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SUMI JEONG AND HWASIN PARK

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CONSTRUCTION OF ELLIPTIC CURVES WITH NONINTEGER TORSION POINTS AND NONCYCLIC TORSION GROUPS

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We study elliptic curves in Weierstrass form (with integer coefficients) that have noninteger torsion points over \mathbb{Q} . After putting the curves in certain normal forms, we find conditions on their coefficients characterizing when the torsion group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}$, for each $N = 1, 2, 3, 4$.

1. Introduction and main results

Let E be an elliptic curve given by the Weierstrass equation

$$(1-1) \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathbb{Z},$$

and assume E has a noninteger torsion point in the projective plane $\mathbb{P}^2(\mathbb{Q})$. Then E has one of the following expressions, up to a translation $(x, y) \mapsto (x - \alpha, y - \beta)$ with $\alpha, \beta \in \mathbb{Z}$:

$$E_1 : y^2 + xy = x^3 + 4(a - 4b)x^2 + ax + b,$$

$$E_2 : y^2 + xy + y = x^3 + 2(2a - 8b - 1)x^2 + ax + b.$$

(See Theorem 1.) Here $a, b \in \mathbb{Z}$. The curve E_1 has a noninteger torsion point at $(-\frac{1}{4}, \frac{1}{8})$, and E_2 one at $(-\frac{1}{4}, -\frac{3}{8})$.

If a curve of the form E_1 or E_2 has a noninteger point in $\mathbb{P}^2(\mathbb{Q})$ apart from $(-\frac{1}{4}, \frac{1}{8})$ or $(-\frac{1}{4}, -\frac{3}{8})$, respectively, that point has infinite order, so the curve has rank at least 1. In Section 2 we give explicit examples.

Since E_1 and E_2 have a 2-torsion point, we may ask for what choices of the coefficients the torsion subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}$, for $N = 1, 2, 3$, or 4 (higher values of N being excluded by [Mazur 1977]). In Sections 3 and 4, we give criteria for determining whether a curve of the form E_1 or E_2 has such a noncyclic torsion subgroup, and construct all possible families of elliptic curves with noninteger torsion points and a noncyclic torsion subgroup. Specifically:

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- Let m, n be integers of the same parity and set

$$(1-2) \quad E_{11} : y^2 + xy = x^3 + 2(m+n)x^2 + (4mn + \frac{1}{2}(m+n))x + mn$$

as a particular case of E_1 . The torsion group of such curves contains a $(\mathbb{Z}/2\mathbb{Z})^2$ (Theorem 8). We also construct a similar family based on E_2 (Theorem 13).

- No curve E_2 can have torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ (Theorem 14). For the case of E_1 , we construct curves with these torsion groups as follows:

(a) Assume that k, s are integers, with $s > 0$. If we choose

$$(1-3) \quad m = -k - 2k^2 \quad \text{and} \quad n = m + 2s^2,$$

then E_{11} has a torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (Theorem 10).

(b) Let (v, w, \square) be a primitive Pythagorean triple with v even, and let t be a nonnegative integer. For $s = v^2(4t + 1)/4$ and $k = (w^2(4t + 1) - 1)/4$ or $s = v^2(4t + 3)/4$ and $k = -(w^2(4t + 3) + 1)/4$, the elliptic curve E_{11} defined by (1-2) and (1-3) has torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ (Theorem 15).

Elliptic curves with torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ have been classified (see [Campbell and Goins 2004, Theorem 6.2], for example). They all can be written in the form just given (Remark 16).

- We construct elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ in Theorems 12(d) and 14(c). Via coordinate transformations these curves take the form

$$y^2 = \left(x + \frac{(s+t)^2}{4}\right) \left(x + \frac{(s-t)^2}{4}\right) \left(x + \frac{(s+t)^2(s-t)^2}{16t^2}\right)$$

where s, t are integers satisfying $s \neq t, -3t, t \mid s^2$ and $s \equiv t \equiv 1$ or $3 \pmod{4}$.

2. Elliptic curves with noninteger torsion points in $\mathbb{P}^2(\mathbb{Q})$ and rank ≥ 1

Given an elliptic curve in Weierstrass form (1-1), we can replace y by $y - a_1x/2$ if a_1 is even, or by $y - (a_1-1)x/2$ if a_1 is odd, to obtain an isomorphic curve in Weierstrass form with $a_1 = 0$ or $a_1 = 1$.

