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STRONGLY r-MATRIX INDUCED TENSORS, KOSZUL COHOMOLOGY, AND ARBITRARY-DIMENSIONAL QUADRATIC POISSON COHOMOLOGY

MOURAD AMMAR, GUY KASS, MOHSEN MASMOUDI AND NORBERT PONCIN

We introduce the concept of strongly r-matrix induced (SRMI) Poisson structure, report on the relation of this property to the stabilizer dimension of the considered quadratic Poisson tensor, and classify the Poisson structures of the Dufour–Haraki classification (DHC) according to their membership in the family of SRMI tensors. A main result is a generic cohomological procedure for classifying SRMI Poisson structures in arbitrary dimension. This approach allows the decomposition of Poisson cohomology into, basically, a Koszul cohomology and a relative cohomology. Also we investigate this associated Koszul cohomology, highlight its tight connections with spectral theory, and reduce the computation of this main building block of Poisson cohomology to a problem of linear algebra. We apply these upshots to two structures of the DHC and provide an exhaustive description of their cohomology. We thus complete our list of data obtained in previous work, and gain fairly good insight into the structure of Poisson cohomology.

1. Introduction

Let $(\mathscr{L}, [\cdot\,,\cdot])$ with $\mathscr{L} = \bigoplus_i \mathscr{L}^i$ be a graded Lie algebra (gLa). Any element with degree 1 that squares to 0 generates a differential graded Lie algebra (dgLa) $(\mathscr{L}, [\cdot\,,\cdot], \partial_\Lambda)$, where $\partial_\Lambda := [\Lambda, \cdot\,]$, and a gLa $H(\mathscr{L}, [\cdot\,,\cdot], \partial_\Lambda)$ in cohomology. Depending on the initial algebra, such a 2-nilpotent degree 1 element is, say, an associative algebra structure, a Lie algebra structure, or a Poisson structure, and the associated cohomology is the adjoint Hochschild, the adjoint Chevalley–Eilenberg, or the Lichnerowicz–Poisson (LP) (or simply Poisson) cohomology, respectively. Recall that the LP-dgLa is implemented by the shifted Grassmann

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algebra $(\mathscr{X}(M)[1], \wedge, [\cdot,\cdot]_{\mathrm{SN}})$, with $\mathscr{X}(M) = \Gamma(\bigwedge TM)$, of polyvectors of a manifold M, endowed with the Schouten–Nijenhuis bracket $[\cdot,\cdot]_{\mathrm{SN}}$. The Hochschild dgLa is generated by the space of multilinear mappings of the underlying vector space endowed with the Gerstenhaber graded Lie bracket, and similarly the Chevalley–Eilenberg dgLa is generated by the space of skew-symmetric multilinear mappings of the underlying vector space endowed with the Nijenhuis–Richardson graded Lie bracket.

Alternatively, LP-cohomology can be viewed as the Lie algebroid (Lad) cohomology of the Lie algebroid $(T^*M, \{\cdot,\cdot\}, \sharp)$ canonically associated with an arbitrary Poisson manifold (M, Λ). The cohomology of a Lad $(E \to M, [\![\cdot,\cdot]\!], \rho)$, or equivalently a Q-structure on a supermanifold, is defined as the cohomology of the Chevalley–Eilenberg subcomplex of the representation $\rho : \Gamma(E) \to \mathrm{Der}(C^\infty(M))$, made up by tensorial cochains. Algebraically, LP-cohomology is defined as the adjoint Chevalley–Eilenberg cohomology of any Poisson–Lie algebra, restricted to the cochain subspace of skew-symmetric multiderivations.

More details about Poisson cohomology can be found, say, in [Lichnerowicz 1977; Vaisman 1994].

The last few decades have seen much work on Poisson cohomology and Poisson homology, starting with [Koszul 1985; Brylinski 1988]. Problems studied include the cohomology of regular Poisson manifolds [Vaisman 1990; Xu 1992], (co)homology and resolutions [Huebschmann 1990], duality [Huebschmann 1999; Xu 1999; Evens et al. 1999], cohomology in low dimensions or specific cases [Nakanishi 1997; Ginzburg 1999; Gammella 2002; Monnier 2002b; 2002a; Roger and Vanhaecke 2002; Roytenberg 2002; Pichereau 2005], and various extensions of Poisson cohomology — for example, the cohomologies Lie algebroid, Jacobi, Nambu–Poisson, double Poisson, and graded Jacobi [de León et al. 1997; Ibáñez et al. 2001; Monnier 2001; Grabowski and Marmo 2003; de León et al. 2003; Nakanishi 2006; Pichereau and Van de Weyer 2008]. In [Masmoudi and Poncin 2007; Ammar and Poncin 2008], we suggest an approach to the cohomology of the Poisson tensors of the Dufour–Haraki classification (DHC).

Here we focus on the formal LP-cohomology associated with the quadratic Poisson tensors (QPTs) Λ of $\mathbb{R}^n$ that read as real linear combinations

$$\Lambda = \sum_{i<j} \alpha^{ij} Y_i \wedge Y_j =: \sum_{i<j} \alpha^{ij} Y_{ij} \quad \text{for } \alpha^{ij} \in \mathbb{R} \tag{1-1}$$

of the wedge products of n commuting linear vector fields $Y_1, \dots, Y_n$, such that $Y_1 \wedge \cdots \wedge Y_n =: Y_{1\dots n} \neq 0$. Let us recall that "formal" means that we substitute the space $\mathbb{R}[\![x_1, \dots, x_n]\!] \otimes \bigwedge \mathbb{R}^n$ of multivectors with coefficients in the formal series for the usual Poisson cochain space $\mathscr{X}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes \bigwedge \mathbb{R}^n$. Furthermore, the reader may think about QPTs of type (1-1) as QPTs implemented by a classical r-matrix in their stabilizer for the canonical matrix action.

Hence, in Section 2, we are interested in characterizing the QPTs that are images of a classical r-matrix. We show that a QPT is induced by an r-matrix if the dimension of its stabilizer is large enough; more precisely, we prove that if the stabilizer of a given QPT Λ of $\mathbb{R}^n$ contains n commuting linear vector fields Y_i such that $Y_{1\dots n} \neq 0$, then Λ is implemented by an r-matrix in its stabilizer; see Corollary 2. We refer to such tensors as strongly r-matrix induced (SRMI) structures and show that any structure of the DHC decomposes into the sum of a maximal SRMI structure and a small compatible (mostly exact) Poisson tensor; see Theorem 4. This decomposition is the foundation of our cohomological techniques proposed in [Masmoudi and Poncin 2007; Ammar and Poncin 2008]. This splitting is in some sense opposite to the one proved in [Liu and Xu 1992], which incorporates the largest possible part of the Poisson tensor into the exact term.

Masmoudi and Poncin [2007] developed a cohomological method in Euclidean three-space that greatly simplified LP-cohomology computations for the SRMI structures of the Dufour–Haraki classification. Section 3 extends this procedure to arbitrary-dimensional vector spaces. In Theorem 10, we inject the space $\mathcal{R}$ of "real" LP-cochains (formal multivector fields) into a larger space $\mathcal{P}$ of "potential" cochains. In Theorems 13 and 15, we identify the natural extension to $\mathcal{P}$ of the LP-differential as the Koszul differential associated with n commuting endomorphisms

$$X_i - (\operatorname{div} X_i)\,\mathrm{id}, \quad \text{where } X_i = \textstyle\sum_j \alpha^{ij} Y_j \text{ and } \alpha^{ji} = -\alpha^{ij},$$

of the space made up by the polynomials on $\mathbb{R}^n$ with some fixed homogeneous degree. We then choose a space $\mathcal{S}$ supplementary to $\mathcal{R}$ in $\mathcal{P}$ and show that the LP-differential induces a differential on $\mathcal{S}$. Eventually, we end up with a short exact sequence of differential spaces and an exact triangle in cohomology. Theorem 16 shows that LP-cohomology ($\mathcal{R}$-cohomology) reduces, essentially, to Koszul cohomology ($\mathcal{P}$-cohomology) and a relative cohomology ($\mathcal{S}$-cohomology).

To take advantage of these results, we investigate in Section 4 the Koszul cohomology associated to n commuting linear operators on a finite-dimensional complex vector space. We prove a homotopy-type formula in Proposition 19 and — using spectral properties — show in Theorem 20 and Corollary 21 that the Koszul cohomology is located inside a primary subspace of the corresponding commuting endomorphisms.

In Section 5, we apply this result to gain insight into the structure of the Koszul cohomology implemented by SRMI tensors, and show that to compute this central part of Poisson cohomology it basically suffices to solve triangular systems of linear equations.

We conclude Section 5 by providing a full description of the LP-cohomology spaces of structures Λ_3 and Λ_9 of the Dufour–Haraki classification.

2. Characterization of strongly r-matrix induced Poisson structures

Stabilizer dimension and r-matrix generation. Poisson structures implemented by an r-matrix are of interest, for example in deformation quantization, especially in view of Drinfeld's method. We next report on an idea for generating quadratic Poisson tensors by classical r-matrices.

Set $G = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$. The Lie algebra isomorphism between $\mathfrak{g}$ and the algebra $\mathcal{X}_0^1(\mathbb{R}^n)$ of linear vector fields extends to a Grassmann algebra and a graded Poisson–Lie algebra homomorphism $J : \bigwedge \mathfrak{g} \to \bigoplus_k (\mathcal{S}^k \mathbb{R}^{n*} \otimes \bigwedge^k \mathbb{R}^n)$. It is known that its restriction

$$J^k : \textstyle\bigwedge^k \mathfrak{g} \to \mathcal{S}^k \mathbb{R}^{n*} \otimes \bigwedge^k \mathbb{R}^n$$

is onto, but has a nontrivial kernel if $k, n \geq 2$. In particular,

$$J^3[r, r]_{\mathrm{SN}} = [J^2 r, J^2 r]_{\mathrm{SN}} \quad \text{for } r \in \mathfrak{g} \wedge \mathfrak{g},$$

where $[\,\cdot\,,\cdot\,]_{\mathrm{SN}}$ is the Schouten–Nijenhuis bracket. It is still an open problem to characterize the quadratic Poisson structures that are implemented by a classical r-matrix, that is, a bimatrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ satisfying the classical Yang–Baxter equation $[r, r]_{\mathrm{SN}} = 0$.

Quadratic Poisson tensors Λ_1 and Λ_2 are equivalent if and only if there is $A \in G$ such that $A_* \Lambda_1 = \Lambda_2$, where $*$ denotes the standard action of G on tensors of $\mathbb{R}^n$. Since J^2 is a G-module homomorphism, that is,

$$A_*(J^2 r) = J^2(\mathrm{Ad}(A) r) \quad \text{for } A \in G \text{ and } r \in \mathfrak{g} \wedge \mathfrak{g},$$

the G-orbit of a quadratic Poisson structure $\Lambda = J^2 r$ is the pointwise J^2-image of the G-orbit of r. Furthermore, the representation Ad acts by graded Lie algebra homomorphisms, that is,

$$\mathrm{Ad}(A)[r, r]_{\mathrm{SN}} = [\mathrm{Ad}(A) r, \mathrm{Ad}(A) r]_{\mathrm{SN}}.$$

Hence, if $\Lambda = J^2 r$, where r is a classical r-matrix, the orbit of this quadratic Poisson tensor consists of r-matrix induced structures.

Any quadratic Poisson tensor Λ is implemented by bimatrices $r \in \mathfrak{g} \wedge \mathfrak{g}$. To determine whether the G-orbit O_Λ of this tensor is generated by r-matrices, we look at the preimage $(J^2)^{-1}(O_\Lambda) = \bigcup_{r \in (J^2)^{-1}\Lambda} O_r$, composed of the G-orbits O_r of all the bimatrices r that are mapped on Λ by J^2. We claim that the bigger the chance that a fiber of this bundle is located inside r-matrices, the smaller is O_Λ.

In other words, the dimension of the isotropy Lie group G_Λ of Λ, or of its Lie algebra, the stabilizer

$$\mathfrak{g}_\Lambda := \{a \in \mathfrak{g} : [\Lambda, Ja]_{\mathrm{SN}} = 0\}$$

of Λ for the corresponding infinitesimal action, should be big enough. For example, in $\mathbb{R}^3$ the Poisson tensor $\Lambda = (x_1^2 + x_2 x_3)\partial_{23}$, where $\partial_{23} := \partial_2 \wedge \partial_3$ and $\partial_i := \partial/\partial_{x_i}$, is not r-matrix induced (see [Manchon et al. 2002]) and the dimension of its stabilizer is $\dim \mathfrak{g}_\Lambda = 2$. More evidence comes from the corollary of the following theorem:

Theorem 1. *Let Λ be an analytic Poisson tensor of $\mathbb{R}^n$. If its stabilizer contains n commuting analytic vector fields Y_i for $i \in \{1, \dots, n\}$ such that $Y_1 \wedge \dots \wedge Y_n \neq 0$, then there exist constants $\alpha^{kl} \in \mathbb{R}$ such that $\Lambda = \sum_{k<l} \alpha^{kl} Y_k \wedge Y_l$.*

Proof. Since $Y_1 \wedge \dots \wedge Y_n \neq 0$, there exists an open subset O of $\mathbb{R}^n$ such that

$$\Lambda = \sum_{k<l} \alpha^{kl}(x) Y_k \wedge Y_l \quad \text{in } O$$

for some local functions $\alpha^{kl} = \alpha^{kl}(x)$. Since for any $i \in \{1, \dots, n\}$, we have

$$0 = [Y_i, \Lambda]_{\mathrm{SN}} = \sum_{k<l} Y_i(\alpha^{kl}) Y_k \wedge Y_l \quad \text{in } O,$$

the α^{kl} are constant in O; the theorem follows by analytic continuation. □

Corollary 2. *Let Λ be a quadratic Poisson tensor of $\mathbb{R}^n$. If its stabilizer $\mathfrak{g}_\Lambda$ contains n commuting linear vector fields Y_i for $i \in \{1, \dots, n\}$ such that $Y_1 \wedge \dots \wedge Y_n$ does not vanish, then Λ is implemented by a classical r-matrix that belongs to the stabilizer, that is, $\Lambda = J^2 a$, where $[a, a]_{\mathrm{SN}} = 0$ and $a \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$.*

Definition 3. If Λ is a quadratic Poisson structure implemented by a classical r-matrix $r \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$, we call Λ a strongly r-matrix induced (SRMI) tensor.

Classification theorem in Euclidean three-space. Two concepts of exact Poisson structure — which are closely related to two special cohomology classes — are used below. Let Λ be a Poisson tensor on a smooth manifold M oriented by a volume element Ω. We say that Λ, which is of course a LP-2-cocycle, is Lichnerowicz–Poisson-exact or LP-exact if

$$\Lambda = [\Lambda, X]_{\mathrm{SN}} \quad \text{for some } X \in \mathcal{X}^1(M).$$

The vector field X is called the Liouville vector field and the cohomology class of Λ is the obstruction to infinitesimal rescaling of Λ. We call Λ Koszul-exact or K-exact if

$$\Lambda = \delta(T) \quad \text{for some } T \in \mathcal{X}^3(M).$$

Here, the operator $\delta := \phi^{-1} \circ d \circ \phi$ is the pullback of the de Rham differential d by the canonical vector space isomorphism $\phi := i_{(\cdot)}\Omega$. Although introduced earlier, the generalized divergence δ defined by $\delta(X) = \mathrm{div}_\Omega X$ for $X \in \mathcal{X}^1(M)$ is usually attributed to J.-L. Koszul. The curl vector field $K(\Lambda) := \delta(\Lambda)$ of Λ[1] is an LP-1-cocycle. $K(\Lambda)$ maps a function to the divergence of its Hamiltonian vector field. The cohomology class of $K(\Lambda)$ is the well-known modular class of Λ. This class is independent of Ω, is the obstruction to existence on M of a measure preserved by all Poisson automorphisms, and is relevant in the classification of Poisson structures [Dufour and Haraki 1991; Grabowski et al. 1993; Liu and Xu 1992] and in Poincaré duality [Evens et al. 1999; Ibáñez et al. 2001]. In $\mathbb{R}^n$ with $n \geq 3$, a Poisson tensor Λ is K-exact if and only if it is irrotational, that is, $K(\Lambda) = 0$, and in $\mathbb{R}^3$, K-exact means function-induced, that is,

$$\Lambda = \Pi_f := \partial_1 f\, \partial_{23} + \partial_2 f\, \partial_{31} + \partial_3 f\, \partial_{12} \quad \text{for } f \in C^\infty(\mathbb{R}^3).$$

The K-exact quadratic Poisson tensors Π_p of $\mathbb{R}^3$, that is, the K-exact Poisson structures induced by a homogeneous polynomial $p \in \mathcal{S}^3\mathbb{R}^{3*}$, represent class 14 of the Dufour–Haraki classification. The cohomology of this class has been studied by Pichereau [2005] (actually Pichereau deals with structures Π_p implemented by a weight-homogeneous polynomial p with an isolated singularity). Hence, we will not examine class 14 here.

Recall that two Poisson tensors Λ_1 and Λ_2 are compatible if their sum is again a Poisson structure, that is, if $[\Lambda_1, \Lambda_2]_{\mathrm{SN}} = 0$.

The next theorem classifies the quadratic Poisson tensors according to their strongly r-matrix induced structure. It also shows that any such tensor is the sum of a "maximal" strongly r-matrix induced tensor and a "small" compatible Poisson structure. The classification makes available the cohomological technique used in [Masmoudi and Poncin 2007], while the splitting[2] is relevant to cohomological approach of [Ammar and Poncin 2008].

Denote the canonical coordinates of $\mathbb{R}^3$ by x, y, z and by x_1, x_2, x_3; denote the corresponding partial derivatives by $\partial_1, \partial_2, \partial_3$. Let $\partial_{ij} = \partial_i \wedge \partial_j$.

Theorem 4. *Let $a, b, c \in \mathbb{R}$, and let Λ_i for $i \in \{1, \dots, 13\}$ be the quadratic Poisson tensors of the Dufour–Haraki classification* [1991].

If $\dim \mathfrak{g}_\Lambda > 3$ (*where the subscript i is omitted*), *there are mutually commuting linear vector fields Y_1, Y_2, Y_3 such that*

$$\Lambda = \alpha Y_{23} + \beta Y_{31} + \gamma Y_{12}, \quad \textit{where } \alpha, \beta, \gamma \in \mathbb{R},$$

[1] If Ω is the standard volume of $\mathbb{R}^3$ and Λ is identified with a vector field $\vec{\Lambda}$ of $\mathbb{R}^3$, then $K(\Lambda)$ coincides with the standard curl $\vec{\nabla} \wedge \vec{\Lambda}$.

[2] This splitting differs from the decomposition used in [Liu and Xu 1992] in that we incorporate as much structure as possible into the strongly induced term.

so that Λ is strongly r-matrix induced (SRMI), that is, implemented by a classical r-matrix in $\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$. In the following classification of the quadratic Poisson tensors by the SRMI property, we decompose each non-SRMI tensor into the sum of a maximal SRMI structure and a smaller compatible quadratic Poisson tensor.

- *Set $Y_1 = x\partial_1,\ Y_2 = y\partial_2$ and $Y_3 = z\partial_3$.*
  (1) $\Lambda_1 = a\,yz\partial_{23} + b\,xz\partial_{31} + c\,xy\partial_{12}$ *is SRMI for all values of the parameters a, b and c, and decomposes as $\Lambda_1 = aY_{23} + bY_{31} + cY_{12}$.*
  (2) $\Lambda_4 = ayz\partial_{23} + axz\partial_{31} + (bxy + z^2)\partial_{12}$ *is SRMI if and only if a and b are both zero. We have $\Lambda_4 = a(Y_{23} + Y_{31}) + bY_{12} + \frac{1}{3}\Pi_{z^3}$.*
- *Set $Y_1 = x\partial_1 + y\partial_2,\ Y_2 = x\partial_2 - y\partial_1$ and $Y_3 = z\partial_3$.*
  (1) $\Lambda_2 = (2ax - by)z\partial_{23} + (bx + 2ay)z\partial_{31} + a(x^2 + y^2)\partial_{12}$ *is SRMI for any a and b. We have $\Lambda_2 = 2aY_{23} + bY_{31} + aY_{12}$.*
  (2) $\Lambda_7 = ((2a + c)x - by)\, z\partial_{23} + (bx + (2a + c)y)\, z\partial_{31} + a(x^2 + y^2)\partial_{12}$ *is SRMI for all a, b, c. We have $\Lambda_7 = (2a + c)Y_{23} + bY_{31} + aY_{12}$.*
  (3) $\Lambda_8 = a\,xz\partial_{23} + a\,yz\partial_{31} + (\frac{1}{2}(a + b)(x^2 + y^2) \pm z^2)\partial_{12}$ *is SRMI if and only if a and b are both zero. We have*
  $$\Lambda_8 = aY_{23} + \tfrac{1}{2}(a + b)Y_{12} \pm \tfrac{1}{3}\Pi_{z^3}.$$
- *Set $Y_1 = x\partial_1 + y\partial_2,\ Y_2 = x\partial_2$ and $Y_3 = z\partial_3$.*
  (1) $\Lambda_3 = (2x - a\,y)z\partial_{23} + a\,xz\partial_{31} + x^2\partial_{12}$ *is SRMI for any a, and we have $\Lambda_3 = 2Y_{23} + aY_{31} + Y_{12}$.*
  (2) $\Lambda_5 = ((2a + 1)x + y)z\partial_{23} - xz\partial_{31} + a\,x^2\partial_{12}$ *is SRMI for any $a \neq -1/2$. We have $\Lambda_5 = (2a + 1)Y_{23} - Y_{31} + aY_{12}$.*
  (3) $\Lambda_6 = a\,yz\partial_{23} - a\,xz\partial_{31} - \frac{1}{2}x^2\partial_{12}$ *is SRMI for any a. The decomposition is $\Lambda_6 = -aY_{31} - \frac{1}{2}Y_{12}$.*
- *Set $Y_1 = \mathscr{E} := x\partial_1 + y\partial_2 + z\partial_3,\ Y_2 = x\partial_2 + y\partial_3$ and $Y_3 = x\partial_3$.*
  (1) $\Lambda_9 = (ax^2 - \frac{1}{3}y^2 + \frac{1}{3}xz)\partial_{23} + \frac{1}{3}xy\partial_{31} - \frac{1}{3}x^2\partial_{12}$ *is SRMI for any a. We have $\Lambda_9 = aY_{23} - \frac{1}{3}Y_{12}$.*
  (2) $\Lambda_{10} = (a\,y^2 - (4a+1)xz)\partial_{23} + (2a+1)xy\partial_{31} - (2a+1)x^2\partial_{12}$ *is SRMI if and only if $a = -1/3$. We have $\Lambda_{10} = -(2a + 1)Y_{12} + (3a + 1)(y^2 - 2xz)\partial_{23}$.*
- *Set $Y_1 = \mathscr{E},\ Y_2 = x\partial_2$ and $Y_3 = (a\,x + (3b + 1)z)\partial_3$.*
  (1) *Set $a = 0$. Then $\Lambda_{11} = (2b + 1)xz\partial_{23} + (bx^2 + cz^2)\partial_{12}$ is SRMI if and only if $c = 0$. We have $\Lambda_{11} = Y_{23} + bY_{12} + \frac{1}{3}c\Pi_{z^3}$.*
  (2) *Set $a = 1$. Then $\Lambda_{12} = (x^2 + (2b + 1)xz)\partial_{23} + (bx^2 + cz^2)\partial_{12}$ is SRMI if and only if $c = 0$. We have $\Lambda_{12} = Y_{23} + bY_{12} + \frac{1}{3}c\Pi_{z^3}$.*
  (3) $\Lambda_{13} = (ax^2 + (2b + 1)xz + z^2)\partial_{23} + (bx^2 + cz^2 + 2xz)\partial_{12}$ *is not SRMI for any a, b, c. We have $\Lambda_{13} = Y_{23} + bY_{12} + \Pi_{cz^3/3+xz^2}$.*

Proof. The basic fields Y_1, Y_2, Y_3 have been read in the stabilizers of the considered Poisson tensors, but for brevity we omit the stabilizer computations. Indeed, once the vector fields Y_i are specified, it is easily checked that, in the SRMI cases, they satisfy the assumptions of Theorem 1. Thus the corresponding Poisson structures are actually SRMI tensors. To show that a quadratic Poisson structure Λ is not SRMI, it suffices to prove that $\Lambda \notin J^2(\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda)$, which we will do below.

All the decompositions above can be directly verified. In most instances, the twist is obviously Poisson, so that compatibility follows. In the case of Λ_{10}, the twist $\Lambda_{10,\mathrm{II}} = (y^2 - 2xz)\partial_{23}$ is a non-K-exact Poisson structure, which follows directly from the fact that $K(\Lambda_{10,\mathrm{II}}) = \vec{\nabla} \wedge \vec{\Lambda}_{10,\mathrm{II}} = -2x\partial_2 - 2y\partial_3 \neq 0$ and the formula $[P, Q]_{\mathrm{SN}} = (-1)^p D(P\wedge Q) - D(P)\wedge Q - (-1)^p P\wedge D(Q)$ for $P \in \mathscr{X}^p(M)$ and $Q \in \mathscr{X}^q(M)$. The main part of this proof will make it obvious why we require that $\dim \mathfrak{g}_\Lambda > 3$.

Denote by E_{ij} for $i, j \in \{1, 2, 3\}$ the canonical basis of $\mathfrak{gl}(3, \mathbb{R})$.

- If $(a, b) \neq (0, 0)$, stabilizer $\mathfrak{g}_{\Lambda_4}$ and the image $J^2(\mathfrak{g}_{\Lambda_4} \wedge \mathfrak{g}_{\Lambda_4})$ are generated by $(\frac{1}{2}E_{11} + E_{22}, \frac{1}{2}E_{11} + E_{33})$ and $yz\partial_{23} - \frac{1}{2}xz\partial_{31} - \frac{1}{2}xy\partial_{12}$, respectively. Hence Λ_4 is not SRMI.
- If $(a, b) \neq (0, 0)$, the generators of $\mathfrak{g}_{\Lambda_8}$ and $J^2(\mathfrak{g}_{\Lambda_8} \wedge \mathfrak{g}_{\Lambda_8})$ are
$$(E_{11} + E_{22} + E_{33}, E_{12} - E_{21}) \quad \text{and} \quad -xz\partial_{23} - yz\partial_{31} + (x^2 + y^2)\partial_{12}.$$
So Λ_8 is not SRMI.
- If $a \neq -1/3$, the generators in the case of Λ_{10} are
$$(E_{11} + E_{22} + E_{33}, E_{12} + E_{23}) \ \text{and}\ (y^2 - xz)\partial_{23} - xy\partial_{31} + x^2\partial_{12}.$$
- For the cases of Λ_{11} and Λ_{12} with $c \neq 0$, and the case Λ_{13}, the generators are
$$(E_{11} + E_{22} + E_{33}, E_{12}, E_{32}) \quad \text{and} \quad -xz\partial_{23} + x^2\partial_{12}, z^2\partial_{23} - xz\partial_{12}. \qquad \square$$

Remarks.

- In the cases Λ_1, Λ_2 and Λ_3, with $c \neq 0$ in the latter two, the dimension of the stabilizer is $\dim \mathfrak{g}_\Lambda = 3$, whereas $J\mathfrak{g}_\Lambda \wedge J\mathfrak{g}_\Lambda \wedge J\mathfrak{g}_\Lambda = \{0\}$. Hence, if the dimension of the stabilizer coincides with the dimension of the space, the Poisson structure is not necessarily a SRMI tensor.
- For Λ_{10}, the decomposition proved in [Liu and Xu 1992] yields
$$\Lambda_{10} = -\tfrac{1}{3}Y_{12} + \Pi_{cz^3/3 + xz^2 + (b+1/3)x^2z + ax^3/3}.$$

3. Poisson cohomology of quadratic structures in a finite-dimensional vector space

Koszul homology and cohomology. Let $\bigwedge = \bigwedge_n\langle\vec{\eta}\rangle$ be the Grassmann algebra on $n \in \mathbb{N}_0$ with generators $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, that is, the algebra over a field $\mathbb{F}$ (here $\mathbb{R}$

or $\mathbb{C}$) of characteristic 0 generated by $\eta_1, \dots, \eta_n$ and subject to the anticommutation relations $\eta_k\eta_\ell + \eta_\ell\eta_k = 0$ for $k, \ell \in \{1, \dots, n\}$. Set $\bigwedge = \bigoplus_{p=0}^n \bigwedge^p$, with obvious notations, and let $\vec{h} = (h_1, \dots, h_n)$ be dual generators defined by $i_{h_k}\eta_\ell = \delta_{k\ell}$. We also need the creation operator $e_{\eta_k} : \bigwedge \to \bigwedge, \omega \mapsto \eta_k\omega$ and the annihilation operator $i_{h_k} : \bigwedge \to \bigwedge, \omega \mapsto i_{h_k}\omega$, where the interior product is defined as usual. Finally, we denote by E a vector space over $\mathbb{F}$ and by $\vec{X} = (X_1, \dots, X_n)$ an n-tuple of commuting linear operators on E.

Definition 5. The Koszul chain complex (or K_*-complex) $K_*(\vec{X}, E)$ associated with $\vec{X}$ on E is the complex

$$0 \to E \otimes_{\mathbb{F}} \bigwedge^n \to E \otimes_{\mathbb{F}} \bigwedge^{n-1} \to \cdots \to E \otimes_{\mathbb{F}} \bigwedge^1 \to E \to 0$$

with differential $\kappa_{\vec{X}} = \sum_{k=1}^n X_k \otimes i_{h_k}$. We denote by $KH_*(\vec{X}, E)$ the corresponding Koszul homology group.

Definition 6. The Koszul cochain complex (or K^*-complex) $K^*(\vec{X}, E)$ associated with $\vec{X}$ on E is the complex

$$0 \to E \to E \otimes_{\mathbb{F}} \bigwedge^1 \to \dots \to E \otimes_{\mathbb{F}} \bigwedge^{n-1} \to E \otimes_{\mathbb{F}} \bigwedge^n \to 0$$

with differential $\mathcal{K}_{\vec{X}} = \sum_{k=1}^n X_k \otimes e_{\eta_k}$. We denote by $KH^*(\vec{X}, E)$ the corresponding Koszul cohomology group.

Since the X_k commute, the anticommutativity of the i_{h_k} and the e_{η_k} imply that $\kappa_{\vec{X}}\kappa_{\vec{X}} = 0$ and $\mathcal{K}_{\vec{X}}\mathcal{K}_{\vec{X}} = 0$, respectively. See [Koszul 1950; 1994].

Example 7. Let $\mathbb{F} = \mathbb{R}$ and $E = C^\infty(\mathbb{R}^3)$. If we choose $\eta_k = dx_k$ and $X_k = \partial_k$, the K^*-complex is the de Rham complex $(\Omega(\mathbb{R}^3), d)$. With $\eta_k = \partial_k = \partial_{x_k}$ and $h_k = dx_k$, the K_*-complex is the dual de Rham complex $(\mathscr{X}(\mathbb{R}^3), \delta)$.

If we identify the subspaces $\Omega^k(\mathbb{R}^3)$ of homogeneous forms with the corresponding spaces of components E, E^3, E^3 and E, this K^*-complex reads

$$0 \to E \xrightarrow{\mathcal{K}=\vec{\nabla}(\cdot)} E^3 \xrightarrow{\mathcal{K}=\vec{\nabla}\wedge(\cdot)} E^3 \xrightarrow{\mathcal{K}=\vec{\nabla}\cdot(\cdot)} E \to 0. \tag{3-1}$$

Example 8. Let $\mathbb{F} = \mathbb{R}$ and $E = \mathcal{S}\mathbb{R}^{3*} = \mathbb{R}[x_1, x_2, x_3]$. For $k \in \{1, 2, 3\}$, let $\eta_k = \partial_k$, $X_k = \mathfrak{m}_{P_k}$, $P_k \in E^{d_k}$ and $d_k \in \mathbb{N}$, where $\mathfrak{m}_{P_k} : E \to E, Q \mapsto P_k Q$. Then the chain spaces of the K_*-complex are the spaces of homogeneous polyvector fields on $\mathbb{R}^3$ with polynomial coefficients, and by identifying these with the corresponding spaces E, E^3, E^3 and E of components, we can write this K_*-complex in the form

$$0 \to E \xrightarrow{\kappa=(\cdot)\vec{P}} E^3 \xrightarrow{\kappa=(\cdot)\wedge\vec{P}} E^3 \xrightarrow{\kappa=(\cdot)\cdot\vec{P}} E \to 0. \tag{3-2}$$

Remarks. First, the Koszul cohomology and homology complexes of Example 7 are exact, except that $KH^0(\vec{\partial}, C^\infty(\mathbb{R}^3)) \simeq KH_3(\vec{\partial}, C^\infty(\mathbb{R}^3)) \simeq \mathbb{R}$.

Second, recall that an R-regular sequence on a module M over a commutative unit ring R is a sequence $(r_1, \dots, r_d) \in R^d$ such that r_k is not a zero divisor on the quotient $M/\langle r_1, \dots, r_{k-1}\rangle M$ for $k \in \{1, \dots, d\}$, and $M/\langle r_1, \dots, r_d\rangle M \neq 0$. In particular, $x_1, \dots, x_d$ is a (maximal length) regular sequence on the polynomial ring $R = \mathbb{F}[x_1, \dots, x_d]$, so that this ring has depth d.

It is well known that the K_*-complex described in Example 8 is exact, except for surjectivity of $\kappa = (\cdot)\cdot\vec{P}$, if the vector $\vec{P} = (P_1, P_2, P_3)$ is regular on $\mathbb{R}[x_1, x_2, x_3]$. If $\vec{P} = \vec{\nabla} p$ for p a homogeneous polynomial with an isolated singularity at the origin, then $\vec{P}$ is regular; see [Pichereau 2005].

Poisson cohomology in dimension 3. Set $E := C^\infty(\mathbb{R}^3)$ and again identify the spaces of homogeneous multivector fields in $\mathbb{R}^3$ with the corresponding component spaces: $\mathcal{X}^0(\mathbb{R}^3) \simeq \mathcal{X}^3(\mathbb{R}^3) \simeq E$ and $\mathcal{X}^1(\mathbb{R}^3) \simeq \mathcal{X}^2(\mathbb{R}^3) \simeq E^3$.

Let $\vec{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3) \in E^3$ be a Poisson tensor, and let $f \in E$, $\vec{X} \in E^3$, $\vec{B} \in E^3$ and $T \in E$ be a 0-, 1-, 2-, and 3-cochain of the LP-complex. By straightforward computations, we get formulas for the LP-coboundary operator $\partial_{\vec{\Lambda}}$:

$$\begin{aligned}
&\partial^0_{\vec{\Lambda}} f = \vec{\nabla} f \wedge \vec{\Lambda},\\
&\partial^1_{\vec{\Lambda}} \vec{X} = (\vec{\nabla}\cdot\vec{X})\vec{\Lambda} - \vec{\nabla}(\vec{X}\cdot\vec{\Lambda}) + \vec{X}\wedge(\vec{\nabla}\wedge\vec{\Lambda}),\\
&\partial^2_{\vec{\Lambda}} \vec{B} = -(\vec{\nabla}\wedge\vec{B})\cdot\vec{\Lambda} - \vec{B}\cdot(\vec{\nabla}\wedge\vec{\Lambda}),\\
&\partial^3_{\vec{\Lambda}} T = 0.
\end{aligned}$$

Recall the differential $\mathcal{K}$ from (3-1), and let κ' and κ'' be the differential in (3-2) when $\vec{P} = \vec{\Lambda}$ and $\vec{P} = \vec{\nabla}\wedge\vec{\Lambda}$, respectively. Then

$$\begin{aligned}
&\partial^0_{\vec{\Lambda}} = \kappa'\mathcal{K},\\
&\partial^1_{\vec{\Lambda}} = \kappa'\mathcal{K} - \mathcal{K}\kappa' + \kappa'', \quad \partial^2_{\vec{\Lambda}} = -\kappa'\mathcal{K} - \kappa'',\\
&\qquad\qquad\qquad\qquad\qquad\quad\; \partial^3_{\vec{\Lambda}} = 0.
\end{aligned} \tag{3-3}$$

Again, this paper investigates only quadratic Poisson tensors and polynomial (or formal) LP-cochains. If the structure $\vec{\Lambda}$ is K-exact, that is, in view of notations due to the elimination of the module basis of multivector fields, $\vec{\Lambda} = \vec{\nabla} p$ for $p \in \mathcal{S}^3\mathbb{R}^{3*}$ if and only if $\vec{\nabla}\wedge\vec{\Lambda} = 0$, homology operator κ'' vanishes. If, moreover, p has an isolated singularity, the K^*-complex associated with $\mathcal{K}$ is exact up to injectivity of $\mathcal{K} = \vec{\nabla}(\,\cdot\,)$, and the K_*-complex associated with κ' is acyclic (see above) up to surjectivity of $\kappa' = (\,\cdot\,)\cdot\vec{\Lambda}$. Pichereau [2005] computed the LP-cohomology for a weight-homogeneous polynomial p with an isolated singularity.

Next we describe a generic cohomological technique for SRMI Poisson tensors in a finite-dimensional vector space. This approach extends (3-3) to dimension n and also reduces the LP-coboundary operator ∂_Λ to a single Koszul differential.

Poisson cohomology in dimension n.

Definition 9. Let $Y_i = \sum_r \ell_{ir}\partial_r$ be n linear vector fields in $\mathbb{R}^n$. Set

$$\mathcal{R} = \bigoplus_{p=0}^{n} \mathcal{R}^p = \bigoplus_{p=0}^{n} \mathbb{R}[\![x_1, \dots, x_n]\!] \otimes \bigwedge_n^p \langle \vec{\partial} \rangle,$$

$$\mathcal{P} = \bigoplus_{p=0}^{n} \mathcal{P}^p = D^{-1} \bigoplus_{p=0}^{n} \mathbb{R}[\![x_1, \dots, x_n]\!] \otimes \bigwedge_n^p \langle \vec{Y} \rangle,$$

where $D = \det \ell$ and $\bigwedge_n^p \langle \vec{\partial} \rangle$ and $\bigwedge_n^p \langle \vec{Y} \rangle$ are the terms of degree p of the Grassmann algebras on generators $\vec{\partial} = (\partial_1, \dots, \partial_n)$ and $\vec{Y} = (Y_1, \dots, Y_n)$, respectively. The spaces $\mathcal{R}$ and $\mathcal{P}$ are respectively the space of *real* and *potential* formal LP-cochains.

For $\boldsymbol{i} = (i_1, \dots, i_m) \in \{1, \dots, n\}^m$, with $i_1 < \cdots < i_m$ and $m \in \{1, \dots, n\}$, we denote by $\boldsymbol{I} = (I_1, \dots, I_{n-m})$ its complement in $\{1, \dots, n\}$. The definition of D gives $Y_1 \wedge \cdots \wedge Y_n = D\, \partial_1 \wedge \cdots \wedge \partial_n$. If we take the interior product of this equation with $dx_{\boldsymbol{I}} = dx_{I_1} \wedge \cdots \wedge dx_{I_{n-m}}$, we get

$$D\partial_{\boldsymbol{i}} = \sum_{\boldsymbol{k}} (-1)^{|\boldsymbol{i}|+|\boldsymbol{k}|} L_{\boldsymbol{ki}} Y_{\boldsymbol{k}},$$

where $\boldsymbol{k}$ is a subscript analogous to $\boldsymbol{i}$, where $\partial_{\boldsymbol{i}}$ and $Y_{\boldsymbol{k}}$ are compact notations similar to $dx_{\boldsymbol{I}}$, where $|\cdot|$ is the sum of the components, and where $L_{\boldsymbol{ki}}$ denotes some homogeneous polynomial. Setting $L^{\boldsymbol{ki}} := L_{\boldsymbol{KI}}$, we have a theorem:

Theorem 10. (i) *There is a canonical nonsurjective injection* $i : \mathcal{R} \to \mathcal{P}$.

(ii) *A homogeneous potential cochain* $D^{-1} \sum_{\boldsymbol{k}} P^{\boldsymbol{k}r} Y_{\boldsymbol{k}}$ (*of bidegree* (p, r), *where* p *is the exterior degree and* r *the polynomial degree*) *is real if and only if the* $n!/p!(n-p)!$ *homogeneous polynomials* $\sum_{\boldsymbol{k}} L^{\boldsymbol{ki}} P^{\boldsymbol{k}r}$ (*of degree* $p + r$) *are divisible by* D; *in case* $p = 0$ *this condition means that* P^r *is divisible by* D.

Remark. The bigrading $\mathcal{P} = \bigoplus_{p=0}^{n} \bigoplus_{r=0}^{\infty} \mathcal{P}^{pr}$, defined on $\mathcal{P}$ by the exterior degree and the polynomial degree, induces a bigrading $\mathcal{R} = \bigoplus_{p=0}^{n} \bigoplus_{r=0}^{\infty} \mathcal{R}^{pr}$ on $\mathcal{R}$.

Consider now a quadratic Poisson tensor Λ in $\mathbb{R}^n$. From now on, we assume that Λ is SRMI, and more precisely that there are n mutually commuting linear vector fields $Y_i = \sum_{r=1}^{n} \ell_{ir}\partial_r$ with $\ell \in \mathfrak{gl}(n, \mathbb{R}^{n*})$ such that $D = \det \ell \neq 0$ and

$$\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}, \quad \text{where } \alpha^{ij} \in \mathbb{R}.$$

Proposition 11. *The determinant* $D = \det \ell \in \mathcal{S}^n \mathbb{R}^{n*} \setminus \{0\}$ *is the unique joint eigenvector of the* Y_i *with eigenvalues* $\operatorname{div} Y_i \in \mathbb{R}$, *that is*, D *is up to multiplication by nonzero constants the unique nonzero polynomial of* $\mathbb{R}^n$ *that satisfies*

$$Y_i D = (\operatorname{div} Y_i) D \quad \textit{for all } i \in \{1, \dots, n\}.$$

Moreover, if $D = D_1 D_2$, where $D_1 \in \mathcal{S}^{n_1}\mathbb{R}^{n*}$ and $D_2 \in \mathcal{S}^{n_2}\mathbb{R}^{n*}$ with $n_1 + n_2 = n$ are two polynomials without common divisor, these factors D_1 and D_2 are also joint eigenvectors. Their eigenvalues λ_i and μ_i satisfy $\lambda_i + \mu_i = \operatorname{div} Y_i$.

Proof. For $i \in \{1, \dots, n\}$,

$$\begin{aligned} 0 = [Y_i, Y_1 \wedge \cdots \wedge Y_n] &= [Y_i, D\partial_1 \wedge \cdots \wedge \partial_n] \\ &= (Y_i D)\partial_1 \wedge \cdots \wedge \partial_n - D(\operatorname{div} Y_i)\partial_1 \wedge \cdots \wedge \partial_n, \end{aligned}$$

so that $Y_i D = (\operatorname{div} Y_i) D$.

For uniqueness, let $P \in \mathcal{S}\mathbb{R}^{n*} \setminus \{0\}$ be another polynomial such that $Y_i P = (\operatorname{div} Y_i) P$ for all $i \in \{1, \dots, n\}$. Then $Y_i\,(P/D) = 0$ in $Z = \{x \in \mathbb{R}^n, D(x) \neq 0\}$, and, reasoning as in the proof of Theorem 1, we conclude there exists $\alpha \in \mathbb{R}^*$ such that $P = \alpha D$.

Finally, because $((\operatorname{div} Y_i) D_1 - Y_i D_1) D_2 = D_1 (Y_i D_2)$ and the polynomials D_1 and D_2 have no common divisor, $Y_i D_2 = P D_2$ and $(\operatorname{div} Y_i) D_1 - Y_i D_1 = Q D_1$, where $P = Q$ is a polynomial. Looking at degrees, we see $P = Q$ is constant. □

Remark. The eigenvalues $\operatorname{div} Y_i$ for $i \in \{1, \dots, n\}$ cannot vanish simultaneously, for otherwise the polynomial $D \in \mathcal{S}^n\mathbb{R}^{n*} \setminus \{0\}$ vanishes everywhere.

Definition 12. The complex $0 \to \mathcal{R}^0 \to \mathcal{R}^1 \to \cdots \to \mathcal{R}^n \to 0$ with differential $\partial_\Lambda = [\Lambda, \cdot]_{\mathrm{SN}}$ is the formal LP-complex of Poisson tensor $\Lambda \in \mathcal{S}^2\mathbb{R}^{n*} \otimes \bigwedge^2 \mathbb{R}^n$. We denote the corresponding cohomology groups by $LH^*(\mathcal{R}, \Lambda)$.

The next theorem shows that if the cochains $C \in \mathcal{R}$ are read as $C = iC \in \mathcal{P}$, the LP-differential simplifies.

Theorem 13. *Set $\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}$, $\alpha^{ji} = -\alpha^{ij}$, and $X_i = \sum_{j \neq i} \alpha^{ij} Y_j$.*

(i) *Let $C = D^{-1} \sum_{\boldsymbol{k}} P^{\boldsymbol{k}r} Y_{\boldsymbol{k}} \in \mathcal{P}^{pr}$ be a homogeneous potential cochain. The LP-coboundary of C is given by*

$$\begin{aligned} \partial_\Lambda C &= \sum_{\boldsymbol{k}i} X_i (D^{-1} P^{\boldsymbol{k}r}) Y_i \wedge Y_{\boldsymbol{k}} \\ &= D^{-1} \sum_{\boldsymbol{k}i} (X_i - \delta_i \,\mathrm{id})(P^{\boldsymbol{k}r}) Y_i \wedge Y_{\boldsymbol{k}} \in \mathcal{P}^{p+1,r}, \end{aligned} \tag{3-4}$$

where $\delta_i = \operatorname{div} X_i \in \mathbb{R}$.

(ii) *The LP-coboundary operator ∂_Λ endows $\mathcal{P}$ with a differential complex structure and preserves the polynomial degree r. This LP-complex of Λ over $\mathcal{P}$ contains the LP-complex $(\mathcal{R}, \partial_\Lambda)$ of Λ over $\mathcal{R}$ as a differential subcomplex.*

Proof. If $C = f\boldsymbol{Y}$, with f a function and $\boldsymbol{Y}$ a wedge product of vector fields Y_k, we get

$$\partial_\Lambda(f\boldsymbol{Y}) = [\Lambda, f\boldsymbol{Y}]_{\mathrm{SN}} = [\Lambda, f]_{\mathrm{SN}} \wedge \boldsymbol{Y}, \tag{3-5}$$

since the Y_k commute. However,

$$
\begin{aligned}
[\Lambda, f]_{\mathrm{SN}} &= \sum_{i<j} \alpha^{ij}((Y_j f)Y_i - (Y_i f)Y_j) \\
&= \sum_i \Big(\sum_{j\neq i} \alpha^{ij}\, Y_j f\Big) Y_i = \sum_i (X_i f) Y_i.
\end{aligned}
\tag{3-6}
$$

By combining (3-5) and (3-6), we get the first part of (3-4), whereas its second part is a consequence of Proposition 11. □

Corollary 14. *The LP-cohomology groups of* Λ *over* $\mathcal{R}$ *and* $\mathcal{P}$ *are bigraded, that is,*

$$
LH(\mathcal{R}, \Lambda) = \bigoplus_{r=0}^{\infty} \bigoplus_{p=0}^{n} LH^{pr}(\mathcal{R}, \Lambda) \quad \text{and} \quad LH(\mathcal{P}, \Lambda) = \bigoplus_{r=0}^{\infty} \bigoplus_{p=0}^{n} LH^{pr}(\mathcal{P}, \Lambda),
$$

where for instance $LH^{pr}(\mathcal{P}, \Lambda)$ *is defined by*

$$
LH^{pr}(\mathcal{P}, \Lambda) = \ker(\partial_\Lambda : \mathcal{P}^{pr} \to \mathcal{P}^{p+1,r}) / \operatorname{im}(\partial_\Lambda : \mathcal{P}^{p-1,r} \to \mathcal{P}^{pr}).
$$

In the following we deal with the terms $LP^{*r}(\mathcal{P}, \Lambda) = \bigoplus_{p=0}^{n} LP^{pr}(\mathcal{P}, \Lambda)$ of the LP-cohomology over $\mathcal{P}$ and with the corresponding part of the LP-cohomology of the subcomplex $\mathcal{R}$.

Theorem 15. *Let* E_r *be the real finite-dimensional vector space* $\mathcal{S}^r\mathbb{R}^{n*}$, *and let* $\vec{X}_\delta := (X_1 - \delta_1\,\mathrm{id}, \ldots, X_n - \delta_n\,\mathrm{id})$, *where* $\delta_i = \operatorname{div} X_i$, *be the* n*-tuple of commuting linear operators* $X_i - \delta_i\,\mathrm{id}$ *on* E_r *defined in Theorem 13. The LP-cohomology space* $LH^{*r}(\mathcal{P}, \Lambda)$ *coincides with the Koszul cohomology space* $KH^*(\vec{X}_\delta, E_r)$.

Proof. This follows from $\partial_\Lambda = \sum_i (X_i - \delta_i\,\mathrm{id}) \otimes e_{Y_i}$, as proved in Theorem 13. □

Since $(\mathcal{R}, \partial_\Lambda)$ is a subcomplex of $(\mathcal{P}, \partial_\Lambda)$, we can use classical techniques (namely, the long exact cohomology sequence) to deduce the LP-cohomology of Λ from the Koszul cohomology associated with $\vec{X}_\delta$. More precisely, consider the relative cohomology $LH(\mathcal{P}, \mathcal{R}, \Lambda)$ of $(\mathcal{P}, \partial_\Lambda)$ with respect to $\mathcal{R}$, that is, the cohomology of the space $(\mathcal{P}/\mathcal{R}, \bar{\partial}_\Lambda)$, and let ϕ be the composition of ∂_Λ with the projection of $\mathcal{P}$ onto $\mathcal{R}$.

Theorem 16. *The LP-cohomology groups of a SRMI Poisson tensor* Λ *over the space* $\mathcal{R}$ *of cochains with coefficients in formal power series are given by*

$$
LH^{pr}(\mathcal{R}, \Lambda) \simeq LH^{pr}(\mathcal{P}, \Lambda) / \ker^{pr} \phi_\sharp \oplus LH^{p-1,r}(\mathcal{P}, \mathcal{R}, \Lambda) / \ker^{p-1,r} \phi_\sharp .
$$

Remark 1. This theorem reduces computing the groups $LH^{pr}(\mathcal{R}, \Lambda)$ to finding the groups $LH^{pr}(\mathcal{P}, \Lambda) \simeq KH^p(\vec{X}_\delta, E_r)$ associated to the operators $\vec{X}_\delta$ on $E_r = \mathcal{S}^r\mathbb{R}^{n*}$ induced by Λ, and to finding the relative cohomology groups $LH^{p-1,r}(\mathcal{P}, \mathcal{R}, \Lambda)$. It thus links Poisson and Koszul cohomology. In [Masmoudi and Poncin 2007], we showed via explicit computations in $\mathbb{R}^3$ that $\mathcal{P}$-cohomology (now identified as Koszul cohomology) and $\mathcal{S}$-cohomology (or relative cohomology) are less intricate than Poisson cohomology.

4. Koszul cohomology in a finite-dimensional vector space

In view of Remark 1, we now turn to the Koszul cohomology space $KH^*(\vec{X}_\lambda, E)$ associated to operators $\vec{X}_\lambda := (X_1 - \lambda_1 \,\mathrm{id}, \ldots, X_n - \lambda_n \,\mathrm{id})$ made up of commuting linear transformations $\vec{X} := (X_1, \ldots, X_n)$ of a finite-dimensional real vector space E and a point $\vec{\lambda} := (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Koszul cohomology is known to be closely connected with spectral theory: A fundamental principle of multivariate operator theory is that all essential spectral properties of operators $\vec{X}$ in a complex space should be understood in terms of properties of the Koszul complex induced by $\vec{X}_\lambda$ for $\vec{\lambda} \in \mathbb{C}^n$. Thus the complex setting is the natural one for investigating Koszul cohomology. To engage this point of view, it suffices to note that, if $\vec{X} \in \mathrm{End}_{\mathbb{R}}(E)$ are commuting $\mathbb{R}$-linear transformations of a real vector space E, and if $\vec{X}^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(E^{\mathbb{C}})$ are the corresponding commuting complexified $\mathbb{C}$-linear transformations of the complexification $E^{\mathbb{C}}$ of E, the cohomology $KH^*(\vec{X}^{\mathbb{C}}, E^{\mathbb{C}})$ of the complexification $K^*(\vec{X}^{\mathbb{C}}, E^{\mathbb{C}})$ of the complex $K^*(\vec{X}, E)$ is isomorphic to the complexification $KH^{*\mathbb{C}}(\vec{X}, E)$ of the cohomology of $K^*(\vec{X}, E)$.

Below, we use the concept of joint spectrum $\sigma(\vec{X})$ of commuting bounded linear operators $\vec{X} = (X_1, \ldots, X_n)$ on a complex vector space E. Such spectra are defined variously in the literature, where E may be a normed space, a Banach space, or a Hilbert space. Here we investigate Koszul cohomology in finite dimension and need the following characterizations of the elements of the joint spectrum $\sigma(\vec{X})$; for a proof, see [Bolotnikov and Rodman 2002].

Proposition 17. *Let $\vec{X} = (X_1, \ldots, X_n)$ be an n-tuple of commuting operators on a finite-dimensional complex vector space E. Then these statements are equivalent for any fixed $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$:*

(a) $\vec{\lambda} \in \sigma(\vec{X})$.

(b) *There is a basis in E in which the matrices representing the X_j are all upper-triangular, and there is an index q in $1 \leq q \leq \dim E$ such that λ_j is the (q, q) entry of the matrix representing X_j for $j \in \{1, \ldots, n\}$.*

(c) *For every basis in E in which matrices for the X_j are all upper-triangular, there is an index q as in* (b).

(d) *There is a nonzero vector x such that $X_j x = \lambda_j x$ for all $j \in \{1, \ldots, n\}$*

(e) *There are no Y_j in the subalgebra of* $\mathrm{End}_{\mathbb{C}}(E)$ *generated by* id *and $\vec{X}$ that satisfy $\sum_{j=1}^n Y_j (X_j - \lambda_j \,\mathrm{id}) = \mathrm{id}$.*

We now supply some results about Koszul cohomology spaces, using the same notation as above. The first is obvious.

Proposition 18. *Let $\bigwedge = \bigwedge_n\langle\vec{\eta}\rangle$ be the exterior algebra on n with generators $\vec{\eta}$ over a field $\mathbb{F}$ of characteristic* 0, *and let $\vec{h}$ be the dual generators, that is, suppose $i_{h_k}\eta_\ell = \partial_{k\ell}$. We then have the homotopy formula $e_{\eta_\ell} i_{h_k} + i_{h_k} e_{\eta_\ell} = \delta_{k\ell}\,\mathrm{id}$, where i_{h_k} and e_{η_ℓ} are the creation and annihilation operators, respectively.*

Proposition 19. *Let $\vec{\mathscr{X}} \in \mathrm{End}_{\mathbb{F}}^{\times n}(E)$ and $\vec{Y} \in \mathrm{End}_{\mathbb{F}}^{\times n}(E)$ be n commuting linear operators $\vec{\mathscr{X}}$ and $\vec{Y}$, respectively, on a vector space E over $\mathbb{F}$. We denote by $\mathscr{K} = \sum_\ell \mathscr{X}_\ell \otimes e_{\eta_\ell}$ and $\kappa = \sum_k Y_k \otimes i_{h_k}$ the respective corresponding Koszul cohomology and homology operators. Then*

$$\mathscr{K}\kappa + \kappa\mathscr{K} = \Big(\sum_\ell Y_\ell \mathscr{X}_\ell\Big) \otimes \mathrm{id} + \sum_{k\ell} [\mathscr{X}_\ell, Y_k] \otimes e_{\eta_\ell} i_{h_k}.$$

Proof. This is a direct consequence of Proposition 18. □

Theorem 20. *Let $\vec{X} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ be n commuting endomorphisms of E, a finite-dimensional complex vector space, and let $\vec{\lambda} \in \mathbb{C}^n$. For splitting $E = E_1 \oplus E_2$, denote by $i_j : E_j \to E$ the injection of E_j into E and by $p_j : E \to E_j$ the projection of E onto E_j.*

If E_1 is stable under the operators X_ℓ, that is, $p_2 X_\ell i_1 = 0$, and if $\vec{\lambda}$ is not in the joint spectrum $\sigma(\vec{X}')$ of the commuting operators $X'_\ell = p_2 X_\ell i_2 \in \mathrm{End}_{\mathbb{C}}(E_2)$, then any cocycle $C \in E \otimes \bigwedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$, where $\vec{X}_\lambda = \vec{X} - \vec{\lambda}\,\mathrm{id}_E$, is cohomologous to a cocycle $C_1 \in E_1 \otimes \bigwedge$, with $\bigwedge = \bigwedge_n\langle\vec{\eta}\rangle$.

Proof. If $q(\vec{X}) \in \mathbb{C}[X_1, \ldots, X_n] \subset \mathrm{End}_{\mathbb{C}}(E)$ denotes a polynomial in the X_ℓ, the map $q(\vec{X})_2 = p_2 q(\vec{X}) i_2$ coincides with the (same) polynomial $q(\vec{X}') \in \mathrm{End}_{\mathbb{C}}(E_2)$ in the X'_ℓ. Indeed, due to stability of E_1, we have

$$p_2 X_\ell X_k i_2 = p_2 X_\ell i_1 p_1 X_k i_2 + p_2 X_\ell i_2 p_2 X_k i_2 = X'_\ell X'_2.$$

This implies that the X'_ℓ commute.

Since $\vec{\lambda} \notin \sigma(\vec{X}')$, Proposition 17(e) implies that there are n operators $\vec{Y}'$ in the subalgebra of $\mathrm{End}_{\mathbb{C}}(E_2)$ generated by id_{E_2} and $\vec{X}'$ such that

$$\sum_\ell Y'_\ell (X'_\ell - \lambda_\ell\,\mathrm{id}_{E_2}) = \mathrm{id}_{E_2}\,. \tag{4-1}$$

Hence $Y'_\ell = Q_\ell(\vec{X}')$ is a polynomial in the X'_k for any ℓ. Set $Y_\ell = Q_\ell(\vec{X}) \in \mathrm{End}_{\mathbb{C}}(E)$.

If applied to operators $\vec{X}_\lambda$ and $\vec{Y}$, Proposition 19 implies that

$$\Big(\sum_\ell Y_\ell (X_\ell - \lambda_\ell\,\mathrm{id}_E)\Big) \otimes \mathrm{id}_\wedge + \sum_{k\ell} [X_\ell - \lambda_\ell\,\mathrm{id}_E, Y_k] \otimes e_{\eta_\ell} i_{h_k} = \mathscr{K}\kappa + \kappa\mathscr{K}, \tag{4-2}$$

where $\mathcal{K}$ and κ are respectively the Koszul cohomology and homology operators associated with $\vec{X}_\lambda$ and $\vec{Y}$ on E. Since Y_k is a polynomial in the commuting endomorphisms X_ℓ, the second term on the left side of (4-2) vanishes. Hence, when evaluating both sides on a cocycle $C = e \otimes w$ of cochain complex $K^*(\vec{X}_\lambda, E)$, we get

$$(Q(\vec{X})(e))w = \mathcal{K}\kappa(e \otimes w),$$

where $Q(\vec{X}) = \sum_\ell Y_\ell(X_\ell - \lambda_\ell \,\mathrm{id}_E) = \sum_\ell Q_\ell(\vec{X})(X_\ell - \lambda_\ell \,\mathrm{id}_E)$ is a polynomial in the X_ℓ. Absent the factor w, the left side reads

$$Q(\vec{X})(e) = p_1 Q(\vec{X}) i_1 p_1(e) + p_2 Q(\vec{X}) i_1 p_1(e) + p_1 Q(\vec{X}) i_2 p_2(e) + p_2 Q(\vec{X}) i_2 p_2(e),$$

where the second term on the right vanishes in view of the stability of E_1, and the last term equals $p_2(e)$, in view of the first sentence of the proof of Theorem 20 and (4-1). Finally, the cocycle $C = e \otimes w$ is cohomologous to

$$C_1 = C - \mathcal{K}\kappa C = \big(p_1(e) - p_1 Q(\vec{X}) i_1 p_1(e) - p_1 Q(\vec{X}) i_2 p_2(e)\big) \otimes w \in E_1 \otimes \bigwedge . \quad \square$$

This theorem has a number of new and partially practical consequences.

First, if $\vec{X} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ are n commuting $\mathbb{C}$-linear endomorphisms, the complex finite-dimensional vector space E on which these operators act has a direct sum decomposition $E = \bigoplus_{\vec{\mu} \in \mathbb{C}^n} E^\mu$, where the primary subspace of E associated with the weight $\vec{\mu}$, namely

$$E^\mu = \bigcap_i E^{\mu_i} = \bigcap_i \bigcup_{n \in \mathbb{N}} \ker(X_i - \mu_i \,\mathrm{id})^n,$$

is stable under the action of the operators $\vec{X}$ [Bourbaki 1975, théorème 1]. Let us also mention that

$$E^{\mu_i} = \bigcup_{n \in \mathbb{N}} \ker(X_i - \mu_i \,\mathrm{id})^n = \ker(X_i - \mu_i \,\mathrm{id})^{m_i},$$

where m_i denotes the multiplicity of the solution μ_i of the characteristic polynomial of X_i, and that $\dim E^{\mu_i} = m_i$. Since the multiplicity $\mathfrak{m}$ of $\vec{0}$ in the joint spectrum of the commuting operators $X_i - \mu_i \,\mathrm{id}$ coincides with its multiplicity in the joint spectrum of the operators $(X_i - \mu_i \,\mathrm{id})^m$, where $m = \sup\{m_i\}$, we easily see that the dimension of $E^\mu = \bigcap_i \ker(X_i - \mu_i \,\mathrm{id})^m$ cannot exceed $\mathfrak{m}$.

Another consequence of Theorem 20 is that Koszul cohomology $KH^*(\vec{X}_\lambda, E)$, roughly speaking, is made up of weak joint eigenvectors with eigenvalues λ_ℓ.

Corollary 21. *Let $\vec{\lambda} \in \mathbb{C}^n$, and let $\vec{X} \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ be n commuting endomorphisms of a finite-dimensional complex vector space E. Denote by $\bigwedge = \bigwedge_n \langle \vec{\eta} \rangle$ the Grassmann algebra with n generators $\vec{\eta}$. Any cocycle $C \in E \otimes \bigwedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle $C_1 \in E^\lambda \otimes \bigwedge$.*

This corollary has a useful variant. Choose a supplementary subspace $E^{(2)}$ in E of the stable subspace $\ker \vec{X}_\lambda := \bigcap_\ell \ker(X_\ell - \lambda_\ell \,\mathrm{id})$ of joint eigenvectors, and denote by $\vec{X}^{(2)}$ the restrictions of the operators $\vec{X}$ to $E^{(2)}$; see Theorem 20. We can iterate this procedure, choosing a supplementary subspace $E^{(3)}$ in $E^{(2)}$ of $\ker \vec{X}^{(2)}_\lambda$, introducing the restrictions $\vec{X}^{(3)}$ of the operators $\vec{X}^{(2)}$ to $E^{(3)}$, and so on, until we get $\ker \vec{X}^{(s+1)}_\lambda = 0$. We will show that the direct sum of these kernels coincides with the space E^λ above, so that Corollary 21 means that any cocycle $C \in E \otimes \bigwedge$ of $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle $C_1 \in \bigoplus_{a=1}^{s} \ker \vec{X}^{(a)}_\lambda \otimes \bigwedge$, where of course $\ker \vec{X}^{(1)}_\lambda = \ker \vec{X}_\lambda$.

Proof of Corollary 21. Let $E_\lambda := \ker \vec{X}_\lambda \subset E^\lambda$ and denote by $E(\lambda)$ a supplementary subspace of E_λ in E^λ. Set

$$E = E^\lambda \oplus \bigoplus_{\vec{\mu} \neq \vec{\lambda}} E^\mu =: E_1 \oplus E_2$$

and use the notation of Theorem 20. Assume that $\vec{\lambda} \in \sigma(\vec{X}')$, that is, that $\ker \vec{X}'_\lambda \neq 0$. If $e \in E_2$ is a nonvanishing joint eigenvector of the $X'_\ell = p_2 X_\ell i_2$ with eigenvalues λ_ℓ, we have $X_\ell e = p_1 X_\ell e + \lambda_\ell e$ for any ℓ. Since E_2 is fixed by the X_ℓ, we get $p_1 X_\ell e = 0$, so that $e \in E_1 \cap E_2 = 0$, a contradiction; finally $\vec{\lambda} \notin \sigma(\vec{X}')$ and we finish using Theorem 20.

On the other hand, set

$$E = E_\lambda \oplus \Big(E(\lambda) \oplus \bigoplus_{\vec{\mu} \neq \vec{\lambda}} E^\mu \Big) =: E_1 \oplus E_2.$$

Note that the operators $\vec{X}'$ in this decomposition coincide with the $\vec{X}^{(2)}$ above. Now $(X_\ell - \lambda_\ell \,\mathrm{id})e = p_1 X_\ell e \in E_1 = \ker \vec{X}_\lambda$. Any $e \in \ker \vec{X}'_\lambda = \ker \vec{X}^{(2)}_\lambda$ belongs to E^λ. Also $\ker \vec{X}_\lambda \oplus \ker \vec{X}^{(2)}_\lambda \subset E^\lambda$; more generally, $\bigoplus_{a=1}^{s} \ker \vec{X}^{(a)}_\lambda \subset E^\lambda$. Since, as is easily checked, the dimension of this direct sum is equal to the multiplicity $\mathfrak{m}$ (which is no less than $\dim E^\lambda$) of $\vec{0}$ in the joint spectrum $\sigma(\vec{X}_\lambda)$, this direct sum coincides with E^λ. □

We next recover a well-known result, and then an important special case.

Corollary 22. *Let $\vec{X} \in \mathrm{End}^{\times n}_{\mathbb{C}}(E)$ be n commuting endomorphisms, and let $\vec{\lambda} \in \mathbb{C}^n$. Then $KH^*(\vec{X}_\lambda, E)$ is trivial if and only if* $\dim(\ker \vec{X}_\lambda) = 0$.

Corollary 23. *Assume the conditions of Corollary 21. If for any $\ell \in \{1, \dots, n\}$ the kernel and image of $X_\ell - \lambda_\ell \,\mathrm{id}$ are supplementary in E, then any cocycle $C \in E \otimes \bigwedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle $C_1 \in \ker \vec{X}_\lambda \otimes \bigwedge$.*

Proofs. First the forward implication in Corollary 22. If there exists $e \in \ker \vec{X}_\lambda \setminus \{0\}$, then

$$\mathcal{K}_{\vec{X}_\lambda} e = \sum_{\ell=1}^{n} (X_\ell - \lambda_\ell \,\mathrm{id}) e \eta_\ell = 0,$$

so that e is a nonbounding 0-cocycle. As for Corollary 23, it follows from the proof of Corollary 21 that if there is a nonzero vector $e \in \ker \vec{X}_\lambda^{(2)} \subset E^{(2)}$, then for every ℓ we have $(X_\ell - \lambda_\ell \,\mathrm{id})e \in \ker \vec{X}_\lambda \cap \mathrm{im}(X_\ell - \lambda_\ell \,\mathrm{id}) = 0$, so that $e \in \ker \vec{X}_\lambda \cap E^{(2)} = 0$, which is a contradiction. □

5. Koszul cohomology associated with Poisson cohomology

We now return to the Koszul cohomology implemented by a SRMI tensor of $\mathbb{R}^n$. Recall the QPT tensor Λ from (1-1) and the conditions under which it is SRMI. Theorems 13 and 15 identify the main building block of the LP-cohomology of Λ as the Koszul cohomology space $KH^*(\vec{X}_\delta, E_r)$. We noted that this cohomology can be deduced from its complex counterpart $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$, which, according to Corollaries 21–23, is closely related to the joint eigenvectors and spectrum of $\vec{X}^{\mathbb{C}}$ or $\vec{X}_\delta^{\mathbb{C}}$. We now further investigate $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$. In particular, we reduce its computation to a problem of linear algebra, and describe the spectrum of the commuting transformations $\vec{X}_\delta^{\mathbb{C}}$.

Proposition 24. *Let $a_j = J^{-1}Y_j \in \mathfrak{gl}(n, \mathbb{R})$ for $j \in \{1, \dots, n\}$. Any basis of $\mathbb{C}^n$ in which the $\vec{a}$ are upper-triangular naturally induces a basis of $E_r^{\mathbb{C}} = \mathcal{S}^r\mathbb{C}^{n*}$ in which the $\vec{X}_\delta^{\mathbb{C}}$ are upper-triangular.*

In what follows, the use of super- and subscripts is dictated by aesthetics and not by contra- or covariance.

Proof. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $z = (z_1, \dots, z_n) \in C^n$. As usual, we set $Y_k = \sum_m \ell_{km}\partial_{x_m} = \sum_{mp} a_k^{mp} x_p \partial_{x_m}$ and use notations as $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ for $\beta \in \mathbb{N}^n$. We complexify

$$E_r = \mathcal{S}^r\mathbb{R}^{n*} = \left\{P \in C^\infty(\mathbb{R}^n) : P(x) = \textstyle\sum_{|\beta|=r} r_\beta x^\beta,\ x \in \mathbb{R}^n,\ r_\beta \in \mathbb{R}\right\}$$

to $E_r^{\mathbb{C}} \simeq \mathcal{S}^r\mathbb{C}^{n*}$ by replacing $\mathbb{R}$ with $\mathbb{C}$. The complexification $Y_k^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(E_r^{\mathbb{C}})$ of $Y_k \in \mathrm{End}_{\mathbb{R}}(E_r)$ is the holomorphic vector field

$$Y_k^{\mathbb{C}} = \sum_{mp} a_k^{mp}\, z_p \partial_{z_m} \in \mathrm{Vect}^{10}(\mathbb{C}^n) \quad \text{of } \mathbb{C}^n.$$

There is a unitary matrix $U \in \mathrm{U}(n, \mathbb{C})$ such that $b_j = U^{-1}a_jU$ is upper-triangular, and there is a corresponding basis $(e'_1, \dots, e'_n)$ of $\mathbb{C}^n$ such that the a_j themselves are upper-triangular. Let $(\varepsilon'_1, \dots, \varepsilon'_n)$ be the corresponding dual basis.

Express z in this basis as $z = \sum_j \mathfrak{z}_j e'_j \in \mathbb{C}^n$. If viewed as a basis of the space $E_r^{\mathbb{C}}$ of degree r homogeneous polynomials of $\mathbb{C}^n$, the induced basis of the space $\mathcal{S}^r\mathbb{C}^{n*}$ of symmetric covariant r-tensors of $\mathbb{C}^n$ is the set $\mathfrak{z}^\beta$ with $\beta \in \mathbb{N}^n$ and $|\beta| = r$.

To find the matrices of the operators $\vec{X}_\delta^{\mathbb{C}}$ in the basis $\mathfrak{z}^\beta$, we arrange its elements by the lexicographic order $\prec$ and perform the coordinate change $z = U\mathfrak{z}$ and put

$\partial_z = \widetilde{\partial_\mathfrak{z} z}^{-1} \partial_\mathfrak{z}$ in the first order differential operators $(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}}$. We get

$$\begin{aligned}(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}} &= \sum_k \alpha^{jk} \sum_{m \le p} b_k^{mp} \mathfrak{z}_p \partial_{\mathfrak{z}_m} - \delta_j \,\mathrm{id}^{\mathbb{C}} \\ &= \sum_{km} \alpha^{jk} b_k^{mm} (\mathfrak{z}_m \partial_{\mathfrak{z}_m} - \mathrm{id}^{\mathbb{C}}) + \sum_k \sum_{m<p} \alpha^{jk} b_k^{mp} \mathfrak{z}_p \partial_{\mathfrak{z}_m},\end{aligned}$$

since $\delta_j = \operatorname{div} X_j = \sum_{km} \alpha^{jk} a_k^{mm} = \sum_{km} \alpha^{jk} b_k^{mm}$. These operators are then upper-triangular in the $\mathfrak{z}^\beta$ basis, since

$$(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}} \mathfrak{z}^\beta = \sum_{km} \alpha^{jk} b_k^{mm} \,(\beta_m - 1)\, \mathfrak{z}^\beta + \sum_k \sum_{m<p} \alpha^{jk} b_k^{mp} \beta_m \, \mathfrak{z}^{\beta - e_m + e_p}, \tag{5-1}$$

where $\mathfrak{z}^{\beta - e_m + e_p} \prec \mathfrak{z}^\beta$. □

With matrices b_j as above, let $B \in \mathfrak{gl}(n, \mathbb{C})$ be the matrix $B_{jk} = b_j^{kk}$.

Theorem 25. *The joint spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ of the $\vec{X}_\delta^{\mathbb{C}}$ is given by*

$$\sigma_r(\vec{X}_\delta^{\mathbb{C}}) = \big\{ \alpha B I : I \in (\mathbb{N} \cup \{-1\})^n, |I| = r - n \big\} \subset \mathbb{C}^n, \quad \textit{where } |I| = \textstyle\sum_j I_j.$$

Proof. This follows directly from Proposition 17 and (5-1). □

Remark. In Proposition 11, we showed $Y_k D = (\operatorname{div} Y_k) D$ that for all k, where $D = \det \ell \in E_n \subset E_n^{\mathbb{C}}$. Then $X_j^{\mathbb{C}} D = X_j D = (\operatorname{div} X_j) D = \delta_j \,\mathrm{id}^{\mathbb{C}} D$ for all j, so that $\vec{0} = (0, \dots, 0) \in \sigma_n(\vec{X}_\delta^{\mathbb{C}})$. This is immediately recovered from Theorem 25.

Set $K_r(\vec{X}_\delta^{\mathbb{C}}) = \{ I \in \ker(\alpha B) : I \in (\mathbb{N} \cup \{-1\})^n, |I| = r - n \}$. Corollary 22 can then be reformulated as follows.

Corollary 26. *$KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ is acyclic if and only if $K_r(\vec{X}_\delta^{\mathbb{C}}) = \varnothing$.*

Proof. $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ is trivial if and only if $\dim(\ker \vec{X}_\delta^{\mathbb{C}}) = 0$, which is true if and only if $\vec{0} \notin \sigma_r(\vec{X}_\delta^{\mathbb{C}})$, that is, if and only if $K_r(\vec{X}_\delta^{\mathbb{C}}) = \varnothing$. □

Example 27. Consider the structure Λ_2 of the Dufour–Haraki classification as in Theorem 4, and assume that $a \neq 0$ and $b = 0$. It is easily checked that the matrix

$$U = \begin{pmatrix} 0 & i/\sqrt{2} & -i/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix}$$

transforms the matrices a_ℓ into upper-triangular matrices b_ℓ. A short computation shows that $K_{3t}(\vec{X}_\delta^{\mathbb{C}})$ for $t \in \mathbb{N}$ contains a single point $I_t = (t-1, t-1, t-1)$, so that the multiplicity μ of $\vec{0}$ in the joint spectrum $\sigma_{3t}(\vec{X}_\delta^{\mathbb{C}})$ equals 1; see the proof of Theorem 25. It follows that the $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_{3t}^{\mathbb{C}})$ are not trivial; see Corollary 26. Furthermore, since the matrices b_ℓ are in fact diagonal in this case, (5-1) implies that $\mathfrak{z}_1^t \mathfrak{z}_2^t \mathfrak{z}_3^t$ belongs to the kernel of the $\vec{X}_\delta^{\mathbb{C}}$ in $E_{3t}^{\mathbb{C}}$. Looking at dimensions, we see that this kernel is $\mathbb{C}\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t$ and that the reduced operators $\vec{X}_\delta^{\mathbb{C}\,(j)}$ for $j \in \{2, \dots, s\}$ do not exist, that is, that $s = 1$. Then, since the change to canonical coordinates is $z = U\mathfrak{z}$ by the proof of Proposition 24, the space $KH^p(\vec{X}_\delta^{\mathbb{C}}, E_{3t}^{\mathbb{C}})$ for $p \in \{0, 1, 2, 3\}$ and $t \in \mathbb{N}$ is contained in

$$\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t \bigoplus_{j_1<\dots<j_p} \mathbb{C}Y_{j_1\dots j_p} = (z_1^2+z_2^2)^t z_3^t \bigoplus_{j_1<\dots<j_p} \mathbb{C}Y_{j_1\dots j_p}.$$

This easy consequence agrees with the results of [Masmoudi and Poncin 2007] modulo slight changes in notation — showing that the new approach is more efficient, though the same results could also be obtained via complexification.

