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THE BLOCK DECOMPOSITION OF FINITE-DIMENSIONAL REPRESENTATIONS OF TWISTED LOOP ALGEBRAS

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Let $L^\sigma(\mathfrak{g})$ be the twisted loop algebra of a simple complex Lie algebra $\mathfrak{g}$ with nontrivial diagram automorphism σ. Although the category $\mathscr{F}^\sigma$ of finite-dimensional representations of $L^\sigma(\mathfrak{g})$ is not semisimple, it can be written as a sum of indecomposable subcategories (the blocks of the category). To describe these summands, we introduce the twisted spectral characters for $L^\sigma(\mathfrak{g})$. These are certain equivalence classes of the spectral characters defined by Chari and Moura for an untwisted loop algebra $L(\mathfrak{g})$, which were used to provide a description of the blocks of finite-dimensional representations of $L(\mathfrak{g})$. Here we adapt this decomposition to parametrize and describe the blocks of $\mathscr{F}^\sigma$ via the twisted spectral characters.

Introduction

In this paper we study the category $\mathscr{F}^\sigma$ of finite-dimensional representations of a twisted loop algebra $L^\sigma(\mathfrak{g})$, where $\mathfrak{g}$ is a simple complex Lie algebra and σ a diagram automorphism of $\mathfrak{g}$. While there is extensive literature on the corresponding category $\mathscr{F}$ of finite-dimensional representations of the untwisted loop algebras $L(\mathfrak{g})$ — see for example [Chari et al. 2008; Chari and Loktev 2006; Chari and Moura 2004; Chari and Pressley 2001; Fourier and Littelmann 2007] — until recently the treatment of $\mathscr{F}^\sigma$ has been neglected, though the simple objects of the category of graded modules for $L^\sigma(\mathfrak{g})$ were described in [Chari and Pressley 1988].

The simple objects of $\mathscr{F}^\sigma$ were described in [Chari et al. 2008]. However, it is not a semisimple category, as there exist objects that are indecomposable but reducible. However we can still write any object uniquely as a direct sum of indecomposables (all objects are finite-dimensional); thus the category $\mathscr{F}^\sigma$ has a decomposition into indecomposable abelian subcategories. In such a decomposition, each indecomposable object will lie in a unique indecomposable abelian subcategory, although such a subcategory may contain many nonisomorphic indecomposables. In this case, when complete reducibility is not at hand, it is natural to search for

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a description of the decomposition of the category. This is a familiar and useful strategy in the BGG category $\mathbb{O}$, for example, where the blocks are parametrized by central characters of the universal enveloping algebra of $\mathfrak{g}$. When the category of representations is semisimple (as is the case, for example, for the finite-dimensional representations of $\mathfrak{g}$), the blocks are parametrized by the isomorphism classes of simple objects.

Some features of the category $\mathscr{F}^\sigma$ can be understood in terms of the corresponding category $\mathscr{F}$ of finite-dimensional representations of $L(\mathfrak{g})$. In particular, any simple object of $\mathscr{F}^\sigma$ can be realized by restricting the action of $L(\mathfrak{g})$ on a suitable simple object of V in $\mathscr{F}$ to the subalgebra $L^\sigma(\mathfrak{g})$. The isomorphism classes of simple objects of $\mathscr{F}$ were classified in [Chari and Pressley 2001], and this classification was used recently in [Chari et al. 2008] to provide the corresponding classification of simple objects in $\mathscr{F}^\sigma$. There the relationship between the irreducibles in $\mathscr{F}$ and in $\mathscr{F}^\sigma$ is understood using the diagram automorphism σ used in the construction of $L^\sigma(\mathfrak{g})$: This automorphism induces a folding on the monoid of Drinfeld polynomials of $\mathfrak{g}$ (this is the map $\mathbf{r}$ constructed in Section 2.4 below), the result of which is a monoid of polynomials that parametrizes the irreducible modules (equivalently, the twisted Weyl modules) of $L^\sigma(\mathfrak{g})$.

The blocks of the category $\mathscr{F}$ have been described as well. For a simple complex Lie algebra $\mathfrak{g}$, we will denote by P the weight lattice of $\mathfrak{g}$ and by Q the root lattice of $\mathfrak{g}$. Chari and Moura [2004] showed that the blocks of $\mathscr{F}$ are parametrized by the spectral characters of $L(\mathfrak{g})$; these are finitely supported functions $\chi : \mathbb{C}^\times \to P/Q$. The set of all such χ forms an additive monoid, denoted by Ξ. The main result of this paper is to show that the methods used in [Chari et al. 2008] to parametrize the simple objects in $\mathscr{F}$ can be extended to parametrize the blocks of $\mathscr{F}^\sigma$. The diagram automorphism σ is used to construct an equivalence relation on Ξ, and we show that the blocks of $\mathscr{F}^\sigma$ are parametrized by the corresponding equivalence classes of spectral characters.

This paper is organized as follows. In Sections 1 and 2, we review the main results concerning the Weyl modules for the algebras $L(\mathfrak{g})$ and $L^\sigma(\mathfrak{g})$ given in [Chari and Pressley 2001] and [Chari et al. 2008]. These are certain maximal finite-dimensional highest weight (in an appropriate sense, described below) modules for the loop algebras. They are in bijective correspondence with the irreducible modules, whose classification plays an important role in the proof of the main theorem. In Section 3, we first review the block decomposition of the category $\mathscr{F}$ by spectral characters carried out in [Chari and Moura 2004]. Then, after defining an equivalence relation $\sim_\sigma$ on the monoid Ξ, we show that the equivalence classes of spectral characters parametrize the blocks of $\mathscr{F}^\sigma$. This is done in two steps: We show that every indecomposable module must have a twisted spectral character, and that any two irreducible modules sharing the same twisted spectral character must lie in the same indecomposable abelian subcategory.

1. The untwisted loop algebras and the modules $W(\pi)$

1.1. *Preliminaries.* Throughout the paper $\mathbb{C}$ and $\mathbb{C}^\times$ respectively denote the set of complex and nonzero complex numbers, and $\mathbb{Z}$ and $\mathbb{Z}_+$ the sets of integers and nonnegative integers. Given a Lie algebra $\mathfrak{a}$, we denote by $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$ and by $L(\mathfrak{a})$ the loop algebra of $\mathfrak{a}$. Specifically, we have

$$L(\mathfrak{a}) = \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}],$$

with commutator given by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} \quad \text{for } x, y \in \mathfrak{a}, r, s \in \mathbb{Z}.$$

We identify $\mathfrak{a}$ with the subalgebra $\mathfrak{a} \otimes 1$ of $L(\mathfrak{a})$.

Let $\mathfrak{g}$ be any finite-dimensional complex simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let W be the corresponding Weyl group, and w_0 the longest element of W. Let R be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let I be an index set for a set of simple roots (and hence also for the fundamental weights). Let R^+ be the set of positive roots. Let Q^+ and P^+ be the $\mathbb{Z}_+$ span of the simple roots and fundamental weights, respectively. Let θ be the highest root in R^+, and let

$$P = P^+ \cup -P^+ \quad \text{and} \quad Q = Q^+ \cup -Q^+.$$

Note P contains Q as a sublattice. Let $\pi : P \to P/Q$ be the canonical projection, and define a partial order $\geq$ on P by setting $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. We will write $\lambda > \mu$ if $\lambda \geq \mu$ and $\lambda \neq \mu$.

Let $\mathfrak{g}_\alpha$ be the root space corresponding to $\alpha \in R$. We have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$. Make $x_\alpha^\pm, h_\alpha$ for $\alpha \in R^+$ a Chevalley basis for $\mathfrak{g}$, and set $x_i^\pm = x_{\alpha_i}^\pm$ and $h_i = h_{\alpha_i}$ for $i \in I$. In particular, $[x_i^+, x_i^-] = h_i$ and $[h_i, x_i^\pm] = \pm 2x_i^\pm$ for $i \in I$.

We collect here some properties of a representation of a Lie algebra on a finite-dimensional complex vector space. If V is a representation of a complex Lie algebra $\mathfrak{a}$ and $V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ is a filtration of submodules of V, we will refer to a quotient module V_i / V_{i+1} as an $\mathfrak{a}$-*constituent* (or just a *constituent*, if the algebra is understood) of V. If each constituent of a filtration is a simple $\mathfrak{a}$-module, we say that the filtration is a composition series. Although composition series are not unique, the Jordan–Hölder theorem guarantees that V has a unique list (up to isomorphism and reordering) of simple constituents. The number of such simple constituents (counting multiplicities) is the *length* of the module V.

1.2. ***Representations of a simple complex Lie algebra.*** If $\mathfrak{g}$ is a simple complex Lie algebra and V is a finite-dimensional representation of $\mathfrak{g}$, we can write

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad \text{where } V_\mu = \{v \in V : h \,.\, v = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

Set $\mathrm{wt}(V) = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$. It is well known that $V_\mu \neq 0$ implies $\mu \in P$ and $w\mu \in \mathrm{wt}(V)$ for all $w \in W$, and that V is isomorphic to a direct sum of irreducible representations; that is, the category of finite-dimensional representations of $\mathfrak{g}$ is semisimple. The set of isomorphism classes of irreducible finite-dimensional $\mathfrak{g}$-modules is in bijection with P^+. For any $\lambda \in P^+$, let $V(\lambda)$ be an element of the corresponding isomorphism class. Then $V(\lambda)$ is generated by an element v_λ satisfying the relations

$$\mathfrak{n}^+ \,.\, v_\lambda = 0, \quad h \,.\, v_\lambda = \lambda(h) v_\lambda, \quad (x_i^-)^{\lambda(h_i)+1} \,.\, v_\lambda = 0. \tag{1-1}$$

The following facts are well known. See for example [Bourbaki 2002].

Proposition 1.3. *Let V be a finite-dimensional representation of $\mathfrak{g}$.*

(i) *For all $w \in W$ and $\mu \in P$, we have $\dim(V_\mu) = \dim(V_{w\mu})$.*

(ii) *Let $V(\lambda)^*$ be the representation of $\mathfrak{g}$ dual to $V(\lambda)$. Then $V(\lambda)^* \cong V(-w_0\lambda)$.*

Proposition 1.4. *Let $\lambda, \mu \in P^+$, and consider $\mathfrak{g}$ as a $\mathfrak{g}$-module via the adjoint representation.*

(i) *If* $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\mu), V(\lambda)) \neq 0$, *then $\lambda - \mu \in Q$.*

(ii) [Chari and Moura 2004, Proposition 1.2] *If $\lambda - \mu \in Q$, then there exists a sequence of weights $\mu_l \in P^+$ for $l = 0, \dots, m$ such that*

  (a) *$\mu_0 = \mu$ and $\mu_m = \lambda$, and*

  (b) $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\mu_l), V(\mu_{l+1})) \neq 0$ *for all* $0 \le l \le m$.

Proof. We give the proof of (i). Since $\mathfrak{g}$ is semisimple, we have

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\mu), V(\lambda)) \neq 0 \quad \text{implies} \quad \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), \mathfrak{g} \otimes V(\mu)) \neq 0.$$

Let ϕ be a nonzero element of $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda), \mathfrak{g} \otimes V(\mu))$, and v_+ a highest weight vector in $V(\lambda)$. Then $\phi(v_+)$ is a weight vector in $\mathfrak{g} \otimes V(\mu)$, and we must have $\phi(v_+) \neq 0$. Therefore $\lambda = \beta + \mu - \eta$, where $\beta \in R$ and $\eta \in Q^+$; hence $\lambda - \mu \in Q$. □

1.5. ***The monoid $\mathcal{P}$.*** Let $\mathcal{P}$ be the monoid of I-tuples $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)$ of polynomials, in an indeterminate u with constant term one, with multiplication being defined componentwise. For $i \in I$, $\lambda \in P^+$, and $a \in \mathbb{C}^\times$, define

$$\boldsymbol{\pi}_{\lambda,a} = \left((1 - au)^{\lambda(h_i)}\right) \in \mathcal{P}.$$

Clearly any $\boldsymbol{\pi} \in \mathcal{P}$ can be written uniquely as a product

$$\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_i, a_i}, \tag{1-2}$$

for some $\lambda_1, \ldots, \lambda_\ell \in P^+ \setminus \{0\}$ and distinct elements $a_1, \ldots, a_\ell \in \mathbb{C}^\times$. We call the scalars a_i the *coordinates* of $\boldsymbol{\pi}$, and the factorization (1-2) the *standard decomposition* of $\boldsymbol{\pi}$. Define a map $\mathcal{P} \to P^+$ by $\boldsymbol{\pi} \mapsto \lambda_{\boldsymbol{\pi}} = \sum_{i \in I} \deg(\pi_i)\omega_i$.

1.6. ***The modules $W(\boldsymbol{\pi})$ and $V(\boldsymbol{\pi})$.***

Definition 1.7. An $L(\mathfrak{g})$-module V is ℓ-highest weight (loop-highest weight) if there exists $v_+ \in V$ such that

$$V = \mathbf{U}(L(\mathfrak{g})) \,.\, v_+, \quad L(\mathfrak{n}^+) \,.\, v_+ = 0, \quad L(\mathfrak{h}) \,.\, v_+ = \mathbb{C}v.$$

For an ℓ-highest weight $L(\mathfrak{g})$-module V and $\lambda \in \mathfrak{h}^*$, we set

$$V_\lambda = \{v \in V : h \,.\, v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\} \quad \text{and} \quad V_\lambda^+ = \{v \in V_\lambda : L(\mathfrak{n}^+) \,.\, v = 0\}.$$

The $\mathfrak{g}$-modules $W(\boldsymbol{\pi})$ we now define are the *Weyl modules* for $\mathfrak{g}$, first introduced and studied in [Chari and Pressley 2001].