By the Lutz–Nagell theorem, if $a_1 = 0$ and $P = (x_p, y_p)$ is a rational torsion point, then x_p and y_p are integers. Because we are interested in finding elliptic curves with *noninteger* torsion points, we therefore restrict our attention to the case $a_1 = 1$; that is, our curve has the form

$$(2-1) \quad E' : y^2 + xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad \text{where } a_2, a_3, a_4, a_6 \in \mathbb{Z}.$$

Theorem 1. *Assume the curve E' in (2-1) has noninteger torsion points. Then, up to a coordinate translation $(x, y) \mapsto (x - \alpha, y - \beta)$ with $\alpha, \beta \in \mathbb{Z}$, the curve is of*

one of the following forms, where a, b are integers:

$$(2-2) \quad \begin{aligned} E_1 : y^2 + xy &= x^3 + 4(a - 4b)x^2 + ax + b, \\ E_2 : y^2 + xy + y &= x^3 + 2(2a - 8b - 1)x^2 + ax + b. \end{aligned}$$

Proof. By the Lutz–Nagell theorem, a noninteger torsion point P of E' in $\mathbb{P}^2(\mathbb{Q})$ must have order two and coordinates $(x_P, y_P) = (m/2^2, n/2^3)$, with $m, n \in \mathbb{Z}$. From $2P = O$, the group law gives

$$(2-3) \quad 2y_P + x_P + a_3 = 0,$$

hence $n + m = -4a_3$. This implies $n + m \equiv 0 \pmod{4}$. We thus have the following possibilities for P , where α and β are integers:

- $m \equiv 0 \pmod{4}, n \equiv 0 \pmod{4} \implies P = (\alpha, \beta + \frac{1}{2})$.
- $m \equiv 1 \pmod{4}, n \equiv 3 \pmod{4} \implies P = (\alpha + \frac{1}{4}, \beta - \frac{1}{8})$ or $P = (\alpha + \frac{1}{4}, \beta + \frac{3}{8})$.
- $m \equiv 2 \pmod{4}, n \equiv 2 \pmod{4} \implies P = (\alpha + \frac{1}{2}, \beta \pm \frac{1}{4})$.
- $m \equiv 3 \pmod{4}, n \equiv 1 \pmod{4} \implies P = (\alpha - \frac{1}{4}, \beta + \frac{1}{8})$ or $P = (\alpha - \frac{1}{4}, \beta - \frac{3}{8})$.

By a coordinate translation, we can assume $\alpha = 0$ and $\beta = 0$, so the possibilities for P after this reduction are $(0, \frac{1}{2}), (\frac{1}{2}, \pm\frac{1}{4}), (\pm\frac{1}{4}, \mp\frac{1}{8})$ and $(\pm\frac{1}{4}, \pm\frac{3}{8})$. However, not all them can occur. If $P = (\frac{1}{2}, \frac{1}{4})$, for example, the equality $2y_P + x_P + a_3 = 0$ gives $a_3 = -1$, so $\frac{1}{16} + \frac{1}{8} - \frac{1}{4} = \frac{1}{8} + \frac{1}{4}a_2 + \frac{1}{2}a_4 + a_6$, which is impossible for $a_2, a_4, a_6 \in \mathbb{Z}$. A similar calculation excludes all but two cases:

- $P = (-\frac{1}{4}, \frac{1}{8}) \implies a_3 = 0, a_2 = 4a_4 - 16a_6,$
- $P = (-\frac{1}{4}, -\frac{3}{8}) \implies a_3 = 1, a_2 = 4a_4 - 16a_6 - 2.$

Setting $a = a_4$ and $b = a_6$ yields the forms in (2-2). □

Remark 2. It follows that, if an elliptic curve of the form E_1 or E_2 has a noninteger point in $\mathbb{P}^2(\mathbb{Q})$ other than $(-\frac{1}{4}, \frac{1}{8})$ or $(-\frac{1}{4}, -\frac{3}{8})$, respectively, that point contributes to the rank of the curve. Similarly, if a curve is *not* isomorphic to either E_1 or E_2 and has a noninteger rational point, this point contributes to the rank, since it is not a torsion point.

Remark 3. The condition (2-3) in the proof of Theorem 1 being also sufficient for P to have order 2, any curve of the form (2-2) with integer a, b —so long as it is nonsingular—has a point of order 2 with coordinates $(-\frac{1}{4}, \frac{1}{8})$ (in the case of E_1) or $(-\frac{1}{4}, -\frac{3}{8})$ (for E_2).

Theorem 4. Consider the elliptic curve $E_1 : y^2 + xy = x^3 + 4(a - 4b)x^2 + ax + b$, where $a, b \in \mathbb{Z}$. If $p/q \in \mathbb{Q}$ is the x -coordinate of a point in $E_1(\mathbb{Q})$, where p, q are relatively prime integers with $q > 0$, then q is a square. Further, if $q = 4$, then $p \equiv 3 \pmod{4}$.

