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We introduce the concept of strongly r-matrix induced (SRMI) Poisson structure, report on the relation of this property to the stabilizer dimension of the considered quadratic Poisson tensor, and classify the Poisson structures of the Dufour–Haraki classification (DHC) according to their membership in the family of SRMI tensors. A main result is a generic cohomological procedure for classifying SRMI Poisson structures in arbitrary dimension. This approach allows the decomposition of Poisson cohomology into, basically, a Koszul cohomology and a relative cohomology. Also we investigate this associated Koszul cohomology, highlight its tight connections with spectral theory, and reduce the computation of this main building block of Poisson cohomology to a problem of linear algebra. We apply these upshots to two structures of the DHC and provide an exhaustive description of their cohomology. We thus complete our list of data obtained in previous work, and gain fairly good insight into the structure of Poisson cohomology.

1. Introduction

Let $(\mathscr{L}, [\cdot, \cdot])$ with $\mathscr{L} = \bigoplus_i \mathscr{L}^i$ be a graded Lie algebra (gLa). Any element with degree 1 that squares to 0 generates a differential graded Lie algebra (dgLa) $(\mathscr{L}, [\cdot, \cdot], \partial_\Lambda)$, where $\partial_\Lambda := [\Lambda, \cdot]$, and a gLa $H(\mathscr{L}, [\cdot, \cdot], \partial_\Lambda)$ in cohomology. Depending on the initial algebra, such a 2-nilpotent degree 1 element is, say, an associative algebra structure, a Lie algebra structure, or a Poisson structure, and the associated cohomology is the adjoint Hochschild, the adjoint Chevalley–Eilenberg, or the Lichnerowicz–Poisson (LP) (or simply Poisson) cohomology, respectively. Recall that the LP-dgLa is implemented by the shifted Grassmann

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algebra $(\mathscr{X}(M)[1], \wedge, [\cdot,\cdot]_{\mathrm{SN}})$, with $\mathscr{X}(M)=\Gamma(\bigwedge TM)$, of polyvectors of a manifold M, endowed with the Schouten–Nijenhuis bracket $[\cdot,\cdot]_{\mathrm{SN}}$. The Hochschild dgLa is generated by the space of multilinear mappings of the underlying vector space endowed with the Gerstenhaber graded Lie bracket, and similarly the Chevalley–Eilenberg dgLa is generated by the space of skew-symmetric multilinear mappings of the underlying vector space endowed with the Nijenhuis–Richardson graded Lie bracket.

Alternatively, LP-cohomology can be viewed as the Lie algebroid (Lad) cohomology of the Lie algebroid $(T^*M, \{\cdot,\cdot\}, \sharp)$ canonically associated with an arbitrary Poisson manifold (M, Λ). The cohomology of a Lad $(E\to M, [\![\cdot,\cdot]\!], \rho)$, or equivalently a Q-structure on a supermanifold, is defined as the cohomology of the Chevalley–Eilenberg subcomplex of the representation $\rho : \Gamma(E) \to \mathrm{Der}(C^\infty(M))$, made up by tensorial cochains. Algebraically, LP-cohomology is defined as the adjoint Chevalley–Eilenberg cohomology of any Poisson–Lie algebra, restricted to the cochain subspace of skew-symmetric multiderivations.

More details about Poisson cohomology can be found, say, in [Lichnerowicz 1977; Vaisman 1994].

The last few decades have seen much work on Poisson cohomology and Poisson homology, starting with [Koszul 1985; Brylinski 1988]. Problems studied include the cohomology of regular Poisson manifolds [Vaisman 1990; Xu 1992], (co)homology and resolutions [Huebschmann 1990], duality [Huebschmann 1999; Xu 1999; Evens et al. 1999], cohomology in low dimensions or specific cases [Nakanishi 1997; Ginzburg 1999; Gammella 2002; Monnier 2002b; 2002a; Roger and Vanhaecke 2002; Roytenberg 2002; Pichereau 2005], and various extensions of Poisson cohomology — for example, the cohomologies Lie algebroid, Jacobi, Nambu–Poisson, double Poisson, and graded Jacobi [de León et al. 1997; Ibáñez et al. 2001; Monnier 2001; Grabowski and Marmo 2003; de León et al. 2003; Nakanishi 2006; Pichereau and Van de Weyer 2008]. In [Masmoudi and Poncin 2007; Ammar and Poncin 2008], we suggest an approach to the cohomology of the Poisson tensors of the Dufour–Haraki classification (DHC).