Definition 1.8. Let $\boldsymbol{\pi} \in \mathcal{P}$ with standard decomposition $\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_i, a_i}$, and $J_{\boldsymbol{\pi}}$ the left ideal of $\mathbf{U}(L(\mathfrak{g}))$ generated by the elements

$$L(\mathfrak{n}^+), \quad (x_i^-)^{\lambda_{\boldsymbol{\pi}}(h_i)+1}, \quad h \otimes t^r - \textstyle\sum_{i=1}^{\ell} a_i^r \lambda_i(h)$$

for all $h \otimes t^k \in L(\mathfrak{h})$ and $i \in I$. Then we define the left $L(\mathfrak{g})$-module $W(\boldsymbol{\pi})$ as

$$W(\boldsymbol{\pi}) = \mathbf{U}(L(\mathfrak{g})) \big/ J_{\boldsymbol{\pi}}.$$

Let $w_{\boldsymbol{\pi}}$ be the image of 1 under the canonical projection $\mathbf{U}(L(\mathfrak{g})) \to W(\boldsymbol{\pi})$.

Proposition 1.9 [Chari and Pressley 2001, Propositions 2.1 and 3.1].

(i) *$W(\boldsymbol{\pi})$ has a unique irreducible quotient, every finite-dimensional irreducible $L(\mathfrak{g})$-module occurs as such a quotient, and for $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$ the irreducible quotients of $W(\boldsymbol{\pi})$ and $W(\boldsymbol{\pi}')$ are nonisomorphic. Therefore the isomorphism classes of simple $L(\mathfrak{g})$-modules are in bijection with $\mathcal{P}$.*

(ii) *Given $\boldsymbol{\pi} \in \mathcal{P}$ with standard decomposition $\prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_i, a_i}$, we have an isomorphism $W(\boldsymbol{\pi}) \cong \bigotimes_{k=1}^{\ell} W(\boldsymbol{\pi}_{\lambda_i, a_i})$ of $L(\mathfrak{g})$-modules.*

(iii) *Let V be any finite-dimensional ℓ-highest weight $L(\mathfrak{g})$-module generated by an element v satisfying*

$$L(\mathfrak{n}^+) \,.\, v = 0 \quad \textit{and} \quad L(\mathfrak{h}) \,.\, v = \mathbb{C}v.$$

Then V is a quotient of $W(\boldsymbol{\pi})$ for some $\boldsymbol{\pi} \in \mathcal{P}$.

We denote by $V(\boldsymbol{\pi})$ an element of the isomorphism class of simple $L(\mathfrak{g})$-modules corresponding to $\boldsymbol{\pi} \in \mathcal{P}$.

2. The twisted algebras $L^\sigma(\mathfrak{g})$ and the modules $W(\boldsymbol{\pi}^\sigma)$

2.1. Here we review the construction of the twisted loop algebra $L^\sigma(\mathfrak{g})$; for further details, see [Kac 1990]. This begins with a *diagram automorphism* σ of $\mathfrak{g}$, which is a Lie algebra automorphism induced by a bijection $\sigma : I \to I$ that preserves all edge relations (and directions, where they occur) on the Dynkin diagram of $\mathfrak{g}$. One can verify by inspection that the only types for which such a nontrivial automorphism occurs are the types A_n, D_n or E_6, and so we assume from here on that $\mathfrak{g}$ is of one of these types. In all types but D_4 there is a unique nontrivial automorphism of order 2, while for type D_4 there are exactly two nontrivial automorphisms (up to relabeling of the nodes of the Dynkin diagram): one of order two and one of order three. We let m be the order of σ, and let G be the cyclic group with elements σ^ϵ for $0 \le \epsilon \le m-1$. For $i \in I$, we denote by G_i the stabilizer of i in G. We also fix a primitive m-th root of unity ζ. The automorphism σ induces a permutation of R given by $\sigma : \sum_{i\in I} n_i\alpha_i \mapsto \sum_{i\in I} n_i\alpha_{\sigma(i)}$, and we have

$$\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}, \quad \sigma(\mathfrak{h}) = \mathfrak{h}, \quad \sigma(\mathfrak{n}^\pm) = \mathfrak{n}^\pm,$$

$$\mathfrak{g} = \bigoplus_{\epsilon=0}^{m-1} \mathfrak{g}_\epsilon, \quad \text{where } \mathfrak{g}_\epsilon = \{x \in \mathfrak{g} : \sigma(x) = \zeta^\epsilon x\}.$$

We also denote by σ the automorphism of $\mathbb{C}^\times$ given by $\sigma : a \mapsto \zeta a$.

Given any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ that is preserved by σ, set $\mathfrak{a}_\epsilon = \mathfrak{g}_\epsilon \cap \mathfrak{a}$. It is known that $\mathfrak{g}_0$ is a simple Lie algebra, $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, and $\mathfrak{g}_\epsilon$ is an irreducible representation of $\mathfrak{g}_0$ for all $0 \le \epsilon \le m-1$. Also,

$$\mathfrak{n}^\pm \cap \mathfrak{g}_0 = \mathfrak{n}_0^\pm = \bigoplus_{\alpha\in R_0^+} (\mathfrak{g}_0)_{\pm\alpha},$$

where we denote by R_0 the set of roots of the Lie algebra $\mathfrak{g}_0$; the sets I_0, $P_0^\pm$ and so on are defined similarly. The set I_0 is in bijection with the set of σ-orbits of I.

Suppose that $\{y_i : i \in I\}$ is one of the sets $\{h_i : i \in I\}$, $\{x_i^+ : i \in I\}$ or $\{x_i^- : i \in I\}$, and assume that $i \neq n$ if $\mathfrak{g}$ is of type A_{2n}. For $0 \le \epsilon \le m-1$, define subsets $\{y_{i,\epsilon} : i \in I_0\}$ of $\mathfrak{g}_\epsilon$ by $y_{i,\epsilon} = |G_i|^{-1} \sum_{j=0}^{m-1} \zeta^{j\epsilon} y_{\sigma^j(i)}$.

If $\mathfrak{g}$ is of type A_{2n}, then we set

$$\begin{aligned} h_{n,0} &= 2(h_n + h_{n+1}), & x_{n,0}^\pm &= \sqrt{2}(x_n^\pm + x_{n+1}^\pm), \\ h_{n,1} &= h_n - h_{n+1}, & x_{n,1}^\pm &= -\sqrt{2}(x_n^\pm - x_{n+1}^\pm), \quad y_{n,1}^\pm = \mp\tfrac{1}{4}[x_{n,0}^\pm, x_{n,1}^\pm]. \end{aligned}$$

Then $\{x_{i,0}^\pm, h_{i,0}\}_{i\in I_0}$ is a Chevalley basis for $\mathfrak{g}_0$, so $\{h_{i,0}\}_{i\in I_0}$ is a basis of $\mathfrak{h}_0$.

The subset P_σ^+ of P_0^+ is defined as[1]

$$P_\sigma^+ = \begin{cases} \lambda \in P_0^+ \text{ such that } \lambda(h_{n,0}) \in 2\mathbb{Z} & \text{if } \mathfrak{g} \text{ is of type } A_{2n}, \\ P_0^+ & \text{otherwise,} \end{cases}$$

and we regard $\lambda \in P_\sigma^+$ as an element of P^+ by

$$\lambda(h_i) = \begin{cases} \lambda(h_{i,0}) & \text{if } i \in I_0 \text{ and } \mathfrak{g} \text{ is not of type } A_{2n}, \\ 0 & \text{if } i \notin I_0 \text{ and } \mathfrak{g} \text{ is not of type } A_{2n}, \\ (1-\delta_{i,n}/2)\lambda(h_{i,0}) & \text{if } \mathfrak{g} \text{ is of type } A_{2n}. \end{cases}$$

Given $\lambda = \sum_{i \in I} m_i \omega_i \in P^+$ and $0 \le \epsilon \le m-1$, define elements $\lambda(\epsilon) \in P_\sigma^+$ by

if $m = 2$ and $\mathfrak{g}$ is not of type A_{2n}, $\quad \lambda(0) = \sum_{i \in I_0} m_i \omega_i$,
$\lambda(1) = \sum_{i \in I_0 : \sigma(i) \ne i} m_{\sigma(i)} \omega_i$;

if $m = 2$ and $\mathfrak{g}$ is of type A_{2n}, $\quad \lambda(0) = \sum_{i \in I_0} (1+\delta_{i,n}) m_i \omega_i$,
$\lambda(1) = \sum_{i \in I_0 : \sigma(i) \ne i} (1+\delta_{\sigma(i),n}) m_{\sigma(i)} \omega_i$;

if $m = 3$, $\quad \lambda(0) = m_1\omega_1 + m_2\omega_2$,
$\lambda(1) = m_3\omega_1$,
$\lambda(2) = m_4\omega_1$.

2.2. Let $\tilde{\sigma} : L(\mathfrak{g}) \to L(\mathfrak{g})$ be the automorphism defined by linearly extending

$$\tilde{\sigma}(x \otimes t^k) = \zeta^k \sigma(x) \otimes t^k \quad \text{for } x \in \mathfrak{g} \text{ and } k \in \mathbb{Z}.$$

Then $\tilde{\sigma}$ is of order m and we let $L^\sigma(\mathfrak{g})$ be the subalgebra of fixed points of $\tilde{\sigma}$. Clearly,

$$L^\sigma(\mathfrak{g}) \cong \bigoplus_{\epsilon=0}^{m-1} \mathfrak{g}_\epsilon \otimes t^{m-\epsilon} \mathbb{C}[t^m, t^{-m}].$$

If $\mathfrak{a}$ is any Lie subalgebra of $\mathfrak{g}$, we set $L^\sigma(\mathfrak{a}) = L(\mathfrak{a}) \cap L^\sigma(\mathfrak{g})$.

2.3. ***The monoid*** $\mathcal{P}^\sigma$. Let $\mathcal{P}^\sigma$ be the monoid of I_0-tuples $\boldsymbol{\pi}^\sigma = (\pi_i)_{i \in I_0}$ of polynomials in an indeterminate u with constant term one, with multiplication being defined componentwise. Let $(\cdot\,,\cdot)$ be the form on $\mathfrak{h}_0^*$ induced by the Killing form

[1]When $\mathfrak{g}$ is of type A_{2n}, the role of λ in the representation theory of $L^\sigma(\mathfrak{g})$ is subject to an unusual constraint. If V is some ℓ-highest weight module generated by $v_+ \in V_\lambda^+$, the element $y_{n,1}^- \otimes t$ of $L^\sigma(\mathfrak{g})$ must act nilpotently on v_+. The $\mathfrak{sl}_2$-subalgebra corresponding to this generator is

$$\mathfrak{sl}_2 \cong \langle y_{n,1}^\pm \otimes t^{\mp 1}, \tfrac{1}{2} h_{n,0} \otimes 1 \rangle \subseteq L^\sigma(\mathfrak{g}).$$

Therefore the usual $\mathfrak{sl}_2$ theory requires $\lambda(h_{n,0}/2) \in \mathbb{Z}$. This constraint motivates the definition of P_σ^+ given above.

of $\mathfrak{g}_0$, normalized so that $(\theta_0, \theta_0) = 2$. Now define elements $\boldsymbol{\pi}^\sigma_{i,a}$, $\boldsymbol{\pi}^\sigma_{\lambda,a} \in \mathcal{P}^\sigma$ as follows: For $i \in I_0$, $a \in \mathbb{C}^\times$, $\lambda \in P_0^+$, and $\mathfrak{g}$ not of type A_{2n}, set

$$\boldsymbol{\pi}^\sigma_{i,a} = ((1 - a^{(\alpha_i,\alpha_i)}u)^{\delta_{ij}} : j \in I_0), \qquad \boldsymbol{\pi}^\sigma_{\lambda,a} = \prod_{i \in I_0} \left(\boldsymbol{\pi}^\sigma_{i,a}\right)^{\lambda(h_i)}.$$

For $i \in I_0$, $a \in \mathbb{C}^\times$, $\lambda \in P_\sigma^+$, and $\mathfrak{g}$ of type A_{2n}, set

$$\boldsymbol{\pi}^\sigma_{i,a} = ((1 - au)^{\delta_{ij}} : j \in I_0), \qquad \boldsymbol{\pi}^\sigma_{\lambda,a} = \prod_{i \in I_0} \left(\boldsymbol{\pi}^\sigma_{i,a}\right)^{(1-\frac{1}{2}\delta_{i,n})\lambda(h_i)}.$$

Define a map $\mathcal{P}_\sigma^+ \to P_\sigma^+$ by

$$\lambda_{\boldsymbol{\pi}^\sigma} = \begin{cases} \sum_{i \in I_0} (\deg \pi_i)\omega_i & \text{if } \mathfrak{g} \text{ is not of type } A_{2n}, \\ \sum_{i \in I_0} (1 + \delta_{i,n})(\deg \pi_i)\omega_i & \text{if } \mathfrak{g} \text{ is of type } A_{2n}. \end{cases}$$

It is clear that any $\boldsymbol{\pi}^\sigma \in \mathcal{P}_\sigma$ can be written (nonuniquely) as a product

$$\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k},$$

where $\mathbf{a} = (a_1, \dots, a_\ell)$ and $\mathbf{a}^m$ have distinct coordinates, that is, $a_i^m \neq a_j^m$ for $i \neq j$. We call any such expression a standard decomposition of $\boldsymbol{\pi}^\sigma$.

