likewise, if we think of the geodesic triangle $(\gamma_{E,q}, \gamma_{E,x_0}, \gamma_{q,x_0})$ and the angle $\gamma = \angle(\dot\gamma_{E,q}, \dot\gamma_{E,x_0})|_E$, then, noting that $d(q,x_0) \ge l_0 \ge \pi - \varepsilon_0$, we have

$$1+\cos\gamma \le \frac{\cosh(k(d(E,q)+d(E,x_0))) - \cosh(kd(q,x_0))}{\sinh(kd(E,q))\sinh(kd(E,x_0))} < \frac{\cosh(k(\pi+3\delta)) - \cosh(k(\pi+\delta))}{\sinh^2(k(\pi/2-\delta/6))}. \tag{14}$$

Now we will conclude the proof of Proposition 1 using the following lemma, whose proof will be postponed.

**Lemma 4.** *For $k > 0$, there exists a positive number $\delta_0 \in (0, 3\pi/5)$ such that $\delta_0$ is a solution of*

$$\cosh(k(\pi+3t)) - \cosh(k(\pi+t)) - (1-\sqrt{3}/2)\sinh^2(k(\pi/2-t/6)) = 0. \tag{15}$$

Take $\delta = \delta_0$ in Lemma 4, take the $\sigma$ from Lemma 2, and let $E$ be the point given by Lemma 3. Obviously, $\sigma < \delta/3$, hence $\sigma + \delta < 4\delta/3 < \pi$. Applying (12)–(14), one immediately deduces

$$\cos\alpha < -\sqrt{3}/2, \quad \cos\beta < -\sqrt{3}/2, \quad \cos\gamma < -\sqrt{3}/2.$$

That is,

$$\alpha > 2\pi/3, \quad \beta > 2\pi/3, \quad \gamma > 2\pi/3.$$

However, since $0 \le \gamma \le 2\pi - (\alpha + \beta)$, we get a contradiction. Thus our hypothesis on $\mathcal{M}$ was wrong, so $\mathcal{M}$ must be homeomorphic to $S^n$. □

In Theorem 1 or Proposition 1, we require that the sectional curvature $K_{\mathcal{M}}$ is in the interval $[-k^2, 1]$ for some $k > 0$. Trivially the result holds if $K_{\mathcal{M}} \in (0, 1]$. In the situation $0 \le K_{\mathcal{M}} \le 1$, we can simplify our proof by comparing against Euclidean space; however the estimates (12)–(14) would need to be changed for the case $k = 0$.

**Theorem 2.** *Suppose* $(\mathcal{M}, g)$ *is a compact connected n-dimensional Riemannian manifold with sectional curvature* $0 \le K_{\mathcal{M}} \le 1$. *Let* $\delta > 0$, *and let*

$$\sigma = \frac{2}{3}\int_0^{\delta/2} (\sin t)^{n-1}\,\mathrm{d}t \quad \text{such that } (2-\sqrt{3})(\pi-\sigma)^2 - 16\delta(\pi-\sigma+2\delta) \ge 0. \tag{16}$$

*Assume also that* $i_{\mathcal{M}} \ge \pi - \sigma$ *and* $0 < V_{\mathcal{M}} \le 3V(\mathcal{B}_{\pi/2-\sigma/2}) + V(\mathcal{B}_{\delta/2})$. *Then* $\mathcal{M}$ *is homeomorphic to* $S^n$.

*Proof.* We prove this result by contradiction. If some manifold $\mathcal{M}$ satisfies the assumptions of Theorem 2 and is not homeomorphic to $S^n$, there is a point $x_0 \in \mathcal{M}$ such that $x_0 \in \mathcal{M} - \mathbb{B}(p, i_{\mathcal{M}}) \cup \mathbb{B}(q, i_{\mathcal{M}})$, with $d(p, q) = d_{\mathcal{M}}$. Assume that $d(q, x_0) \ge d(p, x_0) = l_0 \ge i_{\mathcal{M}}$. By Lemma 3, there exists a point $E \in \partial\mathbb{B}(p, \pi/2 - \sigma/2)$ satisfying (5). By triangle inequality, we get because $K_{\mathcal{M}} \ge 0$ that

$$d(E, q) \ge \pi/2 - \sigma/2 \quad \text{and} \quad d(E, x_0) \ge \pi/2 - \sigma/2. \tag{17}$$

Now consider the geodesic triangle $(\gamma_{p,E}, \gamma_{x_0,E}, \gamma_{p,x_0})$; let $\alpha = \angle(\dot\gamma_{E,p}, \dot\gamma_{E,x_0})|_E$. By Toponogov's comparison theorem,

$$d^2(p, x_0) \le d^2(E, p) + d^2(E, x_0) - 2d(E, p)d(E, x_0)\cos\alpha,$$

so

$$\begin{aligned} 1 + \cos\alpha &\le \frac{(d(E, p) + d(E, x_0))^2 - d^2(p, x_0)}{2d(E, p)d(E, x_0)} \\ &\le \frac{(l_0 + \delta)^2 - l_0^2}{2(\pi/2 - \sigma/2)^2} < \frac{2\delta(\pi - \sigma + 2\delta)}{(\pi/2 - \sigma/2)^2}. \end{aligned} \tag{18}$$