Here we focus on the formal LP-cohomology associated with the quadratic Poisson tensors (QPTs) Λ of $\mathbb{R}^n$ that read as real linear combinations

$$\Lambda = \sum_{i<j} \alpha^{ij} Y_i \wedge Y_j =: \sum_{i<j} \alpha^{ij} Y_{ij} \quad \text{for } \alpha^{ij}\in\mathbb{R} \tag{1-1}$$

of the wedge products of n commuting linear vector fields $Y_1, \dots, Y_n$, such that $Y_1\wedge\cdots\wedge Y_n =: Y_{1\dots n} \neq 0$. Let us recall that “formal” means that we substitute the space $\mathbb{R}[\![x_1,\dots,x_n]\!]\otimes\bigwedge\mathbb{R}^n$ of multivectors with coefficients in the formal series for the usual Poisson cochain space $\mathscr{X}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)\otimes\bigwedge\mathbb{R}^n$. Furthermore, the reader may think about QPTs of type (1-1) as QPTs implemented by a classical r-matrix in their stabilizer for the canonical matrix action.

Hence, in Section 2, we are interested in characterizing the QPTs that are images of a classical r-matrix. We show that a QPT is induced by an r-matrix if the dimension of its stabilizer is large enough; more precisely, we prove that if the stabilizer of a given QPT Λ of $\mathbb{R}^n$ contains n commuting linear vector fields Y_i such that $Y_{1\dots n} \neq 0$, then Λ is implemented by an r-matrix in its stabilizer; see Corollary 2. We refer to such tensors as strongly r-matrix induced (SRMI) structures and show that any structure of the DHC decomposes into the sum of a maximal SRMI structure and a small compatible (mostly exact) Poisson tensor; see Theorem 4. This decomposition is the foundation of our cohomological techniques proposed in [Masmoudi and Poncin 2007; Ammar and Poncin 2008]. This splitting is in some sense opposite to the one proved in [Liu and Xu 1992], which incorporates the largest possible part of the Poisson tensor into the exact term.

Masmoudi and Poncin [2007] developed a cohomological method in Euclidean three-space that greatly simplified LP-cohomology computations for the SRMI structures of the Dufour–Haraki classification. Section 3 extends this procedure to arbitrary-dimensional vector spaces. In Theorem 10, we inject the space $\mathcal{R}$ of "real" LP-cochains (formal multivector fields) into a larger space $\mathcal{P}$ of "potential" cochains. In Theorems 13 and 15, we identify the natural extension to $\mathcal{P}$ of the LP-differential as the Koszul differential associated with n commuting endomorphisms

$$X_i - (\operatorname{div} X_i)\,\mathrm{id}, \quad \text{where } X_i = \textstyle\sum_j \alpha^{ij} Y_j \text{ and } \alpha^{ji} = -\alpha^{ij},$$

of the space made up by the polynomials on $\mathbb{R}^n$ with some fixed homogeneous degree. We then choose a space $\mathcal{S}$ supplementary to $\mathcal{R}$ in $\mathcal{P}$ and show that the LP-differential induces a differential on $\mathcal{S}$. Eventually, we end up with a short exact sequence of differential spaces and an exact triangle in cohomology. Theorem 16 shows that LP-cohomology ($\mathcal{R}$-cohomology) reduces, essentially, to Koszul cohomology ($\mathcal{P}$-cohomology) and a relative cohomology ($\mathcal{S}$-cohomology).

To take advantage of these results, we investigate in Section 4 the Koszul cohomology associated to n commuting linear operators on a finite-dimensional complex vector space. We prove a homotopy-type formula in Proposition 19 and — using spectral properties — show in Theorem 20 and Corollary 21 that the Koszul cohomology is located inside a primary subspace of the corresponding commuting endomorphisms.

In Section 5, we apply this result to gain insight into the structure of the Koszul cohomology implemented by SRMI tensors, and show that to compute this central part of Poisson cohomology it basically suffices to solve triangular systems of linear equations.

We conclude Section 5 by providing a full description of the LP-cohomology spaces of structures Λ_3 and Λ_9 of the Dufour–Haraki classification.

2. Characterization of strongly r-matrix induced Poisson structures

Stabilizer dimension and r-matrix generation. Poisson structures implemented by an r-matrix are of interest, for example in deformation quantization, especially in view of Drinfeld's method. We next report on an idea for generating quadratic Poisson tensors by classical r-matrices.