2.4. ***The map*** $\mathbf{r} : \mathcal{P} \to \mathcal{P}^\sigma$**.** Given $\boldsymbol{\pi} \in \mathcal{P}$ with a standard factorization $\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_k,a_k}$, define a map

$$\mathbf{r} : \mathcal{P} \to \mathcal{P}^\sigma, \quad \boldsymbol{\pi} \mapsto \prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_k(\epsilon),\zeta^\epsilon a_k}$$

(recall the definition of $\lambda_k(\epsilon)$ given in Section 2.1). For any $\boldsymbol{\pi} \in \mathcal{P}$, we have $\lambda_{\mathbf{r}(\boldsymbol{\pi})} = \sum_{\epsilon=0}^{m-1} \lambda_{\boldsymbol{\pi}}(\epsilon)$. Note that $\mathbf{r}$ is well defined (since the choice of the (λ_k, a_k) is unique), and set $\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma) = \{\boldsymbol{\pi} \in \mathcal{P} : \mathbf{r}(\boldsymbol{\pi}) = \boldsymbol{\pi}^\sigma\}$.

Lemma 2.5. (i) *Let $\lambda \in P_\sigma^+$ and $a \in \mathbb{C}^\times$. Then*

$$\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\lambda,a}) = \begin{cases} \left\{\boldsymbol{\pi}_{\lambda+\eta,a}\, \boldsymbol{\pi}_{-\sigma(\eta),-a} : \eta \in (P^+ - \lambda) \cap P^-\right\} & \text{if } m = 2; \\ \left\{\boldsymbol{\pi}_{\lambda+\eta_1+\eta_2,a}\, \boldsymbol{\pi}_{-\sigma^2(\eta_1),\zeta a}\, \boldsymbol{\pi}_{-\sigma(\eta_2),\zeta^2 a} : \right. & \\ \qquad \left. \eta_1, \eta_2 \in P^-,\ (\eta_1 + \eta_2) \in P^+ - \lambda\right\} & \text{if } m = 3. \end{cases}$$

(ii) *Let $m = 2$, and let $\boldsymbol{\pi}^\sigma = \prod_{i=1}^{k} \boldsymbol{\pi}^\sigma_{\lambda_{i,0},a_i} \boldsymbol{\pi}^\sigma_{\lambda_{i,1},-a_i}$ be a standard factorization of $\boldsymbol{\pi}^\sigma \in \mathcal{P}^\sigma$ for $\lambda_{i,\epsilon} \in P_0^+$. Then*

$$\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma) = \prod_{i=1}^{k} \left\{\boldsymbol{\pi}_{(\lambda_{i,0})+\eta_i,a_i}\, \boldsymbol{\pi}_{(\lambda_{i,1})-\sigma(\eta_i),-a_i} : \eta_i \in (P^+ - (\lambda_{i,0})) \cap (P^- + \sigma(\lambda_{i,1}))\right\}.$$

(iii) *Let* $m=3$, *and let* $\boldsymbol{\pi}^\sigma=\prod_{i=1}^k \boldsymbol{\pi}^\sigma_{\lambda_{i,0},a_i}\boldsymbol{\pi}^\sigma_{\lambda_{i,1},\zeta a_i}\boldsymbol{\pi}^\sigma_{\lambda_{i,2},\zeta^2 a_i}$ *be a standard factorization of* $\boldsymbol{\pi}^\sigma\in\mathcal{P}^\sigma$ *for* $\lambda_{i,\epsilon}\in P_0^+$. *Then*

$$\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)=\prod_{i=1}^k\bigl\{\boldsymbol{\pi}_{(\lambda_{i,0})+\eta_i+\nu_i,a_i}\,\boldsymbol{\pi}_{(\lambda_{i,1})-\sigma^2(\eta_i),\zeta a_i}\,\boldsymbol{\pi}_{(\lambda_{i,2})-\sigma(\nu_i),\zeta^2 a_i}: \\ \eta_i+\nu_i\in P^+-(\lambda_{i,0}),\ \sigma^2(\eta_i)\in P^-+(\lambda_{i,1}), \\ \sigma(\nu_i)\in P^-+(\lambda_{i,2})\bigr\},$$

where the product of sets written in (ii) *and* (iii) *is the set of all products.*

Proof. The statements will be proved only for $m=3$. The proof for the remaining cases when $m=2$ is simpler and uniform. The proof begins with the following identity,[2] whose verification is routine (we regard $\lambda,\mu,\gamma\in P_\sigma^+$ as elements of P^+ via the embedding $I_0\subseteq I$ as in Section 2.1): For $\lambda,\mu,\gamma\in P_\sigma^+$, and $\eta,\nu\in P$ such that

$$(\eta+\nu)\in P^+-\lambda,\quad \sigma^2(\eta)\in P^-+\mu,\quad \sigma(\nu)\in P^-+\gamma,$$

we have

$$\mathbf{r}(\boldsymbol{\pi}_{\lambda+\eta+\nu,a}\,\boldsymbol{\pi}_{\mu-\sigma^2(\eta),\zeta a}\,\boldsymbol{\pi}_{\gamma-\sigma(\nu),\zeta^2 a})=\boldsymbol{\pi}^\sigma_{\lambda,a}\,\boldsymbol{\pi}^\sigma_{\mu,\zeta a}\,\boldsymbol{\pi}^\sigma_{\gamma,\zeta^2 a}. \tag{2-1}$$

Now we will prove the $m=3$ case of the identity for $\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\lambda,a})$ given in part (i). The containment $\supseteq$ is immediate from identity (2-1) by taking $\mu=\gamma=0$ and $\eta=\eta_1$ and $\nu=\eta_2$. For the opposite containment, let

$$\boldsymbol{\pi}=\prod_{k=1}^{\ell}\boldsymbol{\pi}_{\rho_k,a_k}\,\boldsymbol{\pi}_{\mu_k,\zeta a_k}\,\boldsymbol{\pi}_{\gamma_k,\zeta^2 a_k}\in\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\lambda,a}).$$

Then we must have $\rho_k=\mu_k=\gamma_k=0$ for all k such that $a_k^3\neq a^3$, and so without loss of generality, $\boldsymbol{\pi}=\boldsymbol{\pi}_{\rho,a}\,\boldsymbol{\pi}_{\mu,\zeta a}\,\boldsymbol{\pi}_{\gamma,\zeta^2 a}$. Then

$$\mathbf{r}(\boldsymbol{\pi})=\boldsymbol{\pi}^\sigma_{\rho(0)+\mu(2)+\gamma(1),a}\,\boldsymbol{\pi}^\sigma_{\rho(1)+\mu(0)+\gamma(2),\zeta a}\,\boldsymbol{\pi}^\sigma_{\rho(2)+\mu(1)+\gamma(0),\zeta^2 a}.$$

The condition $\boldsymbol{\pi}\in\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\lambda,a})$ then forces $\rho=\lambda-\sigma(\mu)-\sigma^2(\gamma)$. Therefore $\boldsymbol{\pi}$ is of the form $\boldsymbol{\pi}_{\lambda+\eta_1+\eta_2,a}\,\boldsymbol{\pi}_{\sigma^2(\eta_1),\zeta a}\,\boldsymbol{\pi}_{-\sigma(\eta_2),\zeta^2 a}$, where $\eta_1=-\sigma(\mu)$ and $\eta_2=-\sigma^2(\gamma)$, and the proof of part (i) is complete.

We continue with the proof of (iii). From the description of $\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)$ given in [Chari et al. 2008, Lemma 3.5], it follows that $\mathbf{r}^{-1}$ is multiplicative in that

$$\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_1\,\boldsymbol{\pi}^\sigma_2)=\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_1)\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_2),$$

[2] The corresponding statement for the cases $m=2$ is as follows: For $m=2$, $\lambda,\mu\in P_\sigma^+$, and $\eta\in(P^+-\lambda)\cap(P^-+\sigma(\mu))$, we have $\mathbf{r}(\boldsymbol{\pi}_{\lambda+\eta,a}\,\boldsymbol{\pi}_{\mu-\sigma(\eta),-a})=\boldsymbol{\pi}^\sigma_{\lambda,a}\,\boldsymbol{\pi}^\sigma_{\mu,-a}$.

where the product of the sets $\mathbf{r}^{-1}(\boldsymbol{\pi}_1^\sigma)\mathbf{r}^{-1}(\boldsymbol{\pi}_2^\sigma)$ is the set of products. Therefore it suffices to prove (iii) for $k = 1$, and the result will now follow from the inclusion

$$\begin{aligned}(2\text{-}2)\quad &\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\lambda,a})\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\mu,\zeta a})\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\gamma,\zeta^2 a})\\ &\subseteq \big\{\boldsymbol{\pi}_{\lambda+\eta+\nu,a}\,\boldsymbol{\pi}_{\mu-\sigma^2(\eta),\zeta a}\,\boldsymbol{\pi}_{\gamma-\sigma(\nu),\zeta^2 a} :\\ &\qquad\qquad \eta+\nu\in(P^+-\lambda),\ \eta\in(P^-+\sigma^2(\mu)),\ \nu\in(P^-+\sigma(\gamma))\big\}.\end{aligned}$$

To prove this, let $\eta_i, \nu_i \in P^+$ for $i = 0, 1, 2$ be such that

$$\begin{aligned}\boldsymbol{\pi}_0 &= \boldsymbol{\pi}_{\lambda+\eta_0+\nu_0,a}\,\boldsymbol{\pi}_{-\sigma^2(\eta_0),\zeta a}\,\boldsymbol{\pi}_{-\sigma(\nu_0),\zeta^2 a} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\lambda,a}),\\ \boldsymbol{\pi}_1 &= \boldsymbol{\pi}_{\mu+\eta_1+\nu_1,\zeta a}\,\boldsymbol{\pi}_{-\sigma^1(\eta_1),\zeta^2 a}\,\boldsymbol{\pi}_{-\sigma(\nu_1),a} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\mu,\zeta a}),\\ \boldsymbol{\pi}_2 &= \boldsymbol{\pi}_{\gamma+\eta_2+\nu_2,\zeta^2 a}\,\boldsymbol{\pi}_{-\sigma^2(\eta_2),a}\,\boldsymbol{\pi}_{-\sigma(\nu_2),\zeta a} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\gamma,\zeta^2 a}).\end{aligned}$$

Then $\boldsymbol{\pi}_0\,\boldsymbol{\pi}_1\,\boldsymbol{\pi}_2$ is equal to

$$\begin{aligned}&\boldsymbol{\pi}_{\lambda+\eta_0+\nu_0-\sigma(\nu_1)-\sigma^2(\eta_2),a}\,\boldsymbol{\pi}_{\mu+\eta_1+\nu_1-\sigma^2(\eta_0)-\sigma(\nu_2),\zeta a}\,\boldsymbol{\pi}_{\gamma+\eta_2+\nu_2-\sigma^2(\eta_1)-\sigma(\nu_0),\zeta^2 a}\\ &\qquad= \boldsymbol{\pi}_{\lambda+\eta'+\nu',a}\,\boldsymbol{\pi}_{\mu-\sigma^2(\eta'),\zeta a}\,\boldsymbol{\pi}_{\gamma-\sigma(\nu'),\zeta^2 a},\end{aligned}$$

where

$$\eta' = \eta_0+\sigma^2(\nu_2)-\sigma(\eta_1)-\sigma(\nu_1) \quad\text{and}\quad \nu' = \nu_0+\sigma(\eta_1)-\sigma^2(\eta_2)-\sigma^2(\nu_2),$$

and it is easily verified that $\lambda+\eta'+\nu'$, $\mu-\sigma^2(\eta')$, $\gamma-\sigma(\nu') \in P^+$.