Example 28. For Λ_3 of the Dufour–Haraki classification with parameter $a = 0$, the multiplicity of $\vec{0}$ in the spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ equals 0 or 1, depending on the value of r. The computations are similar to those of Example 27 except in the case $r = 3$, which generates multiplicity 3. Since for Λ_3 the matrices a_ℓ are already lower-triangular, a coordinate change is not necessary, and it is easily seen that $s = 3$ and

$$\ker_3 \vec{X}_\delta^{\mathbb{C}} = \mathbb{C}z_1^2z_3, \quad \ker_3 \vec{X}_\delta^{\mathbb{C}(2)} = \mathbb{C}z_1z_2z_3, \quad \ker_3 \vec{X}_\delta^{\mathbb{C}(3)} = \mathbb{C}z_2^2z_3.$$

The next two theorems follow from similar computations; no proofs are given. In both, the Y_i are those defined in Theorem 4,

Theorem 29. *If $a \neq 0$, the cohomology spaces of the structure Λ_3 are*

$$\begin{aligned}
LH^{0*}(\mathfrak{R}, \Lambda_3) &= \mathbb{R},\\
LH^{1*}(\mathfrak{R}, \Lambda_3) &= \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Y_3,\\
LH^{2*}(\mathfrak{R}, \Lambda_3) &= \mathbb{R}Y_{23} \oplus \mathbb{R}Y_{31} \oplus \mathbb{R}(2yz\partial_{31} + y^2\partial_{12}),\\
LH^{3*}(\mathfrak{R}, \Lambda_3) &= \mathbb{R}\partial_{123} \oplus \mathbb{R}y^2z\partial_{123}.
\end{aligned}$$

Theorem 30. *If $a \neq 0$, the cohomology spaces of the structure Λ_9 are*

$$LH^{0*}(\mathfrak{R}, \Lambda_9) = \mathbb{R}, \qquad LH^{2*}(\mathfrak{R}, \Lambda_9) = \textstyle\bigoplus_{r\in\mathbb{N}} H_r^2,$$

$$LH^{1*}(\mathfrak{R}, \Lambda_9) = \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Y_3, \qquad LH^{3*}(\mathfrak{R}, \Lambda_9) = \textstyle\bigoplus_{r\in\mathbb{N}} \mathbb{R}z^r\partial_{123},$$

where

$$H_0^2 = \mathbb{R}\partial_{23}, \qquad H_1^2 = \mathbb{R}C_1^0, \qquad H_3^2 = \mathbb{R}C_1^2,$$

$$H_2^2 = \mathbb{R}x^2\partial_{23} + \mathbb{R}xz(\partial_{23} - 2^{-1}\partial_{31}) + \mathbb{R}(xz\partial_{12} - z^2\partial_{23}) + \mathbb{R}(yz\partial_{12} + (-27a^2x^2 - 9axz + 3ay^2 - z^2)\partial_{31}),$$

$$H_{r+1}^2 = \mathbb{R}C_1^r + \mathbb{R}C_2^r \quad \textit{for } r \geq 3,$$

with

$$C_1^r = -a(xz^r + ry^2z^{r-1})\partial_{12} + (9a^2xy^r + a(3r-1)(r+1)^{-1}z^{r+1})\partial_{23} + ayz^r\partial_{31}$$

and

$$C_2^r = \big(9a^2xy^2z^{r-2} - 9ar^{-1}xz^r + 3a(r-3)(r-1)^{-1}y^2z^{r-1} - 3(r-1)r^{-1}(r+1)^{-1}z^{r+1}\big)\partial_{23} + (-a(r-2)y^4z^{r-3} + y^2z^{r-1})\partial_{12} + (6a(r-1)^{-1}xyz^{r-1} - ay^3z^{r-2} - r^{-1}yz^r)\partial_{31},$$

and where the terms that contain a negative power of x, y, or z are ignored.

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A CLASSIFICATION OF SPHERICAL CONJUGACY CLASSES IN GOOD CHARACTERISTIC

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We classify spherical conjugacy classes in a simple algebraic group over an algebraically closed field of good, odd characteristic.

Introduction

When studying a transitive action of a group G, it is particularly interesting to understand when a given subgroup B of G acts with finitely many orbits. An important case of such a situation in the theory of algebraic groups is when B is a Borel subgroup of a connected reductive algebraic group G. The G-spaces for which B acts with finitely many orbits in this case are the so-called spherical homogeneous spaces, and they include important examples such as the flag variety G/B and symmetric varieties. They are precisely those G-spaces for which the B-action has a dense orbit in the Zariski topology [Brion 1986; Grosshans 1992; Knop 1995; Vinberg 1986]. One may want to understand when homogeneous spaces that are relevant in algebraic Lie theory, such as nilpotent orbits in $\mathrm{Lie}(G)$ and conjugacy classes in G for G reductive, are spherical. Spherical nilpotent orbits in simple Lie algebras were classified in [Panyushev 1994; 1999] when the base field is $\mathbb{C}$ and in [Fowler and Röhrle 2008] when it is an algebraically closed field of good characteristic: They are precisely the orbits of type rA_1 for $r \geq 0$ in the simply-laced case and of type $rA_1 + s\tilde{A}_1$ for $r, s \geq 0$ in the multiply-laced case. As for conjugacy classes, it is natural to use the interplay with the Bruhat decomposition, since this has proved to be a fruitful tool in the past. For instance, it is essential in describing regular conjugacy classes [Steinberg 1965], whose intersection with Bruhat cells is the subject of ongoing research [Ellers and Gordeev 2004; 2007]. This approach has led to two characterizations of the spherical conjugacy classes in a connected, reductive algebraic group G over an algebraically closed field of zero or good, odd characteristic [Cantarini et al. 2005; Carnovale 2008; 2009]. The first one is given through a formula relating the dimension of a class $\mathcal{O}$ and the Weyl group element whose associated Bruhat cell intersects $\mathcal{O}$ in a dense subset.

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The second one states that spherical conjugacy classes are exactly those classes intersecting only Bruhat cells corresponding to involutions in the Weyl group of G. These characterizations can be used to give a complete list of the spherical classes in G. This problem can be easily reduced to the case in which G is simple, so we shall make this assumption from now on. The spherical conjugacy classes in a simple algebraic group over $\mathbb{C}$ have been classified in [Cantarini et al. 2005], making use of the classification of spherical nilpotent orbits. Spherical classes in type G_2 in good characteristic have been classified in [Carnovale 2009].

In the present paper, we complete the picture by classifying spherical classes in a simple algebraic group G over a field of good, odd characteristic. In contrast to [Cantarini et al. 2005], this work is independent of the classification of spherical nilpotent orbits existing in the literature. Since Springer isomorphisms exist in good characteristic, it provides an elementary classification of spherical nilpotent orbits alternative to [Fowler and Röhrle 2008], where Kempf–Russeau theory is involved and where a computer program is needed to help deal with the exceptional types. The crucial tools in our method are just those conditions in the characterizations in [Cantarini et al. 2005; Carnovale 2008; 2009], whose proofs are general and rather short. The arguments used for this classification can also be transferred to the characteristic zero situation, providing an alternative, elementary approach to [Panyushev 1994; 1999], although by case-by-case considerations.

After fixing notation and recalling basic notions in Section 1, we introduce spherical conjugacy classes and their characterizations in Section 2. Section 3 provides the list of spherical conjugacy classes through a case-by-case analysis.

The result is as when the base field is $\mathbb{C}$: In the simply-laced case, spherical conjugacy classes are, up to a central element, either semisimple or unipotent, and if G is simply-connected, the centralizers of the semisimple ones are all subgroups of fixed points for an involution on G. By abuse of notation, we say that such classes are symmetric.

In type G_2, spherical conjugacy classes are again either semisimple or unipotent but, as in types B_n and C_n, there are spherical semisimple classes that are not symmetric. Just as in other situations involving spherical homogeneous spaces (for example, in the description of maximal spherical ideals of Borel subalgebras [Panyushev and Röhrle 2005]), the doubly-laced case is slightly more involved. The new phenomenon in the present situation is that there appear spherical classes that are neither semisimple nor unipotent.

1. Notation

Let G be a connected reductive algebraic group over an algebraically closed field k of good odd characteristic [Springer and Steinberg 1970, Section I.4]. In Section 3, we will restrict to the case of simple G. When we consider an integer as an element in k, we mean its image in the prime field of k. We denote by Φ and Φ^+ the root system and the set of positive roots relative to a fixed Borel subgroup B and a maximal torus T of G; denote by $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the corresponding set of simple roots. We number the simple roots as in [Bourbaki 1981, planches I–IX]. Denote the highest positive root by β_1. For a root α, we denote the elements of the associated root subgroup X_α by $x_\alpha(t)$, and we put $X'_\alpha = X_\alpha \setminus \{1\}$. We denote the maximal unipotent subgroup of B by U.

For elements in T in exceptional simple groups, we use the notation in [Steinberg 1968, Lemma 19], that is, every element in T can be expressed as a product of $h_{\alpha_i}(t_i)$ for $i = 1, \dots, n$ and nonzero $t_i \in k$, with uniqueness if the group is simply connected. The $h_{\alpha_i}(t_i)$ satisfy the commutation relations

$$h_{\alpha_i}(t)x_\beta(r)h_{\alpha_i}(t^{-1}) = x_\beta(t^{\langle\beta,\alpha_i\rangle}r) \quad \text{for } \beta \in \Phi \text{ and } t, r \in k,$$

where $\langle\beta, \alpha\rangle = \beta(h_\alpha)$ as usual; see [Steinberg 1968].

When G is simple of type A_n, B_n, C_n or D_n, we work with the corresponding matrix groups, and we choose G and T so that the elements in T are diagonal. Let $X_1, \dots, X_l$ be square matrices of size $n_j \geq 1$ for $j = 1, \dots, l$. By $\operatorname{diag}(X_1, \dots, X_l)$ we mean the square matrix of size $\sum_j n_j$ with the blocks $X_1, \dots, X_l$ along its diagonal. As usual, E_{ij} is a square matrix with the entry 1 in the i-th row and j-th column and all other entries 0. We denote by ${}^tM$ the transpose of a matrix M.

We put $W = N(T)/T$, and s_α indicates the reflection corresponding to the root α. Given an element $w \in W$, we denote by $\dot{w}$ a representative of w in $N(T)$.

Let ℓ denote the usual length function on W, and let $\operatorname{rk}(1 - w)$ denote the rank of the endomorphism $1 - w$ in the geometric representation of W.

We shall frequently use these properties of the Bruhat decomposition of G (see [Bourbaki 1981, IV.2.4]):

$$X'_{-\alpha} \subset X'_\alpha s_\alpha T X'_\alpha \subset B s_\alpha B \quad \text{for all } \alpha \in \Phi^+, \tag{1}$$

$$BwBw'B = Bww'B \quad \text{if } \ell(ww') = \ell(w) + \ell(w'). \tag{2}$$

Given an element $x \in G$, we denote by $\mathcal{O}_x$ the conjugacy class of x in G and by H_x the centralizer of x in $H \leq G$. Denote by $Z(K)$ the center of an algebraic group K and by K° its identity component.

For the dimension of unipotent conjugacy classes in arbitrary good characteristic, see [Carter 1985, Chapter 13] and [Premet 2003, Theorem 2.6].

For a conjugacy class $\mathcal{O}$ in G, we denote by $\mathcal{V}$ the set of its B-orbits.

2. Characterizations through the Bruhat decomposition

Here we introduce our characterizations of spherical conjugacy classes.

Definition 2.1. Let G be a connected reductive algebraic group. A homogeneous G-space X is *spherical* if it has a dense orbit for a Borel subgroup of G.

It is well known [Brion 1986; Grosshans 1992; Knop 1995; Vinberg 1986] that $\mathbb{O}$ is a spherical conjugacy class in G if and only if its set of B-orbits $\mathcal{V}$ is finite.

Since $G = \bigcup_{w \in W} BwB$, for every class $\mathbb{O}$ there is a natural map $\phi\colon \mathcal{V} \to W$ associating to $v \in \mathcal{V}$ the element w in the Weyl group of G for which $v \subset BwB$. Besides, there is a unique $w \in W$ for which $BwB \cap \mathbb{O}$ is dense in $\mathbb{O}$, and this element, which we denote by $w_{\mathbb{O}}$, is maximal in $\mathrm{Im}(\phi)$ with respect to the Bruhat ordering [Cantarini et al. 2005, page 32].

There are two characterizations of spherical classes in G.

Theorem 2.2 [Cantarini et al. 2005, Theorem 25; Carnovale 2008, Theorem 4.4]. *A class $\mathbb{O}$ in a connected reductive algebraic group G over an algebraically closed field of zero or good odd characteristic is spherical if and only if there exists v in $\mathcal{V}$ such that $\ell(\phi(v)) + \mathrm{rk}(1 - \phi(v)) = \dim \mathbb{O}$. If this is the case, v is the dense B-orbit and $\phi(v) = w_{\mathbb{O}}$.*

Theorem 2.3 [Carnovale 2008, Theorem 2.7; Carnovale 2009, Theorem 5.7]. *A class $\mathbb{O}$ in a connected reductive algebraic group G over an algebraically closed field of zero or odd, good characteristic is spherical if and only if* $\mathrm{Im}(\phi)$ *contains only involutions in W.*

Since all Borel subgroups and all maximal tori are G-conjugate, the statement in Theorem 2.3 is independent of the choice of B and T. By abuse of notation, we say that $g \in G$ is spherical if its class $\mathbb{O}_g$ is.

Remark 2.4. Let $g \in G$. The B-orbits in $\mathbb{O}_g$ are in one-to-one correspondence with the (B, G_g)-double cosets in G. Therefore if $x \in G$ is such that $G_x = G_g$, then $\mathbb{O}_g$ is spherical if and only if $\mathbb{O}_x$ is. In particular, if $g^2 \in Z(G)$, then g and x are semisimple. If G is affine, by [Borel 1969, Proposition 9.1] the orbit map is separable, so the symmetric variety $G/G_g = G/G_x$ is G-equivariantly isomorphic to $\mathbb{O}_g$ and $\mathbb{O}_x$. By [Springer 1985, Corollary 4.3], the class $\mathbb{O}_x$ is spherical. Motivated by this, we abuse notation when $G_x = G_g$ and $g^2 \in Z(G)$ by saying that $\mathbb{O}_x$ is a symmetric conjugacy class.

Remark 2.5. Regular classes in a reductive algebraic group whose semisimple quotient is not of type rA_1 cannot be spherical. By [Steinberg 1965, Theorem 8.1], regular classes intersect Bruhat cells corresponding to Coxeter elements.

We will frequently use the following observation.

Lemma 2.6. *Let G be a connected reductive algebraic group, let T be a maximal torus in G, and let H be a closed connected reductive subgroup of G containing T. Let $x \in H$ and suppose that $\mathbb{O}_x$ is spherical. Then the H-conjugacy class of x is spherical.*

Proof. Let B_H be a Borel subgroup of H containing T, and let B be a Borel subgroup of G containing B_H. Let y lie in the H-conjugacy class of x. For some $\dot{w} \in N_H(T) = N(T) \cap H$ and for some $b_1, b_2 \in B_H \leq B$, we have

$$y = b_1 \dot{w} b_2 \in B_H N_H(T) B_H \subset BN(T)B.$$

Since $y \in \mathbb{O}_x$, we have $\dot{w}^2 \in T$ by Theorem 2.3. As this holds for every $y \in H$, the H-class of x satisfies the sufficient condition provided by Theorem 2.3. □

As a first application of Lemma 2.6 we have the following statement.

Lemma 2.7. *Let G be a connected reductive algebraic group. Let $g \in G$ with Jordan decomposition $g = su$. If $\mathbb{O}_g$ is spherical, then $\mathbb{O}_s$ and $\mathbb{O}_u$ are spherical in G and the G_s°-class of u is spherical.*

Proof. It is well known that $G_g = G_s \cap G_u$. Therefore, if for a Borel subgroup B of G there are finitely many (B, G_g) double cosets in G, there are finitely many (B, G_s) double cosets and (B, G_u) double cosets in G. Thus if $\mathbb{O}_g$ is spherical, then $\mathbb{O}_s$ and $\mathbb{O}_u$ are also spherical. For the last statement, by [Humphreys 1995, Section 1.12], we have $u \in G_s^\circ$, and we may apply Lemma 2.6 with $H = G_s^\circ$. □

The next lemma helps show that certain classes in a group are not spherical.

Lemma 2.8. *Let G be a connected reductive algebraic group, let T be a maximal torus in G, and let H be a closed, connected, reductive subgroup of G containing T such that its semisimple part is not of type rA_1. Let $x \in H$ and suppose that the H-conjugacy class of x is regular. Then $\mathbb{O}_x$ is not spherical.*

Proof. This is obtained by combining Lemma 2.6 with Remark 2.5. □

3. The classification

From now on G, will be a simple algebraic group. We aim at a classification of spherical conjugacy classes in G in good odd characteristic. The main tools in our classification will be the sufficient condition in Theorem 2.2 and the necessary condition in Theorem 2.3.

If $\pi : G_1 \to G_2$ is a central isogeny between two simple algebraic groups, a conjugacy class $\mathbb{O}_g$ in G_1 is spherical if and only if $\pi(\mathbb{O}_g)$ is spherical. Indeed, let $x \in G_1$, with $G_{1,x}$ its centralizer in G_1 and $G_{2,x}$ the centralizer of $\pi(x)$ in G_2. Also suppose B_1 is a Borel subgroup of G_1. Then $\pi(B_1)$ is a Borel subgroup of G_2, and the $(B_1, G_{1,x})$-double cosets of G_1 are in one-to-one correspondence with the $(B_2, G_{2,x})$-double cosets of G_2. For this reason it is enough to provide the classification for one representative for each isogeny class of simple groups.

By Remark 2.4, if $x, y \in G$ and xy^{-1} is central, then $\mathbb{O}_x$ is spherical if and only if $\mathbb{O}_y$ is. Thus it is enough to provide the classification up to a central element.

If G is of type G_2, Carnovale [2009, Section 2.1] gives the classification in good characteristic; we provide it here for completeness.

Type G_2.

Theorem 3.1. *Let G be of type G_2. The spherical classes are either semisimple or unipotent. The semisimple ones are represented by $h_{\alpha_1}(-1)$ and $h_{\alpha_1}(\zeta)$ for ζ a fixed primitive third root of* 1. *The unipotent ones are those of type A_1 and $\tilde{A}_1$.*

Type A_n. In this section $G = \mathrm{SL}_{n+1}(k)$, B is the subgroup of upper triangular matrices, T is the subgroup of diagonal matrices in G, and U is the unipotent radical of B. For a positive root $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ we have

$$X_\alpha = \{1 + tE_{i,j+1}, t \in k\} \quad \text{and} \quad X_{-\alpha} = {}^tX_\alpha \quad \text{for every } \alpha \in \Phi.$$

Theorem 3.2. *If $n = 1$, all classes in G are spherical. If $n \geq 2$, the spherical classes in G are either semisimple or unipotent up to a central element. The semisimple ones are those corresponding to matrices with at most two distinct eigenvalues, and they are all symmetric. The unipotent ones are those associated with the partitions $(2^m, 1^{n+1-2m})$ for $m = 1, \ldots, [(n+1)/2]$.*

Proof. If $n = 1$, all Bruhat cells correspond to involutions in W, so every class is spherical by Theorem 2.3.

Unipotent classes. Let $n \geq 2$, and let $\mathcal{O} = \mathcal{O}_u$ be a unipotent class. By Jordan theory, we may assume that $u = x_{\alpha_1}(c_1) \cdots x_{\alpha_n}(c_n)$ with $c_i \in \{0, 1\}$. Then u lies in the connected reductive subgroup H generated by T and by $X_{\pm\alpha_i}$ for all i such that $c_i = 1$. By [Steinberg 1965, Lemma 3.2 and Theorem 3.3], u is regular in H. Lemma 2.8 implies that if $\mathcal{O}_u$ is spherical then $c_i c_{i+1} = 0$, so its associated partition is of type $(2^m, 1^{n+1-2m})$. Conversely, let $\mathcal{O}_j$ be the unipotent class corresponding to $(2^j, 1^{n+1-2j})$, with $2j \leq n+1$. Let $\beta_i = \alpha_i + \cdots + \alpha_{n-i+1}$ for $i = 1, \ldots, j$. The element $x_{-\beta_1}(1) \cdots x_{-\beta_j}(1)$ lies in $\mathcal{O}_j$. By (1) and (2) this element lies in $Bs_{\beta_1} \cdots s_{\beta_j}B$, so its B-orbit satisfies the condition in Theorem 2.2, and thus $\mathcal{O}_j$ is spherical.

Semisimple classes. Let $s = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots, \lambda_l I_{n_l})$ for distinct scalars λ_i. If $l > 2$, then s is conjugate to $t = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, t_1)$ for some invertible diagonal submatrix t_1. Then t lies in the connected reductive subgroup $H = \langle T, X_{\pm\alpha_1}, X_{\pm\alpha_2}\rangle$, and it is regular therein. It follows from Lemma 2.8 that if $\mathcal{O}_s$ is spherical semisimple, then s has at most 2 eigenvalues. Conversely, suppose that $s \in T$ has 2 eigenvalues. We may assume $s = \mathrm{diag}(\lambda I_m, \mu I_{n+1-m})$. Let ζ be a primitive $2(n+1)$-st root of unity if $n+1-m$ is odd, and let $\zeta = 1$ if $n+1-m$ is even. Let also $s_0 = \mathrm{diag}(\zeta I_m, -\zeta I_{n+1-m})$. Then $s_0^2 \in Z(G)$ and $G_s = G_{s_0}$. By Remark 2.4 the class $\mathcal{O}_s$ is symmetric and hence spherical.

Mixed classes. We now show that there is no spherical element x with Jordan decomposition $x = su$ such that $s \notin Z(G)$ and $u \neq 1$. Were this the case, we could assume by Lemma 2.7 that $s = \operatorname{diag}(\lambda I_m, \mu I_{n+1-m})$ with $m \geq 2$ and that $u \in U \cap G_s = \langle X_{\alpha_i}, i \neq m \rangle$.

We could then choose $u = x_{\alpha_1}(t_1) \cdots x_{\alpha_{m-1}}(t_{m-1}) x_{\alpha_{m+1}}(t_{m+1}) \cdots x_{\alpha_n}(t_n)$ with $t_i t_{i+1} = 0$ because u is spherical by Lemma 2.7. If u is nontrivial, we may assume that t_{m-1} or t_{m+1} is nonzero. Put $J = \{i \mid t_i \neq 0\}$ and $H = \langle T, X_{\pm\alpha_m}, X_{\pm\alpha_i} \rangle_{i \in J}$. Then su is regular in H. Since H contains at least a subgroup of type A_2 we may conclude using Lemma 2.8. □

Type C_n. Let us view $G = \mathrm{Sp}_{2n}(k)$ as the subgroup of $\mathrm{GL}_{2n}(k)$ of matrices preserving the bilinear form whose matrix is $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ in the canonical basis of k^{2n}. We choose B as the subgroup of G of matrices of the form $\left(\begin{smallmatrix} A & AX \\ 0 & {}^tA^{-1} \end{smallmatrix}\right)$, where A is an invertible upper triangular matrix, ${}^tA^{-1}$ is its inverse transpose, and X is a symmetric matrix. The torus T is the subgroup of diagonal matrices in B. We have

$$X_{\alpha_i} = \{I + tE_{i,i+1} - tE_{n+i+1,n+i}, t \in k\} \quad \text{for } i = 1, \ldots, n-1,$$
$$X_{\alpha_n} = \{I + tE_{n,2n}, t \in k\},$$

and $X_{-\alpha} = {}^tX_\alpha$ for every $\alpha \in \Phi$. We recall that if $g, h \in \mathrm{Sp}_{2n}(k)$ are $\mathrm{GL}_{2n}(k)$-conjugate they are $\mathrm{Sp}_{2n}(k)$-conjugate [Springer and Steinberg 1970, IV.2.15(ii)]. It is well known that unipotent classes in G are parametrized through Jordan theory by partitions where odd terms occur pairwise [Humphreys 1995, Section 7.11].

Theorem 3.3. *Let $G = \mathrm{Sp}_{2n}(k)$ for $n \geq 2$. The nontrivial spherical semisimple classes are represented by $\sigma_l = \operatorname{diag}(-I_l, I_{n-l}, -I_l, I_{n-l})$ for $l = 1, \ldots, n-1$; by $a_\lambda = \operatorname{diag}(\lambda I_n, \lambda^{-1} I_n)$; and, up to a sign, by $c_\lambda = \operatorname{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$ for $\lambda \in k$ with $\lambda^2 \neq 0, 1$. The unipotent ones are those associated with the partitions $(2^m, 1^{2n-2m})$ for $m = 1, \ldots, n$. The spherical classes that are neither semisimple nor unipotent up to a sign are represented by the elements $\sigma_l u$, where $u \in G_{\sigma_l} \cong \mathrm{Sp}_{2l}(k) \times \mathrm{Sp}_{2n-2l}(k)$ is unipotent and corresponds to the partition $(2, 1^{2n-2})$.*

Proof. Semisimple classes. Let $s \in T$, and let Λ be the set of eigenvalues of s.

Let us first suppose that $|\Lambda| \geq 4$. If $n = 2$, then s is a regular element, and hence it is not spherical. Let $n \geq 3$.

If $\{\pm 1\} \subset \Lambda$, then s is conjugate to $s' = \operatorname{diag}(\lambda, 1, -1, t, \lambda^{-1}, 1, -1, t^{-1})$ for some invertible diagonal submatrix t and some nonzero $\lambda \in k$ with $\lambda^2 \neq 1$.

If $|\{\pm 1\} \cap \Lambda| = 1$, then, since eigenvalues come with their inverse, $|\Lambda| \geq 5$ and s is conjugate to $s' = \operatorname{diag}(\lambda, \mu, \pm 1, t, \lambda^{-1}, \mu^{-1}, \pm 1, t^{-1})$ for some invertible diagonal submatrix t and some $\lambda \neq \mu \in k$ with $\lambda^2 \neq 1 \neq \mu^2$.

If $\{\pm 1\} \cap \Lambda = \varnothing$, then either $|\Lambda| \geq 6$ or there are two reciprocally inverse eigenvalues with multiplicity at least 2. In both cases, the matrix s is conjugate to

$s' = \mathrm{diag}(\lambda, \mu, \nu, t, \lambda^{-1}, \mu^{-1}, \nu^{-1}, t^{-1})$ for some invertible diagonal submatrix t and some distinct $\lambda, \mu, \nu \in k$ with $\lambda^2, \mu^2, \nu^2 \neq 1$ and ν possibly equal to λ^{-1}.

In all these cases, the element s' is regular in $H = \langle T, X_{\pm\alpha_1}, X_{\pm\alpha_2}\rangle$; therefore by Lemma 2.8 the class $\mathbb{O}_s$ cannot be spherical.

Let us now suppose that $|\Lambda| = 3$. Then $\Lambda = \{\eta, \lambda, \lambda^{-1}\}$ with $\eta^2 = 1$ and $\lambda^2 \neq 1$. If the multiplicity of $\lambda^{\pm 1}$ is greater than 1, then s is conjugate to some $r' = \mathrm{diag}(\lambda, \lambda^{-1}, 1, r_1, \lambda^{-1}, \lambda, 1, r_1^{-1})$ with r_1 an invertible, diagonal submatrix. The element r' lies and is regular in the subgroup H above described. By Lemma 2.8 the class $\mathbb{O}_s$ cannot be spherical. On the other hand, if $\Lambda = \{\lambda^{\pm 1}, 1\}$ with the multiplicity of $\lambda^{\pm 1}$ equal to 1, then $\mathbb{O}_s$ is spherical. Indeed, the representative of such a class in [Cantarini et al. 2005, Theorem 15, page 42] works also in odd characteristic and its B-orbit satisfies the condition of Theorem 2.2.

Now assume that $|\Lambda| = 2$. Then either $\Lambda = \{\pm 1\}$ so that $\mathbb{O}_s$ is symmetric, or $\Lambda = \{\lambda, \lambda^{-1}\}$ for $\lambda^2 \neq 1$ so that s is conjugate to $a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$, whose centralizer is independent of λ in the given range. Since $a_\zeta^2 \in Z(G)$ if $\lambda = \zeta$ is a primitive fourth root of 1, we may apply Remark 2.4 and conclude that a_λ is spherical.

Unipotent classes. Let $\mathbb{O}_u$ be a unipotent class and let $\underline{\lambda}$ be its associated partition. Let $\underline{\mu} = (\mu_1, \dots, \mu_l)$ be obtained by taking a representative of each term occurring pairwise in $\underline{\lambda}$ and let $\underline{\nu} = (\nu_1, \dots, \nu_m)$ be obtained by taking the remaining even terms without repetition in $\underline{\lambda}$, so that $2n = |\underline{\nu}| + 2|\underline{\mu}|$. A representative u' of $\mathbb{O}_u$ can be taken in the subgroup isomorphic to

$$\mathrm{Sp}_{2\mu_1}(k) \times \cdots \times \mathrm{Sp}_{2\mu_l}(k) \times \mathrm{Sp}_{\nu_1}(k) \times \cdots \times \mathrm{Sp}_{\nu_m}(k)$$

obtained by repeating the immersion of $\mathrm{Sp}_{2d_1}(k) \times \mathrm{Sp}_{2d_2}(k)$ into $\mathrm{Sp}_{2(d_1+d_2)}(k)$ given by

$$\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} A_1 & & B_1 & \\ & A_2 & & B_2 \\ C_1 & & D_1 & \\ & C_2 & & D_2 \end{pmatrix}.$$

The component of u' in $\mathrm{Sp}_{\nu_j}(k)$ corresponds to the partition (ν_j) and is thus regular in $\mathrm{Sp}_{\nu_j}(k)$, whereas the component of u' in $\mathrm{Sp}_{2\mu_i}(k)$ can be taken to lie and be regular in the subgroup isomorphic to $\mathrm{SL}_{\mu_i}(k)$ obtained by the immersion mapping M to $\mathrm{diag}(M, {}^tM^{-1})$. Therefore, u' is regular in the semisimple group

$$\mathrm{SL}_{\mu_1}(k) \times \cdots \times \mathrm{SL}_{\mu_l}(k) \times \mathrm{Sp}_{\nu_1}(k) \times \cdots \times \mathrm{Sp}_{\nu_m}(k).$$

By Remark 2.5 if u is spherical, we have $\mu_i \leq 2$ and $\nu_j \leq 2$ for every i and j. Conversely, let $\underline{\lambda} = (2^j, 1^{2n-2j})$, and let $\mathbb{O}_j$ be the unipotent class associated with $\underline{\lambda}$. Let $\beta_q = 2\alpha_q + \cdots + 2\alpha_{n-1} + \alpha_n$ for $q = 1, \dots, n-1$ and $\beta_n = \alpha_n$. The element

$$x_{-\beta_1}(1)\cdots x_{-\beta_j}(1) = \begin{pmatrix} I_n & \\ X_j & I_n \end{pmatrix} \quad \text{with } X_j = \operatorname{diag}(I_j, 0_{n-j})$$

lies in $Bs_{\beta_1}\cdots s_{\beta_j}B$ by (1) and (2). Since it also lies in $\mathbb{O}_j$, its B-orbit satisfies the condition in Theorem 2.2 for $\mathbb{O}_j$; see [Cantarini et al. 2005, Theorem 12, page 36]. Thus, $\mathbb{O}_j$ is spherical.

Mixed classes. Let $g = su$ be the Jordan decomposition of a spherical element in G with $s \notin Z(G)$ and $u \neq 1$. Then $\mathbb{O}_s$ is spherical and we may assume s equals a_λ, c_λ or σ_l for some l. The case $s = a_\lambda$ is ruled out because $\dim \mathbb{O}_{a_\lambda u} > \dim \mathbb{O}_{a_\lambda} = \dim B$, so $\mathbb{O}_{a_\lambda u}$ cannot have a dense B-orbit.

Assume that $s = c_\lambda$. Then $u \in G_s \cong k^* \times \mathrm{Sp}_{2n-2}(k)$ and it is spherical therein, so it corresponds to a partition $(2^m, 1^{2n-2-2m})$ for some $m \geq 1$. The class $\mathbb{O}_{c_\lambda u}$ is represented by $c_\lambda x_{\beta_2}(1)\cdots x_{\beta_{m+1}}(1)$, with notation as before. Such an element is regular in the subgroup $H = \langle T, X_{\pm\alpha_1}, X_{\pm\beta_i}, i = 2, \ldots, m+1\rangle$. This case is thus excluded by Lemma 2.8 because the semisimple part of H is of type $C_2 \times (m-1)A_1$.

It follows that $s = \sigma_l$ for some l. Then G_s is generated by $X_{\pm\alpha_i}$ for $i \neq l$ and $X_{\pm\beta_l}$. We have $u = (u_1, u_2) \in G_s \cong \mathrm{Sp}_{2l}(k) \times \mathrm{Sp}_{2n-2l}(k)$, and it is spherical therein. Then u_1 and u_2 are spherical in the respective components. We claim that u_1 and u_2 cannot be both nontrivial. If on the contrary u_1 corresponded to the partition $\underline{\lambda} = (2^a, 1^{2l-2a})$ and u_2 corresponded to the partition $\underline{\mu} = (2^b, 1^{2n-2l-2b})$ with $a, b \geq 1$, the G_s-class of u_1 would be represented by $u_1' = x_{\beta_{l-a+1}}(1)\cdots x_{\beta_l}(1)$ and the G_s-class of u_2 would be represented by $u_2' = x_{\beta_{l+1}}(1)\cdots x_{\beta_{l+b}}(1)$. It is not hard to verify that $\sigma_l u_1' u_2'$ is regular in $\langle T, X_{\pm\alpha_l}, X_{\pm\beta_i}, i = l-a+1, \ldots, l+b\rangle$, whose semisimple part is of type $(a+b-2)A_1 + C_2$. By Lemma 2.8 this option is excluded, and we have $a + b \leq 1$; hence at least one of the u_i is trivial.

There is no loss of generality in assuming that $u_1 = 1$. We claim that the partition $\underline{\mu} = (2^b, 1^{2n-2l-2b})$ associated with u_2 has no repeated 2. Let $b = 2h + j$ with $j = 0, 1$ according to the parity of b, and assume that $h \geq 1$. The G_s-class of u_2 is represented by $u_2' = x_{\alpha_{l+1}}(1)x_{\alpha_{l+3}}(1)\cdots x_{\alpha_{l+2h-1}}(1)x_{\beta_{l+2h+1}}(j)$. The element $\sigma u_2'$ is regular in $\langle T, X_{\pm\alpha_l}, X_{\pm\alpha_{l+1}}, X_{\pm\alpha_{l+3}}, \ldots, X_{\pm\alpha_{2h-1}}, X_{\pm\beta_{2h+1}}(j)\rangle$, whose semisimple part is of type $A_2 \times h\tilde{A}_1 \times jA_1$, where $\tilde{A}_1$ corresponds to a short root. By Lemma 2.8 the claim is proved.

Conversely, for all classes of type $\sigma_l u$ with $u \in G_{\sigma_l}$ corresponding to the partition $(2, 1^{2n-2})$, the representative in [Cantarini et al. 2005, Theorem 21, page 50] is defined in odd characteristic and its B-orbit satisfies the condition of Theorem 2.2.

□

Type D_n. Let $n \geq 4$, and view $\mathrm{O}_{2n}(k)$ as the subgroup of $\mathrm{GL}_{2n}(k)$ of matrices preserving the bilinear form whose matrix is $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ in the canonical basis of k^{2n}, so that $G = \mathrm{SO}_{2n}(k)$ is viewed as the subgroup of such matrices of determinant 1. We choose B as the subgroup of G of matrices of the form

$$\begin{pmatrix} A & AX \\ 0 & {}^tA^{-1} \end{pmatrix},$$

where A is an invertible upper triangular matrix, ${}^tA^{-1}$ is its inverse transpose and X is a skew-symmetric matrix. We fix $T \subset B$ as its subgroup of diagonal matrices.

We have

$$X_{\alpha_i} = \{I + tE_{i,i+1} - tE_{n+i+1,n+i},\, t \in k\} \quad \text{for } i = 1, \ldots, n-1,$$
$$X_{\alpha_n} = \{I + tE_{n-1,2n} - tE_{n,2n-1},\, t \in k\}$$

and ${}^tX_{-\alpha} = X_\alpha$ for every $\alpha \in \Phi$.

We recall that if $g, h \in G$ are $\mathrm{GL}_{2n}(k)$-conjugate, they are $\mathrm{O}_{2n}(k)$-conjugate [Springer and Steinberg 1970, IV.2.15(ii)] but not necessarily G-conjugate. However, conjugation by an element in $\mathrm{O}_{2n}(k)$ determines an automorphism ψ of G, so if $h = \psi(g)$, the class $\mathcal{O}_g$ is spherical if and only if $\psi(\mathcal{O}_g) = \mathcal{O}_{\psi(g)} = \mathcal{O}_h$ is. For this reason, in what follows we will sometimes replace an element $g \in G$ by a $\mathrm{GL}_{2n}(k)$-conjugate h lying in G.

To list a representative for each spherical conjugacy class, we will then have to verify whether an $\mathrm{O}_{2n}(k)$-class splits into two G-classes or not. We recall that such a class splits into two classes if and only if the $\mathrm{O}_{2n}(k)$-centralizer of a representative is contained in G.

It is well known that the even terms occur pairwise in the partition $\underline{\lambda}$ associated with a unipotent conjugacy class in G via Jordan theory. Moreover, a unipotent $\mathrm{O}_{2n}(k)$-class splits into two G-classes only if n is even and the associated partition has only even terms [Humphreys 1995, Section 7.11].

Theorem 3.4. *Let $G = \mathrm{SO}_{2n}(k)$ for $n \geq 4$. The spherical classes in G are either semisimple or unipotent up to a central element. The nontrivial semisimple ones are those represented by*

$$\sigma_l = \mathrm{diag}(-I_l, I_{n-l}, -I_l, I_{n-l}) \quad \textit{for } l = 1, \ldots, n-1;$$
$$c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1}) \quad \textit{for } \lambda^2 \neq 0, 1, \textit{ up to a sign},$$

and the pairs of $\mathrm{SO}_{2n}(k)$-classes into which the $\mathrm{O}_{2n}(k)$-class represented by $a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ splits, for $\lambda^2 \neq 0, 1$. The unipotent ones are those associated with the partitions

$$(2^{2m}, 1^{2n-4m}) \quad \textit{for } m = 1, \ldots, [n/2],$$
$$(3, 2^{2m}, 1^{2n-3-4m}) \quad \textit{for } m = 1, \ldots, [n/2]-1$$

and only $(2^{2(n/2)})$ for n even corresponds to two distinct conjugacy classes.

Proof. Semisimple classes. Let $s \in T$, and let Λ be its set of eigenvalues. Adapting the analysis in type C_n and replacing s by a $\mathrm{GL}_{2n}(k)$-conjugate if necessary, we see that if s is spherical, then $|\Lambda| \leq 3$ and if $|\Lambda| = 3$, then, up to a sign, $\Lambda = \{\lambda, \lambda^{-1}, 1\}$ for some $\lambda^2 \neq 1$ and the multiplicity of λ and λ^{-1} is equal to 1.

On the other hand, if $\Lambda = \{\lambda, \lambda^{-1}, 1\}$ with the multiplicity of λ and λ^{-1} equal to 1, then s is $\mathrm{GL}_{2n}(k)$-conjugate to $c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$. Its centralizer G_{c_λ} is equal to the identity component H° of the centralizer H of the involution $\sigma_1 = \mathrm{diag}(-1, I_{n-1}, -1, I_{n-1})$. By [Borel 1969, Proposition 9.1], we have $\mathcal{O}_{c_\lambda} \cong G/G_{c_\lambda} = G/H^\circ$. Since the index of H° in H is finite, $\mathcal{O}_{c_\lambda}$ is spherical if and only if $G/H \cong \mathrm{O}_{\sigma_1}$ is, and therefore $\mathcal{O}_{c_\lambda}$ is spherical. The centralizer in $\mathrm{O}_{2n}(k)$ of c_λ contains the matrix

$$M = \begin{pmatrix} I_{n-1} & & \mathrm{O}_{n-1} & \\ & 0 & & 1 \\ \mathrm{O}_{n-1} & & I_{n-1} & \\ & 1 & & 0 \end{pmatrix},$$

so each c_λ represents a single spherical $\mathrm{SO}_{2n}(k)$-conjugacy class.

Let now $|\Lambda| = 2$. If $\Lambda = \{\pm 1\}$, then $s^2 = 1$ and $\mathcal{O}_s$ is symmetric. The $\mathrm{GL}_{2n}(k)$-class of s is represented by $\sigma_l = \mathrm{diag}(-I_l, I_{n-l}, -I_l, I_{n-l})$ for some $l = 1, \ldots, n-1$. The centralizer in $\mathrm{O}_{2n}(k)$ of each σ_l contains the matrix M above described, so each σ_l represents a single spherical $\mathrm{SO}_{2n}(k)$-conjugacy class.

If $\Lambda = \{\lambda, \lambda^{-1}\}$ with $\lambda^2 \neq 1$, we may assume that $s = a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ whose centralizer is independent of λ in the given range. Since $a_\zeta^2 \in Z(G)$ for ζ a primitive fourth root of 1, by Remark 2.4 all those classes are symmetric and hence spherical. The $\mathrm{O}_{2n}(k)$-centralizer of a_λ consists of all matrices $\mathrm{diag}(A, {}^t A^{-1})$ for some invertible $n \times n$ matrix A and hence is contained in $\mathrm{SO}_{2n}(k)$. Therefore the $\mathrm{O}_{2n}(k)$-class of each a_λ splits into two spherical $\mathrm{SO}_{2n}(k)$-conjugacy classes.

Unipotent classes. By the discussion of $\mathrm{GL}_{2n}(k)$-conjugacy, it suffices to consider a class for each admissible partition.

Let u be a unipotent element in G, with associated partition $\underline{\lambda}$. Obtain $\underline{\mu} = (\mu_1, \ldots, \mu_l)$ by taking a representative of each term occurring pairwise in $\underline{\lambda}$, and $\underline{\nu} = (\nu_1, \ldots, \nu_m)$ by taking the remaining distinct odd terms so that $2n = 2|\underline{\mu}| + |\underline{\nu}|$. A representative u' of $\mathcal{O}_u$ can be taken in the subgroup isomorphic to $\mathrm{SO}_{\nu_1+\nu_2}(k) \times \cdots \times \mathrm{SO}_{\nu_{m-1}+\nu_m}(k) \times \mathrm{SO}_{2\mu_1}(k) \times \cdots \times \mathrm{SO}_{2\mu_l}(k)$ obtained by repeatedly immersing $\mathrm{SO}_{2d_1}(k) \times \mathrm{SO}_{2d_2}(k)$ into $\mathrm{SO}_{2(d_1+d_2)}(k)$ by

$$\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} A_1 & & B_1 & \\ & A_2 & & B_2 \\ C_1 & & D_1 & \\ & C_2 & & D_2 \end{pmatrix}.$$

The component of u' in $\mathrm{SO}_{\nu_i+\nu_{i+1}}(k)$ is associated with the partition (ν_i, ν_{i+1}), whereas the component of u' in $\mathrm{SO}_{2\mu_i}(k)$ can be chosen to lie and be regular in the subgroup isomorphic to $\mathrm{SL}_{\mu_i}(k)$ obtained by the immersion $A \mapsto \mathrm{diag}(A, {}^tA^{-1})$. Thus u' lies in $\mathrm{SO}_{\nu_1+\nu_2}(k) \times \cdots \times \mathrm{SO}_{\nu_{m-1}+\nu_m}(k) \times \mathrm{SL}_{\mu_1}(k) \times \cdots \times \mathrm{SL}_{\mu_l}(k)$. A class in a semisimple group is spherical if and only if its projection onto each simple component is. By Remark 2.5 applied to $\mathrm{SL}_{\mu_1}(k) \times \cdots \times \mathrm{SL}_{\mu_l}(k)$, we see that if u is spherical, then $\mu_i \leq 2$ for all i. We now show that under the same assumption, $\nu_1 \leq 3$ so that $\underline{\nu}$ is either $(3, 1)$ or the empty partition. It is enough to analyze the $\mathrm{SO}_{\nu_1+\nu_2}(k)$-class $\mathbb{O}$ of the component of u'. Let $\nu_1 = 2l + 1$ and $\nu_2 = 2j - 1$ with $l \geq j \geq 1$, and let $\gamma_1, \ldots, \gamma_{l+j}$ be the simple roots of $\mathrm{SO}_{\nu_1+\nu_2}(k)$. The class $\mathbb{O}$ is represented by $x = \mathrm{diag}(A, {}^tA^{-1})\left(\begin{smallmatrix} I & X \\ & I \end{smallmatrix}\right)$, where

$$A = \begin{pmatrix} 1 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & 1 \\ & & 1 & 1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0_{j-1} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 0_{l-1} \end{pmatrix}.$$

Since $\mathrm{diag}(A, {}^tA^{-1})$ lies in $X'_{-\gamma_1} \cdots X'_{-\gamma_{j+l-1}}$ and $\left(\begin{smallmatrix} I & X \\ & I \end{smallmatrix}\right)$ lies in B, it follows from (1) and (2) that x lies in a cell corresponding to an involution only if $j + l \leq 2$, whence the claim.

Conversely, let $\mathbb{O}_u$ be a unipotent class corresponding to $(2^{2m}, 1^{2n-4m})$ or to $(3, 2^{2m}, 1^{2n-3-4m})$. Cantarini et al. [2005, Theorem 12, pages 37–38] give matrices that represent these classes also when $\mathrm{char}(k)$ is odd and their B-orbits satisfy the condition in Theorem 2.2.

Mixed classes. We show that there is no spherical element with Jordan decomposition $g = su$ with $s \notin Z(G)$ and $u \neq 1$. We may assume that $s = c_\lambda, \sigma_l$, because $\dim B = \dim \mathbb{O}_{a_\lambda} < \dim \mathbb{O}_{a_\lambda u}$.

The subgroup $G^\circ_{c_\lambda}$ is of type $D_{n-1} \times k^*$ and is generated by T and the root subgroups $X_{\pm\alpha_2}, \ldots, X_{\pm\alpha_n}$. The element $u \in G^\circ_{c_\lambda}$ corresponds to a partition $\underline{\pi}$ of $2n - 2$ from which we may construct, as before, the partitions $\underline{\mu}$ and $\underline{\nu}$. Since u is spherical in $G^\circ_{c_\lambda}$, we have $\underline{\mu} = (2^{2a}, 1^{2b})$ with a possibly zero and $\underline{\nu} = (3, 1)$ or trivial. The G_{c_λ}-class of u may be represented by the element

$$u' = x_{\alpha_2}(1)x_{\alpha_4}(1) \cdots x_{\alpha_{2a}}(1)x_{\alpha_{2a+2}}(j)x_{\alpha_{2a+2}+2\alpha_{2a+3}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}(j),$$

where $j = 1$ if $\underline{\nu} = (3, 1)$ and $j = 0$ if $\underline{\nu}$ is trivial. Then $c_\lambda u'$ is regular in $\langle T, X_{\pm\alpha_1}, X_{\pm\alpha_{2l}}, x_{\alpha_{2a+2}}(j), x_{\alpha_{2a+2}+2\alpha_{2a+3}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n}(j)\rangle_{l=1,\ldots,a}$. We may thus apply Lemma 2.8 to deduce that s cannot be equal to c_λ.

Let then $s = \sigma_l$ for some l. The identity component of G_{σ_1} is equal to G_{c_λ}, and we may use the argument above to show that $\sigma_1 u$ cannot be spherical. Let $l \geq 2$.

Then $G^\circ_{\sigma_l} \cong \mathrm{SO}_{2l}(k) \times \mathrm{SO}_{2n-2l}(k)$ and it corresponds to the roots

$$\alpha_1, \dots, \alpha_{l-1}, \alpha_{l+1} + 2\alpha_{l+2} + \cdots + \alpha_{n-1} + \alpha_n, \alpha_{l+1}, \dots, \alpha_n.$$

Let $u = (u_1, u_2) \in \mathrm{SO}_{2l}(k) \times \mathrm{SO}_{2n-2l}(k) = G^\circ_{\sigma_l}$. Since $G_{\sigma_l u} = G_{\sigma_l} \cap G_{u_1} \cap G_{u_2}$ is contained in $G_{\sigma_l u_i}$ for $i = 1, 2$, it is enough to show that $\sigma_l u_i$ is not spherical. We will do so for u_2, the other case being similar. Let $\underline{\lambda}$ be the partition associated with u_2, and let $\underline{\mu}$ and $\underline{\nu}$ be as above. We may find a representative u_2' in the $\mathrm{SO}_{2n-2l}(k)$-class of u_2 lying and being regular in a subgroup H constructed as above for $s = c_\lambda$. If $u_2 \neq 1$, the subgroup H contains the root subgroups $X_{\pm\alpha_{l+1}}$ and $\sigma_l u_2'$ is regular in $H' = \langle T, H, X_{\pm\alpha_l}\rangle$. This proves the claim. □

Type B_n. Let $n \geq 2$. View $\mathrm{O}_{2n+1}(k)$ as the subgroup of $\mathrm{GL}_{2n+1}(k)$ of matrices preserving the bilinear form whose matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$$

in the canonical basis of k^{2n+1}, so that $G = \mathrm{SO}_{2n+1}(k)$ is the subgroup of such matrices with unit determinant. Fix B to be the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & {}^t\gamma \\ -A\gamma & A & AX \\ 0 & 0 & {}^tA^{-1} \end{pmatrix},$$

where A is an invertible upper triangular matrix, γ is a column in k^n and the symmetric part of X is $-(1/2)\gamma\,{}^t\gamma$. We fix $T \subset B$ as its subgroup of diagonal matrices.

We have $X_{\alpha_i} = \{I + tE_{i+1,i+2} - tE_{n+i+2,n+i+1}, t \in k\}$ for $i = 1, \dots, n-1$, $X_{\alpha_n} = \{I + tE_{1,2n+1} - tE_{n+1,1}\}$, and $X_{-\alpha} = {}^tX_\alpha$ for every $\alpha \in \Phi$.

If $g, h \in G$ are $\mathrm{GL}_{2n+1}(k)$-conjugate, then they are also $\mathrm{O}_{2n+1}(k)$-conjugate by [Springer and Steinberg 1970, IV.2.15(ii)]. Thus, they are G-conjugate because $\mathrm{O}_{2n+1}(k) = G \cup (-I_{2n+1})G$. Partitions in which even terms occur pairwise parametrize unipotent conjugacy classes in G [Humphreys 1995, Section 7.11].

We shall frequently use the fact that the group $\mathrm{SO}_{2n}(k)$ may be embedded into G through the map ι defined by $X \mapsto \mathrm{diag}(1, X)$. Denote the image of ι by K.

Theorem 3.5. *Let $G = \mathrm{SO}_{2n+1}(k)$. The spherical semisimple classes in G are represented by*

$$\begin{aligned} \rho_l &= \mathrm{diag}(1, -I_l, I_{n-l}, -I_l, I_{n-l}) \quad \textit{for } l = 1, \dots, n, \\ d_\lambda &= \mathrm{diag}(1, \lambda, I_{n-1}, \lambda^{-1}, I_{n-1}), \\ b_\lambda &= \mathrm{diag}(1, \lambda I_n, \lambda^{-1} I_n) \qquad \textit{with } \lambda^2 \neq 0, 1. \end{aligned}$$

The unipotent ones are those associated with $(2^{2m}, 1^{2n+1-4m})$ *for* $m = 1, \dots, [n/2]$ *and* $(3, 2^{2m}, 1^{2n-2-4m})$ *for* $m = 1, \dots, [(n-1)/2]$. *The spherical classes that are neither semisimple nor unipotent are represented by* $\rho_n u$, *where* $u \in G^{\circ}_{\rho_n} \cong \mathrm{SO}_{2n}(k)$ *is a unipotent element associated with* $(2^{2m}, 1^{2n-4m})$ *for* $m = 1, \dots, [n/2]$.

Proof. Semisimple classes. Let $s \in T$ be a spherical element in G, and let Λ be its set of eigenvalues. By the description of T, we always have $1 \in \Lambda$. By Lemma 2.6 applied to K and Theorem 3.4, we see that $|\Lambda| \leq 4$. We claim that $|\Lambda| < 4$. Assume that $|\Lambda| = 4$. Then $-1 \in \Lambda$ and s is conjugate to $s' = \mathrm{diag}(1, \lambda, -1, t, \lambda^{-1}, -1, t^{-1})$ for some invertible diagonal submatrix t and some scalar λ with $\lambda^2 \neq 1$. Thus s' is regular in $\langle T, X_{\pm\alpha_1}, X_{\pm(\alpha_2+\alpha_3+\cdots+\alpha_n)}\rangle$ whose semisimple part is of type B_2, and by Lemma 2.8 we have the claim. It follows that $|\Lambda| = 2, 3$. If $|\Lambda| = 2$, the element s is conjugate to some involution $\rho_l = \mathrm{diag}(1, -I_l, I_{n-l}, -I_l, I_{n-l})$ for some $l = 1, \dots, n$; hence it is spherical. If $|\Lambda| = 3$, then $\Lambda = \{1, \lambda, \lambda^{-1}\}$ and the multiplicities of λ and 1 cannot be both greater than 1, by Lemma 2.6 applied to K and the discussion in Theorem 3.4 for spherical semisimple elements. Thus, s is conjugate either to $b_\lambda = \mathrm{diag}(1, \lambda I_n, \lambda^{-1} I_n)$ or to $d_\lambda = \mathrm{diag}(1, \lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$, for $\lambda^2 \neq 1, 0$. A representative of $\mathcal{O}_{b_\lambda}$ satisfying the condition in Theorem 2.2 is found in [Cantarini et al. 2005, Theorem 15, page 44] and it is well defined in odd characteristic too, so b_λ is indeed spherical. Moreover, $G_{d_\lambda} = G^{\circ}_{\rho_1}$. Hence d_λ is also spherical because ρ_1 is and the index of $G^{\circ}_{\rho_1}$ in G_{ρ_1} is finite.

Unipotent classes. Let u be a spherical unipotent element in G associated with the partition $\underline{\lambda}$. Let $\underline{\mu}$ and $\underline{\nu}$ be constructed as in Theorem 3.4 with $2n+1 = 2|\underline{\mu}| + |\underline{\nu}|$. We may find a representative u' of $\mathcal{O}_u$ in a subgroup isomorphic to $\mathrm{SO}_{\nu_1}(k) \times \mathrm{SO}_{\nu_2+\nu_3}(k) \times \cdots \times \mathrm{SO}_{\nu_{m-1}+\nu_m}(k) \times \mathrm{SO}_{2\mu_1}(k) \times \cdots \times \mathrm{SO}_{2\mu_l}(k)$. Such a subgroup can be obtained using the embeddings in the proof of Theorem 3.4 and the embedding of $\mathrm{SO}_{2d_1+1}(k) \times \mathrm{SO}_{2d_2}(k)$ into $\mathrm{SO}_{2(d_1+d_2)+1}(k)$ given by

$$\left(\begin{pmatrix} 1 & \alpha_1 & \beta_1 \\ \gamma_1 & A_1 & B_1 \\ \delta_1 & C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} 1 & \alpha_1 & & \beta_1 & \\ \gamma_1 & A_1 & & B_1 & \\ & & A_2 & & B_2 \\ \delta_1 & C_1 & & D_1 & \\ & & C_2 & & D_2 \end{pmatrix}.$$

The component of u' in $\mathrm{SO}_{\nu_1}(k)$ corresponds to (ν_1), so it is regular therein. Hence, its $\mathrm{SO}_{\nu_1}(k)$-class is spherical only if $\nu_1 \leq 3$. Therefore $\underline{\nu} = (3)$ or $\underline{\nu} = (1)$. Moreover, as in Theorem 3.4, the component in $\mathrm{SO}_{2\mu_j}(k)$ can be chosen to lie and be regular in a subgroup isomorphic to $\mathrm{SL}_{\mu_j}(k)$, forcing $\mu_i \leq 2$ for every i. Conversely, for a unipotent class associated with $(2^m, 1^{2n+1-4m})$ or $(3, 2^{2m}, 1^{2n-4m-2})$, the representatives in [Cantarini et al. 2005, Theorem 12, pages 38–39] are well defined in odd characteristic, and the corresponding B-orbits satisfy the condition in Theorem 2.2.

Mixed classes. Let $g = su$ be the Jordan decomposition of a spherical element in G with $s, u \neq 1$. If $s = b_\lambda$ for some λ, then u and b_λ lie in $G_{b_\lambda} \subset K$. By Lemma 2.6 the element su would be spherical in K, but this is excluded by Theorem 3.4.

If $s = \rho_l$ for some l, its centralizer is isomorphic to $\mathrm{SO}_{2l}(k) \times \mathrm{SO}_{2n-2l+1}(k)$ and it corresponds to the roots $\alpha_1, \ldots, \alpha_{l-1}, \alpha_l + 2\alpha_{l+1} + \cdots + 2\alpha_n, \alpha_{l+1}, \ldots, \alpha_n$. Then $u = (u_1, u_2) \in \mathrm{SO}_{2l}(k) \times \mathrm{SO}_{2n-2l+1}(k)$.

First assume that u_2 corresponds to the partition $(3, 2^{2a})$ of $2n - 2l + 1$, so that $n - l$ is odd. We claim that $\rho_l u_2$ is not spherical. Then, since $G_{\rho_l u} \subset G_{\rho_l u_2}$, we may conclude that the class $\mathbb{O}_{\rho_l u}$ cannot be spherical in this case. The $\mathrm{SO}_{2n-2l+1}(k)$-class of u_2 may be represented by the element $u_2' = x_{\alpha_{l+1}}(1) x_{\alpha_{l+3}}(1) \cdots x_{\alpha_n}(1)$ so that $\rho_l u_2'$ is regular in $\langle T, X_{\pm\alpha_l}, X_{\pm\alpha_{l+i}} \rangle_{i \geq 1 \text{ and odd}}$. Invoking Lemma 2.8, we prove the claim.

If u_2 does not correspond to the partition $(3, 2^{2a})$, we may find a representative of $\mathbb{O}_{\rho_l u}$ that lies in K. By Lemma 2.6 and Theorem 3.4, this is possible only if $\rho_l = \iota(t)$ for some $t \in Z(K)$. Therefore $g = \rho_n \mathrm{v} = \mathrm{diag}(1, -I_{2n})\mathrm{v}$ for some spherical unipotent v in $G^\circ_{\rho_n} = K$. We claim that the partition $\underline{\lambda}$ of $2n$ associated with v has no term equal to 3. If $\underline{\lambda} = (3, 2^{2a}, 1^c)$, the K-class of v could be represented by $\mathrm{v}' = x_{\alpha_1}(1) x_{\alpha_3}(1) \cdots x_{\alpha_{2a-1}}(1) x_{\alpha_{n-1}}(1) x_{\alpha_{n-1}+2\alpha_n}(1)$. The element $\rho_l \mathrm{v}'$ is regular in $\langle T, X_{\pm\alpha_1}, X_{\pm\alpha_3}, \ldots X_{\pm\alpha_{2a-1}}, X_{\pm\alpha_{n-1}}, X_{\pm\alpha_n} \rangle$, whose semisimple part is of type $aA_1 \times B_2$; hence the claim follows from Lemma 2.8. Conversely, let $g = \rho_n u$ with u corresponding to $(2^{2m}, 1^{2n-4m})$ for some m. The representative of its class provided in [Cantarini et al. 2005, Theorem 21, page 52] is well defined in odd characteristic and it allows application of Theorem 2.2.

Finally assume that $s = d_\lambda$ for some λ. Then $u \in G_{d_\lambda} = G^\circ_{\rho_1}$ and we may apply the arguments used for $s = \rho_1$ to show that su cannot be spherical. □

Type E_6.

Theorem 3.6. *Let G be simply-connected of type E_6. The spherical classes in G are either semisimple or unipotent up to a central element. The semisimple ones are symmetric and up to a central factor are represented by $p_1 = h_1(-1)h_4(-1)h_6(-1)$ and $p_{2,c} = h_1(c^2)h_2(c^3)h_3(c^4)h_4(c^6)h_5(c^5)h_6(c^4)$ for $c \in k$ with $c^3 \neq 1, 0$. The unipotent ones are those of type A_1, $2A_1$ and $3A_1$.*

Proof. Semisimple classes. Let $s \in T$ be spherical. We may apply [Humphreys 1995, Theorem 2.15] to choose s so that G_s is generated by T and $X_{\pm\alpha}$ for α in a subsystem $\Phi(\Pi) \subset \Phi$ with basis a subset Π of $\Delta \cup \{-\beta_1\}$. By Theorem 2.2 we have $\dim \mathbb{O}_s \leq \ell(w_0) + \mathrm{rk}(1 - w_0)$ and a dimension count shows that Π can only be one of the following subsets:

$$\Pi_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, -\beta_1\}, \quad \Pi_2 = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, -\beta_1\},$$

$$\Pi_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, -\beta_1\} \quad \text{of type } A_5 \times A_1,$$

or

$$\Pi_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \quad \Pi_5 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, -\beta_1\},$$
$$\Pi_6 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \quad \text{of type } D_5.$$

Let us put $H_i = \langle T, X_{\pm\alpha}, \alpha \in \Pi_i \rangle$ for $i = 1, \dots, 6$. The sets Π_i for $i = 1, 2, 3$ are $\mathbb{R}$-bases for the span of Δ, and one may find automorphisms of Φ mapping Π_i for $i = 2, 3$ to $\pm\Pi_1$. On the other hand, $\mathrm{Aut}(\Phi) = \{-w_0\} \ltimes W$ so any element s whose centralizer is H_2 or H_3 is $N(T)$-conjugate to an element whose centralizer is H_1. The elements s for which $G_s = H_1$ are $p_1 = h_1(-1)h_4(-1)h_6(-1)$ and zp_1 for any $z \in Z(G)$. Conjugation by these elements is an involution, so $\mathbb{O}_{p_1}$ is symmetric. This completes the analysis for Π_i with $i \leq 3$.

The subgroups H_4 and H_6 are $\dot{w}_0$-conjugate, so any element whose centralizer is H_4 is $N(T)$-conjugate to an element whose centralizer is H_6. Besides, the automorphism of Φ defined by $\alpha_1 \mapsto -\beta_1$, $\alpha_2 \mapsto \alpha_3$, $\alpha_3 \mapsto \alpha_2$, $\alpha_j \mapsto \alpha_j$ for $j = 4, 5, 6$ maps Π_4 onto Π_5. As before, we may conclude that H_5 is $N(T)$-conjugate to H_4 and any element whose centralizer is H_5 is $N(T)$-conjugate to an element whose centralizer is H_4. The elements whose centralizer is H_4 are $p_{2,c} = h_1(c^2)h_2(c^3)h_3(c^4)h_4(c^6)h_5(c^5)h_6(c^4)$ for $c \in k$ with $c^3 \neq 1, 0$. Multiplying c by a third root of unity yields the same element multiplied by a central one. Since $p_{2,-1}$ is an involution, $\mathbb{O}_{p_{2,c}}$ is spherical by Remark 2.4. We claim that $p_{2,c}$ is not conjugate to $p_{2,d}$ for $c \neq d$. If they were G-conjugate, they would be $N(T)$-conjugate by [Springer and Steinberg 1970, Section 3.1], so there would exist a $\sigma \in W$ such that $\dot{\sigma} p_{2,c} \dot{\sigma}^{-1} = p_{2,d}$. Thus, σ would stabilize $\Phi(\Pi_4)$ and would restrict to an automorphism of $\Phi(\Pi_4)$. Its restriction would therefore be of the form τw, where τ acts an automorphism of the Dynkin diagram of type D_5 and w lies in the Weyl group W' of H_4, which is contained in W. Then σw^{-1} would lie in W, and it would act on Π_4 as τ. Besides, two automorphisms ψ_1, ψ_2 of Φ coinciding on Π_4 are equal. Indeed, for $\alpha = \psi_1 \psi_2^{-1}(\alpha_6)$, we have $\langle \alpha_j, \alpha \rangle = 0$ for $j = 1, 2, 3, 4$ and $\langle \alpha, \alpha_5 \rangle = -1$. Such a root α can only be α_6, so $\psi_1 \psi_2^{-1} = 1$. It follows that σw^{-1} is either the identity, when $\tau = 1$, or it is the automorphism mapping α_j to α_j for $j = 1, 3, 4$, interchanges α_2 and α_5, and maps α_6 to $-\beta_1$. However, one may verify that the second possibility cannot happen because such an automorphism is equal to $s_1s_3s_4s_5s_2s_4s_6s_5s_3s_4s_1s_3s_2s_4s_5s_6(-w_0)$; hence it does not lie in W. Therefore $\tau = 1$ and $\sigma = w \in W'$. Since $G_{p_{2,c}} = H_4$, conjugation by the lift in $N(T)$ of an element in W' does not modify $p_{2,c}$, so $p_{2,c}$ and $p_{2,d}$ represent distinct classes.

Unipotent classes. Let $\mathbb{O}$ be a nontrivial spherical unipotent class. Then $\dim \mathbb{O} \leq \ell(w_0) + \mathrm{rk}(1 - w_0)$ by Theorem 2.2, so $\mathbb{O}$ is of type A_1, $2A_1$ or $3A_1$. Conversely, the arguments in [Cantarini et al. 2005, Theorem 13, pages 39–40] apply in good characteristic and show that the listed orbits have a representative whose B-orbit satisfies the conditions of Theorem 2.2.

Mixed classes. A dimension counting together with Lemma 2.7 shows that no class $\mathcal{O}_{su}$ with $s \notin Z(G)$ and $u \neq 1$ can be spherical. □

Type E_7.

Theorem 3.7. *Let G be simply-connected of type E_7. The spherical classes in G are either semisimple or unipotent up to a central element. The semisimple ones are symmetric and are represented by $q_1 = h_2(\zeta)h_5(-\zeta)h_6(-1)h_7(\zeta)$, where ζ is a fixed primitive fourth root of* 1; $q_2 = h_3(-1)h_5(-1)h_7(-1)$; zq_1, *and* zq_2 *for* $z \in Z(G)$; *and* $q_{3,a} = h_1(a^2)h_2(a^3)h_3(a^4)h_4(a^6)h_5(a^5)h_6(a^4)h_7(a^3)$ *with* $a^2 \neq 1, 0$. *The unipotent ones are those of type* A_1, $2A_1$, $(3A_1)'$, $(3A_1)''$ *and* $4A_1$.

Proof. Semisimple classes. Let $s \in T$ be a spherical element. Proceeding as we did in Theorem 3.6, using that $\dim \mathcal{O}_s \leq \dim B$, we may choose s so that G_s is generated by T and $X_{\pm\alpha}$ for $\alpha \in \Phi(\Pi)$ where Π is one of the following subsets of $\Delta \cup \{-\beta_1\}$:

$$\Pi_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, -\beta_1\} \quad \text{of type } A_7;$$
$$\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, -\beta_1\},$$
$$\Pi_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, -\beta_1\} \quad \text{of type } D_6 \times A_1;$$
$$\Pi_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \quad \text{of type } E_6.$$

Let us put $H_i = \langle T, X_{\pm\alpha}, \alpha \in \Pi_i \rangle$.

There is only one element, up to a central one, whose centralizer is H_1, and this is $q_1 = h_2(\zeta)h_5(-\zeta)h_6(-1)h_7(\zeta)$, where ζ is a fixed primitive fourth root of 1. Since $q_1^2 = h_2(-1)h_5(-1)h_7(-1) \in Z(G)$, the corresponding class is symmetric by Remark 2.4. The root systems generated by Π_2 and Π_3 are mapped onto each other by elements in $\mathrm{Aut}(\Phi) = W$. Thus, each element whose centralizer is H_2 is $N(T)$-conjugate to one whose centralizer is H_3, and it is enough to look at Π_2. The elements whose centralizer is H_2 are $q_2 = h_3(-1)h_5(-1)h_7(-1)$ and zq_2 for $z \in Z(G)$. The corresponding classes are symmetric. The elements whose centralizer is H_4 are $q_{3,a} = h_1(a^2)h_2(a^3)h_3(a^4)h_4(a^6)h_5(a^5)h_6(a^4)h_7(a^3)$ for $a^2 \neq 1, 0$. For ξ a primitive fourth root of unity, we have $q_{3,\xi}^2 \in Z(G)$ and hence all such classes are symmetric. Multiplication of $q_{3,a}$ by the nontrivial central element gives $q_{3,-a}$. We claim that $q_{3,a}$ is never conjugate to $q_{3,b}$ for $a \neq b$. If they were G-conjugate, they would be $N(T)$-conjugate, so there would exist a $\sigma \in W$ for which $\dot{\sigma} q_{3,a} \dot{\sigma}^{-1} = q_{3,b}$. Such a σ would preserve $\Phi(\Pi_4)$, and its restriction to it would be an automorphism. As in the proof of Theorem 3.6, we see that for some w in the Weyl group W' of H_4, the restriction to $\Phi(\Pi_4)$ of $\sigma w^{-1} \in W$ would come from an automorphism of the Dynkin diagram of type E_6. There is no automorphism of Φ whose restriction to E_6 is the nontrivial automorphism. Indeed, if such an automorphism τ existed, for $\alpha = \tau(\alpha_7)$ we would have $\langle \alpha, \alpha_j \rangle = 0$ for $j = 2, 3, 4, 5, 6$ and $\langle \alpha, \alpha_1 \rangle = -1$, but there is no such $\alpha \in \Phi$. Therefore σw^{-1} is the identity on $\Phi(\Pi_4)$. By uniqueness of the extension of an automorphism from E_6 to E_7 we have $\sigma = w \in W'$. Since $G_{q_{3,a}} = H_4$, conjugation by lifts in $N(T)$ of elements in W' preserves $q_{3,a}$.

Unipotent classes. Let $u \neq 1$ be a spherical unipotent element. Then $\dim \mathcal{O}_u \leq \dim B$, so $\mathcal{O}_u$ is either of type rA_1 for some r, or of type A_2. In the latter case, u would be regular in a Levi subgroup of type A_2, so this case cannot occur by Remark 2.5. The arguments in [Cantarini et al. 2005, Theorem 13, pages 39–40] apply also in good characteristic and show that for all unipotent classes of type rA_1, there is a representative whose B-orbit satisfies the condition in Theorem 2.2.

Mixed classes. We claim that there is no spherical element with Jordan decomposition $g = su$ with $s \notin Z(G)$ and $u \neq 1$. Indeed, $\mathcal{O}_s$ would be spherical and u would be spherical in G_s°. A dimensional argument shows that this is possible only if $s \in \mathcal{O}_{q_2}$ up to a central element and u is nontrivial only in the component of type A_1 in G_s. It follows from the discussion of semisimple elements that we may choose s so that $G_s = H_3$ with notation as before, so that we may choose g to be conjugate to $sx_{-\alpha_7}(1)$. Conjugation of g by $x_{-\alpha_6}(1)$ and Chevalley's commutator formula would give $z = sx_{-\alpha_6}(a)x_{-\alpha_7}(1)x_{-\alpha_6-\alpha_7}(b) \in \mathcal{O}_g$ for some nonzero $a, b \in k$. Conjugating z by a suitable element in $X'_{-\alpha_6-\alpha_7}$, we could get rid of the term in $X'_{-\alpha_6-\alpha_7}$, obtaining an element in $\mathcal{O}_g \cap Bs_6s_7B$. By Theorem 2.3, the class $\mathcal{O}_g$ cannot be spherical. □

Type E_8.

Theorem 3.8. *Let G be of type E_8. The spherical classes are either semisimple or unipotent. The semisimple ones are symmetric ,and they are represented by $r_1 = h_2(-1)h_3(-1)$ and $r_2 = h_2(-1)h_5(-1)h_7(-1)$. The unipotent ones are those of type A_1, $2A_1$, $3A_1$ and $4A_1$.*

Proof. Semisimple classes. Let $s \in T$ be a spherical element. Proceeding as we did in Theorems 3.6 and 3.7 we see that, up to $N(T)$-conjugation, the centralizer G_s is generated by T and by the $X_{\pm\alpha}$ for α in a subsystem with basis either $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, -\beta_1\}$ of type D_8 or $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, -\beta_1\}$ of type $E_7 \times A_1$. Then s is conjugate either to $r_1 = h_2(-1)h_3(-1)$ or to $r_2 = h_2(-1)h_5(-1)h_7(-1)$. Since $r_1^2 = r_2^2 = 1$ the corresponding classes are symmetric.

Unipotent classes. Let $\mathcal{O}$ be a nontrivial spherical unipotent class. Then $\dim \mathcal{O} \leq \dim B$, so $\mathcal{O}$ is either of type rA_1 for some r, or it is of type A_2. The latter case is excluded as in the case of G of type E_7. Conversely, the arguments in [Cantarini et al. 2005, Theorem 13, pages 39–40] apply in good characteristic and show that for each orbit of type rA_1 in G, we may find a representative whose B-orbit satisfies the condition in Theorem 2.2.

Mixed classes. We claim that there is no spherical element with Jordan decomposition $g = su$ with $s, u \neq 1$. Indeed, by dimensional reasons, s would be conjugate to r_2 and u would lie in the component of type A_1 in $G_{r_2} = \langle T, X_{\pm\beta_1}, X_{\pm\alpha_i}\rangle_{i=1,\dots,7}$. In other words, we could assume $g = r_2 x_{-\beta_1}(1)$. Let $\gamma = \beta_1 - \alpha_8$. Conjugation of g by $\dot{s}_\gamma$ gives $tx_{-\alpha_8}(a) \in \mathbb{O}_g$ for some nonzero $a \in k$ and some $t \in T$. Since r_2 does not commute with $X'_{\pm(\beta_1-\alpha_8-\alpha_7)}$ and $s_\gamma(\alpha_7+\alpha_8-\beta_1) = \alpha_7$, the element t does not commute with $X'_{\pm\alpha_7}$. Since $s_\gamma(\alpha_7+\alpha_8) = \alpha_7+\alpha_8$ and r_2 does not commute with $X'_{\pm(\alpha_7+\alpha_8)}$, the same holds for t. Then conjugation of $tx_{-\alpha_8}(a)$ by $x_{-\alpha_7}(1)$ would give $tx_{-\alpha_7}(b)x_{-\alpha_8}(a)x_{-\alpha_7-\alpha_8}(c) \in \mathbb{O}_g$ for some nonzero $b, c \in k$. Conjugation by a suitable element in $X'_{-\alpha_7-\alpha_8}$ would yield an element $x \in \mathbb{O}_g \cap TX'_{-\alpha_7}X'_{-\alpha_8}$. By (1) and (2), x would lie in $\mathbb{O}_g \cap Bs_7s_8B$, leading to a contradiction. □

Type F_4.

Theorem 3.9. *Let G be of type F_4. The spherical semisimple classes are symmetric and represented by $f_1 = h_{\alpha_2}(-1)h_{\alpha_4}(-1)$ and $f_2 = h_{\alpha_3}(-1)$. The spherical unipotent ones are those of type $rA_1 + s\tilde{A}_1$ for $r, s \in \{0, 1\}$. There is a spherical class that is neither semisimple nor unipotent, and it is represented by $f_2 x_{\beta_1}(1)$.*

Proof. Semisimple classes. Let $s \in T$ be a spherical element in G. A dimension counting similar to the previous exceptional cases shows that G_s is $N(T)$-conjugate to the subgroup generated by T and the root subgroups corresponding to roots in a subsystem with basis either $\Pi_1 = \{-\beta_1, \alpha_2, \alpha_3, \alpha_4\}$ or $\Pi_2 = \{\alpha_1, \alpha_2, \alpha_3, -\beta_1\}$. They correspond to the involutions $f_1 = h_{\alpha_2}(-1)h_{\alpha_4}(-1)$ and $f_2 = h_{\alpha_3}(-1)$, respectively, which are indeed spherical.

Unipotent classes. Let $\mathbb{O}$ be a nontrivial spherical unipotent class in G. Then $\dim \mathbb{O} \leq \dim B$, so $\mathbb{O}$ is either of type A_1, $\tilde{A}_1$ or $A_1 + \tilde{A}_1$. Conversely, the arguments in [Cantarini et al. 2005, Theorem 13, pages 39–40] hold in good characteristic and show that Theorem 2.2 applies to these three classes.

Mixed classes. Let $g = su$ be the Jordan decomposition of a spherical element with $s, u \neq 1$. Since $\dim \mathbb{O}_{f_1} = \dim B$, we may assume $s = f_2$. Also, G_{f_2} is a reductive group of type B_4. A dimensional argument shows that u lies in the minimal unipotent class in G_{f_2}, so we may assume $g = f_2 x_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}(1) = f_2 x_{-\beta_1}(1)$. We have $\dim \mathbb{O}_g = \dim B$. The proof in [Cantarini et al. 2005, Theorem 23] contains an incorrect argument, which we rectify here.

The element $f_2 = h_{\alpha_3}(-1)$ lies in the subgroup $G_1 = \langle X_{\pm\alpha_i},\ i = 2, 3, 4\rangle$ of type C_3. By looking at the centralizer of f_2 in G_1 we see that, up to an element in $Z(G_1)$, the G_1-conjugacy class of f_2 is represented by σ_1 with notation as in Theorem 3.3. By [Cantarini et al. 2005, Theorem 15, page 42], the G_1-class of σ_1

has a representative in $s_4 s_{\alpha_2+2\alpha_3+\alpha_4} T$ when $k = \mathbb{C}$. The same matrix represents the class in good characteristic. Besides, G_1 centralizes $X_{\pm\beta_1}$, so $f_2 x_{-\beta_1}(1)$ can be represented by an element $z \in s_4 s_{\alpha_2+2\alpha_3+\alpha_4} T X'_{-\beta_1} \subset s_4 s_{\alpha_2+2\alpha_3+\alpha_4} T X'_{\beta_1} s_{\beta_1} X'_{\beta_1} \subset X'_{\beta_1} w_0 s_2 T X'_{\beta_1}$. Conjugating z by $\dot{s}_2 \dot{s}_1$, we obtain an element $z' \in B w_0 s_1 B \cap \mathcal{O}_g$. Thus, $w_{\mathcal{O}_g} \geq w_0 s_2$ and $w_{\mathcal{O}_g} \geq w_0 s_1$, forcing $w_0 = w_{\mathcal{O}_g}$ (notation as in Section 2). Then $\mathcal{O}_g$ has a representative whose B-orbit satisfies the condition in Theorem 2.2 and therefore is spherical. □

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COUNTING CONJUGACY CLASSES IN THE UNIPOTENT RADICAL OF PARABOLIC SUBGROUPS OF $\mathrm{GL}_n(q)$

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Let q be a power of a prime p. Let P be a parabolic subgroup of the general linear group $\mathrm{GL}_n(q)$ that is the stabilizer of a flag in $\mathbb{F}_q^n$ of length at most 5, and let $U = O_p(P)$. We prove that, as a function of q, the number $k(U)$ of conjugacy classes of U is a polynomial in q with integer coefficients.