Set $G = \mathrm{GL}(n,\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$. The Lie algebra isomorphism between $\mathfrak{g}$ and the algebra $\mathscr{X}_0^1(\mathbb{R}^n)$ of linear vector fields extends to a Grassmann algebra and a graded Poisson–Lie algebra homomorphism $J : \bigwedge \mathfrak{g} \to \bigoplus_k (\mathcal{S}^k\mathbb{R}^{n*} \otimes \bigwedge^k \mathbb{R}^n)$. It is known that its restriction

$$J^k : \textstyle\bigwedge^k \mathfrak{g} \to \mathcal{S}^k\mathbb{R}^{n*} \otimes \bigwedge^k \mathbb{R}^n$$

is onto, but has a nontrivial kernel if $k, n \geq 2$. In particular,

$$J^3[r,r]_{\mathrm{SN}} = [J^2r, J^2r]_{\mathrm{SN}} \quad \text{for } r \in \mathfrak{g}\wedge\mathfrak{g},$$

where $[\cdot,\cdot]_{\mathrm{SN}}$ is the Schouten–Nijenhuis bracket. It is still an open problem to characterize the quadratic Poisson structures that are implemented by a classical r-matrix, that is, a bimatrix $r \in \mathfrak{g}\wedge\mathfrak{g}$ satisfying the classical Yang–Baxter equation $[r,r]_{\mathrm{SN}} = 0$.

Quadratic Poisson tensors Λ_1 and Λ_2 are equivalent if and only if there is $A \in G$ such that $A_*\Lambda_1 = \Lambda_2$, where $*$ denotes the standard action of G on tensors of $\mathbb{R}^n$. Since J^2 is a G-module homomorphism, that is,

$$A_*(J^2r) = J^2(\mathrm{Ad}(A)r) \quad \text{for } A \in G \text{ and } r \in \mathfrak{g}\wedge\mathfrak{g},$$

the G-orbit of a quadratic Poisson structure $\Lambda = J^2r$ is the pointwise J^2-image of the G-orbit of r. Furthermore, the representation Ad acts by graded Lie algebra homomorphisms, that is,

$$\mathrm{Ad}(A)[r,r]_{\mathrm{SN}} = [\mathrm{Ad}(A)r, \mathrm{Ad}(A)r]_{\mathrm{SN}}.$$

Hence, if $\Lambda = J^2r$, where r is a classical r-matrix, the orbit of this quadratic Poisson tensor consists of r-matrix induced structures.

Any quadratic Poisson tensor Λ is implemented by bimatrices $r \in \mathfrak{g}\wedge\mathfrak{g}$. To determine whether the G-orbit O_Λ of this tensor is generated by r-matrices, we look at the preimage $(J^2)^{-1}(O_\Lambda) = \bigcup_{r\in(J^2)^{-1}\Lambda} O_r$, composed of the G-orbits O_r of all the bimatrices r that are mapped on Λ by J^2. We claim that the bigger the chance that a fiber of this bundle is located inside r-matrices, the smaller is O_Λ.

In other words, the dimension of the isotropy Lie group G_Λ of Λ, or of its Lie algebra, the stabilizer

$$\mathfrak{g}_\Lambda := \{a \in \mathfrak{g} : [\Lambda, Ja]_{\mathrm{SN}} = 0\}$$

of Λ for the corresponding infinitesimal action, should be big enough. For example, in $\mathbb{R}^3$ the Poisson tensor $\Lambda = (x_1^2 + x_2x_3)\partial_{23}$, where $\partial_{23} := \partial_2 \wedge \partial_3$ and $\partial_i := \partial/\partial_{x_i}$, is not r-matrix induced (see [Manchon et al. 2002]) and the dimension of its stabilizer is $\dim \mathfrak{g}_\Lambda = 2$. More evidence comes from the corollary of the following theorem:

Theorem 1. *Let Λ be an analytic Poisson tensor of $\mathbb{R}^n$. If its stabilizer contains n commuting analytic vector fields Y_i for $i \in \{1, \dots, n\}$ such that $Y_1 \wedge \cdots \wedge Y_n \neq 0$, then there exist constants $\alpha^{kl} \in \mathbb{R}$ such that $\Lambda = \sum_{k<l} \alpha^{kl} Y_k \wedge Y_l$.*

Proof. Since $Y_1 \wedge \cdots \wedge Y_n \neq 0$, there exists an open subset O of $\mathbb{R}^n$ such that

$$\Lambda = \sum_{k<l} \alpha^{kl}(x) Y_k \wedge Y_l \quad \text{in } O$$

for some local functions $\alpha^{kl} = \alpha^{kl}(x)$. Since for any $i \in \{1, \dots, n\}$, we have

$$0 = [Y_i, \Lambda]_{\mathrm{SN}} = \sum_{k<l} Y_i(\alpha^{kl}) Y_k \wedge Y_l \quad \text{in } O,$$

the α^{kl} are constant in O; the theorem follows by analytic continuation. □

Corollary 2. *Let Λ be a quadratic Poisson tensor of $\mathbb{R}^n$. If its stabilizer $\mathfrak{g}_\Lambda$ contains n commuting linear vector fields Y_i for $i \in \{1, \dots, n\}$ such that $Y_1 \wedge \cdots \wedge Y_n$ does not vanish, then Λ is implemented by a classical r-matrix that belongs to the stabilizer, that is, $\Lambda = J^2 a$, where $[a, a]_{\mathrm{SN}} = 0$ and $a \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$.*

Definition 3. If Λ is a quadratic Poisson structure implemented by a classical r-matrix $r \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$, we call Λ a strongly r-matrix induced (SRMI) tensor.