Similarly, consider the triangle $(\gamma_{E,p}, \gamma_{E,q}, \gamma_{p,q})$, with $\beta = \angle(\dot\gamma_{E,p}, \dot\gamma_{E,q})|_E$ and the triangle $(\gamma_{E,q}, \gamma_{E,x_0}, \gamma_{q,x_0}))$, with $\gamma = \angle(\dot\gamma_{E,q}, \dot\gamma_{E,x_0})|_E$. Then

$$(19)\quad \begin{aligned} 1+\cos\beta &\le \frac{(d(E,p)+d(E,q))^2 - d^2(p,q)}{2d(E,p)d(E,q)} \\ &\le \frac{(\pi-\sigma+\delta)^2-(\pi-\sigma)^2}{2(\pi/2-\sigma/2)^2} < \frac{2\delta(\pi-\sigma+2\delta)}{(\pi/2-\sigma/2)^2}, \\ 1+\cos\gamma &\le \frac{(d(E,q)+d(E,x_0))^2 - d^2(q,x_0)}{2d(E,q)d(E,x_0)} \\ &\le \frac{2\delta(l_0+\delta)}{(\pi/2-\sigma/2)^2} < \frac{2\delta(\pi-\sigma+2\delta)}{(\pi/2-\sigma/2)^2}. \end{aligned}$$

Let $\delta$ and $\sigma$ satisfy (16). Then from (18) and (19), one can infer again that

$$\alpha > 2\pi/3, \quad \beta > 2\pi/3, \quad \gamma > 2\pi/3,$$

which is impossible as above. □

*Proof of Lemma 4*. First, we will show that the Equation (15) indeed contains a positive solution $\delta_0$. Define

$$F(t,k) = \cosh(k(\pi+3t)) - \cosh(k(\pi+t)) - (1-\sqrt{3}/2)\sinh^2(k(\pi/2-t/6)).$$

For fixed $k > 0$ and for $t \in [0, 3\pi]$,

$$\frac{dF}{dt} = k\Big\{3\sinh(k(3t+\pi)) - \sinh(k(t+\pi)) + \frac{2-\sqrt{3}}{12}\sinh(k(\pi-t/3))\Big\} > 0,$$

which implies that $F(t,k)$ is increasing with respect to $t$ in $[0,3\pi]$. Moreover, $F(0,k) < 0$ and $F(3\pi,k) > 0$. So (15) has a unique solution $\delta_0 \in (0,3\pi)$ for any $k > 0$. Consider the function $k \mapsto F(3\pi/5,k)$. Then

$$\frac{dF}{dk}\Big(\frac{3\pi}{5},k\Big) = \frac{14\pi}{5}\sinh\Big(\frac{14k\pi}{5}\Big) - \frac{8\pi}{5}\sinh\Big(\frac{8k\pi}{5}\Big) - \frac{(2-\sqrt{3})\pi}{5}\sinh\Big(\frac{4k\pi}{5}\Big).$$

We can check that

$$\frac{14\pi}{5}\sinh\Big(\frac{14k\pi}{5}\Big) - \frac{8\pi}{5}\sinh\Big(\frac{8k\pi}{5}\Big) > \frac{4\pi}{5}e^{8\pi/5} > \frac{(2-\sqrt{3})\pi}{5}\sinh\Big(\frac{4k\pi}{5}\Big),$$

which implies that $F(3\pi/5,k)$ is increasing in $[0,\infty)$. Note that $F(3\pi/5,0) = 0$; thus $F(3\pi/5,k) > 0$ for $k > 0$. This shows there is a solution in $0 < \delta_0 < 3\pi/5$. □

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## References

[Brown 1960] M. Brown, "A proof of the generalized Schoenflies theorem", *Bull. Amer. Math. Soc.* **66** (1960), 74–76. MR 22 #8470b Zbl 0132.20002

[Coghlan and Itokawa 1991] L. Coghlan and Y. Itokawa, "A sphere theorem for reverse volume pinching on even-dimensional manifolds", *Proc. Amer. Math. Soc.* **111**:3 (1991), 815–819. MR 91f:53033 Zbl 0719.53019

[Grove and Shiohama 1977] K. Grove and K. Shiohama, "A generalized sphere theorem", *Ann. Math.* (2) **106**:2 (1977), 201–211. MR 58 #18268 Zbl 0341.53029

[Perelman 1995] G. Perelman, "A diameter sphere theorem for manifolds of positive Ricci curvature", *Math. Z.* **218**:4 (1995), 595–596. MR 96f:53056 Zbl 0831.53033

[Shen 1989] Z. M. Shen, "A sphere theorem for manifolds of positive Ricci curvature", *Indiana Univ. Math. J.* **38**:1 (1989), 229–233. MR 90h:53044 Zbl 0678.53032

[Shiohama 1983] K. Shiohama, "A sphere theorem for manifolds of positive Ricci curvature", *Trans. Amer. Math. Soc.* **275**:2 (1983), 811–819. MR 84c:53041 Zbl 0518.53047

[Suyama 1991] Y. Suyama, "Differentiable sphere theorem by curvature pinching", *J. Math. Soc. Japan* **43**:3 (1991), 527–553. MR 92g:53033 Zbl 0790.53041

[Wen 2004] Y. Wen, "A note on pinching sphere theorem", *C. R. Math. Acad. Sci. Paris* **338**:3 (2004), 229–234. MR 2005k:53040 Zbl 1046.53019

[Wu 1989] J. Y. Wu, "A diameter pinching sphere theorem for positive Ricci curvature", *Proc. Amer. Math. Soc.* **107**:3 (1989), 797–802. MR 90h:53045 Zbl 0676.53045

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