From the inclusion (2-2) we conclude that

$$\begin{aligned}&\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma_{\lambda,a}\,\boldsymbol{\pi}^\sigma_{\mu,\zeta a}\,\boldsymbol{\pi}^\sigma_{\gamma,\zeta^2 a})\\ &\subseteq \big\{\boldsymbol{\pi}_{\lambda+\eta+\nu,a}\,\boldsymbol{\pi}_{\mu-\sigma^2(\eta),\zeta a}\,\boldsymbol{\pi}_{\gamma-\sigma(\nu),\zeta^2 a} :\\ &\qquad\qquad \eta+\nu\in(P^+-\lambda),\ \eta\in(P^-+\sigma^2(\mu)),\ \nu\in(P^-+\sigma(\gamma))\big\},\end{aligned}$$

and part (iii) is established. □

Corollary 2.6. *If $m = 2$ and $\boldsymbol{\pi} = \boldsymbol{\pi}_{\lambda,a}\,\boldsymbol{\pi}_{\mu,-a} \in \mathcal{P}$, then*

$$\mathbf{r}^{-1}(\mathbf{r}(\boldsymbol{\pi})) = \big\{\boldsymbol{\pi}_{\lambda+\eta,a}\,\boldsymbol{\pi}_{\mu-\sigma(\eta),-a} : \eta\in(P^+-\lambda))\cap(P^-+\sigma(\mu))\big\}.$$

If $m = 3$ and $\boldsymbol{\pi} = \boldsymbol{\pi}_{\lambda,a}\,\boldsymbol{\pi}_{\mu,\zeta a}\,\boldsymbol{\pi}_{\gamma,\zeta^2 a} \in \mathcal{P}$, then

$$\begin{aligned}\mathbf{r}^{-1}(\mathbf{r}(\boldsymbol{\pi})) = \big\{&\boldsymbol{\pi}_{\lambda+\eta+\nu,a}\,\boldsymbol{\pi}_{\mu-\sigma^2(\eta),\zeta a}\,\boldsymbol{\pi}_{\gamma-\sigma(\nu),\zeta^2 a} :\\ &(\eta+\nu)\in P^+-\lambda,\ \sigma^2(\eta)\in P^-+\mu,\ \sigma(\nu)\in P^-+\gamma\big\}.\end{aligned}$$

2.7. *The modules $W(\boldsymbol{\pi}^\sigma)$ and $V(\boldsymbol{\pi}^\sigma)$.*

Definition 2.8. An $L^\sigma(\mathfrak{g})$-module V is ℓ-highest weight if there exists $v_+ \in V$ such that

$$V = \mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v_+,\qquad L^\sigma(\mathfrak{n}^+)\,.\,v_+ = 0,\qquad L^\sigma(\mathfrak{h})\,.\,v_+ = \mathbb{C}v.$$

For an ℓ-highest weight $L^\sigma(\mathfrak{g})$-module V and $\lambda \in \mathfrak{h}_0^*$, we set

$$V_\lambda = \{v \in V : h\,.\,v = \lambda(h)v \text{ for all } h \in \mathfrak{h}_0\} \quad \text{and} \quad V_\lambda^+ = \{v \in V_\lambda : L^\sigma(\mathfrak{n}^+)\,.\,v = 0\}.$$

Definition 2.9 (the Weyl modules $W(\boldsymbol{\pi}^\sigma)$). Let $\boldsymbol{\pi}^\sigma \in \mathcal{P}^\sigma$ with a standard factorization $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}$. For $\mathfrak{g}$ not of type A_{2n}, let $J_{\boldsymbol{\pi}^\sigma}$ be the left ideal in $\mathbf{U}(L^\sigma(\mathfrak{g}))$ generated by the elements

$$L^\sigma(\mathfrak{n}^+), \quad (x_i^-)^{\lambda_{\pi^\sigma}(h_i)+1}, \quad (h_{i,\epsilon} \otimes t^{mk-\epsilon}) - \textstyle\sum_{j=1}^{\ell} \lambda_j(h_{i,0}) a_j^{mk-\epsilon},$$

for all $h \otimes t^k \in L^\sigma(\mathfrak{h})$ and $i \in I_0$. For $\mathfrak{g}$ of type A_{2n}, let $J_{\boldsymbol{\pi}^\sigma}$ be the ideal generated by the elements

$$L^\sigma(\mathfrak{n}^+), \quad (x_i^-)^{\lambda_{\pi^\sigma}(h_i)+1}, \quad (h_{i,\epsilon} \otimes t^{mk-\epsilon}) - \textstyle\sum_{j=1}^{\ell} (1 - \frac{1}{2}\delta_{i,n}) \lambda_j(h_{i,\epsilon}) a_j^{mk-\epsilon}$$

for all $h \otimes t^k \in L^\sigma(\mathfrak{h})$ and $i \in I_0$. Then we define the left $L^\sigma(\mathfrak{g})$-module $W(\boldsymbol{\pi}^\sigma)$ as

$$W(\boldsymbol{\pi}^\sigma) = \mathbf{U}(L^\sigma(\mathfrak{g})) \,/\, J_{\boldsymbol{\pi}^\sigma}.$$

Let $w_{\boldsymbol{\pi}^\sigma}$ be the image of 1 under the canonical projection $\mathbf{U}(L^\sigma(\mathfrak{g})) \to W(\boldsymbol{\pi}^\sigma)$.

Since $L^\sigma(\mathfrak{g})$ is a subalgebra of $L(\mathfrak{g})$, any $L(\mathfrak{g})$-module V is an $L^\sigma(\mathfrak{g})$-module via restriction, that is, $L^\sigma(\mathfrak{g}) \hookrightarrow L(\mathfrak{g}) \to \mathrm{End}(V)$. We will denote this restriction to an $L^\sigma(\mathfrak{g})$-action by $V|_{L^\sigma(\mathfrak{g})}$. If two $L(\mathfrak{g})$-modules V and W are isomorphic as $L^\sigma(\mathfrak{g})$-modules, we will write $V \cong_{L^\sigma(\mathfrak{g})} W$. The next two propositions concern the Weyl modules and their irreducible quotients. Proposition 2.10 is the twisted analogue of Proposition 1.9, while Proposition 2.11 describes the relationship between the untwisted and twisted modules.

Proposition 2.10 [Chari et al. 2008, Theorem 2]. (i) *For any $\boldsymbol{\pi}^\sigma \in \mathcal{P}^\sigma$, the module $W(\boldsymbol{\pi}^\sigma)$ has a unique irreducible quotient, which we will denote by $V(\boldsymbol{\pi}^\sigma)$, and each irreducible $L^\sigma(\mathfrak{g})$-module occurs as such a quotient.*

(ii) *Let V be any finite-dimensional ℓ-highest weight $L^\sigma(\mathfrak{g})$-module generated by an element v satisfying $L^\sigma(\mathfrak{n}^+)\,.\,v = 0$ and $L^\sigma(\mathfrak{g})\,.\,v = \mathbb{C}v$. Then V is a quotient of $W(\boldsymbol{\pi}^\sigma)$ for some $\boldsymbol{\pi}^\sigma \in \mathcal{P}^\sigma$.*

(iii) *Let $\boldsymbol{\pi}^\sigma = \prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}$ be a standard decomposition of $\boldsymbol{\pi}^\sigma \in \mathcal{P}^\sigma$. As $L^\sigma(\mathfrak{g})$-modules, we have*

$$W(\boldsymbol{\pi}^\sigma) \cong \bigotimes_{k=1}^{\ell} W\Bigl(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}\Bigr).$$

Let $\mathcal{P}_{\mathrm{Asym}}$ be the subset of $\mathcal{P}$ consisting of $\boldsymbol{\pi} \in \mathcal{P}$ such that, given the standard decomposition $\boldsymbol{\pi} = \prod \boldsymbol{\pi}_{\lambda_i,a_i}$, we have $a_i^m \neq a_j^m$ for $i \neq j$. For any subset $\mathcal{S}$ of $\mathcal{P}$, let $\mathcal{S}_{\mathrm{Asym}} = \mathcal{S} \cap \mathcal{P}_{\mathrm{Asym}}$. The role played by $\mathcal{P}_{\mathrm{Asym}}$ is described in the following proposition.

Proposition 2.11 [Chari et al. 2008, Propositions 4.1, 4.3 and 4.5]. *Let* $\boldsymbol{\pi}^\sigma \in \mathcal{P}^\sigma$ *and* $\boldsymbol{\pi} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)_{\text{Asym}}$.

(i) $W(\boldsymbol{\pi})|_{L^\sigma(\mathfrak{g})} \cong W(\boldsymbol{\pi}^\sigma)$ *and* $V(\boldsymbol{\pi})|_{L^\sigma(\mathfrak{g})} \cong V(\boldsymbol{\pi}^\sigma)$.

(ii) *Denote the representations* $W(\boldsymbol{\pi})$ *of* $L(\mathfrak{g})$ *and* $W(\boldsymbol{\pi}^\sigma)$ *of* $L^\sigma(\mathfrak{g})$ *by*

$$L(\mathfrak{g}) \xrightarrow{\phi_{\boldsymbol{\pi}}} \mathrm{End}(W(\boldsymbol{\pi})) \quad \textit{and} \quad L^\sigma(\mathfrak{g}) \xrightarrow{\phi_{\boldsymbol{\pi}^\sigma}} \mathrm{End}(W(\boldsymbol{\pi}^\sigma)),$$

respectively. Then there exist ideals $I_{\boldsymbol{\pi}} \subseteq L(\mathfrak{g})$ *and* $I_{\boldsymbol{\pi}^\sigma} \subseteq L^\sigma(\mathfrak{g})$ *such that*

(a) *the Lie algebra homomorphism* $\phi_{\boldsymbol{\pi}^\sigma}$ *factors through* $L^\sigma(\mathfrak{g})/I_{\boldsymbol{\pi}^\sigma}$, *giving a representation*

$$L^\sigma(\mathfrak{g})/I_{\boldsymbol{\pi}^\sigma} \xrightarrow{\overline{\phi_{\boldsymbol{\pi}^\sigma}}} \mathrm{End}(W(\boldsymbol{\pi}^\sigma)).$$

(b) *There exists a Lie algebra isomorphism* $\lambda : L(\mathfrak{g})/I_{\boldsymbol{\pi}} \xrightarrow{\sim} L^\sigma(\mathfrak{g})/I_{\boldsymbol{\pi}^\sigma}$, *giving the following diagram of Lie algebra homomorphisms*:

$$\begin{tikzcd} L(\mathfrak{g}) \arrow[d, two heads, "p"'] & L^\sigma(\mathfrak{g}) \arrow[r, "\phi_{\boldsymbol{\pi}^\sigma}"] \arrow[d, two heads, "p_\sigma"] & \mathrm{End}(W(\boldsymbol{\pi}^\sigma)) \\ L(\mathfrak{g})/I_{\boldsymbol{\pi}} \arrow[r, "\lambda"] & L^\sigma(\mathfrak{g})/I_{\boldsymbol{\pi}^\sigma} \arrow[ur, dotted, "\overline{\phi_{\boldsymbol{\pi}^\sigma}}"'] & \end{tikzcd} \tag{2-3}$$

Here p *and* p_σ *are the canonical projections.*

(c) *Let* $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})}$ *denote the action of* $L(\mathfrak{g})$ *on* $W(\boldsymbol{\pi}^\sigma)$ *given* (*as in diagram* (2-3)) *by the composition*

$$x \otimes t^r . w := \overline{\phi_{\boldsymbol{\pi}^\sigma}} \circ \lambda \circ p(x \otimes t^r) . w$$

where $w \in W(\boldsymbol{\pi}^\sigma)$ *and* $x \otimes t^r \in L(\mathfrak{g})$. *Then* $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})} \cong W(\boldsymbol{\pi})$ *and* $V(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})} \cong V(\boldsymbol{\pi})$ *as* $L(\mathfrak{g})$-*modules.*

Remarks. First, it is clear from the diagram (2-3) that the action of $L(\mathfrak{g})$ on $W(\boldsymbol{\pi}^\sigma)$ — and hence the isomorphism $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})} \cong W(\boldsymbol{\pi})$ — depends upon the isomorphism $\lambda : L(\mathfrak{g})/I_{\boldsymbol{\pi}} \xrightarrow{\sim} L^\sigma(\mathfrak{g})/I_{\boldsymbol{\pi}^\sigma}$. So the expression $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})}$ by itself is ambiguous: $W(\boldsymbol{\pi}^\sigma)$ has, up to isomorphism, as many $L(\mathfrak{g})$-module structures (and hence is isomorphic to as many $L(\mathfrak{g})$-Weyl modules) as there are elements $\boldsymbol{\pi} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)_{\text{Asym}}$, the isomorphisms being determined by λ. For this reason, when needed we will write $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})} \cong W(\boldsymbol{\pi})$ to specify which $L(\mathfrak{g})$-module structure we have chosen for $W(\boldsymbol{\pi}^\sigma)$. Several times we will speak of fixing an $L(\mathfrak{g})$-action on some $L^\sigma(\mathfrak{g})$-module; by this we mean making a choice of an isomorphism $\lambda : I_{\boldsymbol{\pi}} \to I_{\boldsymbol{\pi}^\sigma}$ such that $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})} \cong W(\boldsymbol{\pi})$.

Second, if we have the isomorphism $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})} \cong W(\boldsymbol{\pi})$, then we also have $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})}|_{L^\sigma(\mathfrak{g})} \cong W(\boldsymbol{\pi}^\sigma)$. This follows from the commutativity of the diagram

$$\begin{tikzcd} L(\mathfrak{g}) \arrow[d, two heads, "p"'] & L^\sigma(\mathfrak{g}) \arrow[l, hook'] \arrow[r, "\phi_{\boldsymbol{\pi}^\sigma}"] \arrow[d, two heads, "p_\sigma"] & \mathrm{End}(W(\boldsymbol{\pi}^\sigma)). \\ L(\mathfrak{g})/I_{\boldsymbol{\pi}} \arrow[r, "\lambda"] & L^\sigma(\mathfrak{g})/I_{\boldsymbol{\pi}^\sigma} \arrow[ur, dotted, "\overline{\phi_{\boldsymbol{\pi}^\sigma}}"'] & \end{tikzcd}$$

Finally, if $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})} \cong W(\boldsymbol{\pi})$, then the second remark above implies that a subspace U of $W(\boldsymbol{\pi}^\sigma)$ is an $L^\sigma(\mathfrak{g})$-submodule if and only if it is an $L(\mathfrak{g})$-submodule.