Classification theorem in Euclidean three-space. Two concepts of exact Poisson structure — which are closely related to two special cohomology classes — are used below. Let Λ be a Poisson tensor on a smooth manifold M oriented by a volume element Ω. We say that Λ, which is of course a LP-2-cocycle, is Lichnerowicz–Poisson-exact or LP-exact if

$$\Lambda = [\Lambda, X]_{\mathrm{SN}} \quad \text{for some } X \in \mathscr{X}^1(M).$$

The vector field X is called the Liouville vector field and the cohomology class of Λ is the obstruction to infinitesimal rescaling of Λ. We call Λ Koszul-exact or K-exact if

$$\Lambda = \delta(T) \quad \text{for some } T \in \mathscr{X}^3(M).$$

Here, the operator $\delta := \phi^{-1} \circ d \circ \phi$ is the pullback of the de Rham differential d by the canonical vector space isomorphism $\phi := i_{(\cdot)}\Omega$. Although introduced earlier, the generalized divergence δ defined by $\delta(X) = \mathrm{div}_\Omega X$ for $X \in \mathcal{X}^1(M)$ is usually attributed to J.-L. Koszul. The curl vector field $K(\Lambda) := \delta(\Lambda)$ of Λ[1] is an LP-1-cocycle. $K(\Lambda)$ maps a function to the divergence of its Hamiltonian vector field. The cohomology class of $K(\Lambda)$ is the well-known modular class of Λ. This class is independent of Ω, is the obstruction to existence on M of a measure preserved by all Poisson automorphisms, and is relevant in the classification of Poisson structures [Dufour and Haraki 1991; Grabowski et al. 1993; Liu and Xu 1992] and in Poincaré duality [Evens et al. 1999; Ibáñez et al. 2001]. In $\mathbb{R}^n$ with $n \geq 3$, a Poisson tensor Λ is K-exact if and only if it is irrotational, that is, $K(\Lambda) = 0$, and in $\mathbb{R}^3$, K-exact means function-induced, that is,

$$\Lambda = \Pi_f := \partial_1 f\, \partial_{23} + \partial_2 f\, \partial_{31} + \partial_3 f\, \partial_{12} \quad \text{for } f \in C^\infty(\mathbb{R}^3).$$

The K-exact quadratic Poisson tensors Π_p of $\mathbb{R}^3$, that is, the K-exact Poisson structures induced by a homogeneous polynomial $p \in \mathcal{S}^3\mathbb{R}^{3*}$, represent class 14 of the Dufour–Haraki classification. The cohomology of this class has been studied by Pichereau [2005] (actually Pichereau deals with structures Π_p implemented by a weight-homogeneous polynomial p with an isolated singularity). Hence, we will not examine class 14 here.

Recall that two Poisson tensors Λ_1 and Λ_2 are compatible if their sum is again a Poisson structure, that is, if $[\Lambda_1, \Lambda_2]_{\mathrm{SN}} = 0$.

The next theorem classifies the quadratic Poisson tensors according to their strongly r-matrix induced structure. It also shows that any such tensor is the sum of a "maximal" strongly r-matrix induced tensor and a "small" compatible Poisson structure. The classification makes available the cohomological technique used in [Masmoudi and Poncin 2007], while the splitting[2] is relevant to cohomological approach of [Ammar and Poncin 2008].

Denote the canonical coordinates of $\mathbb{R}^3$ by x, y, z and by x_1, x_2, x_3; denote the corresponding partial derivatives by $\partial_1, \partial_2, \partial_3$. Let $\partial_{ij} = \partial_i \wedge \partial_j$.

Theorem 4. *Let $a, b, c \in \mathbb{R}$, and let Λ_i for $i \in \{1, \ldots, 13\}$ be the quadratic Poisson tensors of the Dufour–Haraki classification* [1991].