1. Introduction

Let $\mathrm{GL}_n(q)$ be the finite general linear group defined over the field $\mathbb{F}_q$ of q elements, where q is a power of a prime p. A longstanding conjecture attributed to G. Higman [1960] asserts that the number of conjugacy classes of a Sylow p-subgroup of $\mathrm{GL}_n(q)$ is given by a polynomial in q with integer coefficients. This has been verified by computer calculation by A. Vera-López and J. M. Arregi [2003] for $n \leq 13$. G. R. Robinson [1998] and J. Thompson [2004] have shown much interest in this conjecture. For recent related results, see [Alperin 2006; Evseev 2009; Goodwin and Röhrle 2008; 2009a; 2009b; 2009c].

The following question is precisely Higman's conjecture when $P = B$ is a Borel subgroup of $\mathrm{GL}_n(q)$.

Question 1.1. Let P be a parabolic subgroup of $\mathrm{GL}_n(q)$ and let $U = O_p(P)$. As a function of q, is the number $k(U)$ of conjugacy classes of U a polynomial in q?

Here we recall that $O_p(P)$ is by definition the largest normal p-subgroup of P. In this paper, we give an affirmative answer to Question 1.1 in the following cases.

Theorem 1.2. *Let P be a parabolic subgroup of $\mathrm{GL}_n(q)$ that is the stabilizer of a flag in $\mathbb{F}_q^n$ of length at most 5, and let $U = O_p(P)$. Then, as a function of q, the number $k(U)$ of conjugacy classes of U is a polynomial in q with integer coefficients.*

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We now explain the significance of the hypothesis imposed in Theorem 1.2. Let $\boldsymbol{P}$ be a parabolic subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$, and let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{P}$, where $\overline{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$. All instances when $\boldsymbol{P}$ acts on $\boldsymbol{U}$ with a finite number of orbits were determined in [Hille and Röhrle 1999]; this is precisely the case when $\boldsymbol{P}$ is the stabilizer of a flag in $\overline{\mathbb{F}}_q^n$ of length at most 5. So Theorem 1.2 deals with parabolic subgroups P of $\mathrm{GL}_n(q)$ that correspond to parabolic subgroups $\boldsymbol{P}$ of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ with a finite number of conjugacy classes in $\boldsymbol{U}$. In such cases, it is observed in [Hille and Röhrle 1999, Remark 4.13] that the parameterization of the $\boldsymbol{P}$-conjugacy classes in $\boldsymbol{U}$ is independent of q: This is the crucial point that we require for our proof of Theorem 1.2.

The proof involves a translation of the problem to a representation theoretic setting. More precisely, recall from [Hille and Röhrle 1999, Section 4] that the P-conjugacy classes in U correspond bijectively to the so-called Δ-filtered modules of a certain quasihereditary algebra $\mathscr{A}_t$. This allows us to see that the parameterization of the P-orbits in U is independent of q and that we can choose a set $\mathscr{R}$ of representatives that are matrices with entries equal to 0 or 1. The other key point is that the structures of the centralizers $C_P(x)$ and $C_U(x)$ for $x \in \mathscr{R}$ do not depend on q; this is covered in Propositions 2.2 and 2.4.

We now discuss some natural generalizations of Theorem 1.2. First consider the case of a normal subgroup $\boldsymbol{N}$ of $\boldsymbol{P}$ with $\boldsymbol{N} \subseteq \boldsymbol{U}$. Still assuming that there is only a finite number of $\boldsymbol{P}$-orbits in $\boldsymbol{U}$, we readily derive from the proof of Theorem 1.2 that $k(U, N)$, the number of U-conjugacy classes in $N = \boldsymbol{N} \cap U$, is given by a polynomial in q with integer coefficients. It should also be possible to prove that the number $k(U, N)$ is a polynomial in q with just the assumption that there are finitely many $\boldsymbol{P}$-orbits in $\boldsymbol{N}$. For example, for $\boldsymbol{N} = \boldsymbol{U}^{(l)}$ the l-th member of the descending central series of $\boldsymbol{U}$, there is a classification of all instances when $\boldsymbol{P}$ acts on $\boldsymbol{U}^{(l)}$ with a finite number of orbits; see [Brüstle and Hille 2000]. In such situations a generalization of the proof of Theorem 1.2 would require detailed knowledge of the $\boldsymbol{P}$-conjugacy classes in $\boldsymbol{N}$.

It is also natural to consider the generalization of Question 1.1, where $\mathrm{GL}_n(q)$ is replaced by any finite reductive group G, and also to consider the number $k(P, U)$ of P-conjugacy classes in U rather than $k(U)$. (To avoid degeneracies in the Chevalley commutator relations, it is sensible to only consider these generalizations when q is a power of a good prime for G.)

At present there are no known examples in which $k(U)$ is not given by a polynomial in q, and there are many cases not covered by Theorem 1.2, where $k(U)$ is given by a polynomial in q; see for example [Goodwin and Röhrle 2009b] and [Vera-López and Arregi 2003]. However, it is not necessarily the case that $k(P, U)$ is a polynomial in q. Indeed in [Goodwin 2007, Example 4.6], it shown that in case G is of type G_2, and $P = B$ is a Borel subgroup of G, the number $k(B, U)$ is given by two different polynomials depending on the residue of q modulo 3.

Let $\boldsymbol{P}$ be a parabolic subgroup of a reductive algebraic group $\boldsymbol{G}$ defined over $\mathbb{F}_q$, and suppose that $\boldsymbol{P}$ has finitely many conjugacy classes in $\boldsymbol{U}$; let P and U be the groups of $\mathbb{F}_q$-rational points of $\boldsymbol{P}$ and $\boldsymbol{U}$, respectively. Given the discussion after Theorem 1.2, a natural generalization to consider is whether the number $k(U)$ of conjugacy classes of U is a polynomial in q. Our proof of Theorem 1.2 is dependent on the detailed information about the P-conjugacy classes in U. For this reason the argument does not adapt to the case in which G is any finite reductive group. The main difficulty is that it is not clear whether the parameterization of $\boldsymbol{P}$-orbits in $\boldsymbol{U}$ and the structure of centralizers depends on the characteristic of the underlying ground field. Another problem is that centralizers $C_{\boldsymbol{P}}(u)$ for $u \in U$ need not be connected, so determining the P-classes in U from the $\boldsymbol{P}$-classes in $\boldsymbol{U}$ may be nontrivial.

2. Translation to representation theory

Here, we recall the relationship established in [Hille and Röhrle 1999, Section 4] between adjoint orbits of parabolic subgroups and modules for a certain quasi-hereditary algebra. This relationship is central to our proof of Theorem 1.2. In particular, it is crucial for Propositions 2.2 and 2.4, which describe the structure of certain centralizers. Throughout this section we work in generality over any field, before specializing to finite fields for the proof of Theorem 1.2 in Section 3.

Let K be any field, and let $n, t \in \mathbb{Z}_{\geq 1}$. Let $\boldsymbol{d} = (d_1, \dots, d_t) \in \mathbb{Z}^t_{\geq 0}$ with $d_i \leq d_{i+1}$ and $d_t = n$. We define the parabolic subgroup $P(\boldsymbol{d}) = P_K(\boldsymbol{d})$ of $\mathrm{GL}_n(K)$ to be the stabilizer of the flag $0 \subseteq K^{d_1} \subseteq K^{d_2} \subseteq \dots \subseteq K^{d_t}$ in K^n; any parabolic subgroup of $\mathrm{GL}_n(K)$ is conjugate to $P(\boldsymbol{d})$ for some $\boldsymbol{d}$. We write

$$U(\boldsymbol{d}) = U_K(\boldsymbol{d}) = \{u \in \mathrm{GL}_n(K) \mid (u-1)V_i \subseteq V_{i-1} \text{ for each } i\}$$

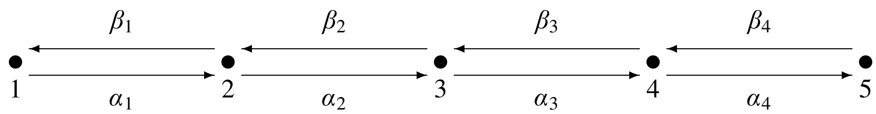
for the unipotent radical of $P(\boldsymbol{d})$, and

$$\mathfrak{u}(\boldsymbol{d}) = \mathfrak{u}_K(\boldsymbol{d}) = \{x \in \mathrm{M}_n(K) \mid xV_i \subseteq V_{i-1} \text{ for each } i\}$$

for the Lie algebra of $U(\boldsymbol{d})$. Then $P(\boldsymbol{d})$ acts on $\mathfrak{u}(\boldsymbol{d})$ via the adjoint action, that is, $g \cdot x = gxg^{-1}$ for $g \in P(\boldsymbol{d})$ and $x \in \mathfrak{u}(\boldsymbol{d})$. For $x \in \mathfrak{u}(\boldsymbol{d})$, we write $P \cdot x$ for the adjoint P-orbit of x and $C_P(x)$ for the centralizer of x in P; we define $U \cdot x$ and $C_U(x)$ analogously.

Though we are primarily interested in the conjugacy classes of $U(\boldsymbol{d})$ and the $P(\boldsymbol{d})$-conjugacy classes in $U(\boldsymbol{d})$, it is more convenient to consider the adjoint $P(\boldsymbol{d})$-orbits in $\mathfrak{u}(\boldsymbol{d})$. The map $x \mapsto 1 + x$ is a $P(\boldsymbol{d})$-equivariant isomorphism between $\mathfrak{u}(\boldsymbol{d})$ and $U(\boldsymbol{d})$, which means that the adjoint $P(\boldsymbol{d})$-orbits in $\mathfrak{u}(\boldsymbol{d})$ are in bijective correspondence with the $P(\boldsymbol{d})$-conjugacy classes in $U(\boldsymbol{d})$; this allows us to work with the adjoint orbits.

The quiver $\mathcal{Q}_t$ is defined to have vertex set $\{1, \dots, t\}$, and there are arrows $\alpha_i : i \to i+1$ and $\beta_i : i+1 \to i$ for $i = 1, \dots, t-1$. Here is an example of the quiver $\mathcal{Q}_t$ for $t = 5$:



Let $I_t = I_{t,K}$ be the ideal of the path algebra $K\mathcal{Q}_t$ of $\mathcal{Q}_t$ generated by the relations

$$\beta_1\alpha_1 = 0 \quad \text{and} \quad \alpha_i\beta_i = \beta_{i+1}\alpha_{i+1} \quad \text{for } i = 1, \dots, t-2. \tag{2-1}$$

The algebra $\mathcal{A}_t = \mathcal{A}_{t,K}$ is defined to be the quotient $K\mathcal{Q}_t/I_t$.

Recall that an $\mathcal{A}_t$-module M is determined by a family of vector spaces $M(i)$ over K for $i = 1, \dots, t$ such that $M = \bigoplus_{i=1}^t M(i)$, and linear maps $M(\alpha_i) : M(i) \to M(i+1)$ and $M(\beta_i) : M(i+1) \to M(i)$ for $i = 1, \dots, t-1$ that satisfy the relations (2-1). The dimension vector $\dim M \in \mathbb{Z}_{\geq 0}^t$ of an $\mathcal{A}_t$-module is defined by $\dim M = (\dim M(1), \dots, \dim M(t))$.

Let $\mathcal{M}_t = \mathcal{M}_{t,K}$ be the category of $\mathcal{A}_t$-modules M such that $M(\alpha_i)$ is injective for all i. Write $\mathcal{M}_t(\boldsymbol{d}) = \mathcal{M}_{t,K}(\boldsymbol{d})$ for the class of modules in $\mathcal{M}_t$ with dimension vector $\boldsymbol{d}$. Hille and Röhrle show in [1999, Section 4] that the orbits of $P(\boldsymbol{d})$ in $\mathfrak{u}(\boldsymbol{d})$ are in bijection with the isoclasses in $\mathcal{M}_t(\boldsymbol{d})$ and moreover, using [Dlab and Ringel 1992, Sections 6 and 7],[1] that there is a unique structure of a quasihereditary algebra on $\mathcal{A}_t$ such that $\mathcal{M}_t$ is the category of Δ-*filtered* $\mathcal{A}_t$-modules.

Suppose for this paragraph that K is infinite. Using the above bijection and the results from [DR], it was proved in [HR, Theorem 4.1] that there is a finite number of $P(\boldsymbol{d})$-orbits in $\mathfrak{u}(\boldsymbol{d})$ if and only if $t \leq 5$. This is deduced from the fact that $\mathcal{A}_t$ has finite Δ-representation type if and only if $t \leq 5$; see [DR, Proposition 7.2].

Let $t \leq 5$. Because the results in [HR, Section 4] are proved for an arbitrary field — see [HR, Remark 4.13] — the parametrization of indecomposable Δ-filtered $\mathcal{A}_t$-modules does not depend on the field K; we explain this more explicitly below. Let $\{I_1, \dots, I_m\}$ be a complete set of representatives of isoclasses of indecomposable Δ-filtered $\mathcal{A}_t$-modules, and write $\boldsymbol{d}_i$ for the dimension vector of I_i. Let $x_i \in \mathfrak{u}(\boldsymbol{d}_i)$ be such that the $P(\boldsymbol{d}_i)$-orbit of x_i corresponds to the isoclass of I_i. As discussed in [HR, Section 7] — see also [Brüstle et al. 1999, Figure 10] — one can choose x_i to be a matrix with entries 0 and 1, and these matrices do not depend on K. In particular, this implies that the modules I_i are absolutely indecomposable.

Another important consequence for us is the following lemma.

[1] These two references are henceforth abbreviated as [HR] and [DR].

Lemma 2.1. *Assume $t \le 5$. We may choose a set $\mathfrak{R}$ of representatives of the adjoint $P(\boldsymbol{d})$-orbits in $\mathfrak{u}(\boldsymbol{d})$ such that each element of $\mathfrak{R}$ is a matrix with all entries equal to* 0 *or* 1. *Moreover, the elements of $\mathfrak{R}$ do not depend on the field K, that is, the positions of entries equal to* 1 *do not depend on K.*

We still assume that $t \le 5$, and let $\boldsymbol{d} \in \mathbb{Z}^t_{\ge 0}$. Let $P = P(\boldsymbol{d})$, $U = U(\boldsymbol{d})$ and $x \in \mathfrak{u} = \mathfrak{u}(\boldsymbol{d})$. For the proof of Theorem 1.2 we need information about the structure of the centralizers $C_P(x)$ and $C_U(x)$; this is given by Propositions 2.2 and 2.4.

Let M be a Δ-filtered $\mathscr{A}_t$-module (with dimension vector $\boldsymbol{d}$) whose isoclass corresponds to the P-orbit of x. Extending the arguments of [HR, Section 4], one can show that the automorphism group $\mathrm{Aut}_{\mathscr{A}_t}(M)$ of M is isomorphic to $C_P(x)$. Below we explain the structure of $\mathrm{End}_{\mathscr{A}_t}(M)$ and $\mathrm{Aut}_{\mathscr{A}_t}(M)$; this uses standard arguments that we outline here for convenience. We proceed to explain how $C_U(x)$ is related to $\mathrm{End}_{\mathscr{A}_t}(M)$.

As above, let $\{I_1, \dots, I_m\}$ be a complete set of representatives of isoclasses of indecomposable Δ-filtered $\mathscr{A}_t$-modules. We may decompose M as a direct sum of indecomposable modules

$$M \cong \bigoplus_{i=1}^{m} n_i I_i, \quad \text{where } n_i \in \mathbb{Z}_{\ge 0}. \tag{2-2}$$

Then

$$\mathrm{End}_{\mathscr{A}_t}(M) \cong \bigoplus_{i,j=1}^{m} n_i n_j \,\mathrm{Hom}_{\mathscr{A}_t}(I_i, I_j)$$

as a vector space and composition is defined in the obvious way.

We observed above that I_i is absolutely indecomposable, which means that $\mathrm{End}_{\mathscr{A}_t}(I_i)$ is a local ring, and that we have the decomposition $\mathrm{End}_{\mathscr{A}_t}(I_i) = K \oplus \mathfrak{m}_i$, where K is acting by scalars and $\mathfrak{m}_i$ is the maximal ideal. Therefore,

$$n_i^2 \,\mathrm{End}_{\mathscr{A}_t}(I_i) \cong \mathrm{M}_{n_i}(K) \oplus \mathrm{M}_{n_i}(\mathfrak{m}_i),$$

where $\mathrm{M}_{n_i}(K)$ is a subalgebra and $\mathrm{M}_{n_i}(\mathfrak{m}_i)$ is an ideal. In fact, $\mathrm{M}_{n_i}(\mathfrak{m}_i)$ is the Jacobson radical of $n_i^2 \,\mathrm{End}_{\mathscr{A}_t}(I_i)$.

Now one can see that the Jacobson radical of $\mathrm{End}_{\mathscr{A}_t}(M)$ is

$$J(\mathrm{End}_{\mathscr{A}_t}(M)) \cong \bigoplus_{i=1}^{m} \mathrm{M}_{n_i}(\mathfrak{m}_i) \oplus \bigoplus_{i \ne j} n_i n_j \,\mathrm{Hom}_{\mathscr{A}_t}(I_i, I_j).$$

There is a complement to $J(\mathrm{End}_{\mathscr{A}_t}(M))$ in $\mathrm{End}_{\mathscr{A}_t}(M)$ denoted by $C(\mathrm{End}_{\mathscr{A}_t}(M))$ with

$$C(\mathrm{End}_{\mathscr{A}_t}(M)) \cong \bigoplus_{i=1}^{m} \mathrm{M}_{n_i}(K).$$

We can now describe the automorphism group $\mathrm{Aut}_{\mathcal{A}_t}(M)$. We have

$$\mathrm{Aut}_{\mathcal{A}_t}(M) \cong U(C(\mathrm{End}_{\mathcal{A}_t}(M))) \ltimes (1_M + J(\mathrm{End}_{\mathcal{A}_t}(M))),$$

with $U(C(\mathrm{End}_{\mathcal{A}_t}(M)))$ the group of units of $C(\mathrm{End}_{\mathcal{A}_t}(M))$ and $1_M + J(\mathrm{End}_{\mathcal{A}_t}(M))$ the unipotent group $\{1_M + \phi \mid \phi \in J(\mathrm{End}_{\mathcal{A}_t}(M))\}$. We have $U(C(\mathrm{End}_{\mathcal{A}_t}(M))) \cong \prod_{i=1}^m \mathrm{GL}_{n_i}(K)$, and therefore

$$\mathrm{Aut}_{\mathcal{A}_t}(M) \cong \prod_{i=1}^{m} \mathrm{GL}_{n_i}(K) \ltimes N,$$

where N is a split unipotent group over K. By saying N is a *split unipotent group*, we mean that N has a normal series with all quotients isomorphic to the additive group K. The dimension of N is

$$\delta := \sum_{i=1}^{m} n_i^2(\dim \mathrm{End}_{\mathcal{A}_t}(I_i) - 1) + \sum_{i \neq j} n_i n_j \dim \mathrm{Hom}_{\mathcal{A}_t}(I_i, I_j). \tag{2-3}$$

One can compute all Hom-groups $\mathrm{Hom}_{\mathcal{A}_t}(I_i, I_j)$ from the underlying Auslander–Reiten quivers of $\mathcal{A}_t$ in [DR, pages 221 and 222]; see also [Brüstle et al. 1999, Appendix A]. The dimensions $\dim \mathrm{Hom}_{\mathcal{A}_t}(I_i, I_j)$ are independent of K. Therefore, the positive integer δ is also independent of K.

We said above that $\mathrm{Aut}_{\mathcal{A}_t}(M)$ is isomorphic to $C_P(x)$, so we have the following proposition.

Proposition 2.2. *The Levi decomposition of $C_P(x)$ is given by*

$$C_P(x) \cong \prod_{i=1}^{m} \mathrm{GL}_{n_i}(K) \ltimes N,$$

where N, the unipotent radical of $C_P(x)$, is a split unipotent group over K of dimension δ.

Remark 2.3. It is natural to ask whether Proposition 2.2 still holds if $t > 5$. The arguments above do apply in case K is assumed to be algebraically closed. It would be interesting to know what happens in general, and also if Corollary 3.1 holds for $t > 5$.

We now wish to give the structure of the centralizer $C_U(x)$. By further extending the arguments in [HR, Section 4], one sees that there is an isomorphism

$$C_U(x) \cong 1_M + \mathrm{End}'_{\mathcal{A}_t}(M),$$

where

$$\mathrm{End}'_{\mathcal{A}_t}(M) := \{\phi \in \mathrm{End}_{\mathcal{A}_t}(M) \mid \phi M(l) \subseteq M(l-1) \text{ for all } l\};$$

here we are identifying $M(l-1)$ with its image in $M(l)$ under $M(\alpha_{l-1})$. We have that $\mathrm{End}'_{\mathcal{A}_t}(M)$ is a nilpotent ideal of $\mathrm{End}_{\mathcal{A}_t}(M)$. We define

$$\mathrm{Hom}'_{\mathcal{A}_t}(I_i, I_j) := \{\phi \in \mathrm{Hom}_{\mathcal{A}_t}(I_i, I_j) \mid \phi I_i(l) \subseteq I_j(l-1) \text{ for all } l\}.$$

Then we have the isomorphism

$$\mathrm{End}'_{\mathcal{A}_t}(M) \cong \bigoplus_{i,j=1}^{m} n_i n_j \,\mathrm{Hom}'_{\mathcal{A}_t}(I_i, I_j).$$

We write

$$\delta' := \dim \mathrm{End}'_{\mathcal{A}_t}(M) = \sum_{i,j=1}^{m} n_i n_j \dim \mathrm{Hom}'_{\mathcal{A}_t}(I_i, I_j). \tag{2-4}$$

From the Auslander–Reiten quivers of $\mathcal{A}_t$ exhibited in [DR, pages 221 and 222], one can compute the dimensions $\dim \mathrm{Hom}'_{\mathcal{A}_t}(I_i, I_j)$. These integers are independent of K, so that δ' is also independent of K. The discussion above proves the following proposition.

Proposition 2.4. *The centralizer $C_U(x)$ is a δ'-dimensional split unipotent group over K.*

3. Proof of Theorem 1.2

Let q be a prime power and let $K = \mathbb{F}_q$ be the field of q elements. Let $t \leq 5$ and let $\boldsymbol{d} \in \mathbb{Z}^t_{\geq 0}$. Let $P = P(\boldsymbol{d})$, $U = U(\boldsymbol{d})$ and $\mathfrak{u} = \mathfrak{u}(\boldsymbol{d})$ be as in the previous section, so that P is a parabolic subgroup of $\mathrm{GL}_n(q)$.

The following corollary is a key step in our proof of Theorem 1.2. It follows immediately from Propositions 2.2 and 2.4 along with the elementary fact that the order of a general linear group over $\mathbb{F}_q$ is given by a polynomial in q. The positive integers in the statement are determined in (2-2), (2-3) and (2-4).

Corollary 3.1. *Let $x \in \mathfrak{u}$. Then there are positive integers $n_1, \dots, n_m$, δ and δ' independent of q such that*

$$|C_P(x)| = \prod_{i=1}^{m} |\mathrm{GL}_{n_i}(q)| \cdot q^{\delta} \quad \text{and} \quad |C_U(x)| = q^{\delta'}.$$

In particular, $|C_P(x)|$ and $|C_U(x)|$ are polynomials in q with integer coefficients.

Proof of Theorem 1.2. We must prove that $k(U)$ is given by a polynomial in q. As discussed in the previous section $k(U)$ is equal to $k(U, \mathfrak{u})$, the number of adjoint U-orbits in $\mathfrak{u}$. We will prove that $k(U, \mathfrak{u})$ is a polynomial in q with integer coefficients.

We may choose a set of representatives $\mathcal{R}$ of the adjoint P-orbits in $\mathfrak{u}$, as in Lemma 2.1, and consider $\mathcal{R}$ to be independent of q. We have

$$k(U,\mathfrak{u}) = \sum_{x\in\mathcal{R}} k(U, P\cdot x),$$

where $k(U, P\cdot x)$ is the number of U-orbits contained in $P\cdot x$. For $x \in \mathfrak{u}$ and $g \in P$, we have $C_U(g\cdot x) = gC_U(x)g^{-1}$. Therefore, we get $|U\cdot x| = |U\cdot(g\cdot x)|$ and $k(U, P\cdot x) = |P\cdot x|/|U\cdot x|$. It follows that

$$k(U,\mathfrak{u}) = \sum_{x\in\mathcal{R}} k(U, P\cdot x) = \sum_{x\in\mathcal{R}} \frac{|P\cdot x|}{|U\cdot x|} = \frac{|P|}{|U|}\sum_{x\in\mathcal{R}} \frac{|C_U(x)|}{|C_P(x)|} = |L|\sum_{x\in\mathcal{R}} \frac{|C_U(x)|}{|C_P(x)|},$$

where L is a Levi subgroup of P. Since $|L|$ is a polynomial in q, Corollary 3.1 and the fact that $\mathcal{R}$ is independent of q imply $k(U,\mathfrak{u}) = k(U)$ is a rational function in q. Since $k(U)$ takes integer values for all prime powers, standard arguments show that $k(U)$ is in fact a polynomial in q with rational coefficients; see for example [Goodwin and Röhrle 2009a, Lemma 2.11].

Let $\boldsymbol{P}$ be the subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ corresponding to P and let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{P}$. The *commuting variety of* $\boldsymbol{U}$ is the closed subvariety of $\boldsymbol{U}\times\boldsymbol{U}$ defined by

$$\mathcal{C}(\boldsymbol{U}) = \{(u,u') \in \boldsymbol{U}\times\boldsymbol{U} \mid uu' = u'u\}.$$

Setting $\mathcal{C}(U) = \mathcal{C}(\boldsymbol{U}) \cap (U\times U)$ and using the Burnside counting formula, we get

$$|\mathcal{C}(U)| = \sum_{x\in U} |C_U(x)| = |U|\cdot k(U).$$

Since $|U| = q^{\dim \boldsymbol{U}}$ and $k(U)$ is a polynomial in q with rational coefficients, so is $|\mathcal{C}(U)|$. Now using the Grothendieck trace formula applied to $\mathcal{C}(\boldsymbol{U})$ (see [Digne and Michel 1991, Theorem 10.4]), standard arguments prove that the coefficients of this polynomial are integers; see for example [Reineke 2006, Propopsition 6.1]. Thus, it follows that $k(U)$ is a polynomial function in q with integer coefficients, as claimed. □

Remark 3.2. Let $t \le 5$ and $\boldsymbol{d}, \boldsymbol{d}' \in \mathbb{Z}^t_{\ge 0}$ with $d_t = d'_t = n$. Suppose that $P = P(\boldsymbol{d})$ and $Q = P(\boldsymbol{d}')$ are *associated parabolic subgroups* of $\mathrm{GL}_n(\mathbb{F}_q)$, that is, P and Q have Levi subgroups that are conjugate in $\mathrm{GL}_n(q)$. This means that there is a $\sigma \in S_n$ such that $d_i - d_{i-1} = d'_{\sigma(i)} - d'_{\sigma(i)-1}$ for all $i = 1, \dots, t$, with the convention that $d_0 = d'_0 = 0$. Let $U = U(\boldsymbol{d})$ and $V = U(\boldsymbol{d}')$. A consequence of [HR, Corollary 4.7] is that the number $k(P, U)$ of P-conjugacy classes in U is the same as $k(Q, V)$; see [Goodwin and Röhrle 2009a, Corollary 4.8] for similar phenomena. However, it is not always the case that the number of conjugacy classes of U is the same as the number of conjugacy classes of V. For example, take $t = 3$ and consider the dimension vectors $\boldsymbol{d} = (2, 3, 4)$ and $\boldsymbol{d}' = (1, 3, 4)$. Then $P(\boldsymbol{d})$ and $P(\boldsymbol{d}')$ are associated parabolic subgroups of $\mathrm{GL}_4(q)$. Let $U = U(\boldsymbol{d})$ and $V = U(\boldsymbol{d}')$. Then by direct calculation one can check that

$$\begin{aligned} k(U) &= (q-1)^3 + 6(q-1)^2 + 5(q-1) + 1 \\ &\neq (q-1)^4 + 4(q-1)^3 + 6(q-1)^2 + 5(q-1) + 1 = k(V). \end{aligned}$$

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THREE CLASSES OF PSEUDOSYMMETRIC CONTACT METRIC 3-MANIFOLDS

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We study the class of pseudosymmetric contact metric 3-manifolds satisfying $Q\xi = \rho\xi$, where ρ is a smooth function constant along the characteristic flow. We classify the complete pseudosymmetric contact metric 3-manifolds of constant type satisfying $Q\xi = \rho\xi$, where ρ is a smooth function, and we also classify the complete (κ, μ, ν)-contact metric pseudosymmetric 3-manifolds of constant type.

1. Introduction

A Riemannian manifold (M^m, g) is said to be *semisymmetric* if its curvature tensor R satisfies the condition $R(X, Y) \cdot R = 0$ for all vector fields X, Y on M, where the dot means that $R(X, Y)$ acts as a derivation on R [Szabó 1982; 1985]. Semisymmetric Riemannian manifolds were first studied by E. Cartan. Obviously, locally symmetric spaces (those with $\nabla R = 0$) are semisymmetric, but the converse is not true, as was proved by H. Takagi [1972].

According to R. Deszcz [1992], a Riemannian manifold (M^m, g) is pseudosymmetric if its curvature tensor R satisfies $R(X, Y) \cdot R = L((X \wedge Y) \cdot R)$, where L is a smooth function and the endomorphism field $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y \tag{1-1}$$

for all vectors fields X, Y, Z on M, and $X \wedge Y$ similarly acts as a derivation on R.

The condition $R(X, Y) \cdot R = L((X \wedge Y) \cdot R)$ arose in the study of totally umbilical submanifolds of semisymmetric manifolds, as well as in the study of geodesic mappings of semisymmetric manifolds [Deszcz 1992]. If L is constant, M is called a pseudosymmetric manifold of constant type. Obviously, pseudosymmetric spaces generalize the semisymmetric ones where $L = 0$. In dimension 3, the pseudosymmetry condition of constant type is equivalent to the condition that the eigenvalues ρ_1, ρ_2, ρ_3 of the Ricci tensor satisfy $\rho_1 = \rho_2$ (up to numeration) and $\rho_3 = \text{constant}$ [Deprez et al. 1989; Kowalski and Sekizawa 1996b].

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Three-dimensional pseudosymmetric spaces of constant type have been studied by O. Kowalski and M. Sekizawa [1996b; 1996a; 1997; 1998]. N. Hashimoto and M. Sekizawa [2000] classified 3-dimensional conformally flat pseudosymmetric spaces of constant type, while G. Calvaruso [2006] gave the complete classification of conformally flat pseudosymmetric spaces of constant type for dimensions greater than two. J. T. Cho and J. Inoguchi [2005] studied pseudosymmetric contact homogeneous 3-manifolds. Finally, M. Belkhelfa, R. Deszcz and L. Verstraelen [Belkhelfa et al. 2005] studied pseudosymmetric Sasakian space forms in arbitrary dimension.

This article studies 3-dimensional pseudosymmetric contact metric manifolds, and is organized as follows. In Section 2, we give some preliminaries on pseudosymmetric manifolds and contact manifolds as well. In Section 3, we give the necessary conditions for a 3-dimensional contact metric manifold to be pseudosymmetric. In the remaining sections, we use the results of Section 3 to study 3-dimensional contact metric manifolds that satisfy one of the following:

- M is pseudosymmetric with $Q\xi = \rho\xi$, where ρ is a smooth function on M constant along the characteristic flow.
- M is pseudosymmetric of constant type with $Q\xi=\rho\xi$, where ρ a smooth function on M.
- M is pseudosymmetric of constant type and its curvature satisfies the (κ, μ, ν)-condition.

2. Preliminaries

Let (M^m, g) for $m \geq 3$ be a connected Riemannian smooth manifold. We denote by ∇ the Levi-Civita connection of M^m and by R the corresponding Riemannian curvature tensor with $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$.

A Riemannian manifold (M^m, g) for $m \geq 3$ was called *pseudosymmetric* by R. Deszcz [1992] if at every point of M the curvature tensor satisfies

$$(R(X, Y)\cdot R)(X_1, X_2, X_3) = L(((X \wedge Y)\cdot R)(X_1, X_2, X_3))$$

or equivalently

$$\begin{aligned}(2\text{-}1)\quad R(X, Y)(R(X_1, X_2)X_3) &- R(R(X, Y)X_1, X_2)X_3 \\ &- R(X_1, R(X, Y)X_2)X_3 - R(X_1, X_2)(R(X, Y)X_3) \\ = L\big((X \wedge Y)(R(X_1, X_2)X_3) &- R((X \wedge Y)X_1, X_2)X_3 \\ &- R(X_1, (X \wedge Y)X_2)X_3 - R(X_1, X_2)((X \wedge Y)X_3)\big)\end{aligned}$$

for all vectors fields X, Y, X_1, X_2, X_3 on M, where $X \wedge Y$ is given by (1-1) and L is a smooth function. For details and examples of pseudosymmetric manifolds, see [Belkhelfa et al. 2002; Deszcz 1992].

A contact manifold is a smooth manifold M^{2n+1} endowed with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Then there is an underlying contact metric structure (η, ξ, ϕ, g), where g is a Riemannian metric (the *associated metric*), ϕ is a global tensor of type (1, 1), and ξ is a unique global vector field (the *characteristic* or *Reeb vector field*). These structure tensors satisfy

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(X) = g(X, \xi), \qquad \eta(\xi) = 1,$$
$$d\eta(X, Y) = g(X, \phi Y), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2-2}$$

The associated metrics can be constructed by the polarization of $d\eta$ on the contact subbundle defined by $\eta = 0$. Denoting by L the Lie differentiation, we define the tensors

$$h = \tfrac{1}{2} L_\xi \phi, \quad \tau = L_\xi g, \quad l = R(\,\cdot\,, \xi)\xi. \tag{2-3}$$

These tensors satisfy the formulas

$$\begin{aligned}
&\phi\xi = h\xi = l\xi = 0, \qquad \eta \circ \phi = \eta \circ h = 0, \qquad d\eta(\xi, X) = 0,\\
&\operatorname{Tr} h = \operatorname{Tr} h\phi = 0, \qquad \nabla_X \xi = -\phi X - \phi h X, \qquad h\phi = -\phi h,\\
&hX = \lambda X \quad \text{implies} \quad h\phi X = -\lambda \phi X,\\
&\nabla_\xi h = \phi - \phi l - \phi h^2, \quad \phi l \phi - l = 2(\phi^2 + h^2),\\
&\nabla_\xi \phi = 0, \qquad \operatorname{Tr} l = g(Q\xi, \xi) = 2n - \operatorname{Tr} h^2.
\end{aligned} \tag{2-4}$$

Now $\tau = 0$ (or equivalently $h = 0$) if and only if ξ is Killing, and then M is called K-contact. If the structure is normal, it is Sasakian. A K-contact structure is Sasakian only in dimension 3, and this fails in higher dimensions. For details about contact manifolds, see [Blair 2002].

Let (M, ϕ, ξ, η, g) be a 3-dimensional contact metric manifold. Let U be the open subset of points $p \in M$ such that $h \neq 0$ in a neighborhood of p, and let U_0 be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. Because h is a smooth function on M, the set $U \cup U_0$ is an open and dense subset of M; thus a property that is satisfied in $U_0 \cup U$ is also satisfied in M. For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p (a ϕ-basis). On U, we put $he = \lambda e$, where λ is a nonvanishing smooth function that is supposed positive. From the third line of (2-4), we have $h\phi e = -\lambda \phi e$.

Lemma 2.1 [Gouli-Andreou and Xenos 1998a]. *On U we have*

$$\begin{aligned} &\nabla_\xi e = a\phi e, && \nabla_e e = b\phi e, && \nabla_{\phi e} e = -c\phi e + (\lambda-1)\xi, \\ &\nabla_\xi \phi e = -ae, && \nabla_e \phi e = -be + (1+\lambda)\xi, && \nabla_{\phi e}\phi e = ce, \\ &\nabla_\xi \xi = 0, && \nabla_e \xi = -(1+\lambda)\phi e, && \nabla_{\phi e}\xi = (1-\lambda)e, \end{aligned}$$

where a is a smooth function and

$$\begin{aligned} b &= \frac{1}{2\lambda}((\phi e\cdot\lambda) + A), \quad \text{with } A = S(\xi, e), \\ c &= \frac{1}{2\lambda}((e\cdot\lambda) + B), \quad \text{with } B = S(\xi, \phi e). \end{aligned} \tag{2-5}$$

From Lemma 2.1 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$ we can prove that

$$\begin{aligned} [e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi, \\ [e, \xi] &= \nabla_e \xi - \nabla_\xi e = -(a+\lambda+1)\phi e, \\ [\phi e, \xi] &= \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a-\lambda+1)e, \end{aligned} \tag{2-6}$$

and from (1-1) we estimate

$$\begin{aligned} &(e\wedge\phi e)e = -\phi e, && (e\wedge\xi)e = -\xi, && (\phi e\wedge\xi)\xi = \phi e, \\ &(e\wedge\phi e)\phi e = e, && (e\wedge\xi)\xi = e, && (\phi e\wedge\xi)\phi e = -\xi, \end{aligned} \tag{2-7}$$

while $(X\wedge Y)Z = 0$ whenever $X \neq Y \neq Z \neq X$ and $X, Y, Z \in \{e, \phi e, \xi\}$.

By direct computations we calculate the nonvanishing independent components of the Riemannian $(1, 3)$ curvature tensor field R to be

$$\begin{aligned} &R(\xi, e)\xi = -Ie - Z\phi e, && R(e, \phi e)e = -C\phi e - B\xi, \\ &R(\xi, \phi e)\xi = -Ze - D\phi e, && R(\xi, e)\phi e = -Ke + Z\xi, \\ &R(e, \phi e)\xi = Be - A\phi e, && R(\xi, \phi e)\phi e = He + D\xi, \\ &R(\xi, e)e = K\phi e + I\xi, && R(e, \phi e)\phi e = Ce + A\xi, \\ &R(\xi, \phi e)e = -H\phi e + Z\xi, \end{aligned} \tag{2-8}$$

where

$$\begin{aligned} C &= -b^2 - c^2 + \lambda^2 - 1 + 2a + (e\cdot c) + (\phi e\cdot b), \\ H &= b(\lambda - a - 1) + (\xi\cdot c) + (\phi e\cdot a), \\ K &= c(\lambda + a + 1) + (\xi\cdot b) - (e\cdot a), \\ I &= -2a\lambda - \lambda^2 + 1, \\ D &= 2a\lambda - \lambda^2 + 1, \\ Z &= \xi\cdot\lambda. \end{aligned} \tag{2-9}$$

Setting $X = e$, $Y = \phi e$ and $Z = \xi$ in the Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ and using (2-6), we get

$$\begin{aligned} b(a+\lambda+1) - (\xi\cdot c) - (\phi e\cdot\lambda) - (\phi e\cdot a) &= 0,\\ c(a-\lambda+1) + (\xi\cdot b) + (e\cdot\lambda) - (e\cdot a) &= 0, \end{aligned} \tag{2-10}$$

or equivalently $A = H$ and $B = K$.

The components of the Ricci operator Q with respect to a ϕ-basis are

$$\begin{aligned} Qe &= (\tfrac{1}{2}r - 1 + \lambda^2 - 2a\lambda)e + Z\phi e + A\xi,\\ Q\phi e &= Ze + (\tfrac{1}{2}r - 1 + \lambda^2 + 2a\lambda)\phi e + B\xi,\\ Q\xi &= Ae + B\phi e + 2(1-\lambda^2)\xi, \end{aligned} \tag{2-11}$$

where

$$r = \operatorname{Tr} Q = 2(1 - \lambda^2 - b^2 - c^2 + 2a + (e\cdot c) + (\phi e\cdot b)). \tag{2-12}$$

The relations (2-9) and (2-12) yield

$$C = -b^2 - c^2 + \lambda^2 - 1 + 2a + (e\cdot c) + (\phi e\cdot b) = 2\lambda^2 - 2 + \tfrac{1}{2}r, \tag{2-13}$$

and the relation on the last line of (2-4) gives $\operatorname{Tr} l = 2(1-\lambda^2)$.

Definition 2.2 [Gouli-Andreou et al. 2008]. Let M^3 be a 3-dimensional contact metric manifold and $h = \lambda h^+ - \lambda h^-$ the spectral decomposition of h on U. If

$$\nabla_{h^- X} h^- X = [\xi, h^+ X]$$

for all vector fields X on M^3 and all points of an open subset W of U, and if $h = 0$ on the points of M^3 that do not belong to W, then the manifold is said to be a *semi-K-contact* manifold.

From Lemma 2.1 and the relations (2-6), the condition above leads to $[\xi, e] = 0$ when $X = e$ and to $\nabla_{\phi e}\phi e = 0$ when $X = \phi e$. Hence on a semi-K-contact manifold, we have $a + \lambda + 1 = c = 0$. If we apply the deformation

$$e \to \phi e, \quad \phi e \to e, \quad \xi \to -\xi, \quad \lambda \to -\lambda, \quad b \to c, \quad c \to b,$$

the contact metric structure remains the same. Hence a 3-dimensional contact metric manifold is semi-K-contact if $a - \lambda + 1 = b = 0$.

Definition 2.3. In [Koufogiorgos et al. 2008], a (κ, μ, ν)-contact metric manifold is a contact metric manifold $(M^{2n+1}, \eta, \xi, \phi, g)$ on which the curvature tensor satisfies for every $X, Y \in X(M)$ the condition

$$\begin{aligned} R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) &+ \mu(\eta(Y)hX - \eta(X)hY)\\ &+ \nu(\eta(Y)\phi hX - \eta(X)\phi hY), \end{aligned} \tag{2-14}$$

where κ, μ, ν are smooth functions on M. If $\nu = 0$, we have a generalized (κ, μ)-contact metric manifold [Koufogiorgos and Tsichlias 2000], and if also κ, μ are constants, then M is a contact metric (κ, μ)-space [Blair et al. 1995; Boeckx 2000].

In [Koufogiorgos et al. 2008], it was proved that for a (κ, μ, ν)-contact metric manifold M^{2n+1} of dimension greater than 3, the functions κ and μ are constants and ν is the zero function; in [Koufogiorgos and Tsichlias 2000], this was proved for generalized (κ, μ)-contact metric manifolds M^{2n+1} of dimension greater than 3.

Remark 2.4. If $M^3 = U_0$, the case treated in [Gouli-Andreou and Xenos 1998b], then Lemma 2.1 is expressed in a similar form with $\lambda = 0$, e is a unit vector field belonging to the contact distribution, and the functions A, B, D, H, I, K and Z satisfy $A = B = Z = H = K = 0$, $I = D = 1$ and $C = r/2 - 2$.

Proposition 2.5. *In a* 3*-dimensional contact metric manifold*, *we have*

$$Q\phi = \phi Q \quad \textit{if and only if} \quad \xi \cdot \lambda = 2b\lambda - (\phi e \cdot \lambda) = 2c\lambda - (e \cdot \lambda) = a\lambda = 0. \tag{2-15}$$

Proof. The relations (2-11) by (2-2), (2-5), (2-9) and (2-13) yield

$$\begin{aligned}(Q\phi - \phi Q)e &= 2Ze + 4a\lambda\phi e + B\xi,\\ (Q\phi - \phi Q)\phi e &= 4a\lambda e - 2Z\phi e - A\xi,\\ (Q\phi - \phi Q)\xi &= Be - A\phi e,\end{aligned}$$

from which the proposition follows. □

3. Pseudosymmetric contact metric 3-manifolds

Let (M, η, g, ϕ, ξ) be a contact metric 3-manifold. In case $M = U_0$, that is, (ξ, η, ϕ, g) is a Sasakian structure, then M is a pseudosymmetric space of constant type [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis as in Lemma 2.1.

Lemma 3.1. *A contact metric* 3*-manifold* (M, η, g, ϕ, ξ) *is pseudosymmetric if and only if*

$$\begin{cases} B(\xi\cdot\lambda)+(-2a\lambda-\lambda^2+1)A = LA,\\ A(\xi\cdot\lambda)+(2a\lambda-\lambda^2+1)B = LB,\\ (\xi\cdot\lambda)(\tfrac{1}{2}r+2\lambda^2-2)+AB = L(\xi\cdot\lambda),\\ A^2-|(\xi\cdot\lambda)|^2+(2a\lambda-\lambda^2+1)(-2a\lambda-3\lambda^2+3-\tfrac{1}{2}r)\\ \qquad\qquad = L(-2a\lambda-3\lambda^2+3-\tfrac{1}{2}r),\\ B^2-|(\xi\cdot\lambda)|^2+(-2a\lambda-\lambda^2+1)(2a\lambda-3\lambda^2+3-\tfrac{1}{2}r)\\ \qquad\qquad = L(2a\lambda-3\lambda^2+3-\tfrac{1}{2}r),\end{cases} \tag{3-1}$$

where L *is the function in the pseudosymmetry definition* (2-1).

Proof. Setting $X_1 = e,\ X_2 = \phi e$ and $X_3 = \xi$ in (2-1), we obtain

$$(R(X,Y)\cdot R)(e,\phi e,\xi) = L\big(((X\wedge Y)\cdot R)(e,\phi e,\xi)\big).$$

First we set $X = e$ and $Y = \phi e$. Then by virtue of (2-7) and (2-8), we obtain

$$(B(\xi\cdot\lambda)+(-2a\lambda-\lambda^2+1)A)e+(A(\xi\cdot\lambda)+(2a\lambda-\lambda^2+1)B)\phi e = L(Ae+B\phi e),$$

from which the first two equations of (3-1) follow at once.

Similarly, setting $X = \phi e$, $Y = \xi$ we obtain

$$\begin{aligned}&\big(A^2-|(\xi\cdot\lambda)|^2+(2a\lambda-\lambda^2+1)(-2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\big)e\\ &\quad+\big((\xi\cdot\lambda)(\tfrac{1}{2}r+2\lambda^2-2)+AB\big)\phi e = L\big((-2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)e+(\xi\cdot\lambda)\phi e\big),\end{aligned}$$

from which we get the next two equations of (3-1).

Finally, setting $X = e$ and $Y = \xi$, we have

$$\begin{aligned}&\big(B^2-|(\xi\cdot\lambda)|^2+(-2a\lambda-\lambda^2+1)(2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\big)\phi e\\ &\qquad+\big((\xi\cdot\lambda)(\tfrac{1}{2}r+2\lambda^2-2)+AB\big)e = L\big((2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\phi e+(\xi\cdot\lambda)e\big),\end{aligned}$$

from which we obtain the last equation of (3-1). Using the equations (2-9) and (2-13), the system (3-1) takes the convenient form

$$\begin{aligned}ZB+IA &= LA,\\ ZA+DB &= LB,\\ ZC+AB &= LZ,\\ A^2-Z^2+D(I-C) &= L(I-C),\\ B^2-Z^2+I(D-C) &= L(D-C).\end{aligned}\tag{3-2}$$

□

Remark 3.2. If $L = 0$, the manifold is semisymmetric and the system (3-2) is in accordance with [Calvaruso and Perrone 2002, equations (3.1)–(3.5)].

Remark 3.3. If the manifold M^3 is Sasakian and we work in a similar way, then (3-2) is reduced to the equation $(C-1)(L-1) = 0$. Cho and Inoguchi [2005] proved that M is a pseudosymmetric space of constant type. Hence, a Sasakian 3-manifold satisfying the condition $R(X,Y)\cdot R = L((X\wedge Y)\cdot R)$ with $L\neq 1$ is a space of constant scalar curvature $r = 6$, where L is some constant function on M^3.

Proposition 3.4. *Let* M^3 *be a* 3*-dimensional contact metric manifold satisfying* $Q\phi = \phi Q$*. Then* M^3 *is a pseudosymmetric space of constant type.*

Proof. Cho and Inoguchi [2005] have proved that contact metric 3-manifolds satisfying $Q\phi = \phi Q$ are pseudosymmetric. We know from [Blair et al. 1990] that in these manifolds the Ricci operator has the form $QX = \alpha X + \beta\eta(X)\xi$ or equivalently the Ricci tensor is given by the equation

$$S = \alpha g + \beta\eta \otimes \eta,$$

where $\alpha = \frac{1}{2}(r - \operatorname{Tr} l)$ and $\beta = \frac{1}{2}(3\operatorname{Tr} l - r)$, and the functions of the ϕ-sectional curvature and $\operatorname{Tr} l$ are constants. By [Koufogiorgos 1995], the ϕ-sectional curvature is given by $r/2 - \operatorname{Tr} l$. Hence in contact metric 3-manifolds with $Q\phi = \phi Q$, the function $r = \operatorname{Tr} Q$ is also constant; obviously the functions α and β in the equations above are constants as well. The manifold is quasi-Einstein and hence pseudosymmetric, and because β is constant it is pseudosymmetric of constant type, that is, L is constant. □

Remark 3.5. In dimension 3, the pseudosymmetry condition is equivalent to the Ricci-pseudosymmetry condition $R(X, Y)\cdot S = L((X \wedge Y)\cdot S)$, so (3-2) is also valid for the Ricci-pseudosymmetric contact metric 3-manifolds [Arslan et al. 1997].

4. Pseudosymmetric contact metric 3-manifolds with $Q\xi = \rho\xi$ and ρ constant in the direction of ξ

Theorem 4.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold such that $Q\xi = \rho\xi$, where ρ is a smooth function on M^3 constant along the characteristic direction ξ. Then there are at most six open subsets of M^3 for which their union is an open and dense subset inside of the closure of M^3 and each of them as an open submanifold of M^3 is either*

(a) *a Sasakian manifold*,

(b) *flat*,

(c) *locally isometric to one of the Lie groups* SU(2) *or* SL(2, $\mathbb{R}$) *equipped with a left invariant metric*,

(d) *pseudosymmetric of constant type L and of constant scalar curvature r equal to* $2(1 - \lambda^2 + 2a)$,

(e) *semi-K contact with* $L = -3a^2 - 4a$, *or*

(f) *semi-K contact with* $L = a^2$.

Proof. We consider these next open subsets of M:

$$U_0 = \{p \in M : \lambda = 0 \quad \text{in a neighborhood of p}\},$$
$$U = \{p \in M : \lambda \neq 0 \quad \text{in a neighborhood of p}\},$$

where $U_0 \cup U$ is open and dense subset of M.

If $M = U_0$, then M is a pseudosymmetric space of constant type [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis.

The assumption $Q\xi = \rho\xi$ and (2-11) imply

$$\phi e \cdot \lambda = 2b\lambda, \tag{4-1}$$
$$e \cdot \lambda = 2c\lambda, \tag{4-2}$$
$$\rho = 2(1-\lambda^2), \tag{4-3}$$

where the smooth function ρ satisfies

$$\xi \cdot \rho = 0. \tag{4-4}$$

From (2-10), (4-1) and (4-2), we have

$$\xi \cdot c = -(\phi e \cdot a) + b(a - \lambda + 1), \tag{4-5}$$
$$\xi \cdot b = (e \cdot a) - c(\lambda + a + 1). \tag{4-6}$$

Under the conditions (4-1) and (4-2), the system (3-2) becomes

$$\begin{aligned} (C-L)Z &= 0,\\ -Z^2 + (D-L)(I-C) &= 0,\\ -Z^2 + (I-L)(D-C) &= 0, \end{aligned} \tag{4-7}$$

where Z, C, I, D are given by (2-9) and (2-13) and L is the smooth function of the pseudosymmetry condition.

From equations (4-3) and (4-4) we can deduce everywhere in U that

$$\xi \cdot \lambda = 0. \tag{4-8}$$

Differentiating the equations (4-1) and (4-2) with respect to e and ϕe respectively and subtracting, we get

$$[e, \phi e]\lambda = 2b(e \cdot \lambda) + 2\lambda(e \cdot b) - 2c(\phi e \cdot \lambda) - 2\lambda(\phi e \cdot c),$$

or because of (2-6), (4-1), (4-2) and (4-8), we obtain

$$e \cdot b = \phi e \cdot c. \tag{4-9}$$

Differentiating Equations (4-1) and (4-8) with respect to ξ and ϕe respectively and subtracting, we obtain $[\xi, \phi e]\lambda = 2\lambda(\xi \cdot b)$ or because of (2-6), (4-2) and (4-6)

$$\xi \cdot b = c(\lambda - a - 1), \tag{4-10}$$
$$e \cdot a = 2c\lambda. \tag{4-11}$$

Differentiating (4-2) and (4-8) with respect to ξ and e respectively and subtracting we obtain $[\xi, e]\lambda = 2\lambda(\xi \cdot c)$ or because of (2-6), (4-1) and (4-5)

$$\xi \cdot c = b(\lambda + a + 1), \tag{4-12}$$

$$\phi e \cdot a = -2b\lambda. \tag{4-13}$$

Differentiating (4-11) and (4-13) with respect to ϕe and e respectively and subtracting, we get

$$[\phi e, e]a = 2b(e \cdot \lambda) + 2\lambda(e \cdot b) + 2c(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot c)$$

or because of (2-6), (4-1), (4-2), (4-9), (4-11) and (4-13)

$$\xi \cdot a = -2\lambda(e \cdot b) - 2bc\lambda \tag{4-14}$$

Under the condition (4-8) everywhere in U the system (4-7) becomes

$$\begin{cases} (I - C)(D - L) = 0, \\ (D - C)(I - L) = 0. \end{cases}$$

or equivalently

$$\begin{cases} (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L) = 0, \\ (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(-2a\lambda - \lambda^2 + 1 - L) = 0. \end{cases}$$

To study this system we consider the open subsets

$$V = \{p \in U : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \text{ in a neighborhood of } p\},$$

$$V' = \{p \in U : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0 \text{ in a neighborhood of } p\},$$

where $V \cup V'$ is open and dense in the closure of U. We also have the equation

$$(-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L) = 0.$$

Hence we consider the open subsets

$$V_1 = \{p \in V : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \text{ in a neighborhood of } p\},$$

$$V_2 = \{p \in V : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0 \text{ in a neighborhood of } p\},$$

where the set $V_1 \cup V_2$ is open and dense in the closure of V. For V', in which $-2a\lambda - \lambda^2 + 1 - L = 0$, we consider the open subsets

$$V_3 = \{p \in V' : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \quad \text{in a neighborhood of } p\},$$
$$V_4 = \{p \in V' : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0 \quad \text{in a neighborhood of } p\},$$

where $V_3 \cup V_4$ is open and dense in the closure of V'. We describe the previous sets more precisely as

$$\begin{aligned} V_1 = \{p \in V \subseteq U :\ & -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \\ & \text{in a neighborhood of } p\}, \\ V_2 = \{p \in V \subseteq U :\ & 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & 2a\lambda - \lambda^2 + 1 - L = 0 \\ & \text{in a neighborhood of } p\}, \\ V_3 = \{p \in V' \subseteq U :\ & -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & -2a\lambda - \lambda^2 + 1 - L = 0 \\ & \text{in a neighborhood of } p\}, \\ V_4 = \{p \in V' \subseteq U :\ & -2a\lambda - \lambda^2 + 1 - L = 0, \\ & 2a\lambda - \lambda^2 + 1 - L = 0 \quad \text{in a neighborhood of } p\}, \end{aligned}$$

and the set $\bigcup V_i$ is open and dense in the closure of U.

In V_1, we have

$$\begin{aligned} -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) &= 0, \\ 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) &= 0. \end{aligned}$$

Subtracting these two equations we find that $a = 0$ in $V_1 \subset U$. Hence we conclude that the structure has the property $Q\phi = \phi Q$ (Proposition 2.5), that L is constant (Proposition 3.4) and the classification results from [Blair et al. 1990] and [Blair and Chen 1992] hold.

In V_2, we have

$$\begin{aligned} 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) &= 0, \\ -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) &\neq 0, \end{aligned}$$

(hence $a \neq 0$) or equivalently

$$2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \tag{4-15}$$

$$2a\lambda - \lambda^2 + 1 - L = 0. \tag{4-16}$$

Differentiating (4-15) with respect to ξ and using (4-8), (4-10), (4-12) and (4-14), we obtain

$$\xi \cdot e \cdot c + \xi \cdot \varphi e \cdot b = -4bc\lambda^2 + 8bc\lambda - 4\lambda^2(e \cdot b) + 4\lambda(e \cdot b). \tag{4-17}$$

Differentiating (4-10) and (4-12) with respect to ϕe and e respectively, we use (4-1), (4-2), (4-9), (4-11), (4-13), and adding we obtain

$$\phi e \cdot \xi \cdot b + e \cdot \xi \cdot c = 2\lambda(e \cdot b) + 8bc\lambda. \tag{4-18}$$

Subtract (4-17) and (4-18) and using (2-6), (4-9) and (4-14), we obtain

$$e \cdot b = \phi e \cdot c = -bc, \tag{4-19}$$

$$\xi \cdot a = 0. \tag{4-20}$$

Differentiating (4-20) and (4-13) with respect to ϕe and ξ respectively and subtracting, we obtain $[\phi e, \xi]a = 2\lambda(\xi \cdot b)$, or because of (2-6), (4-10), (4-11) and since $\lambda \neq 0$ in U, we have

$$c(a - \lambda + 1) = 0. \tag{4-21}$$

Differentiating (4-20) and (4-11) with respect to e and ξ respectively and subtracting, we obtain $[\xi, e]a = 2\lambda(\xi \cdot c)$, or because of (2-6), (4-12), (4-13) and since $\lambda \neq 0$ in U, we have

$$b(a + \lambda + 1) = 0. \tag{4-22}$$

Differentiating (4-16) with respect to ξ, ϕe and e and using (4-1), (4-2), (4-8), (4-11), (4-13) and (4-20) we obtain respectively

$$\xi \cdot L = 0, \tag{4-23}$$

$$\phi e \cdot L = 4ab\lambda - 8b\lambda^2, \tag{4-24}$$

$$e \cdot L = 4ac\lambda. \tag{4-25}$$

To study the system (4-21) and (4-22), we consider the open subsets

$$G = \{p \in V_2 : b = 0 \text{ in a neighborhood of } p\},$$

$$G' = \{p \in V_2 : b \neq 0 \text{ in a neighborhood of } p\},$$

where $G \cup G'$ is open and dense in the closure of V_2. Having also $c(\lambda - a - 1) = 0$ we consider the open subsets

$$G_1 = \{p \in G : c = 0 \text{ in a neighborhood of } p\},$$
$$G_2 = \{p \in G : c \neq 0 \text{ in a neighborhood of } p\},$$

where $G_1 \cup G_2$ is open and dense in the closure of G. The set G' (where $b \neq 0$ or equivalently $\lambda + a + 1 = 0$) is decomposed similarly as

$$G_3 = \{p \in G' : c = 0 \text{ in a neighborhood of } p\},$$
$$G_4 = \{p \in G' : c \neq 0 \text{ in a neighborhood of } p\},$$

where $G_3 \cup G_4$ is open and dense in the closure of G'. The sets G_1, G_2, G_3 and G_4 are described more specifically as

$$G_1 = \{p \in G \subset V_2 : b = c = 0 \text{ in a neighborhood of } p\},$$
$$G_2 = \{p \in G \subset V_2 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_3 = \{p \in G' \subset V_2 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_4 = \{p \in G' \subset V_2 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$

The set $\bigcup G_i$ is open and dense subset of V_2. We have $V_2 \subset U$, where $\lambda \neq 0$; hence $G_4 = \varnothing$.

In G_1, we have $b = 0$ and $c = 0$. From (4-1), (4-2), (4-8), (4-11), (4-13), (4-14), (4-23), (4-24) and (4-25), we find that λ, a and L are constant in G_1 with $\lambda, a \neq 0$; hence from (2-12) the scalar curvature $r = 2(1 - \lambda^2 + 2a)$ is also constant.

In G_2, we have $b = 0$ and $\lambda - a - 1 = 0$. Hence we have a semi-K contact structure. Then (4-16) and $a = \lambda - 1$ give $L = (\lambda - 1)^2 = a^2 \neq 0$.

In G_3, we have $c = 0$ and $\lambda + a + 1 = 0$. Similarly, we have a semi-K contact structure with $L = -3\lambda^2 - 2\lambda + 1 = -3a^2 - 4a$, with $a \neq 0$.

In V_3,

$$-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \tag{4-26}$$
$$-2a\lambda - \lambda^2 + 1 - L = 0. \tag{4-27}$$

We similarly obtain the system of (4-21) and (4-22) with $a \neq 0$, while for the function L, we have (4-23) as well as $\phi e \cdot L = -4ab\lambda$ and $e \cdot L = -4ac\lambda - 8c\lambda^2$.

We consider the open subsets

$$G'_1 = \{p \in V_3 : b = c = 0 \text{ in a neighborhood of p}\},$$
$$G'_2 = \{p \in V_3 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of p}\},$$
$$G'_3 = \{p \in V_3 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of p}\},$$
$$G'_4 = \{p \in V_3 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of p}\}.$$

The set $\bigcup G'_i$ is open and dense subset of V_3. We have $V_3 \subset U$, where $\lambda \neq 0$; hence G'_4 is empty.

In G'_1, we have $b = 0$ and $c = 0$. As in case of G_1, the functions λ, a, L and r are constants.

In G'_2, we have $b = 0$ and $\lambda - a - 1 = 0$. Hence we have a semi-K contact structure with $L = -3\lambda^2 + 2\lambda + 1 = -3a^2 - 4a$, with $a \neq 0$.

In G'_3, we have $c = 0$ and $\lambda + a + 1 = 0$. We have a semi-K contact structure with $L = (\lambda + 1)^2 = a^2 \neq 0$.

In V_4 we have $-2a\lambda - \lambda^2 + 1 - L = 0$ and $2a\lambda - \lambda^2 + 1 - L = 0$. Subtracting these two equations we obtain $a = 0$ in $V_4 \subset U$, and hence as in case of V_1 we have the structure $Q\phi = \phi Q$.

Finally, the sets U_0, V_1 and V_4, G_1 and G'_1, G_3 and G'_2, G_2 and G'_3 satisfy the structures a, b and c, d, e and f respectively of Theorem 4.1. □

5. Pseudosymmetric contact metric 3-manifolds of constant type with $Q\xi = \rho\xi$

Theorem 5.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold of constant type such that $Q\xi = \rho\xi$, where ρ is a smooth function on M^3. Then ρ is constant. If M^3 is also complete then it is either a Sasakian manifold (meaning* $\operatorname{Tr} l = 2$*) or locally isometric to one of the following Lie groups equipped with a left invariant metric*: SU(2); SO(3); SL(2, $\mathbb{R}$); $E(2)$, *the rigid motions of Euclidean 2-space*; $E(1,1)$, *the rigid motions of Minkowski 2-space*; *or* $O(1,2)$, *the Lorentz group of linear maps preserving the quadratic form $t^2 - x^2 - y^2$.*

Proof. We consider open subsets

$$U_0 = \{p \in M : \lambda = 0 \text{ in a neighborhood of } p\},$$

$$U = \{p \in M : \lambda \neq 0 \text{ in a neighborhood of } p\},$$

where $U_0 \cup U$ is open and dense subset of M.

If $M = U_0$, then it is a pseudosymmetric space of constant type; see [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis. The assumption $Q\xi = \rho\xi$ and (2-11) imply

$$\phi e \cdot \lambda = 2b\lambda, \tag{5-1}$$

$$e \cdot \lambda = 2c\lambda, \tag{5-2}$$

$$\rho = 2(1 - \lambda^2), \tag{5-3}$$

where ρ is a smooth function on M. From (2-10), (5-1) and (5-2) we have

$$\xi \cdot c = -(\phi e \cdot a) + b(a - \lambda + 1), \tag{5-4}$$

$$\xi \cdot b = (e \cdot a) - c(\lambda + a + 1). \tag{5-5}$$

Under the conditions (5-1) and (5-2) the system (3-2) becomes

$$
\begin{aligned}
(C-L)Z&=0,\\
-Z^2+(D-L)(I-C)&=0,\\
-Z^2+(I-L)(D-C)&=0,
\end{aligned}
\tag{5-6}
$$

where Z, C, I and D are given by (2-9) and (2-13) and L is the constant of the pseudosymmetry condition.

We work in the open subset U and suppose that there is a point p in U where $Z=\xi\cdot\lambda\neq 0$. The function Z is smooth, so because of its continuity there is an open neighborhood U_1 of p such that $U_1\subset U$ and $Z=\xi\cdot\lambda\neq 0$ everywhere in U_1. From the first equation of (5-6), we get $C=L$ in U_1, or equivalently

$$
(e\cdot c)+(\phi e\cdot b)=L+b^2+c^2-\lambda^2+1-2a. \tag{5-7}
$$

Differentiating (5-7) with respect to ξ, we get

$$
\xi\cdot e\cdot c+\xi\cdot\phi e\cdot b=2b(\xi\cdot b)+2c(\xi\cdot c)-2\lambda(\xi\cdot\lambda)-2(\xi\cdot a),
$$

which because of (5-4) and (5-5) becomes

$$
\xi\cdot e\cdot c+\xi\cdot\phi e\cdot b=2b(e\cdot a)-2c(\phi e\cdot a)-2\lambda(\xi\cdot\lambda)-2(\xi\cdot a)-4bc\lambda. \tag{5-8}
$$

Next, we differentiate (5-4) and (5-5) with respect to e and ϕe, respectively. Adding the results, we have

$$
\begin{aligned}
e\cdot\xi\cdot c+\phi e\cdot\xi\cdot b=-[e,\phi e]a-(a+\lambda+1)(\phi e\cdot c)+(a-\lambda+1)(e\cdot b)&\\
-c(\phi e\cdot a)+b(e\cdot a)-4bc\lambda.&
\end{aligned}
$$

Subtracting this from (5-8), we get

$$
\begin{aligned}
[\xi,e]c+[\xi,\phi e]b=b(e\cdot a)-c(\phi e\cdot a)-2(\xi\cdot a)-2\lambda(\xi\cdot\lambda)+[e,\phi e]a&\\
+(a+\lambda+1)(\phi e\cdot c)-(a-\lambda+1)(e\cdot b),&
\end{aligned}
$$

or because of (2-6),

$$
\begin{aligned}
&(a+\lambda+1)(\phi e\cdot c)+(\lambda-a-1)(e\cdot b)\\
&\qquad=b(e\cdot a)-c(\phi e\cdot a)-2(\xi\cdot a)-2\lambda(\xi\cdot\lambda)-b(e\cdot a)\\
&\qquad\quad+c(\phi e\cdot a)+2(\xi\cdot a)+(\lambda+a+1)(\phi e\cdot c)+(\lambda-a-1)(e\cdot b).
\end{aligned}
$$

Equivalently, $\lambda(\xi\cdot\lambda)=0$, and because we work in $U_1\subset U$, we have $\xi\cdot\lambda=0$, which is a contradiction. Hence, we can deduce everywhere in U that

$$
\xi\cdot\lambda=0. \tag{5-9}
$$

Working as previously, we obtain the equations

$$e \cdot b = \phi e \cdot c, \tag{5-10}$$

$$\xi \cdot b = c(\lambda - a - 1), \tag{5-11}$$

$$e \cdot a = 2c\lambda, \tag{5-12}$$

$$\xi \cdot c = b(\lambda + a + 1), \tag{5-13}$$

$$\phi e \cdot a = -2b\lambda. \tag{5-14}$$

Under the condition (5-9) everywhere in U the system (5-6) becomes

$$\begin{cases} (I - C)(D - L) = 0, \\ (D - C)(I - L) = 0, \end{cases}$$

or equivalently

$$\begin{cases} (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L) = 0, \\ (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(-2a\lambda - \lambda^2 + 1 - L) = 0. \end{cases}$$

To study this system, we consider (as previously) the open subsets

$$\begin{aligned} V_1 = \{p \in U :\ & -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \\ & \qquad\qquad\qquad \text{in a neighborhood of } p\}, \\ V_2 = \{p \in U :\ & 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & 2a\lambda - \lambda^2 + 1 - L = 0 \qquad \text{in a neighborhood of } p\}, \\ V_3 = \{p \in U :\ & -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & -2a\lambda - \lambda^2 + 1 - L = 0 \qquad \text{in a neighborhood of } p\}, \\ V_4 = \{p \in U :\ & -2a\lambda - \lambda^2 + 1 - L = 0, \quad 2a\lambda - \lambda^2 + 1 - L = 0, \\ & \qquad\qquad\qquad \text{in a neighborhood of } p\}. \end{aligned}$$

The set $\bigcup V_i$ is open and dense in the closure of U. We shall prove that the functions λ and a are constants at V_i for $i = 1, 2, 3, 4$.

In V_1, we have

$$-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0,$$
$$2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0.$$

Subtracting these two equations we can deduce that $a = 0$ in $V_1 \subset U$. Hence from (5-12) and (5-14), we have $c = b = 0$, and from (5-1) and (5-2), we have $\phi e \cdot \lambda = e \cdot \lambda = 0$, which together with (5-9) give $\lambda =$ constant in V_1. Moreover, if we put $a = b = c = 0$ in one of the equations of the set V_1, we finally get $\lambda^2 = 1$.

In V_2,

$$2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \tag{5-15}$$

$$2a\lambda - \lambda^2 + 1 - L = 0. \tag{5-16}$$

Differentiating (5-16) with respect to ξ, ϕe and e and using (5-9), (5-12) and (5-14), we obtain respectively

$$\xi \cdot a = 0, \tag{5-17}$$

$$b(a - 2\lambda) = 0, \quad ac = 0.$$

Differentiating (5-12) and (5-17) with respect to ξ and e respectively and subtracting, we obtain $[\xi, e]a = 2\lambda(\xi \cdot c)$ or because of (2-6), (5-13) and (5-14)

$$b(\lambda + a + 1) = 0. \tag{5-18}$$

Similarly, differentiating (5-14) with respect to ξ and (5-17) with respect to ϕe and subtracting, we have $[\xi, \phi e]a = -2\lambda(\xi \cdot b)$ or because of (2-6), (5-11) and (5-12)

$$c(\lambda - a - 1) = 0. \tag{5-19}$$

We study the system of (5-18) and (5-19). As in the previous section, we consider open subsets

$$G_1 = \{p \in V_2 : b = c = 0 \text{ in a neighborhood of } p\},$$
$$G_2 = \{p \in V_2 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_3 = \{p \in V_2 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_4 = \{p \in V_2 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$

The set $\bigcup G_i$ is open and dense subset of V_2. We have $V_2 \subset U$ where $\lambda \neq 0$; hence G_4 is empty.

In G_1, we have $b = 0$ and $c = 0$. From (5-1) and (5-2) we can conclude $\phi e \cdot \lambda = e \cdot \lambda = 0$, which together with (5-9) implies λ is constant in G_1. Similarly from (5-12), (5-14) and (5-17), a is constant.

In G_2, we have $b = 0$ and $\lambda - a - 1 = 0$. The second of these together with (5-16) gives $\lambda^2 - 2\lambda + 1 - L = 0$. If we assume $e \cdot \lambda \neq 0$, we differentiate this equation twice with respect to e, and we obtain $e \cdot \lambda = 0$, which contradicts our assumption. Hence, $e \cdot \lambda = 0$ (and $c = 0$) and (5-1) gives $\phi e \cdot \lambda = 0$, or finally λ is constant in G_2 and $a = \lambda - 1$ is also constant.

In G_3, we have $c = 0$ and $\lambda + a + 1 = 0$. The first equation gives $e \cdot \lambda = 0$ by (5-2), while the second together with (5-16) gives $-3\lambda^2 - 2\lambda + 1 - L = 0$. Differentiating this equation with respect to ϕe, we get $(3\lambda + 1)(\phi e \cdot \lambda) = 0$. Suppose there is a point $p \in G_3$ at which $\phi e \cdot \lambda \neq 0$. Then, there is a neighborhood F of p in which

$\phi e \cdot \lambda \neq 0$. In that neighborhood we must have $\lambda = -1/3$ by the last equation; hence $\phi e \cdot \lambda = 0$, a contradiction. Thus $\phi e \cdot \lambda = 0$ everywhere in G_3, which gives $b = 0$. In G_3, we note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$, so λ is constant in G_3. Obviously a is also constant because $a = -\lambda - 1$. Moreover, if we put $b = c = 0$ and $a = -\lambda - 1$ in (5-15), we get $\lambda^2 = 1$.

We have proved that λ is constant at every G_i for $i = 1, 2, 3$, while the set $G_1 \cup G_2 \cup G_3$ is an open and dense subset of V_2; hence λ is constant in V_2 and the equations $b(a - 2\lambda) = 0$ and $ac = 0$ are satisfied because $b = c = 0$.

In V_3,

$$-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \tag{5-20}$$

$$-2a\lambda - \lambda^2 + 1 - L = 0. \tag{5-21}$$

Working as we did for the set V_2, we get again the first equation of (5-17), and

$$ab = 0 \quad \text{and} \quad c(a + 2\lambda) = 0$$

and the system of (5-18) and (5-19). We similarly consider the open subsets

$$G'_1 = \{p \in V_3 : b = c = 0 \text{ in a neighborhood of } p\},$$
$$G'_2 = \{p \in V_3 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$
$$G'_3 = \{p \in V_3 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$
$$G'_4 = \{p \in V_3 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$

The set $\bigcup G'_i$ is open and dense subset of V_3. We have $V_3 \subset U$ where $\lambda \neq 0$; hence G'_4 is empty.

In G'_1, we have $b = 0$ and $c = 0$. From (5-1) and (5-2), we can conclude $\phi e \cdot \lambda = e \cdot \lambda = 0$, which together with (5-9) implies λ is constant in G'_1. From (5-12), (5-14) and (5-17) we obtain that a constant in G'_1.

In G'_2, we have $b = 0$ and $\lambda - a - 1 = 0$. The first equation gives $\phi e \cdot \lambda = 0$ from (5-1), while the second together with (5-21) gives $-3\lambda^2 + 2\lambda + 1 - L = 0$. Differentiating this equation with respect to e, we get $(-3\lambda + 1)(e \cdot \lambda) = 0$. Suppose that there is a point $p \in G'_2$ at which $e \cdot \lambda \neq 0$. Then, there is a neighborhood F' of p in which $e \cdot \lambda \neq 0$. In that neighborhood we must have from the last equation that $\lambda = 1/3$ and $e \cdot \lambda = 0$, a contradiction. Hence $e \cdot \lambda = 0$ everywhere in G'_2, which gives $c = 0$. In G'_2, we note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$, so λ is constant in G'_2. Obviously a is also constant because $a = \lambda - 1$. Moreover, if we put $b = c = 0$ and $a = \lambda - 1$ in (5-20) we get $\lambda^2 = 1$.

In G'_3, we have $c = 0$ and $\lambda + a + 1 = 0$. The second equation together with (5-21) gives $\lambda^2 + 2\lambda + 1 - L = 0$. Assuming $\phi e \cdot \lambda \neq 0$, we differentiate this equation twice with respect to ϕe and obtain $\phi e \cdot \lambda = 0$, a contradiction. Thus, $\phi e \cdot \lambda = 0$

everywhere in G'_3, which gives $b = 0$. From (5-2), we get $e \cdot \lambda = 0$. We note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$, so λ is constant in G'_3 and obviously so is $a = -\lambda - 1$.

We have proved that λ is constant in every G'_i for $i = 1, 2, 3$ while the set $G'_1 \cup G'_2 \cup G'_3$ is open and dense in the closure of V_3; hence λ is constant at V_3 and the equations $ab = 0$ and $c(a + 2\lambda) = 0$ are satisfied because $b = c = 0$.

In V_4, we have $2a\lambda - \lambda^2 + 1 - L = 0$ and $-2a\lambda - \lambda^2 + 1 - L = 0$. Subtracting these two equations, we can deduce that $a = 0$ in $V_4 \subset U$. Hence from (5-12) and (5-14), we have $c = b = 0$, and from (5-1) and (5-2), we have $\phi e \cdot \lambda = e \cdot \lambda = 0$, which together with (5-9) implies λ is constant in V_4. Moreover, if we put $a = 0$ in one of the equations of the set V_4, we finally obtain $\lambda^2 = 1 - L \geq 0$.

We have proved that λ is constant in every V_i for $i = 1, 2, 3, 4$. The set $V_1 \cup V_2 \cup V_3 \cup V_4$ is open and dense inside of the closure of U; hence λ is constant at U and because of (5-3) the function ρ is constant at U. Finally if the manifold M^3 is complete, we may use the main theorem of [Koufogiorgos 1995] to complete the proof. □

6. Pseudosymmetric (κ, μ, ν)-contact metric 3-manifolds of constant type

Theorem 6.1. *A 3-dimensional (κ, μ, ν)-contact metric pseudosymmetric manifold of constant type is either a Sasakian manifold or a (κ, μ)-contact metric manifold. In the second case, if M^3 is also complete, then it is locally isometric to one of the following Lie groups equipped with a left invariant metric*: SU(2); SO(3); SL(2, $\mathbb{R}$); $E(2)$, *the rigid motions of Euclidean 2-space*; $E(1, 1)$, *the rigid motions of Minkowski 2-space*; *or* $O(1, 2)$, *the Lorentz group consisting of linear transformations preserving the quadratic form $t^2 - x^2 - y^2$*).