Lemma 2.12 [Chari and Moura 2004, Proposition 3.3]. *Let $V(\boldsymbol{\pi})$ be an irreducible $L(\mathfrak{g})$-constituent of $W(\boldsymbol{\pi}_{\lambda,a})$. Then $\boldsymbol{\pi} = \boldsymbol{\pi}_{\mu,a}$, where $\mu \le \lambda$.*

Proposition 2.13. *Let $\boldsymbol{\pi}^\sigma = \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}$ and $\boldsymbol{\pi}_{\lambda,a} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)_{\mathrm{Asym}}$. Then any irreducible $L^\sigma(\mathfrak{g})$-constituent of $W(\boldsymbol{\pi}^\sigma)$ is isomorphic to some $V(\boldsymbol{\pi}_{\mu,a})|_{L^\sigma(\mathfrak{g})}$, where $\mu \le \lambda$.*

Proof. We fix an $L(\mathfrak{g})$-action $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})} \cong W(\boldsymbol{\pi}_{\lambda,a})$. Let V be an irreducible $L^\sigma(\mathfrak{g})$-constituent of $W(\boldsymbol{\pi}^\sigma)$. Then $V_{L(\mathfrak{g})}$ is isomorphic to an irreducible $L(\mathfrak{g})$-constituent of $W(\boldsymbol{\pi}_{\lambda,a})$. Therefore $V_{L(\mathfrak{g})} \cong V(\boldsymbol{\pi}_{\mu,a})$ for $\mu \le \lambda$ (by Lemma 2.12), and so $V \cong (V_{L(\mathfrak{g})})|_{L^\sigma(\mathfrak{g})} \cong V(\boldsymbol{\pi}_{\mu,a})|_{L^\sigma(\mathfrak{g})}$. □

3. Block decomposition of the category $\mathcal{F}^\sigma$

3.1. ***Block decomposition of a category.*** Let $\mathfrak{a}$ be any Lie algebra, and $\mathcal{M}$ the category of its finite-dimensional representations. Then $\mathcal{M}$ is an abelian tensor category. Any object in M can be written uniquely as a direct sum of indecomposables, and we recall the following:

Definition 3.2. Two indecomposable objects $V_1, V_2 \in \mathcal{M}$ are *linked*, and written $V_1 \sim V_2$, if there do not exist subcategories $\mathcal{M}_1, \mathcal{M}_2$ such that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $V_1 \in \mathcal{M}_1$ and $V_2 \in \mathcal{M}_2$. More generally, two objects $U, V \in \mathcal{M}$ are linked if every indecomposable summand of U is linked to every indecomposable summand of V. We will say that a single object V in $\mathcal{M}$ is linked if there exists some other object W such that $V \sim W$. The relation $\sim$, when restricted to the collection of linked objects,[3] is an equivalence relation.

A *block* of $\mathcal{M}$ is an equivalence class of linked objects.

Proposition 3.3 [Etingof and Moura 2003, Proposition 1.1]. *The category $\mathcal{M}$ has a unique decomposition into a direct sum of indecomposable abelian subcategories, that is, $\mathcal{M} = \bigoplus_{\alpha\in\Lambda} \mathcal{M}_\alpha$.*

[3]The relation $\sim$ of linkage is symmetric and transitive, but it is not reflexive. For example, if W_1 and W_2 are two objects in $\mathcal{M}$ that are *not* linked, then $W = W_1 \oplus W_2$ is not linked to itself; in fact W is linked to nothing at all.

In fact the indecomposable abelian subcategories of this decomposition consist of the equivalence classes of linked objects. The goal of the rest of the paper is to describe these blocks using data from the Lie algebra $L^\sigma(\mathfrak{g})$.

Definition 3.4. Let $U, V \in \mathcal{M}$ be indecomposable. We say that U is strongly linked to V if there exist indecomposable $\mathfrak{a}$-modules $U_1, \dots, U_\ell$, with $U_1 = U$, $U_\ell = V$ and either $\mathrm{Hom}_{\mathfrak{a}}(U_k, U_{k+1}) \neq 0$ or $\mathrm{Hom}_{\mathfrak{a}}(U_{k+1}, U_k) \neq 0$ for all $1 \leq k < \ell$. We extend this to all of $\mathcal{M}$ by saying that two modules U and V are strongly linked if and only if every indecomposable summand of U is strongly linked to every indecomposable summand of V.

Lemma 3.5 [Chari and Moura 2004, Lemma 2.2, Lemma 2.5]. (i) *Suppose V_1 and V_2 are indecomposable objects in $\mathcal{M}$. Then $V_1 \sim V_2$ if and only if they contain submodules $U_k \subseteq V_k$ for $k = 1, 2$ with $U_1 \sim U_2$.*

(ii) *Two modules $U, V \in \mathcal{M}$ are linked if and only if they are strongly linked.*

Let $\mathcal{F}$ and $\mathcal{F}^\sigma$ be the category of finite-dimensional $L(\mathfrak{g})$- and $L^\sigma(\mathfrak{g})$-modules, respectively. From here on we fix a Lie algebra $\mathfrak{g}$ of type A, D or E_6, although any of the following results stated for untwisted loop algebras are true for the loop algebra $L(\mathfrak{g})$ of any simple Lie algebra.

3.6. ***The blocks of the category $\mathcal{F}$.***

Definition 3.7 (the monoid Ξ). Let Ξ be the set of all functions $\chi : \mathbb{C}^\times \to P/Q$ with finite support. Given $\lambda \in P^+$ and $a \in \mathbb{C}^\times$, let $\chi_{\lambda,a} \in \Xi$ be defined by

$$\chi_{\lambda,a}(z) = \delta_a(z)\overline{\lambda},$$

where $\overline{\lambda}$ is the image of λ in P/Q and $\delta_a(z)$ is the characteristic function of $a \in \mathbb{C}^\times$.

Clearly Ξ has the structure of an additive monoid under pointwise addition. For $\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_k,a_k} \in \mathcal{P}$, we set $\chi_{\boldsymbol{\pi}} := \sum_{k=1}^{\ell} \chi_{\lambda_k,a_k}$. It is immediate from the definition that the map $\boldsymbol{\pi} \mapsto \chi_{\boldsymbol{\pi}}$ is a monoid homomorphism (from a multiplicative monoid to an additive monoid). The elements of Ξ are the *spectral characters* of $L(\mathfrak{g})$.

Definition 3.8. We say that a module $V \in \mathcal{F}$ has spectral character $\chi \in \Xi$ if $\chi\boldsymbol{\pi}_i = \chi$ for every irreducible constituent $V(\boldsymbol{\pi}_i)$ of V. Let $\mathcal{F}_\chi$ be the abelian subcategory consisting of all modules $V \in \mathcal{F}$ with spectral character χ.

Theorem 1 [Chari and Moura 2004, Theorem 1 (main theorem)]. *The blocks of the category $\mathcal{F}$ are in bijective correspondence with the spectral characters $\chi \in \Xi$. In particular,*

$$\mathcal{F} = \bigoplus_{\chi \in \Xi} \mathcal{F}_\chi,$$

and each $\mathcal{F}_\chi$ is a block.

3.9. ***The blocks of the category*** $\mathcal{F}^\sigma$**.** Here we will define the *twisted spectral characters* of the twisted loop algebra $L^\sigma(\mathfrak{g})$. These will be equivalence classes of spectral characters under a certain equivalence relation $\sim_\sigma$, defined below. First we need several technical results.

The relation $\mathbf{r}(\boldsymbol{\pi}) = \boldsymbol{\pi}^\sigma$ will be illustrated with the diagram $\boldsymbol{\pi} \xrightarrow{\mathbf{r}} \boldsymbol{\pi}^\sigma$. We will also write $\boldsymbol{\pi}_1 \xleftrightarrow{\mathbf{r}} \boldsymbol{\pi}_2$ if $\mathbf{r}(\boldsymbol{\pi}_1) = \mathbf{r}(\boldsymbol{\pi}_2)$. If $\chi_{\boldsymbol{\pi}_1} = \chi_{\boldsymbol{\pi}_2}$, we will write $\boldsymbol{\pi}_1 \sim_\chi \boldsymbol{\pi}_2$. This relation $\sim_\chi$ is clearly an equivalence, and will be illustrated with the diagram $\boldsymbol{\pi}_1 \xleftrightarrow{\chi} \boldsymbol{\pi}_2$.

Lemma 3.10. *Let $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}$, with $\boldsymbol{\pi} = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_k,a_k}$. Then $\boldsymbol{\pi}' \sim_\chi \boldsymbol{\pi}$ if and only if $\boldsymbol{\pi}'$ is of the form*

$$\boldsymbol{\pi}' = \prod_{k=1}^{\ell} \boldsymbol{\pi}_{\lambda_k+\nu_k,a_k}\,\tilde{\boldsymbol{\pi}},$$

where $\nu_k \in Q$ such that $\lambda_k + \nu_k \in P^+$ and $\tilde{\boldsymbol{\pi}} \in \mathcal{P}$ such that $\chi_{\tilde{\boldsymbol{\pi}}} = 0$.

Proof. Because χ is a monoid homomorphism, it suffices to prove the lemma in the case $\ell = 1$, in which it states $\boldsymbol{\pi} = \boldsymbol{\pi}_{\lambda,a}$. If $\lambda \in Q$, then we take $\nu_1 = 0$ and $\tilde{\boldsymbol{\pi}} = 1$. Now suppose $\lambda \notin Q$ and $\chi_{\boldsymbol{\pi}} = \chi_{\boldsymbol{\pi}'}$. Let us write $\boldsymbol{\pi}' = \prod_{k=1}^{r} \boldsymbol{\pi}_{\mu_k,b_k} \prod_{k=1}^{s} \boldsymbol{\pi}_{\gamma_k,c_k}$, where all b_j are pairwise distinct, $\mu_k \notin Q$ and $\gamma_k \in Q$. Then

$$\chi_{\boldsymbol{\pi}} = \chi_{\boldsymbol{\pi}'} \quad \text{implies} \quad \delta_a(z)\overline{\lambda} = \sum_{k=1}^{r} \delta_{b_k}(z)\overline{\mu_k}.$$

Assume $r > 1$. Evaluating this expression at $z = a$ forces $b_j = a$ for some $1 \le j \le r$ and $\overline{\lambda} = \overline{\mu_j}$; hence $\mu_j - \lambda \in Q$. Therefore $\mu_j = \lambda + \nu$ for some $\nu \in Q$. Next, evaluating the equality at any $z = b_k$ with $k \neq j$ gives us $\overline{\mu_k} = 0$; hence $\mu_k \in Q$, a contradiction. So we must have $r = 1$. Setting $\tilde{\boldsymbol{\pi}} = \prod_{k=1}^{s} \boldsymbol{\pi}_{\gamma_k,c_k}$, we have $\chi_{\tilde{\boldsymbol{\pi}}} = 0$ and $\boldsymbol{\pi}_{\lambda,a} = \boldsymbol{\pi}_{\mu_j,b_j}\,\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi}_{\lambda+\nu,a}\,\tilde{\boldsymbol{\pi}}$, as desired. □

Definition 3.11. We define a relation $\sim_\sigma$ on Ξ by saying $\chi_1 \sim_\sigma \chi_2$ if there exist $\boldsymbol{\pi}_i \in \mathcal{P}$ for $i = 1, 2$ such that $\chi_i = \chi_{\boldsymbol{\pi}_i}$ and $\mathbf{r}(\boldsymbol{\pi}_1) = \mathbf{r}(\boldsymbol{\pi}_2)$.

It is routine to show that $\chi_1 \sim_\sigma \chi_2$ if and only if

$$\sum_{\epsilon=0}^{m-1} \sigma^\epsilon \circ \chi_1 \circ \sigma^{-\epsilon} = \sum_{\epsilon=0}^{m-1} \sigma^\epsilon \circ \chi_2 \circ \sigma^{-\epsilon}, \tag{3-1}$$

(where we regard σ as an automorphism of $\mathbb{C}^\times$ via $a \mapsto \zeta a$ and as an automorphism of P/Q via $\overline{\omega_i} \mapsto \overline{\sigma(\omega_i)}$) and therefore that $\sim_\sigma$ is an equivalence relation on Ξ.

Definition 3.12. The twisted spectral characters Ξ^σ of $L^\sigma(\mathfrak{g})$, denoted Ξ^σ, are the equivalence classes of Ξ with respect to the equivalence $\sim_\sigma$, that is, $\Xi^\sigma = \Xi/\!\sim_\sigma$. If $\boldsymbol{\pi}_\sigma \in \mathcal{P}^\sigma$, we define $\chi_{\boldsymbol{\pi}^\sigma} := \overline{\chi_{\boldsymbol{\pi}}}$, where $\boldsymbol{\pi} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)$. Using the relation (3-1), we can see that the binary operation $\overline{\chi_1} + \overline{\chi_2} = \overline{\chi_1 + \chi_2}$ is well defined; hence Ξ^σ is an abelian monoid.

Definition 3.13. We say that an $L^\sigma(\mathfrak{g})$-module V has spectral character $\bar{\chi}$ if, for every irreducible $L^\sigma(\mathfrak{g})$-constituent $V(\boldsymbol{\pi}^\sigma)$ of V, we have $\chi_{\boldsymbol{\pi}^\sigma} = \bar{\chi}$. Let $\mathcal{F}^\sigma(\bar{\chi})$ be the abelian subcategory of all $L^\sigma(\mathfrak{g})$-modules with spectral character $\bar{\chi}$.

The main result of this paper is the following theorem.

Theorem 2. *The blocks of $\mathcal{F}^\sigma$ are in bijection with Ξ^σ. In particular,*

$$\mathcal{F}^\sigma = \bigoplus_{\bar{\chi} \in \Xi^\sigma} \mathcal{F}^\sigma(\bar{\chi}),$$

and each $\mathcal{F}^\sigma$ is a block.