Proof. Setting $x = p/q$ in the equation for E_1 , we see y is rational if and only if

$$(2-4) \quad q(4p+q)(p^2+4apq-16bpq+4bq^2) \text{ is a square,}$$

or equivalently, with $q = q'r^2$, where $r \in \mathbb{Z}$ and q' is a square-free positive integer,

$$(2-5) \quad q'(4p+q)(p^2+4apq-16bpq+4bq^2) \text{ is a square.}$$

Assume this is the case. Then $q' \mid (4p+q)(p^2+4apq-16bpq+4bq^2)$, which is the same as $q' \mid 4p^3$. But $q' \nmid p^3$, since $\gcd(p, q) = 1$, so we get $q' = 1$ or $q' = 2$. If $q' = 2$, then p is odd and $(2p+r^2)(p^2+8apr^2-32bpr^2+16br^4)$ is a square, by (2-5). But this cannot be so, because this expression is congruent (mod 4) to $(2p+r^2)p^2 \equiv (2+r^2) \cdot 1 \equiv 2$ or 3 . This contradiction shows that $q' = 1$; that is, q is a square.

If $q = 4$, again p is odd and (2-5) implies that $(p+1)(p^2+16ap-64bp+64b)$ is a square. This reduces (mod 4) to $(p+1)p^2 \equiv p+1$. Hence $p \equiv 3 \pmod{4}$. \square

We can use Remark 2 and Theorem 4 to construct a family of elliptic curves with rank at least 1.

Example 5. For $k, t \in \mathbb{Z}$, consider the elliptic curve

$$E : y^2 + xy = x^3 - (16k^2 + 12k + 16t)x^2 + (12k^2 + 9k + 8t)x + (4k^2 + 3k + 3t).$$

$E(\mathbb{Q})$ contains the point $(\frac{3}{4}, \frac{1}{8}(3+16k))$ and its additive inverse $(\frac{3}{4}, \frac{1}{8}(-9-16k))$; being noninteger and distinct from $(-\frac{1}{4}, \frac{1}{8})$, these points have infinite order. Hence the curve has rank at least 1.

To see how the example is obtained, we need only consider the conditions on $a, b \in \mathbb{Z}$ such that E_1 in Theorem 4 has a rational point (x, y) with $x = \frac{3}{4}$. Substitution gives $y = \frac{1}{8}(-3 \pm 2\sqrt{9+48a-128b})$; that is, we must find conditions on a, b ensuring that $9+48a-128b = A^2$ for some $A \in \mathbb{Z}$. Since $16(3a-8b) = A^2-9$, we have $16 \mid (A^2-9)$, or, upon replacing A by $-A$ if necessary, $A \equiv 3 \pmod{8}$. Put $A = 8k+3$ for some $k \in \mathbb{Z}$; then $3a-8b = 4k^2+3k$. Since $(3, 8) = 1$, we can write $a = 3(4k^2+3k) + 8t$ and $b = (4k^2+3k) + 3t$ for some $t \in \mathbb{Z}$. We find the value $8k+3$ for the radical and hence the values of y . Working backwards, or simply checking by substitution, we see that any $k, t \in \mathbb{Z}$ will work.

Theorem 6. Consider the elliptic curve $E_2 : y^2 + xy + y = x^3 + 2(2a-8b-1)x^2 + ax + b$, where $a, b \in \mathbb{Z}$. If $p/q \in \mathbb{Q}$ is the x -coordinate of a point in $E_2(\mathbb{Q})$, where p, q are relatively prime integers with $q > 0$, then q is a square. Further, if $q = 4$, then $p \equiv 3 \pmod{4}$.

The proof is similar to that of Theorem 4. Moreover, a reasoning similar to that used to justify Example 5 gives rise to our second example family:

Example 7. Consider the elliptic curve

$$E : y^2 + xy + y = x^3 - (2 + 4k + 16k^2 + 16t)x^2 + (12k^2 + 3k + 8t)x + (4k^2 + k + 3t),$$

where $k, t \in \mathbb{Z}$. It has noninteger points $(\frac{3}{4}, \frac{1}{8}(-5 + 16k))$, $(\frac{3}{4}, \frac{1}{8}(-9 - 16k))$ distinct from $(-\frac{1}{4}, -\frac{3}{8})$, so $E(\mathbb{Q})$ has rank at least 1.

We now turn to the torsion groups of the curves E_1 and E_2 . We know from Remark 3 that there is always a point of order 2; we wish to find conditions on the coefficients a and b that characterize when the torsion group is noncyclic — which, by [Mazur 1977], means isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}$, for $N = 1, 2, 3, 4$.

3. Torsion subgroups on $E_1 : y^2 + xy = x^3 + 4(a - 4b)x^2 + ax + b$

Recall that E_1 has a torsion point of order 2 at $(-\frac{1}{4}, \frac{1}{8})$ so long as it is nonsingular, a condition that boils down to $(a - 4b)^2 - b \neq 0$, since the discriminant of E_1 factors as $(1 - 16a + 128b)^2((a - 4b)^2 - b)$, and the square factor is clearly nonzero.