Proof. We work as in the previous section. If $M = U_0$, then (ξ, η, ϕ, g) is a Sasakian structure that is a pseudosymmetric space of constant type with $\kappa = 1$, $\mu \in \mathbb{R}$ and $h = 0$. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis. From (2-14) we can calculate these components of the Riemannian curvature tensor:

$$R(\xi, e)\xi = -(\kappa + \lambda\mu)e - \lambda\nu\phi e, \qquad R(e, \phi e)\xi = 0,$$
$$R(\xi, \phi e)\xi = -\lambda\nu e - (\kappa - \lambda\mu)\phi e.$$

By virtue of (2-8), we can conclude that

$$A = B = 0, \quad Z = \lambda\nu, \quad D = \kappa - \lambda\mu, \quad I = \kappa + \lambda\mu, \tag{6-1}$$

and hence the system (3-2) gives again the system (5-6). First we get $Z = \xi \cdot \lambda = 0$ or equivalently $\nu = 0$ and then that λ, a are constants. Finally from (2-9) and (6-1) we have $\kappa = 1 - \lambda^2$ and $\mu = -2a$, and from the main theorem of [Koufogiorgos 1995] and [Boeckx 2000, Theorem 3], we can complete the proof. □

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SCOTT AND SWARUP'S REGULAR NEIGHBORHOOD AS A TREE OF CYLINDERS

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Let G be a finitely presented group. Scott and Swarup have constructed a canonical splitting of G that encloses all almost invariant sets over virtually polycyclic subgroups of a given length. We give an alternative construction of this regular neighborhood by showing that it is the tree of cylinders of a JSJ splitting.

1. Introduction

Scott and Swarup [2003] have constructed a canonical graph of groups decomposition (or splitting) of a finitely presented group G; this splitting encloses all almost invariant sets over virtually polycyclic subgroups of a given length n (the VPC_n groups), and in particular over virtually cyclic subgroups for $n = 1$.

Almost invariant sets generalize splittings: Whereas a splitting is analogous to an embedded codimension-one submanifold of a manifold M, an almost invariant set is analogous to an immersed codimension-one submanifold.

Two splittings are *compatible* if they have a common refinement, in that both can be obtained from the refinement by collapsing some edges. For example, two splittings induced by disjoint embedded codimension-one submanifolds are compatible.

Enclosing is a generalization of this notion to almost invariant sets: Take, in the analogy above, two codimension-one submanifolds F_1 and F_2 of M with F_1 immersed and F_2 embedded. Then F_1 is enclosed in a connected component of $M \setminus F_2$ if one can isotope F_1 into this component.

Scott and Swarup's construction is called the *regular neighborhood* of all almost invariant sets over VPC_n subgroups. This is analogous to the topological regular neighborhood of a finite union of (nondisjoint) immersed codimension-one submanifolds: It defines a splitting that encloses the initial submanifolds.

One main virtue of their splitting is that it is canonical: It is invariant under automorphisms of G. Because of this, it is often quite different from usual JSJ

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splittings, which are unique only up to deformation. There the canonical object is the JSJ deformation space [Forester 2003; Guirardel and Levitt 2009].

The main reason for this rigidity is that the regular neighborhood is defined in terms of enclosing. Enclosing, like compatibility of splittings, is more rigid than domination, which is the basis for usual JSJ theory. For instance, any two splittings in Culler–Vogtmann's outer space dominate each other, but they are compatible if and only if they lie in a common simplex.

We have shown a general construction producing a canonical splitting T_c from a canonical deformation space: the *tree of cylinders* [Guirardel and Levitt 2008]. It also enjoys strong compatibility properties. In this paper, we show that the splitting constructed by Scott and Swarup is a subdivision of the tree of cylinders of the usual JSJ deformation space.

More precisely, let T_J be the Bass–Serre tree of a JSJ splitting of G over VPC_n groups, as constructed for instance in [Dunwoody and Sageev 1999]. To construct the tree of cylinders, say that two edges are in the same cylinder if their stabilizers are commensurable. Cylinders are subtrees, and the tree T_c dual to the covering of T_J by cylinders is *the tree of cylinders* of T_J; see [Guirardel and Levitt 2008] or Section 2b below.

Theorem 4.1. *Let G be a finitely presented group, and let $n \geq 1$. Assume that G does not split over a VPC_{n-1} subgroup, and that G is not VPC_{n+1}. Let T_J be a JSJ tree of G over VPC_n subgroups, and let T_c be its tree of cylinders for the commensurability relation.*

Then the Bass–Serre tree of Scott and Swarup's regular neighborhood of all almost invariant subsets over VPC_n subgroups is equivariantly isomorphic to a subdivision of T_c.

This gives a new proof that this regular neighborhood is a tree. Deriving the regular neighborhood from a JSJ splitting, instead of building it from an abstract betweenness relation, seems to greatly simplify the construction, by completely avoiding the notion of good or good-enough position for almost invariant subsets.

There are two ingredients in our approach, to be found in Sections 3 and 4. (Section 2 recalls basic material about trees of cylinders, almost invariant sets, cross-connected components, and regular neighborhoods.)

The first ingredient is a general fact about almost invariant sets that are *based* on a given tree T. Consider any simplicial tree T with an action of G. Any edge e separates T into two half-trees, and this defines almost invariant sets Z_e and Z_e^* (see Section 3a). The collection $\mathcal{B}(T)$ of almost invariant subsets *based* on T is then defined by taking Boolean combinations of such sets Z_e.

Following Scott and Swarup, one defines cross-connected components of $\mathcal{B}(T)$ by using *crossing* of almost invariant sets. The set of cross-connected components is then endowed with a betweenness relation that allows one to construct a bipartite graph $RN(\mathcal{B}(T))$ associated to $\mathcal{B}(T)$. This is the *regular neighborhood* of $\mathcal{B}(T)$; see Definition 2.2.

Theorem 3.3. *Let G be a finitely generated group, and T a tree with a minimal action of G. Assume that no two groups commensurable to edge stabilizers are contained in each other with infinite index.*

Then the regular neighborhood $RN(\mathcal{B}(T))$ is equivariantly isomorphic to a subdivision of T_c, the tree of cylinders of T for the commensurability relation; in particular, $RN(\mathcal{B}(T))$ is a tree.

The hypothesis about edge stabilizers holds in particular if all edge stabilizers of T are VPC_n for a fixed n.

This theorem remains true if one enlarges $\mathcal{B}(T)$ to $\mathcal{B}(T) \cup \mathrm{QH}(T)$ by including almost invariant sets enclosed by quadratically hanging vertices of T. Geometrically, such a vertex is associated to a fiber bundle over a 2-dimensional orbifold $\mathbb{O}$. Any simple closed curve on $\mathbb{O}$ gives a way to blow up T by creating new edges and therefore new almost invariant sets. These sets are in $\mathrm{QH}(T)$, as well as those associated to immersed curves on $\mathbb{O}$. Under the same hypotheses as Theorem 3.3, we show in Theorem 3.11 that the regular neighborhood $RN(\mathcal{B}(T) \cup \mathrm{QH}(T))$ also is a subdivision of T_c.

The second ingredient, specific to the VPC_n case, is due to (but not explicitly stated by) Scott and Swarup [2003]. We believe it is worth emphasizing this statement, as it gives a very useful description of almost invariant sets over VPC_n subgroups in terms of a JSJ splitting T_J. In plain words, it says that any almost invariant set over a VPC_n subgroup is either dual to a curve in a QH subgroup, or is a Boolean combination of almost invariant sets dual to half-trees of T_J.

Theorem 4.2 [Dunwoody and Swenson 2000; Scott and Swarup 2003]. *Let G and T_J be as in Theorem 4.1.*

For any almost invariant subset X over a VPC_n *subgroup, the equivalence class $[X]$ belongs to $\mathcal{B}(T_J) \cup \mathrm{QH}(T_J)$.*

Theorem 4.2 is essentially another take on the proof of Scott and Swarup's [2003, Theorem 8.2], and makes a crucial use of algebraic torus theorems of [Dunwoody and Swenson 2000; Dunwoody and Roller 1993]. We give a proof in Section 4.

Theorem 4.1 is a direct consequence of Theorems 4.2 and 3.11.

2. Preliminaries

Let G be a fixed finitely generated group, which, in Section 4, is finitely presented.

2a. ***Trees.*** If Γ is a graph, we denote by $V(\Gamma)$ its set of vertices and by $E(\Gamma)$ its set of (closed) nonoriented edges.

A tree always means a simplicial tree T on which G acts without inversions. Given a family $\mathcal{E}$ of subgroups of G, an *$\mathcal{E}$-tree* is a tree whose edge stabilizers belong to $\mathcal{E}$. We denote by G_v or G_e the stabilizer of a vertex v or an edge e.

Given a subtree A, we denote by pr_A the projection onto A, mapping x to the point of A closest to x. If A and B are disjoint, or intersect in at most one point, then $\mathrm{pr}_A(B)$ is a single point, and we define the *bridge* between A and B as the segment joining $\mathrm{pr}_A(B)$ to $\mathrm{pr}_B(A)$.

A tree T is *nontrivial* if there is no global fixed point, and *minimal* if there is no proper G-invariant subtree.

An element or a subgroup of G is *elliptic* in T if it has a global fixed point. An element that is not elliptic is *hyperbolic*. It has an axis on which it acts as a translation. If T is minimal, then it is the union of all translation axes of elements of G. In particular, if $Y \subset T$ is a subtree, then any connected component of $T \setminus Y$ is unbounded.

A subgroup A consisting only of elliptic elements fixes a point if it is finitely generated, a point or an end in general. If a finitely generated subgroup A is not elliptic, there is a unique minimal A-invariant subtree.

A tree T *dominates* a tree T' if there is an equivariant map $f : T \to T'$. Equivalently, any subgroup that is elliptic in T is also elliptic in T'. Having the same elliptic subgroups is an equivalence relation on the set of trees, and the equivalence classes are called *deformation spaces*; see [Forester 2002; Guirardel and Levitt 2007] for details.

2b. ***Trees of cylinders.*** Two subgroups A and B of G are *commensurable* if $A \cap B$ has finite index in both A and B.

Definition 2.1. We fix a conjugacy-invariant family $\mathcal{E}$ of subgroups of G such that

- any subgroup A commensurable with some $B \in \mathcal{E}$ lies in $\mathcal{E}$, and
- if $A, B \in \mathcal{E}$ are such that $A \subset B$, then $[B : A] < \infty$.

An $\mathcal{E}$-tree is a tree whose edge stabilizers belong to $\mathcal{E}$.

For instance, $\mathcal{E}$ may consist of all subgroups of G that are virtually $\mathbb{Z}^n$ for some fixed n, or all subgroups that are virtually polycyclic of Hirsch length exactly n.

In [Guirardel and Levitt 2008], we associated a tree of cylinders T_c to any $\mathcal{E}$-tree T, as follows. Two (nonoriented) edges of T are equivalent if their stabilizers are commensurable. A *cylinder* of T is an equivalence class Y. We identify Y with the union of its edges, which is a subtree of T.

Two distinct cylinders meet in at most one point. One can then define the tree of cylinders of T as the tree T_c dual to the covering of T by its cylinders, as in

[Guirardel 2004, Definition 4.8]. Formally, T_c is the bipartite tree with vertex set $V(T_c) = V_0(T_c) \sqcup V_1(T_c)$ defined as follows:

(1) $V_0(T_c)$ is the set of vertices x of T belonging to (at least) two distinct cylinders;

(2) $V_1(T_c)$ is the set of cylinders Y of T;

(3) there is an edge $\varepsilon = (x, Y)$ between $x \in V_0(T_c)$ and $Y \in V_1(T_c)$ if and only if x (viewed as a vertex of T) belongs to Y (viewed as a subtree of T).

Alternatively, one can define the *boundary* ∂Y of a cylinder Y as the set of vertices of Y belonging to another cylinder, and obtain T_c from T by replacing each cylinder by the cone on its boundary.

All edges of a cylinder Y have commensurable stabilizers, and we denote by $\mathscr{C} \subset \mathscr{E}$ the corresponding commensurability class. We sometimes view $V_1(T_c)$ as a set of commensurability classes.

2c. *Almost invariant subsets.* Given a subgroup $H \subset G$, consider the action of H on G by left multiplication. A subset $X \subset G$ is *H-finite* if it is contained in the union of finitely many H-orbits. Two subsets X and Y are *equivalent* if their symmetric difference $X + Y$ is H-finite. We denote by $[X]$ the equivalence class of X, and by X^* the complement of X.

An *H-almost invariant subset* (or an almost invariant subset over H) is a subset $X \subset G$ that is invariant under the (left) action of H and equivalent to the right-translate Xs for all $s \in G$. An H-almost invariant subset X is *nontrivial* if neither X nor its complement X^* is H-finite. Given $H < G$, the set of equivalence classes of H-almost invariant subsets is a *Boolean algebra* $\mathscr{B}_H$ for the usual operations.

If H contains H' with finite index, then any H-almost invariant subset X is also H'-almost invariant. Furthermore, two sets X and Y are equivalent over H' if and only if they are equivalent over H. It follows that, given a commensurability class $\mathscr{C}$ of subgroups of G, the set of equivalence classes of almost invariant subsets over subgroups in $\mathscr{C}$ is a *Boolean algebra* $\mathscr{B}_{\mathscr{C}}$.

Two almost invariant subsets X over H and Y over K are *equivalent* if their symmetric difference $X+Y$ is H-finite. By [Scott and Swarup 2003, Remark 2.9], this is a symmetric relation: $X + Y$ is H-finite if and only if it is K-finite. If X and Y are nontrivial, equivalence implies that H and K are commensurable.

The algebras $\mathscr{B}_{\mathscr{C}}$ are thus disjoint, except for the (trivial) equivalence classes of $\varnothing$ and G that belong to every $\mathscr{B}_{\mathscr{C}}$. We denote by $\mathscr{B}$ the union of the algebras $\mathscr{B}_{\mathscr{C}}$. It is the set of equivalence classes of all almost invariant sets, but it is not a Boolean algebra in general. There is a natural action of G on $\mathscr{B}$ induced by left translation (or conjugation).

2d. *Cross-connected components and regular neighborhoods.* Let X be an H-almost invariant subset, and Y a K-almost invariant subset. One says that X

crosses Y, or the pair $\{X, X^*\}$ crosses $\{Y, Y^*\}$, if none of the four sets $X^{(*)} \cap Y^{(*)}$ is H-finite (we denote by $X^{(*)} \cap Y^{(*)}$ the four possible intersections $X \cap Y$, $X^* \cap Y$, $X \cap Y^*$, and $X^* \cap Y^*$). By [Scott 1998], this is a symmetric relation. Note that X and Y do not cross if they are equivalent, and that crossing depends only on the equivalence classes of X and Y. Following [Scott and Swarup 2003], we will say that $X^{(*)} \cap Y^{(*)}$ is *small* if it is H-finite (or equivalently K-finite).

Now let $\mathcal{X}$ be a subset of $\mathcal{B}$. Let $\overline{\overline{\mathcal{X}}}$ be the set of nontrivial unordered pairs $\{[X], [X^*]\}$ for $[X] \in \mathcal{X}$. A *cross-connected component* (CCC) of $\mathcal{X}$ is an equivalence class C for the equivalence relation generated on $\overline{\overline{\mathcal{X}}}$ by crossing. We often say that X, rather than $\{[X], [X^*]\}$, belongs to C, or represents C. We denote by $\mathcal{H}$ the set of cross-connected components of $\mathcal{X}$.

Given three distinct cross-connected components C_1, C_2, C_3, we say that C_2 is *between* C_1 and C_3 if there are representatives X_i of C_i satisfying $X_1 \subset X_2 \subset X_3$.

A *star* is a subset $\Sigma \subset \mathcal{H}$ containing at least two elements, and maximal for the property that, given $C, C' \in \Sigma$, no $C'' \in \mathcal{H}$ is between C and C'. We denote by $\mathcal{S}$ the set of stars.

Definition 2.2. Let $\mathcal{X} \subset \mathcal{B}$ be a collection of almost invariant sets. Its *regular neighborhood* $RN(\mathcal{X})$ is the bipartite graph whose vertex set is $\mathcal{H} \sqcup \mathcal{S}$ (a vertex is either a cross-connected component or a star), and whose edges are pairs $(C, \Sigma) \in \mathcal{H} \times \mathcal{S}$ with $C \in \Sigma$. If $\mathcal{X}$ is G-invariant, then G acts on $RN(\mathcal{X})$.

This definition is motivated by the following remark, whose proof we leave to the reader.

Remark 2.3. Let T be any simplicial tree. Suppose that $\mathcal{H} \subset T$ meets any closed edge in a nonempty finite set. Define betweenness in $\mathcal{H}$ by $C_2 \in [C_1, C_3] \subset T$. Then the bipartite graph defined as above is isomorphic to a subdivision of T.

In the situation of Scott and Swarup [2003], a main result is that $RN(\mathcal{X})$ is a tree. We will reprove this fact by identifying $RN(\mathcal{X})$ with a subdivision of the tree of cylinders.

3. Regular neighborhoods as trees of cylinders

Now we fix a family $\mathcal{E}$ as in Definition 2.1. It is stable under commensurability, and a group of $\mathcal{E}$ cannot contain another with infinite index. Let T be an $\mathcal{E}$-tree.

In Section 3a, we define the set $\mathcal{B}(T)$ of almost invariant sets based on T, and we state the main result, Theorem 3.3: The regular neighborhood $RN(\mathcal{B}(T))$ of $\mathcal{B}(T)$ is up to subdivision the tree of cylinders T_c. In Section 3b, we represent elements of $\mathcal{B}(T)$ by special subforests of T. We then study the cross-connected components of $\mathcal{B}(T)$. We prove Theorem 3.3 in Section 3d by constructing a map Φ from the set of cross-connected components to T_c. In Section 3e we generalize Theorem 3.3 to

Theorem 3.11 by including almost invariant sets enclosed by quadratically hanging vertices of T.

3a. *Almost invariant sets based on a tree.* We fix a basepoint $v_0 \in V(T)$. If e is an edge of T, we denote by $\mathring{e}$ the open edge. Let T_e and T_e^* be the connected components of $T \setminus \mathring{e}$. The set of $g \in G$ such that $gv_0 \in T_e$ (respectively $gv_0 \in T_e^*$) is an almost invariant set Z_e (respectively Z_e^*) over G_e. Up to equivalence, it is independent of v_0. When we need to distinguish between Z_e and Z_e^*, we orient e and declare that the terminal vertex of e belongs to T_e.

Now consider a cylinder $Y \subset T$ and the corresponding commensurability class $\mathscr{C}$. Any Boolean combination of the Z_e for $e \in E(Y)$ is an almost invariant set over some subgroup $H \in \mathscr{C}$.

Definition 3.1. Given a cylinder Y, associated to a commensurability class $\mathscr{C}$, the *Boolean algebra of almost invariant subsets based on Y* is the subalgebra $\mathscr{B}_\mathscr{C}(T)$ of $\mathscr{B}_\mathscr{C}$ generated by the classes $[Z_e]$ for $e \in E(Y)$.

The set of *almost invariant subsets based on T* is the union $\mathscr{B}(T) = \bigcup_\mathscr{C} \mathscr{B}_\mathscr{C}(T)$, a subset of $\mathscr{B} = \bigcup_\mathscr{C} \mathscr{B}_\mathscr{C}$; just like $\mathscr{B}$, it is a union of Boolean algebras but not itself a Boolean algebra.

Proposition 3.2. *Let T and T' be minimal $\mathscr{E}$-trees. Then $\mathscr{B}(T) = \mathscr{B}(T')$ if and only if T and T' belong to the same deformation space.*

More precisely, T dominates T' if and only if $\mathscr{B}(T') \subset \mathscr{B}(T)$.

Proof. Suppose T dominates T'. After subdividing T (this does not change $\mathscr{B}(T)$), we may assume that there is an equivariant map $f : T \to T'$ sending every edge to a vertex or an edge. We claim that, given $e' \in E(T')$, there are only finitely many edges $e_i \in E(T)$ such that $f(e_i) = e'$. To see this, we may restrict to a G-orbit of edges of T, since there are finitely many such orbits. If e and ge both map onto e', then $g \in G_{e'}$. Because of the hypotheses on $\mathscr{E}$, the stabilizer G_e is contained in $G_{e'}$ *with finite index*. The claim follows.

Choose basepoints $v \in T$ and $v' = f(v) \in T'$. Then $Z_{e'}$ (defined using v') is a Boolean combination of the sets Z_{e_i} (defined using v), so $\mathscr{B}(T') \subset \mathscr{B}(T)$.

Conversely, assume $\mathscr{B}(T') \subset \mathscr{B}(T)$. Let $K \subset G$ be a subgroup elliptic in T. We show that it is also elliptic in T'.

If not, we can find an edge $e' = [v', w'] \subset T'$, and sequences $g_n \in G$ and $k_n \in K$, such that the sequences $g_n v'$ and $g_n k_n v'$ have no bounded subsequence, and $e' \subset [g_n v', g_n k_n v']$ for all n. (If K contains a hyperbolic element k, we choose e' on its axis, and we define $g_n = k^{-n}$ and $k_n = k^{2n}$; if K fixes an end ω, we want $g_n^{-1} e' \subset [v', k_n v']$, so we choose e' and g_n such that all edges $g_n^{-1} e'$ are contained on a ray ρ going out to ω, and then we choose k_n.) Defining $Z_{e'}$ using the vertex v' and a suitable orientation of e', we have $g_n \in Z_{e'}$ and $g_n k_n \notin Z_{e'}$.

Using a vertex of T fixed by K to define the almost invariant sets Z_e, we see that any element of $\mathcal{B}(T)$ is represented by an almost invariant set X satisfying $XK = X$. In particular, since $\mathcal{B}(T') \subset \mathcal{B}(T)$, there exist finite sets F_1 and F_2 such that $Z = (Z_{e'} \setminus G_{e'}F_1) \cup G_{e'}F_2$ is K-invariant on the right. For every n, one has $g_n k_n \in G_{e'}F_2$ (if $g_n, g_n k_n \in Z$) or $g_n \in G_{e'}F_1$ (if not), so one of the sequences $g_n k_n v'$ or $g_n v'$ has a bounded subsequence (because $G_{e'}$ is elliptic), a contradiction. □

Remark. The only fact used in the proof is that no edge stabilizer of T has infinite index in an edge stabilizer of T'.

Theorem 3.3. *Let T be a minimal $\mathcal{E}$-tree, with $\mathcal{E}$ as in Definition 2.1, and T_c its tree of cylinders for the commensurability relation. Let $\mathcal{X} = \mathcal{B}(T)$ be the set of almost invariant subsets based on T.*

Then $RN(\mathcal{X})$ is equivariantly isomorphic to a subdivision of T_c.

By Proposition 3.2, and [Guirardel and Levitt 2008, Theorem 1], $RN(\mathcal{X})$ and T_c only depend on the deformation space of T.

To prove the version of Theorem 3.3 stated in the introduction, one takes $\mathcal{E}$ to be the family of subgroups commensurable to an edge stabilizer of T.

The theorem will be proved in the next three subsections. We always fix a base vertex $v_0 \in T$.

3b. ***Special forests.*** Let S and S' be subsets of $V(T)$. We say that S and S' are *equivalent* if their symmetric difference is finite; we say S is trivial if it is equivalent to $\varnothing$ or $V(T)$.

The *coboundary* δS is the set of edges having one endpoint in S and one in S^* (the complement of S in $V(T)$). We shall be interested in sets S with finite coboundary. Since $\delta(S \cap S') \subset \delta S \cup \delta S'$, they form a Boolean algebra.

We also view such an S as a *subforest* of T, by including all edges whose endpoints are both in S; we can then consider the (connected) *components* of S. The set of edges of T is partitioned into edges in S, edges in S^*, and edges in $\delta S = \delta S^*$. Note that S is equivalent to the set of endpoints of its edges. In particular, S is finite (as a set of vertices) if and only if it contains finitely many edges.

We say that S is a *special forest* based on a cylinder Y if $\delta S = \{e_1, \dots, e_n\}$ is finite and contained in Y. If nonempty, S contains at least one vertex of Y. Each component of S (viewed as a subforest) is a component of $T \setminus \{\mathring{e}_1, \dots, \mathring{e}_n\}$, and S^* is the union of the other components of $T \setminus \{\mathring{e}_1, \dots, \mathring{e}_n\}$.

We define $\mathcal{B}_Y$ as the Boolean algebra of equivalence classes of special forests based on Y.

Given a special forest S based on Y, we define $X_S = \{g \mid gv_0 \in S\}$. It is an almost invariant set over $H = \bigcap_{e \in \delta S} G_e$, a subgroup of G belonging to the commensurability class $\mathcal{C}$ associated to Y; we denote its equivalence class by $[X_S]$. Every element of $\mathcal{B}(T)$ may be represented in this form. More precisely:

Lemma 3.4. *Let Y be a cylinder associated to a commensurability class $\mathcal{C}$. Then the map $S \mapsto [X_S]$ induces an isomorphism of Boolean algebras between $\mathcal{B}_Y$ and $\mathcal{B}_{\mathcal{C}}(T)$.*

Proof. It is easy to check that $S \mapsto [X_S]$ is a morphism of Boolean algebras. It is onto because the set T_e used to define the almost invariant set Z_e is a special forest (based on the cylinder containing e). It remains to determine the "kernel", namely to show that X_S is H-finite if and only if S is finite (where H denotes any group in $\mathcal{C}$).

First suppose that S is finite. Then S is contained in Y since it contains any connected component of $T \setminus Y$ that it intersects. Since δS is finite, no vertex x of S has infinite valence in T. In particular, for each vertex $x \in S$, the group G_x is commensurable with H. It follows that $\{g \in G \mid g.v_0 = x\}$ is H-finite, and X_S is H-finite.

If S is infinite, one of its components is infinite, and by minimality of T there exists a hyperbolic element $g \in G$ such that $g^n v_0 \in S$ for all $n \geq 0$. Thus $g^n \in X_S$ for $n \geq 0$. If X_S is H-finite, one can find a sequence n_i going to infinity, and $h_i \in H$, such that $g^{n_i} = h_i g^{n_0}$. Since H is elliptic in T, the sequence $h_i g^{n_0} v_0$ is bounded, a contradiction. □

Lemma 3.5. *Let S and S' be special forests.*

(1) *If S and S' are infinite and based on distinct cylinders, and if $S \cap S'$ is finite, then $S \cap S' = \varnothing$.*

(2) *If X_S crosses $X_{S'}$, then S and S' are based on the same cylinder.*

(3) *$X_S \cap X_{S'}$ is small if and only if $S \cap S'$ is finite.*

Proof. For part (1), assume that S and S' are infinite and based on $Y \neq Y'$, and that $S \cap S'$ is finite. Let $[u, u']$ be the bridge between Y and Y' (with $u = u'$ if Y and Y' intersect in a point). Since u and u' lie in more than one cylinder, they have infinite valence in T.

Assume first that $u \in S$. Then S contains all components of $T \setminus \{u\}$, except finitely many of them (which intersect Y). In particular, S contains Y'. If S' contains u', it contains u by the same argument, and $S \cap S'$ contains infinitely many edges incident on u, a contradiction. If S' does not contain u', it is contained in S, also a contradiction.

We may therefore assume $u \notin S$ and $u' \notin S'$. It follows that S (respectively S') is contained in the union of the components of $T \setminus \{u\}$ (respectively $T \setminus \{u'\}$) which intersect Y (respectively Y'), so S and S' are disjoint.

Part (2) is a consequence of [Scott and Swarup 2003, Proposition 13.5], but here is a direct argument. Assume that S and S' are based on $Y \neq Y'$, and let $[u, u']$

be as above. Up to replacing S and S' by their complements, we have $u \notin S$ and $u' \notin S'$. The argument above shows that $S \cap S' = \varnothing$, so X_S does not cross $X_{S'}$.

For part (3), first suppose that $S \cap S'$ is finite. If, say, S is finite, then X_S is H-finite by Lemma 3.4, so $X_S \cap X_{S'}$ is small. Assume therefore that S and S' are infinite. If they are based on distinct cylinders, then $X_S \cap X_{S'} = \varnothing$ by part (1). If they are based on the same cylinder, then $S \cap S'$ is itself a finite special forest, so $X_S \cap X_{S'} = X_{S \cap S'}$ is small by Lemma 3.4. Conversely, if $S \cap S'$ is infinite, one shows that $X_S \cap X_{S'}$ is not H-finite as in the proof of Lemma 3.4, using g such that $g^n v \in S \cap S'$ for all $n \geq 0$. □

Remark 3.6. If S and S' are infinite and $X_S \cap X_{S'}$ is small, then S and S' are equivalent to disjoint special forests. This follows from the lemma if they are based on distinct cylinders. If not, one replaces S' by $S' \cap S^*$.

3c. ***Peripheral cross-connected components.*** Theorem 3.3 is trivial if T is a line, so we can assume that each vertex of T has valence at least 3 (we now allow G to act with inversions). We need to understand cross-connected components. By Lemma 3.5(2), every such component is based on a cylinder, so we focus on a given Y. We first define peripheral special forests and almost invariant sets.

Recall that ∂Y is the set of vertices of Y that belong to another cylinder. Let $v \in \partial Y$ be a vertex whose valence in Y is finite. Let $e_1, \dots, e_n$ be the edges of Y containing v, oriented towards v. Let $S_{v,Y} = T_{e_1} \cap \dots \cap T_{e_n}$ (recall that T_e denotes the component of $T \setminus \mathring{e}$ containing the terminal point of e). It is a subtree satisfying $S_{v,Y} \cap Y = \{v\}$, with coboundary $\delta S_{v,Y} = \{e_1, \dots, e_n\}$. We say that $S_{v,Y}$, and any special forest equivalent to it, is *peripheral* (but $S^*_{v,Y}$ is not peripheral in general).

We denote by $X_{v,Y}$ the almost invariant set corresponding to $S_{v,Y}$, and we say that X is peripheral if it is equivalent to some $X_{v,Y}$. Both $S_{v,Y}$ and $S^*_{v,Y}$ are infinite, so $X_{v,Y}$ is nontrivial by Lemma 3.4.

We claim that $C_{v,Y} = \{\{[X_{v,Y}], [X^*_{v,Y}]\}\}$ is a complete cross-connected component of $\mathcal{B}(T)$, called a peripheral CCC. Indeed, assume that $X_{v,Y}$ crosses some X_S. Then S is based on Y by Lemma 3.5, but since $S_{v,Y}$ contains no edge of Y, it is contained in S_X or S^*_X, which prevents crossing.

Note that if $C_{v,Y} = C_{v',Y'}$, then $Y = Y'$ (because an H-almost invariant subset determines the commensurability class of H), and $v = v'$ except when Y is a single edge vv', in which case $X_{v,Y} = X^*_{v',Y}$.

Lemma 3.7. *Let Y be a cylinder. There is at most one nonperipheral cross-connected component C_Y based on Y. There is exactly one if and only if $|\partial Y| \neq 2, 3$.*

Proof. The proof is in three parts.

We first claim that, given any infinite connected nonperipheral special forest S based on Y, there is an edge $e \subset S \cap Y$ such that both connected components of $S \setminus \{\mathring{e}\}$ are infinite.

Assume there is no such e. Then $S \cap Y$ is locally finite: Given $v \in S$, all but finitely many components of $S \setminus \{v\}$ are infinite, so infinitely many edges incident on v satisfy the claim if v has infinite valence in $S \cap Y$.

Since S is infinite and nonperipheral, $S \cap Y$ is not reduced to a single point. We orient every edge e of $S \cap Y$ so that $S \cap T_e$ is infinite and $S \cap T_e^*$ is finite. If a vertex v of $S \cap Y$ is terminal in every edge of $S \cap Y$ that contains it, S is peripheral. We may therefore find an infinite ray $\rho \subset S \cap Y$ consisting of positively oriented edges. Since every vertex of T has valence ≥ 3, every vertex of ρ is the projection onto ρ of an edge of δS, contradicting the finiteness of δS. This proves the claim.

Secondly, to show that there is at most one nonperipheral cross-connected component, we fix two nontrivial forests S and S' based on Y, and we show that X_S and $X_{S'}$ are in the same CCC if they do not belong to peripheral CCCs. We can assume that $X_S \cap X_{S'}$ is small, and by Remark 3.6 that $S \cap S'$ is empty. We may also assume that every component of S and S' is infinite.

Since S is not peripheral, it contains two disjoint infinite special forests S_1 and S_2 based on Y: This is clear if S has several components, and follows from the claim otherwise. Construct S'_1 and S'_2 similarly. Then $X_{S_1} \cup X_{S'_1}$ crosses both X_S and $X_{S'}$, so X_S and $X_{S'}$ are in the same cross-connected component.

Finally, we discuss the existence of C_Y. If $|\partial Y| \geq 4$, choose $v_1, \ldots, v_4 \in \partial Y$, and consider edges e_1, e_2, e_3 of Y such that each v_i belongs to a different component S_i of $T \setminus \{\mathring{e}_1, \mathring{e}_2, \mathring{e}_3\}$. These components are infinite because $v_i \in \partial Y$, and $X_{S_1 \cup S_2}$ belongs to a nonperipheral CCC.

If ∂Y is empty, then $Y = T$ and existence is clear. If ∂Y is nonempty, minimality of T implies that Y is the convex hull of ∂Y (replacing every cylinder by the convex hull of its boundary yields an invariant subtree). From this we deduce that $|\partial Y| \neq 1$, and every CCC based on Y is peripheral if $|\partial Y|$ equals 2 or 3. There is one peripheral CCC if $|\partial Y| = 2$ (that is, Y is a single edge) and three if $|\partial Y| = 3$. □

Remark 3.8. The proof shows that, if $|\partial Y| \geq 4$, then for all $u \neq v$ in ∂Y, the nonperipheral CCC is represented by a special forest S such that $u \in S$ and $v \in S^*$.

3d. ***Proof of Theorem 3.3.*** From now on we assume that T has more than one cylinder; otherwise there is exactly one cross-connected component, and both $RN(\mathcal{X})$ and T_c are points.

It will be helpful to distinguish between a cylinder $Y \subset T$ or a point $\eta \in \partial Y$, and the corresponding vertex of T_c. We therefore denote by Y_c or η_c the vertex of T_c corresponding to Y or η.

Recall that $\mathcal{H}$ denotes the set of cross-connected components of $\mathcal{X} = \mathcal{B}(T)$. Consider the map $\Phi : \mathcal{H} \to T_c$ defined as follows:

- If $C = C_Y$ is a nonperipheral CCC, then $\Phi(C) = Y_c \in V_1(T_c)$.

- If $C = C_{v,Y}$ is peripheral, and $\#\partial Y \geq 3$, then $\Phi(C)$ is the midpoint of the edge $\varepsilon = (v_c, Y_c)$ of T_c.
- If $\#\partial Y = 2$, and C is the peripheral CCC based on Y, then $\Phi(C) = Y_c$.

In all cases, the distance between $\Phi(C)$ and Y_c is at most $1/2$. If C is peripheral, $\Phi(C)$ has valence 2 in T_c.

Clearly, Φ is one-to-one. By Remark 2.3, it now suffices to show that the image of Φ meets every closed edge, and Φ preserves betweenness: For $C_1, C_2, C_3 \in \mathcal{H}$, C_2 is between C_1 and C_3 if and only if $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$.

The first fact is clear because $\Phi(\mathcal{H})$ contains all vertices $Y_c \in V_1(T_c)$ with $|\partial Y| \neq 3$ and the three points at distance $1/2$ from Y_c if $|\partial Y| = 3$. To control betweenness, we need a couple of technical lemmas.

If S is a nontrivial special forest, we denote by $[\![S]\!]$ the cross-connected component represented by the almost invariant set X_S.

Let $Y \subset T$ be a cylinder. We denote by $\mathrm{pr}_Y : T \to Y$ the projection. If Y' is another cylinder, then $\mathrm{pr}_Y(Y')$ is a single point. This point belongs to two cylinders and hence defines a vertex of $V_0(T_c)$ that is at distance 1 from Y_c on the segment of T_c joining Y_c to Y'_c.

Let Y be a cylinder with $|\partial Y| \geq 4$. For each nontrivial special forest S' that is either based on some $Y' \neq Y$, or based on Y and peripheral, we define a point $\eta_Y(S') \in Y \subset T$ as follows. If S' is based on $Y' \neq Y$, we define $\eta_Y(S')$ to be $\mathrm{pr}_Y(Y')$. If S' is equivalent to some $S_{v,Y}$, we define $\eta_Y(S') = v$; note that in this case $\eta_Y(S'^*)$ is not defined.

Lemma 3.9. *Let Y be a cylinder with $|\partial Y| \geq 4$. Consider two nontrivial special forests S, S' with $[\![S']\!] \neq C_Y$ and $[\![S]\!] = C_Y$, and assume $S' \subset S$.*

Then $\eta = \eta_Y(S') \in Y$ is defined, $\eta \in S$, and S' contains an equivalent subforest S'' with $S'' \subset \mathrm{pr}_Y^{-1}(\{\eta\}) \subset S$.

Moreover, $\Phi([\![S']\!])$ lies in the connected component of $T_c \setminus \{Y_c\}$ containing η_c.

Proof. Let Y' be the cylinder on which S' is based.

If $Y' = Y$, then S'^* is not peripheral, so S' is peripheral. Thus η is defined, and S' is equivalent to its subforest $S'' = S_{Y,\eta}$. Then $S'' = \mathrm{pr}_Y^{-1}(\{\eta\}) \subset S$. In this case $\Phi([\![S']\!])$ is the midpoint of the edge (η_c, Y_c) of T_c.

Assume that $Y' \neq Y$. Then $\eta = \mathrm{pr}_Y(Y') \in S$; otherwise Y' would be disjoint from S and hence from S', a contradiction. It follows that $\mathrm{pr}_Y^{-1}(\{\eta\}) \subset S$. If $\eta \in S'$, then S' contains the complement of $\mathrm{pr}_Y^{-1}(\{\eta\})$, so $S = T$, a contradiction. Thus $\eta \notin S'$ and therefore $S' \subset \mathrm{pr}_Y^{-1}(\{\eta\})$. The "moreover" is clear in this case since η_c is between Y_c and Y'_c, and $\Phi([\![S']\!])$ is at distance $\leq 1/2$ from Y'_c. □

Lemma 3.10. *Let $S = S_{Y,u}$ be peripheral, and let S' be a nontrivial special forest with $[\![S']\!] \neq [\![S]\!]$. Recall that u_c is the vertex of T_c associated to u.*

(1) *If* $S' \subset S$, *then* $\Phi([\![S']\!])$ *belongs to the component of* $T_c \setminus \{\Phi([\![S]\!])\}$ *that contains* u_c.

(2) *If* $S \subset S'$, *then* $\Phi([\![S']\!])$ *belongs to the component of* $T_c \setminus \{\Phi([\![S]\!])\}$ *that does not contain* u_c.

Proof. If $S' \subset S$, then S' is based on some $Y' \neq Y$. Since $S' \subset S = \mathrm{pr}_Y^{-1}(\{u\})$, we have $Y' \subset \mathrm{pr}_Y^{-1}(\{u\})$ and u_c is between Y_c and Y'_c in T_c. The result follows since $\Phi([\![S]\!])$ is $1/2$-close to Y_c and $\Phi([\![S']\!])$ is $1/2$-close to Y'_c.

If $S \subset S'$ and $Y \neq Y'$, we have $\mathrm{pr}_Y(Y') \neq u$ because $S' \neq T$, and the lemma follows. If $Y = Y'$, the lemma is immediate. □

We can now show that Φ preserves betweenness. Consider three distinct cross-connected components $C_1, C_2, C_3 \in \mathcal{H}$. Let Y_2 be the cylinder on which C_2 is based. Note that $|\partial Y_2| \geq 4$ if C_2 is nonperipheral.

First assume that C_2 is between C_1 and C_3. By definition, there exist almost invariant subsets X_i representing C_i such that $X_1 \subset X_2 \subset X_3$. By Lemma 3.4, one can find special forests S_i with $[X_{S_i}] = [X_i]$. By Remark 3.6, since the C_i are distinct, one can assume $S_1 \subset S_2 \subset S_3$ (if necessary, replace S_2 by $S_2 \cap S_3$, and then S_1 by $S_1 \cap S_2 \cap S_3$).

If S_2 is peripheral, $\Phi(C_1)$ and $\Phi(C_3)$ are in distinct components of $T_c \setminus \{\Phi(C_2)\}$ by Lemma 3.10, so $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$. If S_2^* is peripheral, we apply the same argument using $S_3^* \subset S_2^* \subset S_1^*$.

Assume therefore that C_2 is nonperipheral. Lemma 3.9 implies that the points $\eta_1 = \eta_{Y_2}(S_1)$ and $\eta_3 = \eta_{Y_2}(S_3^*)$ are defined, and $\eta_1 \in S_2$ and $\eta_3 \in S_2^*$. In particular, we have $\eta_1 \neq \eta_3$. By the "moreover", we get $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$ since $\Phi(C_2) = (Y_2)_c$.

Now assume that C_2 is not between C_1 and C_3, and choose S_i with $[\![S_i]\!] = C_i$. By Remark 3.6, we may assume that for each $i \in \{1, 3\}$ some inclusion $S_i^{(*)} \subset S_2^{(*)}$ holds. Since C_2 is not between C_1 and C_3, we may assume after changing S_i to S_i^* if needed that $S_1 \subset S_2$ and $S_3 \subset S_2$.

If S_2 or S_2^* is peripheral, Lemma 3.10 implies that $\Phi(C_1)$ and $\Phi(C_3)$ lie in the same connected component of $T_c \setminus \{\Phi(C_2)\}$, so $\Phi(C_2)$ is not between $\Phi(C_1)$ and $\Phi(C_3)$.

Assume therefore that C_2 is nonperipheral. Lemma 3.9 says that the points $\eta_1 = \eta_{Y_2}(S_1)$ and $\eta_3 = \eta_{Y_2}(S_3)$ are defined, and we may assume $S_i \subset \mathrm{pr}_{Y_2}^{-1}(\{\eta_i\})$. If $\eta_1 = \eta_3$, then $\Phi(C_2)$ does not lie between $\Phi(C_1)$ and $\Phi(C_3)$ by the "moreover" of Lemma 3.9. If $\eta_1 \neq \eta_3$, consider $\tilde{S}_2$ with $[\![\tilde{S}_2]\!] = C_2$ such that $\eta_1 \in \tilde{S}_2$ and $\eta_3 \in \tilde{S}_2^*$ (it exists by Remark 3.8). Then $S_1 \subset \mathrm{pr}_{Y_2}^{-1}(\eta_1) \subset \tilde{S}_2$ and $S_3 \subset \mathrm{pr}_{Y_2}^{-1}(\eta_3) \subset \tilde{S}_2^*$, so C_2 lies between C_1 and C_3, a contradiction.

This ends the proof of Theorem 3.3. □

3e. ***Quadratically hanging vertices.*** We say that a vertex stabilizer G_v of T is a *QH-subgroup* if there is an exact sequence $1 \to F \to G_v \xrightarrow{\pi} \Sigma \to 1$, where $\Sigma = \pi_1(\mathbb{O})$ is a hyperbolic 2-orbifold group and every incident edge group G_e is peripheral: It is contained with finite index in the preimage by π of a boundary subgroup $B = \pi_1(C)$, with C a boundary component of $\mathbb{O}$. We say that v is a *QH-vertex* of T.

We now define almost invariant sets based on v. They will be included in our description of the regular neighborhood.

We view Σ as a convex cocompact Fuchsian group acting on $\mathbb{H}^2$. Let $\overline{H}$ be any nonperipheral maximal two-ended subgroup of Σ (represented by an immersed curve or 1-suborbifold). Let γ be the geodesic invariant by $\overline{H}$. It separates $\mathbb{H}^2$ into two half-spaces $P^{\pm}$, which may be interchanged by certain elements of $\overline{H}$.

Let $\overline{H}_0$ be the stabilizer of P^+, which has index at most 2 in $\overline{H}$, and let x_0 be a basepoint. We define an $\overline{H}_0$-almost invariant set $\overline{X} \subset \Sigma$ as the set of $g \in \Sigma$ such that $gx_0 \in P^+$. (If $\overline{H}$ is the fundamental group of a two-sided simple closed curve on $\mathbb{O}$, there is a one-edge splitting of Σ over $\overline{H}$, and $\overline{X}$ is a Z_e as in Section 3a.)

The preimage of $\overline{X}$ in G_v is an almost invariant set X_v over the preimage H_0 of $\overline{H}_0$. We extend it to an almost invariant set X of G as follows. Let S' be the set of vertices $u \neq v$ of T such that, denoting by e the initial edge of the segment $[v, u]$, the geodesic of $\mathbb{H}^2$ invariant under $G_e \subset G_v$ lies in P^+; note that it lies in either P^+ or P^-. Then X is the union of X_v with the set of $g \notin G_v$ such that $gv \in S'$.

Starting from $\overline{H}$, we have thus constructed an almost invariant set X, which is well defined up to equivalence and complementation (because of the choices of x_0 and $P^{\pm}$). We say that X is a *QH-almost invariant subset* based on v. We let $\mathrm{QH}_v(T)$ be the set of equivalence classes of QH-almost invariant subsets obtained from v as above (varying $\overline{H}$), and we let $\mathrm{QH}(T)$ be the union of all $\mathrm{QH}_v(T)$ when v ranges over all QH-vertices of T.

Theorem 3.11. *With $\mathscr{E}$ and T as in Theorem 3.3, let $\hat{\mathscr{X}} = \mathscr{B}(T) \cup \mathrm{QH}(T)$. Then RN$(\hat{\mathscr{X}})$ is isomorphic to a subdivision of T_c.*

Proof. The proof is similar to that of Theorem 3.3.

If X is a QH-almost invariant subset as constructed above, we call $S = S' \cup \{v\}$ the *QH-forest* associated to X. We say that it is based on v. The coboundary of S is infinite, but all its edges contain v. We may therefore view S as a subtree of T (the union of v with certain components of $T \setminus \{v\}$). It is a union of cylinders. We let $S^* = (T \setminus S) \cup \{v\}$, so that $S \cap S^* = \{v\}$.

Note that S cannot contain a peripheral special forest $S_{v,Y}$, with Y a cylinder containing v (this is because the subgroup $\overline{H} \subset \Sigma$ was chosen nonperipheral).

Conversely, given a QH-forest S, one can recover H_0, which is the stabilizer of S, and the equivalence class of X. In other words, there is a bijection between $\mathrm{QH}_v(T)$

and the set of QH-forests based on v. We denote by X_S the almost invariant set X corresponding to S (it is well defined up to equivalence). Note that X_S is not a subset of $\{g \in G \mid gv \in S\}$, and these sets have the same intersection with $G \setminus G_v$.

The following fact is analogous to Lemma 3.5.

Lemma 3.12. *Let S be a QH-forest based on v. Let S' be a nontrivial special forest, or a QH-forest based on $v' \neq v$.*

(1) *$X_S \cap X_{S'}$ is small if and only if $S \cap S' = \varnothing$.*

(2) *X_S and $X_{S'}$ do not cross.*

Proof. When S' is a special forest, we use v as a basepoint to define $X_{S'}$ as the set $\{g \mid gv \in S'\}$. Beware that X_S is properly contained in $\{g \mid gv \in S\}$.

We claim that if S' is a special forest with $v \notin S'$ and $S \cap S' \neq \varnothing$, then $X_{S'} \subset X_S$. Let Y' be the cylinder on which S' is based. Since each connected component of S' contains a point in Y', there is a point $w \neq v$ in $S \cap Y'$. Since S is a union of cylinders, S contains Y'. All connected components of S' therefore contain a point of S and so are contained in $S \setminus \{v\}$ since $v \notin S'$. We deduce $X_{S'} \subset X_S$.

We now prove (1). If $S \cap S' = \varnothing$, then $X_S \cap X_{S'} = \varnothing$. We assume $S \cap S' \neq \varnothing$, and we show that $X_S \cap X_{S'}$ is not small. If S' is a QH-forest, then $v \in S'$ or $v' \in S$. If for instance $v \in S'$, then $X_S \cap X_{S'}$ is not small because it contains $X_S \cap G_v$. Now assume that S' is a special forest. If $v \in S'$, the same argument applies, so assume that $v \notin S'$. The claim implies $X_{S'} \subset X_S$, so $X_S \cap X_{S'}$ is not small.

To prove (2), first consider the case where S' is a QH-forest. Up to changing S and S' to S^* or S'^*, one can assume $S \cap S' = \varnothing$, so X_S does not cross $X_{S'}$. If S' is a special forest, we can assume $v \notin S'$ by changing S' to S'^*. By the claim, X_S does not cross $X_{S'}$. □

The lemma implies that no element of $\mathrm{QH}(T)$ crosses an element of $\mathcal{B}(T)$, and elements of $\mathrm{QH}_v(T)$ do not cross elements of $\mathrm{QH}_{v'}(T)$ for $v \neq v'$.

Since $\mathrm{QH}_v(T)$ is a cross-connected component, the set $\hat{\mathcal{H}}$ of cross-connected components of $\mathcal{B}(T) \cup \mathrm{QH}(T)$ is therefore the set of cross-connected components of $\mathcal{B}(T)$, together with one new cross-connected component $\mathrm{QH}_v(T)$ for each QH-vertex v.

One extends the map Φ defined in the proof of Theorem 3.3 to a map $\hat{\Phi} : \hat{\mathcal{H}} \to T_c$ by sending $\mathrm{QH}_v(T)$ to v (viewed as a vertex of $V_0(T_c)$ since a QH-vertex belongs to infinitely many cylinders). We need to prove that $\hat{\Phi}$ preserves betweenness.

Lemmas 3.9 and 3.10 extend immediately to the case where S' is a QH-forest: one just needs to define $\eta_Y(S') = \mathrm{pr}_Y(v')$ for S' based on v', so that v' plays the role of Y' in the proofs. (In the proof of Lemma 3.9, the assertion that $\eta \notin S'$ should be replaced by the fact that $S' \cap Y$ contains no edge; this holds since otherwise S' would contain Y.) This allows to prove that, if C_2 is not a component $\mathrm{QH}_v(T)$, then $\Phi(C_2)$ is between $\Phi(C_1)$ and $\Phi(C_3)$ if and only if C_2 lies between C_1 and C_3.

To treat the case when $C_2 = \mathrm{QH}_v(T)$, we need a cylinder-valued projection η_v. Let Y be a cylinder or a QH-vertex distinct from v. We define $\eta_v(Y)$ as the cylinder of T containing the initial edge of $[v, x]$ for any $x \in Y$ different from v. Equivalently, $\eta_v(Y)$ is Y if $v \in Y$, the cylinder containing the initial edge of the bridge joining x to Y otherwise.

If v lies in a cylinder Y^0, denote by $\eta_v^{-1}(Y^0)$ the union of cylinders Y such that $\eta_v(Y) = Y^0$. Equivalently, this is the set of points $x \in T$ such that $x = v$ or $[x, v]$ contains an edge of Y^0.

As before, $[\![S]\!]$ denotes the cross-connected component represented by X_S.

Lemma 3.13. *Let S be a QH-forest based on v. Let S' be a nontrivial special forest, or a QH-forest based on $v' \neq v$. Let Y' be the cylinder or QH-vertex on which S' is based, and let $Y'^0 = \eta_v(Y')$.*

If $S' \subset S$, then $S' \subset \eta_v^{-1}(Y'^0) \subset S$.

Moreover, $\Phi([\![S']\!])$ and $Y_c'^0$ lie in the same component of $T_c \setminus \{\Phi([\![S]\!])\}$.

We leave the proof of this lemma to the reader.

Assume now that $S_1 \subset S_2 \subset S_3$ with $[\![S_i]\!] = C_i$ and S_2 based on v. For $i = 1, 3$, let $Y_i^0 = \eta_v(Y_i)$. Then $S_1 \subset \eta_v^{-1}(Y_1^0) \subset S_2$ and $S_3^* \subset \eta_v^{-1}(Y_3^0) \subset S_2^*$. In particular, $Y_1^0 \neq Y_3^0$. Since $(Y_1^0)_c$ and $(Y_3^0)_c$ are neighbors of v_c, they lie in distinct components of $T_c \setminus \{\Phi(C_2)\}$. By Lemma 3.13, so do $\Phi([\![S_1]\!])$ and $\Phi([\![S_3]\!])$.

Conversely, assume that C_2 does not lie between C_1 and C_3, and consider $S_1 \subset S_2$ and $S_3 \subset S_2$ with $[\![S_i]\!] = C_i$. For $i = 1, 3$, let Y_i^0 be as above. If $Y_1^0 = Y_3^0$, then $\Phi(C_2)$ is not between $\Phi(C_1)$ and $\Phi(C_3)$ by Lemma 3.13, and we are done. If $Y_1^0 \neq Y_3^0$, these cylinders correspond to distinct peripheral subgroups of G_v, with invariant geodesics $\gamma_1 \neq \gamma_3$. There exists a nonperipheral group $\overline{H} \subset \Sigma$, as in the beginning of this subsection, whose invariant geodesic separates γ_1 and γ_3. Let S_2' be the associated QH-forest. Then $[\![S_2']\!] = C_2$ and, up to complementation, $\eta_v^{-1}(Y_1^0) \subset S_2'$ and $\eta_v^{-1}(Y_3^0) \subset S_2'^*$. It follows that $S_1 \subset S_2'$ and $S_3^* \subset S_2'^*$, so C_2 lies between C_1 and C_3, contradicting our assumptions. □

4. The regular neighborhood of Scott and Swarup

A group is VPC$_n$ if some finite index subgroup is polycyclic of Hirsch length n. For instance, VPC$_0$ groups are finite groups, VPC$_1$ groups are virtually cyclic groups, and VPC$_2$ groups are virtually $\mathbb{Z}^2$ (but not all VPC$_n$ groups are virtually abelian for $n \geq 3$).

Fix $n \geq 1$. We assume that G is finitely presented and does not split over a VPC$_{n-1}$ subgroup. We also assume that G itself is not VPC$_{n+1}$. All trees we consider here are assumed to have VPC$_n$ edge stabilizers.

A subgroup $H \subset G$ is *universally elliptic* if it is elliptic in every tree. A tree is universally elliptic if all its edge stabilizers are.

A tree is a *JSJ tree* (over VPC_n subgroups) if it is universally elliptic, and maximal for this property: it dominates every universally elliptic tree. JSJ trees exist (because G is finitely presented) and belong to the same deformation space, called the JSJ deformation space; see [Guirardel and Levitt 2009].

A vertex stabilizer G_v of a JSJ tree is *flexible* if it is not VPC_n and is not universally elliptic. It follows from [Dunwoody and Sageev 1999] that a flexible vertex stabilizer is a QH-subgroup, as defined in Section 3e: There is an exact sequence $1 \to F \to G_v \to \Sigma \to 1$, where $\Sigma = \pi_1(\mathbb{O})$ is the fundamental group of a hyperbolic 2-orbifold, F is VPC_{n-1}, and every incident edge group G_e is peripheral. Note that the QH-almost invariant subsets X constructed in Section 3e are over VPC_n subgroups.

We can now describe the regular neighborhood of all almost invariant subsets of G over VPC_n subgroups as a tree of cylinders.

Theorem 4.1. *Let G be a finitely presented group, and let $n \geq 1$. Assume that G does not split over a* VPC_{n-1} *subgroup and that G is not* VPC_{n+1}. *Let T be a JSJ tree over* VPC_n *subgroups, and let T_c be its tree of cylinders for the commensurability relation.*

Then Scott and Swarup's regular neighborhood of all almost invariant subsets over VPC_n *subgroups is equivariantly isomorphic to a subdivision of T_c.*

This is immediate from Theorem 3.11 and Theorem 4.2, which says one can read any almost invariant set over a VPC_n subgroup in a JSJ tree T, and which follows from [Dunwoody and Swenson 2000] and [Scott and Swarup 2003, Theorem 8.2].

Theorem 4.2. *Let G and T be as above.*

For any almost invariant subset X over a VPC_n *subgroup, the equivalence class $[X]$ belongs to $\mathcal{B}(T) \cup \mathrm{QH}(T)$.*

Proof. We essentially follow the proof by Scott and Swarup [2003],[1] and we adopt their definitions. All trees considered here have VPC_n edge stabilizers.

Let X be a nontrivial almost invariant subset over a VPC_n subgroup H. We first consider the case where X crosses strongly some other almost invariant subset. Then by [Dunwoody and Swenson 2000, Proposition 4.11], H is contained as a nonperipheral subgroup in a QH-vertex stabilizer W of some tree T'. When acting on T, the group W fixes a QH-vertex $v \in T$; see [Guirardel and Levitt 2009, Remark 7.20].

Note that H is not peripheral in G_v, because it is not peripheral in W. Since (G, H) only has 2 coends [2003, Proposition 13.8], there are (up to equivalence) only two almost invariant subsets over subgroups commensurable with H (namely X and X^*), and therefore $[X] \in \mathrm{QH}_v(T)$.

[1] From here on, [2003] refers to [Scott and Swarup 2003].

From now on, we assume that X crosses strongly no other almost invariant subset over a VPC_n subgroup. Then, by [Dunwoody and Roller 1993] and [Dunwoody and Swenson 2000, Section 3], there is a nontrivial tree T_0 with one orbit of edges and an edge stabilizer H_0 commensurable with H.

Since X crosses strongly no other almost invariant set, H and H_0 are universally elliptic; see [Guirardel 2005, Lemme 11.3]. In particular, T dominates T_0. It follows that there is an edge of T with stabilizer contained in H_0 (necessarily with finite index). This edge is contained in a cylinder Y associated to the commensurability class of H.

The main case is when T has no edge e such that Z_e crosses X. (See Section 3a for the definition of Z_e.) The following lemma implies that X is enclosed in some vertex v of T.

Lemma 4.3. *Suppose G is finitely generated. Let $X \subset G$ be a nontrivial almost invariant set over a finitely generated subgroup H. Let T be a tree with an action of G. If X crosses no Z_e, then X is enclosed in some vertex $v \in T$.*

Proof. The argument follows a part of the proof of [2003, Proposition 5.7].

Given two almost invariant subsets, we use the notation $X \geq Y$ when $Y \cap X^*$ is small. The noncrossing hypothesis says that each edge e of T may be oriented so that $Z_e \geq X$ or $Z_e \geq X^*$. If one can choose both orientations for some e, then X is equivalent to Z_e, so X is enclosed in both endpoints of e and we are done.

We orient each edge of T in this manner. We color the edge blue or red according to whether $Z_e \geq X$ or $Z_e \geq X^*$. No edge can have both colors. If e is an oriented edge, and if e' lies in T_e^*, then e' is oriented towards e, so that $Z_e \subset Z_{e'}$, and e' has the same color as e. In particular, given a vertex v, either all edges containing v are oriented towards v, or there exists exactly one edge containing v and oriented away from v, and all edges containing v have the same color.

If v is as in the first case, X is enclosed in v by definition. If there is no such v, then all edges have the same color and are oriented towards an end of T. By [2003, Lemma 2.31], G is contained in the R-neighborhood of X for some $R > 0$, so X is trivial, a contradiction. □

Let v be a vertex of T enclosing X. In particular, $H \subset G_v$. The set $X_v = X \cap G_v$ is an H-almost invariant subset of G_v (note that G_v is finitely generated). By [2003, Lemma 4.14], there is a subtree $S \subset T$ containing v, with $S \setminus \{v\}$ a union of components of $T \setminus \{v\}$, such that X is equivalent to $X_v \cup \{g \mid g.v \in S \setminus \{v\}\}$.

Lemma 4.4. *The H-almost invariant subset X_v of G_v is trivial.*

Proof. Otherwise, by [Dunwoody and Roller 1993; Dunwoody and Swenson 2000], there is a G_v-tree T_1 with one orbit of edges and an edge stabilizer H_1 commensurable with H, and an edge $e_1 \subset T_1$, such that Z_{e_1} lies up to equivalence in the Boolean algebra generated by the orbit of X_v under the commensurator of H in G_v.

Note that G_e is elliptic in T_1 for each edge e of T incident to v: By symmetry of strong crossing [2003, Proposition 13.3], G_e does not cross strongly any translate of X, and thus does not cross strongly Z_{e_1}, so G_e is elliptic in T_1 [Guirardel 2005, lemme 11.3]. This ellipticity allows us to refine T by creating new edges with stabilizer conjugate to H_1. Since H_1 is universally elliptic, this contradicts the maximality of the JSJ tree T. □

After replacing X by an equivalent almost invariant subset or its complement, and possibly changing S to $(T \setminus S) \cup \{v\}$, we can assume that $X = \{g \mid g.v \notin S\}$. Recall that Y is the cylinder defined by the commensurability class of H.

Lemma 4.5. *The coboundary δS, consisting of edges vw with $w \notin S$, is a finite set of edges of Y.*

This implies that $[X] \in \mathcal{B}(T)$, ending the proof when X crosses no Z_e.

Proof of Lemma 4.5. Let E be the set of edges of δS, oriented so that $X = \bigsqcup_{e \in E} Z_e$ (we use v as a basepoint to define Z_e). Let A be a finite generating system of G such that, for all $a \in A$, the open segment (av, v) does not meet the orbit of v. One can construct such a generating system from any finite generating system by iteratively replacing $\{a\}$ by the pair $\{g, g^{-1}a\}$ if (av, v) contains some $g.v$.

Let Γ be the Cayley graph of (G, A). For any subset $Z \subset G$, denote by δZ the set of edges of Γ having one endpoint in Z and the other endpoint in $G \setminus Z$. By our choice of A, no edge joins a vertex of Z_e to a vertex of $Z_{e'}$ for $e \neq e'$. It follows that $\delta X = \bigsqcup_{e \in E} \delta Z_e$.

Since δX is H-finite, the set δZ_e is H-finite for each $e \in E$, and E is contained in a finite union of H-orbits. Let $e \in E$. Since δZ_e is G_e-invariant and H-finite, $G_e \cap H$ has finite index in G_e. Since G_e and H are both VPC_n, they are commensurable, so the H-orbit of e is finite. It follows that $E \subset Y$ and that E is finite. □

We now turn to the case when X crosses some of the Z_e. For each $e \in E(T)$, the intersection number $i(Z_e, X)$ is finite [Scott 1998], which means that there are only finitely many edges e' in the orbit of e such that $Z_{e'}$ crosses X. Since T/G is finite, let $e_1, e_1^{-1}, e_2, e_2^{-1}, \dots, e_n, e_n^{-1}$ be the finite set of oriented edges e such that Z_e crosses X, where we denote by $e \mapsto e^{-1}$ the orientation-reversing involution. Note that $e_i \subset Y$ by [2003, Proposition 13.5]. Now X is a finite union of sets of the form $X' = X \cap Z_{e_1^{\pm 1}} \cap \dots \cap Z_{e_n^{\pm 1}}$. Since X' does not cross any Z_e, its equivalence class lies in $\mathcal{B}(T)$ by the argument above and so does $[X]$. □

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QUANTIZATION OF POISSON–HOPF STACKS ASSOCIATED WITH GROUP LIE BIALGEBRAS

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Let G be a simply connected Poisson–Lie group and $\mathfrak{g}$ its Lie bialgebra. Suppose that $\mathfrak{g}$ is a group Lie bialgebra. This means that there is an action of a discrete group Γ on G deforming the Poisson structure into coboundary equivalent ones. This induces the existence of a Poisson–Hopf algebra structure on the direct sum over Γ of formal functions on G, with Poisson structures translated by Γ. A quantization of this algebra can be obtained by taking the linear dual of a quantization of the Γ Lie bialgebra $\mathfrak{g}$, which is the infinitesimal of a Γ Poisson–Lie group. In this paper we find out an interesting structure on the dual Lie group G^*. We prove that we can construct a stack of Poisson–Hopf algebras and prove the existence of the associated deformation quantization of it. This stack can be viewed as the function algebra on "the formal Poisson group" dual to the original Γ Poisson–Lie group. To quantize this stack, we apply Drinfeld functors to quantization of the associated Γ Lie bialgebra.

Introduction

In this paper, we study examples of Poisson–Hopf stacks and their quantization. Enriquez and Halbout [2008] considered quantization of a Γ Lie bialgebra (**LBA**). As an outcome, they constructed a functor from the category of Γ Lie bialgebra to the category of Γ quantized universal enveloping algebras (**QUE**). Our goal here is to study the objects dual to Γ Lie bialgebras and their quantizations.

There are two kinds of duality map we can apply to a Γ Lie bialgebra: One is to consider the algebra of functions on G. We obtain a direct sum $\bigoplus_{\gamma\in\Gamma}\mathbb{O}_\gamma$ of formal functions on G, with Poisson structures translated by Γ. When Γ is not a finite group, the coproduct Δ maps $\mathbb{O}_\gamma$ to an infinite sum. In general, $\bigoplus_{\gamma\in\Gamma}\mathbb{O}_\gamma$ is a Poisson algebra but does not have a Hopf algebra structure because an infinite sum appears in the coproduct. Nevertheless, we will still call $\bigoplus_{\gamma\in\Gamma}\mathbb{O}_\gamma$ a Γ Poisson–Hopf algebra. (We do have a collection of Poisson algebras and Poisson morphisms

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$\Delta_{\gamma,\gamma'} : \mathbb{O}_{\gamma\gamma'} \to \mathbb{O}_{\gamma} \otimes \mathbb{O}_{\gamma'}$ that satisfy coassociativity rules). A quantization of such a Γ Poisson–Hopf algebra defines the function algebra on a Γ quantum group. We refer to [Majid and Soĭbel'man 1994] for examples of quantum Weyl groups, and [Enriquez and Halbout 2008] for quantization in the general case.

In this paper, we apply a duality map different from the function dual above. We study the structures on the dual group G^* by applying the Drinfeld functor to a Γ universal enveloping algebra. We discover a stack of Poisson formal series Hopf algebras (**PFSHA** as defined in Section 1) dual to a Γ Lie bialgebra; this duality is similar to the one between Lie bialgebras and Poisson–Lie groups. Then we study deformation quantization of this stack. We construct the deformation quantization by applying the Drinfeld functor to a Γ quantized universal enveloping algebra, and obtain a stack of quantized formal series Hopf algebras (**QFSHA**). We summarize our results in a commutative diagram:

$$\begin{array}{ccc} \Gamma\text{-}\mathbf{LBA} & \xrightarrow{\text{EH}} & \Gamma\text{-}\mathbf{QUE} \\ \updownarrow \approx & & \updownarrow \text{Dr} \\ \Gamma\text{-}\mathbf{PFSHA} & \xrightarrow{\text{Quant}} & \Gamma\text{-}\mathbf{QFSHA} \end{array}$$

Let Γ be a discrete group, G a simply connected Lie group and $\mathfrak{g}$ its Lie algebra. Suppose that $\mathfrak{g}$ is a Γ Lie bialgebra (or equivalently that G is a Γ Poisson group), that is, a Lie algebra $(\mathfrak{g}, \mu_{\mathfrak{g}})$ together with a Lie cobracket δ_e, an action of Γ, $\theta : \Gamma \to \operatorname{Aut}(\mathfrak{g}, \mu_{\mathfrak{g}})$ and $f : \Gamma \to \bigwedge^2(\mathfrak{g})$ a map satisfying compatibility rules such that Γ acts on the double. Precise definitions and equivalent categories corresponding to these objects will be recalled in Section 1. Examples of Γ Lie bialgebras arise when G is a Poisson–Lie group with Lie bialgebra $(\mathfrak{g}, \mu_{\mathfrak{g}}, \delta_{\mathfrak{g}})$, and $\Gamma \subset G$ is a discrete subgroup. Another example is when $\mathfrak{g}$ is a Kac–Moody Lie algebra $\mathfrak{g}$, and Γ is a covering of the Weyl group of $\mathfrak{g}$. In the latter case, a quantization was given [Majid and Soĭbel'man 1994]. Quantization of a general Γ Lie bialgebra was done in [Enriquez and Halbout 2008], as we will review in Section 1.

What structure does one get on the corresponding dual groups? Considering the function algebra of a formal group, we get a trivial stack of Poisson–Hopf algebras. In Section 3, we prove that we get a nontrivial stack of Poisson algebras of functions on the formal Poisson–Lie group G^* dual to a Γ Poisson–Lie group G. To do so, we will construct "lifts" of the elements $(f(\gamma))_{\gamma\in\Gamma}$ in the function algebra on G^*. In Section 2, we recall basic definitions of stacks and explain our main results.

In Section 4, we construct quantization of these nontrivial Poisson–Hopf stacks. To do so we use quantization [Enriquez and Halbout 2008] of a Γ Lie bialgebra. To deduce from it a quantization of a nontrivial Poisson–Hopf stack, we use the

Drinfeld functor and prove that quantization of the elements $(f(\gamma))_{\gamma\in\Gamma}$ can be made "admissible", that is, they will give quantizations of the corresponding "lifts".

Finally, in Section 5, we give an explicit example corresponding to the case where G is a simple Lie group and Γ is a covering of the corresponding Weyl group. In this case, quantization of Majid and Soĭbel'man [1994] will lead to an explicit quantization of the nontrivial Poisson–Hopf stack.

Our results fit very well within Bressler, Gorokhovsky, Nest and Tsygan's framework [Bressler et al. 2007] of deformation quantization of gerbes. On one hand, our results provide interesting examples of quantization of stacks; on the other, the problems we deal with in this paper are more special and complicated because we need to treat Hopf algebra structure. In [Kirillov and Reshetikhin 1990] and [Soĭbel'man 1991] quantum Weyl groups are used to study R-matrices, and we hope that the results in this paper will shed a light on the general Γ R-matrices.

1. Γ Lie bialgebras and equivalent categories

We recall some results of [Enriquez and Halbout 2008].

Γ Lie algebras. A group Lie algebra is a triple $(\Gamma, \mathfrak{g}, \theta_{\mathfrak{g}})$, where Γ is a group, $\mathfrak{g}$ is a Lie algebra and $\theta_{\mathfrak{g}} : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ is a group morphism. It is the infinitesimal version of a Γ action on a group G. Group Lie algebras form a category.

If Γ is a discrete group, a Γ Lie algebra is a pair $(\mathfrak{g}, \theta_{\mathfrak{g}})$ such that $(\Gamma, \mathfrak{g}, \theta_{\mathfrak{g}})$ is a group Lie algebra. Γ Lie algebras form a subcategory of group Lie algebras. Such a Γ Lie algebra will be said to be the infinitesimal of a Γ group G.

A group cocommutative bialgebra is a triple (Γ, U, i), where Γ is a group, U is a cocommutative bialgebra, $U = \bigoplus_{\gamma\in\Gamma} U_\gamma$ is a decomposition of U and $i : \mathbf{k}\Gamma \to U$ is a bialgebra morphism, such that $U_\gamma U_{\gamma'} \subset U_{\gamma\gamma'}$, $\Delta_U(U_\gamma) \subset U_\gamma^{\otimes 2}$ and i is compatible with the Γ grading.

We then define a Γ cocommutative bialgebra as a pair (U, i) such that (Γ, U, i) is a group cocommutative bialgebra. Γ cocommutative bialgebras form a category.

The category of group (or Γ) cocommutative bialgebras contains as a full subcategory the category of group (respectively Γ) universal enveloping algebras, where (U, Γ, i) satisfies the additional requirement that U_e is a universal enveloping algebra.

Let $\mathbb{O}$ be a commutative algebra (in a symmetric monoidal category $\mathcal{S}$) with a decomposition $\mathbb{O} = \bigoplus_{\gamma\in\Gamma} \mathbb{O}_\gamma$. Suppose that $\mathbb{O}_\gamma \mathbb{O}_{\gamma'} = 0$ for $\gamma \neq \gamma'$ and that we have algebra morphisms

$$\Delta_{\gamma'\gamma''} : \mathbb{O}_{\gamma'\gamma''} \to \mathbb{O}_{\gamma'} \otimes \mathbb{O}_{\gamma''}, \quad \eta : \mathbf{k} \to \mathbb{O}_e, \quad \varepsilon : \mathbb{O}_e \to \mathbf{k}$$

satisfying axioms such that these morphisms add up to a bialgebra structure on $\mathbb{O}$ when Γ is finite. Then we define a group commutative bialgebra (in a symmetric monoidal category $\mathcal{S}$) as a triple $(\Gamma, \mathbb{O}, j)$, where Γ is a group and $j : \mathbb{O} \to \mathbf{k}^\Gamma$ is a morphism of commutative algebras compatible with the Γ gradings and the maps $\Delta_{\gamma'\gamma''}$ on both sides. We define Γ commutative bialgebras as above.

Define the category of group (or Γ) formal series Hopf (FSH) algebras as a full subcategory of the category of group (respectively Γ) commutative bialgebras in $\mathcal{S} = \{\text{provector spaces}\}$ by the condition that $\mathbb{O}_e$ (or equivalently, each $\mathbb{O}_\gamma$) is a formal series algebra. Such an FSH algebra corresponds to functions on the formal dual group of a Γ group G.

Proposition 1.1 [Enriquez and Halbout 2008]. (1) *We have* (*anti*)*equivalences of categories*

$$\{\textit{group Lie algebras}\} \leftrightarrow \{\textit{group universal enveloping algebras}\}$$
$$\leftrightarrow \{\textit{group FHS algebras}\},$$

where the last map is an antiequivalence.

(2) *If* Γ *is a group*, *these* (*anti*)*equivalences restrict to*

$$\{\Gamma\textit{-Lie algebras}\} \leftrightarrow \{\Gamma\textit{-universal enveloping algebras}\} \leftrightarrow \{\Gamma\textit{-FHS algebras}\}.$$

We denote the Γ universal enveloping algebra corresponding to a Γ Lie algebra $(\Gamma, \mathfrak{g}, \theta_\mathfrak{g})$ as $U(\mathfrak{g}) \rtimes \Gamma$. It is isomorphic to $U(\mathfrak{g}) \otimes \mathbf{k}\Gamma$ as a vector space. If we denote by $x \mapsto [x]$ and $\gamma \mapsto [\gamma]$ the natural maps $\mathfrak{g} \to U(\mathfrak{g}) \rtimes \Gamma$ and $\Gamma \to U(\mathfrak{g}) \rtimes \Gamma$, then the bialgebra structure of $U(\mathfrak{g}) \rtimes \Gamma$ is given by

$$[\gamma][x][\gamma^{-1}] = [\theta_\gamma(x)], \quad [\gamma][\gamma'] = [\gamma\gamma'], \quad \Delta([\gamma]) = [\gamma] \otimes [\gamma],$$
$$[x][x'] - [x'][x] = [\![x, x']\!], \qquad [e] = 1, \qquad \Delta([x]) = [x] \otimes 1 + 1 \otimes [x].$$

When Γ is finite, the corresponding Γ FSH algebra is then $(U(\mathfrak{g}) \rtimes \mathbf{k}\Gamma)^*$, and in general, this is $\bigoplus_{\gamma\in\Gamma}(U(\mathfrak{g}) \otimes \mathbf{k}\gamma)^*$.

Γ Lie bialgebras.

Definition 1.2. A group Lie bialgebra is a 5-uple $(\Gamma, \mathfrak{g}, \theta_\mathfrak{g}, \delta_\mathfrak{g}, f)$, where $(\Gamma, \mathfrak{g}, \theta_\mathfrak{g})$ is a group Lie algebra, $\delta_\mathfrak{g} : \mathfrak{g} \to \bigwedge^2(\mathfrak{g})$ is[1] such that $(\mathfrak{g}, \delta_\mathfrak{g})$ is a Lie bialgebra, and $f : \Gamma \to \bigwedge^2(\mathfrak{g})$ is a map $\gamma \mapsto f_\gamma$ such that

(a) $\bigwedge^2(\theta_\gamma) \circ \delta \circ \theta_\gamma^{-1}(x) = \delta(x) + [f_\gamma, x \otimes 1 + 1 \otimes x]$ for any $x \in \mathfrak{g}$,

(b) $f_{\gamma\gamma'} = f_\gamma + \bigwedge^2(\theta_\gamma)(f_{\gamma'})$,

(c) $(\delta \otimes \mathrm{id})(f_\gamma) + [f_\gamma^{1,3}, f_\gamma^{2,3}] + \text{cyclic permutations} = 0$.

[1]We view $\bigwedge^2(V)$ as a subspace of $V^{\otimes 2}$.

Group Lie bialgebras form a category. When Γ is fixed, one defines the category of Γ Lie bialgebras as above.

A co-Poisson structure on a group cocommutative bialgebra (Γ, U, i) is a co-Poisson structure $\delta_U : A \to \bigwedge^2(U)$ such that $\delta_U(U_\gamma) \subset \bigwedge^2(U_\gamma)$. Co-Poisson group cocommutative bialgebras form a category.

Co-Poisson group universal enveloping algebras form a full subcategory of the latter category. One defines the full subcategories of co-Poisson Γ cocommutative bialgebras and co-Poisson Γ enveloping algebras as above.

A Poisson structure on a group commutative bialgebra $(\Gamma, \mathbb{O}, j)$ is a Poisson bialgebra structure $\{\cdot,\cdot\} : \bigwedge^2(\mathbb{O}) \to \mathbb{O}$ such that $\{\mathbb{O}_\gamma, \mathbb{O}_\gamma\} \subset \mathbb{O}_\gamma$ and $\{\mathbb{O}_\gamma, \mathbb{O}_{\gamma'}\} = 0$ if $\gamma \neq \gamma'$. Poisson group bialgebras form a category, and Poisson group FSH algebras form a full subcategory when $\mathcal{S} = \{\text{provector spaces}\}$. One defines the full subcategories of Poisson Γ bialgebras and Poisson Γ FSH algebras as above.