The theorem follows from the next two propositions:

Proposition 3.14. *Any two irreducible modules in $\mathcal{F}^\sigma(\bar{\chi})$ are linked.*

Proposition 3.15. *Every indecomposable $L^\sigma(\mathfrak{g})$-module has a twisted spectral character.*

3.16. ***Proof of Proposition 3.14.***

Lemma 3.17. *Let $\boldsymbol{\pi}_{\lambda,a} \in \mathcal{P}$, and suppose $\boldsymbol{\pi} \in \mathcal{P}$ such that $\chi_{\lambda,a} = \chi_{\boldsymbol{\pi}}$. Then there exists some $\tilde{\boldsymbol{\pi}} \in \mathcal{P}_{\mathrm{Asym}}$ such that $\chi_{\tilde{\boldsymbol{\pi}}} = \chi_{\lambda,a}$ and $\mathbf{r}(\boldsymbol{\pi}) = \mathbf{r}(\tilde{\boldsymbol{\pi}})$.*

Proof. Since $\boldsymbol{\pi}_{\lambda,a} \sim_\chi \boldsymbol{\pi}$, by Lemma 3.10 $\boldsymbol{\pi}$ must be of the form

$$\boldsymbol{\pi} = \boldsymbol{\pi}_{\lambda+\lambda',a} \prod_{\epsilon=1}^{m-1} \boldsymbol{\pi}_{\eta_\epsilon,\zeta^\epsilon a} \prod_{i=1}^{\ell} \prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}_{\nu_{\epsilon,i},\zeta^\epsilon b_i},$$

for $\lambda', \eta_\epsilon, \nu_{\epsilon,i} \in Q$, $\lambda + \lambda' \in P^+$, $\eta_\epsilon, \nu_{\epsilon,i} \in P^+$, and $b_i^m \neq b_j^m \neq a^m$ for all $1 \leq i \neq j \leq \ell$. Define

$$\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi}_{\lambda+\lambda'+\sum_{\epsilon=1}^{m-1} \sigma^{m-\epsilon}(\eta_\epsilon),a} \prod_{i=1}^{\ell} \boldsymbol{\pi}_{\nu_{0,i}+\sum_{\epsilon=1}^{m-1} \sigma^{m-\epsilon}(\nu_{\epsilon,i}),b_i}.$$

Then $\tilde{\boldsymbol{\pi}} \in \mathcal{P}_{\mathrm{Asym}}$, $\chi_{\tilde{\boldsymbol{\pi}}} = \chi_{\boldsymbol{\pi}_{\lambda,a}}$ and $\mathbf{r}(\boldsymbol{\pi}) = \mathbf{r}(\tilde{\boldsymbol{\pi}})$. □

Lemma 3.18. (i) *Let $\boldsymbol{\pi} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)_{\mathrm{Asym}}$. Then $V(\boldsymbol{\pi})_{L^\sigma(\mathfrak{g})} \in \mathcal{F}^\sigma(\overline{\chi_{\boldsymbol{\pi}}})$.*

(ii) *Let $V(\boldsymbol{\pi}^\sigma) \cong V(\boldsymbol{\pi})_{L^\sigma(\mathfrak{g})}$ and $\boldsymbol{\pi} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)_{\mathrm{Asym}}$. Then $V(\boldsymbol{\pi}^\sigma) \in \mathcal{F}^\sigma(\overline{\chi_{\boldsymbol{\pi}}})$.*

Proof. The lemma follows directly from the definitions. For the first, note that $\chi_{\boldsymbol{\pi}^\sigma} = \overline{\chi_{\boldsymbol{\pi}}}$, and $V(\boldsymbol{\pi})_{L^\sigma(\mathfrak{g})} = V(\mathbf{r}(\boldsymbol{\pi})) = V(\boldsymbol{\pi}^\sigma)$. The second is immediate from the first. □

Proposition 3.19. *For $k = 1, 2$, let $V(\boldsymbol{\pi}_k^\sigma) \in \mathcal{F}^\sigma(\overline{\chi_k})$ for some $\chi_k \in \Xi$. Then*

$$V(\boldsymbol{\pi}_1^\sigma) \otimes V(\boldsymbol{\pi}_2^\sigma) \in \mathcal{F}^\sigma(\overline{\chi_1} + \overline{\chi_2}).$$

Proof. For $k=1,2$, let $V(\boldsymbol{\pi}_k^\sigma)\in\mathscr{F}_{\overline{\chi_k}}$ for some $\chi_k\in\Xi$. Choose $\boldsymbol{\pi}_i\in\mathbf{r}^{-1}(\boldsymbol{\pi}_i^\sigma)_{\mathrm{Asym}}$; therefore $\overline{\chi_i}=\overline{\chi_{\boldsymbol{\pi}_i}}$ and $(V(\boldsymbol{\pi}_1)\otimes V(\boldsymbol{\pi}_2))_{L^\sigma(\mathfrak{g})}\cong V(\boldsymbol{\pi}_1^\sigma)\otimes V(\boldsymbol{\pi}_2^\sigma)$. Fix $L(\mathfrak{g})$-actions

$$V(\boldsymbol{\pi}_i^\sigma)_{L(\mathfrak{g})}\cong V(\boldsymbol{\pi}_i)\quad\text{and}\quad (V(\boldsymbol{\pi}_1^\sigma)\otimes V(\boldsymbol{\pi}_2^\sigma))_{L(\mathfrak{g})}\cong V(\boldsymbol{\pi}_1)\otimes V(\boldsymbol{\pi}_2).$$

Let V be an irreducible $L^\sigma(\mathfrak{g})$-constituent of $V(\boldsymbol{\pi}_1^\sigma)\otimes V(\boldsymbol{\pi}_2^\sigma)$. Then $V_{L(\mathfrak{g})}$ is some irreducible $L(\mathfrak{g})$-constituent $V(\boldsymbol{\pi})$ of $(V(\boldsymbol{\pi}_1^\sigma)\otimes V(\boldsymbol{\pi}_2^\sigma))_{L(\mathfrak{g})}\cong V(\boldsymbol{\pi}_1)\otimes V(\boldsymbol{\pi}_2)$. We know from the untwisted affine case that $V(\boldsymbol{\pi})$ has spectral character $\chi_{\boldsymbol{\pi}_1}+\chi_{\boldsymbol{\pi}_2}$ (and hence $\chi_{\boldsymbol{\pi}}=\chi_{\boldsymbol{\pi}_1}+\chi_{\boldsymbol{\pi}_2}$), and V has character $\overline{\chi_{\boldsymbol{\pi}}}=\overline{\chi_{\boldsymbol{\pi}_1}+\chi_{\boldsymbol{\pi}_2}}=\overline{\chi_{\boldsymbol{\pi}_1}}+\overline{\chi_{\boldsymbol{\pi}_2}}=\overline{\chi_1}+\overline{\chi_2}$ by Lemma 3.18(ii). □

Corollary 3.20. *For all $\chi_1,\chi_2\in\Xi$, we have $\mathscr{F}^\sigma(\overline{\chi_1})\otimes\mathscr{F}^\sigma(\overline{\chi_2})\subseteq\mathscr{F}^\sigma(\overline{\chi_1}+\overline{\chi_2})$.*

Proposition 3.21. $W(\boldsymbol{\pi}^\sigma)\in\mathscr{F}^\sigma(\chi_{\boldsymbol{\pi}^\sigma})$.

Proof. By Corollary 3.20, it suffices to prove the claim when $\boldsymbol{\pi}^\sigma=\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}$ for $a\in\mathbb{C}^\times$ and $\lambda_\epsilon\in P_\sigma^+$. Let $\boldsymbol{\pi}_{\lambda,a}\in\mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)_{\mathrm{Asym}}$, so that $\chi_{\boldsymbol{\pi}^\sigma}=\overline{\chi_{\lambda,a}}$, and fix an isomorphism $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})}\cong W(\boldsymbol{\pi}_{\lambda,a})$. Now let $V=V(\boldsymbol{\pi}_1^\sigma)$ be an irreducible $L^\sigma(\mathfrak{g})$-constituent of $W(\boldsymbol{\pi}^\sigma)$. We will show that $\chi_{\boldsymbol{\pi}^\sigma}=\chi_{\boldsymbol{\pi}_1^\sigma}$.

Now $V_{L(\mathfrak{g})}$ is an irreducible $L(\mathfrak{g})$-constituent of $W(\boldsymbol{\pi}^\sigma)_{L(\mathfrak{g})}\cong W(\boldsymbol{\pi}_{\lambda,a})$. Since $W(\boldsymbol{\pi}_{\lambda,a})\in\mathscr{F}_{\chi_{\lambda,a}}$ [Chari and Moura 2004, Lemma 5.1], $V_{L(\mathfrak{g})}\cong V(\boldsymbol{\pi}_1)$ for some $\boldsymbol{\pi}_1\in\mathcal{P}$ such that $\chi_{\boldsymbol{\pi}_1}=\chi_{\lambda,a}$. Since $V(\boldsymbol{\pi}_1)$ is an irreducible $L(\mathfrak{g})$-constituent of $W(\boldsymbol{\pi}_{\lambda,a})$, it must be of the form $V(\boldsymbol{\pi}_{\mu,a})$ for some $\mu\le\lambda$ [Chari and Pressley 2001, Proposition 3.3]. Therefore $V(\boldsymbol{\pi}_1^\sigma)=(V(\boldsymbol{\pi}_1^\sigma)_{L(\mathfrak{g})})_{L^\sigma(\mathfrak{g})}=V(\boldsymbol{\pi}_1)_{L^\sigma(\mathfrak{g})}=V(\mathbf{r}(\boldsymbol{\pi}_1))$; hence $\boldsymbol{\pi}_1\in\mathbf{r}^{-1}(\boldsymbol{\pi}_1^\sigma)$.

Therefore $\chi_{\boldsymbol{\pi}_1^\sigma}=\overline{\chi_{\boldsymbol{\pi}_1}}=\overline{\chi_{\boldsymbol{\pi}}}=\chi_{\boldsymbol{\pi}^\sigma}$. □

The following proposition provides a strong linking between certain irreducible $L^\sigma(\mathfrak{g})$-modules.

Proposition 3.22. *Let $a\in\mathbb{C}^\times$, $\lambda_\epsilon,\mu_\epsilon\in P_\sigma^+$, and let $\lambda=\sum_{\epsilon=0}^{m-1}\sigma^{m-\epsilon}(\lambda_\epsilon)$ and $\mu=\sum_{\epsilon=0}^{m-1}\sigma^{m-\epsilon}(\mu_\epsilon)$, so that*

$$\boldsymbol{\pi}_{\lambda,a}\in\mathbf{r}^{-1}\Bigl(\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}\Bigr)_{\mathrm{Asym}}\quad\textit{and}\quad\boldsymbol{\pi}_{\mu,a}\in\mathbf{r}^{-1}\Bigl(\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\mu_\epsilon,\zeta^\epsilon a}\Bigr)_{\mathrm{Asym}}.$$

Assume there exists a nonzero homomorphism $p:\mathfrak{g}\otimes V(\lambda)\to V(\mu)$ of $\mathfrak{g}$-modules. The formula

$$x\otimes t^k(v,w)=(a^kxv,\,a^kxw+ka^{k-1}p(x\otimes v)),\tag{3-2}$$

defines an action of an $L^\sigma(\mathfrak{g})$-module on $V(\lambda)\oplus V(\mu)$, where $x\in\mathfrak{g}_{\bar{k}}$, $v\in V(\lambda)$ and $w\in V(\mu)$. Denote this $L^\sigma(\mathfrak{g})$-module by $V(\lambda,\mu,a)$. Then

$$0\to V\Bigl(\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}\Bigr)\to V(\lambda,\mu,a)\to V\Bigl(\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\mu_\epsilon,\zeta^\epsilon a}\Bigr)\to 0$$

is a nonsplit short exact sequence of $L^\sigma(\mathfrak{g})$-modules. If $\lambda > \mu$, then there exists a canonical surjective homomorphism $W(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}) \to V(\lambda,\mu,a)$ of $L^\sigma(\mathfrak{g})$-modules.

Proof. For brevity we will write

$$V\left(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}\right) = V(\mathbf{r}(\boldsymbol{\pi}_{\lambda,a})) \quad \text{and} \quad V\left(\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\mu_\epsilon,\zeta^\epsilon a}\right) = V(\mathbf{r}(\boldsymbol{\pi}_{\mu,a})).$$

It is routine to check that (3-2) gives an $L^\sigma(\mathfrak{g})$-action, and that the sequence is exact. To prove that the sequence is nonsplit, assume that $V(\lambda,\mu,a) = W_1 \oplus W_2$ is a nontrivial decomposition of $V(\lambda,\mu,a)$ into $L^\sigma(\mathfrak{g})$-submodules. It is immediate from its construction that the length of $V(\lambda,\mu,a)$ is 2, with constituents $V(\mathbf{r}(\boldsymbol{\pi}_{\lambda,a}))$ and $V(\mathbf{r}(\boldsymbol{\pi}_{\mu,a}))$. Therefore we can assume without loss of generality that $W_1 \cong V(\mathbf{r}(\boldsymbol{\pi}_{\lambda,a}))$. But it is clear from the description of the action of $L^\sigma(\mathfrak{g})$ on $V(\lambda,\mu,a)$ that $V(\mathbf{r}(\boldsymbol{\pi}_{\lambda,a}))$ is not a submodule of $V(\lambda,\mu,a)$. Therefore $V(\lambda,\mu,a)$ must be indecomposable.