Theorem 8. *Consider the curve E_{11} with equation (1-2), obtained as a particular case of E_1 with coefficients $a = 4mn + \frac{1}{2}(m + n)$ and $b = mn$, where m and n are integers satisfying $m \equiv n \pmod{2}$ and $m < n$. The 2-torsion subgroup of E_{11} is $E_{11, \text{tors}}(\mathbb{Q})[2] = \{O, (-\frac{1}{4}, \frac{1}{8}), (-2m, m), (-2n, n)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

All other curves E_1 not of this form have $E_{1, \text{tors}}(\mathbb{Q})[2] = \{O, (-\frac{1}{4}, \frac{1}{8})\} \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Let E_1 have a torsion point P of order 2 other than $(-\frac{1}{4}, \frac{1}{8})$. From the proof of Theorem 1, and especially (2-3), we know that P has the form $(q/4, -q/8)$ for $q \in \mathbb{Z}$. From the curve's equation we get $(1 + q)(q^2 + 16(a - 4b)q + 64b) = 0$. But $q \neq -1$ by assumption, so

$$(3-1) \quad q = -8(a - 4b) \pm 8\sqrt{(a - 4b)^2 - b}.$$

That is, $(a - 4b)^2 - b = A^2$ for some integer A , which must be nonzero by the observation at the start of this section. Setting $B = a - 4b$, $m = B - A$ and $n = B + A$, so that $b = B^2 - A^2 = mn$ and $a = 4mn + (m + n)/2$, we obtain the equation of E_{11} in the theorem, with the side conditions on m and n . (We know that $m \neq n$ since $A \neq 0$, and we can interchange m and n if necessary to ensure that $m < n$.)

This shows the last assertion of the theorem, and confirms that there cannot be more than three points of order 2, since there are only two choices of q in (3-1). There remains to note that for any m and n as in the statement of the theorem (equivalently, for any integers A, B with $A \neq 0$) we do indeed get torsion points of order 2 via (3-1), arising from $q = -8m$ and $q = -8n$. \square

We next recall a classical result; see for example [Knapp 1992, Theorem 4.2].

Lemma 9. *Consider the elliptic curve*

$$(3-2) \quad E : y^2 = (x - x_1)(x - x_2)(x - x_3).$$

For $(x, y) \in E(\mathbb{Q})$, there exists $P \in E(\mathbb{Q})$ with $2P = (x, y)$ if and only if $x - x_1, x - x_2, x - x_3$ are squares.

Theorem 10. *Let k and s be integers, with $s > 0$. If the curve E_{11} of Theorem 8 has coefficients $m = -k - 2k^2$ and $n = -k - 2k^2 + 2s^2$, then it has a torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.*

Proof. The coordinate change $y = \eta - x/2$ followed by $(x, \eta) = (X/4, Y/8)$ transforms E_{11} into the curve

$$(3-3) \quad E'_{11} : Y^2 = (X + 1)(X + 8m)(X + 8n).$$

The points $(-\frac{1}{4}, \frac{1}{8})$, $(-2m, m)$ and $(-2n, n)$ of order 2 in $E_{11}(\mathbb{Q})$ correspond to $(-1, 0)$, $(-8m, 0)$ and $(-8n, 0)$ in $E'_{11}(\mathbb{Q})$. We now ask which, if any, of these points can be the double of some point P in $E'_{11}(\mathbb{Q})$.

If $2P = (-1, 0)$, the differences $-1 - (-1)$, $-1 - (-8m)$ and $-1 - (-8n)$ must be squares, by Lemma 9. But $8m - 1$ is certainly not a square, because it has residue 3 (mod 4). Therefore $(-1, 0)$ is not a double. Similarly, $(-8n, 0)$ is not a double because $-8n - (-8m) < 0$ is not a square. Finally $(-8m, 0)$ is a double if and only if $-8m - (-1)$ and $-8m - (-8n)$ are squares; that is, if and only if $8(n - m) = S^2$ and $1 - 8m = K^2$, where S, K are integers. Clearly $S \equiv 0 \pmod{4}$ and K is odd; by interchanging K and $-K$ we can ensure that $K \equiv 1 \pmod{4}$. So $E'_{11}(\mathbb{Q})$, and hence also $E_{11}(\mathbb{Q})$, has torsion points of order 4 if and only if

$$n - m = 2s^2 \text{ with } s \in \mathbb{Z}, s > 0 \quad \text{and} \quad 1 - 8m = (4k + 1)^2 \text{ with } k \in \mathbb{Z},$$

or equivalently if $m = -k - 2k^2$ and $n = -k - 2k^2 + 2s^2$ for $k, s \in \mathbb{Z}$ with $s > 0$. \square

Remark 11. Here is the explicit form of the points of order 4 in E'_{11} :

$$(3-4) \quad \begin{cases} P_1 = (4(2k + 4k^2 - s(1 + 4k)), & 4s(1 + 4k)(1 + 4k - 4s)) \\ P_2 = (4(2k + 4k^2 - s(1 + 4k)), & -4s(1 + 4k)(1 + 4k - 4s)) \\ P_3 = (4(2k + 4k^2 + s(1 + 4k)), & 4s(1 + 4k)(1 + 4k + 4s)) \\ P_4 = (4(2k + 4k^2 + s(1 + 4k)), & -4s(1 + 4k)(1 + 4k + 4s)) \end{cases}$$

We now recapitulate and complement our results for E_1 , giving criteria for the occurrence of each torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, $n = 1, 2, 3, 4$.