If $\dim \mathfrak{g}_\Lambda > 3$ (*where the subscript i is omitted*), *there are mutually commuting linear vector fields Y_1, Y_2, Y_3 such that*

$$\Lambda = \alpha Y_{23} + \beta Y_{31} + \gamma Y_{12}, \quad \textit{where } \alpha, \beta, \gamma \in \mathbb{R},$$

[1]If Ω is the standard volume of $\mathbb{R}^3$ and Λ is identified with a vector field $\vec{\Lambda}$ of $\mathbb{R}^3$, then $K(\Lambda)$ coincides with the standard curl $\vec{\nabla} \wedge \vec{\Lambda}$.

[2]This splitting differs from the decomposition used in [Liu and Xu 1992] in that we incorporate as much structure as possible into the strongly induced term.

so that Λ is strongly r-matrix induced (SRMI), that is, implemented by a classical r-matrix in $\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$. In the following classification of the quadratic Poisson tensors by the SRMI property, we decompose each non-SRMI tensor into the sum of a maximal SRMI structure and a smaller compatible quadratic Poisson tensor.

- *Set $Y_1 = x\partial_1$, $Y_2 = y\partial_2$ and $Y_3 = z\partial_3$.*
  (1) $\Lambda_1 = a\,yz\partial_{23} + b\,xz\partial_{31} + c\,xy\partial_{12}$ *is SRMI for all values of the parameters a, b and c, and decomposes as $\Lambda_1 = aY_{23} + bY_{31} + cY_{12}$.*
  (2) $\Lambda_4 = ayz\partial_{23} + axz\partial_{31} + (bxy + z^2)\partial_{12}$ *is SRMI if and only if a and b are both zero. We have $\Lambda_4 = a(Y_{23} + Y_{31}) + b\,Y_{12} + \frac{1}{3}\Pi_{z^3}$.*
- *Set $Y_1 = x\partial_1 + y\partial_2$, $Y_2 = x\partial_2 - y\partial_1$ and $Y_3 = z\partial_3$.*
  (1) $\Lambda_2 = (2ax - by)z\partial_{23} + (bx + 2ay)z\partial_{31} + a(x^2 + y^2)\partial_{12}$ *is SRMI for any a and b. We have $\Lambda_2 = 2a\,Y_{23} + b\,Y_{31} + a\,Y_{12}$.*
  (2) $\Lambda_7 = ((2a + c)x - by)\,z\partial_{23} + (bx + (2a + c)y)\,z\partial_{31} + a(x^2 + y^2)\partial_{12}$ *is SRMI for all a, b, c. We have $\Lambda_7 = (2a + c)Y_{23} + b\,Y_{31} + a\,Y_{12}$.*
  (3) $\Lambda_8 = a\,xz\partial_{23} + a\,yz\partial_{31} + (\frac{1}{2}(a + b)(x^2 + y^2) \pm z^2)\partial_{12}$ *is SRMI if and only if a and b are both zero. We have*
  $$\Lambda_8 = aY_{23} + \tfrac{1}{2}(a + b)Y_{12} \pm \tfrac{1}{3}\Pi_{z^3}.$$
- *Set $Y_1 = x\partial_1 + y\partial_2$, $Y_2 = x\partial_2$ and $Y_3 = z\partial_3$.*
  (1) $\Lambda_3 = (2x - a\,y)z\partial_{23} + a\,xz\partial_{31} + x^2\partial_{12}$ *is SRMI for any a, and we have $\Lambda_3 = 2Y_{23} + a\,Y_{31} + Y_{12}$.*
  (2) $\Lambda_5 = ((2a + 1)x + y)z\partial_{23} - xz\partial_{31} + a\,x^2\partial_{12}$ *is SRMI for any $a \neq -1/2$. We have $\Lambda_5 = (2a + 1)Y_{23} - Y_{31} + a\,Y_{12}$.*
  (3) $\Lambda_6 = a\,yz\partial_{23} - a\,xz\partial_{31} - \frac{1}{2}x^2\partial_{12}$ *is SRMI for any a. The decomposition is $\Lambda_6 = -a\,Y_{31} - \frac{1}{2}Y_{12}$.*
- *Set $Y_1 = \mathscr{E} := x\partial_1 + y\partial_2 + z\partial_3$, $Y_2 = x\partial_2 + y\partial_3$ and $Y_3 = x\partial_3$.*
  (1) $\Lambda_9 = (ax^2 - \frac{1}{3}y^2 + \frac{1}{3}xz)\partial_{23} + \frac{1}{3}xy\partial_{31} - \frac{1}{3}x^2\partial_{12}$ *is SRMI for any a. We have $\Lambda_9 = a\,Y_{23} - \frac{1}{3}Y_{12}$.*
  (2) $\Lambda_{10} = (a\,y^2 - (4a+1)xz)\partial_{23} + (2a+1)xy\partial_{31} - (2a+1)x^2\partial_{12}$ *is SRMI if and only if $a = -1/3$. We have $\Lambda_{10} = -(2a + 1)Y_{12} + (3a + 1)(y^2 - 2xz)\partial_{23}$.*
- *Set $Y_1 = \mathscr{E}$, $Y_2 = x\partial_2$ and $Y_3 = (a\,x + (3b + 1)z)\partial_3$.*
  (1) *Set $a = 0$. Then $\Lambda_{11} = (2b + 1)xz\partial_{23} + (bx^2 + cz^2)\partial_{12}$ is SRMI if and only if $c = 0$. We have $\Lambda_{11} = Y_{23} + b\,Y_{12} + \frac{1}{3}c\Pi_{z^3}$.*
  (2) *Set $a = 1$. Then $\Lambda_{12} = (x^2 + (2b + 1)xz)\partial_{23} + (bx^2 + cz^2)\partial_{12}$ is SRMI if and only if $c = 0$. We have $\Lambda_{12} = Y_{23} + b\,Y_{12} + \frac{1}{3}c\Pi_{z^3}$.*
  (3) $\Lambda_{13} = (ax^2 + (2b + 1)xz + z^2)\partial_{23} + (bx^2 + cz^2 + 2xz)\partial_{12}$ *is not SRMI for any a, b, c. We have $\Lambda_{13} = Y_{23} + b\,Y_{12} + \Pi_{cz^3/3+xz^2}$.*