Example. Let G be a Poisson–Lie (for example, algebraic) group, and let $\Gamma \subset G$ be a subgroup (which we view as an abstract group). We define $\theta_\gamma := \mathrm{Ad}(\gamma)$, where $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ is the adjoint action. If $P : G \to \bigwedge^2(\mathfrak{g})$ is the Poisson bivector satisfying $P(gg') = P(g') + \bigwedge^2(\mathrm{Ad}(g'))(P(g))$, then we set $f_\gamma := -P(\gamma)$. Then $(\mathfrak{g}, \Gamma, f)$ is a Γ Lie bialgebra.

Example. Let $(\mathfrak{g}, r_\mathfrak{g})$ be a quasitriangular Lie bialgebra and let $\theta : \Gamma \to \mathrm{Aut}(\mathfrak{g}, t_\mathfrak{g})$ be an action of Γ on $\mathfrak{g}$ by Lie algebra automorphisms preserving $t_\mathfrak{g} := r_\mathfrak{g} + r_\mathfrak{g}^{2,1}$. If we set $f_\gamma := \theta_\gamma^{\otimes 2}(r) - r$, then $(\mathfrak{g}, \theta, f)$ is a Γ Lie bialgebra (we call this a quasitriangular Γ Lie bialgebra). For example, $\mathfrak{g}$ is a Kac–Moody Lie algebra, and $\Gamma = \widetilde{W}$ is a covering of the Weyl group of $\mathfrak{g}$; see [Majid and Soĭbel'man 1994].

Proposition 1.3 [Enriquez and Halbout 2008]. (1) *We have category (anti)equivalences*

$$\{\textit{group bialgebras}\} \leftrightarrow \{\textit{co-Poisson group universal enveloping algebras}\}$$
$$\leftrightarrow \{\textit{Poisson group FSH algebras}\}.$$

(2) *The (anti)equivalences above restrict to category (anti)equivalences*

$$\{\Gamma\textit{-bialgebras}\} \leftrightarrow \{\textit{co-Poisson } \Gamma \textit{ universal enveloping algebras}\}$$
$$\leftrightarrow \{\textit{Poisson } \Gamma \textit{ FSH algebras}\}.$$

If $(\mathfrak{g}, \theta_\mathfrak{g}, \delta_\mathfrak{g})$ is a Γ Lie bialgebra, then the co-Poisson structure on $U := U(\mathfrak{g}) \rtimes \Gamma$ is given by $\delta_U([x]) = [\delta_\mathfrak{g}(x)]$ and $\delta_U([\gamma]) = -[f_\gamma]([\gamma] \otimes [\gamma])$. Here we also denote by $x \mapsto [x]$ the natural map $\bigwedge^2(\mathfrak{g}) \to \bigwedge^2(U(\mathfrak{g}) \rtimes \Gamma)$.

Quantization of* Γ *Lie bialgebras. In a symmetric monoidal category $\mathcal{S}$, let a Γ graded bialgebra be a bialgebra A (in $\mathcal{S}$) equipped with a grading $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, such that $A_\gamma A_{\gamma'} \subset A_{\gamma\gamma'}$ and $\Delta_A(A_\gamma) \subset A_\gamma^{\otimes 2}$.

Assume that A is a Γ graded bialgebra in the category of topologically free $\mathbf{k}[[\hbar]]$-modules and that A is quasicocommutative in that $A_0 := A/\hbar A$ is cocommutative. Then we get a co-Poisson structure on A_0. It is Γ graded in that $\delta_{A_0}((A_0)_\gamma) \subset \bigwedge^2((A_0)_\gamma)$. We therefore get a classical limit functor, class, from {Γ-graded quasicocommutative bialgebras} to {Γ-graded co-Poisson bialgebras}.

Definition 1.4. A quantization functor for Γ Lie bialgebras is a functor

{co-Poisson Γ universal enveloping algebras}

$\to$ {Γ-graded quasicocommutative bialgebras},

which is right inverse to class.

Assume that $(\mathfrak{g}, \theta, f)$ is a Γ Lie bialgebra. Let $(U_e, *, \Delta_e)$ be the Etingof–Kazhdan quantization of $(\mathfrak{g}, \delta)$; we will denote the multiplication by m_e. We get this from [Enriquez and Halbout 2008]:

Proposition 1.5. *There exist sets* $(\mathrm{F}_{\gamma,\gamma\gamma'})_{\gamma,\gamma'\in\Gamma}$ *of elements in* $U^{\otimes 2}$, *with* $\mathrm{F}_{\gamma,\gamma\gamma'} = 1 + \hbar\,\mathrm{f}_1 + O(\hbar^2)$ *and* $\mathrm{Alt}(\mathrm{f}_1) = \bigwedge^2(\theta_\gamma)(f_{\gamma'})$, *sets* $(\mathrm{v}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})_{\gamma,\gamma',\gamma''\in\Gamma}$ *of elements in* $1 + \hbar^2 U$, *sets* $(U_\gamma, m_\gamma, \Delta_\gamma)_{\gamma\in\Gamma}$ *of bialgebras, and sets* $(\mathrm{i}_{\gamma,\gamma\gamma'})_{\gamma,\gamma'\in\Gamma}$ *of algebra morphisms from* (U_γ, m_γ) *to* $(U_{\gamma\gamma'}, m_{\gamma\gamma'})$, *such that*

- $\Delta_\gamma = \mathrm{i}_{e,\gamma}^{\otimes 2} \circ \mathrm{Ad}(\mathrm{F}_{e,\gamma}) \circ \Delta_e \circ \mathrm{i}_{e,\gamma}^{-1}$,
- $(\mathrm{F}_{e,\gamma} \otimes 1) * (\Delta_e \otimes \mathrm{id})(\mathrm{F}_{e,\gamma}) = (1 \otimes \mathrm{F}_{e,\gamma}) * (\mathrm{id} \otimes \Delta_e)(\mathrm{F}_{e,\gamma})$,
- $\mathrm{F}_{e,\gamma\gamma'} = \mathrm{v}_{e,\gamma,\gamma\gamma'}^{\otimes 2} * (\mathrm{i}_{e,\gamma}^{\otimes 2})^{-1}(\mathrm{F}_{\gamma,\gamma\gamma'}) * \mathrm{F}_{e,\gamma} * \Delta_e(\mathrm{v}_{e,\gamma,\gamma\gamma'})^{-1}$,
- $\mathrm{i}_{e,\gamma\gamma'} = \mathrm{i}_{\gamma,\gamma\gamma'} \circ \mathrm{i}_{e,\gamma} \circ \mathrm{Ad}(\mathrm{v}_{e,\gamma,\gamma\gamma'}^{-1})$,
- $\mathrm{v}_{e,\gamma\gamma',\gamma\gamma'\gamma''} * \mathrm{v}_{e,\gamma,\gamma\gamma'} = \mathrm{v}_{e,\gamma,\gamma\gamma'\gamma''} * \mathrm{i}_{e,\gamma}^{-1}(\mathrm{v}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$.

Here to make the formulas shorter we have chosen to write the above equations with e being the unit of the group Γ; however, the formulas are still valid if we replace e by any other element of the group and multiply γ, $\gamma\gamma'$ and $\gamma\gamma'\gamma''$ on the left by this element.

We then get a quantization of the Γ Lie bialgebra by setting $U = S(\mathfrak{g}) \otimes \mathbf{k}\Gamma[[\hbar]]$ and putting $[x\,|\,\gamma] := x \otimes \gamma$ and $[x \otimes x'\,|\,\gamma, \gamma'] := (x \otimes \gamma) \otimes (x' \otimes \gamma') \in U^{\otimes 2}$.

There are unique linear maps $m : U^{\otimes 2} \to U$ and $\Delta : U \to U^{\otimes 2}$ such that

$$m : [x\,|\,\gamma][x'\,|\,\gamma'] \mapsto [x * \mathrm{i}_{e,\gamma}^{-1}(\theta_\gamma(x')) * \mathrm{v}_{e,\gamma,\gamma\gamma'}^{-1}\,|\,\gamma\gamma'],$$
$$\Delta : [x\,|\,\gamma] \mapsto [\Delta_e(x) * \mathrm{F}_{e,\gamma}^{-1}\,|\,\gamma, \gamma].$$

The unit for U is $[1\,|\,e]$, and the counit is the map $[x\,|\,\gamma] \mapsto \delta_{\gamma,e}\varepsilon(x)$.

Proposition 1.6 [Enriquez and Halbout 2008]. *This defines a bialgebra structure on U, quantizing the co-Poisson bialgebra structure induced by* $(\mathfrak{g}, \theta, f)$.

2. Stacks and main results

It is well known that the semiclassical limit of a quantum group is a Poisson–Lie group. In this paper, we attempt to answer, What is the semiclassical limit corresponding to a "Γ coboundary quantum group"? We hope to say that this semiclassical object is a "stack" of Poisson–Lie groups G^* over the classifying stack $B\Gamma$ of the group Γ. Toward this, we construct a stack of Poisson–Hopf algebras over the groupoid $\Gamma \rtimes \Gamma$ (a transformation groupoid associated to the Γ right action on Γ) and also a quantization of such a stack. Because of the existence of the twisted cocycle, we expect that such an algebroid stack is not trivial. We also hope that our construction will eventually lead to a complete description of the semiclassical limit of a Γ-coboundary quantum group.

Definition 2.1. A stack on M is

- an open cover of $M = \bigcup U_i$,
- a sheaf of rings A_i on every U_i,
- an isomorphism of sheaves of rings $G_{ij}\colon A_j|_{U_i \cap U_j} \to A_i|_{U_i \cap U_j}$ for every i, j,
- an invertible element $c_{ijk} \in A_i|_{U_i \cap U_j \cap U_k}$ for every i, j, k satisfying
  (1) $G_{ij}G_{jk} = \mathrm{Ad}(c_{ijk})G_{ik}$ and
  (2) $c_{ijk}c_{ikl} = G_{ij}(c_{jkl})c_{ijl}$ for every i, j, k, l.

If two such data $(U'_i, A'_i, G'_{ij}, c'_{ijk})$ and $(U''_i, A''_i, G''_{ij}, c''_{ijk})$ are given on M, an isomorphism between them is

- an open cover $M = \bigcup U_i$ refining both $\{U'_i\}$ and $\{U''_i\}$,
- isomorphisms $H_i\colon A'_i \to A''_i$ on U_i, and
- invertible elements b_{ij} of $A'_i|_{U_i \cap U_j}$ such that
  (1) $G''_{ij} = H_i \,\mathrm{Ad}(b_{ij})G'_{ij}H_j^{-1}$ and
  (2) $H_i^{-1}(c''_{ijk}) = b_{ij}G'_{ij}(b_{jk})c_{ijk}b_{ik}^{-1}$.

Inspired by Definition 2.1, we define a stack over a discrete groupoid. Let $\mathfrak{G}$ be a discrete groupoid with its unit space $\mathfrak{G}_0$.

Definition 2.2. A stack on $\mathfrak{G}$ consists of

- a collection of rings A_x on every point x of $\mathfrak{G}_0$,
- an isomorphism of sheaves of rings $T_g : A_{t(g)} \to A_{s(g)}$ for every arrow $g \in \mathfrak{G}$, where s, t are the source and target maps of $\mathfrak{G}$, and
- an invertible element $c_{g_1,g_2} \in A_{s(g_1)}$ for every pair of composable arrows in $\mathfrak{G}$ such that
  (1) $T_{g_1} \circ T_{g_2} = \mathrm{Ad}(c_{g_1,g_2})T_{g_1g_2}$, where by $\mathrm{Ad}(c_{g_1,g_2})$ we mean the conjugation operator on $A_{s(g_1)}$ associated to the invertible element c_{g_1,g_2}, and

(2) $c_{g_1,g_2}c_{g_1g_2,g_3} = T_{g_1}(c_{g_2,g_3})c_{g_1,g_2g_3}$ for every triple of composable arrows g_1, g_2, g_3 in $\mathfrak{G}$.

One can generalize the equivalence between stacks of Definition 2.1, but we omit the details here.

Our main result will involve the notion of a stack of Poisson–Hopf algebras.

Definition 2.3. A stack of Poisson–Hopf algebras over a discrete groupoid $\mathfrak{G}$ is

- a collection A_x of Poisson–Hopf algebras $(A_x, m_x, \Delta_x, \{\cdot,\cdot\}_x)_{x\in\mathfrak{G}_0}$,
- a Poisson morphism $T_g : A_{t(g)} \to A_{s(g)}$ for $g \in \mathfrak{G}$, and
- an invertible element $c_{g_1,g_2} \in A_{s(g_1)}$ for every pair of composable arrows in $\mathfrak{G}$ such that
  (1) $T_{g_1} \circ T_{g_2} = \mathrm{Ad}(c_{g_1,g_2})T_{g_1g_2}$, where by $\mathrm{Ad}(c_{g_1,g_2})$, we mean the conjugation operator on $A_{s(g_1)}$ associated to the invertible element c_{g_1,g_2}, and
  (2) $c_{g_1,g_2}c_{g_1g_2,g_3} = T_{g_1}(c_{g_2,g_3})c_{g_1,g_2g_3}$ for every triple of composable arrows g_1, g_2, g_3.

In what follows, we consider the groupoid $\Gamma \rtimes \Gamma$ defined by the action of Γ on Γ itself by right multiplication. As Γ is discrete, $\Gamma \rtimes \Gamma$ is a discrete groupoid. We will use $(\gamma, \gamma\gamma')$ to denote an arrow in $\Gamma \rtimes \Gamma$ mapping from γ to $\gamma\gamma'$. The product of a pair of composable arrows $(\gamma, \gamma\gamma')$ and $(\gamma\gamma', \gamma\gamma'\gamma'')$ in $\Gamma \rtimes \Gamma$ is $(\gamma, \gamma\gamma'\gamma'')$. For our main results, we associate an Poisson–Hopf algebra $\mathbb{O}_{G^*_\gamma}$ to each point γ in the unit space of $\Gamma \rtimes \Gamma$, and we will prove the existence of a stack of Poisson–Hopf algebras over the groupoid $\Gamma \rtimes \Gamma$.

Theorem 2.4. *Associated to a coboundary Lie bialgebra* $(\Gamma, \mathfrak{g}, \theta_{\mathfrak{g}}, \delta_{\mathfrak{g}}, f)$, *there is a stack of Poisson–Hopf algebras over the groupoid* $\Gamma \rtimes \Gamma$.

To be compatible with the result of Proposition 1.5, we will mainly prove the following results.

- There is a set $(\mathbb{O}_{G^*_\gamma})_{\gamma\in\Gamma}$ of Poisson–Hopf algebras $(\mathbb{O}_{G^*_\gamma}, m_\gamma, \Delta_\gamma, \{\cdot,\cdot\}_\gamma)_{\gamma\in\Gamma}$.
- Associated to each arrow $(\gamma, \gamma\gamma')$ in $\Gamma \rtimes \Gamma$, there is a Poisson morphism $j_{\gamma,\gamma\gamma'}$ from $\mathbb{O}_{G^*_\gamma}$ to $\mathbb{O}_{G^*_{\gamma\gamma'}}$.
- Associated to a pair of composable arrows $(\gamma, \gamma\gamma')$ and $(\gamma\gamma', \gamma\gamma'\gamma'')$ in $\Gamma \rtimes \Gamma$, there is an element $u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ of $\mathbb{O}_{G^*_\gamma}$ satisfying relations
  (1) $j_{\gamma,\gamma\gamma'\gamma''} = j_{\gamma\gamma',\gamma\gamma'\gamma''} \circ j_{\gamma,\gamma\gamma'} \circ \mathrm{Ad}_{\star_\gamma}(u^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$, where $\mathrm{Ad}_{\star_\gamma}(u^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$ is the conjugation operator associated to $u^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ with respect to the Baker–Campbell–Hausdorff product $\star_\gamma$, and
  (2) $u_{\gamma,\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''} \star_\gamma u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} = u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''\gamma'''} \star_\gamma j^{-1}_{\gamma,\gamma\gamma'}(u_{\gamma\gamma',\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''})$.

With the above data $((\mathbb{O}_{G^*_\gamma}, m_\gamma, \Delta_\gamma, \{\cdot,\cdot\}_\gamma)_{\gamma\in\Gamma}, j_{\gamma,\gamma\gamma'}, u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$, if we define $T_{\gamma,\gamma\gamma'}: \mathbb{O}_{G^*_{\gamma\gamma'}} \to \mathbb{O}_{G^*_\gamma}$ by $T_{\gamma,\gamma\gamma'} = j^{-1}_{\gamma,\gamma\gamma'}$ and $c_{(\gamma,\gamma\gamma'),(\gamma\gamma',\gamma\gamma'\gamma'')} \in \mathbb{O}_{G^*_\gamma}$ to be $u^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$, then we can easily check that we do obtain a stack of Poisson–Hopf algebras over $\Gamma \rtimes \Gamma$ satisfying Definition 2.2.

We will then prove the existence of a stack of Hopf algebras quantizing this stack of Poisson–Hopf algebras:

Theorem 2.5. *There is a stack of Hopf algebras quantizing the stack of Poisson–Hopf algebras over $\Gamma \rtimes \Gamma$ defined in Theorem 2.4. Namely, we have a collection of the following data:*[2]

- $(\mathbb{A}_\gamma, *_\gamma)_{\gamma\in\Gamma}$, *which are deformation quantizations*[3] *of the Poisson algebras* $(\mathbb{O}_{G^*_\gamma}, \{\cdot,\cdot\}_\gamma)_{\gamma\in\Gamma}$,
- *algebra morphisms* $i_{\gamma,\gamma\gamma'}: \mathbb{A}_\gamma \to \mathbb{A}_{\gamma\gamma'}$, *and*
- *elements* $v_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ *of* $\mathbb{A}_\gamma$ *such that* $ev_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} := \exp(v_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}/\hbar)$ *satisfy relations*
  (1) $i_{\gamma,\gamma\gamma'\gamma''} = i_{\gamma\gamma',\gamma\gamma'\gamma''} \circ i_{\gamma,\gamma\gamma'} \circ \mathrm{Ad}(ev^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$ *and*
  (2) $ev_{\gamma,\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''} *_\gamma ev_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} = ev_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''\gamma'''} *_\gamma i^{-1}_{\gamma,\gamma\gamma'}(ev_{\gamma\gamma',\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''})$.

From our setup in this section, one can see that the facts that Γ is a group and that we have a transformation groupoid $\Gamma \rtimes \Gamma$ are not crucial in our construction. It is natural to expect a more general theory for quantization of a $\mathfrak{G}$-coboundary Lie bialgebras with $\mathfrak{G}$ a discrete groupoid or even just category.

3. A stack of Poisson bialgebras of functions on the formal group G^*

Let $(\mathfrak{g}, \theta_\mathfrak{g}, \delta_\mathfrak{g}, f)$ be a Γ Lie bialgebra. In this section we will construct a stack of Poisson bialgebras of functions on a formal Poisson group G^*.

Notations. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. Let $(U(\mathfrak{g}), \Delta_0, \delta)$ be its corresponding cocommutative co-Poisson bialgebra, which can be seen as the dual of the function algebra of the formal Poisson–Lie group G corresponding to $(\mathfrak{g}, \delta)$. In the same way, we will define $\mathbb{O}_{G^*}$ as the commutative Poisson–Hopf algebra of functions of the formal Poisson–Lie group G^* corresponding to the dual Lie bialgebra $\mathfrak{g}^*$. We define by $\mathfrak{m}_{G^*} \subset \mathbb{O}_{G^*}$ the maximal ideal of this ring. If k is an integer ≥ 1, we denote by $\mathbb{O}_{(G^*)^k}$ the ring of formal functions on $(G^*)^k$, by $\mathfrak{m}_{(G^*)^k}$ its maximal ideal, and by $\mathfrak{m}^i_{(G^*)^k}$ the i-th power of this ideal.

If $f, g \in \mathfrak{m}^2_{(G^*)^k}$, then the series

$$f \star g = f + g + \tfrac{1}{2}\{f, g\} + \cdots + B_n(f, g) + \cdots$$

[2]Similarly to what we did for Theorem 2.4, we will take the inverse of $i_{\gamma,\gamma\gamma'}$ and $ev_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ to construct the corresponding data for the stack of Hopf algebras.

[3]Deformation quantization here means that $\mathbb{A}_\gamma/\hbar\mathbb{A}_\gamma = \mathbb{O}_{G^*_\gamma}$, and $\frac{1}{\hbar}[\cdot,\cdot]_{*_\gamma} = \{\cdot,\cdot\}_\gamma + O(\hbar)$.

is convergent, where $\sum_{i\geq 1} B_i(x, y)$ is the Baker–Campbell–Hausdorff (BCH) series specialized to the Poisson bracket of $\mathfrak{m}^2_{(G^*)^k}$. The $\star$ product defines a group structure on $\mathfrak{m}^2_{(G^*)^k}$.

A useful technical lemma was proved in [Enriquez et al. 2003, page 2477] for $m_{\mathfrak{g}^*}$ and is still true for m_{G^*}:

Lemma 3.1. *For any $k \geq 1$ and $n \geq 2$, $f, h \in \mathfrak{m}^2_{(G^*)^k}$ and $g \in \mathfrak{m}^n_{(G^*)^k}$, one has*

$$f \star (h+g) = f \star h + g \quad \text{and} \quad (f+g)\star h = f\star h + g \text{ modulo } \mathfrak{m}^{n+1}_{(G^*)^k}.$$

When $(\mathfrak{g}, \theta_{\mathfrak{g}}, \delta_{\mathfrak{g}}, f)$ is a Γ Lie bialgebra we thus get a collection of Lie bialgebras and so a collection $(\mathbb{O}_{G^*_\gamma}, m_\gamma, \Delta_\gamma, \{\cdot,\cdot\}_\gamma)_{\gamma\in\Gamma}$ of Poisson bialgebras. We will denote by $\star_\gamma$ the corresponding BCH products.

"Lifts" and functional equations. We will now construct "lifts" $\tilde{f}_{\gamma,\gamma\gamma'} \in \mathfrak{m}^{\hat{\otimes}2}_{G^*}$ of the elements $\bigwedge^2(\theta_\gamma)(f_{\gamma'})$, $\gamma, \gamma' \in \Gamma$ that will satisfy similar relation as $\mathrm{F}_{\gamma,\gamma\gamma'}$ in Proposition 1.5. The proof in this subsection is a direct generalization of the results in [Enriquez and Halbout 2007], and some parts are transcribed almost verbatim.

If $f \in \mathbb{O}^{\hat{\otimes}n}_{G^*}$ and $P_1,\dots,P_m$ are disjoint subsets of $\{1,\dots,m\}$, one defines $f^{P_1,\dots,P_n}$ using the coproduct of $\mathbb{O}_{G^*}$:

Definition 3.2. If $I_1,\dots,I_m$ are disjoint ordered subsets of $\{1,\dots,n\}$, (U,Δ) is a Hopf algebra, and $a \in U^{\otimes m}$, we define

$$a^{I_1,\dots,I_n} = \sigma_{I_1,\dots,I_m} \circ (\Delta^{|I_1|} \otimes \cdots \otimes \Delta^{|I_n|})(a),$$

with $\Delta^{(1)} = \mathrm{id}$, $\Delta^{(2)} = \Delta$, and $\Delta^{(n+1)} = (\mathrm{id}^{\otimes n-1} \otimes \Delta)\circ\Delta^{(n)}$. Here

$$\sigma_{I_1,\dots,I_m} : U^{\otimes \sum_i |I_i|} \to U^{\otimes n}$$

is the morphism corresponding to the map $\{1,\dots,\sum_i |I_i|\} \to \{1,\dots,n\}$ taking $(1,\dots,|I_1|)$ to I_1, $(|I_1|+1,\dots,|I_1|+|I_2|)$ to I_2, and so on.

When U is cocommutative, this definition depends only on the underlying sets $I_1,\dots,I_m$.

Proposition 3.3. *Let γ, γ' be in Γ. Then there exists $\tilde{f}_{\gamma,\gamma\gamma'}$ in $m^{\hat{\otimes}2}_{G^*}$, the image of which in $\mathfrak{g}^{\otimes 2}$ under the square of the projection $\mathfrak{m}_{G^*} \to \mathfrak{m}_{G^*}/\mathfrak{m}^2_{G^*} = \mathfrak{g}$ equals $\bigwedge^2(\theta_\gamma)(f_{\gamma'})$, and such that*

$$(1) \qquad (\tilde{f}_{\gamma,\gamma\gamma'} \otimes 1) \star_\gamma (\Delta_\gamma \otimes \mathrm{id})(\tilde{f}_{\gamma,\gamma\gamma'}) = (1 \otimes \tilde{f}_{\gamma,\gamma\gamma'}) \star_\gamma (\mathrm{id} \otimes \Delta_\gamma)(\tilde{f}_{\gamma,\gamma\gamma'}).$$

The element $\tilde{f}_{\gamma,\gamma\gamma'}$ is unique up to the action of $\mathfrak{m}^2_{G^*}$ by $\lambda\cdot\tilde{f} = \lambda^1 \star_\gamma \lambda^2 \star_\gamma \tilde{f} \star_\gamma (-\lambda)^{12}$. We will call $\tilde{f}$ a twist for Δ_γ.

Proof. Let us construct $\tilde{f}_{\gamma,\gamma\gamma'}$ by induction, by constructing a convergent sequence $\tilde{f}^N \in \mathfrak{m}_{G^*}^{\hat{\otimes}2}$ ($N \geq 2$) satisfying (1) in $\mathfrak{m}_{G^*}^{\hat{\otimes}3}/(\mathfrak{m}_{G^*}^{\hat{\otimes}3} \cap \mathfrak{m}_{(G^*)^3}^N)$, where $\mathfrak{m}_{(G^*)^3}^N$ is the N-th power of $\mathfrak{m}_{(G^*)^3}$. When $N = 3$, we take $\tilde{f}^2$ to be any lift of $\bigwedge^2(\theta_\gamma)(f_{\gamma'})$ to $\mathfrak{m}_{G^*}^{\hat{\otimes}2}$; then (1) is automatically satisfied.

Let N be an integer no less than 3; assume that we have constructed $\tilde{f}^N$ in $\mathfrak{m}_{G^*}^{\hat{\otimes}2}$ satisfying (1) in $\mathfrak{m}_{G^*}^{\hat{\otimes}3}/(\mathfrak{m}_{G^*}^{\hat{\otimes}3} \cap \mathfrak{m}_{(G^*)^3}^N)$. Set $\alpha_{1,2,3}^N := \tilde{f}_{1,2}^N \star_\gamma \tilde{f}_{12,3}^N - \tilde{f}_{2,3}^N \star_\gamma \tilde{f}_{1,23}^N$. Then $\alpha_{1,2,3}^N$ belongs to $\mathfrak{m}_{G^*}^{\hat{\otimes}3} \cap \mathfrak{m}_{(G^*)^3}^N$, and in $\mathfrak{m}_{G^*}^{\hat{\otimes}4}/(\mathfrak{m}_{G^*}^{\hat{\otimes}4} \cap \mathfrak{m}_{(G^*)^4}^{N+1})$, we have the equalities

$$\begin{aligned}
\alpha_{12,3,4}^N &= \tilde{f}_{1,2}^N \star_\gamma \alpha_{12,3,4}^N = \tilde{f}_{1,2}^N \star_\gamma \tilde{f}_{12,3}^N \star_\gamma \tilde{f}_{123,4}^N - \tilde{f}_{1,2}^N \star_\gamma \tilde{f}_{3,4}^N \star_\gamma \tilde{f}_{12,34}^N \\
&= \alpha_{1,2,3}^N + \tilde{f}_{2,3}^N \star_\gamma \tilde{f}_{1,23}^N \star_\gamma \tilde{f}_{123,4}^N - \tilde{f}_{3,4}^N \star_\gamma \tilde{f}_{1,2}^N \star_\gamma \tilde{f}_{12,34}^N \\
&\hphantom{=} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(using Lemma 3.1)} \\
&= \alpha_{1,2,3}^N + \tilde{f}_{2,3}^N \star_\gamma \tilde{f}_{1,23}^N \star \tilde{f}_{123,4}^N - \tilde{f}_{3,4}^N \star (\tilde{f}_{2,34}^N \star_\gamma \tilde{f}_{1,234}^N + \alpha_{1,2,34}^N) \\
&\hphantom{=} \qquad\qquad \text{(using Lemma 3.1 and the definition of } \alpha_{1,2,34}^N) \\
&= \alpha_{1,2,3}^N + \tilde{f}_{2,3}^N \star_\gamma (\alpha_{1,23,4}^N + \tilde{f}_{23,4}^N \star_\gamma \tilde{f}_{1,234}^N) \\
&\quad - \alpha_{1,2,34}^N - \tilde{f}_{3,4}^N \star_\gamma \tilde{f}_{2,34}^N \star_\gamma \tilde{f}_{1,234}^N \\
&\hphantom{=} \qquad\qquad \text{(using the definition of } \alpha_{1,23,4}^N \text{ and Lemma 3.1)} \\
&= \alpha_{1,2,3}^N + \alpha_{1,23,4}^N + (\tilde{f}_{3,4}^N \star_\gamma \tilde{f}_{2,34}^N + \alpha_{2,3,4}^N) \star_\gamma \tilde{f}_{1,234}^N \\
&\quad - \alpha_{1,2,34}^N - \tilde{f}_{3,4}^N \star_\gamma \tilde{f}_{2,34}^N \star_\gamma \tilde{f}_{1,234}^N \\
&\hphantom{=} \qquad\qquad \text{(using the definition of } \alpha_{2,3,4}^N \text{ and Lemma 3.1)} \\
&= \alpha_{1,2,3}^N + \alpha_{1,23,4}^N - \alpha_{1,2,34}^N + \alpha_{2,3,4}^N \qquad\qquad \text{(using Lemma 3.1).}
\end{aligned}$$

Let us denote by $\bar{\alpha}^N$ the image of α^N in $(\mathfrak{m}_{\mathfrak{g}^*}^{\hat{\otimes}3} \cap \mathfrak{m}_{(\mathfrak{g}^*)^3}^N)/(\mathfrak{m}_{\mathfrak{g}^*}^{\hat{\otimes}3} \cap \mathfrak{m}_{(\mathfrak{g}^*)^3}^{N+1}) = (S^{>0}(\mathfrak{g})^{\otimes 3})_N$. Then we get

$$\bar{\alpha}_{12,3,4}^N + \bar{\alpha}_{1,2,34}^N = \bar{\alpha}_{1,2,3}^N + \bar{\alpha}_{1,23,4}^N + \bar{\alpha}_{2,3,4}^N,$$

meaning that $\bar{\alpha}$ is a cocycle for the subcomplex $(S^{>0}(\mathfrak{g})^{\otimes\cdot}, d)$ of the co-Hochschild complex. By using [Drinfeld 1989, Proposition 3.11], one proves that the k-th cohomology group of this subcomplex is $\bigwedge^k(\mathfrak{g})$, and that the antisymmetrization map coincides with the canonical projection from the space of cocycles to the cohomology group. For $N = 3$, the equations of Definition 1.2 imply $\mathrm{Alt}(\bar{\alpha}^3) = 0$, and hence $\bar{\alpha}^3$ is the coboundary of an element $\bar{\beta}^3 \in (S^{>0}(\mathfrak{g})^{\otimes 2})^3$, and $\bar{\alpha}^N$ for $N > 3$ is the coboundary of an element $\bar{\beta}^N \in (S^{>0}(\mathfrak{g})^{\otimes 2})^N$ since the degree N part of the cohomology vanishes. We then set $\tilde{f}^{N+1} := \tilde{f}^N + \beta^N$, where $\beta^N \in \mathfrak{m}_{G^*}^{\hat{\otimes}2} \cap \mathfrak{m}_{(G^*)^2}^N$ is a representative of $\bar{\beta}^N$. Then this $\tilde{f}^{N+1}$ satisfies (1) in $\mathfrak{m}_{G*}^{\hat{\otimes}3}/(\mathfrak{m}_{G^*}^{\hat{\otimes}3} \cap \mathfrak{m}_{(G^*)^3}^{N+1})$.

The sequence $(\tilde{f}^N)^{N \geq 2}$ has a limit $\tilde{f}$, which then satisfies (1).

The second part of the theorem can be proved in the same way or by analyzing the choices for $\bar{\beta}^N$ in the proof above. □

Isomorphism of formal Poisson groups $G^*_\gamma \simeq G^*_{\gamma\gamma'}$.

Proposition 3.4. *Let* $\gamma, \gamma' \in \Gamma$ *and let* G^*_γ *and* $G^*_{\gamma\gamma'}$ *be the formal Poisson–Lie groups associated to the corresponding Lie cobrackets. There exists an isomorphism of Poisson algebras* $j_{\gamma,\gamma\gamma'}\colon \mathbb{O}_{G^*_\gamma} \simeq \mathbb{O}_{G^*_{\gamma\gamma'}}$.

Proof. Let $P : \bigwedge^2(\mathbb{O}_{G^*_\gamma}) \to \mathbb{O}_{G^*_\gamma}$ be the Poisson bracket on $\mathbb{O}_{G^*_\gamma}$ corresponding to the Lie–Poisson Poisson structure on G^*_γ. Then $(\mathbb{O}_{G^*_\gamma}, m_0, P, \Delta_\gamma)$ is a Poisson formal series Hopf (PFSH) algebra; it corresponds to the formal Poisson–Lie group G^*_γ equipped with its Lie–Poisson structure.

Set $\tilde{f}^\star_{\gamma,\gamma\gamma'}\Delta_\gamma(a) = \tilde{f}_{\gamma,\gamma\gamma'} \star_\gamma \Delta_\gamma(a) \star_\gamma (-\tilde{f}_{\gamma,\gamma\gamma'})$ for any $a \in \mathbb{O}_{G^*_\gamma}$. It follows from the fact that $\tilde{f}_{\gamma,\gamma\gamma'}$ satisfies Equation (1) that $(\mathbb{O}_{G^*_\gamma}, m_0, P, \tilde{f}^\star_{\gamma,\gamma\gamma'}\Delta_\gamma)$ is a PFSH algebra.

Let us denote by **PFSHA** and **LBA** the categories of PSFH algebras and Lie bialgebras. We have a category equivalence c : **PFSHA** $\to$ **LBA**, taking $(\mathbb{O}, m, P, \Delta)$ to the Lie bialgebra $(\mathfrak{c}, \mu, \delta)$, where $\mathfrak{c} := \mathfrak{m}/\mathfrak{m}^2$ (here $\mathfrak{m} \subset \mathbb{O}$ is the maximal ideal), the Lie cobracket of $\mathfrak{c}$ is induced by $\Delta - \Delta^{2,1} : \mathfrak{m} \to \bigwedge^2(\mathfrak{m})$, and the Lie bracket of $\mathfrak{c}$ is induced by the Poisson bracket $P : \bigwedge^2(\mathfrak{m}) \to \mathfrak{m}$. The inverse of the functor c takes $(\mathfrak{c}, \mu, \delta)$ to $\mathbb{O} = \hat{S}(\mathfrak{c})$ equipped with its usual product; Δ depends only on δ and P depends on (μ, δ).

Then c restricts to a category equivalence c_{fd} : $\mathbf{PFSHA}_{\mathrm{fd}} \to \mathbf{LBA}_{\mathrm{fd}}$ of subcategories of finite-dimensional objects (in the case of **PFSH**, we say that $\mathbb{O}$ is finite-dimensional if and only if $\mathfrak{m}/\mathfrak{m}^2$ is).

Let dual : $\mathbf{LBA}_{\mathrm{fd}} \to \mathbf{LBA}_{\mathrm{fd}}$ be the duality functor. It is a category antiequivalence; we have $\mathrm{dual}(\mathfrak{g}, \mu, \delta) = (\mathfrak{g}^*, \delta^t, \mu^t)$. Then $\mathrm{dual} \circ c_{\mathrm{fd}}$: $\mathbf{PFSHA}_{\mathrm{fd}} \to \mathbf{LBA}_{\mathrm{fd}}$ is a category antiequivalence. Its inverse is the usual functor $\mathfrak{g} \mapsto U(\mathfrak{g})^*$. If G is the formal Poisson–Lie group with Lie bialgebra $\mathfrak{g}$, one sets $\mathbb{O}_G = U(\mathfrak{g})^*$.

Let us apply the functor c to $(\mathbb{O}_{G^*_\gamma}, m_0, P, \tilde{f}^\star_{\gamma,\gamma\gamma'}\Delta_\gamma)$. We obtain $\mathfrak{c} = \mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}$; the Lie bracket is unchanged with respect to the case $\tilde{f}_{\gamma,\gamma\gamma'} = 0$, so it is the Lie bracket of $\mathfrak{g}$; the Lie cobracket is $\delta_{\gamma\gamma'}(x) = \delta_\gamma + [\bigwedge^2(\theta_\gamma)(f_{\gamma'}), x \otimes 1 + 1 \otimes x]$ since the reduction of $\tilde{f}_{\gamma,\gamma\gamma'}$ modulo $(\mathfrak{m}_{G^*_\gamma})^2 \widehat{\otimes} \mathfrak{m}_{G^*_\gamma} + \mathfrak{m}_{G^*_\gamma} \widehat{\otimes} (\mathfrak{m}_{G^*_\gamma})^2$ is equal to $\bigwedge^2(\theta_\gamma)(f_{\gamma'})$.

Then applying $\mathrm{dual} \circ c_{\mathrm{fd}}$ to $(\mathbb{O}_{G^*_\gamma}, m_0, P, \tilde{f}^\star_{\gamma,\gamma\gamma'}\Delta_\gamma)$, we obtain the Lie bialgebra $(\mathfrak{g}^*, \delta_{\gamma\gamma'})$. So this PFSH algebra is isomorphic to the PFSH algebra of the formal Poisson–Lie group $G^*_{\gamma\gamma'}$. Let us call such a PFSH algebra morphism $j_{\gamma,\gamma\gamma'}$.

In particular, the Poisson algebras $\mathbb{O}_{G^*_\gamma}$ and $\mathbb{O}_{G^*_{\gamma\gamma'}}$ are isomorphic. □

Remark 3.5. It is easy to check that the map $\mathfrak{g} = \mathfrak{m}_{G^*_\gamma}/\mathfrak{m}^2_{G^*_\gamma} \to \mathfrak{m}_{G^*_{\gamma\gamma'}}/\mathfrak{m}^2_{G^*_{\gamma\gamma'}} = \mathfrak{g}$ induced by the isomorphism $j_{\gamma,\gamma\gamma'}$ is the identity.

Remark 3.6. We have proved a result stronger than the existence of a Poisson algebra morphism $j_{\gamma,\gamma\gamma'}\colon \mathbb{O}_{G^*_\gamma} \simeq \mathbb{O}_{G^*_{\gamma\gamma'}}$. This morphism intertwines the coproducts as

$$\Delta_{\gamma\gamma'} = j^{\otimes 2}_{\gamma,\gamma\gamma'} \circ \tilde{f}^\star_{\gamma,\gamma\gamma'}\Delta_\gamma \circ j^{-1}_{\gamma,\gamma\gamma'}.$$

Composition of equivalences.

Lemma 3.7. *For γ, γ' in Γ, the element $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''}) \star_\gamma \tilde{f}_{\gamma,\gamma\gamma'}$ is a solution of the equation*

$$(\tilde{g}\otimes 1)\star_\gamma (\Delta_\gamma\otimes \mathrm{id})(\tilde{g}) = (1\otimes\tilde{g})\star_\gamma (\mathrm{id}\otimes\Delta_\gamma)(\tilde{g}). \tag{2}$$

Proof. One can check this directly. Notice that $\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''}$ is a twist for $\Delta_{\gamma\gamma'}$. Therefore $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''})$ is a twist for

$$(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}\circ\Delta_{\gamma\gamma'}\circ j_{\gamma,\gamma\gamma'} = \tilde{f}^{\star}_{\gamma,\gamma\gamma'}\Delta_\gamma.$$

Accordingly the element $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''}) \star_\gamma \tilde{f}_{\gamma,\gamma\gamma'}$ is a twist for Δ_γ. □

Note that the image of $(j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''}) \star_\gamma \tilde{f}_{\gamma,\gamma\gamma'}$ under the square of the projection $\mathfrak{m}_{G^*}\to\mathfrak{m}_{G^*}/\mathfrak{m}^2_{G^*}=\mathfrak{g}$ equals

$$\textstyle\bigwedge^2(\theta_\gamma)(f_{\gamma'}) + \bigwedge^2(\theta_{\gamma\gamma'})(f_{\gamma''}) = \bigwedge^2(\theta_\gamma)(f_{\gamma'}+\bigwedge^2(\theta_{\gamma'})(f_{\gamma''})) = \bigwedge^2(\theta_\gamma)(f_{\gamma'\gamma''}).$$

Thanks to Proposition 3.3, there exists an element $u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ in $1+\mathfrak{m}^2_{G^*}$ such that

$$\tilde{f}_{\gamma,\gamma\gamma'\gamma''} = u^{\otimes 2}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}\star_\gamma (j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''})\star_\gamma \tilde{f}_{\gamma,\gamma\gamma'}\star_\gamma \Delta_\gamma(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})^{-1}.$$

Finally, in the previous section, we defined $j_{\gamma,\gamma\gamma'}$, $j_{\gamma\gamma',\gamma\gamma'\gamma''}$ and $j_{\gamma,\gamma\gamma'\gamma''}$ such that

$$\begin{aligned}
\Delta_{\gamma\gamma'\gamma''} &= j^{\otimes 2}_{\gamma,\gamma\gamma'\gamma''}\circ\tilde{f}^{\star}_{\gamma,\gamma\gamma'\gamma''}\Delta_\gamma\circ j^{-1}_{\gamma,\gamma\gamma'\gamma''}\\
&= j^{\otimes 2}_{\gamma,\gamma\gamma'\gamma''}\\
&\quad\circ(u^{\otimes 2}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}\star_\gamma (j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''})\star_\gamma \tilde{f}_{\gamma,\gamma\gamma'}\star_\gamma \Delta_\gamma(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})^{-1})^{\star}\Delta_\gamma\\
&\qquad\circ j^{-1}_{\gamma,\gamma\gamma'\gamma''}\\
&= (j_{\gamma,\gamma\gamma'\gamma''}\circ\mathrm{Ad}_{\star_\gamma}(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}))^{\otimes 2}\circ((j^{\otimes 2}_{\gamma,\gamma\gamma'})^{-1}(\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''})\star_\gamma \tilde{f}_{\gamma,\gamma\gamma'})^{\star}\Delta_\gamma\\
&\qquad\circ(j_{\gamma,\gamma\gamma'\gamma''}\circ\mathrm{Ad}_{\star_\gamma}(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}))^{-1}\\
&= (j_{\gamma,\gamma\gamma'\gamma''}\circ\mathrm{Ad}_{\star_\gamma}(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})\circ j^{-1}_{\gamma,\gamma\gamma'}\circ j^{-1}_{\gamma\gamma',\gamma\gamma'\gamma''})^{\otimes 2}\circ\Delta_{\gamma\gamma'\gamma''}\\
&\qquad\circ(j_{\gamma,\gamma\gamma'\gamma''}\circ\mathrm{Ad}_{\star_\gamma}(u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})\circ j^{-1}_{\gamma,\gamma\gamma'}\circ j^{-1}_{\gamma\gamma',\gamma\gamma'\gamma''})^{-1}.
\end{aligned}$$

By the equivalence c_{fd} between the category $\mathbf{PFSHA}_{\mathrm{fd}}$ and $\mathbf{LBA}_{\mathrm{fd}}$, we get

$$j_{\gamma,\gamma\gamma'\gamma''} = j_{\gamma\gamma',\gamma\gamma'\gamma''}\circ j_{\gamma,\gamma\gamma'}\circ\mathrm{Ad}_{\star_\gamma}(u^{-1}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}).$$

Cocycle relation for the $u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$.

Proposition 3.8. *For any γ, γ', γ'', γ''' in Γ, we have*

$$u_{\gamma,\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''}\star_\gamma u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} = u_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''\gamma'''}\star_\gamma j^{-1}_{\gamma,\gamma\gamma'}(u_{\gamma\gamma',\gamma\gamma'\gamma'',\gamma\gamma'\gamma''\gamma'''}).$$

Proof. To shorten the notation, we will write $\tilde{f}_{1,2}$ for $\tilde{f}_{\gamma,\gamma\gamma'}$, $\tilde{f}_{2,3}$ for $\tilde{f}_{\gamma\gamma',\gamma\gamma'\gamma''}$ and so on, and the same for the $j_{\cdot,\cdot}$ and the $u_{\cdot,\cdot,\cdot}$. We will omit the BCH product $\star_\gamma$ and write $\star$ for the product $\star_{\gamma\gamma'}$, Δ_0 for the coproduct Δ_γ, and Δ for the coproduct $\Delta_{\gamma\gamma'}$. We will also write $j(\cdot)$ instead of $j^{\otimes 2}(\cdot)$ when no confusion is possible.

We have by definition $\tilde{f}_{1,4}\Delta_0 u_{1,3,4} = u^{\otimes 2}_{1,3,4} j^{-1}_{1,3}(\tilde{f}_{3,4})\tilde{f}_{1,3}$. Multiplying this on the right by $\Delta_0 u_{1,2,3}$ and using the fact that $\tilde{f}_{1,3}\Delta_0 u_{1,2,3} = u^{\otimes 2}_{1,2,3} j^{-1}_{1,2}(\tilde{f}_{2,3})\tilde{f}_{1,2}$, we get

$$\tilde{f}_{1,4}\Delta_0 u_{1,3,4}\Delta_0 u_{1,2,3} = u^{\otimes 2}_{1,3,4} j^{-1}_{1,3}(\tilde{f}_{3,4}) u^{\otimes 2}_{1,2,3} j^{-1}_{1,2}(\tilde{f}_{2,3})\tilde{f}_{1,2}.$$

Using now that $j^{-1}_{1,3}(\cdot)u_{1,2,3} = u_{1,2,3} j^{-1}_{1,2}\circ j^{-1}_{2,3}(\cdot)$, we get

$$\tilde{f}_{1,4}\Delta_0 u = u^{\otimes 2} j^{-1}_{1,2}\circ j^{-1}_{2,3}(\tilde{f}_{3,4}) j^{-1}_{1,2}(\tilde{f}_{2,3})\tilde{f}_{1,2}, \tag{3}$$

where $u = u_{1,3,4}u_{1,2,3}$. On the other hand, we have

$$\tilde{f}_{2,4}\star\Delta u_{2,3,4} = u^{\otimes 2}_{2,3,4}\star j^{-1}_{2,3}(\tilde{f}_{3,4})\star\tilde{f}_{2,3}.$$

Using the Poisson algebra morphism $j_{1,2}$ and that $j^{-1}_{1,2}\circ\Delta = \tilde{f}_{1,2}\Delta_0(j^{-1}_{1,2}(\cdot))\tilde{f}^{-1}_{1,2}$, we get

$$j^{-1}_{1,2}(\tilde{f}_{2,4})\tilde{f}_{1,2}\Delta_0(j^{-1}_{1,2}(u_{2,3,4}))\tilde{f}^{-1}_{1,2} = j^{-1}_{1,2}(u^{\otimes 2}_{2,3,4}) j^{-1}_{1,2}\circ j^{-1}_{2,3}(\tilde{f}_{3,4}) j^{-1}_{1,2}(\tilde{f}_{2,3}). \tag{4}$$

From $\tilde{f}_{1,4}\Delta_0 u_{1,2,4} = u^{\otimes 2}_{1,2,4} j^{-1}_{1,2}(\tilde{f}_{2,4})\tilde{f}_{1,2}$, using (4) we get

$$\tilde{f}_{1,4}\Delta_0(u') = (u')^{\otimes 2} j^{-1}_{1,2}\circ j^{-1}_{2,3}(\tilde{f}_{3,4}) j^{-1}_{1,2}(\tilde{f}_{2,3})\tilde{f}_{1,2}, \tag{5}$$

where $u' = u_{1,2,4} j^{-1}_{1,2}(u_{2,3,4})$. Then (3) and (5) imply that if $w = u(u')^{-1}$, then $\tilde{f}_{1,4}\Delta_0(w) = w\tilde{f}_{1,4}$, and so if $w' = j_{1,4}(w)$, then $\Delta_0(w') = w'$. Recall that $w' \in 1 + m^2_{G^*}$ by similar properties of $u_{i,j,k}$. Suppose that $w' \neq 1$ and set $i \geq 2$ the largest possible i such that w' is in $1 + m^i_{G^*}$ but not in $1 + m^{i+1}_{G^*}$. Let $\overline{w}'$ be the projection of w' in $m^i_{G^*}/m^{i+1}_{G^*}$. The relation $\Delta_0(w') = w'$ implies that $\overline{w}'$ is in $\mathfrak{g}$ and so in $m^1_{G^*}$ which is a contradiction. Thus we have proved that $w = w' = 1$ and so that $u = u'$. □

4. Quantization

Duality of QUE and QFSH algebras. In this subsection, we recall some facts from [Drinfeld 1987], whose proofs can be found in [Gavarini 2002]. Let us denote by **QUE** the category of quantized universal enveloping (QUE) algebras and by **QFSH** the category of quantized formal series Hopf (QFSH) algebras. We denote by $\mathbf{QUE}_{\mathrm{fd}}$ and $\mathbf{QFSH}_{\mathrm{fd}}$ the subcategories corresponding to finite dimensional Lie bialgebras.

We have contravariant functors

$$\mathbf{QUE}_{\mathrm{fd}} \to \mathbf{QFSH}_{\mathrm{fd}}, \quad U \mapsto U^* \quad \text{and} \quad \mathbf{QFSH}_{\mathrm{fd}} \to \mathbf{QUE}_{\mathrm{fd}}, \quad \mathbb{O} \mapsto \mathbb{O}^\circ.$$

These functors are inverse to each other. Here U^* is the full topological dual of U, that is, the space of all continuous (for the $\hbar$-adic topology) $\mathbb{K}[[\hbar]]$-linear maps $U \to \mathbb{K}[[\hbar]]$, and $\mathbb{O}^\circ$ is the space of continuous $\mathbb{K}[[\hbar]]$-linear forms $\mathbb{O} \to \mathbb{K}[[\hbar]]$, where $\mathbb{O}$ is equipped with the $\mathfrak{m}$-adic topology (here $\mathfrak{m} \subset \mathbb{O}$ is the maximal ideal).

We also have covariant functors

$$\mathbf{QUE} \to \mathbf{QFSH}, \quad U \mapsto U' \quad \text{and} \quad \mathbf{QFSH} \to \mathbf{QUE}, \quad \mathbb{O} \mapsto \mathbb{O}^\vee.$$

These functors are also inverse to each other. Here U' is a subalgebra of U, while $\mathbb{O}^\vee$ is the $\hbar$-adic completion of $\sum_{k\geq 0} \hbar^{-k}\mathfrak{m}^k \subset \mathbb{O}[1/\hbar]$.

We also have canonical isomorphisms $(U')^\circ \simeq (U^*)^\vee$ and $(\mathbb{O}^\vee)^* \simeq (\mathbb{O}^\circ)'$.

If $\mathfrak{a}$ is a finite-dimensional Lie bialgebra and $U = U_\hbar(\mathfrak{a})$ is a QUE algebra quantizing $\mathfrak{a}$, then $U^* = \mathbb{O}_{A,\hbar}$ is a QFSH algebra quantizing the Poisson–Lie group A, with Lie bialgebra $\mathfrak{a}$, and $U' = \mathbb{O}_{A^*,\hbar}$ is a QFSH algebra quantizing the Poisson–Lie group A^*, with Lie bialgebra $\mathfrak{a}^*$. If now $\mathbb{O} = \mathbb{O}_{A,\hbar}$ is a QFSH algebra quantizing A, then $\mathbb{O}^\circ = U_\hbar(\mathfrak{a})$ is a QUE algebra quantizing $\mathfrak{a}$, and $\mathbb{O}^\vee = U_\hbar(\mathfrak{a}^*)$ is a QFSH algebra quantizing $\mathfrak{a}^*$.

We now compute these functors explicitly in the case of cocommutative QUE and commutative QFSH algebras. If $U = U(\mathfrak{a})[[\hbar]]$ with cocommutative coproduct (where $\mathfrak{a}$ is a Lie algebra), then U' is a completion of $U(\hbar\mathfrak{a}[[\hbar]])$; this is a flat deformation of $\hat{S}(\mathfrak{a})$ equipped with its linear Lie–Poisson structure. If G is a formal group with function ring $\mathbb{O}_G$, then $\mathbb{O} := \mathbb{O}_G[[\hbar]]$ is a QFSH algebra, and $\mathbb{O}^\vee$ is a commutative QUE algebra; it is a quantization of $S(\mathfrak{g}^*)$, with a commutative product, a cocommutative coproduct, and a co-Poisson structure induced by the Lie bracket of $\mathfrak{g}$.

Proof that "twists" can be made admissible.

Definition 4.1. An element x in a QUE algebra U is admissible if $x \in 1 + \hbar U$, and if $\hbar \log x$ is in $U' \subset U$.

In this subsection, we will prove that for γ, γ' in Γ, the twist $\mathrm{F}_{\gamma,\gamma\gamma'}$ defined in Proposition 1.5 is twist equivalent to an admissible one.

Proposition 4.2. *Let* $\mathrm{F}_{\gamma,\gamma\gamma'}$ *be the element in* $U^{\otimes 2}$ *introduced in Proposition 1.5. Then there exists elements* $\mathrm{b}_{\gamma,\gamma\gamma'}$ *in* U *such that*

$$\mathrm{b}^{\maltese}_{\gamma,\gamma\gamma'}\mathrm{F}_{\gamma,\gamma\gamma'} := \mathrm{b}^{\otimes 2}_{\gamma,\gamma\gamma'}\mathrm{F}_{\gamma,\gamma\gamma'}\Delta_\gamma(\mathrm{b}^{-1}_{\gamma,\gamma\gamma'})$$

is admissible.

Proof. Let us denote $\mathrm{F}_0 = \mathrm{F}_{\gamma,\gamma\gamma'}$. We will follow the proof of [Enriquez and Halbout 2007, Proposition 5.2]. Let us construct $\mathrm{b} = \mathrm{b}_{\gamma,\gamma\gamma'}$ as a product $\cdots \mathrm{b}_2\mathrm{b}_1$, where $\mathrm{b}_n \in 1 + \hbar^n U_0$, so that if $\mathrm{F}_n := \mathrm{b}_n \cdots \mathrm{b}_1^{\maltese}\mathrm{F}_0$, then $\hbar \log(\mathrm{F}_n) \in U_0'^{\hat\otimes 2} + \hbar^{n+2} U_0^{\hat\otimes 2}$; here U_0 denotes the augmentation ideal.

We have already $\hbar\log(\mathrm{F}_0) \in \hbar^2 U_0^{\hat\otimes 2}$.

Expand $\mathrm{F}_0 = 1^{\otimes 2} + \hbar\,\mathrm{f}_1 + \cdots$. Then $\mathrm{Alt}(\mathrm{f}_1) = r$. Moreover, the coefficient of $\hbar$ in $\mathrm{F}_0^{1,2}\mathrm{F}_0^{12,3} = \mathrm{F}_0^{2,3}\mathrm{F}_0^{1,23}$ yields $d(\mathrm{f}_1) = 0$, where $d : U(\mathfrak{g})_0^{\otimes 2} \to U(\mathfrak{g})_0^{\otimes 3}$ is the co-Hochschild differential. It follows that $\mathrm{f}_1 = r + d(a_1)$ for some $a_1 \in U(\mathfrak{g})_0$. Then if we set $\mathrm{b}_1 := \exp(\hbar a_1)$ and $\mathrm{F}_1 = \mathrm{b}_1^{\maltese}\mathrm{F}_0$, we get $\mathrm{F}_1 \in 1^{\otimes 2} + \hbar r + \hbar^2 U_0^{\hat\otimes 2}$. Then $\hbar\log(\mathrm{F}_1) \in \hbar^2 r + \hbar^3 U_0^{\hat\otimes 2} \subset U_0'^{\hat\otimes 2} + \hbar^3 U_0^{\hat\otimes 3}$.

Assume that for $n \geq 2$, we have constructed $\mathrm{b}_1, \dots, \mathrm{b}_{n-1}$ such that

$$\alpha_{n-1} := \hbar\log(\mathrm{F}_{n-1}) \in U_0'^{\hat\otimes 2} + \hbar^{n+1} U_0^{\hat\otimes 2}.$$

Let us recall two technical lemmas from [Enriquez and Halbout 2007]:

Lemma 4.3. *The quotient $(U' + \hbar^n U)/(U' + \hbar^{n+1} U)$ identifies with $U(\mathfrak{g})/U(\mathfrak{g})_{\leq n}$. In the same way, the quotient $(U_0'^{\hat\otimes k} + \hbar^n U_0^{\hat\otimes k})/(U_0'^{\hat\otimes k} + \hbar^{n+1} U_0^{\hat\otimes k})$ identifies with $U(\mathfrak{g})_0^{\otimes k}/(U(\mathfrak{g})_0^{\otimes k})_{\leq n}$ and the quotient $(U_0'^{\hat\otimes k} + \hbar^n U_0^{\hat\otimes k})^{\mathfrak{g}}/(U_0'^{\hat\otimes k} + \hbar^{n+1} U_0^{\hat\otimes k})^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant subspaces identifies with $(U(\mathfrak{g})_0^{\otimes k})^{\mathfrak{g}}/(U(\mathfrak{g})_0^{\otimes k})^{\mathfrak{g}}_{\leq n}$.*

Lemma 4.4. *Assume that $n \geq 2$. If $f_1, f_2 \in (U_0')^2 + \hbar^{n+1} U_0$ and $g, h \in \hbar^n U_0$, then $(f_1 + g) \star_\hbar (f_2 + h) = g + h$ modulo $(U_0')^2 + \hbar^{n+1} U_0$, where $\star_\hbar$ is the CBH product for the Lie bracket $[a, b]_\hbar = [a, b]/\hbar$.*

Let us denote by $\bar\alpha$ the image of the class of α_{n-1} in $U(\mathfrak{g})_0^{\otimes 2}/(U(\mathfrak{g})_0^{\otimes 2})_{\leq n+1}$ under the isomorphism of this space with

$$(U_0'^{\hat\otimes 2} + \hbar^{n+1} U_0^{\hat\otimes 2})/(U_0'^{\hat\otimes 2} + \hbar^{n+2} U_0^{\hat\otimes 2})$$

(see Lemma 4.3). Let $\alpha \in U(\mathfrak{g})_0^{\otimes 2}$ be a representative of $\bar\alpha$. Then $\alpha_{n-1} = \alpha' + \hbar^{n+1}\alpha$, where $\alpha' \in U_0'^{\hat\otimes 2} + \hbar^{n+2} U_0^{\hat\otimes 2}$. Then the twist equation gives

$$(-\alpha' - \hbar^{n+1}\alpha)^{1,23} \star_\hbar (-\alpha' - \hbar^{n+1}\alpha)^{2,3} \star_\hbar (\alpha' + \hbar^{n+1}\alpha)^{1,2} \star_\hbar (\alpha' + \hbar^{n+1}\alpha)^{12,3} = 0.$$

By Lemma 4.4, the image of this equality in $(U^{\hat\otimes 3} + \hbar^{n+1} U'^{\hat\otimes 3})/(U^{\hat\otimes 3} + \hbar^{n+2} U'^{\hat\otimes 3}) \simeq U(\mathfrak{g})^{\otimes 3}/(U(\mathfrak{g})^{\otimes 3})_{\leq n+1}$ is $d(\bar\alpha) = 0$, where d is the co-Hochschild differential on the quotient $U(\mathfrak{g})_0^{\otimes \cdot}/(U(\mathfrak{g})_0^{\otimes \cdot})_{\leq n+1}$. Since $n \geq 2$, the relevant cohomology group vanishes, so $\bar\alpha = d(\bar\beta)$, where $\bar\beta \in U(\mathfrak{g})_0/(U(\mathfrak{g})_0)_{\leq n+1}$. Let $\beta \in U(\mathfrak{g})_0$ be a representative of $\bar\beta$ and set

$$\mathrm{b}_n := \exp(\hbar^n\beta), \quad \mathrm{F}_n := \mathrm{b}_n^{\maltese}\mathrm{F}_{n-1}, \quad \alpha_n := \hbar\log(\mathrm{F}_n).$$

Then $\alpha_n = (\hbar^{n+1}\beta)^1 \star_\hbar (\hbar^{n+1}\beta)^2 \star_\hbar \alpha_{n-1} \star_\hbar (-\hbar^{n+1}\beta)^{12}$. According to Lemma 4.4, the image of α_n in

$$(U_0^{\hat\otimes 2} + \hbar^{n+1} U_0'^{\hat\otimes 2})/(U_0^{\hat\otimes 2} + \hbar^{n+2} U_0'^{\hat\otimes 2}) \simeq U(\mathfrak{g})_0^{\otimes 2}/(U(\mathfrak{g})_0^{\otimes 2})_{\leq n+1}$$

is $\bar\alpha - d(\bar\beta) = 0$. So α_n belongs to $U_0^{\hat\otimes 2} + \hbar^{n+2} U_0'^{\hat\otimes 2}$, as required. This proves the induction step. □

The proof of Theorem 2.5. Thanks to the previous subsection, we now know that there exists an element $\mathrm{b}_{\gamma,\gamma\gamma'}$ in U such that $\mathrm{b}^{\maltese}_{\gamma,\gamma\gamma'}\mathrm{F}_{\gamma,\gamma\gamma'} := \mathrm{b}^{\otimes 2}_{\gamma,\gamma\gamma'}\mathrm{F}_{\gamma,\gamma\gamma'}\,\Delta_\gamma(\mathrm{b}^{-1}_{\gamma,\gamma\gamma'})$ is admissible. Let us define

$$\mathrm{F}'_{\gamma,\gamma\gamma'} = \mathrm{b}^{\maltese}_{\gamma,\gamma\gamma'}\mathrm{F}_{\gamma,\gamma\gamma'}, \quad \mathrm{i}'_{\gamma,\gamma\gamma'} = \mathrm{i}_{\gamma,\gamma\gamma'} \circ \mathrm{Ad}(\mathrm{b}^{-1}_{\gamma,\gamma\gamma'})$$

and

$$\mathrm{v}'_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''} = \mathrm{b}_{\gamma,\gamma\gamma'\gamma''}\mathrm{v}_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}\mathrm{i}^{-1}_{\gamma,\gamma\gamma'}(\mathrm{b}^{-1}_{\gamma\gamma',\gamma\gamma'\gamma''})\mathrm{b}^{-1}_{\gamma,\gamma\gamma'}.$$

Clearly, $\mathrm{F}'_{\gamma,\gamma\gamma'}$, $\mathrm{i}'_{\gamma,\gamma\gamma'}$ and $\mathrm{v}'_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ still satisfy the conclusion of Proposition 1.5.

Applying the functor **QUE** $\to$ **QFSH** explained on page 112 to the algebras $(U_{\gamma,\gamma}, \Delta_\gamma)$, we get algebras $(U'_\gamma, *_\gamma, \Delta_\gamma)$, which are quantizations of the Poisson algebras $(\mathbb{O}_{G^*_\gamma}, \{\cdot,\cdot\}_\gamma)$. Since the twists $\mathrm{F}'_{\gamma,\gamma\gamma'}$ are admissible, the algebra morphisms $\mathrm{i}'_{\gamma,\gamma\gamma'}$ restrict to the QFSH algebras U'_γ. Then Theorem 2.5 will follow from this:

Proposition 4.5. *The elements* $\mathrm{v}'_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ *are admissible.*

Proof. Let us denote $\mathrm{v} = \mathrm{v}'_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$. Suppose v is not admissible and let n be the bigger i such that $\alpha_0 := \hbar\log(\mathrm{v}) \in U_0 + \hbar^{n+1}U_0$. By the assumption on v, we know that $n \geq 2$. Let us denote by $\bar{\alpha}$ the image of the class of α_0 in $U(\mathfrak{g})_0/(U(\mathfrak{g})_0)_{\leq n+1}$ under the isomorphism of this space with $(U_0 + \hbar^{n+1}U_0)/(U_0 + \hbar^{n+2}U_0)$; see Lemma 4.3. Let $\alpha \in U(\mathfrak{g})_0$ be a representative of $\bar{\alpha}$. Then $\alpha_0 = \alpha' + \hbar^{n+1}\alpha$, where $\alpha' \in U_0 + \hbar^{n+2}U_0$. Let f, f' and f'' be respectively the $\hbar$ logs of $\mathrm{F}'_{\gamma,\gamma\gamma'}$, $\mathrm{F}'_{\gamma\gamma',\gamma\gamma'\gamma''}$ and $\mathrm{F}_{\gamma,\gamma\gamma'\gamma''}$. Then the compatibility equation for composition of twists gives

$$f'' = (\alpha' + \hbar^{n+1}\alpha)^{\otimes 2} \star_\hbar \mathrm{i}^{-1}_{\gamma,\gamma\gamma'}(f') \star_\hbar f \star_\hbar (-\alpha' - \hbar^{n+1}\alpha)^{12} = 0.$$

According to Lemma 4.4, the image of this equation in

$$(U^{\hat{\otimes}2} + \hbar^{n+1}U'^{\hat{\otimes}2})/(U^{\hat{\otimes}2} + \hbar^{n+2}U'^{\hat{\otimes}2}) \simeq U(\mathfrak{g})^{\otimes 2}/(U(\mathfrak{g})^{\otimes 2})_{\leq n+1}$$

is $d(\bar{\alpha}) = 0$. So $\bar{\alpha} \in \mathfrak{g}$, which is a contradiction with $n \geq 2$. □

5. Example of simple group with action of the Weyl group

Quantization of Majid and Soĭbel'man. We start by briefly recalling Majid and Soĭbel'man's approach [1994] to the quantum Weyl group. Let $\mathfrak{g}$ be a complex simple Lie algebra, and let $U_\hbar(\mathfrak{g})$ be the natural deformation of the universal enveloping algebra $U(\mathfrak{g})$. Lustig [1990] and Soĭbel'man [1991] first independently noticed that a simple reflection w in the Weyl group W of $\mathfrak{g}$ defines an automorphism α_w on $U_\hbar(\mathfrak{g})$. Then one can extend $U_\hbar(\mathfrak{g})$ by elements $\overline{w}$ with $\alpha_w(g) = \overline{w}g\overline{w}^{-1}$ for all simple reflections in W. The extended algebra is called the "quantum Weyl group" and denoted by $\widetilde{U_\hbar(\mathfrak{g})}$. In [Kirillov and Reshetikhin 1990] and [Soĭbel'man 1991], this algebra is used to construct explicit solutions to the Yang–Baxter equation.

Majid and Soĭbel'man also discovered the bicrossed product structure on $\widetilde{U_\hbar(\mathfrak{g})}$. For $1 \le i, j \le \mathrm{rank}(\mathfrak{g})$, let w_i be simple reflections in W, and let t_j be elements in the maximal torus corresponding to $\phi_j\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ with $\phi_j : sl_2 \hookrightarrow \mathfrak{g}$ embedding to the j-th vertex of the Dynkin diagram. Then define $\widetilde{W}$ to be the group generated by w_i and t_j, which is a covering of the Weyl group W with the kernel isomorphic to the direct sum of k-copies of $\mathbb{Z}_2$, where $k = \mathrm{rank}(\mathfrak{g})$. The quantum Weyl group $\widetilde{U_\hbar(\mathfrak{g})}$ is proved in [Majid and Soĭbel'man 1994, Corollary 3.4] to be isomorphic to the bicrossed product

$$k\widetilde{W}^{\psi} \bowtie_{\alpha,\chi} U_\hbar(\mathfrak{g}),$$

defined in terms of linear maps

$$\begin{aligned}
\alpha &: U_\hbar(\mathfrak{g}) \otimes k\widetilde{W} \to U_q(g), && a \otimes wt \mapsto t^{-1}\alpha_w(a)t,\\
\chi &: k\widetilde{W} \otimes k\widetilde{W} \to U_\hbar(\mathfrak{g}), && w_1t_1 \otimes w_2t_2 \mapsto x^{-1},\\
\psi &: k\widetilde{W} \to U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g}), && wt \mapsto (\overline{w}^{-1} \otimes \overline{w}^{-1})\Delta\overline{w}.
\end{aligned}$$

Here, x is an element in $U_\hbar(\mathfrak{g})$ such that $\alpha_{w_1w_2(\alpha_{w_1}(t_1)t_2)} = \alpha_{w_1t_1}\alpha_{w_2t_2}\mathrm{Ad}_{x^{-1}}$ with $x \in U_\hbar(\mathfrak{g})$.

Proposition 5.1. *The quantum Weyl group $\widetilde{U_\hbar(\mathfrak{g})}$ is a quantization of the $\Gamma = \widetilde{W}$ Lie bialgebra $(\mathfrak{g}, [\,\cdot\,,\cdot\,], \delta)$, where $(\mathfrak{g}, [\,\cdot\,,\cdot\,], \delta)$ is the Lie bialgebra structure on $\mathfrak{g}$ corresponding to the deformation $U_\hbar(\mathfrak{g})$, and $\widetilde{W}$ acts on $\mathfrak{g}$ as the Weyl group (t acts on $\mathfrak{g}$ by adjoint action), and $f_\gamma = \bigwedge^2(\gamma) \circ \delta \circ \gamma^{-1} - \delta$ for $\gamma \in \widetilde{W}$.*

Proof. Inspired by the above bicrossed product structure on $\widetilde{U_\hbar(\mathfrak{g})}$, we introduce the Γ quantized universal enveloping algebras for $\Gamma = \widetilde{W}$ generated as follows:

- Set $(U_\hbar(\mathfrak{g})_\gamma, m_\gamma, \Delta_\gamma) = (U_\hbar(\mathfrak{g}), m, \Delta_\gamma)$, where m is the canonical multiplication on $U_\hbar(\mathfrak{g})$ and $\Delta_\gamma = \alpha(\,\cdot\,,\gamma)^{\otimes 2} \circ \mathrm{Ad}(\psi(\gamma)) \circ \Delta \circ \alpha^{-1}(\,\cdot\,,\gamma)$ with Δ the canonical coproduct on $U_\hbar(\mathfrak{g})$.
- Define $i_{\gamma,\gamma\gamma'} : (U_\hbar(\mathfrak{g}), m_\gamma) \to (U_\hbar(\mathfrak{g}), m_{\gamma\gamma'})$ by $i_{e,\gamma} = \alpha(\,\cdot \otimes \gamma) : U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{g})$ and $i_{\gamma,\gamma\gamma'} = i_{e,\gamma'}$.
- Set $F_{e,\gamma} \in U_\hbar(\mathfrak{g})^{\otimes 2}$ equal to $\psi(\gamma)$ and put $F_{\gamma,\gamma\gamma'} = F_{e,\gamma'}$. By [Majid and Soĭbel'man 1994, Lemma 3.3], we have

$$F_{e,w_it} = \psi(w_i) = e^{\frac{1}{2}\hbar H_i \otimes H_i/(\alpha_i,\alpha_i)}(\mathcal{R}_i)^{-1}_{12} = 1 + \hbar f_1 + O(\hbar^2)$$

  for any reflection $w_i \in W$. (Here (H_i, X_i^+, X_i^-) corresponds to the embedding $\phi_i : sl_2 \hookrightarrow \mathfrak{g}$ for the i-th root α_i with normal (α_i, α_i).) Because the first order part of $e^{\frac{1}{2}\hbar H_i \otimes H_i/(\alpha_i,\alpha_i)}$ is symmetric, the antisymmetrization of f_1 is equal to the antisymmetrization of the first order term of $(\mathcal{R}_i)^{-1}_{21}$, which is equal to the definition of f_{w_i} by the asymptotic expansion of $\mathcal{R}_i$. This result extends to an arbitrary element γ simply because w_i generates W.

- Set $\upsilon_{e,\gamma,\gamma\gamma'} = \chi(\gamma, \gamma\gamma') \in U_\hbar(\mathfrak{g})^{\otimes 2}$. By the definition of $\chi(\gamma, \gamma\gamma')$, we can choose υ to be an element in $1 + \hbar^2 U_\hbar(\mathfrak{g})$ because the α action is associative up to the $\hbar$-linear terms by [Kirillov and Reshetikhin 1990, Formula (13)] and [Levendorskiĭ and Soĭbel'man 1990, Prop 1.4.10].

It is easy to check that the cocycle conditions for α, χ, ψ, and their compatibilities are equivalent to the conditions for $(U_\hbar, m, \Delta_\gamma, i_{\gamma,\gamma\gamma'}, F_{\gamma,\gamma\gamma}, \upsilon_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''})$ to be a $\Gamma = \widetilde{W}$ quantized universal enveloping algebra. Therefore, the corresponding Γ quantized universal enveloping algebra is isomorphic to $\widetilde{U_\hbar(\mathfrak{g})}$. □

Admissibility of the twists.

Corollary 5.2. *The twists $F_{\gamma,\gamma\gamma'}$ and $\upsilon_{\gamma,\gamma\gamma',\gamma\gamma'\gamma''}$ defined in Proposition 5.1 are admissible. Therefore, the quantum Weyl group defines a stack of formal series Hopf algebras quantizing the corresponding stack of Poisson–Hopf algebras dual to $(\widetilde{W}, \mathfrak{g}, [\cdot,\cdot], \delta, f_\gamma)$.*

Proof. We look at the formulas for $F_{e,wt}$. By the one for ψ, if w_i is a simple reflection, then $F_{e,w_i t} = e^{\frac{1}{2}\hbar H_i \otimes H_i/(\alpha_i,\alpha_i)} (\mathcal{R}_i)_{12}^{-1}$. Taking $\hbar$ log on $F_{e,w}$, we have

$$\hbar^2 \tfrac{1}{2} H_i \otimes H_i/(\alpha_i, \alpha_i) + \hbar \log((\mathcal{R}_i)_{12}^{-1}).$$

The first term is primitive as H_i is primitive, and the second term $\hbar \log((\mathcal{R}_i)_{12}^{-1})$ is primitive because $\hbar \log(\mathcal{R}_i)$ is primitive, which was proved in [Enriquez and Halbout 2003, Theorem 0.1]. Therefore, we conclude that $F_{e,w_i t}$ is admissible when w is a simple reflection. This property extends to a general element γ directly by products.

By Proposition 4.5, we also know that υ is admissible because F is admissible. We conclude the corollary by Theorem 2.5. □

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THE KAUFFMAN BRACKET SKEIN MODULE OF SURGERY ON A (2, 2b) TORUS LINK

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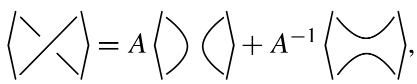
We show that the Kauffman bracket skein modules of certain manifolds obtained from integral surgery on a (2, 2b) torus link are finitely generated, and list the generators for select examples.

1. Introduction

Kauffman [1988] presented an elegant construction of the Jones polynomial, an invariant of oriented links in S^3, by constructing a new invariant, the Kauffman bracket polynomial. The Kauffman bracket is an invariant of unoriented framed links in S^3, defined by the skein relations

(1) 

(2) $\langle L \cup \text{unknot}\rangle = (-A^{-2} - A^2)\langle L\rangle$.

For the invariant to be well defined, one also must normalize it by choosing a value for the empty link, for instance $\langle\text{empty link}\rangle = 1$.

Alternatively, we can use the skein relations to construct a module of equivalence classes of links in S^3, or, for that matter, in any oriented 3-manifold. See Przytycki [1991] and Turaev [1988].

Definition 1. Let N be an oriented 3-manifold, and let R be a commutative ring with identity, with a specified unit A. The Kauffman bracket skein module of N, denoted $S(N; R, A)$, or simply $S(N)$, is the free R-module generated by the framed isotopy classes of unoriented links in N, including the empty link, quotiented by the skein relations that define the Kauffman bracket.

Since every crossing and unknot can be eliminated from a link in S^3 by the skein relations, $S(S^3)$ is generated by the empty link. Kauffman's argument that his bracket polynomial is well defined shows that $S(S^3)$ is free on the empty link.

For $R = \mathbb{Z}[A^{\pm 1}]$, Hoste and Przytycki have computed the skein modules of all of the closed, oriented manifolds of genus 1: In [1993], they computed $S(L(p, q))$,

MSC2000: 57M27.
Keywords: skein theory, Poincaré homology sphere, quaternionic manifold.

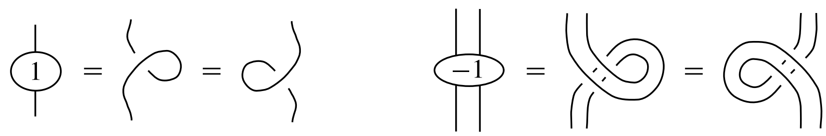
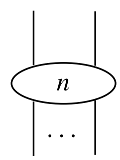


Figure 1. Examples of twist notation.

which is free on $\lfloor p/2 \rfloor + 1$ generators, and in [1995] they computed $S(S^1 \times S^2) \cong \mathbb{Z}[A^{\pm 1}] \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z}[A^{\pm 1}]/(1 - A^{2i+4}))$. Over $\mathbb{Z}[A^{\pm 1}]$, localized by inverting all of the cyclotomic polynomials, Gilmer and the author have computed the skein module of the quaternionic manifold [Gilmer and Harris 2007].

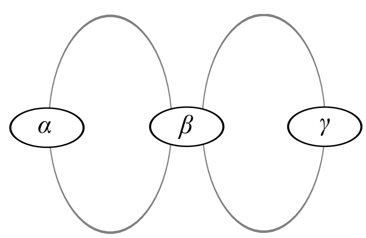
Additionally, Bullock [1997a] has determined whether or not the skein module obtained from integral surgery on a trefoil is finitely generated. In this paper, we obtain a similar result for integral surgery on a $(2, 2b)$ torus link.

Notation 2. For any integer n, let



denote n full twists in the depicted strands. For example, see Figure 1.

Definition 3. We define $M(\alpha, \beta, \gamma)$ to be the manifold obtained by surgery on the torus link



with the blackboard framing.