Theorem 12. *Consider the elliptic curve $E_1 : y^2 + xy = x^3 + 4(a - 4b)x^2 + ax + b$, where $a, b \in \mathbb{Z}$. and recall that $(-\frac{1}{4}, \frac{1}{8})$ is a point of order 2 in $E_1(\mathbb{Q})$.*

- (a) $E_1(\mathbb{Q})$ has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if and only if the equation $t^2 - 2(a - 4b)t + b = 0$ in t has two integer solutions $m < n$, in which case E_1 has the form E_{11} of (1-2): $y^2 + xy = x^3 + 2(m+n)x^2 + (4mn + \frac{1}{2}(m+n))x + mn$.

(b) Assuming the condition in (a) is met, $E_{11}(\mathbb{Q})$ has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ if and only if $1 - 8m$ and $\frac{1}{2}(n - m)$ are square integers.

(c) In the situation of (b), the full torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ if and only if

$$(3-5) \quad \sqrt{\frac{n-m}{2}} \left(4\sqrt{\frac{n-m}{2}} + \sqrt{1-8m} \right) \quad \text{and} \quad \sqrt{1-8m} \left(4\sqrt{\frac{n-m}{2}} + \sqrt{1-8m} \right)$$

are square integers.

(d) Assuming the condition in (a) is met, the full torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ if and only if there exist integers α, β such that $|\alpha| < \beta$, $\gamma := -\alpha\beta/(\alpha + \beta)$ is an integer, $m = \frac{1}{8}(\alpha^2 - \gamma^2 + 1)$, and $n = \frac{1}{8}(\beta^2 - \gamma^2 + 1)$. In this situation the points of order 3 have x -coordinate $\frac{1}{4}(\gamma^2 - 1)$; moreover γ is odd and $\beta \equiv \alpha \equiv 0 \pmod{4}$.

Proof. Part (a) is just a restatement of Theorem 8, apart from the easily checked equivalence between the conditions $a = 4mn + \frac{1}{2}(m+n)$ and $b = mn$ in that theorem and m, n being the roots of the quadratic equation $t^2 - 2(a - 4b)t + b = 0$.

(b) If $1 - 8m$ and $(n - m)/2$ are squares, the quantities $s = \sqrt{(n - m)/2}$ and $k = (-1 + \sqrt{1 - 8m})/4$ or $k = (-1 - \sqrt{1 - 8m})/4$ satisfy the conditions of Theorem 10. (We choose whichever definition of k yields an integer; note that $\sqrt{1 - 8m}$ is odd.) Therefore in this situation E_{11} has a torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Conversely, if E_{11} has a torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, we have $m = -k - 2k^2$ and $n = -k - 2k^2 + 2s^2$ for some integers k, s with $s > 0$.

(c) Let s and k be as above, and recall the short form E'_{11} of the curve given in (3-3). We ask when the points P_1, P_2, P_3, P_4 of order four listed in (3-4) are doubles. Consider first the condition imposed by Lemma 9 for P_1 to be a double. It is that the following three differences be squares:

$$\begin{aligned} 4(2k + 4k^2 - s(1 + 4k)) + 1 &= (1 + 4k)(1 + 4k - 4s), \\ 4(2k + 4k^2 - s(1 + 4k)) + 8m &= -4s(1 + 4k), \\ 4(2k + 4k^2 - s(1 + 4k)) + 8n &= -4s(1 + 4k - 4s). \end{aligned}$$

Clearly if any two are squares, so is the third. We discard the middle line and rewrite the other two right-hand sides in terms of m and n , using the expressions in (b). We must take the minus sign in $k = (-1 \pm \sqrt{1 - 8m})/4$, since s is positive and $-4s(1 + 4k)$ is a square. It follows that the condition for P_1 to be a double is that the quantities in (3-5) be square integers. The same holds for P_2 , since it has the same x -coordinate as P_1 .

A similar argument shows that the condition on the integers $s > 0$ and k for P_3 (or P_4) to be a double is that $s(1 + 4k + 4s)$ and $(1 + 4k)(1 + 4k + 4s)$ be squares. Substituting $k = (-1 + \sqrt{1 - 8m})/4$ and s leads to the same expressions (3-5) in terms of m and n . Thus there is a point of order 8 in the curve if and only if

both quantities in (3-5) are square integers. (Because of the different relationship between k and m in each case, P_1 and P_2 being doubles is mutually exclusive with P_3 and P_4 being doubles, as expected from the structure of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.)