Proof. The basic fields Y_1, Y_2, Y_3 have been read in the stabilizers of the considered Poisson tensors, but for brevity we omit the stabilizer computations. Indeed, once the vector fields Y_i are specified, it is easily checked that, in the SRMI cases, they satisfy the assumptions of Theorem 1. Thus the corresponding Poisson structures are actually SRMI tensors. To show that a quadratic Poisson structure Λ is not SRMI, it suffices to prove that $\Lambda \notin J^2(\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda)$, which we will do below.

All the decompositions above can be directly verified. In most instances, the twist is obviously Poisson, so that compatibility follows. In the case of Λ_{10}, the twist $\Lambda_{10,\mathrm{II}} = (y^2 - 2xz)\partial_{23}$ is a non-K-exact Poisson structure, which follows directly from the fact that $K(\Lambda_{10,\mathrm{II}}) = \vec{\nabla} \wedge \vec{\Lambda}_{10,\mathrm{II}} = -2x\partial_2 - 2y\partial_3 \neq 0$ and the formula $[P, Q]_{\mathrm{SN}} = (-1)^p D(P \wedge Q) - D(P) \wedge Q - (-1)^p P \wedge D(Q)$ for $P \in \mathscr{X}^p(M)$ and $Q \in \mathscr{X}^q(M)$. The main part of this proof will make it obvious why we require that $\dim \mathfrak{g}_\Lambda > 3$.

Denote by E_{ij} for $i, j \in \{1, 2, 3\}$ the canonical basis of $\mathfrak{gl}(3, \mathbb{R})$.

- If $(a, b) \neq (0, 0)$, stabilizer $\mathfrak{g}_{\Lambda_4}$ and the image $J^2(\mathfrak{g}_{\Lambda_4} \wedge \mathfrak{g}_{\Lambda_4})$ are generated by $(\frac{1}{2}E_{11} + E_{22}, \frac{1}{2}E_{11} + E_{33})$ and $yz\partial_{23} - \frac{1}{2}xz\partial_{31} - \frac{1}{2}xy\partial_{12}$, respectively. Hence Λ_4 is not SRMI.
- If $(a, b) \neq (0, 0)$, the generators of $\mathfrak{g}_{\Lambda_8}$ and $J^2(\mathfrak{g}_{\Lambda_8} \wedge \mathfrak{g}_{\Lambda_8})$ are
$$(E_{11} + E_{22} + E_{33}, E_{12} - E_{21}) \quad \text{and} \quad -xz\partial_{23} - yz\partial_{31} + (x^2 + y^2)\partial_{12}.$$
So Λ_8 is not SRMI.
- If $a \neq -1/3$, the generators in the case of Λ_{10} are
$$(E_{11} + E_{22} + E_{33}, E_{12} + E_{23}) \;\text{ and }\; (y^2 - xz)\partial_{23} - xy\partial_{31} + x^2\partial_{12}.$$
- For the cases of Λ_{11} and Λ_{12} with $c \neq 0$, and the case Λ_{13}, the generators are
$$(E_{11} + E_{22} + E_{33}, E_{12}, E_{32}) \quad \text{and} \quad -xz\partial_{23} + x^2\partial_{12}, z^2\partial_{23} - xz\partial_{12}. \qquad \square$$

Remarks.