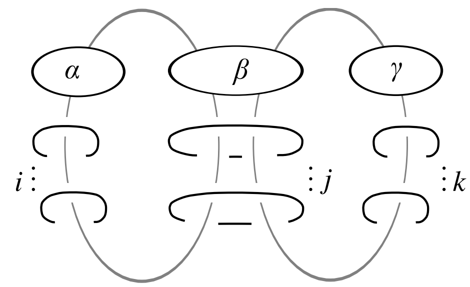
Theorem 4. *For all integers α, β, and γ such that*

$$a = |\alpha| > 1, \qquad b = |\beta| > 1, \qquad c = |\gamma| > 1,$$
$$a^{-1} < b^{-1} + c^{-1}, \quad b^{-1} < a^{-1} + c^{-1}, \quad c^{-1} < a^{-1} + b^{-1},$$

$S(M(\alpha, \beta, \gamma))$ is finitely generated.

For specific values of α, β, and γ, we can use brute-force computation to refine our result, explicitly listing generating sets for $S(M(\alpha, \beta, \gamma))$.

Notation 5. We refer to the collection of loops



in $M(\alpha, \beta, \gamma)$ using the algebraic notation $x^i y^j z^k$.

In particular, we obtain the following for $S(M(2, -2, 2))$, $S(M(3, -2, 3))$, and $S(M(3, -2, 5))$, which are respectively the skein modules of the 3-, 4-, and 5-fold branched cyclic coverings of S^3 over the trefoil, as listed by Rolfsen [1976].

α	β	γ	fundamental group	generators
2	−2	2	quaternion group	$1, z, z^2, y, x$
3	−2	3	binary tetrahedral group	$1, z, z^2, z^3, y, x, x^2$
3	−2	5	binary icosahedral group	$1, z, z^2, z^3, z^4, z^5, y, x, x^2$

Note that the generating set for the skein module of the quaternionic manifold essentially coincides with what was shown in [Gilmer and Harris 2007] over the ring R' obtained from $\mathbb{Z}[A^{\pm 1}]$ by inverting the multiplicative set generated by the elements of the set $\{A^n - 1 \mid n \in \mathbb{Z}^+\}$. Since any dependence relation over $\mathbb{Z}[A^{\pm 1}]$ would hold over R' and since $S(M(2, -2, 2); R', A)$ is a free module of rank 5, we obtain the following:

Corollary 6. $S(M(2, -2, 2); \mathbb{Z}[A^{\pm 1}], A)$ *is a free module of rank* 5.

This result was conjectured in [Gilmer and Harris 2007]. The quaternionic manifold is the first closed, irreducible, genus two 3-manifold whose Kauffman bracket skein module has been computed.

2. Twists and loops

Twists have many useful properties, a few of which are listed in Figure 2. Note that, to obtain clearer diagrams, we represent a fixed but arbitrary number of parallel strands with a thick line.

We are most interested in using skein relations and isotopy to rewrite one strand, twisted with others, as a linear combination involving loops encircling the others, as in Figure 3.

In fact, by repeating the steps performed in Figure 3, we obtain this:

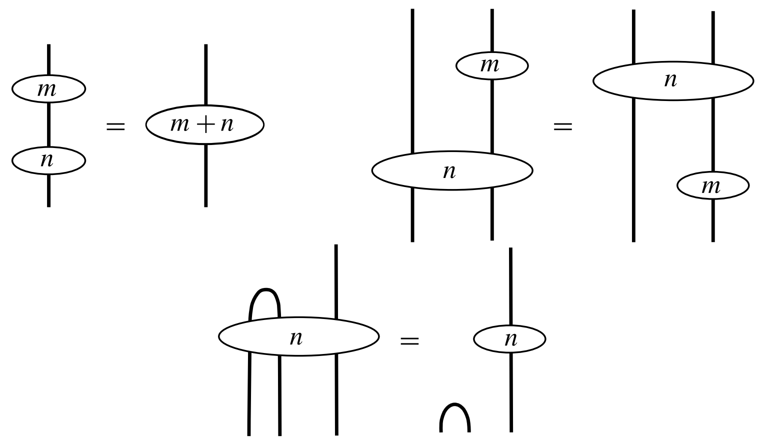
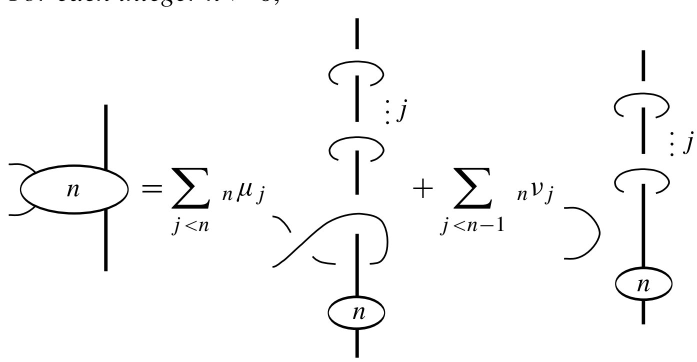
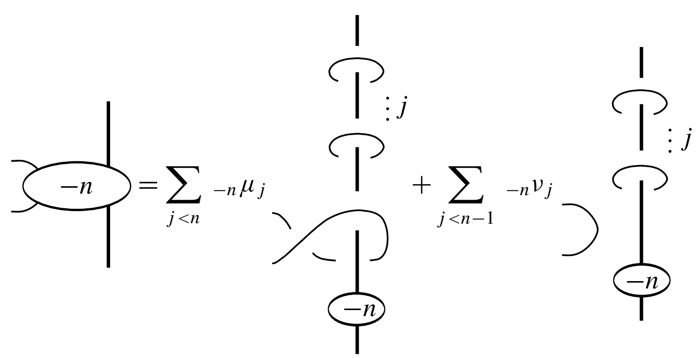


Figure 2. Useful properties of twists.

Lemma 7. *For each integer $n > 0$,*



for some ${}_n\mu_0, \dots, {}_n\mu_{n-1}, {}_n\nu_0, \dots, {}_n\nu_{n-2} \in R$ *with* ${}_n\mu_{n-1} = A^{n-1}$*, and*



for some ${}_{-n}\mu_0, \dots, {}_{-n}\mu_{n-1}, {}_{-n}\nu_0, \dots, {}_{-n}\nu_{n-2} \in R$ *with* ${}_{-n}\mu_{n-1} = A^{1-n}$.

Proof. For $n = 1$ and $n = 2$, the result is obtained in Figure 3.

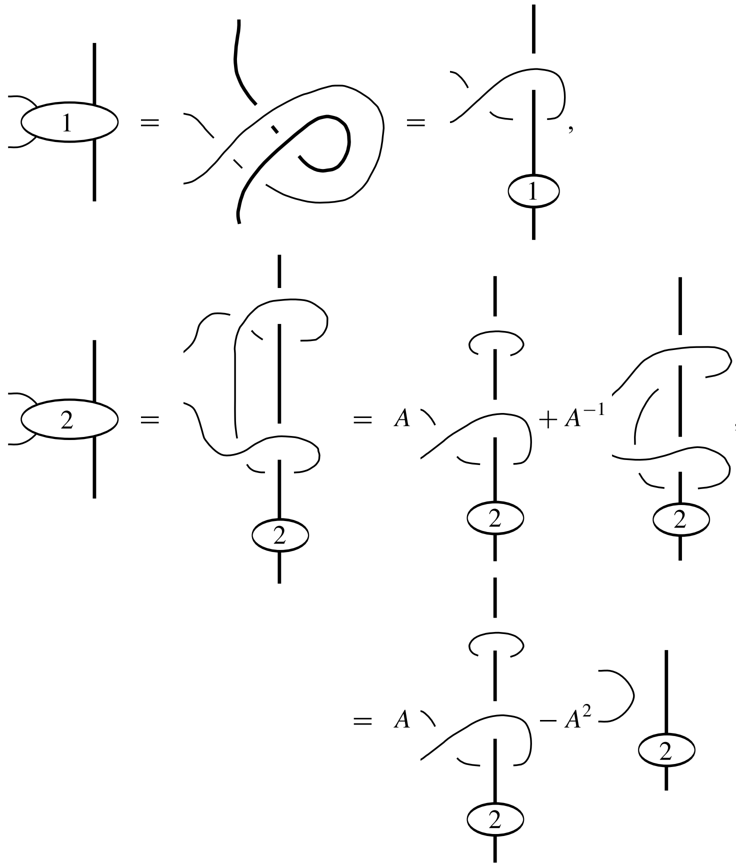
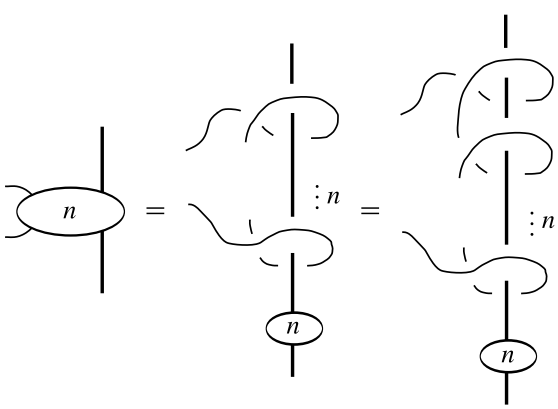
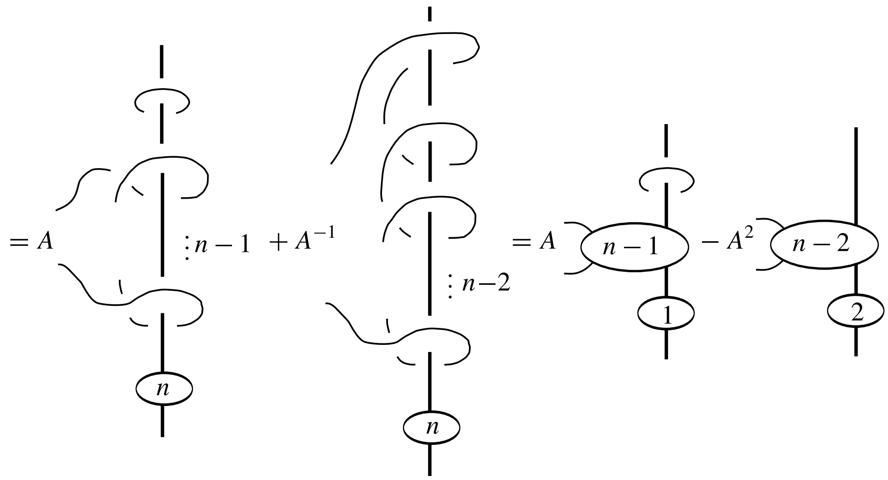


Figure 3. Examples of rewriting twists.

Let $n > 2$, and suppose that the result holds for all $k < n$. Then

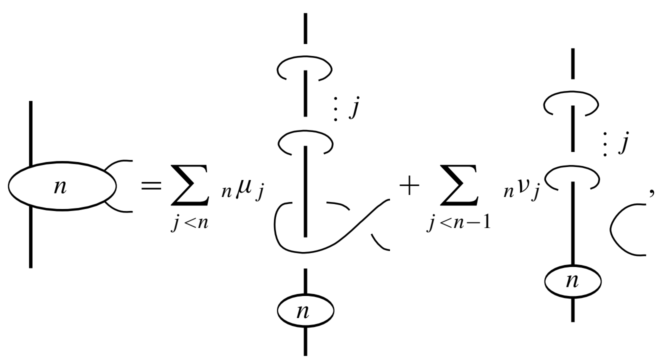




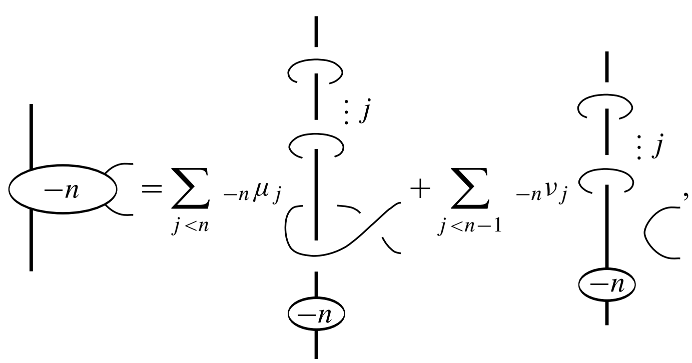
Hence, the first equation follows by induction on n. The second equation can be obtained by reversing all of the crossings in the first. □

By rotating the diagrams in Lemma 7 by 180 degrees, we obtain another:

Lemma 8. *For each integer $n > 0$,*



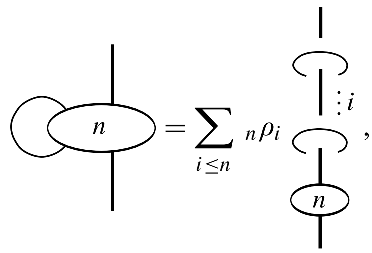
where ${}_n\mu_{n-1} = A^{n-1}$, and



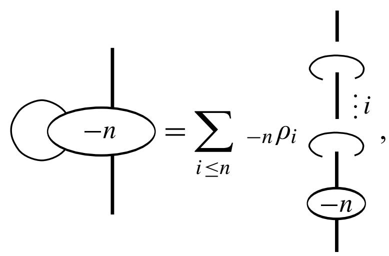
where ${}_{-n}\mu_{n-1} = A^{1-n}$.

In particular, if a component of a link is only twisted about one set of other strands, we obtain an immediate corollary of Lemma 7:

Lemma 9. *For each integer* $n > 0$,



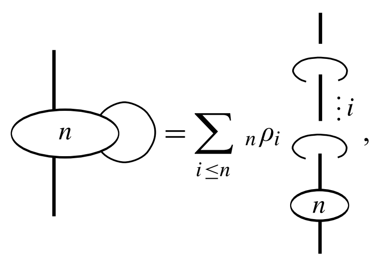
for some ${}_n\rho_0, \ldots, {}_n\rho_n \in R$ *with* ${}_n\rho_n = -A^{n+2}$, *and*



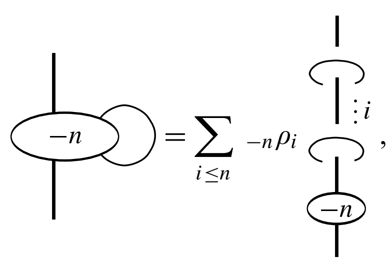
for some ${}_{-n}\rho_0, \ldots, {}_{-n}\rho_n \in R$ *with* ${}_{-n}\rho_n = -A^{-n-2}$.

Similarly, the corollary of Lemma 8:

Lemma 10. *For each integer* $n > 0$,



where ${}_n\rho_n = -A^{n+2}$, *and*



where ${}_{-n}\rho_n = -A^{-n-2}$.

Remark 11. We only need the explicitly computed coefficients in Lemmas 7–10 for the proofs that follow, but the other coefficients are not too difficult to compute explicitly as well: For $n > 0$, we have ${}_n\rho_j = -A^3\, {}_n\mu_{j-1} + (-A^{-2} - A^2)\, {}_n\nu_j$, and for $n > 2$, we have ${}_n\mu_j = A\, {}_{n-1}\mu_{j-1} - A^2\, {}_{n-2}\mu_j$ and ${}_n\nu_j = A\, {}_{n-1}\nu_{j-1} - A^2\, {}_{n-2}\nu_j$, yielding

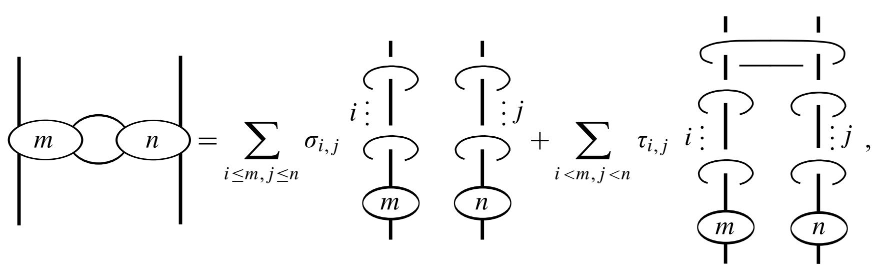
$$ {}_n\mu_j = \begin{cases} (-1)^{(n-j-1)/2}\binom{(n+j-1)/2}{j} A^{n-1} & \text{for } n+j \text{ odd and } 0 \le j < n, \\ 0 & \text{otherwise} \end{cases} $$

and

$$ {}_n\nu_j = \begin{cases} (-1)^{(n-j)/2}\binom{(n+j-2)/2}{j} A^{n} & \text{for } n+j \text{ even and } 0 \le j < n-1, \\ 0 & \text{otherwise.} \end{cases} $$

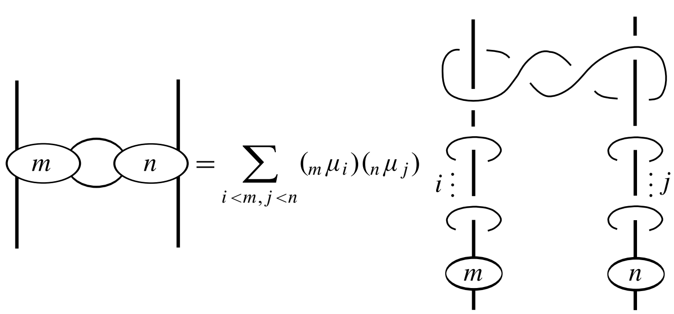
Suppose that a component of a link is twisted with two sets of strands. While more complicated than in the cases previously considered, it is still possible to rewrite the component as a linear combination of loops around the other strands:

Lemma 12. *For all integers $m, n > 0$,*

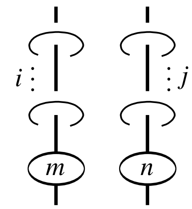


for some $\sigma_{0,0}, \ldots, \sigma_{m,n}, \tau_{0,0}, \ldots, \tau_{m-1,n-1} \in R$ with $\sigma_{m,n} = -A^{m+n+2}$.

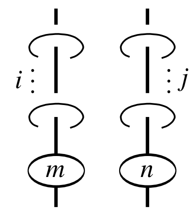
Proof. Applying Lemma 7, and then applying Lemma 8 to each diagram of the resulting linear combination, we have



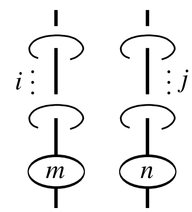
$$+\sum_{i\leq m,\, j<n-1} -A^3({}_m\mu_i)({}_n\nu_j)$$



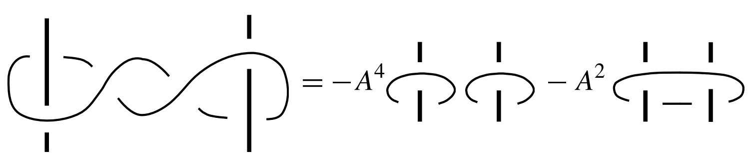
$$+\sum_{i<m-1,\, j\leq n} -A^3({}_m\nu_i)({}_n\mu_j)$$



$$+\sum_{i<m-1,\, j<n-1} (-A^2-A^{-2})({}_m\nu_i)({}_n\nu_j)$$

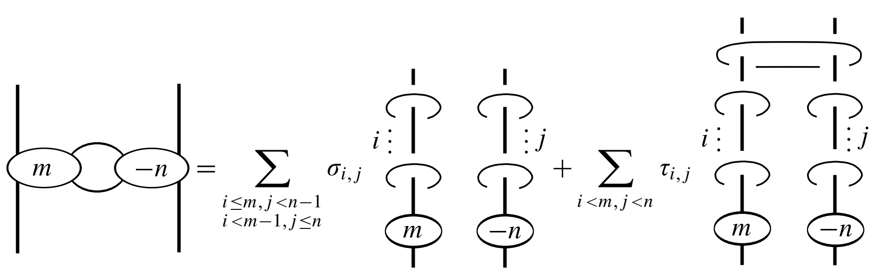


Since



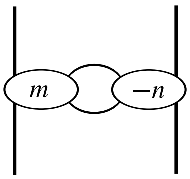
the result follows. □

Lemma 13. *For all integers $m, n > 0$,*

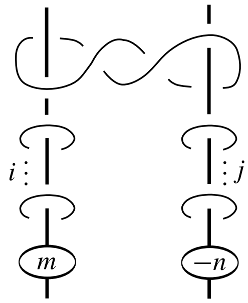


where $\tau_{m-1,n-1} = A^{m-n}$.

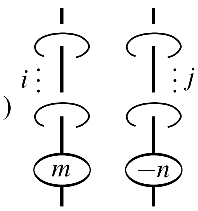
Proof. Applying Lemma 7, and then applying Lemma 8 to each diagram of the resulting linear combination, we get



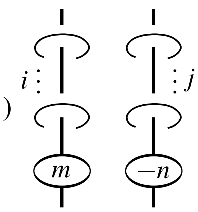
$$= \sum_{i<m,\, j<n} ({}_m\mu_i)({}_{-n}\mu_j)$$



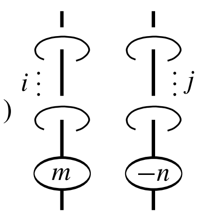
$$+ \sum_{i\le m,\, j<n-1} -A^3({}_m\mu_i)({}_{-n}\nu_j)$$



$$+ \sum_{i<m-1,\, j\le n} -A^{-3}({}_m\nu_i)({}_{-n}\mu_j)$$



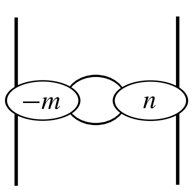
$$+ \sum_{i<m-1,\, j<n-1} (-A^2 - A^{-2})({}_m\nu_i)({}_{-n}\nu_j)$$



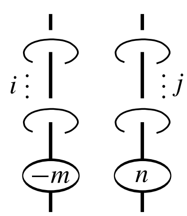
□

By an argument similar to that for Lemma 13, we obtain this:

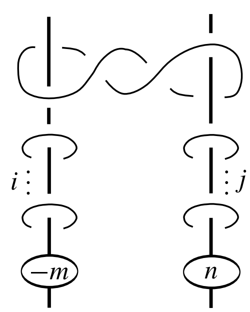
Lemma 14. *For all integers $m, n > 0$,*



$$= \sum_{\substack{i\le m,\, j<n-1 \\ i<m-1,\, j\le n}} \sigma_{i,j}$$



$$+ \sum_{i<m,\, j<n} \tau_{i,j}$$

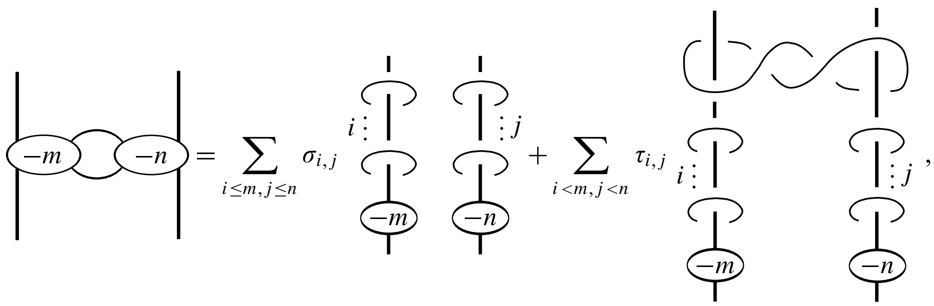


,

where $\tau_{m-1,n-1} = A^{n-m}$.

By an argument similar to that for Lemma 12, we obtain this:

Lemma 15. *For all integers $m, n > 0$,*



where $\sigma_{m,n} = -A^{-m-n-2}$.

Remark 16. In Lemmas 12–15, the coefficients depend on the number of twists as in Lemmas 7–10: for example, $\sigma_{i,j}$ would be written more precisely as ${}_{m,n}\sigma_{i,j}$ in Lemma 12. Since we do not need to refer to the coefficients by name in the following sections, we have simplified the notation for the sake of readability.

3. Finitely generating the skein module

Since all links in the exterior of the surgery description of $M(\alpha, \beta, \gamma)$ can be isotoped into a genus two handlebody and since the skein relations allow us to remove all crossings in a diagram, $S(M(\alpha, \beta, \gamma))$ is generated by $\{x^i y^j z^k\}$.

Definition 17. For $a = |\alpha|$, $b = |\beta|$, $c = |\gamma| > 0$, we define a strict linear ordering on the generating set $\{x^i y^j z^k\}$ of $M(\alpha, \beta, \gamma)$. We say $x^i y^j z^k < x^m y^n z^p$ if any of the following hold:

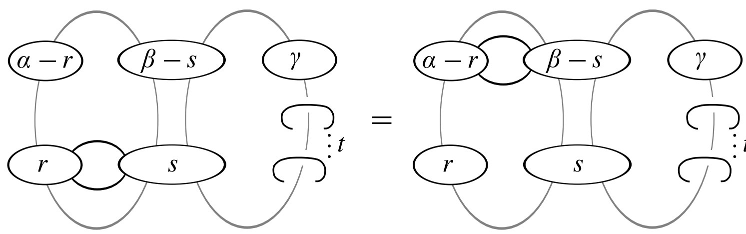
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} < \frac{m}{a} + \frac{n}{b} + \frac{p}{c}$.
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c}, \quad i(k+1) < m(p+1)$.
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c}, \quad i(k+1) = m(p+1),$
$$\max\left(\frac{j}{b}, \frac{k}{c}\right) < \max\left(\frac{n}{b}, \frac{p}{c}\right).$$
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c}, \quad i(k+1) = m(p+1),$
$$\max\left(\frac{j}{b}, \frac{k}{c}\right) = \max\left(\frac{n}{b}, \frac{p}{c}\right), \quad j < n.$$
- $\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{m}{a} + \frac{n}{b} + \frac{p}{c}, \quad i(k+1) = m(p+1),$
$$\max\left(\frac{j}{b}, \frac{k}{c}\right) = \max\left(\frac{n}{b}, \frac{p}{c}\right), \quad j = n, \quad k < p.$$

Suppose that

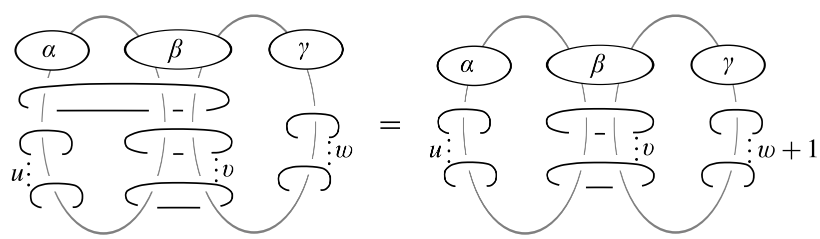
$$a, b, c > 1, \quad a^{-1} < b^{-1} + c^{-1}, \quad b^{-1} < a^{-1} + c^{-1}, \quad c^{-1} < a^{-1} + b^{-1}.$$

By sliding over an attached 2-handle, we obtain useful relations:

Definition 18. The Type I relation is



First, note that by Lemmas 12–15, each side of the relation can be written as a linear combination of loops of the form $x^i y^j z^k$, since for all nonnegative integers u, v, and w,



Note that when $r \geq 0$ and $s \geq 0$, the greatest term appearing on the left side of the Type I relation, rewritten as a linear combination of loops, is $x^r y^s z^t$:

When $r, s > 0$, by Lemma 12, $x^r y^s z^t$ and $x^{r-1} y^{s-1} z^{t+1}$ appear as the greatest terms of their respective types.

Since $c^{-1} < a^{-1} + b^{-1}$,

$$\frac{r}{a} + \frac{s}{b} + \frac{t}{c} > \left(\frac{r}{a} + \frac{s}{b} + \frac{t}{c}\right) + \left(-\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right) = \frac{r-1}{a} + \frac{s-1}{b} + \frac{t+1}{c}.$$

When either $r = 0$ or $s = 0$, the claim follows by Lemma 9 or Lemma 10. When both are 0, the claim follows trivially.

Also note that as long as $r > 0$ or $s > 0$, the leading coefficient is $-A^{r+s+2}$.

Similarly, when $r \leq 0$ and $s \leq 0$, the greatest term appearing on the left side of the Type I relation is $x^{-r} y^{-s} z^t$, and as long as both are nonzero, its coefficient is $-A^{r+s-2}$.

When $r > 0$ and $s < 0$, the greatest term appearing on the left side of the Type I relation is $x^{r-1} y^{-s-1} z^{t+1}$:

By Lemma 13, $x^{r-1}y^{-s-1}z^{t+1}$, $x^{r-2}y^{-s}z^{t}$, and $x^{r}y^{-s-2}z^{t}$ appear as the greatest terms of their respective types. Since $b^{-1} < a^{-1}+c^{-1}$,

$$\frac{r-1}{a}+\frac{-s-1}{b}+\frac{t+1}{c} > \Bigl(\frac{r-1}{a}+\frac{-s-1}{b}+\frac{t+1}{c}\Bigr)+\Bigl(-\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\Bigr)=\frac{r-2}{a}+\frac{-s}{b}+\frac{t}{c}.$$

Since $a^{-1} < b^{-1}+c^{-1}$,

$$\frac{r-1}{a}+\frac{-s-1}{b}+\frac{t+1}{c} > \Bigl(\frac{r-1}{a}+\frac{-s-1}{b}+\frac{t+1}{c}\Bigr)+\Bigl(\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\Bigr)=\frac{r}{a}+\frac{-s-2}{b}+\frac{t}{c}.$$

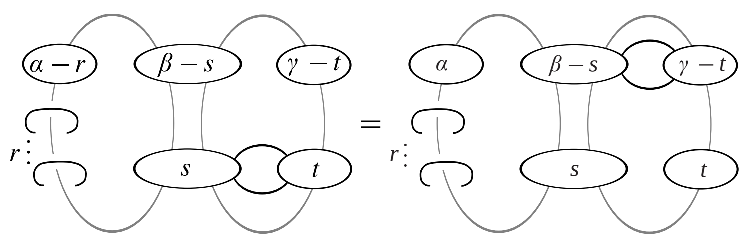
Also note that in this case, the leading coefficient is A^{r+s}.

Similarly, when $r < 0$ and $s > 0$, the greatest term appearing on the left side is $x^{-r-1}y^{s-1}z^{t+1}$, with coefficient A^{r+s}.

Likewise, the greatest term on the right side is $x^{|\alpha-r|-1}y^{|\beta-s|-1}z^{t+1}$, when $\alpha-r$ and $\beta-s$ are nonzero with different signs, and the greatest term on the right side is $x^{|\alpha-r|}y^{|\beta-s|}z^{t}$ otherwise.

By sliding over the other attached 2-handle, we obtain additional relations:

Definition 19. The Type II relation is



As with the Type I relation, each side of the relation can be rewritten as a linear combination of loops of the form $x^{i}y^{j}z^{k}$.

Also, as with the Type I relation, the greatest term appearing on the left side of the Type II relation is $x^{r+1}y^{|s|-1}z^{|t|-1}$ when the signs of s and t differ, with coefficient A^{s+t}. Otherwise, the greatest term appearing on the left side is $x^{r}y^{|s|}z^{|t|}$, and as long as one of s and t are nonzero, the leading coefficient is $-A^{s+t\pm 2}$.

Finally, as with the Type I relation, the greatest term on the right side of the Type II relation is $x^{r+1}y^{|\beta-s|-1}z^{|\gamma-t|-1}$ when the signs of $\beta-s$ and $\gamma-t$ differ, and the greatest term appearing on the left side is $x^{r}y^{|\beta-s|}z^{|\gamma-t|}$ otherwise.

Theorem 20. *For all integers $a, b, c > 1$ such that*

$$a^{-1} < b^{-1}+c^{-1}, \qquad b^{-1} < a^{-1}+c^{-1}, \qquad c^{-1} < a^{-1}+b^{-1},$$

$S(M(a,b,c))$ is finitely generated.

Proof. We show that with respect to our previously defined ordering, $x^{i}y^{j}z^{k}$ can be rewritten as linear combinations of lesser terms whenever $i \geq a$, $j \geq b$, or $k \geq c$. We accomplish this by choosing a Type I or Type II relation in which

$x^i y^j z^k$ appears as the greatest term on the left side, as in the previous discussion. We then show that $x^i y^j z^k$ is greater than the greatest term on the right side of the relation. Hence, by subtracting all of the terms less than $x^i y^j z^k$ from both sides of the equation and dividing both sides by the (invertible, as previously discussed) coefficient of $x^i y^j z^k$, we successfully rewrite $x^i y^j z^k$.

Case 1. Suppose $i \geq a$. Let $r = i$, $s = j$, and $t = k$. Since $r > 0$ and $s \geq 0$, the greatest term on the left of the Type I relation is $x^i y^j z^k$. Since $a - r = a - i \leq 0$, the greatest term on the right side is $x^{i-a} y^{j-b} z^k$ if $j \geq b$ or $i = a$, and $x^{i-a-1} y^{b-j-1} z^{k+1}$ if $j < b$ and $i > a$.

Case 1.1. Suppose $j \geq b$ or $i = a$. Then

$$\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \frac{i-a}{a} + \frac{j-b}{b} + \frac{k}{c},$$

and thus $x^i y^j z^k > x^{i-a} y^{j-b} z^k$.

Case 1.2. Suppose $j < b$ and $i > a$. Then

$$\begin{aligned}\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \frac{i}{a} - \frac{j}{b} + \frac{k}{c} &= \frac{i-a}{a} + \frac{b-j}{b} + \frac{k}{c} \\ &> \left(\frac{i-a}{a} + \frac{b-j}{b} + \frac{k}{c}\right) + \left(-\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right) \\ &= \frac{i-a-1}{a} + \frac{b-j-1}{b} + \frac{k+1}{c}.\end{aligned}$$

Hence, $x^i y^j z^k > x^{i-a-1} y^{b-j-1} z^{k+1}$.

Case 2. Suppose $i < a$ and $j \geq b$. Let $r = i$, $s = j$, and $t = k$. Since $r \geq 0$ and $s > 0$, the greatest term on the left of the Type I relation is $x^i y^j z^k$. Since $a - r = a - i > 0$ and $b - s = b - j \leq 0$, the greatest term on the right side is $x^{a-i-1} y^{j-b-1} z^{k+1}$ if $j > b$, and $x^{a-i} z^k$ if $j = b$.

Case 2.1. Suppose $j > b$. Then

$$\begin{aligned}\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > -\frac{i}{a} + \frac{j}{b} + \frac{k}{c} &= \frac{a-i}{a} + \frac{j-b}{b} + \frac{k}{c} \\ &> \left(\frac{a-i}{a} + \frac{j-b}{b} + \frac{k}{c}\right) + \left(-\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right) \\ &= \frac{a-i-1}{a} + \frac{j-b-1}{b} + \frac{k+1}{c},\end{aligned}$$

and thus $x^i y^j z^k > x^{a-i-1} y^{j-b-1} z^{k+1}$.

Case 2.2. Suppose $j=b$. Then

$$\frac{i}{a}+\frac{j}{b}+\frac{k}{c}=\frac{i}{a}+1+\frac{k}{c}>-\frac{i}{a}+1+\frac{k}{c}=\frac{a-i}{a}+\frac{k}{c},$$

and hence $x^i y^j z^k > x^{a-i} z^k$.

Case 3. Suppose $i<a$, $j<b$, and $k\ge c$. Let $r=i$, $s=j$, and $t=k$. Since $s\ge 0$ and $t>0$, the greatest term on the left of the Type II relation is $x^i y^j z^k$. Since $c-t=c-k\le 0$, the greatest term on the right side is $x^{i+1}y^{b-j-1}z^{k-c-1}$ if $k>c$, and $x^i y^{b-j}$ if $k=c$.

Case 3.1. Suppose $k>c$. Then

$$\begin{aligned}\frac{i}{a}+\frac{j}{b}+\frac{k}{c}>\frac{i}{a}-\frac{j}{b}+\frac{k}{c}=\frac{i}{a}+\frac{b-j}{b}+\frac{k-c}{c}&\\ >\Big(\frac{i}{a}+\frac{b-j}{b}+\frac{k-c}{c}\Big)+\Big(\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\Big)&\\ =\frac{i+1}{a}+\frac{b-j-1}{b}+\frac{k-c-1}{c},&\end{aligned}$$

and thus $x^i y^j z^k > x^{i+1}y^{b-j-1}z^{k-c-1}$.

Case 3.2. Suppose $k=c$. Then

$$\frac{i}{a}+\frac{j}{b}+\frac{k}{c}=\frac{i}{a}+\frac{j}{b}+1>\frac{i}{a}-\frac{j}{b}+1=\frac{i}{a}+\frac{b-j}{b},$$

and so $x^i y^j z^k > x^i y^{b-j}$. □

Remark 21. Note that we can refine the generating set obtained in the proof above, through additional applications of the Type I and Type II relations. For instance, we can rewrite $x^i y^j z^k$ when

- $i<a$, $j<b$, and $i/a+j/b>1$;
- $i<a$, $j<b$, $i/a+j/b=1$, and $i>a/2$;
- $j<b$, $k<c$, and $j/b+k/c>1$; or
- $j<b$, $k<c$, $j/b+k/c=1$, and $k>c/2$.

Theorem 22. *For all integers $a,b,c>1$ such that*

$$a^{-1}<b^{-1}+c^{-1},\qquad b^{-1}<a^{-1}+c^{-1},\qquad c^{-1}<a^{-1}+b^{-1},$$

$S(M(a,-b,c))$ is finitely generated.

Proof. We show that with respect to our previously defined ordering, $x^i y^j z^k$ can be rewritten as linear combinations of lesser terms whenever $i\ge a$, $j\ge b$, or $k>c(2-2/b)$. As in the previous proof, we accomplish this by choosing a Type I or Type II relation in which $x^i y^j z^k$ appears as the greatest term on the left side, and then show that $x^i y^j z^k$ is greater than the greatest term on the right side of the relation. Here, however, the task is a bit more difficult, since the difference in signs prevents us from proceeding in a completely straightforward manner.

Case 1. Suppose $i \geq a$. Let $r = i$, $s = j$, and $t = k$. Since $r > 0$ and $s \geq 0$, the greatest term on the left of the Type I relation is $x^i y^j z^k$. Then $a - r = a - i \leq 0$ and $-b - s = -b - j < 0$, and thus $x^{i-a} y^{b+j} z^k$ is the greatest term on the right. Then

$$\frac{i}{a} + \frac{j}{b} + \frac{k}{c} = \frac{i-a}{a} + \frac{b+j}{b} + \frac{k}{c} \quad \text{and} \quad i(k+1) > (i-a)(k+1),$$

so $x^i y^j z^k > x^{i-a} y^{b+j} z^k$.

Case 2. Suppose $i < a$ and $j \geq b$.

Case 2.1. Suppose $k > 0$. Let $r = i + 1$, $s = -j - 1$, and $t = k - 1$. Since $r > 0$ and $s < 0$, the greatest term on the left of the Type I is relation $x^i y^j z^k = x^{(i+1)-1} y^{-(-j-1)-1} z^{(k-1)+1}$. Since $a-r = a-i-1 \geq 0$ and $-b-s = -b+j+1 > 0$, $x^{a-i-1} y^{-b+j+1} z^{k-1}$ is the greatest term on the right. Then

$$\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \left(\frac{i}{a} + \frac{j}{b} + \frac{k}{c}\right) + \left(-\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) = \frac{a-i-1}{a} + \frac{-b+j+1}{b} + \frac{k-1}{c},$$

and thus $x^i y^j z^k > x^{a-i-1} y^{-b+j+1} z^{k-1}$.

Case 2.2. Suppose $k = 0$.

Case 2.2.1. Suppose $i > 0$. Let $r = i - 1$, $s = -j - 1$, and $t = 1$. Since $s < 0$ and $t > 0$,the greatest term on the left of the Type II relation is $x^i y^j$. Then $-b - s = -b + j + 1 > 0$ and $c - t = c - 1 > 0$, and thus $x^{i-1} y^{-b+j+1} z^{c-1}$ is the greatest term on the right. Then

$$\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \left(\frac{i}{a} + \frac{j}{b} + \frac{k}{c}\right) + \left(-\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) = \frac{i-1}{a} + \frac{-b+j+1}{b} + \frac{c-1}{c},$$

and thus $x^i y^j > x^{i-1} y^{-b+j+1} z^{c-1}$.

Case 2.2.2. Suppose $i = 0$. Let $r = 0$, $s = -j$, and $t = 0$. Since $t = 0$, the greatest term on the left of the Type II relation is y^j. Then

$$-b - s = -b + j \geq 0 \quad \text{and} \quad c - t = c > 0,$$

and thus $y^{-b+j} z^c$ is the greatest term on the right. Then $j/b = (-b+j)/b + c/c$ and $0(0+1) = 0(c+1)$. When $j > b$, we have $\max(j/b, 0) > \max(-b + j/b, c/c)$, and when $j = b$, we have $\max(j/b, 0) = 1 = \max((-b + j)/b, c/c)$ and also $j = b > 0 = -b + j$. Hence $y^j > y^{-b+j} z^c$.

Case 3. Suppose $i < a$, $j < b$, and $k > c(2 - 2/b)$. (Hence, $k > c$.)

Case 3.1. Suppose $i > 0$. Let $r = i - 1$, $s = -j - 1$, and $t = k + 1$. Since $s < 0$ and $t > 0$, the greatest term on the left of the Type II relation is $x^i y^j z^k$. Since $-b - s = -b + j + 1 \leq 0$ and $c - t = c - k - 1 < 0$, the greatest term on the right is $x^{i-1} y^{b-j-1} z^{k-c+1}$. Then

$$\frac{i}{a} + \frac{j}{b} + \frac{k}{c} > \left(\frac{i}{a} + \frac{j}{b} + \frac{k}{c}\right) + \left(-\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right) = \frac{i-1}{a} + \frac{b-j-1}{b} + \frac{k-c+1}{c},$$

and thus $x^i y^j > x^{i-1} y^{b-j-1} z^{k-c+1}$.

Case 3.2. Suppose $i = 0$.

Case 3.2.1. Suppose $j = b - 1$. Let $r = 1$, $s = -b$, and $t = k - 1$. Since $r > 0$ and $s < 0$, the greatest term on the left of the Type I relation is $y^{b-1} z^k$. Then $a - r = a - 1 > 0$ and $-b - s = 0$, and thus $x^{a-1} z^{k-1}$ is the greatest term on the right. Since

$$\frac{b-1}{b} + \frac{k}{c} > \left(\frac{b-1}{b} + \frac{k}{c}\right) + \left(-\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) = \frac{a-1}{a} + \frac{k-1}{c},$$

$y^{b-1} z^k > x^{a-1} z^{k-1}$.

Case 3.2.2. Suppose $j < b - 1$. Let $r = 0$, $s = j$, and $t = k$. Since $s \geq 0$ and $t > 0$, the greatest term on the left of the Type II relation is $y^j z^k$. $-b - s = -b - j < 0$ and $c - k < 0$, and thus, $y^{b+j} z^{k-c}$ is the greatest term on the right. Then

$$\frac{j}{b} + \frac{k}{c} = \frac{b+j}{b} + \frac{k-c}{c}, \qquad 0(k+1) = 0(k-c+1),$$

and $\max(j/b, k/c) = k/c > \max(b + j/(b), (k-c)/c)$ since $k > c((2b-2)/b) \geq c((b+j)/b)$. Hence $y^j z^k > y^{b+j} z^{k-c}$. □

Proof of Theorem 4. If α, β and γ are all positive, the result follows by Theorem 20. If α, β and γ are all negative, the result follows as well, since $S(M(\alpha, \beta, \gamma))$ is isomorphic to $S(M(-\alpha, -\beta, -\gamma))$.

Suppose that exactly one of α, β and γ is negative. If $\beta < 0$, the result follows by Theorem 22. If $\alpha < 0$, by sliding the right handle over the left and performing isotopy, we see that $M(\alpha, \beta, \gamma)$ is identical to $M(\gamma, \alpha, \beta)$, and so the result follows. Similarly, if $\gamma < 0$, by sliding the left handle over the right, $M(\alpha, \beta, \gamma)$ is seen to be identical to $M(\beta, \gamma, \alpha)$, and so again the result follows.

If exactly one of α, β, and γ is positive, then $S(M(-\alpha, -\beta, -\gamma))$ is finitely generated, and thus $S(M(\alpha, \beta, \gamma))$ is finitely generated as well. □

4. Examples

While the previous proofs yield a finite set of generators for $S(M(\alpha, \beta, \gamma))$, they do not exploit the full potential of the Type I and Type II relations. Using the following Python code, we can refine our results for $S(M(a, -b, c))$.

```
def greaterthan(a,b,c,i,j,k,m,n,p):
   if i*b*c + j*a*c + k*a*b > m*b*c + n*a*c + p*a*b:
      return True
   elif i*b*c + j*a*c + k*a*b == m*b*c + n*a*c + p*a*b:
      if i*(k+1) > m*(p+1):
         return True
      elif i*(k+1) == m*(p+1):
         if max(j*c,k*b) > max(n*c,p*b):
            return True
         elif max(j*c,k*b) == max(n*c,p*b):
            if j > n:
               return True
            elif j == n:
               if k > p:
                  return True
   return False

def left1(i,j,k):
   L = []
   if i > 0 or j > 0:
      L.append([i,j,k])
      L.append([-i,-j,k])
   if k > 0:
      L.append([i+1,-j-1,k-1])
      L.append([-i-1,j+1,k-1])
   return L

def left2(i,j,k):
   L = []
   if j > 0 or k > 0:
      L.append([i,j,k])
      L.append([i,-j,-k])
   if i > 0:
      L.append([i-1,j+1,-k-1])
      L.append([i-1,-j-1,k+1])
   return L

def right1(a,b,c,r,s,t):
   if (a-r > 0 and -b-s < 0) or (a-r < 0 and -b-s > 0):
      return [abs(a-r)-1,abs(-b-s)-1,t+1]
   return [abs(a-r),abs(-b-s),t]
```

```
def right2(a,b,c,r,s,t):
   if (-b-s > 0 and c-t < 0) or (-b-s < 0 and c-t > 0):
      return [r+1,abs(-b-s)-1,abs(c-t)-1]
   return [r,abs(-b-s),abs(c-t)]

def generatingset(a,b,c):
   GS = []
   MGS = []
   for i in range(a):
      for j in range(b):
         k = 0
         while b*k <= 2*c*(b-1):
            GS.append([i,j,k])
            k += 1
   for T in GS:
      rewrite = False
      for L in left1(T[0],T[1],T[2]):
         R = right1(a,b,c,L[0],L[1],L[2])
         if greaterthan(a,b,c,T[0],T[1],T[2],R[0],R[1],R[2]):
            rewrite = True or rewrite
      for L in left2(T[0],T[1],T[2]):
         R = right2(a,b,c,L[0],L[1],L[2])
         if greaterthan(a,b,c,T[0],T[1],T[2],R[0],R[1],R[2]):
            rewrite = True or rewrite
      if not rewrite:
         MGS.append(T)
   return MGS
```

Using this code, we obtain the generating sets listed in the introduction for $S(M(2,-2,2))$, $S(M(3,-2,3))$, and $S(M(3,-2,5))$, and we find that our generating set is minimal for $S(M(2,-2,2); \mathbb{Z}[A^{\pm 1}], A)$.

As for getting minimality of our generating sets for $S(M(3,-2,3); R[A^{\pm 1}], A)$ and $S(M(3,-2,5); R[A^{\pm 1}], A)$, we might consider $S(M(3,-2,3); R, -1)$ and $S(M(3,-2,5); R, -1)$, as they are isomorphic to the skein algebras of their fundamental groups, which are generated by representatives of conjugacy classes. For $S(M(3,-2,3); R[A^{\pm 1}], A)$, however, this will not help, as only three of the conjugacy classes of the binary tetrahedral group are self-inversive, and hence $S(M(3,-2,3); R, -1)$ can be generated by five elements. See [Przytycki and Sikora 2000].

Still, for $S(M(3,-2,5); R[A^{\pm 1}], A)$, we can hope to gain some insight, as its conjugacy classes are self-inversive, and since we have the following result:

Proposition 23. *Suppose that a set* $L = \{L_1, \dots, L_n\}$ *of links in* M *represents a generating set for* $S(M;\, R[A^{\pm 1}],\, A)$.

(1) *If* L *yields a minimal generating set for* $S(M;\, R, -1)$, *then* L *represents a minimal generating set for* $S(M;\, R[A^{\pm 1}],\, A)$.

(2) *If* L *yields a linearly independent set for* $S(M;\, R, -1)$ *and* $S(M;\, R[A^{\pm 1}],\, A)$ *has no* $(A+1)$ *torsion, then* L *represents a basis for* $S(M;\, R[A^{\pm 1}],\, A)$.

(3) *If* L *yields a linearly independent set for* $S(M;\, R, -1)$ *and* $S(M;\, R[A^{\pm 1}],\, A)$ *has torsion, then* $S(M;\, R[A^{\pm 1}],\, A)$ *has* $(A+1)$ *torsion.*

Proof. (1) Suppose that $L_n = f_1(A)L_1 + \cdots + f_{n-1}(A)L_{n-1}$ in $S(M;\, R[A^{\pm 1}],\, A)$. Then $L_n = f_1(-1)L_1 + \cdots + f_{n-1}(-1)L_{n-1}$ in $S(M;\, R, -1)$, a contradiction.

(2) Suppose that $f_1(A)L_1 + \cdots + f_n(A)L_n = 0$ in $S(M;\, R[A^{\pm 1}],\, A)$. Then in $S(M;\, R, -1)$, we have $f_1(-1)L_1 + \cdots + f_n(-1)L_n = 0$. Now $L_1, \dots, L_n$ is a basis of $S(M;\, R, -1)$, so $f_i(-1) = 0$ for each i, and thus $(A+1) \mid f_i$ for each i. Hence, $(A+1)(g_1(A)L_1 + \cdots + g_n(A)L_n) = 0$ for some $g_1, \dots g_n$. $S(M;\, R[A^{\pm 1}],\, A)$ has no $(A+1)$ torsion, so $g_1(A)L_1 + \cdots + g_n(A)L_n = 0$. Hence, $S(M;\, R[A^{\pm 1}],\, A)$ is free.

(3) If L yields a linearly independent set for $S(M;\, R, -1)$, and $S(M;\, R[A^{\pm 1}],\, A)$ has torsion, then L cannot represent a basis; and hence $S(M;\, R[A^{\pm 1}],\, A)$ must have $(A+1)$ torsion by (2). □

Remark 24. The existence of torsion is a topic of particular interest in skein theory. For example, see the study of $(A+1)$ torsion in [McLendon 2006].

Let G be the binary icosahedral group, with presentation $\langle r, s \mid r^5 = s^3 = (rs)^2 \rangle$. Since G is finite, the skein algebra of G over $\mathbb{C}$ is isomorphic to $\mathbb{C}[X(G)]$, the $\mathrm{SL}(2, \mathbb{C})$ character variety of G, a result of [Przytycki and Sikora 2000]; see also [Bullock 1997b].

Let σ_0 be the trivial 2-dimensional representation of G, let σ_1 be the representation of G that sends r and s to

$$A_1 = \frac{1}{5}\begin{bmatrix} -3e_5 - e_5^2 + e_5^3 - 2e_5^4 & e_5 - 3e_5^2 - 2e_5^3 - e_5^4 \\ e_5 + 2e_5^2 + 3e_5^3 - e_5^4 & -2e_5 + e_5^2 - e_5^3 - 3e_5^4 \end{bmatrix}$$

and

$$B_1 = \frac{1}{5}\begin{bmatrix} -e_5 - 2e_5^2 - 3e_5^3 - 4e_5^4 & 2e_5 - e_5^2 + e_5^3 - 2e_5^4 \\ 2e_5 - e_5^2 + e_5^3 - 2e_5^4 & -4e_5 - 3e_5^2 - 2e_5^3 - e_5^4 \end{bmatrix},$$

respectively, and let σ_2 be the representation of G that sends r and s to

$$A_2 = \begin{bmatrix} e_5 - e_5^2 & -e_5^2 - e_5^4 \\ -e_5 - e_5^4 & -e_5 - e_5^3 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 1 & -e_5^3 \\ e_5^2 & 0 \end{bmatrix},$$

respectively, where $e_5 = e^{2\pi i/5}$.

Using GAP [GAP 2007], we can see that σ_0, σ_1, and σ_2 are $\mathrm{SL}(2,\mathbb{C})$ representations of G, and any $\mathrm{SL}(2,\mathbb{C})$ representation σ of G is equivalent to one of them: If irreducible, σ is equivalent to σ_1 or σ_2, and if reducible, σ is equivalent to σ_0, since G is perfect. See [Culler and Shalen 1983].

Let χ_0, χ_1, and χ_2 be the characters of σ_0, σ_1, and σ_2, respectively, and for each $g \in G$, let τ_g be the evaluation map defined on the characters of G by $\tau_g(\chi) = \chi(g)$. Note that since 1, r, r^2, r^3, r^4, r^5, rs, s, and s^2 represent the conjugacy classes of G, $\mathbb{C}[X(G)]$ is generated by τ_1, τ_r, τ_{r^2}, τ_{r^3}, τ_{r^4}, τ_{r^5}, τ_{rs}, τ_s, and τ_{s^2}.

	τ_1	τ_r	τ_{r^2}	τ_{r^3}	τ_{r^4}	τ_{r^5}	τ_{rs}	τ_s	τ_{s^2}
χ_0	2	2	2	2	2	2	2	2	2
χ_1	2	$-e_5 - e_5^4$	$e_5^2 + e_5^3$	$-e_5^2 - e_5^3$	$e_5 + e_5^4$	-2	0	1	-1
χ_2	2	$-e_5^2 - e_5^3$	$e_5 + e_5^4$	$-e_5 - e_5^4$	$e_5^2 + e_5^3$	-2	0	1	-1

From the table, we can see that the following relations hold in $\mathbb{C}[X(G)]$:

$$\tau_{s^2} = 3\tau_s - 2\tau_1, \qquad \tau_{rs} = 2\tau_s - \tau_1, \qquad \tau_{r^5} = 4\tau_s - 3\tau_1,$$
$$\tau_{r^4} = 4\tau_s - \tau_r - 2\tau_1, \quad \tau_{r^3} = 3\tau_s - \tau_r - \tau_1, \quad \tau_{r^2} = \tau_s + \tau_r - \tau_1$$

Furthermore, $\{\tau_1, \tau_r, \tau_s\}$ are linearly independent in $\mathbb{C}[X(G)]$, since the matrix

$$\begin{bmatrix} \tau_1(\chi_0) & \tau_r(\chi_0) & \tau_s(\chi_0) \\ \tau_1(\chi_1) & \tau_r(\chi_1) & \tau_s(\chi_1) \\ \tau_1(\chi_2) & \tau_r(\chi_2) & \tau_s(\chi_2) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & -e_5 - e_5^4 & 1 \\ 2 & -e_5^2 - e_5^3 & 1 \end{bmatrix}$$

is invertible.

Thus, $S(M(3, -2, 5); \mathbb{C}, -1)$ is 3-dimensional, and therefore we cannot use Proposition 23 to show that our generating set for $S(M(3, -2, 5); \mathbb{C}[A^{\pm 1}], A)$ is minimal. Hence, we are left with the following:

Question. For some ring R and unit A, is $\{1, z, z^2, z^3, z^4, z^5, y, x, x^2\}$ a minimal generating set for $S(M(3, -2, 5); R, A)$? If not, it is generated by $\{1, z, x\}$ for every ring R and unit A?

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TRANSITIVE ACTIONS AND EQUIVARIANT COHOMOLOGY AS AN UNSTABLE $\mathscr{A}^*$-ALGEBRA

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A graded $\mathbb{F}_p$-algebra A with action of the Steenrod algebra $\mathscr{A}^*$ is said to be Steenrod presentable if there is a polynomial ring $P = \mathbb{F}_p[u_1, \dots, u_n]$ with an action of $\mathscr{A}^*$ and an $\mathscr{A}^*$-invariant ideal $I \subset P$ such that $A = P/I$ and the induced action of $\mathscr{A}^*$ on P/I is the given one. It is shown that an action φ of a simple compact Lie group G on a homogeneous Kähler manifold $X = G/H$ has a Steenrod presentable equivariant cohomology for almost all primes p if and only if φ is conjugate to the standard action by left translation. Application to the case $H = T$ a maximal torus reproduces a former result of the author: namely, that every topological G-action on G/T is conjugate to the standard action by left translation with isotropy group a maximal torus.

1. Introduction

Suppose X to be a space, and let $A = H^*(X; \mathbb{F}_p)$ be its cohomology with coefficients in the prime field $\mathbb{F}_p$. Then on A there is an unstable action of the p-Steenrod algebra $\mathscr{A}^*$. On the other hand, given a presentation $A = P/I$, for an ideal $I \subset P$ where P is the polynomial algebra $P = \mathbb{F}_p[h_1, \dots, h_n]$, with deg $h_i = d_i$, one might ask whether the given action of $\mathscr{A}^*$ is induced by an action of $\mathscr{A}^*$ on the polynomial algebra that leaves the defining ideal stable. In the case $p \neq 2$ and d_i prime to p for all i, a necessary condition condition is given by a theorem of Adams and Wilkerson [1980]; see also [Smith 1995, Theorem 10.5.1]. In particular it follows from this theorem that the polynomial ring P must be the invariant ring

$$P = \mathbb{F}_p[x_1, \dots, x_n]^W, \quad \deg x_i = 2,$$

where $W \subset \mathrm{GL}(n, \mathbb{F}_p)$ is a finite group of order $d_1 \dots d_n$ generated by pseudoreflections acting on $\mathbb{F}_p[x_1, \dots, x_n] = \mathbb{F}_p[V]$ in the standard way [Smith 1995; 1997].

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This implies that the degrees d_i can only assume certain precise values, which are exactly the Weyl–Coxeter degrees of the group W; see for example [Smith 1995, p. 199].

In the following, we say that the $\mathbb{F}_p$-algebra A with an unstable action of $\mathcal{A}^*$ is *Steenrod presentable* if there is a polynomial ring $P = \mathbb{F}_p[x_1, \dots, x_n]^W$ with the standard action of the Steenrod algebra $\mathcal{A}^*$ and an $\mathcal{A}^*$-stable ideal $I \subset P$ such that $A \cong P/I$ with the induced $\mathcal{A}^*$-module structure.

As the main example of Steenrod presentable $\mathbb{F}_p$-algebras, we consider the cohomology of homogeneous spaces $X = G/H$, where $H \subset G$ is a closed connected maximal rank subgroup of a compact connected Lie group G. Then there is the standard fibration

$$G/H \longrightarrow B_H \longrightarrow B_G,$$

where B_K is the classifying space for the topological group K. If $H \subset G$ is a subgroup of maximal rank and if nor G neither H have p-torsion, the ring $H^*(G/H; \mathbb{F}_p)$ has a presentation

$$H^*(G/H; \mathbb{F}_p) \cong \frac{H^*(B_H; \mathbb{F}_p)}{H^*_+(B_G; \mathbb{F}_p) \cdot H^*(B_H; \mathbb{F}_p)}$$

such that the action of the Steenrod algebra $\mathcal{A}^*$ on $H^*(G/H; \mathbb{F}_p)$ is induced by the standard action of $\mathcal{A}^*$ on the ring $H^*(B_H; \mathbb{F}_p)$.

So, throughout this note we shall assume that $p \neq 2$ and that B_G and B_H do not have p-torsion for all primes to be considered.

Suppose a compact connected Lie group K is acting in a reasonable way on $X = G/H$. Then X is totally nonhomologous to zero in the fibration

$$X \longrightarrow X_K \longrightarrow B_K,$$

where $X_K = E_K \times_K X$ is the Borel construction. Write $H^*(X; \mathbb{F}_p) = P/I_0$, $P = \mathbb{F}_p[h_1, \dots, h_n]$, where the ideal $I_0 \subset P$ is generated by a set $g_1, \dots, g_n$ of multiplicative generators of the invariant ring $R_G = H^*(B_G; \mathbb{F}_p) \subset H^*(B_H; \mathbb{F}_p)$. As can be shown in the same way as in the proof of [Hauschild 1986, Theorem 1.1], the equivariant cohomology $H^*_K(X; \mathbb{F}_p) = H^*(X_K; \mathbb{F}_p)$ is a graded algebra over $R = H^*(B_K; \mathbb{F}_p)$, which can be written as $H^*_K(X; \mathbb{F}_p) = P_R/I$, where $P_R = R \otimes P$ and I is an ideal generated by homogeneous elements of the form $1 \otimes g_j - r_j$, where the r_j are elements of the ideal $R_+ P_R$ generated by the augmentation ideal of R. On the ring $P_R = H^*(B_K; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(B_H; \mathbb{F}_p)$ there is the natural unstable $\mathcal{A}^*$-module structure and the equivariant cohomology is *Steenrod presentable* if I is stable under this $\mathcal{A}^*$-action inducing the given $\mathcal{A}^*$-action on the quotient. Moreover, since the isomorphism $H^*(X; \mathbb{F}_p) \cong H^*_K(X; \mathbb{F}_p)/H^*_+(B_K; \mathbb{F}_p)H^*_K(X; \mathbb{F}_p)$ is induced by the inclusion $i \colon X \to X_K$ of the fiber, the Steenrod presentation of

$H^*_K(X;\mathbb{F}_p)$ induces the Steenrod presentation of $H^*(X;\mathbb{F}_p)$. For more information on Steenrod powers acting on equivariant cohomology, see [Allday and Puppe 1993; Quillen 1971].

2. Steenrod powers and rational cohomology

Observe that $X = G/H$ is now the fiber of two fibrations, and that in both fibrations it is totally nonhomologous to zero. Consequently there is the canonical epimorphism $j^*\colon H^*(B_H;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$ induced by the inclusion $j\colon X \to B_H$ of the fiber. Moreover, let i^* be induced by the inclusion $i\colon X \to X_K$ of the fiber in the Borel fibration. Both maps commute of course with the respective $\mathcal{A}^*$-module structures.

Observation 1. *The equivariant cohomology $H^*_K(X;\mathbb{F}_p)$ is Steenrod presentable if and only if there is a homomorphism $J\colon H^*(B_H;\mathbb{F}_p) \to H^*_K(X;\mathbb{F}_p)$ making the following diagram commute*:

$$\begin{array}{ccc} & & H^*_K(X;\mathbb{F}_p) \\ & \overset{J}{\nearrow} & \big\downarrow i^* \\ H^*(B_H;\mathbb{F}_p) & \underset{j^*}{\longrightarrow} & H^*(X;\mathbb{F}_p) \end{array}$$

Proof. Let $\pi : X_K \to B_K$ be the projection in the Borel fibration, and then consider the homomorphism $\pi^* \otimes J\colon H^*(B_K;\mathbb{F}_p) \otimes H^*(B_H;\mathbb{F}_p) \to H^*_K(X;\mathbb{F}_p)$. This map is surjective and commutes with the respective $\mathcal{A}^*$-actions. Let $I = \operatorname{Ker}(\pi^* \otimes J)$; then $H^*_K(X;\mathbb{F}_p) = (H^*(B_K;\mathbb{F}_p) \otimes H^*(B_H;\mathbb{F}_p))/I$ is a Steenrod presentation.

On the other hand, given a Steenrod presentation

$$H^*_K(X;\mathbb{F}_p) = (H^*(B_K;\mathbb{F}_p) \otimes H^*(B_H;\mathbb{F}_p))/I,$$

and $J\colon H^*(B_H;\mathbb{F}_p) \to H^*_K(X;\mathbb{F}_p)$ given by

$$H^*(B_H;\mathbb{F}_p) \ni \xi_H \mapsto 1 \otimes \xi_H \mod I,$$

then J commutes with the $\mathcal{A}^*$-actions and $i^* \circ J = j^*$. □

Let X, X' be spaces such that the rational cohomology rings $H^*(X;\mathbb{Q})$ and $H^*(X';\mathbb{Q})$ are finitely generated as graded $\mathbb{Q}$-algebras. Then we have to define what it means for a homomorphism $\theta\colon H^*(X;\mathbb{Q}) \leftarrow H^*(X';\mathbb{Q})$ to commute with Steenrod powers for almost all primes p. Let $y_1,\dots,y_m \in H^*(X';\mathbb{Q})$ be a set of multiplicative generators; similarly, let $x_1,\dots,x_n \in H^*(X;\mathbb{Q})$ be a set of multiplicative generators. Then $\theta(y_i) = p_i(x_1,\dots,x_n) \in H^*(X;\mathbb{Q})$ are polynomials. Let Prime_θ be the (finite) subset of primes which appear as divisors of the denominators of the coefficients of the p_i. Then for all $p \notin \mathrm{Prime}_\theta$ there are unique homomorphisms $\theta_p, \bar{\theta}_p$ which make the following diagram commute [Adams and Mahmud 1976]:

$$\begin{array}{ccc} H^*(X;\mathbb{Q}) & \xleftarrow{\ \theta\ } & H^*(X';\mathbb{Q}) \\ \uparrow & & \uparrow \\ H^*(X;\mathbb{Z}_{(p)}) & \xleftarrow{\ \theta_p\ } & H^*(X';\mathbb{Z}_{(p)}) \\ \downarrow & & \downarrow \\ H^*(X;\mathbb{F}_p) & \xleftarrow{\ \bar{\theta}_p\ } & H^*(X';\mathbb{F}_p) \end{array}$$

Here the vertical maps are induced by the canonical maps $\mathbb{Z}_{(p)} \to \mathbb{Q}$ and $\mathbb{Z}_{(p)} \to \mathbb{F}_p$ respectively. We say that θ commutes with the Steenrod powers for almost all primes p if the $\bar{\theta}_p$ commute with Steenrod powers for $p \notin \mathrm{Prime}_\theta$.

Definition 2. Let K be a compact Lie group acting on $X = G/H$. Then we say that the rational equivariant cohomology $H^*_K(X;\mathbb{Q})$ is Steenrod presentable if there is a lifting J of the edge homomorphism j^*

$$\begin{array}{ccc} & & H^*_K(X;\mathbb{Q}) \\ & \overset{J}{\nearrow} & \downarrow i^* \\ H^*(B_H;\mathbb{Q}) & \xrightarrow[j^*]{} & H^*(X;\mathbb{Q}) \end{array}$$

such that $\bar{J}_p \colon H^*(B_H;\mathbb{F}_p) \to H^*_K(X;\mathbb{F}_p)$ commutes with Steenrod powers for almost all p.

A homogeneous space G/H such that $\operatorname{rank} G = \operatorname{rank} H$ is Kähler if and only if $H = Z(K)$ is the centralizer of a (not necessarily maximal) torus K, or, equivalently, if H is conjugate to an isotropy group of the adjoint representation [Besse 1987, Chapter 8].

Here is the main theorem of this article.

Theorem 3. *Let G be a simple compact connected Lie group and $H \subset G$ be a closed connected subgroup of maximal rank such that $X = G/H$ is Kähler and let G act topologically on $X = G/H$. Then the following statements are equivalent.*

(i) *The equivariant cohomology $H^*_G(X;\mathbb{Q})$ is Steenrod presentable.*

(ii) *The group G acts transitively on X with an isotropy group conjugate to K, where K is a maximal rank subgroup of G isomorphic to H by an automorphism of G which is inner with the possible exception of the even Spin groups.*

(iii) *There is an isomorphism $H^*_G(X;\mathbb{Q}) \cong R_H$ as R_G-algebras.*

As a corollary, we recover an earlier result from [Hauschild 1985]. (See also [Hauschild 1986] and the introduction of [Hauschild 2006], where the uniqueness problem for locally smooth $\mathrm{SU}(n+1)$-actions on $\mathrm{SU}(n+1)/S(U(n-1)\times U(2))$ is considered.)

Theorem 4. *Let G be a simple compact connected Lie group, and let $T\subset G$ be a maximal torus. Let G act nontrivially on $X=G/T$ via φ. Then up to conjugacy, φ is the standard transitive G-action on X with isotropy group conjugate to T.*

Proof (for a proof using obstruction theory, see the Appendix). Write $H^*(B_T;\mathbb{Q})=\mathbb{Q}[x_1,\dots,x_n]$, $\deg x_i=2$. Let $R_G=H^*(B_G;\mathbb{Q})$ and write

$$H_G^*(X;\mathbb{Q})=\frac{R_G[X_1,\dots,X_n]}{I},\quad \deg X_i=2.$$

Define $J(x_i)=\overline{X}_i$, where the $\overline{X}_i$ is the class of X_i. Let p be a prime such that J_p and $\bar{J}_p$ are defined.

The values of the Steenrod powers $\mathcal{P}^k(x_i)$ and $\mathcal{P}^k(\overline{X}_i)$ are completely determined by the instability conditions, that is, we have $\mathcal{P}^k(x_i)=x_i^p$ for $k=1$ and $\mathcal{P}^k(x_i)=0$ for $k>1$. The same holds in $H_G^*(X;\mathbb{F}_p)$; that is, $\mathcal{P}^k(\overline{X}_i)=\overline{X}_i^p$ for $k=1$ and $\mathcal{P}^k(\overline{X}_i)=0$ for $k>1$. It follows that $\mathcal{P}^k\bar{J}_p(x_i)=\bar{J}_p\mathcal{P}^k(x_i)$ for all i. By simple induction using the Cartan rule, one gets the relation $\mathcal{P}^k\circ\bar{J}_p=\bar{J}_p\circ\mathcal{P}^k$ for all $k\geq 0$ and almost all primes p. So, the equivariant cohomology is Steenrod presentable and the result follows from Theorem 3. □

3. A proof of the main theorem

The following definitions synthesize certain cohomological properties of symplectic manifolds and are taken from the paper [Allday 1998]. We consider cohomology with coefficients in a field $\mathbb{Q}$, with $\operatorname{char}\mathbb{Q}=0$. As a coefficient field of cohomology, the symbol $\mathbb{Q}$ will be omitted in this paragraph.

Definition 5. Let X be a Poincaré duality space over $\mathbb{Q}$ with formal dimension $2n$.

(i) The space X is said to be c-symplectic (that is, cohomologically symplectic) if there is $w\in H^2(X)$ such that $w^n\neq 0$.

(ii) If X is c-symplectic, for $0\leq j\leq n$, consider the map $w^j\colon H^{n-j}(X)\to H^{n+j}(X)$, defined as $a\mapsto w^ja$, for all $a\in H^{n-j}(X)$. Then X is said to satisfy the hard Lefschetz condition if w^j is an isomorphism for all j. In this case X is also said to be *c*-Kähler.

Let X be a c-symplectic space with $w\in H^2(X)$ as in the definition above. Let G be a compact connected Lie group acting on X. Then $g^*(w)=w$ for all $g\in G$. In this way any action of a compact connected Lie group on a c-symplectic space is considered to be a cohomologically symplectic action.

Definition 6. Let X be a c-symplectic space with c-symplectic class $w \in H^2(X)$. Let a torus G act on X. Then the action is said to be cohomologically Hamiltonian (c-Hamiltonian) if $w \in \mathrm{Im}\{i^*\colon H^2_G(X) \to H^2(X)\}$, where $i\colon X \to X_G$ is the inclusion of the fiber in the bundle $X_G \to B_G$.

The main reason we have restricted ourselves to homogeneous spaces G/H with the Kähler property is the following result, which can be considered a generalization of a theorem of Atiyah [1983] (see also [Guillemin and Sternberg 1982; Audin 1991, Corollary 4.2.3]). For the definition of uniformity see [Allday and Puppe 1993, Definition 3.6.17]. For other consequences of the Kähler property, see [Allday et al. 2002].

Theorem 7 [Allday 1998]. *Let the r-torus $G = T^r$ act on a closed c-symplectic manifold X in an effective, uniform, c-Hamiltonian way. Then X^G has at least $r+1$ connected components.*

The conditions of the theorem are always satisfied if X is totally nonhomologous to zero in the Borel fibration [Allday and Puppe 1993]. Let G be a torus and suppose G is acting on a c-symplectic manifold X with vanishing odd cohomology. As we have seen before, the equivariant cohomology can be written as $H^*_G(X) = R_G[h_1, \dots, h_n]/I$ where $R_G = H^*(B_G)$ and the $h_1, \dots, h_n$ is a system of homogeneous multiplicative generators, I the defining ideal. Let $X^G = F_1 + F_2 + \cdots + F_s$ be the decomposition of the fixed space X^G into its connected components. Then for every α, $1 \le \alpha \le s$, we choose a point $p_\alpha \in F_\alpha$ and define a prime ideal P_α as the kernel of the composed homomorphism $R_G[h_1, \dots, h_n] \to H^*_G(X) \to H^*_G(p_\alpha) \cong R_G$. Here the first homomorphism is the natural projection and the second is given by restricting equivariant cohomology classes to $E_G \times_G \{p_\alpha\}$. Then the radical of I is given by $\sqrt{I} = \bigcap_\alpha P_\alpha$. Moreover there is a natural bijection between the primary components of the ideal I and the connected components of X^G. For more details on these standard facts on equivariant cohomology see [Allday and Puppe 1993; Hsiang 1975]. The following lemma is an immediate consequence of the result of Allday.

Lemma 8. *Let the r-torus $G = T^r$ act on a closed c-symplectic manifold X with vanishing odd cohomology. Suppose G is acting on X in an effective, uniform, c-Hamiltonian way. Then there exists a connected component F of X^G such that the prime ideal $P \subset R[h_1, \dots, h_n]$ belonging to F is of the kind $P = (h_1 - \beta_1, \dots, h_n - \beta_n)$ with $\beta_i \in R^{\deg h_i}$ and some $\beta_i \neq 0$.*

Proof of the main theorem. (i) $\Rightarrow$ (ii): Let $R_G = H^*(B_G)$ and let $R_H = H^*(B_H) \cong \mathbb{Q}[h_1, \dots, h_n]$. Suppose $H^*_G(X) = (R_G \otimes_{\mathbb{Q}} R_H)/I_G$ to be a Steenrod presentation. Let $T \subset G$ be a maximal torus; then the equivariant cohomology of the induced T-action is given by $H^*_T(X) \cong H^*_G(X) \otimes_{R_G} R_T$. Let $I_T \subset R_G \otimes_{\mathbb{Q}} R_T$ be the ideal generated by I_G, that is, $I_T = I_G \cdot (R_T \otimes_{\mathbb{Q}} R_H)$; then $H_T^*(X) \cong R_T[h_1, \dots, h_n]/I_T$. By the previous lemma there is a connected component $F \subset X^T$ of the fixed set X^T such that the corresponding prime ideal has the form $P = (h_1 - \beta_1, \dots, h_n - \beta_n)$ with $(\beta_1, \dots, \beta_n) \neq 0$. In particular, the restriction homomorphism $H_T^*(X) \to H_T^*(\{p\}) \cong R_T$, $p \in F$ is nontrivial. Let $G_p \subset G$ be the isotropy group of p. It follows from the commutativity of the diagram

$$\begin{array}{ccc} H_T^*(X) & \longrightarrow & H_T^*(\{p\}) \cong R_T \\ \cup\uparrow & & \cup\uparrow \\ H_G^*(X) & \underset{\mathrm{res}_p}{\longrightarrow} & H_G^*(G(p)) \cong R_{G_p} \end{array}$$

that the restriction homomorphism

$$\mathrm{res}_p \colon H_G^*(X) \to H_G^*(G(p)) \cong R_{G_p}$$

must also be nontrivial. Let $U = G_p^o$ be the connected component of the unit element in G_p, and let $\eta \colon H^*(B_{G_p}) \to H^*(B_U)$ be the homomorphism induced by the inclusion $U \subset G_p$. Then consider the composition $\theta = \eta \circ \mathrm{res}_p \circ J$

$$\theta \colon H^*(B_H) \xrightarrow{J} H_G^*(X) \xrightarrow{\mathrm{res}_p} H^*(B_{G_p}) \xrightarrow{\eta} H^*(B_U).$$

It follows from the construction and the hypothesis that θ commutes with the Steenrod powers in $\mathscr{A}^*$ for almost all primes p. Let LT be the Lie algebra of the maximal torus T. Let $\Sigma \subset LT$ be the kernel of the projection $LT \to T$. After [Adams and Mahmud 1976, Theorem 1.5] there is an $\mathbb{R}$-linear map $\phi \colon LT \to LT$ carrying $\Sigma \otimes \mathbb{Q}$ into $\Sigma \otimes \mathbb{Q}$ such that the following diagram is commutative.

$$\begin{array}{ccc} H^*(B_H) & \xrightarrow{\theta} & H^*(B_U) \\ \downarrow & & \downarrow \\ H^*(B_T) & \xrightarrow{\phi^*} & H^*(B_T) \end{array}$$

Here ϕ^* is the graded ring homomorphism induced by the linear map ϕ. The existence of this map is a consequence of [Adams and Mahmud 1976, Lemma 1.2]. The vertical maps are the homomorphisms induced by the standard fibrations $B_T \to B_H$ and $B_T \to B_U$. It follows from our assumption that θ is nontrivial, which implies that ϕ^* is also nontrivial. Observe that the map θ induces exactly the homomorphism $\bar{\theta} \colon H^*(G/H) \to H^*(G/U)$ induced by the map $G/U \to G/G_p \cong$

$G(p) \subset X = G/H$. This means that we have a commutative diagram

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\bar{\theta}} & H^*(G/U) \\ \uparrow & & \uparrow \\ H^*(B_H) & \xrightarrow{\theta} & H^*(B_U) \end{array}$$

where the vertical maps are the edge homomorphisms for the fibrations $B_H \to B_G$ and $B_U \to B_G$, respectively. It follows that θ sends the ideal

$$H^*_+(B_G) \cdot H^*(B_H) \subset H^*(B_H)$$

generated by the invariants of the Weyl group in $H^*(B_H)$ into the ideal

$$H^*_+(B_G) \cdot H^*(B_U) \subset H^*(B_U)$$

generated by the same invariants in $H^*(B_U)$. Then ϕ^* sends the ideal

$$H^*_+(B_G) \cdot H^*(B_T) \subset H^*(B_T)$$

into the ideal

$$H^*_+(B_G) \cdot H^*(B_T) \subset H^*(B_T),$$

therefore inducing a graded and nontrivial homomorphism

$$\overline{\phi^*} \colon H^*(G/T) \longrightarrow H^*(G/T).$$

Since G is a simple Lie group we can apply [Hauschild 1985, Lemma 4.1]. Therefore $\overline{\phi^*}$ must be a surjective map and consequently must be an isomorphism. Now the commutative diagrams above induce a commutative diagram

$$\begin{array}{ccc} H^*(G/H) & \xrightarrow{\bar{\theta}} & H^*(G/U) \\ \downarrow & & \downarrow \\ H^*(G/T) & \xrightarrow{\overline{\phi^*}} & H^*(G/T) \end{array}$$

where the vertical maps are the respective inclusions of invariants under the Weyl groups WH, WU respectively. It follows that the homomorphism $\bar{\theta}$ must be injective which implies dimensions $cd_{\mathbb{Q}}(X) \leq cd_{\mathbb{Q}}(G/U)$ for the respective rational cohomology. But G/H and G/U are closed oriented manifolds and therefore $\dim X \leq \dim G/U$, which implies $\dim X = \dim G(p)$. It follows that $X = G/G_p$, that is, the action is transitive. Now X is 1-connected and therefore G_p must be connected, that is, $G_p = G^o_p = U$. It follows that $G/H = G/U$ and $\bar{\theta}$ is an isomorphism. By a theorem of Papadima [1986], the isomorphism $\overline{\phi^*}$ is induced by an automorphism of the root system of G. This implies that the root systems of the maximal rank subgroups H and U are conjugate by such an automorphism, and consequently, the groups H and U are conjugate by an automorphism which is inner with the possible exception of the Spin groups.