Let v_+ be a highest-weight vector of $V(\mathbf{r}(\boldsymbol{\pi}_{\lambda,a}))$. Then $\mathbf{U}(L^\sigma(\mathfrak{g})).(v_+,0)$ must be isomorphic to $V(\mathbf{r}(\boldsymbol{\pi}_{\mu,a}))$ or $V(\lambda,\mu,a)$. If we assume that $\lambda > \mu$, then by weight considerations we cannot have $\mathbf{U}(L^\sigma(\mathfrak{g})).(v_+,0) \cong V(\mathbf{r}(\boldsymbol{\pi}_{\mu,a}))$. Therefore if $\lambda > \mu$, then $V(\lambda,\mu,a)$ is cyclically generated by $(v_+,0)$. Since this element is also highest weight with $L^\sigma(\mathfrak{g})$-weights given by $\mathbf{r}(\boldsymbol{\pi}_{\lambda,a})$, it follows that $V(\lambda,\mu,a)$ is a quotient of $W(\mathbf{r}(\boldsymbol{\pi}_{\lambda,a}))$. □

Corollary 3.23. *Let*

$$\boldsymbol{\pi}_{\lambda,a} \in \mathbf{r}^{-1}\big(\textstyle\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}\big)_{\mathrm{Asym}} \quad \textit{and} \quad \boldsymbol{\pi}_{\mu,a} \in \mathbf{r}^{-1}\big(\textstyle\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\mu_\epsilon,\zeta^\epsilon a}\big)_{\mathrm{Asym}}$$

be as in the proposition, and assume that there exists a nonzero homomorphism $p : \mathfrak{g} \otimes V(\lambda) \to V(\mu)$ of $\mathfrak{g}$-modules. Then the $L^\sigma(\mathfrak{g})$-modules

$$V\big(\textstyle\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}\big) \quad \textit{and} \quad V\big(\textstyle\prod_{\epsilon=0}^{m-1} \boldsymbol{\pi}^\sigma_{\mu_\epsilon,\zeta^\epsilon a}\big)$$

are strongly linked.

The following is [Chari and Moura 2004, Proposition 2.3]. We prove it here to clarify the proof of the analogous statement for the twisted case.

Proposition 3.24. *Any two irreducible $L(\mathfrak{g})$-modules $V(\boldsymbol{\pi}_1)$ and $V(\boldsymbol{\pi}_2)$ belonging to $\mathscr{C}_\chi$ are strongly linked.*

Proof. Since $V(\boldsymbol{\pi}_1), V(\boldsymbol{\pi}_2) \in \mathscr{C}_\chi$, there exist $\lambda_i, \mu_i \in P^+,\ a_i \in \mathbb{C}$ for $1 \le i \le \ell$ such that $\lambda_i - \mu_i \in Q,\ a_i \neq a_j$ and

$$\boldsymbol{\pi}_1 = \prod_{i=1}^{\ell} \boldsymbol{\pi}_{\lambda_i,a_i} \quad \text{and} \quad \boldsymbol{\pi}_2 = \prod_{i=1}^{\ell} \boldsymbol{\pi}_{\mu_i,a_i}.$$

Let us assume for simplicity here that $\ell = 2$; the more general case extends straightforwardly. By Proposition 1.4(ii), there exist weight sequences $\{\nu_i\}_{i=0}^q$ and $\{\eta_i\}_{i=0}^r$ with $\nu_0 = \lambda_1,\ \nu_q = \mu_1,\ \eta_0 = \lambda_2$ and $\eta_q = \mu_2$, such that

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\nu_i), V(\nu_{i+1})) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\eta_j), V(\eta_{j+1})) \neq 0$$

for $0 \le i \le q-1$ and $0 \le j \le r-1$. Fix some i satisfying $1 \le i \le q$. By Proposition 1.4(i), either $\nu_i \ge \nu_{i+1}$ or $\nu_{i+1} \ge \nu_i$. If $\nu_i \ge \nu_{i+1}$, we can conclude by [Chari and Moura 2004, Proposition 3.4] that $V(\boldsymbol{\pi}_{\nu_i,a_1})$ and $V(\boldsymbol{\pi}_{\nu_{i+1},a_1})$ are both irreducible constituents of some quotient M_i of $W(\boldsymbol{\pi}_{\nu_i,a_1})$.

If $\nu_{i+1} \ge \nu_i$, we use the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\nu_i), V(\nu_{i+1})) &\cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\nu_{i+1})^*, V(\nu_i)^*) \\ &\cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(-w_0(\nu_{i+1})), V(-w_0(\nu_i)))\end{aligned}$$

to conclude that $V(\boldsymbol{\pi}_{\nu_i,a_1})$ and $V(\boldsymbol{\pi}_{\nu_{i+1},a_1})$ are both irreducible constituents of some quotient M_i of $W(\boldsymbol{\pi}_{\nu_{i+1},a_1})$.

We may now assume without loss that $\nu_i \ge \nu_{i+1}$ for $0 \le i \le q-1$. For such i, $V(\boldsymbol{\pi}_{\nu_i,a_1}) \otimes V(\boldsymbol{\pi}_{\lambda_2,a_2})$ and $V(\boldsymbol{\pi}_{\nu_{i+1},a_1}) \otimes V(\boldsymbol{\pi}_{\lambda_2,a_2})$ are simple constituents of $M_i \otimes V(\boldsymbol{\pi}_{\lambda_2,a_2})$. This module, in turn, is a quotient of $W(\boldsymbol{\pi}_{\nu_i,a_1}) \otimes W(\boldsymbol{\pi}_{\lambda_2,a_2}) \cong W(\boldsymbol{\pi}_{\nu_i,a_1}\boldsymbol{\pi}_{\lambda_2,a_2})$ and is hence indecomposable. Therefore $V(\boldsymbol{\pi}_{\nu_i,a_1}) \otimes V(\boldsymbol{\pi}_{\lambda_2,a_2})$ and $V(\boldsymbol{\pi}_{\nu_{i+1},a_1}) \otimes V(\boldsymbol{\pi}_{\lambda_2,a_2})$ are strongly linked, and so $V(\boldsymbol{\pi}_{\lambda_1,a_1}) \otimes V(\boldsymbol{\pi}_{\lambda_2,a_2})$ and $V(\boldsymbol{\pi}_{\mu_1,a_1}) \otimes V(\boldsymbol{\pi}_{\lambda_2,a_2})$ are strongly linked. To complete the proof, we show similarly that $V(\boldsymbol{\pi}_{\mu_1,a_1}) \otimes V(\boldsymbol{\pi}_{\lambda_2,a_2})$ and $V(\boldsymbol{\pi}_{\mu_1,a_1}) \otimes V(\boldsymbol{\pi}_{\mu_2,a_2})$ are strongly linked. ☐

Proposition 3.25. *Let $\{\lambda_i\}_{i=1}^{\ell}, \{\mu_i\}_{i=1}^{\ell} \subseteq P^+$ such that $\lambda_i - \mu_i \in Q$ for all i, and suppose $\mathcal{P}_{\mathrm{Asym}}$ contains*

$$\boldsymbol{\pi}_1 = \prod_{i=1}^{\ell} \boldsymbol{\pi}_{\lambda_i,a_i} \quad \textit{and} \quad \boldsymbol{\pi}_2 = \prod_{i=1}^{\ell} \boldsymbol{\pi}_{\mu_i,a_i}.$$

Then the $L^\sigma(\mathfrak{g})$-modules $V(\mathbf{r}(\boldsymbol{\pi}_1))$ and $V(\mathbf{r}(\boldsymbol{\pi}_2))$ are strongly linked.

Proof. Again it suffices to prove the claim for $\ell = 2$. Since $\lambda_1 - \mu_1 \in Q$, by Proposition 1.4(ii), there exists a sequence $\{\nu_i\}_{i=0}^q \subseteq P^+$ with $\nu_0 = \lambda_1$ and $\nu_q = \mu_1$ such that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\nu_i), V(\nu_{i+1})) \neq 0$ for all $0 \le i \le q-1$. Fix some i. By Proposition 1.4(i), either $\nu_i \ge \nu_{i+1}$ or $\nu_{i+1} \ge \nu_i$. In either case, $V(\mathbf{r}(\boldsymbol{\pi}_{\nu_i,a_1}))$ and $V(\mathbf{r}(\boldsymbol{\pi}_{\nu_{i+1},a_1}))$ are both simple constituents of an indecomposable module M_i, which is in turn a quotient of $W(\mathbf{r}(\boldsymbol{\pi}_{\nu_i,a_1}))$ for $\nu_i \ge \nu_{i+1}$ or $W(\mathbf{r}(\boldsymbol{\pi}_{\nu_{i+1},a_1}))$ for $\nu_i \le \nu_{i+1}$. We may assume without loss that $\nu_i \ge \nu_{i+1}$. Therefore $V(\mathbf{r}(\boldsymbol{\pi}_{\nu_i,a_1})) \otimes V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_2,a_2}))$ and $V(\mathbf{r}(\boldsymbol{\pi}_{\nu_{i+1},a_1})) \otimes V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_2,a_2}))$ are both simple constituents of $M_i \otimes V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_2,a_2}))$, which is in turn a quotient of $W(\mathbf{r}(\boldsymbol{\pi}_{\nu_i,a_1})) \otimes W(\mathbf{r}(\boldsymbol{\pi}_{\lambda_2,a_2})) \cong W(\mathbf{r}(\boldsymbol{\pi}_{\nu_i,a_1}\boldsymbol{\pi}_{\lambda_2,a_2}))$, and hence is indecomposable. Therefore the modules

$$V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_1,a_1}))\otimes V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_2,a_2}))\cong V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_1,a_1}\,\boldsymbol{\pi}_{\lambda_2,a_2})),$$
$$V(\mathbf{r}(\boldsymbol{\pi}_{\mu_1,a_1}))\otimes V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_2,a_2}))\cong V(\mathbf{r}(\boldsymbol{\pi}_{\mu_1,a_1}\,\boldsymbol{\pi}_{\lambda_2,a_2}))$$

are strongly linked. Similarly we can show that the modules $V(\mathbf{r}(\boldsymbol{\pi}_{\mu_1,a_1}\,\boldsymbol{\pi}_{\lambda_2,a_2}))$ and $V(\mathbf{r}(\boldsymbol{\pi}_{\mu_1,a_1}\,\boldsymbol{\pi}_{\mu_2,a_2}))$ are strongly linked, completing the proof. □

Corollary 3.26. *Let $\boldsymbol{\pi}\in\mathcal{P}_{\mathrm{Asym}}$ and $\chi_{\boldsymbol{\pi}}=0$. Then $V(\mathbf{r}(\boldsymbol{\pi}))$ is strongly linked to $\mathbb{C}$.*

Proof. The result follows from Proposition 3.25 and two observations. First, if $\chi_{\boldsymbol{\pi}}=0$, then $\boldsymbol{\pi}$ is of the form $\prod\boldsymbol{\pi}_{\lambda_i,a_i}$ with $\lambda_i\in Q\cap P^+$. Second, $V(\mathbf{r}(\boldsymbol{\pi}_{\lambda,a}))$ is strongly linked to $\mathbb{C}$ for $\lambda\in Q\cap P^+$ and $a\in\mathbb{C}^*$. □

Corollary 3.27. *Any $V(\boldsymbol{\pi}_1^\sigma)$ and $V(\boldsymbol{\pi}_2^\sigma)$ belonging to $\mathcal{F}^\sigma(\bar{\chi})$ are strongly linked.*

Proof. Let $\boldsymbol{\pi}_i\in\mathbf{r}^{-1}(\boldsymbol{\pi}_i^\sigma)_{\mathrm{Asym}}$ for $i=1,2$. Then we have

$$\begin{tikzcd}[column sep=small]
 & \boldsymbol{\pi}_1 \arrow[r, leftrightarrow, "\chi"] \arrow[dl, "\mathbf{r}"] & \tilde{\boldsymbol{\pi}}_1 \arrow[dr, "\mathbf{r}"] & & \tilde{\boldsymbol{\pi}}_2 \arrow[r, leftrightarrow, "\chi"] \arrow[dl, "\mathbf{r}"] & \boldsymbol{\pi}_2 \arrow[dr, "\mathbf{r}"] & \\
\boldsymbol{\pi}_1^\sigma & & & \tilde{\boldsymbol{\pi}}^\sigma & & & \boldsymbol{\pi}_2^\sigma .
\end{tikzcd}$$

By Lemma 3.17, we can assume without loss that $\tilde{\boldsymbol{\pi}}_1,\tilde{\boldsymbol{\pi}}_2\in\mathcal{P}_{\mathrm{Asym}}$. It suffices now to show that $V(\boldsymbol{\pi}_1^\sigma)$ is strongly linked to $V(\tilde{\boldsymbol{\pi}}^\sigma)$.