- In the cases Λ_1, Λ_2 and Λ_3, with $c \neq 0$ in the latter two, the dimension of the stabilizer is $\dim \mathfrak{g}_\Lambda = 3$, whereas $J\mathfrak{g}_\Lambda \wedge J\mathfrak{g}_\Lambda \wedge J\mathfrak{g}_\Lambda = \{0\}$. Hence, if the dimension of the stabilizer coincides with the dimension of the space, the Poisson structure is not necessarily a SRMI tensor.
- For Λ_{10}, the decomposition proved in [Liu and Xu 1992] yields
$$\Lambda_{10} = -\tfrac{1}{3}Y_{12} + \Pi_{cz^3/3 + xz^2 + (b+1/3)x^2z + ax^3/3}.$$

3. Poisson cohomology of quadratic structures in a finite-dimensional vector space

Koszul homology and cohomology. Let $\bigwedge = \bigwedge_n \langle \vec{\eta} \rangle$ be the Grassmann algebra on $n \in \mathbb{N}_0$ with generators $\vec{\eta} = (\eta_1, \dots, \eta_n)$, that is, the algebra over a field $\mathbb{F}$ (here $\mathbb{R}$

or $\mathbb{C}$) of characteristic 0 generated by $\eta_1, \dots, \eta_n$ and subject to the anticommutation relations $\eta_k\eta_\ell + \eta_\ell\eta_k = 0$ for $k, \ell \in \{1, \dots, n\}$. Set $\bigwedge = \bigoplus_{p=0}^{n} \bigwedge^p$, with obvious notations, and let $\vec{h} = (h_1, \dots, h_n)$ be dual generators defined by $i_{h_k}\eta_\ell = \delta_{k\ell}$. We also need the creation operator $e_{\eta_k} : \bigwedge \to \bigwedge, \omega \mapsto \eta_k\omega$ and the annihilation operator $i_{h_k} : \bigwedge \to \bigwedge$, $\omega \mapsto i_{h_k}\omega$, where the interior product is defined as usual. Finally, we denote by E a vector space over $\mathbb{F}$ and by $\vec{X} = (X_1, \dots, X_n)$ an n-tuple of commuting linear operators on E.

Definition 5. The Koszul chain complex (or K_*-complex) $K_*(\vec{X}, E)$ associated with $\vec{X}$ on E is the complex

$$0 \to E \otimes_{\mathbb{F}} \textstyle\bigwedge^n \to E \otimes_{\mathbb{F}} \bigwedge^{n-1} \to \cdots \to E \otimes_{\mathbb{F}} \bigwedge^1 \to E \to 0$$

with differential $\kappa_{\vec{X}} = \sum_{k=1}^{n} X_k \otimes i_{h_k}$. We denote by $KH_*(\vec{X}, E)$ the corresponding Koszul homology group.

Definition 6. The Koszul cochain complex (or K^*-complex) $K^*(\vec{X}, E)$ associated with $\vec{X}$ on E is the complex

$$0 \to E \to E \otimes_{\mathbb{F}} \textstyle\bigwedge^1 \to \dots \to E \otimes_{\mathbb{F}} \bigwedge^{n-1} \to E \otimes_{\mathbb{F}} \bigwedge^n \to 0$$

with differential $\mathcal{K}_{\vec{X}} = \sum_{k=1}^{n} X_k \otimes e_{\eta_k}$. We denote by $KH^*(\vec{X}, E)$ the corresponding Koszul cohomology group.

Since the X_k commute, the anticommutativity of the i_{h_k} and the e_{η_k} imply that $\kappa_{\vec{X}}\kappa_{\vec{X}} = 0$ and $\mathcal{K}_{\vec{X}}\mathcal{K}_{\vec{X}} = 0$, respectively. See [Koszul 1950; 1994].

Example 7. Let $\mathbb{F} = \mathbb{R}$ and $E = C^\infty(\mathbb{R}^3)$. If we choose $\eta_k = dx_k$ and $X_k = \partial_k$, the K^*-complex is the de Rham complex $(\Omega(\mathbb{R}^3), d)$. With $\eta_k = \partial_k = \partial_{x_k}$ and $h_k = dx_k$, the K_*-complex is the dual de Rham complex $(\mathcal{X}(\mathbb{R}^3), \delta)$.