(ii) $\Rightarrow$ (i): We have $X_G = E_G \times_G G/U \cong E_G/U = B_U$. But $H \cong U$ and so $X_G \cong B_H$ and therefore $H^*_G(X) \cong R_H$ as R_G-algebras.

(iii) $\Rightarrow$ (i): Take $J = \mathrm{Id}\colon R_H \to R_H$. $\square$

Appendix

Proof of Theorem 4 using obstruction theory. Let $\pi : X_G \to B_G$ be the projection, let $b \in B_G$, and let $X_b = \pi^{-1}(b) \subset X_G$ be the fiber over b. Let $i_b\colon X_b \to X_G$ be the corresponding inclusion. Then consider the extension problem

$$\begin{array}{ccc} X_b & \xrightarrow{\;j\;} & B_T \\ {\scriptstyle i_b}\big\downarrow & \nearrow{\scriptstyle j'} & \\ X_G & & \end{array}$$

The obstruction to extend the inclusion $j\colon X_b \to B_T$ to a map $j'\colon X_G \to B_T$ is to be found in the group $H^3(X_G, X_b; \pi_2(B_T))$. Consider the following piece of the long exact cohomology sequence of the pair (X_G, X_b).

$$H^2(X_G;\mathbb{Z}) \to H^2(X_b;\mathbb{Z}) \to H^3(X_G, X_b;\mathbb{Z}) \to H^3(X_G,\mathbb{Z}) \to \dots$$

Now the first arrow, induced by the inclusion of the fiber, is surjective whereas $H^3(X_G;\mathbb{Z}) = 0$. It follows $H^3(X_G, X_b;\mathbb{Z}) = 0$ and so $H^3(X_G, X_b;\mathbb{Z}^n) = 0$. We thus have a lifting $J = j'^*$ which gives rise to the commutative diagram

$$\begin{array}{ccc} & & H^*_G(X;\mathbb{Z}) \\ & \overset{J}{\nearrow} & \big\downarrow{\scriptstyle i_b^*} \\ H^*(B_T;\mathbb{Z}) & \xrightarrow[j^*]{} & H^*(X;\mathbb{Z}). \end{array}$$

By the definition of J as a map induced geometrically, we conclude that $H^*_G(X;\mathbb{Q})$ is Steenrod presentable. Using the equivalence between (i) and (ii) in Theorem 3 and the standard fact that two maximal tori are conjugate, the result follows. $\square$

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HORIZONTAL HEEGAARD SPLITTINGS OF SEIFERT FIBERED SPACES

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We show that if an orientable Seifert fibered space M with an orientable genus g base space admits a strongly irreducible horizontal Heegaard splitting, then there is a one-to-one correspondence between isotopy classes of strongly irreducible horizontal Heegaard splittings and elements of $\mathbb{Z}^{2g}$. This correspondence is determined by the slopes of intersection of each Heegaard splitting with a set of $2g$ incompressible tori in M. We also show there are Seifert fibered spaces with infinitely many nonisotopic Heegaard splittings that determine Nielsen equivalent generating systems for the fundamental group of M.

1. Introduction

Certain closed Seifert fibered spaces are known to admit a type of Heegaard splitting called a horizontal Heegaard splitting. Bachman and Derby-Talbot [2006] showed that any Seifert fibered space that admits a strongly irreducible horizontal splitting admits infinitely many isotopy classes of horizontal splittings. We improve their analysis to show the following:

Let M be an orientable Seifert fibered space with base space an orientable genus g surface, and let $T_1, \dots, T_{2g}$ be vertical tori in M such that $T_i \cap T_j$ is a single loop for i odd and $j = i + 1$ (or vice versa), and empty otherwise. The complement in M of a regular neighborhood of these tori is a Seifert fibered space over a g-times punctured sphere.

Theorem 1. *If M admits a strongly irreducible horizontal Heegaard splitting and M is not a circle bundle, then for every $2g$-tuple of integers $(s_1, \dots, s_{2g}) \in \mathbb{Z}^{2g}$, there is a unique (up to isotopy) strongly irreducible, horizontal Heegaard splitting that intersects each T_i in a family of essential loops with slope s_i. Moreover, Heegaard splittings that define distinct $2g$-tuples of slopes are not isotopic.*

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A *Heegaard splitting* for a compact, closed, orientable 3-manifold M is a triple (Σ, H_1, H_2), where $\Sigma \subset M$ is a compact, closed, two-sided surface and the subsets H_1 and H_2 of M are handlebodies (that is, homeomorphic copies of closed regular neighborhoods of connected, finite graphs in S^3) with $\partial H_1 = \Sigma = \partial H_2$ and $H_1 \cup H_2 = M$.

A Heegaard splitting (Σ, H_1, H_2) is *strongly irreducible* if every essential, properly embedded disk in H_1 intersects every essential, properly embedded disk in H_2. We will describe the construction of a horizontal Heegaard splitting in Section 3.

Given a Heegaard splitting (Σ, H_1, H_2) of M, there is a smooth function f from M to the interval [0, 1] such that the preimage of each point in (0, 1) is a surface isotopic to Σ and the preimages of $\{0\}$ and $\{1\}$ are graphs (called *spines*) in H_1 and H_2. Such a function is called a *sweep-out* [Johnson 2005] and the restriction of f to a vertical torus in M is (generically) a Morse function. A Morse function on a torus always has level sets that are essential in the torus. Level sets of a Morse function are pairwise disjoint and disjoint essential loops in a torus are parallel, so f determines a unique isotopy class of simple closed curves in the torus.

We will describe below how a simple closed curve in a vertical torus determines a rational number called its slope. Different sweep-outs will restrict to different Morse functions on T, so a Heegaard splitting may determine more than one slope. We will show that in many cases if two sweep-outs come from the same Heegaard splitting, then they will determine the same slope on the vertical torus. In particular, for M a Seifert fibered space with orientable base space and $T_1, \dots, T_{2g}$ vertical tori in M as above, we show the following:

Lemma 2. *If a strongly irreducible Heegaard splitting of a Seifert fibered space M determines more than one slope in a vertical torus T_i, then M is a circle bundle.*

This is proved in Section 3, based on techniques developed in Section 2, and shows one direction of Theorem 1. The other direction follows from the construction of horizontal Heegaard splittings and is also proved in Section 3.

Weidmann has shown, in the appendix of [Bachman and Derby-Talbot 2006], that every circle bundle contains a unique irreducible Heegaard splitting (up to isotopy). The only circle bundles with strongly irreducible Heegaard splittings are circle bundles over the circle (all of which are lens spaces) and the circle bundle over the torus with Euler number one.

In Sections 4 and 5, we consider the generating set for the fundamental group of M. Two generating sets are called *Nielsen equivalent* if one can be changed to the other by a finite number of type-one Tietze moves (that is, by replacing the i-th generator with its inverse or with the product of the i-th and the j-th generator for some $i \neq j$).

The fundamental group of each handlebody in a Heegaard splitting is a free group. The inclusion of its fundamental group into $\pi_1(M)$ determines a generating set for $\pi_1(M)$. If the handlebodies of two Heegaard splittings determine generating sets for $\pi_1(M)$ that are not Nielsen equivalent, then these two splittings can not be isotopic. Lustig and Moriah have used Nielsen equivalence to distinguish vertical Heegaard splittings of Seifert fibered spaces [1991] as well as Heegaard splittings of certain hyperbolic 3-manifolds [1997]. We show that, unfortunately, Nielsen class does not always distinguish nonisotopic Heegaard splittings. In particular, we describe in Section 5 a family of Seifert fibered space over the torus with two singular fibers such that each admits infinitely many nonisotopic Heegaard splittings whose handlebodies determine Nielsen equivalent generating sets in $\pi_1(M)$.

2. Toroidal summands

Let M be a compact, closed, orientable, irreducible 3-manifold (not necessarily a Seifert fibered space), and let $N \subset M$ be a submanifold homeomorphic to $T \times S^1$, where T is a once punctured torus. Assume ∂N is incompressible in M. (If ∂N is compressible in M, then it compresses to a sphere in the complement of N, so because M is irreducible, M must be a solid torus.)

Two canonical simple closed curves in ∂N are picked out by the topology: a meridian μ that is the boundary of an incompressible torus $T \times \{y\}$ in $T \times S^1$ (for some $y \in S^1$) and a longitude λ that is the slope of a vertical loop $\{x\} \times S^1$ $(y \in \partial T)$. The meridian μ is the unique (up to isotopy) loop in ∂N that is homology trivial in N, so it is determined independently of the product structure on N. Every essential annulus properly embedded in N has boundary parallel to λ, so this loop is also independent of the product structure. Any simple closed curve in ∂N is a sum $p\mu + q\lambda$, and thus determines a fraction $p/q \in \mathbb{Q} \cup \{1/0\}$, called its *slope*.

For any essential, simple closed curve ℓ in T, the subset $\ell \times S^1 \subset T \times S^1$ is a nonseparating incompressible torus in M. We can define slopes $\mu' = \ell \times \{y\}$ for $y \in S^1$ and $\lambda' = \{x\} \times S^1$ for $x \in \ell$, so again each loop in $\ell \times S^1$ determines a slope p/q. In this case, the loop μ' is determined by the product structure of N, not the topology alone. A different product structure will imply a different μ'. For our purposes, it suffices to fix a product structure on N, since we will always be dealing with these slopes in a relative way.

Let (Σ, H_1, H_2) be a Heegaard splitting for M. Let $f : M \to [0, 1]$ be a sweep-out such that each level surface of f is isotopic to Σ. Let $S = \ell \times S^1$ be a vertical torus in N. After an arbitrarily small isotopy of f, the restriction of f to S will be a Morse function. As mentioned above, a Morse function on a torus always has an essential level set and the essential levels define a single isotopy class of simple closed curves. Thus f determines a unique slope in S.

We will say that Σ *determines* a slope p/q on S if there is a sweep-out f with level sets isotopic to Σ such that the restriction of f to S has a level set in S with slope p/q. As noted above, a Heegaard splitting may determine more than one slope. If the intersection of Σ with S contains an essential loop of slope p/q, then f can be chosen so that Σ is a level set of f (rather than just isotopic to one), so Σ determines the slope p/q. Conversely, for any sweep-out for Σ, each level surface is isotopic to Σ. Thus Σ determines a slope p/q in S if and only if Σ can be isotoped so that the intersection contains a loop with that slope.

Lemma 3. *If a strongly irreducible genus g Heegaard splitting (Σ, H_1, H_2) for M determines more than one slope in a vertical torus S in N, then Σ can be isotoped so that the closure of $\Sigma \setminus N$ in the closure of $M \setminus N$ is a properly embedded incompressible genus $g-3$ surface whose boundary is a pair of loops in ∂N, each with slope* 1 *or* -1.

Before we begin the proof, recall that a smooth function f is *Morse* if every critical point is nondegenerate and no two critical points are in the same level. A function is *near-Morse* if either all but one of its critical points are nondegenerate and all are in distinct levels, or all its critical points are nondegenerate and all but two are in distinct levels.

Proof. If (Σ, H_1, H_2) determines more than one slope in S, then there are sweep-outs f and f' such that Σ is isotopic to both a level surface of f and of f' and such that the essential level sets of $f|_S$ and $f'|_S$ determine different slopes. Because f and f' are sweep-outs for the same Heegaard splitting, there is an isotopy of M taking a level surface of f' to a level surface of f. In particular, there is a family of sweep-outs $\{f_t \mid t \in [0, 1]\}$ such that f_0 determines the same slope in S as f' and f_1 determines the same slope in S as f.

Assume the family of sweep-outs is generic with respect to S, that is, that $f_t|_S$ is Morse for all but finitely many values of t. At the finitely many non-Morse values, the restriction will fail to be Morse because either two critical points pass through the same level, as in Figure 1, or there is a single degenerate critical point. For any value t_0 such that $f_{t_0}|_S$ is a Morse function, there is a neighborhood of t_0 in $[0, 1]$ such that for any t in this neighborhood, $f_t|_S$ is isotopic (in S) to $f_{t_0}|_S$. Thus the slope of the essential levels can only change at the near-Morse values of t.

If two essential loops in a torus are disjoint, then they are parallel, and thus define the same slope. Thus if the essential slope changes at a near-Morse value t_0, then the regular levels of $f_{t_0}|_S$ must all be trivial in S. This is the case if and only if each component of the complement of the critical levels is contained in an open disk in S. If $f_{t_0}|_S$ is a near-Morse function with a degenerate critical point (but its critical points are in distinct levels), then the complement of the critical levels must still contain an essential level loop. The only type of intermediate function

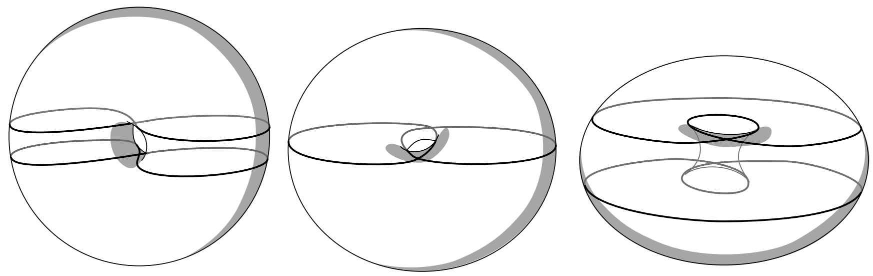

Figure 1. The slope of the level loops in a Morse function on a torus changes when two saddle singularities pass through the same level. The surface is embedded so that the Morse function is a height function.

that does not contain an essential regular level is one in which $f_{t_0}|_S$ has two saddles at the same level, and this level set cuts S into disks. This is shown in Figure 1. Thus if the slope changes, there must be such an intermediate function.

The critical level containing the two saddle singularities is a graph with two valence four vertices and thus four edges. There are exactly two (homeomorphism classes of) connected graphs with four edges and two valence four vertices: Let Γ_0 be a two-vertex graph in which two edges pass between the two vertices and one edge goes from each vertex back to itself. Let Γ_1 be a two-vertex graph in which each edge goes from one vertex to the other.

Let Γ be a critical level set of a near-Morse function on an oriented surface S such that Γ is homeomorphic to Γ_0 or Γ_1. Given an orientation for an edge of Γ, the orientation of S defines a transverse orientation. Choose an orientation for each edge so that the transverse orientation points in the direction in which the near-Morse function is increasing. The embedding of Γ suggests a cyclic ordering of the ends of the edges that enter each vertex. Because each vertex is at a saddle singularity, the edges must alternate whether they point towards the vertex or away.

If Γ is homeomorphic to Γ_0, then for each edge that passes from a vertex to itself, one end points towards the vertex and the other away. Thus the ends of each such edge are adjacent in the cyclic ordering around the vertex. This implies that a regular neighborhood in S of Γ is a planar subsurface. If S is a torus, then the complement of Γ must contain a component that is not contained in a disk in S.

Thus if the slope defined by the Morse function changes, the level containing two saddles must be homeomorphic to Γ_1. There is a unique (up to homeomorphism) way that such a graph can be embedded in a torus so that its complement is a collection of disks. This is shown at the bottom left of Figure 2. The top left picture shows the intersection of this level set with a square whose sides are glued

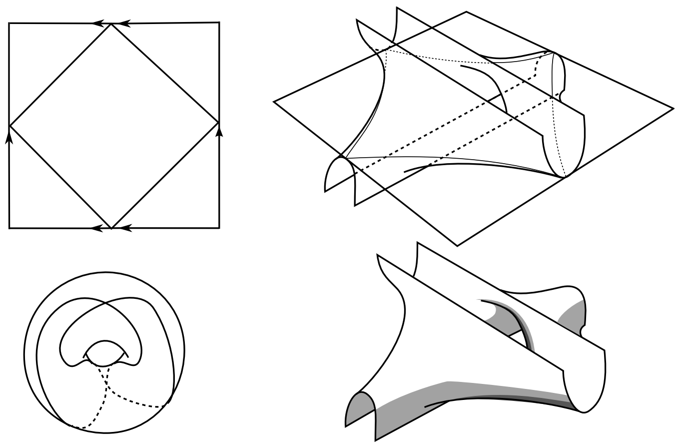

Figure 2. Extending a level surface of f_{t_0} locally from a critical level in S with two saddles produces a sphere with four punctures.

to construct the torus, chosen so that the two vertices of Γ are in the two edges of the glued square.

Let R be a regular neighborhood of S, and let F be the level surface of f_{t_0} that contains the critical level. The surface F intersects R as shown on the right of Figure 2. Because of the identifications at the edges of the square, this intersection is a sphere with four punctures, which we will call U, and a (possibly empty) collection of annuli. The boundary loops of U (and thus the boundary loops of $F \setminus R$) determine slopes in S that intersect at one point. Note that F is isotopic to Σ because it is a level surface of a sweep-out for (Σ, H_1, H_2).

Claim. *The intersection $F \cap S$ consists of the graph $U \cap S$ and a (possibly empty) collection of loops that are trivial in both S and F.*

Proof. It suffices to show that the curves in $F \cap S$ other than $U \cap S$ are trivial in F. Let g be the restriction of the sweep-out f_{t_0} to the surface S. Each level set of g is the intersection of S with a level surface of f_{t_0}. There is a canonical way (up to isotopy) to identify this level surface with Σ, so each loop component of each level set of g determines an isotopy class of simple closed curves in Σ. At a central singularity in g, a loop corresponding to a trivial loop in Σ is added or removed. At a level where there is a single saddle singularity in g, one loop is turned into two, or vice versa by a band summing operation.

For t near 0, these simple closed curves bound disks in H_1, and near 1 they bound disks in H_2. For any regular level of g, consisting of a number of simple closed curves, the corresponding isotopy classes of loops in Σ are pairwise disjoint. Because Σ is strongly irreducible, a fixed level set of g cannot determine essential loops in Σ bounding disks on both sides. Every regular level of g contains a trivial loop in S, so one of the loops in Σ determined by this regular level bounds a disk in H_1 or H_2.

The disks cannot switch from one side to the other at a critical level with a single saddle since the loops in Σ before and after the band summing are disjoint. Thus the switch occurs at the single level of g with two saddle singularities. In particular, the loops that limit onto this level set bound disks in opposite handlebodies (though these disks are not disjoint), so the remaining loops of intersection must be trivial in F. □

To complete the proof, isotope F so as to remove any loops that are trivial in both F and S. Bachman and Derby-Talbot [2006] pointed out that after these trivial loops are removed, $S \setminus F$ is a pair of compressing disks for F whose boundaries, when made transverse, intersect at four points. These compressing disks are on opposite sides of F and are contained in the regular neighborhood R. Any compressing disk for $F \setminus R$ is disjoint from each of the disks in $S \setminus F$. Because (Σ, H_1, H_2) is strongly irreducible and F is isotopic to Σ, the surface $\Sigma \setminus R$ must be incompressible in $M \setminus R$.

The manifold $N \setminus R$ is homeomorphic to a pair of pants cross S^1. Any incompressible surface in $N \setminus R$ is one of the following forms: a vertical torus or annulus isotopic to an essential loop or arc cross S^1. A horizontal pair of pants is a properly embedded surface that intersects each vertical S^1 transversely at a single point. The surface $F \cap (N \setminus R)$ has boundary, so it is not a vertical torus and must consist of some number of horizontal pairs of pants. The pairs of pants intersect R in the loops ∂U. As noted above, these loops have slopes in the boundary of the closure of R that, when projected into S, intersect at one point.

The first homology group of $N \setminus R$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}^2$, where the first $\mathbb{Z}$ is generated by the S^1 factor of the pair of pants cross a circle. The three boundary loops of a component of $F \cap (N \setminus R)$ bound a pair of pants, so the sum of the homology elements they generate is zero. The first coordinates of the two loops in ∂R differ by exactly one (since they intersect at a single point in S), so the first coordinate of the third loop must be 1 or -1. In other words, the third cuff of each pair of pants must have slope 1 or -1 in ∂N.

The surface $F \cap N$ is the union of a four times punctured sphere $F \cap R$ and two pairs of pants $F \cap (N \setminus R)$, so $F \cap N$ is a twice punctured genus two surface. Thus $F \setminus N$ is an incompressible, twice punctured genus $g - 3$ surface whose boundary has slope 1 or -1 in ∂N. □

3. Seifert fibered spaces

Let M be a Seifert fibered space, and let $c \subset M$ be a critical fiber. The complement in M of a regular neighborhood U of c is a surface bundle. Let F be a leaf of this bundle, and assume that ∂F consists of a single loop in ∂U that is a longitude of the solid torus U. Let Σ be union of two disjoint leaves parallel to F and an annulus in U connecting the boundaries of these leaves. Each component of the complement in M of Σ is homeomorphic to the union of $F \times (0, 1)$ and a regular neighborhood of $\partial F \times (0, 1)$.

Because F is a surface with boundary, this set is a handlebody, so Σ determines a Heegaard splitting (Σ, H_1, H_2). A Heegaard splitting constructed in this way is called a *horizontal Heegaard splitting*. Recall the collection $\{T_i\}$ of vertical tori defined in Section 1. The slope that (Σ, H_1, H_2) determines on each T_i is precisely the slope of intersection between F and T_i.

Sedgwick [1999] showed that a horizontal Heegaard splitting is irreducible if and only if the multiplicity of c is greater than the least common multiple of the multiplicities of the other critical fibers. In particular, if Σ is strongly irreducible, then the winding number of c must be the largest over all the critical fibers in M. Thus if (Σ', H_1', H_2') is a second strongly irreducible horizontal Heegaard splitting of M, then Σ' is constructed starting from the same fiber c. The incompressible surface F is uniquely determined (up to isotopy) by the slopes of intersection between F and each T_i. Thus if Σ and Σ' determine the same slope in each T_i, then they were constructed from the same c and F, and are therefore isotopic.

Proof of Lemma 2. The discussion above shows that if two strongly irreducible, horizontal Heegaard splittings determine the same slope with each T_i, then they are isotopic. We will prove the converse. Without loss of generality, assume i is odd, so that $T_i \cap T_{i+1}$ is a single simple closed curve.

A regular neighborhood N of $T_i \cup T_{i+1}$ is homeomorphic to a punctured torus cross a circle. Because T_i and T_{i+1} are each isotopic to a union of regular fibers in M, we can assume that N is also a union of regular fibers. The complement in M of N is a Seifert fibered space, so every incompressible surface in $M \setminus N$ is either a vertical torus or a horizontal incompressible surface. The only one of these surfaces that has boundary in ∂N is a horizontal surface.

Assume for contradiction that Σ determines more than one slope in T_i. Then by Lemma 3 there is an incompressible surface F in the complement of N that intersects the boundary in two parallel loops with slope ± 1. A horizontal incompressible surface in a Seifert fibered space is nonseparating, so F (which is separating) must be a union of two horizontal surfaces. The complement $M \setminus N$ is a Seifert fibered space whose fibers in ∂M match the fibers in N, and thus have slope ∞ in ∂N.

Each boundary component of F has slope ± 1, so each regular fiber of the fibrations intersects each component of F at a single point. The number of intersections of a singular fiber with a horizontal surface is a proper integral fraction of the number of intersections with the nearby regular fibers, so $M \setminus N$ contains no singular fibers. This implies that N contains no singular fibers, so M must be a circle bundle. □

Proof of Theorem 1. By Lemma 2, a strongly irreducible horizontal Heegaard splitting is uniquely determined by the $2g$-tuple of slopes it determines with the incompressible tori $\{T_i\}$. To show a one-to-one correspondence to $\mathbb{Z}^{2g}$, we need only show that if M admits a strongly irreducible horizontal Heegaard splitting, then for any $2g$-tuple there is a strongly irreducible horizontal Heegaard splitting that determines this $2g$-tuple of slopes.

If M has a strongly irreducible, horizontal Heegaard splitting (Σ, H_1, H_2), then Σ was constructed from some critical fiber c and an incompressible surface F in the complement of c that intersects each regular fiber at two points. The critical fiber c can always be taken to be disjoint from each T_i.

Given positive integers n and $i \leq g$, consider n parallel copies T_{2i}. Because F intersects each regular fiber at two points and the torus T_{2i} is a union of regular fibers, the surface F will intersect each copy T_{2i}^j of T_{2i} in two simple closed curves. Let U be a regular neighborhood of a component of $F \cap T_{2i}^j$. The intersection of $F \cup T_{2i}^j$ with U is the union of a pair of annuli that intersect in a common essential loop. There are two ways to replace these two intersecting annuli with two disjoint annuli. If we make this replacement in the same way in each neighborhood, the resulting surface will have slope either n or $-n$ in T_{2i+1}. For every other T_j, the slopes of $F \cap T_j$ and $F' \cap T_j$ agree. (This operation is called a *Haken sum*.) We say that the surface with slope n is the result of *spinning* F around T_{2i} n times.

Similarly, spinning F around T_{2i+1} changes its slope with T_{2i} but not with the other vertical tori. Thus by spinning F around the vertical tori, one can construct a horizontal surface F' that intersects the vertical tori $\{T_i\}$ in any $2g$-tuple. This F' has the same boundary as F in $\partial N(c)$, so $F' \cup A$ is a horizontal Heegaard surface Σ' for M. There are two ways to see that Σ' is a strongly irreducible, horizontal Heegaard surface. First, the reader can check that there is a homeomorphism from M to itself taking Σ onto Σ'. Second, both Heegaard splittings are constructed from the same critical fiber in M, so by Sedgwick's results [1999], both are strongly irreducible. The Heegaard surface Σ' determines the same $2g$-tuple of integers as F', so for each $2g$-tuple of integers, there is a strongly irreducible, horizontal Heegaard splitting whose slopes in $\{T_i\}$ realize those values. □

4. Double primitive knots

Here, we will construct a family of 3-manifolds with infinitely many nonisotopic genus three Heegaard splittings. In the next section, we will show that for certain Seifert fibered spaces over the torus with two critical fibers, the Heegaard splittings all determine the same Nielsen classes of generators for the fundamental group.

Let X be a compact, closed, orientable, irreducible 3-manifold with a genus two Heegaard splitting (Σ', H_1', H_2'), and let $\ell \subset \Sigma'$ be a simple closed curve such that ℓ intersects some essential, properly embedded disk $D_i \subset H_i'$ at a single point for $i = 1, 2$. Such a loop is called *double primitive*. (A knot in S^3 that is isotopic to a double primitive loop in a genus two Heegaard splitting is called a *Berge knot*.)

Let $Y \subset X$ be a regular neighborhood of a double primitive loop $\ell \subset \Sigma'$. The intersection $\partial Y \cap \Sigma'$ is a pair of loops ℓ_1' and ℓ_2' in the torus ∂Y. Define $N = T \times S^1$, where T is a once punctured torus, and let x_1 and x_2 be points in S^1. Let M be the result of gluing $X \setminus Y$ to N by a map that sends ℓ_i' to $\partial T \times \{x_i\}$ for $i = 1, 2$.

Lemma 4. *$\Sigma = \Sigma' \cup (T \times \{x_1, x_2\})$ is a genus three Heegaard surface for M.*

Proof. The complement in N of $(T \times \{x_1, x_2\})$ consists of two components, each of whose closure is a genus two handlebody $T \times I$, where $I \subset S^1$ is one of the two intervals with endpoints x_1 and x_2. Let N_1 and N_2 be these handlebodies. Then each of $N_1 \cap \partial N$ and $N_2 \cap \partial N$ is an annulus.

In M, the complement $M_1 = H_1' \setminus Y$ is a handlebody. Because ℓ is double primitive, the intersection of M_1 with the closure of Y is an annular neighborhood A of a loop in ∂M_1 that intersects some properly embedded, essential disk $D \subset M_1$ at a single point. A closed regular neighborhood U in M_1 of $A \cup D$ is a solid torus such that U intersects the closure of $M_1 \setminus U$ in a disk. In M, the set U is a regular neighborhood of an annulus in the boundary of N_1. Thus $N_1 \cup U$ is a handlebody.

The set $M_1 \cup N_1$ is the union of the closure of $M_1 \setminus U$ (which is a handlebody) and the handlebody $N_1 \cup U$. The two handlebodies intersect in a disk, so their union is a handlebody H_1. A similar argument for M_2 implies that $N_2 \cup M_2$ is a handlebody H_2. Thus $\Sigma = \partial H_1 = \partial H_2$ is a Heegaard surface for M. □

The Heegaard surface $\Sigma = \Sigma' \cup (T \times \{x_1, x_2\})$ determines a Heegaard splitting (Σ, H_1, H_2) such that $H_1' \subset H_1$ and $H_2' \subset H_2$. Lemma 4 requires that for $s \in S^1$, $\partial T \times \{s\}$ is sent to the same slope in ∂Y as ℓ_1', but there is no requirement for the slope that a loop $\{t\} \times S^1$ (where $t \in \partial T$) is glued to. Thus there are infinitely many gluings that will produce a manifold with a genus three Heegaard splitting.

Lemma 5. *If (Σ, H_1, H_2) is weakly reducible, then $X \setminus Y$ is a solid torus.*

Proof. Because M has Heegaard genus at most three, it cannot be a connect sum of T^3 with a nontrivial manifold. If ∂N is compressible, then it compresses down to a sphere, which must bound a ball. Thus if ∂N is compressible, $X \setminus Y$ is a solid torus. We will therefore assume that ∂N is incompressible.

Assume Σ be weakly reducible. By Casson and Gordon's theorem [1987], if (Σ, H_1, H_2) is weakly reducible, Σ is reducible or compresses to a separating incompressible surface S in M. In the second case, each component of the complement of S has a Heegaard splitting that comes from compressing Σ.

The 3-manifold M does not admit a genus two Heegaard splitting because by the main theorem of [Kobayashi 1984], if a closed 3-manifold M contains a separating incompressible torus and a genus two Heegaard splitting, then each piece of the complement is either a Seifert fibered space over a disk, an annulus or a Möbius band, the complement of a (1, 1) knot in a lens space, or the complement of a two-bridge knot in S^3. (In fact, the theorem is much stronger than this, but that's all we need.) The component N is not one of these three types, so M does not admit a genus two Heegaard splitting. Because Σ is not reducible, the weak reduction must determine a separating incompressible surface $S \subset M$.

Because S is the result of compressing the genus three surface Σ at least twice, S must consist of one, two or three tori. Because each component of $M \setminus S$ has a Heegaard splitting that comes from compressing the genus three surface Σ, each component of $M \setminus S$ has Heegaard genus at most two. Any submanifold of M containing N has Heegaard genus at least three (for the same reason that M has Heegaard genus at least three), so S must intersect N.

Any incompressible surface in N is either a vertical torus or a horizontal once punctured torus. If $S \cap N$ contains a horizontal punctured torus, then $S \setminus N$ contains a disk, so ∂N is compressible into X, which contradicts the assumption on ∂N.

Thus S consists of vertical tori in N. Because it is separating, S must consist of a union $U \subset N$ of two parallel vertical tori (each of which is nonseparating). Each component of $M \setminus U$ has a Heegaard splitting induced from Σ and such a splitting has genus at most two. One component is homeomorphic to a torus cross an interval. The other is the union of $X \setminus Y$ and a pair of pants cross an interval.

Let Z be this second component. Note that if $X \setminus Y$ is not a solid torus, then the fundamental group of $X \setminus Y$ has rank at least two. The fundamental group of a pair of pants cross an interval is the direct product of $\mathbb{Z}$ and a free group $\mathbf{F}^2$ on two generators. By Van Kampen's theorem, the fundamental group of Z is the quotient of the free product $\pi_1(X \setminus Y) * (\mathbf{F}^2 \times \mathbb{Z})$ by two relations, one that equates an element of $\mathbf{F}^2$ to an element of $\pi_1(X \setminus Y)$, and the other that equates a generator of $\mathbb{Z}$ to an element of $\pi_1(X \setminus Y)$. There is thus a homomorphism from $\pi_1(Z)$ onto the direct product $\pi_1(X \setminus Y) \times \mathbb{Z}$. If Z admits a genus two Heegaard splitting, then $\pi_1(Z)$ has rank at most 2, so $\pi_1(X \setminus Y)$ has rank at most one, implying $X \setminus Y$ is a solid torus. □

Let α and β be essential simple closed curves in T whose intersection is a single point. Define $S_\alpha = \alpha \times S^1$ and $S_\beta = \beta \times S^1$. Because Σ contains $T \times x_1$ and $T \times x_2$, it determines the slope 0 in both S_α and S_β. Let Σ_i be the result of spinning Σ i times around S_β as in Section 3. If two such surfaces Σ_i and Σ_j (for $i \neq j$) are isotopic, then Σ_i determines both the slope i and the slope j. By Lemma 3, this implies that Σ can be isotoped to intersect $X \setminus Y$ in an incompressible, twice punctured, genus zero surface, that is, an annulus.

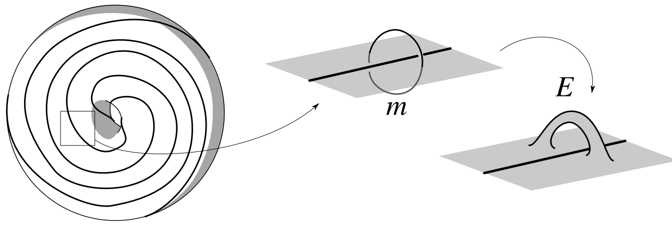

Figure 3. The construction of the surface E.

Assume that Σ can be isotoped to intersect $X \setminus Y$ in an incompressible, properly embedded annulus A. If A is boundary compressible, then because X is irreducible, A must be boundary parallel. Isotoping A out of $X \setminus Y$ makes Σ disjoint from $\partial(X \setminus Y)$. This implies that $\partial(X \setminus Y)$ must be compressible because a Heegaard surface cannot be made disjoint from a closed incompressible surface. Thus we have proved the following:

Lemma 6. *If $\partial(X \setminus Y)$ is incompressible in M and $X \setminus Y$ does not contain a properly embedded, essential (that is, incompressible and boundary incompressible) annulus, then M admits infinitely many nonisotopic Heegaard splittings.*

Note that if M is the complement of a knot K in S^3, then Lemma 6 holds whenever K is not a torus knot or a cable knot.

5. Nielsen equivalence

Let (O, B_1, B_2) be a genus one Heegaard splitting of S^3, and let $K \subset O$ be a simple closed curve that does not bound a disk in S^3. (The curve K is a nontrivial torus knot.) We can give each solid torus B_1 and B_2 a Seifert fibration such that $K \subset \partial B_i$ is a fiber. This defines a Seifert fibration of S^3 such that there is a regular neighborhood Y of K consisting of a union of fibers. Thus the complement in S^3 of Y is a Seifert fibered space over the disk with two singular fibers. A regular fiber in ∂Y determines the same slope as $O \cap \partial Y$.

Let m be the boundary of a small disk D that intersects K in a single point and O in a single arc, as in Figure 3. Let U be an open regular neighborhood of m. Define E to be the union of the twice punctured torus $O \setminus U$ and the annulus $\partial\overline{U} \cap B_1$. (Here $\overline{U}$ is the closure of the open set U.) Define E' to be the union of $O \setminus U$ and $\partial\overline{U} \cap B_2$.

Because m bounds a disk that intersects K at a single point, there is a homeomorphism from S^3 to the result of 1 Dehn surgery on m that takes K onto itself. Let F be the image in this homeomorphism of E, and let F' be the image of E'. In other words, F and F' are the result of "twisting" E and E', respectively, about the meridian m. The differences $E' \setminus E$ and $E \setminus E'$ are annuli whose union bounds the solid torus $\overline{U}$. The annuli meet in meridians of the solid torus. After the Dehn surgery, $F' \setminus F$ and $F \setminus F'$ again bound the solid torus, but this time they meet along a longitude of the solid torus. Thus there is an isotopy from $F' \setminus F$ to $F \setminus F'$. Extending this isotopy to all of F' takes F' onto F, so F and F' are isotopic in S^3 fixing $F' \cap Y = F \cap Y$.

The surface F is the result of adding a trivial handle to the genus one Heegaard surface O. Thus F defines a genus two Heegaard splitting (Σ', H'_1, H'_2) for S^3 where $\Sigma' = F$, $H'_1 = B_1 \setminus U$, and $H'_2 = B_2 \cup \overline{U}$. The intersection $D \cap H'_2 \cap B_1$ is an essential disk properly embedded in H'_2 and intersects K at a single point. Thus K is primitive in H'_2. Because F' is isotopic to F, a similar argument for F' implies that K is also primitive in H'_1. (The reader can check that F results from taking the standard unknotting graph consisting of a core for B_1 and a short arc to K, then pushing K into the resulting Heegaard surface in a way that makes it double primitive.)

Let T be a once punctured torus, let s_1 and s_2 be points in S^1, and let t_1 and t_2 be points in ∂T. Each component of $O \cap \partial Y$ intersects each component of $\Sigma' \cap \partial Y$ at a single point, so these four loops cut ∂Y into four squares. The loops $t_i \times S^1$ and $\partial T \times s_i$ intersect at one point for each pair, so they form a homeomorphic pattern in $\partial T \times S^1$. Let M be the result of gluing $S^3 \setminus Y$ to $T \times S^1$ so that the loops $O \cap \partial Y$ are sent to $t_1 \times S^1$ and $t_2 \times S^1$ while the loops $\Sigma' \cap \partial Y$ are sent to $\partial T \times s_1$ and $\partial T \times s_2$.

Since K is double primitive in Σ, Lemma 4 implies $\Sigma = (\Sigma' \setminus Y) \cup (T \times (s_1 \cup s_2))$ is the surface in a genus three Heegaard splitting (Σ, H_1, H_2) for M. By Lemma 5, this Heegaard splitting is strongly irreducible. Moreover, because loops of the Seifert fibration in $\partial(S^3 \setminus Y)$ are glued to vertical loops in $T \times S^1$ (which can be thought of as loops of a Seifert fibration for N), M is a Seifert fibered space.

Let α and β be simple closed curves in T that intersect in a single point. As in the previous section, we can spin Σ around the vertical torus $S_\beta = \beta \times S^1$ to construct an infinite family of Heegaard splittings $\{(\Sigma^i, H^i_1, H^i_2)\}$ such that Σ^i determines the slope i in the vertical torus $S_\alpha = \alpha \times S^1$.

Lemma 7. *The Heegaard splittings (Σ^i, H^i_1, H^i_2) and (Σ^j, H^j_1, H^j_2) are isotopic if and only if $i = j$. However, the generating set for $\pi_1(M)$ defined by the inclusion map $\pi_1(H^i_1) \to \pi_1(M)$ is Nielsen equivalent to that defined by $\pi_1(H^j_1) \to \pi_1(M)$ for all i and j. The generating set determined by $\pi_1(H^i_2) \to \pi_1(M)$ is Nielsen equivalent that defined by $\pi_1(H^j_2) \to \pi_1(M)$ as well.*

Proof. There is an essential annulus properly embedded in $S^3 \setminus Y$, so Lemma 6 is not enough to distinguish Heegaard splittings by their slopes. However, this annulus intersects ∂Y in the same slope as $O \cap \partial Y$, which determines the slope ∞ in $\partial T \times S^1$. Since there is no incompressible surface with slope ± 1, Lemma 3 implies that Σ^i determines a unique slope on any vertical torus in $T \times S^1$. Since Σ_i determines slope i and Σ_j determines slope j, we conclude Σ^i and Σ^j are isotopic if and only if $i = j$.

All that remains is to show that the Nielsen classes of the generators for $\pi_1(M)$ determined by these Heegaard splittings are all equivalent. We will show this for Σ^0 and Σ^1. A similar argument works for any Σ^i and Σ^{i+1} and the general result follows by induction on $|i|$.

We will choose as the base point for $\pi_1(M)$ a point $p = (a, b) \in \partial T \times S^1$. The fundamental group of the punctured torus $T \times \{b\}$ (with base point $b \in \partial T$) is a free group on two generators. Let x and y be the inclusion into $\pi_1(M)$ of these generators. We can choose x and y so that an arc representing x intersects S_β at a single point and is disjoint from S_α. Similarly, we can assume that an arc representing y intersects S_α at a single point and is disjoint from S_β.

Let z be the element of $\pi_1(M)$ defined by the loop $a \times S^1$. Let t be the element of $\pi_1(M)$ defined by a path that follows a short arc into $B_1 \subset S^3$, then follows a core of B_1 disjoint from the disk D, and then follows the short arc back to p. Because z is determined by a regular fiber and t is determined by a singular fiber of order t, we have $z = t^p$ for some integer p.

The fundamental group of H_1^0 is generated by x, y and t, so it induces the Nielsen class $[x, y, t]$ for $\pi_1(M)$. The only generator for H_1^0 that intersects S_β is x. Spinning Σ^0 around S_β replaces x with $xz = zx$ or $xz^{-1} = z^{-1}x$, while fixing y and t. Without loss of generality, we will assume it replaces x with xt. Thus H_1^1 determines the Nielsen class $[xz, y, t]$. We noted above that $z = t^p$, so the new generating set is in fact $[xt^p, y, t]$, which is Nielsen equivalent to $[x, y, t]$, the generating set for H_1^0. The generating sets induced by $\pi_1(H_1^0)$ and $\pi_1(H_1^1)$, and by induction of any $\pi_1(H_1^i)$, are Nielsen equivalent.

Above, we constructed Σ^0 from the surface F in the knot complement $M \setminus Y$. Switching the roles of B_1 and B_2 in this construction switches F and F', so the resulting Heegaard splitting would be constructed from F'. However, we noted that F' is isotopic to F in $M \setminus Y$. Thus the Heegaard splitting that results from switching the roles of B_1 and B_2 is isotopic to (Σ^0, H_2^0, H_1^0), that is, the same Heegaard surface, but with the order of the handlebodies switched. We can thus apply the argument above to H_2^0 and H_2^1, implying that the generating sets induced by $\pi_1(H_2^i)$ and $\pi_1(H_2^j)$ are Nielsen equivalent for all i, j. □

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REPRESENTATIONS OF MULTILOOP ALGEBRAS

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We describe the finite-dimensional simple modules of all the (twisted and untwisted) multiloop algebras and classify them up to isomorphism.

1. Introduction

Multiloop algebras are multivariable generalizations of the loop algebras appearing in affine Kac–Moody theory. The study of these algebras and their extensions includes a substantial literature on (twisted and untwisted) multiloop, toroidal, and extended affine Lie algebras. This paper describes the finite-dimensional simple modules of multiloop algebras and classifies them up to isomorphism.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over an algebraically closed field F of characteristic zero. Suppose that $\sigma_1, \ldots, \sigma_N : \mathfrak{g} \to \mathfrak{g}$ are commuting automorphisms of finite orders $m_1, \ldots, m_N$, respectively. For each i, fix a primitive m_i-th root of unity $\xi_i \in F$. Then $\mathfrak{g}$ decomposes into common eigenspaces relative to these automorphisms:

$$\mathfrak{g} = \bigoplus_{\bar{k} \in G} \mathfrak{g}_{\bar{k}},$$

where $\mathfrak{g}_{\bar{k}} = \{x \in \mathfrak{g} \mid \sigma_i x = \xi_i^{k_i} x\}$ and $\bar{k}$ is the image of each $k \in \mathbb{Z}^N$ under the canonical map $\mathbb{Z}^N \to G = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_N\mathbb{Z}$. The *multiloop algebra* of $\mathfrak{g}$, relative to these automorphisms, is the Lie algebra

$$\mathscr{L} = \mathscr{L}(\mathfrak{g}; \sigma_1, \ldots, \sigma_N) = \bigoplus_{k \in \mathbb{Z}^N} \mathfrak{g}_{\bar{k}} \otimes F t^k,$$

where Ft^k is the span of $t^k = t_1^{k_1} \cdots t_N^{k_N}$, and multiplication is defined pointwise. If the automorphisms $\sigma_1, \ldots, \sigma_N$ are all trivial, $\mathscr{L}$ is called an *untwisted multiloop algebra*. Otherwise, it is a *twisted multiloop algebra.*

In the one variable case (*untwisted* and *twisted loop algebras*), a proof of the classification of the finite-dimensional simple modules appears in [Chari 1986;

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Chari and Pressley 1986; 1987; 1988]. A complete list of these modules also appears explicitly in [Rao 1993], and a very recent paper [Chari et al. 2008] gives a detailed discussion of this problem in the twisted loop case.

A description of the finite-dimensional simple modules of the untwisted multiloop Lie algebras was first given by S. E. Rao [2001]. Subsequent work by P. Batra [2004] provided a complete (but redundant) list of the finite-dimensional simple modules when σ_1 is a diagram automorphism and the other automorphisms $\sigma_2, \dots, \sigma_N$ are all trivial.

In the one variable case, every twisted loop algebra $\mathscr{L}(\mathfrak{g}; \tau \circ \gamma)$ defined by an inner automorphism τ and a diagram automorphism γ is isomorphic to $\mathscr{L}(\mathfrak{g}; \gamma)$ [Kac 1990, Proposition 8.5]. It thus suffices to consider twists only by diagram automorphisms in this setting. Unfortunately, this is far from true in the multivariable case. See [Gille and Pianzola 2007, Remark 5.9], for instance. It has recently been shown that the centreless core of almost every extended affine Lie algebra is a multiloop algebra [Allison et al. 2009], using results of [Allison et al. 2008; Neher 2004]. Even for these multiloop algebras, any number of the automorphisms σ_i can be nontrivial, and any number of them can fail to be diagram automorphisms.

In this paper, we consider an arbitrary (twisted or untwisted) multiloop algebra $\mathscr{L}$. From any ideal $\mathscr{I}$ of $\mathscr{L}$, we construct a G-graded ideal $I = I(\mathscr{I})$ of the ring $R = F[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$ of Laurent polynomials. If $\mathscr{I}$ is the kernel of a finite-dimensional irreducible representation, the $\bar{0}$-component $I_{\bar{0}}$ of I turns out to be a radical ideal of the $\bar{0}$-component of R. The resulting decomposition of $I_{\bar{0}}$ into an intersection of a finite number of maximal ideals produces an isomorphism

$$\psi_a : \mathscr{L}/\mathscr{I} \to \mathfrak{g} \oplus \cdots \oplus \mathfrak{g} \quad (r \text{ copies})$$

whose composition with the quotient map $\pi : \mathscr{L} \to \mathscr{L}/\mathscr{I}$ is evaluation at an r-tuple $a = (a_1, \dots, a_r)$ of points $a_i \in (F^\times)^N$:

$$\psi_a \circ \pi : x \otimes f(t) \mapsto (f(a_1)x, \dots, f(a_r)x)$$

for any $x \otimes f(t) \in \mathscr{L}$. Since the finite-dimensional simple modules of the semisimple Lie algebra $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ are the tensor products of finite-dimensional simple modules for $\mathfrak{g}$, we obtain a complete (but redundant) list of the finite-dimensional irreducible representations of $\mathscr{L}$ (Corollary 4.4). Namely, any finite-dimensional simple module for $\mathscr{L}$ is of the form

$$V(\lambda, a) = V_{\lambda_1}(a_1) \otimes \cdots \otimes V_{\lambda_r}(a_r),$$

where V_{λ_i} is the $\mathfrak{g}$-module of dominant integral highest weight λ_i, and $V_{\lambda_i}(a_i)$ is the $\mathscr{L}$-module obtained by evaluating elements of $\mathscr{L}$ at the point a_i, and then letting the resulting element of $\mathfrak{g}$ act on V_{λ_i}. The r-tuples $a = (a_1, \dots, a_r)$ that occur in this process must satisfy the condition that the points $m(a_i)$ are all distinct, where $m(a_i) = (a_{i1}^{m_1}, \dots, a_{iN}^{m_N})$ is determined by the orders $m_1, \dots, m_N$ of the automorphisms $\sigma_1, \dots, \sigma_N$. Conversely, the $\mathscr{L}$-module $V(\lambda, a)$ is finite-dimensional and simple if the a_i satisfy this condition (Theorem 4.5).

In the second half of the paper, we establish necessary and sufficient conditions for $\mathscr{L}$-modules $V(\lambda, a)$ and $V(\mu, b)$ to be isomorphic. Namely, we "pull back" a triangular decomposition $N_- \oplus H \oplus N_+$ of $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ to a triangular decomposition $\psi_a^{-1}(N_-) \oplus \psi_a^{-1}(H) \oplus \psi_a^{-1}(N_+)$ of $\mathscr{L}/\mathscr{I}$. The modules $V(\lambda, a)$ and $V(\mu, b)$ are highest weight with respect to this decomposition of $\mathscr{L}/\mathscr{I}$, and they are isomorphic if and only if they have the same highest weights. We conclude with three equivalent criteria for isomorphism in terms of an explicit formula in Theorem 5.4, orbits under a group action in Corollary 5.9, and equivariant maps in Corollary 5.10. These are the first such isomorphism results for modules in any multiloop setting.

Interestingly, the triangular decomposition $N_- \oplus H \oplus N_+$ is replaced with a new triangular decomposition $\psi_b \psi_a^{-1}(N_-) \oplus \psi_b \psi_a^{-1}(H) \oplus \psi_b \psi_a^{-1}(N_+)$ of $\mathfrak{g}^{\oplus r}$ in the computation of the highest weight of $V(\mu, b)$. Unlike diagram automorphisms, arbitrary finite-order automorphisms σ_i often fail to stabilize *any* triangular decomposition of a finite-dimensional semisimple Lie algebra. This fact is reflected in the change of triangular decomposition on $\mathfrak{g}^{\oplus r}$, and it is one of the reasons that past work considered only twists by diagram automorphisms.

Another novelty in this classification is the passage from twists by a single nontrivial automorphism σ_1 to a family of nontrivial automorphisms $\sigma_1, \dots, \sigma_N$. Here the major obstacle to past approaches was reliance on the representation theory of the fixed point subalgebra $\mathfrak{g}_{\bar{0}}$ under the action of the automorphisms. While this was a great success when working with twists by a single automorphism, it cannot be used when considering twists by more than one automorphism, since the algebra $\mathfrak{g}_{\bar{0}}$ is then often 0. We avoid this pitfall by using a new approach that does not rely on the usual Dynkin diagram folding arguments.

We expect the methods of this paper to be widely applicable. For example, the arguments given here classify the finite-dimensional simple modules of the Lie algebra $\mathrm{Map}(X, \mathfrak{g})$ of $\mathfrak{g}$-valued regular functions on any affine variety X; namely, they are tensor products of evaluation modules at distinct points of X. Since the release of earlier versions of this paper, our approach has already been adapted to classify modules for Lie algebras $\mathfrak{g} \otimes A$ of $\mathfrak{g}$-valued functions on affine schemes $\mathrm{Spec}(A)$ and their invariants under more general finite group actions [Chari et al. 2009; Neher et al. 2009]. Another promising direction is the classification of $\mathbb{Z}^N$-graded-simple modules of $L(\mathfrak{g}; \sigma_1, \dots, \sigma_N)$ with finite-dimensional graded components. See [Pal and Batra 2008; Rao 2001] for partial results.

Notation. Throughout this paper, F will be an algebraically closed field of characteristic zero. All Lie algebras, linear spans, and tensor products will be taken over F unless otherwise indicated. We will denote the integers by $\mathbb{Z}$, the nonnegative integers by $\mathbb{Z}_+$, and the nonzero elements of F by $F^\times$.

2. Multiloop algebras and their ideals

The following proposition is an immediate consequence of general facts about reductive Lie algebras.

Proposition 2.1. *Let L be a perfect Lie algebra over F, and let $\phi : L \to \operatorname{End} V$ be a finite-dimensional irreducible representation. Then $L/\ker\phi$ is a semisimple Lie algebra.*

Proof. The representation ϕ descends to a faithful representation of $L/\ker\phi$. By [Bourbaki 1960, Proposition 6.4.5], any Lie algebra with a faithful finite-dimensional irreducible representation is reductive. Also, L is perfect. Therefore, $L/\ker\phi$ is perfect and reductive, and hence semisimple. □

We now focus our attention on multiloop algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over F, and let $R = F[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$ be the commutative algebra of Laurent polynomials in N variables. The *untwisted multiloop algebra* is the Lie algebra $\mathfrak{g} \otimes R$ with (bilinear) pointwise multiplication given by

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg \quad \text{for all } x, y \in \mathfrak{g} \text{ and } f, g \in R.$$

Suppose that $\mathfrak{g}$ is equipped with N commuting automorphisms $\sigma_1, \dots, \sigma_N : \mathfrak{g} \to \mathfrak{g}$ of finite orders $m_1, \dots, m_N$, respectively. For each i, fix $\xi_i \in F$ to be a primitive m_i-th root of 1. Then $\mathfrak{g}$ has a common eigenspace decomposition $\mathfrak{g} = \bigoplus_{\bar{k} \in G} \mathfrak{g}_{\bar{k}}$, where $\bar{k}$ is the image of $k = (k_1, \dots, k_N) \in \mathbb{Z}^N$ under the canonical map

$$\mathbb{Z}^N \to G = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_N\mathbb{Z},$$

and

$$\mathfrak{g}_{\bar{k}} = \{x \in \mathfrak{g} \mid \sigma_i x = \xi_i^{k_i} x \text{ for } i = 1, \dots, N\}.$$

The (*twisted*) *multiloop algebra* $\mathcal{L} = \mathcal{L}(\mathfrak{g}; \sigma_1, \dots, \sigma_N)$ is the Lie subalgebra

$$\mathcal{L} = \bigoplus_{k \in \mathbb{Z}^N} \mathfrak{g}_{\bar{k}} \otimes Ft^k \subseteq \mathfrak{g} \otimes R,$$

where $t^k = t_1^{k_1} \cdots t_N^{k_N}$ is multiindex notation.

Note that R has a G-grading

$$R = \bigoplus_{\bar{k} \in G} R_{\bar{k}}, \tag{2.2}$$

where $R_{\bar{0}} = F[t_1^{\pm m_1}, \dots, t_N^{\pm m_N}]$ and $R_{\bar{k}} = t^k R_{\bar{0}}$ for every $k \in \mathbb{Z}^N$. In this notation,

$$\mathscr{L} = \bigoplus_{\bar{k} \in G} (\mathfrak{g}_{\bar{k}} \otimes R_{\bar{k}}).$$

Fix an F-basis

$$\{x_{\bar{k}j} \mid j = 1, \dots, \dim \mathfrak{g}_{\bar{k}}\} \tag{2.3}$$

of $\mathfrak{g}_{\bar{k}}$ for all $\bar{k} \in G$. Then

$$\mathscr{L} = \bigoplus_{\bar{k} \in G} \bigoplus_{j=1}^{\dim \mathfrak{g}_{\bar{k}}} (F x_{\bar{k}j} \otimes R_{\bar{k}}). \tag{2.4}$$

Since $\mathfrak{g}$ is simple (hence perfect) and graded, each $x_{\bar{k}j}$ can be expressed as a sum of brackets of homogeneous elements $y, z \in \mathfrak{g}$, with $\deg y + \deg z = \bar{k}$. For each such $k \in \mathbb{Z}^N$ and pair y, z, there exist $a, b \in \mathbb{Z}^N$ with $\deg y = \bar{a}$, $\deg z = \bar{b}$, and $a + b = k$. Then the sum of the brackets $[y \otimes t^a, z \otimes t^b]$ will be $x_{\bar{k}j} \otimes t^k$. Since these elements span $\mathscr{L}$, it is clear that $\mathscr{L}$ is perfect. See also [Allison et al. 2006, Lemma 4.9].

Let $\pi_{\bar{k}j}$ be the projection $\pi_{\bar{k}j} : \mathscr{L} \to F x_{\bar{k}j} \otimes R_{\bar{k}}$ relative to the decomposition (2.4). We will view $\pi_{\bar{k}j}$ as a projection $\mathscr{L} \to R_{\bar{k}}$ by identifying $x_{\bar{k}j} \otimes f$ with f for all $f \in R_{\bar{k}}$. Let $\mathscr{I}$ be an ideal of the Lie algebra $\mathscr{L}$, and let $I = I(\mathscr{I})$ be the ideal of R generated by

$$\bigcup_{\bar{k} \in G} \bigcup_{j=1}^{\dim \mathfrak{g}_{\bar{k}}} \pi_{\bar{k}j}(\mathscr{I}).$$

Note that the definition of I is independent of the choice of homogeneous basis $\{x_{\bar{k}j}\}$ of $\mathfrak{g}$, and the ideal I is G-graded since its generators are homogeneous with respect to the G-grading of R. That is,

$$I = \bigoplus_{\bar{k} \in G} I_{\bar{k}}, \quad \text{where } I_{\bar{k}} = I \cap R_{\bar{k}}.$$

Moreover, $t^{\ell-k} I_{\bar{k}} \subseteq I \cap R_{\bar{\ell}} = I_{\bar{\ell}} = t^{\ell-k}(t^{k-\ell} I_{\bar{\ell}}) \subseteq t^{\ell-k} I_{\bar{k}}$, so

$$I_{\bar{\ell}} = t^{\ell-k} I_{\bar{k}} \quad \text{for all } k, \ell \in \mathbb{Z}^N. \tag{2.5}$$

We will use the following technical lemma to show that $\mathscr{I} = \mathscr{L} \cap (\mathfrak{g} \otimes I)$.

Lemma 2.6. *Let*

$$Y = \sum_{\bar{r} \in G} \sum_{n=1}^{\dim \mathfrak{g}_{\bar{r}}} x_{\bar{r}n} \otimes \pi_{\bar{r}n}(Y) \in \mathscr{I}.$$

Then $x_{\bar{k}i} \otimes t^{k-\ell} \pi_{\bar{\ell}j}(Y) \in \mathscr{I}$ *for all* $k, \ell \in \mathbb{Z}^N$, $1 \le i \le \dim \mathfrak{g}_{\bar{k}}$, *and* $1 \le j \le \dim \mathfrak{g}_{\bar{\ell}}$.

Proof. The finite-dimensional simple Lie algebra $\mathfrak{g}$ is a finite-dimensional simple $\mathfrak{g}$-module (and hence a finite-dimensional simple $\mathfrak{U}(\mathfrak{g})$-module) under the adjoint action of $\mathfrak{g}$. Fix $k, \ell \in \mathbb{Z}^N$, $i \in \{1, \dots, \dim \mathfrak{g}_{\bar{k}}\}$ and $j \in \{1, \dots, \dim \mathfrak{g}_{\bar{\ell}}\}$. By the Jacobson density theorem, there exists $u \in \mathfrak{U}(\mathfrak{g})$ such that

$$u \,.\, x_{\bar{r}n} = \begin{cases} x_{\bar{k}i} & \text{if } \bar{r} = \bar{\ell} \text{ and } n = j, \\ 0 & \text{otherwise} \end{cases}$$

for all $\bar{r} \in G$ and $n \in \{1, \dots, \mathfrak{g}_{\bar{r}}\}$. By the Poincaré–Birkhoff–Witt theorem, we can write $u = \sum_{s=1}^{a} p_s$, where $a \geq 1$ and each p_s is a monomial in the variables in $\{x_{\bar{r}n} \mid \bar{r} \in G,\ n = 1, \dots, \dim \mathfrak{g}_{\bar{r}}\}$. Considering the induced G-grading of $\mathfrak{U}(\mathfrak{g})$, we can assume that each p_s is homogeneous of degree $\overline{k - \ell}$. Write

$$p_s = c_s \prod_{\bar{r} \in G} \prod_{n=1}^{\dim \mathfrak{g}_{\bar{r}}} (x_{\bar{r}n})^{b_{\bar{r}n}^{(s)}}, \quad \text{where } c_s \in F \text{ and } b_{\bar{r}n}^{(s)} \in \mathbb{Z}_+.$$

Since p_s is homogeneous of degree $\overline{k - \ell}$ in the G-grading of $\mathfrak{U}(\mathfrak{g})$, we can choose $\alpha(s, \bar{r}, n, 1), \alpha(s, \bar{r}, n, 2), \dots, \alpha(s, \bar{r}, n, b_{\bar{r}n}^{(s)}) \in \mathbb{Z}^N$ for each $s \in \{1, \dots, a\}$, $\bar{r} \in G$, and $n \in \{1, \dots, \dim \mathfrak{g}_{\bar{r}}\}$ so that

(i) $\bar{r} = \overline{\alpha(s, \bar{r}, n, 1)} = \cdots = \overline{\alpha(s, \bar{r}, n, b_{\bar{r}n}^{(s)})}$ and

(ii) $\displaystyle\sum_{\bar{r} \in G} \sum_{n=1}^{\dim \mathfrak{g}_{\bar{r}}} \sum_{b=1}^{b_{\bar{r}n}^{(s)}} \alpha(s, \bar{r}, n, b) = k - \ell.$

Then

$$\widetilde{p}_s = c_s \prod_{\bar{r} \in G} \prod_{n=1}^{\dim \mathfrak{g}_{\bar{r}}} \prod_{b=1}^{b_{\bar{r}n}^{(s)}} (x_{\bar{r}n} \otimes t^{\alpha(s, \bar{r}, n, b)})$$

is in the universal enveloping algebra $\mathfrak{U}(\mathscr{L})$ of $\mathscr{L}$, which acts on $\mathscr{I}$ via the adjoint action of $\mathscr{L}$ on $\mathscr{I}$, and $\sum_{s=1}^{a} \widetilde{p}_s \,.\, Y = x_{\bar{k}i} \otimes t^{k-\ell} \pi_{\bar{\ell}j}(Y)$, so $x_{\bar{k}i} \otimes t^{k-\ell} \pi_{\bar{\ell}j}(Y) \in \mathscr{I}$. □

Proposition 2.7. *In the notation introduced above*,

$$\mathscr{I} = \mathscr{L} \cap (\mathfrak{g} \otimes I) \tag{2.8}$$

$$= \bigoplus_{\bar{k} \in G} \mathfrak{g}_{\bar{k}} \otimes I_{\bar{k}}. \tag{2.9}$$

Proof. The second equality (2.9) and the inclusion $\mathscr{I} \subseteq \mathscr{L} \cap (\mathfrak{g} \otimes I)$ are clear, so it remains only to verify the reverse inclusion $\mathscr{L} \cap (\mathfrak{g} \otimes I) \subseteq \mathscr{I}$. In light of (2.9), it suffices to show that $x_{\bar{k}i} \otimes f \in \mathscr{I}$ for all $\bar{k} \in G$, $i \in \{1, \dots, \dim \mathfrak{g}_{\bar{k}}\}$, and $f \in I_{\bar{k}}$.

By the definition of I, there exist $Y_{\bar{\ell}j} \in \mathcal{I}$ and $f_{\bar{\ell}j} \in R_{\overline{k-\ell}}$ such that

$$f = \sum_{\bar{\ell} \in G} \sum_{j=1}^{\dim \mathfrak{g}_{\bar{\ell}}} f_{\bar{\ell}j} \pi_{\bar{\ell}j}(Y_{\bar{\ell}j}).$$

By Lemma 2.6, $x_{\bar{k}i} \otimes t^r \pi_{\bar{\ell}j}(Y_{\bar{\ell}j}) \in \mathcal{I}$ for all $r, \ell \in \mathbb{Z}^N$ satisfying $\bar{r} = \overline{k-\ell}$. Since each $f_{\bar{\ell}j} \in R_{\overline{k-\ell}}$ is an F-linear combination of $\{t^r \mid \bar{r} = \overline{k-\ell}\}$, we see that

$$x_{\bar{k}i} \otimes f_{\bar{\ell}j} \pi_{\bar{\ell}j}(Y_{\bar{\ell}j}) \in \mathcal{I}$$

for all $\bar{\ell} \in G$ and $j = 1, \ldots, \dim \mathfrak{g}_{\bar{\ell}}$. Thus $x_{\bar{k}i} \otimes f \in \mathcal{I}$. □

We close this section by considering the structure of $I_{\bar{0}} \subseteq R_{\bar{0}}$ in the case where $\mathcal{I}$ is the kernel of an irreducible finite-dimensional representation of $\mathcal{L}$. Clearly $I_{\bar{0}}$ is an ideal of $R_{\bar{0}}$. Moreover, it is a radical ideal:

Proposition 2.10. *Let $\phi : \mathcal{L} \to \operatorname{End} V$ be a finite-dimensional irreducible representation of the multiloop algebra $\mathcal{L}$, and let $\mathcal{I} = \ker \phi$. Define $I = I(\mathcal{I}) \subseteq R$ as above. Then the graded component $I_{\bar{0}}$ is a radical ideal of $R_{\bar{0}}$.*

Proof. Suppose p is an element of $\sqrt{I_{\bar{0}}}$, the radical of the ideal $I_{\bar{0}} = I \cap R_{\bar{0}}$ of $R_{\bar{0}}$. Choose $k \in \mathbb{Z}^N$ so that $\mathfrak{g}_{\bar{k}} \neq 0$, and let $x \in \mathfrak{g}_{\bar{k}}$ be a nonzero element.

For $y \otimes f \in \mathcal{L}$, let $\langle y \otimes f \rangle \subseteq \mathcal{L}$ be the ideal (of $\mathcal{L}$) generated by $y \otimes f$. Let $J = \langle x \otimes t^k p \rangle$, and note that the n-th term $J^{(n)}$ in the derived series of J satisfies $J^{(n)} \subseteq \mathcal{L} \cap (\mathfrak{g} \otimes \langle p^n \rangle)$, where $\langle p^n \rangle$ is the principal ideal of R generated by p^n. Since $I_{\bar{\ell}} = t^\ell I_{\bar{0}}$ for all $\ell \in \mathbb{Z}^N$ by (2.5), and since $p^n \in I_{\bar{0}}$ for n sufficiently large, we see that $J^{(n)} \subseteq \mathcal{L} \cap (\mathfrak{g} \otimes I)$ for $n \gg 0$. Then by Proposition 2.7, $J^{(n)} \subseteq \mathcal{I}$, so

$$\frac{J + \mathcal{I}}{\mathcal{I}} \subseteq \operatorname{Rad}(\mathcal{L}/\mathcal{I}).$$

Since $\operatorname{Rad}(\mathcal{L}/\mathcal{I}) = 0$ by Proposition 2.1, we see that $x \otimes t^k p \in \mathcal{I}$. That is, $p = t^{-k}(t^k p) \in t^{-k} I_{\bar{k}} = I_{\bar{0}}$, and thus $\sqrt{I_{\bar{0}}} = I_{\bar{0}}$. □

3. Some commutative algebra

In this short section, we recall some basic commutative algebra that will be useful for classifying modules for multiloop algebras. Let F, $F^\times$, and R be as before. For any ideal $I \subseteq R$, let $\mathcal{V}(I) = \{x \in (F^\times)^N \mid f(x) = 0 \text{ for all } f \in I\}$ be the (quasiaffine) variety corresponding to I, and let $\operatorname{Poly}(S) = \{g \in R \mid g(s) = 0 \text{ for all } s \in S\}$ be the ideal associated with any subset $S \subseteq (F^\times)^N$.

Proposition 3.1. *Let I be an ideal of $R = F[t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$. Then*

$$\operatorname{Poly}(\mathcal{V}(I)) = \sqrt{I}.$$

Proof. It is straightforward to verify that the usual proofs of the Hilbert Nullstellensatz (see [Atiyah and Macdonald 1969, page 85] for instance) also hold for this Laurent polynomial analogue. □

The following crucial lemma is an easy consequence of Proposition 3.1:

Lemma 3.2. *Let J be a radical ideal of R for which the quotient R/J is a finite-dimensional vector space over F. Then there exist distinct points $a_1, \dots, a_r \in (F^\times)^N$ such that*

$$J = \mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r},$$

where $\mathfrak{m}_{a_i} = \langle t_1 - a_{i1}, \dots, t_N - a_{iN} \rangle$ is the maximal ideal corresponding to $a_i = (a_{i1}, \dots, a_{iN})$ for $i = 1, \dots, r$. Moreover, the set $\{a_1, \dots, a_r\}$ is unique.

Proof. Clearly, $a \in \mathcal{V}(J)$ implies that $J \subseteq \mathfrak{m}_a$, so $J \subseteq \bigcap_{a \in \mathcal{V}(J)} \mathfrak{m}_a$. Conversely, if $f \in \bigcap_{a \in \mathcal{V}(J)} \mathfrak{m}_a$ and $x \in \mathcal{V}(J)$, then $f(x) = 0$ and $f \in \mathrm{Poly}(\mathcal{V}(J)) = \sqrt{J} = J$. Hence $J = \bigcap_{a \in \mathcal{V}(J)} \mathfrak{m}_a$.

Since $J \subseteq \mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r}$ for all subsets $\{a_1, \dots, a_r\} \subseteq \mathcal{V}(J)$, we see that the ($F$-vector space) dimension of $R/(\mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r})$ is bounded by $\dim_F(R/J)$. Take a finite collection $\{a_1, \dots, a_r\}$ of points in $\mathcal{V}(J)$ for which this dimension is maximal. Then $\mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r} \cap \mathfrak{m}_{a_{r+1}} = \mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r}$ for all points $a_{r+1} \in \mathcal{V}(J)$, so

$$J = \bigcap_{b \in \mathcal{V}(J)} \mathfrak{m}_b = \mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r} \cap \Big(\bigcap_{b \in \mathcal{V}(J)} \mathfrak{m}_b \Big) = \mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r}.$$

To see that $\{a_1, \dots, a_r\} \subseteq (F^\times)^N$ is uniquely determined, suppose that $J = \mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r} = \mathfrak{m}_{b_1} \cap \cdots \cap \mathfrak{m}_{b_s}$ for some $a_1, \dots, a_r, b_1, \dots, b_s \in (F^\times)^N$. Then

$$\begin{aligned} \{a_1, \dots, a_r\} &= \mathcal{V}(\mathfrak{m}_{a_1} \cap \cdots \cap \mathfrak{m}_{a_r}) \\ &= \mathcal{V}(J) \\ &= \mathcal{V}(\mathfrak{m}_{b_1} \cap \cdots \cap \mathfrak{m}_{b_s}) = \{b_1, \dots, b_s\}. \end{aligned}$$ □

Note that the ideal $I_{\bar{0}} \subseteq R_{\bar{0}}$ of Proposition 2.10 is radical and cofinite. Viewing $R_{\bar{0}} = F[t_1^{\pm m_1}, \dots, t_N^{\pm m_N}]$ as the ring of Laurent polynomials in the variables $t_1^{m_1}, \dots, t_N^{m_N}$, we see that

$$I_{\bar{0}} = M_{a_1} \cap \cdots \cap M_{a_r},$$

where $\{a_1, \dots, a_r\} = \mathcal{V}(I_{\bar{0}})$ is a set of distinct points in $(F^\times)^N$, and

$$M_{a_i} = \langle t_1^{m_1} - a_{i1}, \dots, t_N^{m_N} - a_{iN} \rangle_{R_{\bar{0}}}$$

is the maximal ideal of $R_{\bar{0}}$ corresponding to the point $a_i = (a_{i1}, \dots, a_{iN})$. Then by the Chinese remainder theorem, we have the following corollary:

Corollary 3.3. *Let $I_{\bar{0}}$ and $R_{\bar{0}}$ be as in Proposition 2.10. Then there exist unique* (*up to reordering*) *points $a_1, \dots, a_r \in (F^\times)^N$ such that the canonical map*

$$R_{\bar{0}}/I_{\bar{0}} \to R_{\bar{0}}/M_{a_1} \times \cdots \times R_{\bar{0}}/M_{a_r}, \quad f + I_{\bar{0}} \mapsto (f + M_{a_1}, \dots, f + M_{a_r})$$

is a well-defined F-algebra isomorphism.

4. Classification of simple modules

We now return to classifying the finite-dimensional simple modules of multiloop algebras. As in Section 2, let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let $\phi : \mathscr{L} \to \operatorname{End} V$ be a finite-dimensional irreducible representation of a multiloop algebra $\mathscr{L} = \mathscr{L}(\mathfrak{g}; \sigma_1, \dots, \sigma_N)$ defined by commuting automorphisms $\sigma_1, \dots, \sigma_N$ of order $m_1, \dots, m_N$, respectively.

Define $\mathscr{I}$, I, G, and R as in Section 2. Then we see that

$$\mathscr{L} = \bigoplus_{\bar{k} \in G} \mathfrak{g}_{\bar{k}} \otimes R_{\bar{k}} \quad \text{and} \quad \mathscr{I} = \bigoplus_{\bar{k} \in G} \mathfrak{g}_{\bar{k}} \otimes I_{\bar{k}},$$

by Proposition 2.7. Since $\mathscr{I}$ is a G-graded ideal of $\mathscr{L}$, we have

$$\mathscr{L}/\mathscr{I} = \bigoplus_{\bar{k} \in G} ((\mathfrak{g}_{\bar{k}} \otimes R_{\bar{k}})/(\mathfrak{g}_{\bar{k}} \otimes I_{\bar{k}})) = \bigoplus_{\bar{k} \in G} \mathfrak{g}_{\bar{k}} \otimes (R_{\bar{k}}/I_{\bar{k}}).$$

Each graded component $R_{\bar{k}}/I_{\bar{k}}$ of R/I is an $R_{\bar{0}}$-module, and it is easy to check that the map

$$\mu_k : R_{\bar{0}}/I_{\bar{0}} \to R_{\bar{k}}/I_{\bar{k}}, \quad f + I_{\bar{0}} \mapsto t^k f + I_{\bar{k}}$$

is a well-defined $R_{\bar{0}}$-module homomorphism for each $k \in \mathbb{Z}^N$ and $f \in R_{\bar{0}}$. By (2.2) and (2.5), $R_{\bar{k}} = t^k R_{\bar{0}}$ and $t^{-k} I_{\bar{k}} = I_{\bar{0}}$, so the map μ_k is both surjective and injective. Hence the following lemma holds:

Lemma 4.1. *Let $k \in \mathbb{Z}^N$. Then the map $\mu_k : R_{\bar{0}}/I_{\bar{0}} \to R_{\bar{k}}/I_{\bar{k}}$ is a well-defined isomorphism of $R_{\bar{0}}$-modules. In particular*, *each graded component $R_{\bar{k}}/I_{\bar{k}}$ has the same dimension* (*as a vector space*), *that is*, $\dim(R_{\bar{k}}/I_{\bar{k}}) = \dim(R_{\bar{0}}/I_{\bar{0}})$.

Let $a_1, \dots, a_r \in (F^\times)^N$ be the (unique) points defined by Corollary 3.3, and let $b_i = (b_{i1}, \dots, b_{iN})$ be a point in $(F^\times)^N$ such that $b_{ij}^{m_j} = a_{ij}$ for all $1 \le i \le r$ and $1 \le j \le N$. Recall that $I_{\bar{k}} = t^k I_{\bar{0}}$ for all $k \in \mathbb{Z}^N$, and $I_{\bar{0}}$ is contained in the ideal M_{a_i} of $R_{\bar{0}}$ for $i = 1, \dots, r$. Therefore, the map

$$\psi = \psi_b : \mathscr{L} \to \mathfrak{g} \oplus \cdots \oplus \mathfrak{g} \quad (r \text{ copies}),$$
$$x \otimes f \mapsto (f(b_1)x, \dots, f(b_r)x)$$

descends to a well-defined Lie algebra homomorphism

$$\overline{\psi} : \mathscr{L}/\ker\phi \to \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}. \tag{4.2}$$

Theorem 4.3. *The map* $\overline{\psi} : \mathscr{L}/\ker\phi \to \mathfrak{g}\oplus\cdots\oplus\mathfrak{g}$ *in* (4.2) *is a Lie algebra isomorphism.*

Proof. Let $k \in \mathbb{Z}^N$, and let $\overline{\psi}_{\bar{k}} : \mathfrak{g}_{\bar{k}} \otimes (R_{\bar{k}}/I_{\bar{k}}) \to \mathfrak{g}_{\bar{k}} \oplus\cdots\oplus \mathfrak{g}_{\bar{k}}$ be the restriction of $\overline{\psi}$ to the graded component $\mathfrak{g}_{\bar{k}} \otimes (R_{\bar{k}}/I_{\bar{k}})$ of $\mathscr{L}/\ker\phi$.