(d) Assume the curve has a point of order 3, and let (X_P, Y_P) be its coordinates in the alternate equation (3-3) introduced in the proof of Theorem 10. Because this point is the double of a generator of $\mathbb{Z}/6\mathbb{Z}$, we can apply Lemma 9 to conclude that $A := X_P + 8m, B := X_P + 8n, G := X_P + 1$ are all square integers.

Meanwhile, the standard algebraic constraint for a point on an elliptic curve of the form (3-2) to have order three (easy to derive using the characterization of such a point as an inflection point) amounts in this case to

$$(3-6) \quad 4ABG(A + B + G) - (BG + AB + AG)^2 = 0.$$

If we let α, β, γ be the nonnegative square roots of A, B, G (note that $\alpha < \beta$ since $m < n$), we can write the left-hand side of (3-6) as

$$(\alpha\beta + \beta\gamma + \alpha\gamma)(-\alpha\beta + \beta\gamma + \alpha\gamma)(\alpha\beta - \beta\gamma + \alpha\gamma)(\alpha\beta + \beta\gamma - \alpha\gamma),$$

so one of these factors vanishes. By changing the sign of α and/or γ we can ensure that $\alpha\beta + \beta\gamma + \alpha\gamma = 0$, while β remains positive and greater than $|\alpha|$. Thus $\gamma = -\alpha\beta/(\alpha + \beta)$. Substitution also gives, successively,

$$X_P = \gamma^2 - 1, \quad m = \frac{1}{8}(\alpha^2 - \gamma^2 + 1), \quad n = \frac{1}{8}(\beta^2 - \gamma^2 + 1).$$

Recalling that the x -coordinate in E_{11} is related to the X -coordinate in E'_{11} by $x = X/4$, we deduce that $x = \frac{1}{4}(\gamma^2 - 1)$. The divisibility conditions on γ, α, β follow since x, m, n are integers. This concludes one direction of the proof.

The other direction is a matter of checking (using the same algebra) that, given integers α, β, γ with $\gamma = -\alpha\beta/(\alpha + \beta), m = \frac{1}{8}(\alpha^2 - \gamma^2 + 1)$, and $n = \frac{1}{8}(\beta^2 - \gamma^2 + 1)$, the points on E_{11} with $x = \frac{1}{4}(\gamma^2 - 1)$ have order 3. □

4. Torsion subgroups on $E_2 : y^2 + xy + y = x^3 + 2(2a - 8b - 1)x^2 + ax + b$

Recall that E_2 has a torsion point of order 2 at $(-\frac{1}{4}, -\frac{3}{8})$ so long as it is nonsingular, a condition equivalent to $(a - 4b)^2 - a + 3b \neq 0$, since the discriminant of E_2 factors as $(25 - 16a + 128b)^2((a - 4b)^2 - a + 3b)$, and the square factor is clearly nonzero.

Theorem 13. *Consider the curve*

$$E_{22} : y^2 + xy + y = x^3 + 2(m + n)x^2 + (4mn + \frac{1}{2}(m + n - 1))x + mn - \frac{1}{4}$$

*obtained as a particular case of E_2 with coefficients $a = 4mn + \frac{1}{2}(m + n) - \frac{1}{2}$ and $b = mn - \frac{1}{4}$, where m and n are **half-integers** (that is, $2m, 2n$ are odd integers)*

satisfying $m - n \equiv 0 \pmod{2}$ and $m < n$. Then E_{22} does not have torsion points of order 4, and the 2-torsion subgroup of E_{22} is

$$E_{22, \text{tors}}(\mathbb{Q})[2] = \left\{ O, \left(-\frac{1}{4}, -\frac{3}{8}\right), \left(-2m, m - \frac{1}{2}\right), \left(-2n, n - \frac{1}{2}\right) \right\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

All other curves E_2 not of this form have $E_{2, \text{tors}}(\mathbb{Q})[2] = \left\{ O, \left(-\frac{1}{4}, -\frac{3}{8}\right) \right\} \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Assume that E_2 has a torsion point P of order 2 other than $\left(-\frac{1}{4}, -\frac{3}{8}\right)$, and write $P = \left(\frac{q}{4}, -\frac{q}{8} - \frac{1}{2}\right)$ with $q \in \mathbb{Z}$. From the equation of E_2 we obtain $(1 + q)(q^2 + 8q(2a - 8b - 1) + 16(4b + 1)) = 0$, so

$$(4-1) \quad q = -4(2a - 8b - 1) \pm 4\sqrt{(2a - 8b - 1)^2 - (4b + 1)}.$$

That is, $(2a - 8b - 1)^2 - (4b + 1) = A^2$ for some even integer A , which must be nonzero by the observation at the start of this section (note that the radicand in (4-1) equals $(a - 4b)^2 - a + 3b$). Setting $B = 2a - 8b - 1$, $m = \frac{1}{2}(B + A)$ and $n = \frac{1}{2}(B - A)$, so that $b = \frac{1}{4}(B^2 - A^2 - 1) = mn - \frac{1}{4}$ and $a = \frac{1}{2}(B + 8b + 1) = \frac{1}{2}(m + n) + (4mn - 1) + \frac{1}{2}$, we obtain the equation of E_{22} in the theorem, with the side conditions on m and n .