Let

$$\boldsymbol{\pi}_1^\sigma=\prod_{i=1}^{\ell}\prod_{\epsilon=0}^{m-1}\boldsymbol{\pi}^\sigma_{\lambda_{i,\epsilon},\zeta^\epsilon a_i}\quad\text{for } a_i^m\neq a_j^m$$

be a factorization of $\boldsymbol{\pi}_1^\sigma$. Then we have $\boldsymbol{\pi}_1=\prod_{i=1}^{\ell}\boldsymbol{\pi}_{\lambda_i,a_i}$ for some $\lambda_i\in P^+$, and $\tilde{\boldsymbol{\pi}}_1=\prod_{i=1}^{\ell}\boldsymbol{\pi}_{\lambda_i+\lambda_i',a_i}\,\tilde{\boldsymbol{\pi}}$, where $\lambda_i'\in Q$ such that $\lambda_i+\lambda_i'\in P^+$, where $\chi_{\tilde{\boldsymbol{\pi}}}=0$, and where the coordinates $\{b_i\}$ of $\tilde{\boldsymbol{\pi}}$ all satisfy $b^m\neq a^m$. Furthermore

$$V(\tilde{\boldsymbol{\pi}}^\sigma)=V(\mathbf{r}(\tilde{\boldsymbol{\pi}}_1))=V\big(\mathbf{r}\big(\textstyle\prod_{i=1}^{\ell}\boldsymbol{\pi}_{\lambda_i+\lambda_i',a_i}\,\tilde{\boldsymbol{\pi}}\big)\big)\cong\displaystyle\bigotimes_{i=1}^{\ell}V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_i+\lambda_i',a_i}))\otimes V(\mathbf{r}(\tilde{\boldsymbol{\pi}})).$$

Since $\chi_{\tilde{\boldsymbol{\pi}}}=0$, we can conclude from Proposition 3.25 and its corollary that the modules

$$\bigotimes_{i=1}^{\ell}V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_i+\lambda_i',a_i}))\cong\bigotimes_{i=1}^{\ell}V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_i+\lambda_i',a_i}))\otimes\mathbb{C},$$
$$\bigotimes_{i=1}^{\ell}V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_i+\lambda_i',a_i}))\otimes V(\mathbf{r}(\tilde{\boldsymbol{\pi}}))\cong V(\tilde{\boldsymbol{\pi}}^\sigma)$$

are strongly linked, as are $V(\boldsymbol{\pi}_1^\sigma)\cong V(\mathbf{r}(\boldsymbol{\pi}_1))$ and $\bigotimes_{i=1}^{\ell}V(\mathbf{r}(\boldsymbol{\pi}_{\lambda_i+\lambda_i',a_i}))$. □

3.28. ***Proof of Proposition 3.15.*** We first prove Lemma 3.30(ii), an important result concerning $\mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U, V)$ for modules $U, V \in \mathcal{F}^\sigma$: It says distinct spectral characters have no nontrivial extensions. We'll first need a lemma. In the following, w_0 is the longest element of the Weyl group of $\mathfrak{g}$, and for a standard decomposition $\boldsymbol{\pi} = \prod_{i=1}^{\ell} \boldsymbol{\pi}_{\lambda_i, a_i}$, we define $\boldsymbol{\pi}^* = \prod_{i=1}^{\ell} \boldsymbol{\pi}_{-w_0\lambda_i, a_i}$. Then $V(\boldsymbol{\pi})^* \cong V(\boldsymbol{\pi}^*)$; this is [Chari and Moura 2004], Proposition 3.2). Also it is easy to see that $\lambda_{\boldsymbol{\pi}} = \sum_{i=1}^{\ell} \lambda_i$. For any irreducible $L^\sigma(\mathfrak{g})$-module $V(\boldsymbol{\pi}^\sigma)$, let $(\boldsymbol{\pi}^\sigma)^*$ be the element of $\mathcal{P}^\sigma$ such that $V((\boldsymbol{\pi}^\sigma)^*) \cong V(\boldsymbol{\pi}^\sigma)^*$.

Lemma 3.29. (i) $V(\mathbf{r}(\boldsymbol{\pi}))^* \cong V(\mathbf{r}(\boldsymbol{\pi}^*))$.

(ii) $\lambda_{(\boldsymbol{\pi}^\sigma)^*} = \lambda_{\boldsymbol{\pi}^\sigma}$.

Proof. For any $\boldsymbol{\pi} \in \mathcal{P}$, we have $V(\boldsymbol{\pi})^* \cong V(\boldsymbol{\pi}^*)$. Therefore

$$V(\mathbf{r}(\boldsymbol{\pi}))^* \cong (V(\boldsymbol{\pi})|_{L^\sigma(\mathfrak{g})})^* \cong (V(\boldsymbol{\pi})^*)|_{L^\sigma(\mathfrak{g})} \cong V(\boldsymbol{\pi}^*)|_{L^\sigma(\mathfrak{g})} \cong V(\mathbf{r}(\boldsymbol{\pi}^*)).$$

For the proof of (ii), let $\boldsymbol{\pi} \in \mathbf{r}^{-1}(\boldsymbol{\pi}^\sigma)$. Then $\lambda_{\boldsymbol{\pi}^\sigma} = \sum_{i=0}^{\epsilon} \lambda_{\boldsymbol{\pi}}(\epsilon)$. For $\mathfrak{g}$ of type A, D or E_6, we have either $-w_0 = \mathrm{Id}$ or $-w_0 = \sigma$; see for example [Bourbaki 2002]. In either case, for any $\lambda \in P^+$ we have $\sum_{\epsilon=0}^{m-1} -w_0\lambda(\epsilon) = \sum_{\epsilon=0}^{m-1} \lambda(\epsilon)$. Also for any $\lambda, \mu \in P^+$, we have $(\lambda + \mu)(\epsilon) = \lambda(\epsilon) + \mu(\epsilon)$ for $0 \le \epsilon \le m-1$. Therefore $\lambda_{(\boldsymbol{\pi}^\sigma)^*} = \lambda_{\mathbf{r}(\boldsymbol{\pi}^*)} = \sum_{\epsilon=0}^{m-1} \lambda_{\boldsymbol{\pi}^*}(\epsilon) = \sum_{\epsilon=0}^{m-1} \lambda_{\boldsymbol{\pi}}(\epsilon) = \lambda_{\boldsymbol{\pi}^\sigma}$, where for the first equality we have used part (i) of the lemma. $\square$

The proof of Lemma 3.30 is adapted from a proof in [Chari and Moura 2004].

Lemma 3.30. (i) *Let $U \in \mathcal{F}^\sigma(\bar{\chi})$, and let $\boldsymbol{\pi}^\sigma \in \mathcal{P}^\sigma$ such that $\bar{\chi} \neq \chi_{\boldsymbol{\pi}^\sigma}$. Then* $\mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U, V(\boldsymbol{\pi}^\sigma)) = 0$.

(ii) *Assume $V_j \in \mathcal{F}^\sigma(\overline{\chi_j})$ for $j = 1, 2$ and that $\overline{\chi_1} \neq \overline{\chi_2}$. Then* $\mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(V_1, V_2) = 0$.

Proof. Since Ext^1 preserves direct sums, to prove the lemma it suffices to consider the case when U is indecomposable. Consider an extension

$$0 \to V(\boldsymbol{\pi}_1^\sigma) \longrightarrow V \longrightarrow U \to 0.$$

We prove by induction on the length of U that this extension must be trivial. So first suppose that $U = V(\boldsymbol{\pi}_2^\sigma)$ for some $\boldsymbol{\pi}_2^\sigma \in \mathcal{P}^\sigma$ and that $\chi_{\boldsymbol{\pi}_2^\sigma} \neq \chi_{\boldsymbol{\pi}_1^\sigma}$. Then we have

$$0 \to V(\boldsymbol{\pi}_1^\sigma) \xrightarrow{\iota} V \xrightarrow{\pi} V(\boldsymbol{\pi}_2^\sigma) \longrightarrow 0.$$

For the remainder of the proof, let $\lambda_i = \lambda_{\boldsymbol{\pi}_i^\sigma} \in P_0^+$. We must have either

(1) $\lambda_2 < \lambda_1$, or

(2) $\lambda_1 - \lambda_2 \notin (Q_0^+ - \{0\})$.

If we are in case (1), then dualizing the exact sequence above takes us to

$$0 \to V(\boldsymbol{\pi}_2^\sigma)^* \longrightarrow V^* \longrightarrow V(\boldsymbol{\pi}_1^\sigma)^* \to 0,$$

which, by Lemma 3.29, takes us to case (2). Thus we can assume without loss that we are in case (2). The exact sequence always splits as a sequence of $\mathfrak{g}_0$-modules, so we have

$$V \cong_{\mathfrak{g}_0} V(\boldsymbol{\pi}_1^\sigma)_{\mathfrak{g}_0} \oplus V(\boldsymbol{\pi}_2^\sigma)_{\mathfrak{g}_0}.$$

Therefore $V_{\lambda_2} \cong V(\boldsymbol{\pi}_1^\sigma)_{\lambda_2} \oplus V(\boldsymbol{\pi}_2^\sigma)_{\lambda_2}$. Since we are in case (2), we know that $\lambda_2 \notin wt(V(\boldsymbol{\pi}_1^\sigma))$, and therefore $V_{\lambda_2} \cong V(\boldsymbol{\pi}_2^\sigma)_{\lambda_2}$. Hence $L^\sigma(\mathfrak{n}^+)V_{\lambda_2} = 0$. On the other hand, since V_{λ_2} maps onto $V(\boldsymbol{\pi}_2^\sigma)_{\lambda_2}$, there must be some nonzero vector $v \in V_{\lambda_2}$ with $L^\sigma(\mathfrak{h})$-eigenvalue $\boldsymbol{\pi}_2^\sigma$. Therefore the submodule $\mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v$ of V must be a quotient of $W(\boldsymbol{\pi}_2^\sigma)$, and hence $\mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v \in \mathscr{F}^\sigma(\chi_{\boldsymbol{\pi}_2^\sigma})$. If $\mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v = V$, then V has spectral character $\chi_{\boldsymbol{\pi}_2^\sigma}$, but $V(\boldsymbol{\pi}_1^\sigma)$ is a submodule of V and $\chi_{\boldsymbol{\pi}_2^\sigma} \neq \chi_{\boldsymbol{\pi}_1^\sigma}$. Therefore $\mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v$ must be a proper nontrivial submodule of V. But then $l(V) = 2$ implies that either

$$\mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v \cong V(\boldsymbol{\pi}_1^\sigma) \quad \text{or} \quad \mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v \cong V(\boldsymbol{\pi}_2^\sigma),$$

and $\mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v \cong V(\boldsymbol{\pi}_2^\sigma)$ since $\chi_{\boldsymbol{\pi}_2^\sigma} \neq \chi_{\boldsymbol{\pi}_1^\sigma}$. Also $\iota(V(\boldsymbol{\pi}_1^\sigma)) \cap \mathbf{U}(L^\sigma(\mathfrak{g}))\,.\,v = 0$; hence $V \cong V(\boldsymbol{\pi}_1^\sigma) \oplus V(\boldsymbol{\pi}_2^\sigma)$, and the induction begins.

Now assume that U is indecomposable of length ≥ 1 and $U \in \mathscr{F}^\sigma(\bar{\chi})$. Let U_1 be a proper nontrivial submodule of U and consider the short exact sequence $0 \to U_1 \to U \to U_2 \to 0$, where $U_2 = U/U_1$. Since U belongs to $\mathscr{F}^\sigma(\bar{\chi})$, so does U_i. Then the induction hypothesis gives us $\mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U_i, Vr(\boldsymbol{\pi}_1^\sigma)) = 0$, and the result follows by using the exact sequence

$$0 \to \mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U_2, V(\boldsymbol{\pi}_1^\sigma)) \longrightarrow \mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U, V(\boldsymbol{\pi}_1^\sigma)) \longrightarrow \mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U_1, V(\boldsymbol{\pi}_1^\sigma)) \to 0.$$

Part (ii) is now immediate by using a similar induction on the length of V_2. ☐

We now finish the proof of Proposition 3.15. Let V be an indecomposable $L^\sigma(\mathfrak{g})$-module. By an induction on the length of V, we will show that there exists a $\chi \in \Xi$ such that $V \in \mathscr{F}^\sigma(\bar{\chi})$. If V is irreducible, the result is immediate. Now assume V is reducible, and let $V(\boldsymbol{\pi}^\sigma)$ be an irreducible submodule of V; let $U = V/V(\boldsymbol{\pi}^\sigma)$. Then we have an extension $0 \to V(\boldsymbol{\pi}^\sigma) \longrightarrow V \longrightarrow U \to 0$. Now let $U = \bigoplus_{j=1}^r U_j$, where U_j is indecomposable. Clearly $l(U_j) < l(V)$. Therefore the induction hypothesis ensures that $U_j \in \mathscr{F}^\sigma(\overline{\chi_j})$ for some $\chi_j \in \Xi$ with $1 \leq j \leq r$. Now we would like to argue that $\overline{\chi_j} = \chi_{\boldsymbol{\pi}^\sigma}$ for all j, for if so, then $U_j \in \mathscr{F}^\sigma(\chi_{\boldsymbol{\pi}^\sigma})$ for all j and hence $U \in \mathscr{F}^\sigma(\chi_{\boldsymbol{\pi}^\sigma})$.

Suppose instead there is some j_0 such that $\overline{\chi_{j_0}} \neq \chi_{\boldsymbol{\pi}^\sigma}$. Then Lemma 3.30 gives

$$\mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U, V(\boldsymbol{\pi}^\sigma)) \cong \bigoplus_{j=1}^r \mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U_j, V(\boldsymbol{\pi}^\sigma)) \cong \bigoplus_{j \neq j_0} \mathrm{Ext}^1_{L^\sigma(\mathfrak{g})}(U_j, V(\boldsymbol{\pi}^\sigma)).$$

That is, the sequence $0 \to V(\boldsymbol{\pi}^\sigma) \to V \to U \to 0$ is equivalent to one of the form

$$0 \to V(\boldsymbol{\pi}^\sigma) \longrightarrow U_{j_0} \oplus V' \longrightarrow U_{j_0} \bigoplus_{j \neq j_0} U_j \to 0,$$

where $0 \to V(\boldsymbol{\pi}^\sigma) \to V' \to \bigoplus_{j \neq j_0} U_j \to 0$ is in $\bigoplus_{j \neq j_0} \operatorname{Ext}^1_{L^\sigma(\mathfrak{g})}(U_j, V(\boldsymbol{\pi}^\sigma))$. This contradicts the indecomposability of V. Hence $\overline{\chi_j} = \chi_{\boldsymbol{\pi}^\sigma}$ for all $1 \le j \le r$ and $V \in \mathscr{F}^\sigma(\chi_{\boldsymbol{\pi}^\sigma})$.

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