The map $\overline{\psi}$ is injective if each $\overline{\psi}_{\bar{k}}$ is injective. In the notation of (2.3), if

$$u = \sum_{j=1}^{\dim \mathfrak{g}_{\bar{k}}} x_{\bar{k}j} \otimes (t^k f_j(t) + I_{\bar{k}})$$

is in the kernel of $\overline{\psi}_{\bar{k}}$ for some collection of $f_j \in R_{\bar{0}}$, then $b_i^k f_j(b_i) = 0$ for all i and j. Then for all i and j, we have $f_j(b_i) = 0$ and $f_j \in M_{a_i}$, where M_{a_i} is the ideal of $R_{\bar{0}}$ generated by $\{t_\ell^{m_\ell} - a_{i\ell} \mid \ell = 1, \dots, N\}$. Hence $f_j \in \bigcap_{i=1}^r M_{a_i} = I_{\bar{0}}$, so $t^k f_j(t) \in t^k I_{\bar{0}} = I_{\bar{k}}$, and

$$\sum_{j=1}^{\dim \mathfrak{g}_{\bar{k}}} x_{\bar{k}j} \otimes t^k f_j(t) \in \mathfrak{g}_{\bar{k}} \otimes I_{\bar{k}} \subseteq \ker\phi.$$

Hence $u = 0$ in $\mathscr{L}/\ker\phi$, so $\overline{\psi}_{\bar{k}}$ (and thus $\overline{\psi}$) is injective.

By Lemma 4.1, $\dim(R_{\bar{\ell}}/I_{\bar{\ell}}) = \dim(R_{\bar{0}}/I_{\bar{0}})$ for all $\ell \in \mathbb{Z}^N$. Therefore,

$$\dim(\mathscr{L}/\ker\phi) = \sum_{\bar{\ell}\in G} (\dim \mathfrak{g}_{\bar{\ell}})(\dim(R_{\bar{\ell}}/I_{\bar{\ell}})) = \dim(R_{\bar{0}}/I_{\bar{0}}) \dim \mathfrak{g}.$$

Since F is algebraically closed, $R_{\bar{0}}/M_{a_i} \cong F$ for every i, so the (F-vector space) dimensions satisfy

$$\dim(R_{\bar{0}}/I_{\bar{0}}) = \dim(R_{\bar{0}}/M_{a_1} \times\cdots\times R_{\bar{0}}/M_{a_r}) = r,$$

by Corollary 3.3. Therefore, $\overline{\psi}$ is an injective homomorphism between two Lie algebras of equal dimension, so $\overline{\psi}$ is an isomorphism. □

The finite-dimensional simple modules over direct sums of copies of the Lie algebra $\mathfrak{g}$ are tensor products of finite-dimensional simple modules over $\mathfrak{g}$. (See [Bourbaki 1958, section 7, numéro 7] for instance.) We can thus conclude that the finite-dimensional simple modules for multiloop algebras are pullbacks (under ψ) of tensor products of finite-dimensional simple modules over $\mathfrak{g}$.

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a base Δ of simple roots, and weights $\lambda_i \in \mathfrak{h}^*$ for $i = 1, \dots, r$. Then we will write $V_{\lambda_i}(b_i)$ for the simple $\mathfrak{g}$-module V_{λ_i} of highest weight λ_i, equipped with the $\mathscr{L}$-action given by

$$(x \otimes f(t)).v = f(b_i)xv \quad \text{for all } x \otimes f \in \mathscr{L} \text{ and } v \in V_{\lambda_i}.$$

The tensor product of such a family of evaluation modules will be denoted

$$V(\lambda, b) = V_{\lambda_1}(b_1) \otimes \cdots \otimes V_{\lambda_r}(b_r),$$

and we will write $m(b_i)$ for the point $(b_{i1}^{m_1}, \ldots, b_{iN}^{m_N}) \in (F^\times)^N$ for $i = 1, \ldots, r$. We have now proved one of our main results:

Corollary 4.4. *Let V be a finite-dimensional simple module for the multiloop algebra $\mathcal{L}$. Then there exist $b_1, \ldots, b_r \in (F^\times)^N$ and $\lambda_1, \ldots, \lambda_r$ dominant integral weights for $\mathfrak{g}$ such that $V \cong V(\lambda, b)$, where $m(b_i) \neq m(b_j)$ whenever $i \neq j$.* □

Conversely, if the points $m(b_i) \in (F^\times)^N$ are pairwise distinct, then such a tensor product of evaluation modules is simple:

Theorem 4.5. *Let $\lambda_1, \ldots, \lambda_r$ be dominant integral weights for $\mathfrak{g}$, and suppose $b_1, \ldots, b_r \in (F^\times)^N$ satisfy the property that $m(b_i) \neq m(b_j)$ whenever $i \neq j$. Then $V(\lambda, b)$ is a finite-dimensional simple $\mathcal{L}$-module.*

Proof. Let $I_{\bar{0}}$ be the intersection $\bigcap_{i=1}^r M_{a_i}$ of the maximal ideals M_{a_i} of $R_{\bar{0}}$ that correspond to the points $a_i = m(b_i)$. For any $k, \ell \in \mathbb{Z}^N$, we see that $t^{k-\ell} I_{\bar{0}} = I_{\bar{0}}$ if $\bar{k} = \bar{\ell}$ as elements of $G = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_N\mathbb{Z}$. Thus $t^k I_{\bar{0}} = t^\ell I_{\bar{0}}$ if $\bar{k} = \bar{\ell}$, so we can unambiguously define $I_{\bar{k}} = t^k I_{\bar{0}}$ for any $k \in \mathbb{Z}^N$.

Since $a_1, \ldots, a_r$ are pairwise distinct points in $(F^\times)^N$, the proof of Theorem 4.3 (in particular, the appeal to Corollary 3.3) shows that the map

$$\psi : \mathcal{L} \to \mathfrak{g} \oplus \cdots \oplus \mathfrak{g} \quad (r \text{ copies}),$$
$$x \otimes f(t) \mapsto (f(b_1)x, \ldots, f(b_r)x)$$

is surjective. Then since each V_{λ_i} is a simple $\mathfrak{g}$-module, we see that the tensor product $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ is a simple module over $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$, and the pullback $V(\lambda, b)$ is a simple $\mathcal{L}$-module. □

Remark 4.6. It is not difficult to verify that if $m(b_i) = m(b_j)$ for some $i \neq j$ for which λ_i and λ_j are both nonzero, then $V(\lambda, b)$ is *not* simple. However, as we do not need this fact for the classification of simple modules, we will omit its proof.

5. Isomorphism classes of simple modules

By Corollary 4.4 and Theorem 4.5, the finite-dimensional simple modules of the multiloop algebra $\mathcal{L}(\mathfrak{g}; \sigma_1, \ldots, \sigma_N)$ are precisely the tensor products

$$V(\lambda, a) = V_{\lambda_1}(a_1) \otimes \cdots \otimes V_{\lambda_r}(a_r) \tag{5.1}$$

for which all the $\lambda_i \in \mathfrak{h}^*$ are dominant integral, and $m(a_i) \neq m(a_j)$ whenever $i \neq j$. If $\lambda_i = 0$ for some i, then $V_{\lambda_i}(a_i)$ is the trivial module, and (up to isomorphism) this term can be omitted from the tensor product (5.1). With the convention that empty tensor products of $\mathscr{L}$-modules are the 1-dimensional trivial module, we may assume that every λ_i is a *nonzero* dominant integral weight in (5.1).

To proceed further, we will need a lemma about how highest weights depend on triangular decompositions.

Let L be a finite-dimensional semisimple Lie algebra with Cartan subalgebra H and base of simple roots $\Delta \subset H^*$. The group $\operatorname{Aut} L$ of automorphisms of L is (canonically) a semidirect product of the group $\operatorname{Int} L$ of inner automorphisms and the group $\operatorname{Out} L$ of diagram automorphisms with respect to (H, Δ):

$$\operatorname{Aut} L = \operatorname{Int} L \rtimes \operatorname{Out} L.$$

See [Jacobson 1962, IX.4] for instance. Every automorphism θ can therefore be decomposed as $\theta = \tau \circ \gamma$ with an *inner part* $\tau \in \operatorname{Int} L$ and *outer part* $\gamma \in \operatorname{Out} L$.

Lemma 5.2. *Let H be a Cartan subalgebra of a finite-dimensional semisimple Lie algebra L, and let $\Delta \subset H^*$ be a base of simple roots. Suppose that V is a finite-dimensional simple L-module of highest weight λ with respect to (H, Δ), and $\theta \in \operatorname{Aut} L$. Write $\theta = \tau \circ \gamma$ for some $\tau \in \operatorname{Int} L$ and $\gamma \in \operatorname{Out} L$.*

Then $\Delta \circ \theta^{-1} = \{\alpha \circ \theta^{-1} \mid \alpha \in \Delta\}$ is a base of simple roots for L, relative to the Cartan subalgebra $\theta(H) \subset L$, and V has highest weight $\lambda \circ \tau^{-1}$ with respect to $(\theta(H), \Delta \circ \theta^{-1})$.

Proof. Any diagram automorphism with respect to (H, Δ) will preserve H and Δ, so V has highest weight λ with respect to $(\gamma(H), \Delta \circ \gamma^{-1}) = (H, \Delta)$. Therefore, it is enough to prove the lemma for the case where $\theta = \tau$ is an inner automorphism. Since inner automorphisms are products of automorphisms of the form $\exp(\operatorname{ad} x)$ for ad-nilpotent elements $x \in L$, we may also assume without loss of generality that $\tau = \exp(\operatorname{ad} u)$ for some ad-nilpotent element u.

Let $\rho : L \to \operatorname{End} V$ be the homomorphism describing the action of L on V. Then for any $v \in V$,

$$\tau(h) . v = (\exp(\operatorname{ad} u)(h)) . v = e^{\rho(u)} \rho(h) e^{-\rho(u)} v, \tag{5.3}$$

where $e^{\rho(u)}$ denotes the matrix exponential of the endomorphism $\rho(u)$.

The map $e^{\rho(u)}$ is invertible, so for any nonzero element

$$w \in V_\alpha^H := \{v \in V \mid h . v = \alpha(h) v \text{ for all } h \in H\},$$

we see that $e^{\rho(u)} w \neq 0$, and using (5.3),

$$\tau(h) . e^{\rho(u)} w = e^{\rho(u)} \rho(h) e^{-\rho(u)} e^{\rho(u)} w = \alpha(h) e^{\rho(u)} w.$$

That is,

$$e^{\rho(u)} V_\alpha^H \subseteq V_{\alpha \circ \tau^{-1}}^{\tau(H)} := \{v \in V \mid h . v = \alpha \circ \tau^{-1}(h) . v \text{ for all } h \in \tau(H)\}.$$

The reverse inclusion follows similarly by considering $\tau^{-1} = \exp(-\operatorname{ad} u)$, so

$$e^{\rho(u)} V_\alpha^H = V_{\alpha\circ\tau^{-1}}^{\tau(H)}$$

for all $\alpha \in H^*$. In the case where V is the adjoint module L, we now see that α is a root relative to H if and only if $\alpha \circ \tau^{-1}$ is a root relative to $\tau(H)$. It follows easily that $\Delta \circ \tau^{-1}$ is a base of simple roots for L, with respect to the Cartan subalgebra $\tau(H)$.

The second part also follows easily, since $V_{\lambda\circ\tau^{-1}}^{\tau(H)} = e^{\rho(u)} V_\lambda^H$ is nonzero, but $V_{\lambda\circ\tau^{-1}+\alpha\circ\tau^{-1}}^{\tau(H)} = e^{\rho(u)} V_{\lambda+\alpha}^H = 0$ for all $\alpha \in \Delta$. That is, the highest weight of V is $\lambda \circ \tau^{-1}$, relative to $(\tau(H), \Delta \circ \tau^{-1}) = (\theta(H), \Delta \circ \theta^{-1})$. □

Fix a base Δ of simple roots with respect to a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. The next theorem gives necessary and sufficient conditions for modules of the form $V(\lambda, a)$ to be isomorphic.

Theorem 5.4. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ be sequences of nonzero dominant integral weights with respect to Δ. Suppose that $a = (a_1, \dots, a_r)$ and $b = (b_1, \dots, b_s)$ are sequences of points in $(F^\times)^N$ such that $m(a_i) \neq m(a_j)$ and $m(b_i) \neq m(b_j)$ whenever $i \neq j$.*

Then the finite-dimensional simple $\mathscr{L}$-modules $V(\lambda, a)$ and $V(\mu, b)$ are isomorphic if and only if $r = s$ and there is a permutation $\pi \in S_r$ satisfying the conditions

$$m(a_i) = m(b_{\pi(i)}) \quad \text{and} \quad \lambda_i = \mu_{\pi(i)} \circ \gamma_i$$

for $i = 1, \dots, r$, where γ_i is the outer part of the automorphism $\omega_i : \mathfrak{g} \to \mathfrak{g}$ defined by $\omega_i(x) = (b_{\pi(i)}^k / a_i^k)x$ for all $k \in \mathbb{Z}^N$ and $x \in \mathfrak{g}_{\bar{k}}$.

Proof. Let $\phi_{\lambda,a} : \mathscr{L} \to \operatorname{End} V(\lambda, a)$ and $\phi_{\mu,b} : \mathscr{L} \to \operatorname{End} V(\mu, b)$ be the Lie algebra homomorphisms defining the representations $V(\lambda, a)$ and $V(\mu, b)$. By Theorem 4.3, the kernel of $\phi_{\lambda,a}$ is equal to the kernel of the evaluation map ψ_a, defined by

$$\psi_a : \mathscr{L} \to \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}, \quad x \otimes f \mapsto (f(a_1)x, \dots, f(a_r)x)$$

for all $x \otimes f \in \mathscr{L}$. Similarly, $\ker \phi_{\mu,b} = \ker \psi_b$.

If the $\mathscr{L}$-modules $V(\lambda, a)$ and $V(\mu, b)$ are isomorphic, then $\ker \phi_{\lambda,a} = \ker \phi_{\mu,b}$, so $\ker \psi_a = \ker \psi_b$. But $\ker \psi_a = \bigoplus_{\bar{k}\in G} \mathfrak{g}_{\bar{k}} \otimes I_{\bar{k}}$, where $I_{\bar{k}} = t^k I_{\bar{0}}$ for all $k \in \mathbb{Z}^N$, and

$$I_{\bar{0}} = M_{m(a_1)} \cap \cdots \cap M_{m(a_r)},$$

where $M_{m(a_i)} = \langle t_1^{m_1} - a_{i1}^{m_1}, \dots, t_N^{m_N} - a_{iN}^{m_N} \rangle_{R_{\bar{0}}}$ is the maximal ideal of $R_{\bar{0}}$ that corresponds to the point $m(a_i) = (a_{i1}^{m_1}, \dots, a_{iN}^{m_N})$. Since $\ker \psi_a = \ker \psi_b$, we see that (in the notation of Section 3)

$$\begin{aligned}\{m(a_1), \dots, m(a_r)\} &= \mathcal{V}(M_{m(a_1)} \cap \cdots \cap M_{m(a_r)}) \\ &= \mathcal{V}(I_{\bar{0}}) \\ &= \mathcal{V}(M_{m(b_1)} \cap \cdots \cap M_{m(b_s)}) = \{m(b_1), \dots, m(b_s)\}.\end{aligned}$$

Hence $r = s$, and there is a permutation $\pi \in S_r$ such that $m(a_i) = m(b_{\pi(i)})$ for $i = 1, \dots, r$. We will write $\pi(b) = (b_{\pi(1)}, \dots, b_{\pi(r)})$.

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$ relative to Δ. Assuming that $r = s$ and $m(a_i) = m(b_{\pi(i)})$ for all i, view $V_\lambda = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ and $V_{\pi(\mu)} = V_{\mu_{\pi(1)}} \otimes \cdots \otimes V_{\mu_{\pi(r)}}$ as highest weight modules of the semisimple Lie algebra $\mathfrak{g}^{\oplus r}$ relative to the triangular decomposition

$$\mathfrak{g}^{\oplus r} = (\mathfrak{n}_-^{\oplus r}) \oplus (\mathfrak{h}^{\oplus r}) \oplus (\mathfrak{n}_+^{\oplus r}). \tag{5.5}$$

The highest weights of V_λ and $V_{\pi(\mu)}$ are λ and $\pi(\mu) = (\mu_{\pi(1)}, \dots, \mu_{\pi(r)})$, respectively, where $\lambda(h_1, \dots, h_r) = \sum_i \lambda_i(h_i)$ for all $(h_1, \dots, h_r) \in \mathfrak{h}^{\oplus r}$, and $\pi(\mu) \in (\mathfrak{h}^{\oplus r})^*$ is defined analogously.

We can pull back the triangular decomposition (5.5) via the isomorphism $\overline{\psi}_a : \mathcal{L}/\ker \psi_a \to \mathfrak{g}^{\oplus r}$ defined in (4.2). Then $V(\lambda, a)$ and $V(\mu, b)$ are irreducible highest weight modules of the semisimple Lie algebra $\mathcal{L}/\ker \psi_a$ relative to the triangular decomposition

$$\mathcal{L}/\ker \psi_a = \overline{\psi}_a^{-1}(\mathfrak{n}_-^{\oplus r}) \oplus \overline{\psi}_a^{-1}(\mathfrak{h}^{\oplus r}) \oplus \overline{\psi}_a^{-1}(\mathfrak{n}_+^{\oplus r}). \tag{5.6}$$

The $\mathcal{L}$-modules $V(\lambda, a)$ and $V(\mu, b)$ are isomorphic if and only if they have the same highest weights relative to the decomposition (5.6). Since $\overline{\psi}_a$ maps the decomposition (5.6) to the decomposition (5.5), the highest weight of $V(\lambda, a)$ is clearly $\lambda \circ \overline{\psi}_a : \overline{\psi}_a^{-1}(\mathfrak{h}^{\oplus r}) \to F$.

The highest weight of $V(\mu, b)$ is $\nu \circ \overline{\psi}_{\pi(b)}$, where $\nu \in (\overline{\psi}_{\pi(b)} \overline{\psi}_a^{-1}(\mathfrak{h}^{\oplus r}))^*$ is the highest weight of $V_{\pi(\mu)}$ relative to the new triangular decomposition

$$\mathfrak{g}^{\oplus r} = \overline{\psi}_{\pi(b)} \overline{\psi}_a^{-1}(\mathfrak{n}_-^{\oplus r}) \oplus \overline{\psi}_{\pi(b)} \overline{\psi}_a^{-1}(\mathfrak{h}^{\oplus r}) \oplus \overline{\psi}_{\pi(b)} \overline{\psi}_a^{-1}(\mathfrak{n}_+^{\oplus r}).$$

Let $\overline{\psi}_{\pi(b)} \overline{\psi}_a^{-1} = \tau \circ \gamma$ be a decomposition into an inner automorphism τ and a diagram automorphism γ with respect to $(\mathfrak{h}^{\oplus r}, \Delta)$. By Lemma 5.2, we have $\nu = \pi(\mu) \circ \tau^{-1}$, so the two modules $V(\lambda, a)$ and $V(\mu, b)$ are isomorphic if and only if $\lambda \circ \overline{\psi}_a = \pi(\mu) \circ \tau^{-1} \circ \overline{\psi}_{\pi(b)}$ on $\overline{\psi}_a^{-1}(\mathfrak{h}^{\oplus r})$. That is, $V(\lambda, a) \cong V(\mu, b)$ if and only if

$$\lambda = \pi(\mu) \circ \gamma$$

on $\mathfrak{h}^{\oplus r}$. To finish the proof, it is enough to write down an explicit formula for the automorphism $\overline{\psi}_{\pi(b)} \overline{\psi}_a^{-1} = \tau \circ \gamma$ of $\mathfrak{g}^{\oplus r}$.

For each $x \in \mathfrak{g}$, let $x^i = (0, \ldots, x, \ldots, 0) \in \mathfrak{g}^{\oplus r}$, where x is in the i-th position. If $k \in \mathbb{Z}^N$ and $x \in \mathfrak{g}_{\bar{k}}$, then we see that

$$\overline{\psi}_a^{-1}(x^i) = a_i^{-k} x \otimes t^k f_i(t) + \ker \psi_a$$

in $\mathcal{L}/\ker \psi_a$, for any $f_i(t) \in R_{\bar{0}}$ with $f_i(a_j) = \delta_{ij}$ for all $j = 1, \ldots, r$. Since $f_i \in R_{\bar{0}} = F[t_1^{\pm m_1}, \ldots, t_N^{\pm m_N}]$ and $m(a_j) = m(b_{\pi(j)})$ for all j, we see that $f_i(b_{\pi(j)}) = \delta_{ij}$, and

$$\overline{\psi}_{\pi(b)} \overline{\psi}_a^{-1}(x^i) = (b_{\pi(i)}^k / a_i^k) x^i. \qquad \square$$

Theorem 5.4 may also be interpreted in terms of a group action on the space of parameters (λ, a) defining the finite-dimensional simple modules of $\mathcal{L}$. Let $G^r = G \times \cdots \times G$ (r factors), where G is the finite abelian group $G = \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_N \rangle$ as before. Note that G acts on $(F^\times)^N$ via the primitive m_i-th roots of unity ξ_i used in the definition of $\mathcal{L}$:

$$(\sigma_1^{c_1}, \ldots, \sigma_N^{c_N}) . (d_1, \ldots, d_N) = (\xi_1^{c_1} d_1, \ldots, \xi_N^{c_N} d_N)$$

for any $(c_1, \ldots, c_N) \in \mathbb{Z}^N$ and $(d_1, \ldots, d_N) \in (F^\times)^N$. Form the semidirect product $G^r \rtimes S_r$ by letting the symmetric group S_r act on G^r (on the left) by permuting the factors of G^r. That is,

$${}^{\pi}(\rho_1, \ldots, \rho_r) = (\rho_{\pi(1)}, \ldots, \rho_{\pi(r)}) \quad \text{for all } \pi \in S_r \text{ and } \rho_i \in G.$$

This semidirect product acts on the space of ordered r-tuples of points in the torus $(F^\times)^N$ by letting G^r act diagonally and letting S_r permute the points:

$$\text{(5.7)} \qquad {}^{\rho\pi}a = (\rho_1 . a_{\pi(1)}, \ldots \rho_r . a_{\pi(r)}),$$

for all $\rho = (\rho_1, \ldots, \rho_r) \in G^r$, $\pi \in S_r$, and r-tuples $a = (a_1, \ldots, a_r)$ of points $a_i \in (F^\times)^N$.

The group $G^r \rtimes S_r$ also acts on the space of r-tuples λ of nonzero dominant integral weights. For each $\rho = (\rho_1, \ldots, \rho_r) \in G^r$, write $\rho_i = (\sigma_1^{\rho_{i1}}, \ldots, \sigma_N^{\rho_{iN}})$ for some nonnegative integers ρ_{ij}. Let the ρ_i act on $\mathfrak{g}$ by

$$\rho_i(x) = \sigma_1^{\rho_{i1}} \cdots \sigma_N^{\rho_{iN}} x$$

for all $x \in \mathfrak{g}$, and on the weights λ_i by

$$\rho_i(\lambda_i) = \lambda_i \circ \gamma(\rho_i^{-1}),$$

where $\gamma(\rho_i^{-1})$ is the outer part of the automorphism $\rho_i^{-1} : \mathfrak{g} \to \mathfrak{g}$. Then $G^r \rtimes S_r$ acts on each $\lambda = (\lambda_1, \ldots, \lambda_r)$ by

$$\text{(5.8)} \qquad {}^{\rho\pi}\lambda = (\lambda_{\pi(1)} \circ \gamma(\rho_1^{-1}), \ldots, \lambda_{\pi(r)} \circ \gamma(\rho_r^{-1})).$$

Combining (5.7) and (5.8) gives an action of $G^r \rtimes S_r$ on the set of pairs (λ, a), where λ is an r-tuple of nonzero dominant integral weights λ_i and a is an r-tuple of points $a_i \in (F^\times)^N$. Namely, let

$$^{\rho\pi}(\lambda, a) = (^{\rho\pi}\lambda, {}^{\rho\pi}a).$$

In terms of this action, the isomorphism classes of the finite-dimensional simple $\mathscr{L}$-modules are labeled by orbits of the groups $G^r \rtimes S_r$.

Corollary 5.9. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ be sequences of nonzero dominant integral weights with respect to Δ. Suppose $a = (a_1, \dots, a_r)$ and $b = (b_1, \dots, b_s)$ are sequences of points in $(F^\times)^N$ with $m(a_i) \neq m(a_j)$ and $m(b_i) \neq m(b_j)$ whenever $i \neq j$. Then $V(\lambda, a)$ and $V(\mu, b)$ are isomorphic if and only if $r = s$ and $(\lambda, a) = {}^{\rho\pi}(\mu, b)$ for some $(\rho, \pi) \in G^r \rtimes S_r$.*

Proof. Note that $m(a_i) = m(b_{\pi(i)})$ if and only if the coordinates a_{ij} of $a_i = (a_{i1}, \dots, a_{iN})$ differ from the coordinates $b_{\pi(i)j}$ of $b_{\pi(i)}$ by an m_j-th root of unity. Since ξ_j is a primitive m_j-th root of unity, this happens if and only if there are integers ρ_{ij} such that $a_{ij} = \xi_j^{\rho_{ij}} b_{\pi(i)j}$. In terms of group actions, this is precisely the existence of $\rho_i = (\sigma_1^{\rho_{i1}}, \dots, \sigma_N^{\rho_{iN}}) \in G$ with $a_i = \rho_i . b_{\pi(i)}$. In other words, $m(a_i) = m(b_{\pi(i)})$ for all i if and only if $a = {}^{\rho\pi}b$ for some $\rho \in G^r$ and $\pi \in S_r$.

Since $\xi_j^{\rho_{ij}} = a_{ij}/b_{\pi(i)j}$, we see that

$$\rho_i^{-1}(x) = \sigma_1^{-\rho_{i1}} \cdots \sigma_N^{-\rho_{iN}} x = \xi_1^{-\rho_{i1}k_1} \cdots \xi_N^{-\rho_{iN}k_N} x = (b^k_{\pi(i)}/a_i^k)x$$

for all $k \in \mathbb{Z}^N$ and $x \in \mathfrak{g}_{\bar{k}}$. Therefore, the automorphism ω_i of Theorem 5.4 is equal to ρ_i^{-1}, and $\lambda = {}^{\rho\pi}\mu$ is equivalent to the condition that $\lambda_i = \mu_{\pi(i)} \circ \gamma_i$ for every i. □

For any diagram automorphism σ_1, the finite-dimensional simple modules for the twisted (single) loop algebra $\mathscr{L}(\mathfrak{g}; \sigma_1)$ were classified in [Chari et al. 2008]. Recently, E. Neher, A. Savage, and P. Senesi [Senesi 2009] have reinterpreted this work in terms of finitely supported σ_1-equivariant maps $F^\times \to P_+$, where P_+ is the set of nonzero dominant integral weights of $\mathfrak{g}$ with respect to a fixed Cartan subalgebra and base of simple roots. Theorem 5.4 and Corollary 5.9 can be used to extend this perspective to the multiloop setting.

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ and $a = (a_1, \dots, a_r)$ be as in Theorem 5.4. Each evaluation module $V_{\lambda_i}(a_i)$ corresponds to a map

$$\chi_{\lambda_i, a_i} : (F^\times)^N \to P_+, \quad x \mapsto \delta_{x, a_i}\lambda_i.$$

The isomorphism class $[\lambda, a]$ of the tensor product $V(\lambda, a)$ can then be identified with the sum of all the characters χ_{η_0, c_0} for which $(\eta_0, c_0) = (\mu_1, b_1)$ for some $\mu = (\mu_1, \dots, \mu_r)$ and $b = (b_1, \dots, b_r)$ with (μ, b) in the $G^r \rtimes S_r$-orbit of (λ, a).

That is, we let

$$\chi_{[\lambda,a]} = \sum_{g\in G}\sum_{i=1}^{r} \chi_{\lambda_i\circ\gamma(g^{-1}),g\,.a_i}.$$

Thus to each isomorphism class of finite-dimensional simple $\mathscr{L}(\mathfrak{g};\sigma_1,\dots,\sigma_N)$-modules, we associate a finitely supported G-equivariant map

$$\chi_{[\lambda,a]} : (F^\times)^N \to P_+.$$

From Corollary 5.9 and the construction of $\chi_{[\lambda,a]}$, it is easy to see that distinct isomorphism classes get sent to distinct functions.

Conversely, any finitely supported G-equivariant map $f : (F^\times)^N \to P_+$ corresponds to an isomorphism class $[\lambda, a]$ of finite-dimensional simple $\mathscr{L}$-modules, as follows. By G-equivariance, the support $\operatorname{supp} f$ of f decomposes into a disjoint union of G-orbits. Choose representatives $a_1,\dots,a_r \in (F^\times)^N$ to label each G-orbit in $\operatorname{supp} f$. Since the G-orbits are disjoint, $m(a_i) \neq m(a_j)$ whenever $i \neq j$, and by definition of f, the r-tuple $\lambda := (f(a_1),\dots, f(a_r))$ consists of nonzero dominant integral weights. Then by Theorem 4.5, $V(\lambda, a)$ is a finite-dimensional simple $\mathscr{L}$-module, and by Corollary 5.9, the isomorphism class $[f] := [\lambda, a]$ of this module is independent of the choice of orbit representatives $a_1,\dots,a_r$. It is now straightforward to verify that $\chi_{[f]} = f$ for all finitely supported G-equivariant maps $f : (F^\times)^N \to P_+$.

Corollary 5.10. *The isomorphism classes of finite-dimensional simple $\mathscr{L}$-modules are in bijection with the finitely supported G-equivariant maps $(F^\times)^N \to P_+$.* □

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GRADIENT AND HARNACK INEQUALITIES ON NONCOMPACT MANIFOLDS WITH BOUNDARY

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By using the reflecting diffusion process and a conformal change of metric, a generalized maximum principle is established for (unbounded) time-space functions on a class of noncompact Riemannian manifolds with (nonconvex) boundary. As applications, Li–Yau-type gradient and Harnack inequalities are derived for the Neumann semigroup on a class of noncompact manifolds with (nonconvex) boundary. These generalize some previous ones obtained for the Neumann semigroup on compact manifolds with boundary. As a byproduct, the gradient inequality for the Neumann semigroup derived by Hsu on a compact manifold with boundary is confirmed on these noncompact manifolds.

1. Introduction

Suppose M is a d-dimensional connected complete Riemannian manifold, and let $L = \Delta + Z$, where Z is a C^1 vector field satisfying the curvature-dimension condition of Bakry and Émery [1984] given by

$$\Gamma_2(f, f) := \tfrac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle \geq \frac{(Lf)^2}{m} - K|\nabla f|^2 \quad \text{for } f \in C^\infty(M) \tag{1-1}$$

for some constants $K \geq 0$ and $m > d$. By [Qian 1998, page 138], this condition is equivalent to

$$\mathrm{Ric} - \nabla Z - \frac{Z \otimes Z}{m-d} \geq -K. \tag{1-2}$$

When $Z = 0$ and M is either without boundary or compact and with a convex boundary ∂M, Li and Yau [1986] found a now-famous gradient estimate for the (Neumann) semigroup P_t generated by L:

$$|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \leq \frac{d\alpha^2}{2t} + \frac{d\alpha^2 K}{4(\alpha-1)} \quad \text{for } t > 0 \text{ and } \alpha > 1 \tag{1-3}$$

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for all positive $f \in C_b(M)$. We note that in [Li and Yau 1986] the second term on the right side of (1-3) is $d\alpha^2 K/(\sqrt{2}(\alpha-1))$, but $\sqrt{2}$ here can be replaced by 4 according to a refined calculation; see for example [Davies 1989].

As an application, (1-3) implies a parabolic Harnack inequality for P_t:

$$P_t f(x) \le \left(\frac{t+s}{t}\right)^{d\alpha/2} (P_{t+s} f(y)) \exp\left(\frac{\alpha\rho(x,y)^2}{4s} + \frac{\alpha K d s}{4(\alpha-1)}\right) \quad \text{for } t>0 \text{ and } x,y\in M, \tag{1-4}$$

where $\alpha > 1$ and $f \in C_b(M)$ is positive. From this Harnack inequality, one obtains Gaussian-type heat kernel bounds for P_t; see [Li and Yau 1986; Davies 1989].

The gradient estimate (1-3) has been extended and improved in several papers. See for example [Bakry and Qian 1999] for an improved version for $\alpha = 1$ with $Z \neq 0$ and $\partial M = \varnothing$, and see [Wang 1997] for an extension to a compact manifold with nonconvex boundary. The aim of this paper is to investigate the gradient and Harnack inequalities for P_t on noncompact manifolds with (nonconvex) boundary.

Recall that the key step of Li and Yau's argument for the gradient estimate (1-3) is to apply the maximum principle to the reference function

$$G(t,x) := t(|\nabla \log P_t f|^2 - \alpha\partial_t \log P_t f)(x) \quad \text{for } t\in[0,T] \text{ and } x\in M.$$

When M is compact without boundary, the maximum principle says that for any smooth function G on $[0,T]\times M$ with $G(0,\cdot)\le 0$ and $\sup G > 0$, there exists a maximal point of G at which $\nabla G = 0$, $\partial_t G \ge 0$, and $\Delta G \le 0$. When M is compact with a convex boundary, the same assertion holds for the above specified function G, as observed in [Li and Yau 1986, proof of Theorem 1.1]. In [1997], J. Wang extended this maximum principle on a compact manifold with nonconvex boundary by taking

$$G(t,x) = t(\phi|\nabla \log P_t f|^2 - \alpha\partial_t \log P_t f)(x) \quad \text{for } t\in[0,T] \text{ and } x\in M$$

for a nice function ϕ compensating the concavity of the boundary.

As for a noncompact manifold without boundary, Li and Yau [1986] established the gradient estimate by applying the maximal principle to a sequence of functions with compact support that approximate the original function G. An alternative is to apply directly the following generalized maximum principle:

Lemma 1.1 [Yau 1975]. *For any bounded smooth function G on $[0,T]\times M$ with $G(0,\cdot)\le 0$ and $\sup G>0$, there exists a sequence $\{(t_n,x_n)\}_{n\ge1}\subset[0,T]\times M$ such that*

(i) $0 < G(t_n,x_n) \uparrow \sup G$ *as* $n\uparrow\infty$, *and*

(ii) *for any* $n\ge 1$,

$$LG(t_n,x_n)\le 1/n, \quad |\nabla G(t_n,\cdot)(x_n)|\le 1/n, \quad \partial_t G(t_n,x_n)\ge 0.$$

To apply this generalized maximal principle for the gradient estimate, one has to first confirm the boundedness of $G(t,\cdot) := t(|\nabla \log P_t f|^2 - \alpha\partial_t \log P_t f)$ on $[0,T]\times M$ for $T>0$.

Since the boundedness of this type of reference function is unknown when M is noncompact with a nonconvex boundary, we shall establish a generalized maximum principle on a class of noncompact manifolds with boundary for not necessarily bounded functions. Applying this principle to a suitable reference function G, we derive the Li–Yau-type gradient and Harnack inequalities for Neumann semigroups. To establish such a maximum principle, we adopt a localization argument so that the classical maximum principle can be applied.

For M noncompact without boundary, Li and Yau [1986] used such a localization argument to apply the maximal principle to functions with compact support; they then passed to the desired global estimate by taking a limit. To do this, they constructed cut-off functions using ρ_o, the Riemannian distance function to a fixed point $o\in M$. It turns out that this argument works also when ∂M is convex; see Section 2.1. For the nonconvex case, we will use the conformal change of metric introduced in [Wang 2007] to make a nonconvex boundary convex; see Section 2.2.

Assumption A. The manifold M is connected and complete with boundary ∂M and such that either

(1) ∂M is convex, or

(2) the second fundamental form of ∂M is bounded, the sectional curvature of M is bounded from above, and the injectivity radius $\mathrm{i}_{\partial M}$ of ∂M is positive.

Recall that the Riemannian distance function $\rho_{\partial M}$ to the boundary is smooth on the set $\{\rho_{\partial M} < \mathrm{i}_{\partial M}\}$.

Let N be the inward unit normal vector field on ∂M. The second fundamental form of ∂M is

$$\mathrm{II}(X,Y) = -\langle \nabla_X N, Y\rangle \quad \text{for } X, Y\in T\partial M.$$

The boundary ∂M is called convex if $\mathrm{II}\geq 0$. We are now ready to state our generalized maximal principle for possibly unbounded functions.

Theorem 1.2. *Let M satisfy A, and let L satisfy* (1-2). *Let $T>0$, and let G be a smooth function on* $[0,T]\times M$ *such that* $NG|_{\partial M}\geq 0$, $G(0,\cdot)\leq 0$ *and* $\sup G>0$. *Then for any $\varepsilon>0$, there exists a sequence* $\{(t_n,x_n)\}_{n\geq 1}\subset(0,T]\times M$ *such that Lemma 1.1(i) holds and for any $n\geq 1$*

$$LG(t_n,x_n)\leq \frac{G(t_n,x_n)^{1+\varepsilon}}{n},\qquad |\nabla G(t_n,x_n)|\leq \frac{G(t_n,x_n)^{1+\varepsilon}}{n},$$
$$\partial_t G(t_n,x_n)\geq 0.$$

Applying Theorem 1.2 to a proper choice of function G, we will derive the Li–Yau-type gradient estimate (1-5). We shall prove that the reflecting diffusion process X_t generated by L on M is non explosive, so that the corresponding Neumann semigroup P_t can be formulated as

$$P_t f(x) = \mathsf{E}^x f(X_t) \quad \text{for } t \geq 0,\ x \in M, \text{ and } f \in C_b(M),$$

where E^x is the expectation taken for $X_0 = x$.

Theorem 1.3. *Let M satisfy A, and suppose L satisfies* (1-2) *with $\|Z\|_\infty < \infty$. Then the reflecting L-diffusion process on M is nonexplosive and the corresponding Neumann semigroup P_t satisfies these assertions*:

(i) *If ∂M is convex, then* (1-3) *holds with m in place of d.*

(ii) *If ∂M is nonconvex with $\mathrm{II} \geq -\sigma$ for some $\sigma > 0$, then for any bounded $\phi \in C^\infty(M)$ with $\phi \geq 1$ and $N \log \phi|_{\partial M} \geq 2\sigma$, the gradient inequality*

$$|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \leq \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K(\phi,\varepsilon,\alpha)}{4(\alpha - \|\phi\|_\infty)} \tag{1-5}$$

holds for all positive $f \in C_b(M)$, $\alpha > \|\phi\|_\infty$, $t > 0$, $\varepsilon \in (0,1)$ and

$$K(\phi,\varepsilon,\alpha) := \frac{1+\varepsilon}{1-\varepsilon}\biggl(K + \tfrac{1}{\varepsilon}\|\nabla \log \phi\|_\infty^2 + \tfrac{1}{2}\sup(-\phi^{-1}L\phi) + \frac{m\alpha^2\|\nabla \log \phi\|_\infty^2(1+\varepsilon)}{8(\alpha - \|\phi\|_\infty)^2\varepsilon(1-\varepsilon)}\biggr).$$

We emphasize that the results in Theorem 1.3 are new for noncompact manifolds with boundary. When M is compact with a convex boundary, the first assertion was proved in [Li and Yau 1986] by using the classical maximum principle on compact manifolds, while when M is compact with a nonconvex boundary, an inequality similar to (1-5) was proved in [Wang 1997] by using the "interior rolling R-ball" condition.

These two theorems will be proved in Sections 2 and 3. By a standard argument due to Li and Yau [1986], the gradient estimate (1-5) implies a Harnack inequality. Let $\rho(x, y)$ be the Riemannian distance between $x, y \in M$, that is, the infimum of the length of all smooth curves in M that link x and y.

Corollary 1.4. *In the situation of Theorem 1.3 the Neumann semigroup P_t satisfies*

$$P_t f(x) \leq \Bigl(\frac{t+s}{t}\Bigr)^{m(1+\varepsilon)\alpha/2(1-\varepsilon)} (P_{t+s} f(y)) \exp\Bigl(\frac{\alpha\rho(x,y)^2}{4s} + \frac{\alpha m K(\phi,\varepsilon,\alpha)s}{4(\alpha - \|\phi\|_\infty)}\Bigr) \tag{1-6}$$

for all positive $f \in C_b(M)$, $t, \varepsilon \in (0,1)$, $\alpha > \|\phi\|_\infty$ and $x, y \in M$. In particular, if ∂M is convex, then (1-4) *holds with m in place of d and for all $\alpha > 1$.*

To derive explicit inequalities for the nonconvex case, we shall take a specific choice of ϕ as in [Wang 2007]. Let $\mathrm{i}_{\partial M}$ be the injectivity radius of ∂M, and let $\rho_{\partial M}$ be the Riemannian distance to the boundary. We shall take $\phi = \varphi \circ \rho_{\partial M}$ for a nice reference function φ on $[0, \infty)$. More precisely, let the sectional curvature satisfy $\mathrm{Sect}_M \le k$ and $-\sigma \le \mathrm{II} \le \gamma$ for some $k, \sigma, \gamma > 0$. Let

$$h(s) = \cos(\sqrt{k}\, s) - (\gamma/\sqrt{k}) \sin(\sqrt{k}\, s) \quad \text{for } s \ge 0.$$

Then h is the unique solution to the differential equation $h'' + kh = 0$ with boundary conditions $h(0) = 1$ and $h'(0) = -\gamma$. By the Laplacian comparison theorem for $\rho_{\partial M}$ (see [Kasue 1984, Theorem 0.3] or [Wang 2007]),

$$\Delta \rho_{\partial M} \ge \frac{(d-1)h'}{h}(\rho_{\partial M}) \quad \text{and} \quad \rho_{\partial M} < \mathrm{i}_{\partial M} \wedge h^{(-1)}(0), \tag{1-7}$$

where $h^{(-1)}(0) = (1/\sqrt{k}) \arcsin(\sqrt{k}/\sqrt{k+\gamma^2})$ is the first zero point of h. Fix a positive number $r_0 \le \mathrm{i}_{\partial M} \wedge h^{(-1)}(0)$, and let

$$\delta = \frac{2\sigma(1-h(r_0))^{d-1}}{\int_0^{r_0} (h(s)-h(r_0))^{d-1}\,\mathrm{d}s},$$

$$\varphi(r) = 1 + \delta \int_0^r (h(s)-h(r_0))^{1-d}\,\mathrm{d}s \int_{s\wedge r_0}^{r_0} (h(u)-h(r_0))^{d-1}\,\mathrm{d}u.$$

It is easy to see that $\varphi \circ \rho_{\partial M}$ is differentiable with a Lipschitzian gradient. By a simple approximation argument, we may apply Theorem 1.3 and Corollary 1.4 to $\phi = \varphi \circ \rho_{\partial M}$; see [Wang 2007, page 1436].

Obviously, (1-7) and $N = \nabla \rho_{\partial M}$ imply

$$\Delta \varphi \circ \rho_{\partial M} \ge -\delta \quad \text{and} \quad N \log \varphi \circ \rho_{\partial M}|_{\partial M} = \varphi'(0)/\varphi(0) = 2\sigma.$$

Moreover, by [Wang 2007, (20)] we have

$$\delta \le 2\sigma d r_0^{-1} \quad \text{and} \quad \varphi(r_0) \le 1 + \sigma d r_0.$$

Thus, for $\phi := \varphi \circ \rho_{\partial M}$ we have

$$-\phi^{-1} L\phi \le 2\sigma d r_0^{-1} + 2\sigma \|Z\|_\infty, \quad \|\nabla \log \phi\|_\infty^2 \le 4\sigma^2,$$
$$\|\phi\|_\infty \le \varphi(r_0) \le 1 + \sigma d r_0.$$

Combining these with Theorem 1.3 and Corollary 1.4, we obtain these explicit inequalities on a class of nonconvex and noncompact manifolds:

Corollary 1.5. *Let* $\mathrm{i}_{\partial M} > 0$, *and suppose* $\gamma \ge \mathrm{II} \ge -\sigma$ *and* $\mathrm{Sect}_M \le k$ *for some* $\gamma, \sigma, k > 0$. *If* (1-2) *holds and* $\|Z\|_\infty < \infty$, *then for any positive number*

$$r_0 \le \min\{\mathrm{i}_{\partial M}, (1/\sqrt{k}) \arcsin(\sqrt{k}/\sqrt{k+\gamma^2})\},$$

the inequalities

$$|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \le \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K_\varepsilon}{4(\alpha - 1 - \sigma d r_0)}$$

and

$$P_t f(x) \le \Big(\frac{t+s}{t}\Big)^{m(1+\varepsilon)\alpha/2(1-\varepsilon)} (P_{t+s} f(y)) \exp\Big(\frac{\alpha\rho(x,y)^2}{4s} + \frac{m\alpha K_\varepsilon s}{4(\alpha-1-\sigma d r_0)}\Big) \quad \text{for } x, y \in M$$

hold for all positive $f \in C_b(M)$, $t > 0$, $\varepsilon \in (0,1)$, $\alpha > 1 + \sigma d r_0$, *and*

$$K_\varepsilon = \frac{1+\varepsilon}{1-\varepsilon}\Big(K + \frac{4\sigma^2}{\varepsilon} + \frac{\sigma d}{r_0} + \sigma \|Z\|_\infty + \frac{m\alpha^2\sigma^2(1+\varepsilon)}{2(\alpha-1-\sigma d r_0)^2 \varepsilon(1-\varepsilon)}\Big).$$

Combining our gradient estimate with an approximation and a probabilistic argument, we can derive the gradient estimate (1-9) for a class of noncompact manifolds:

Theorem 1.6. *Let M satisfy A, and let L satisfy* (1-2) *with* $\|Z\|_\infty < \infty$. *Let* κ_1 *and* κ_2 *be positive elements of* $C_b(M)$ *such that*

$$\text{Ric} - \nabla Z \ge -\kappa_1 \quad \textit{and} \quad \text{II} \ge -\kappa_2 \tag{1-8}$$

hold on M and ∂M, respectively. Then

$$|\nabla P_t f|(x) \le \mathsf{E}^x\Big(|\nabla f|(X_t) \exp\Big(\int_0^t \kappa_1(X_s)\,\mathrm{d}s + \int_0^t \kappa_2(X_s)\,\mathrm{d}l_s\Big)\Big) \tag{1-9}$$

holds for all $f \in C_b^1(M)$, $t > 0$, *and* $x \in M$.

Inequality (1-9) was first derived by Hsu [2002] on a compact manifold with boundary. In [2002, Theorem 3.7], Hsu applied the Itô formula to $F(U_t, T-t) := U_t^{-1} \nabla P_{T-t} f(X_t)$, where U_t is the horizontal lift of X_t on the frame bundle $O(M)$. Since M is compact, the (local) martingale part of this process is a real martingale (it may not be for noncompact M). Then the desired gradient estimate followed immediately from [2002, Corollary 3.6]. In Section 4, we will prove the boundedness of $\nabla P_{(\cdot)} f$ on $[0, T] \times M$ for any $T > 0$ and $f \in C_b^1(M)$, which leads to a simple proof of (1-9) for a class of noncompact manifolds.

2. Proof of Theorem 1.2

We consider the convex case and pass to the nonconvex case using the conformal change of metric constructed in [Wang 2007]. Without loss of generality, we may assume that $\sup G := \sup_{[0,T]\times M} > 1$. (Otherwise, we simply replace G by mG for a sufficiently large $m > 0$.)

2.1. ***Convex*** ∂M**.** Fix $o \in M$, and let ρ_o be the Riemannian distance to the point o. Since ∂M is convex, there exists a minimal geodesic in M of length $\rho(x, y)$ that links any x and y in M; see for example [Wang 2005a, Proposition 2.1.5]. So, by (1-2) and a comparison theorem (see [Qian 1998])

$$L\rho_o \le \sqrt{K(m-1)}\ \coth\bigl(\sqrt{K/(m-1)}\,\rho_o\bigr)$$

holds outside $\{o\}\cup\text{cut}(o)$, where $\text{cut}(o)$ is the cut locus of o. In the sequel, we will set $L\rho_o = 0$ on $\text{cut}(o)$ so that this implies

$$L\sqrt{1+\rho_o^2} \le c_1 \quad \text{on } M \tag{2-1}$$

for some constant $c_1 > 0$.

Let $h \in C_0^\infty([0,\infty))$ be decreasing such that

$$h(r) = \begin{cases} 1 & \text{if } r \le 1,\\ \exp(-(3-r)^{-1}) & \text{if } r \in [2,3),\\ 0 & \text{if } r \ge 3.\end{cases}$$

Obviously, for any $\varepsilon > 0$ we have

$$\sup_{[0,\infty)} \bigl\{|h^{\varepsilon-1}h''| + |h^{\varepsilon-1}h'|\bigr\} < \infty. \tag{2-2}$$

Let $W = \sqrt{1+\rho_o^2}$, and take $\varphi_n = h(W/n)$ for $n \ge 1$. Then

$$\{\varphi_n = 1\} \uparrow M \quad \text{as } n \uparrow \infty. \tag{2-3}$$

So, according to (2-1) and (2-2),

$$\begin{aligned} &|\nabla \log \varphi_n| \le \frac{c}{n\varphi_n^\varepsilon},\\ &\varphi_n^{-1} L\varphi_n = \frac{h'(W/n)}{nh(W/n)} LW + \frac{h''(W/n)}{n^2 h(W/n)} |\nabla W|^2 \ge -\frac{c}{n\varphi_n^\varepsilon} \end{aligned} \tag{2-4}$$

holds for some constant $c > 0$ and all $n \ge 1$.

Let $G_n(t,x) = \varphi_n(x)G(t,x)$ for $t \in [0,T]$ and $x \in M$. Since G_n is continuous with compact support, there exists $(t_n, x_n) \in [0,T] \times M$ such that

$$G_n(t_n, x_n) = \max_{[0,T]\times M} G_n.$$

By (2-3) and that $\sup G > 1$, we have $\lim_{n\to\infty} G(t_n, x_n) = \sup G > 1$. By renumbering from a sufficient large n_0, we may assume that $G_n(t_n, x_n)$ is greater than 1 and is increasing in n. In particular, Lemma 1.1(i) holds and

$$\varphi_n(x_n) \ge 1/G(t_n, x_n) \quad \text{for } n \ge 1. \tag{2-5}$$

Moreover, since $G_n(0,\cdot)\le 0$, we have $t_n>0$ and $\partial_t G(t_n,x_n)\ge 0$ for $n\ge 1$. Thus, it remains to confirm that

$$\begin{aligned} |\nabla G(t_n,x_n)| &\le cG(t_n,x_n)^{1+\varepsilon}/n \quad \text{and}\\ LG(t_n,x_n) &\le cG(t_n,x_n)^{1+\varepsilon}/n \quad \text{for } n\ge 1 \end{aligned} \tag{2-6}$$

for some constant $c>0$. Indeed, by using a subsequence $\{(t_{mn},x_{mn})\}_{n\ge1}$ for $m\ge c$ to replace $\{(t_n,x_n)\}_{n\ge1}$, one may reduce (2-6) with some $c>0$ to that with $c=1$.

Since x_n is the maximal point of G_n, we have $\nabla G_n(t_n,x_n)=0$ if $x_n\in M\setminus\partial M$. If $x_n\in\partial M$, we have $NG_n(t_n,x_n)\le 0$. Recall that $NG(t_n,\cdot)\ge 0$ and $G(t_n,x_n)>0$. Then, noting that $N\rho_0\le 0$ together with $h'\le 0$ implies $N\varphi_n\ge 0$, we conclude that $NG_n(t_n,x_n)\ge 0$. Hence, $NG_n(t_n,x_n)=0$. Moreover, since x_n is the maximal point of $G_n(t_n,\cdot)$ on the closed manifold ∂M, we have $UG_n(t_n,x_n)=0$ for all $U\in T\partial M$. Therefore, $\nabla G_n(t_n,x_n)=0$ also holds for $x_n\in\partial M$. Combining this with (2-4) and (2-5), we obtain

$$|\nabla G(t_n,x_n)|\le \frac{G(t_n,x_n)}{\varphi_n(x_n)}|\nabla\varphi_n|\le\frac{cG(t_n,x_n)^{1+\varepsilon}}{n},$$

which proves the first inequality in (2-6).

Finally, by (2-4), the inequality

$$\varphi_n L_n G + GL_n\varphi_n + 2\langle\nabla G,\nabla\varphi_n\rangle \ge \varphi_n L_n G - \frac{c\varphi_n^{1-\varepsilon}}{n}G - \frac{2c\varphi_n^{1-\varepsilon}}{n}|\nabla G| =: \Phi$$

holds on $\{G_n>0\}\setminus\mathrm{cut}(o)$. By Lemma 2.1 below we obtain at the point (t_n,x_n) that

$$LG\le\frac{c}{n\varphi_n^{\varepsilon}}G+\frac{2c}{n\varphi_n}|\nabla G|.$$

Combining this with (2-5) and the first inequality in (2-6), we get

$$LG(t_n,x_n)\le\frac{c}{n}G^{1+2\varepsilon}(t_n,x_n)$$

for some constant $c>0$ and all $n\ge 1$. Since $\varepsilon>0$ is arbitrary, we may replace ε by $\varepsilon/2$ (recall that $G(t_n,x_n)\ge 1$). This proves the second inequality in (2-6).

Lemma 2.1. *The reflecting L-diffusion process is nonexplosive, and for any Φ in $C_b(M)$ such that*

$$\Phi\le LG_n = GL\varphi_n+\varphi_n LG+2\langle\nabla\varphi_n,\nabla G\rangle \quad \text{on } \{G_n>0\}\setminus\mathrm{cut}(o),$$

we have $\Phi(t_n,x_n)\le 0$ for all $n\ge 1$.

Proof. Let X_t be the reflecting L-diffusion process generated by L, and let U_t be its horizontal lift on the frame bundle $O(M)$. By the Itô formula for $\rho_o(X_t)$

found by Kendall [1987] for $\partial M = \varnothing$ and by the fact that $N\rho_o|_{\partial M} \le 0$ when ∂M is nonempty but convex, we have

$$\mathrm{d}\rho_o(X_t) = \sqrt{2}\langle \nabla \rho_o(X_t), U_t \,\mathrm{d}B_t\rangle + L\rho_o(X_t)\,\mathrm{d}t - \mathrm{d}l_t + \mathrm{d}l'_t, \tag{2-7}$$

where B_t is the d-dimensional Brownian motion, where $L\rho_o$ is taken to be zero on $\{o\} \cup \mathrm{cut}(o)$, and where l_t and l'_t are two increasing processes such that l'_t increases only when $X_t = o$, while l_t increases only when $X_t \in \mathrm{cut}(o) \cup \partial M$ (note that $l'_t = 0$ for $d \ge 2$). Combining this with (2-1) we obtain

$$\mathrm{d}\sqrt{1+\rho_o^2(X_t)} \le \mathrm{d}M_t + L\sqrt{1+\rho_o^2(X_t)}\,\mathrm{d}t \le \mathrm{d}M_t + c_1\,\mathrm{d}t$$

for some martingale M_t. This implies immediately that X_t does not explode.

Now, let us take $X_0 = x_n$. Since $h' \le 0$, it follows from (2-7) that

$$\mathrm{d}\varphi_n(X_t) \ge \sqrt{2}\langle \nabla \varphi_n(X_t), U_t \,\mathrm{d}B_t\rangle + L\varphi_n(X_t)\,\mathrm{d}t, \tag{2-8}$$

where we set $L\varphi_n = 0$ on $\mathrm{cut}(o)$ as above.

On the other hand, since $NG(t_n, \cdot) \ge 0$, we may apply the Itô to $G(t_n, X_t)$ to obtain

$$\mathrm{d}G(t_n, X_t) \ge \sqrt{2}\langle \nabla G(t_n, X_t), U_t \,\mathrm{d}B_t\rangle + LG(t_n, X_t)\,\mathrm{d}t. \tag{2-9}$$

Because $G_n(t_n, x_n) > 0$, there exists an $r > 0$ such that $G_n > 0$ on $B(x_n, r)$, the geodesic ball in M centered at x_n with radius r. Let

$$\tau = \inf\{t \ge 0 : \ X_t \notin B(x_n, r)\}.$$

Then (2-8) and (2-9) imply

$$\mathrm{d}G_n(t_n, X_t) \ge \mathrm{d}M_t + LG_n(t_n, \cdot)(X_t)\,\mathrm{d}t \ge \mathrm{d}M_t + \Phi(t_n, X_t)\,\mathrm{d}t \quad \text{for } t \le \tau$$

for some martingale M_t. Since $G_n(t_n, X_t) \le G_n(t_n, x_n)$ and $X_0 = x_n$, this implies that

$$0 \ge \mathsf{E}G_n(t_n, X_{t\wedge\tau}) - G_n(t_n, x_n) \ge \mathsf{E}\int_0^{t\wedge\tau} \Phi(t_n, X_s)\,\mathrm{d}s.$$

Therefore, the continuity of Φ implies that

$$\Phi(t_n, x_n) = \lim_{t\to 0} \frac{1}{\mathsf{E}(t\wedge\tau)} \mathsf{E}\int_0^{t\wedge\tau} \Phi(t_n, X_s)\,\mathrm{d}s \le 0. \qquad \square$$

2.2. ***Nonconvex*** $\boldsymbol{\partial M}$**.** Under our assumptions on M, there exists a constant $R > 1$ and a function $\phi \in C^\infty(M)$ such that

$$1 \le \phi \le R, \quad |\nabla\phi| \le R, \quad N\log\phi|_{\partial M} \ge \sigma.$$

By [Wang 2007, Lemma 2.1], the boundary ∂M is convex under the new metric $\langle\cdot,\cdot\rangle' := \phi^{-2}\langle\cdot,\cdot\rangle$. Let $L' = \phi^2 L$. By [Wang 2007, Equation (9)], the vector

$U' := \phi U$ is unit under the new metric for any unit vector $U \in TM$, and the corresponding Ricci curvature satisfies

$$\text{(2-10)}\quad \mathrm{Ric}'(U', U') \ge \phi^2 \,\mathrm{Ric}(U, U) + \phi\Delta\phi - (d-3)|\nabla\phi|^2 - 2(U\phi)^2 + (d-2)\phi \,\mathrm{Hess}_\phi(U, U).$$

Let Δ' be the Laplacian induced by the new metric. By [Wang 2007, Lemma 2.2], we have

$$L' := \phi^2 L = \Delta' + (d-2)\phi\nabla\phi + \phi^2 Z =: \Delta' + Z'.$$

Noting that

$$\nabla'_X Y = \nabla_X Y - \langle X, \nabla \log\phi\rangle Y - \langle Y, \nabla\log\phi\rangle X + \langle X, Y\rangle \nabla\log\phi \quad \text{for } X, Y \in TM,$$

we have

$$\begin{aligned}\langle \nabla_{U'} Z', U'\rangle' &= \langle \nabla_U Z', U\rangle - \langle Z', \nabla\log\phi\rangle \\ &= \phi^2 \langle \nabla_U Z, U\rangle + (U\phi^2)\langle Z, U\rangle + (d-2)(U\phi)^2 \\ &\qquad + (d-2)\phi\,\mathrm{Hess}_\phi(U, U) - \langle Z', \nabla\log\phi\rangle.\end{aligned}$$

Combining this with (2-10) and the properties of ϕ mentioned above, we find a constant $c_1 > 0$ such that

$$\text{(2-11)}\quad \mathrm{Ric}'(U,' U') - \langle \nabla'_{U'} Z', U'\rangle' \ge \phi^2(\mathrm{Ric} - \nabla Z)(U, U) - c_1 \quad \text{for } |U| = 1.$$

Moreover, since

$$\begin{aligned}(Z' \otimes' Z')(U', U') := (\langle Z', U'\rangle')^2 = \phi^{-2}\langle Z', U\rangle^2 &\le 2(d-2)^2\langle\nabla\phi, U\rangle^2 + 2\phi^2\langle Z, U\rangle^2 \\ &\le 2(d-2)^2 R^2 + 2\phi^2 (Z\otimes Z)(U, U),\end{aligned}$$

it follows from (1-2) and (2-11) that

$$\mathrm{Ric}' - \nabla' Z' - \frac{Z' \otimes' Z'}{2(m-d)} \ge -\phi^2 K - c_2 \ge -K'$$

holds for the metric $\langle\cdot,\cdot\rangle'$ and some constants $c_2, K' > 0$. Therefore, we may apply Lemma 2.1 to L' on the convex Riemannian manifold $(M, \langle\cdot,\cdot\rangle')$ to conclude that the desired sequence $\{(t_n, x_n)\}$ exists.

3. Proofs of Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3. When ∂M is convex, Lemma 2.1 ensures that X_t does not explode. If ∂M is nonconvex, this can be confirmed by reparametrizing the time of the process. More precisely, let X'_t be the reflecting diffusion process on M generated by $L' := \phi^2 L$ constructed in Section 2.2. Since $L' = \Delta' + Z'$

satisfies (1-2) for some $K > 0$ on the convex manifold $(M, \langle\cdot,\cdot\rangle')$, the process X'_t generated by L' is nonexplosive by Lemma 2.1. Since $X_t = X'_{\xi^{-1}(t)}$, where ξ^{-1} is the inverse of

$$t \mapsto \xi(t) = \int_0^t \phi^2(X'_s)\,\mathrm{d}s,$$

we have $t\|\phi\|_\infty^{-2} \le \xi^{-1}(t) \le t$, and the process X_t is nonexplosive as well.

Let $f \in C_b^1(M)$ be strictly positive, and let $u(t,x) = \log P_t f(x)$. For a fixed number $T > 0$, we will apply Theorem 1.2 to the reference function

$$G(t,x) = t\big\{\phi(x)|\nabla u|^2(t,x) - \alpha u_t(t,x)\big\} \quad \text{for } t \in [0,T] \text{ and } x \in M.$$

Note that $\mathrm{II} \ge -\sigma$ and $N\log\phi \ge 2\sigma$ imply

$$\begin{aligned} &N\phi \ge 2\sigma\phi,\\ &N|\nabla P_t f|^2 = 2\,\mathrm{Hess}_{P_t f}(N, \nabla P_t f) = 2\mathrm{II}(\nabla P_t f, \nabla P_t f) \ge -2\sigma|\nabla P_t f|^2. \end{aligned}$$

Since $P_t f$ and hence u_t satisfy the Neumann boundary condition, this implies that

$$NG = t\left\{(N\phi)|\nabla u|^2 + \frac{\phi}{(P_t f)^2} N|\nabla P_t f|^2\right\} \ge t\{2\sigma\phi|\nabla u|^2 - 2\sigma\phi|\nabla u|^2\} = 0 \quad \text{on } \partial M.$$

According to [Ledoux 2000, (1.14)], inequality (1-2) implies

$$L|\nabla u|^2 - 2\langle\nabla Lu, \nabla u\rangle \ge -2K|\nabla u|^2 + \frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}. \tag{3-1}$$

By multiplying this inequality by ε and (1-1) by $2(1-\varepsilon)$ and by combining the results, we obtain

$$L|\nabla u|^2 \ge 2\langle\nabla Lu, \nabla u\rangle - 2K|\nabla u|^2 + \frac{2(1-\varepsilon)(Lu)^2}{m} + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}.$$

It is also easy to check that $Lu = u_t - |\nabla u|^2$ and $\partial_t|\nabla u|^2 = 2\langle\nabla u, \nabla u_t\rangle$. Then we arrive at

$$\begin{aligned} &(L-\partial_t)|\nabla u|^2\\ &\quad\ge \frac{2(1-\varepsilon)}{m}(|\nabla u|^2 - u_t)^2 + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2} - 2\langle\nabla u, \nabla|\nabla u|^2\rangle - 2K|\nabla u|^2. \end{aligned} \tag{3-2}$$

On the other hand,

$$\begin{aligned} -\alpha(L-\partial_t)u_t &= 2\alpha\langle\nabla u, \nabla u_t\rangle = 2\langle\nabla u, \nabla(\phi|\nabla u|^2 - t^{-1}G)\rangle\\ &= 2\phi\langle\nabla u, \nabla|\nabla u|^2\rangle + 2|\nabla u|^2\langle\nabla u, \nabla\phi\rangle - 2t^{-1}\langle\nabla u, \nabla G\rangle. \end{aligned}$$

Combining this with (3-2), we obtain

$$\begin{aligned}(L-\partial_t)G &= -\frac{G}{t}+t\big(\phi(L-\partial_t)|\nabla u|^2+|\nabla u|^2L\phi+2\langle\nabla\phi,\nabla|\nabla u|^2\rangle\big)\\ &\qquad+t\big(2\phi\langle\nabla u,\nabla|\nabla u|^2\rangle+2|\nabla u|^2\langle\nabla u,\nabla\phi\rangle-2t^{-1}\langle\nabla u,\nabla G\rangle\big)\\ &\ge -\frac{G}{t}+\frac{2(1-\varepsilon)\phi t}{m}(|\nabla u|^2-u_t)^2+\frac{\varepsilon\phi t|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}-2K\phi t|\nabla u|^2\\ &\qquad-2|\nabla u|\cdot|\nabla G|-2t|\nabla u|^3|\nabla\phi|-2t|\nabla\phi|\cdot|\nabla|\nabla u|^2|+t|\nabla u|^2L\phi.\end{aligned}$$

Noting that

$$\frac{\varepsilon\phi t|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}-2t|\nabla\phi|\cdot|\nabla|\nabla u|^2|\ge-\frac{2t|\nabla\phi|^2|\nabla u|^2}{\varepsilon\phi},$$

we get

$$\begin{aligned}(L-\partial_t)G\ge-\frac{G}{t}+\frac{2(1-\varepsilon)\phi t}{m}(|\nabla u|^2-u_t)^2-2K\phi t|\nabla u|^2-2|\nabla u|\cdot|\nabla G|\\ -2t|\nabla u|^3|\nabla\phi|+t|\nabla u|^2L\phi-\frac{2t|\nabla\phi|^2|\nabla u|^2}{\varepsilon\phi}.\end{aligned}\tag{3-3}$$

We assume that $\sup G>0$, otherwise the proof is done. Since $G(0,\cdot)=0$ and $NG|_{\partial M}\ge 0$, we can apply Theorem 1.2. Let $\{(t_n,x_n)\}$ be fixed in Theorem 1.2 with, for example, $\varepsilon=1/2$. Then,

$$(L-\partial_t)G(t_n,x_n)\le\frac{G^{3/2}(t_n,x_n)}{n}\quad\text{and}\quad|\nabla G|(t_n,x_n)\le\frac{G^{3/2}(t_n,x_n)}{n}.\tag{3-4}$$

From now on, we evaluate functions at the point (t_n,x_n), so that $t=t_n$.

Let $\mu=|\nabla u|^2/G$. We have

$$|\nabla u|^2-u_t=\Big(\mu-\frac{(\mu t-1)\phi}{\alpha t}\Big)G=\frac{\mu t(\alpha-\phi)+\phi}{\alpha t}G.$$

Combining this with (3-3) and (3-4), we arrive at

$$\begin{aligned}&\frac{2(1-\varepsilon)\phi(\mu t(\alpha-\phi)+\phi)^2}{m\alpha^2t}G^2\\ &\le\frac{G^{3/2}}{n}+\frac{G}{t}+\frac{2\sqrt{\mu}G^2}{n}+2t|\nabla\phi|(\mu G)^{3/2}+(2k\phi+2\varepsilon^{-1}\phi^{-1}|\nabla\phi|^2-L\phi)\mu tG.\end{aligned}\tag{3-5}$$

Since it is easy to see that

$$(\mu t(\alpha-\phi)+\phi)^2\ge\max\big\{\phi^2,4\mu t(\alpha-\phi)\phi,(2t(\alpha-\phi))^{3/2}\sqrt{\phi}\mu^{3/2}\big\},$$

we may multiply both sides of (3-5) by $t(\mu t(\alpha-\phi)+\phi)^{-2}G^{-2}$ to obtain

$$\begin{aligned}\frac{2(1-\varepsilon)\phi}{m\alpha^2} &\le \frac{c't}{n(1\wedge\sqrt{G})}+\frac{1}{\phi^2 G}+\frac{2K+2\varepsilon^{-1}|\nabla\log\phi|^2-\phi^{-1}L\phi}{4(\alpha-\phi)G}t \\ &\qquad\qquad +\frac{|\nabla\log\phi|\sqrt{t\phi}}{(\alpha-\phi)^{3/2}\sqrt{2G}} \\ &\le \frac{c't}{n(1\wedge\sqrt{G})}+\frac{1}{\phi^2 G}+\frac{2K+2\varepsilon^{-1}|\nabla\log\phi|^2-\phi^{-1}L\phi}{4(\alpha-\phi)G} \\ &\qquad\qquad +\frac{|\nabla\log\phi|^2 m\alpha^2(1+\varepsilon)t}{16(\alpha-\phi)^3\varepsilon(1-\varepsilon)G}+\frac{2(1-\varepsilon)\varepsilon\phi}{m\alpha^2(1+\varepsilon)}\end{aligned}$$

for some constant $c'>0$. Taking $n\to\infty$ and noting that $\phi\ge 1$, we conclude that $\theta:=\sup G$ satisfies

$$\begin{aligned}\frac{2(1-\varepsilon)}{m\alpha^2(1+\varepsilon)} \le \frac{1}{\theta}\bigg(1&+\frac{2K+2\varepsilon^{-1}\|\nabla\log\phi\|_\infty^2+\sup(-\phi^{-1}L\phi)}{4(\alpha-\|\phi\|_\infty)}T \\ &+\frac{\|\nabla\log\phi\|_\infty^2 m\alpha^2(1+\varepsilon)T}{16(\alpha-\|\phi\|_\infty)^3\varepsilon(1-\varepsilon)}\bigg).\end{aligned}$$

Combining this with $\theta\ge G(T,x)=T(\phi(x)|\nabla u|^2(T,x)-\alpha u_t(T,x))$ for $x\in M$, we obtain

$$\begin{aligned}&\phi(x)|\nabla u|^2(T,x)-\alpha u_t(T,x) \\ &\quad\le \frac{m\alpha^2(1+\varepsilon)}{2(1-\varepsilon)}\Big(\frac{1}{T}+\frac{2K+2\varepsilon^{-1}\|\nabla\log\phi\|_\infty^2+\sup(-\phi^{-1}L\phi)}{4(\alpha-\|\phi\|_\infty)} \\ &\qquad\qquad +\frac{\|\nabla\log\phi\|_\infty^2 m\alpha^2(1+\varepsilon)}{16(\alpha-\|\phi\|_\infty)^3\varepsilon(1-\varepsilon)}\Big)\end{aligned}$$

for all $x\in M$. Then the proof is completed since $T>0$ is arbitrary. □

Proof of Corollary 1.4. By Theorem 1.3, the proof is standard according to [Li and Yau 1986]. For $x, y\in M$, let $\gamma:[0,1]\to M$ be the shortest curve in M linking x and y such that $|\dot\gamma|=\rho(x,y)$. Then, for any $s,t>0$ and $f\in C_b^\infty(M)$, it follows from (1-5) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\log P_{t+rs}f(\gamma_r) &= s\partial_u\log P_u f(\gamma_r)|_{u=t+rs}+\langle\dot\gamma_r,\nabla P_{t+rs}f(\gamma_r)\rangle \\ &\ge \frac{s}{\alpha}|\nabla\log P_{t+rs}f|^2(\gamma_r)-\rho(x,y)|\nabla\log f|(\gamma_r) \\ &\qquad -s\Big(\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)}+\frac{m\alpha K(\phi,\varepsilon,\alpha)}{4(\alpha-1\|\phi\|_\infty)}\Big) \\ &\ge -\frac{\alpha}{4s}-s\Big(\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)}+\frac{m\alpha K(\phi,\varepsilon,\alpha)}{4(\alpha-\|\phi\|_\infty)}\Big).\end{aligned}$$

We complete the proof by integrating with respect to dr over [0, 1]. ☐

4. Proof of Theorem 1.6

We first provide a simple proof of (1-9) under an extra assumption that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$ for any $T > 0$; we then drop this assumption by an approximation argument.

Lemma 4.1. *If that $f \in C_b^1(M)$ is such that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$ for any $T > 0$, then* (1-9) *holds.*

Proof. For any $\varepsilon > 0$, let $\eta_s = \sqrt{\varepsilon + |\nabla P_{t-s} f|^2}(X_s)$ for $s \le t$. By the Itô formula, we have

$$\begin{aligned} \mathrm{d}\eta_s = \mathrm{d}M_s + \frac{L|\nabla P_{t-s} f|^2 - 2\langle \nabla L P_{t-s} f, \nabla P_{t-s} f\rangle}{2\sqrt{\varepsilon + |\nabla P_{t-s} f|^2)^2}}(X_s)\,\mathrm{d}s \\ - \frac{|\nabla |\nabla P_{t-s} f|^2|^2}{4(\varepsilon + |\nabla P_{t-s} f|^2)^{3/2}}(X_s)\,\mathrm{d}s + \frac{N|\nabla P_{t-s} f|^2}{2\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s)\,\mathrm{d}l_s \end{aligned}$$

for $s \le t$, where M_s is a local martingale. Combining this with (1-8) and (3-1), with κ_1 in place of K_0, we obtain

$$\begin{aligned} \mathrm{d}\eta_s &\ge \mathrm{d}M_s - \frac{\kappa_1 |\nabla P_{t-s} f|^2}{\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s)\,\mathrm{d}s - \frac{\kappa_2 |\nabla P_{t-s} f|^2}{\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s)\,\mathrm{d}l_s \\ &\ge \mathrm{d}M_s - \kappa_1(X_s)\eta_s\,\mathrm{d}s - \kappa_2(X_s)\eta_s\,\mathrm{d}l_s \quad \text{for } s \le t. \end{aligned}$$

Now η_s is bounded on $[0, t]$, and by the proof of [Wang 2005b, Lemma 2.1] we have $\mathsf{E}e^{\lambda l_t} < \infty$ for all $\lambda > 0$. This implies that

$$[0, t] \ni s \mapsto \sqrt{\varepsilon + |\nabla P_{t-s} f|^2(X_s)}\exp\Big(\int_0^s \kappa_1(X_s)\,\mathrm{d}s + \int_0^s \kappa_2(X_s)\,\mathrm{d}l_s\Big)$$

is a submartingale for any $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we conclude that

$$[0, t] \ni s \mapsto |\nabla P_{t-s} f|(X_s)\exp\Big(\int_0^s \kappa_1(X_s)\,\mathrm{d}s + \int_0^s \kappa_2(X_s)\,\mathrm{d}l_s\Big)$$

is a submartingale as well. ☐

According to Lemma 4.1, it suffices to confirm the boundedness of $|\nabla P_{(\cdot)} f|$ on $[0, T] \times M$ for any $T > 0$ and $f \in C_b^1(M)$. We shall start from $f \in C_0^\infty(M)$ with $Nf|_{\partial M} = 0$, then pass to $f \in C_b^1(M)$ by combining an approximation argument and Lemma 4.1.

Case a. Let $f \in C_0^\infty(M)$ with $Nf|_{\partial M} = 0$. We have

$$P_t f = f + \int_0^t P_s L f\,\mathrm{d}s. \tag{4-1}$$

Since Lf is bounded, there is a $c > 0$ such that $Lf + c \geq 1$. Applying Corollary 1.5 with for example $\alpha = 2 + \sigma d r_0$ and $\varepsilon = 1/2$, but using $Lf + c$ in place of f, we obtain

$$\begin{aligned}|\nabla P_s Lf| &= |\nabla P_s(Lf+c)|\\ &\leq \|Lf+c\|_\infty \Big(\alpha \|P_s L^2 f\|_\infty + \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)s} + \frac{m\alpha^2 K_\varepsilon}{4(\alpha-1-\sigma d r_0)}\Big)^{1/2}\\ &\leq c_1/\sqrt{s} \quad \text{for } s \leq T\end{aligned}$$

for some constant $c_1 > 0$. Combining this with (4-1) we conclude that, for some constant $c_2 > 0$,

$$|\nabla P_t f| \leq |\nabla f| + \int_0^t \frac{c_1}{\sqrt{s}} \mathrm{d}s \leq c_2 \quad \text{for } t \leq T.$$

Case b. Let $f \in C_0^\infty(M)$. There exists a sequence of functions $\{f_n\}_{n\geq 1} \subset C_0^\infty(M)$ such that $Nf_n|_{\partial M} = 0$, $f_n \to f$ uniformly as $n \to \infty$, and $\|\nabla f_n\|_\infty \leq 1 + \|\nabla f\|_\infty$ holds for any $n \geq 1$; see for example [Wang 1994]. By Case a and Lemma 4.1, (1-9) holds with f_n in place of f, so that

$$\frac{|P_t f_n(x) - P_t f_n(y)|}{\rho(x,y)} \leq C \quad \text{for } t \leq T,\ n \geq 1,\ x \neq y$$

for some constant $C > 0$. Letting first $n \to 0$ and then $y \to x$, we conclude that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$.

Case c. Let $f \in C_b^\infty(M)$. Let $\{g_n\}_{n\geq 1} \subset C_0^\infty)$ be such that $0 \leq g_n \leq 1$, $|\nabla g_n| \leq 2$ and $g_n \uparrow 1$ as $n \uparrow \infty$. By Case b and Lemma 4.1, we may apply (1-9) to $g_n f$ in place of f such that

$$\frac{|P_t(g_n f)(x) - P_t(g_n f)(y)|}{\rho(x,y)} \leq C \quad \text{for } t \leq T,\ n \geq 1,\ x \neq y$$

for some constant $C > 0$. By the same reason as in Case b, we conclude that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$.

Case d. Finally, for $f \in C_b^1(M)$, there exist $\{f_n\}_{n\geq 1} \subset C_b^\infty(M)$ such that $f_n \to f$ uniformly as $n \to \infty$ and $\|\nabla f_n\|_\infty \leq \|\nabla f\|_\infty + 1$ for any $n \geq 1$. The proof is completed by the same reason as in Cases b and c. □

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