This shows the last assertion of the theorem, and also that there cannot be more than three points of order 2, since there are only two choices of q in (4-1). Further, for any m and n as in the statement of the theorem (equivalently, for any nonzero even integer A and any odd integer B) we do indeed get torsion points of order 2 via (4-1), arising from $q = -8m$ and $q = -8n$.

There remains to show that $E_{22}(\mathbb{Q})$ has no torsion of order 4. To do this, apply the coordinate change $y = \eta - (x + 1)/2$ followed by $(x, \eta) = (X/4, Y/8)$. This transforms E_{22} into the curve

$$(4-2) \quad E'_{22} : Y^2 = (X + 1)(X + 8m)(X + 8n),$$

which is the same as E'_{11} of (3-3). The points of order two listed above map become $(-1, 0)$, $(-8m, 0)$, and $(-8n, 0)$. We then proceed as in the proof of Theorem 10, with the difference that here m, n are half-integers. First, $(-1, 0)$ is not a double in $E'_{22}(\mathbb{Q})$ because $-1 - (-8m) \equiv 3 \pmod{8}$ is not a square. Nor can $(-8m, 0)$ be a double, since $-8m - (-1) \equiv 5 \pmod{8}$. Similarly, $(-8n, 0)$ cannot be a double. \square

Theorem 14. Consider the elliptic curve $E_2 : y^2 + xy + y = x^3 + 2(2a - 8b - 1)x^2 + ax + b$, where $a, b \in \mathbb{Z}$. and recall that $\left(-\frac{1}{4}, -\frac{3}{8}\right)$ is a point of order 2 in $E_2(\mathbb{Q})$.

(a) If the equation $t^2 - (2a - 8b - 1)t + b + \frac{1}{4} = 0$ in t has two distinct half-integer solutions $m < n$, then E_2 can be written as

$$E_{22} : y^2 + xy + y = x^3 + 2(m + n)x^2 + \left(4mn + \frac{1}{2}(m + n - 1)\right)x + mn - \frac{1}{4}$$

and $E_2(\mathbb{Q})$ has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(b) Never does $E_2(\mathbb{Q})$ have a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

(c) Assuming the condition in (a) is met, the full torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ if and only if there exist integers α, β such that $|\alpha| < \beta$, $\gamma := -\alpha\beta/(\alpha + \beta)$ is an integer, $m = \frac{1}{8}(\alpha^2 - \gamma^2 + 1)$, and $n = \frac{1}{8}(\beta^2 - \gamma^2 + 1)$. In this situation the points of order 3 have x -coordinate $\frac{1}{4}(\gamma^2 - 1)$; moreover γ is odd and $\beta \equiv \alpha \equiv 2 \pmod{4}$.

Proof. Parts (a) and (b) restate Theorem 13, apart from the easily checked equivalence between the conditions $a = 4mn + \frac{1}{2}(m+n-1)$ and $b = mn - \frac{1}{4}$ in that theorem and m, n being the roots of the quadratic equation $t^2 - (2a - 8b - 1)t + b + \frac{1}{4} = 0$.

The proof of part (c) is verbatim the same as that of Theorem 12(d). (The condition $m, n \in \mathbb{Z}$ was not used in that proof except to show that α, β were divisible by 4. Here the condition $m, n \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ gives $\beta \equiv \alpha \equiv 2 \pmod{4}$ instead.) \square

5. Characterization of curves with torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ in terms of Pythagorean triples

We now give a family of curves E_{11} whose torsion subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, and show that it is exhaustive.

Theorem 15. *Let t be a nonnegative integer and (v, w, z) a primitive Pythagorean triple with v even. For the integers*

$$s = \frac{v^2(4t + 1)}{4}, k = \frac{w^2(4t + 1) - 1}{4} \text{ or } s = \frac{v^2(4t + 3)}{4}, k = -\frac{w^2(4t + 3) + 1}{4},$$

the conditions in part (c) of Theorem 12 are satisfied, so the elliptic curve E_{11} written there has a torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

Conversely, if E_{11} in Theorem 12 has a torsion point of order 8, we are in the situation of part (c) of that theorem, with s and k arising from a primitive Pythagorean triple as above.

Proof. Recall from the proof of Theorem 12(c) that the condition for the existence of torsion of order 8 is that there should be integers B, C satisfying either

$$(5-1) \quad B^2 = s(1+4k+4s) \quad \text{and} \quad C^2 = (1+4k)(1+4k+4s)$$

or

$$(5-2) \quad B^2 = -s(1+4k-4s) \quad \text{and} \quad C^2 = (1+4k)(1+4k-4s).$$

Substituting $s = \frac{1}{4}v^2(4t + 1)$ and $k = \frac{1}{4}(w^2(4t + 1) - 1)$ on the left-hand sides of equalities (5-1) leads to

$$\frac{1}{4}v^2(4t + 1)^2(v^2 + w^2), \quad w^2(4t + 1)^2(v^2 + w^2),$$

which are squares because v is even and $v^2 + w^2 = z^2$. Similarly, substituting $s = \frac{1}{4}v^2(4t + 3)$ and $k = -\frac{1}{4}(w^2(4t + 3) + 1)$ in (5-2) also yields squares.

Conversely, suppose (5-1) is satisfied; our job is to find a Pythagorean triple as in the statement of Theorem 15. Combining the two equations (5-1) we get $4B^2 + C^2 = (1 + 4k + 4s)^2$, so

$$(2B, C, 1 + 4k + 4s)$$

is a Pythagorean triple. Let c be the gcd of the three members and (v, w, z) the corresponding primitive triple, so $2B = cv$, $C = cw$, $1 + 4k + 4s = cz$. This last equation shows that c is odd, so v is even, so w is odd. Now note that

$$4s(1 + 4k + 4s) = c^2v^2, \quad (1 + 4k)(1 + 4k + 4s) = c^2w^2.$$

Since $cz = 1 + 4k + 4s$ divides both c^2v^2 and c^2w^2 , and since v, w, z are relatively prime, we conclude that z divides c , that is, $c = zu$ for some odd integer u . Hence

$$(5-3) \quad 4s = v^2u, \quad 1 + 4k = w^2u.$$

This last equation gives $u \equiv 1 \pmod{4}$ since $w^2 \equiv 1 \pmod{4}$. Thus we can write $u = 4t + 1$ for some integer $t \geq 0$, and from (5-3) we get

$$s = \frac{v^2(4t + 1)}{4} \quad \text{and} \quad k = \frac{w^2(4t + 1) - 1}{4},$$

as needed.

A wholly analogous reasoning shows that when (5-2) is satisfied, instead of (5-1), there is a Pythagorean triple (v, w, z) (with v even) and an integer $t \geq 0$ such that

$$s = \frac{v^2(4t + 3)}{4} \quad \text{and} \quad k = -\frac{w^2(4t + 3) + 1}{4}. \quad \square$$

Remark 16. Every elliptic curve over \mathbb{Q} with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ is isomorphic to one of those in Theorem 15. Indeed, it is known (see [Campbell and Goins 2004, Theorem 6.2], for example) that every such curve has an equation of the form

$$y^2 = x(x + u^2)(x + u^{-2}), \quad \text{for } u = \frac{T^2 - 1}{2T} \text{ with } T \in \mathbb{Q} \setminus \{0, 1, -1\}.$$

It's easy to see that there is a primitive Pythagorean triple (v, w, z) with $v/w = u$, and by interchanging v and w if necessary we can ensure that v is even. To go from the form

$$y^2 = x \left(x + \frac{w^2}{v^2} \right) \left(x + \frac{v^2}{w^2} \right)$$

to the desired form of E_{11} , we apply affine coordinate changes with rational coefficients: the scaling $(x, y) \mapsto (4xv^{-2}w^{-2}, 8yv^{-3}w^{-3})$, followed by the change of

parameters $s = \frac{1}{4}v^2$ and $k = \frac{1}{4}(w^2 - 1)$ (case $t = 0$ in the first set of substitution in Theorem 15), gives

$$y^2 = x(x + (2s)^2)(x + (2k + \frac{1}{2})^2);$$

further inserting the values $m = -k - 2k^2$ and $n = -k - 2k^2 + 2s^2$ and applying the coordinate change $x \mapsto x + 2m$ followed by $y \mapsto y + x/2$ leads to the canonical form of E_{11} .

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SUMI JEONG

DEPARTMENT OF MATHEMATICS AND INSTITUTE OF PURE AND APPLIED MATHEMATICS
CHONBUK NATIONAL UNIVERSITY
JEONJU, JEONBUK 561-756
REPUBLIC OF KOREA
sumi@chonbuk.ac.kr

HWASIN PARK

DEPARTMENT OF MATHEMATICS AND INSTITUTE OF PURE AND APPLIED MATHEMATICS
CHONBUK NATIONAL UNIVERSITY
JEONJU, JEONBUK 561-756
REPUBLIC OF KOREA
park@chonbuk.ac.kr