If we identify the subspaces $\Omega^k(\mathbb{R}^3)$ of homogeneous forms with the corresponding spaces of components E, E^3, E^3 and E, this K^*-complex reads

$$0 \to E \xrightarrow{\mathcal{K}=\vec{\nabla}(\cdot)} E^3 \xrightarrow{\mathcal{K}=\vec{\nabla}\wedge(\cdot)} E^3 \xrightarrow{\mathcal{K}=\vec{\nabla}\cdot(\cdot)} E \to 0. \tag{3-1}$$

Example 8. Let $\mathbb{F} = \mathbb{R}$ and $E = \mathcal{S}\mathbb{R}^{3*} = \mathbb{R}[x_1, x_2, x_3]$. For $k \in \{1, 2, 3\}$, let $\eta_k = \partial_k$, $X_k = \mathfrak{m}_{P_k}$, $P_k \in E^{d_k}$ and $d_k \in \mathbb{N}$, where $\mathfrak{m}_{P_k} : E \to E$, $Q \mapsto P_k Q$. Then the chain spaces of the K_*-complex are the spaces of homogeneous polyvector fields on $\mathbb{R}^3$ with polynomial coefficients, and by identifying these with the corresponding spaces E, E^3, E^3 and E of components, we can write this K_*-complex in the form

$$0 \to E \xrightarrow{\kappa=(\cdot)\vec{P}} E^3 \xrightarrow{\kappa=(\cdot)\wedge\vec{P}} E^3 \xrightarrow{\kappa=(\cdot)\cdot\vec{P}} E \to 0. \tag{3-2}$$

Remarks. First, the Koszul cohomology and homology complexes of Example 7 are exact, except that $KH^0(\vec{\partial}, C^\infty(\mathbb{R}^3)) \simeq KH_3(\vec{\partial}, C^\infty(\mathbb{R}^3)) \simeq \mathbb{R}$.

Second, recall that an R-regular sequence on a module M over a commutative unit ring R is a sequence $(r_1, \dots, r_d) \in R^d$ such that r_k is not a zero divisor on the quotient $M/\langle r_1, \dots, r_{k-1}\rangle M$ for $k \in \{1, \dots, d\}$, and $M/\langle r_1, \dots, r_d\rangle M \neq 0$. In particular, $x_1, \dots, x_d$ is a (maximal length) regular sequence on the polynomial ring $R = \mathbb{F}[x_1, \dots, x_d]$, so that this ring has depth d.

It is well known that the K_*-complex described in Example 8 is exact, except for surjectivity of $\kappa = (\cdot)\cdot \vec{P}$, if the vector $\vec{P} = (P_1, P_2, P_3)$ is regular on $\mathbb{R}[x_1, x_2, x_3]$. If $\vec{P} = \vec{\nabla} p$ for p a homogeneous polynomial with an isolated singularity at the origin, then $\vec{P}$ is regular; see [Pichereau 2005].

Poisson cohomology in dimension 3. Set $E := C^\infty(\mathbb{R}^3)$ and again identify the spaces of homogeneous multivector fields in $\mathbb{R}^3$ with the corresponding component spaces: $\mathscr{X}^0(\mathbb{R}^3) \simeq \mathscr{X}^3(\mathbb{R}^3) \simeq E$ and $\mathscr{X}^1(\mathbb{R}^3) \simeq \mathscr{X}^2(\mathbb{R}^3) \simeq E^3$.

Let $\vec{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3) \in E^3$ be a Poisson tensor, and let $f \in E$, $\vec{X} \in E^3$, $\vec{B} \in E^3$ and $T \in E$ be a 0-, 1-, 2-, and 3-cochain of the LP-complex. By straightforward computations, we get formulas for the LP-coboundary operator $\partial_{\vec{\Lambda}}$:

$$\begin{aligned}
\partial^0_{\vec{\Lambda}} f &= \vec{\nabla} f \wedge \vec{\Lambda},\\
\partial^1_{\vec{\Lambda}} \vec{X} &= (\vec{\nabla} \cdot \vec{X})\vec{\Lambda} - \vec{\nabla}(\vec{X} \cdot \vec{\Lambda}) + \vec{X} \wedge (\vec{\nabla} \wedge \vec{\Lambda}),\\
\partial^2_{\vec{\Lambda}} \vec{B} &= -(\vec{\nabla} \wedge \vec{B}) \cdot \vec{\Lambda} - \vec{B} \cdot (\vec{\nabla} \wedge \vec{\Lambda}),\\
\partial^3_{\vec{\Lambda}} T &= 0.
\end{aligned}$$

Recall the differential $\mathscr{K}$ from (3-1), and let κ' and κ'' be the differential in (3-2) when $\vec{P} = \vec{\Lambda}$ and $\vec{P} = \vec{\nabla} \wedge \vec{\Lambda}$, respectively. Then

$$\begin{aligned}
&\partial^0_{\vec{\Lambda}} = \kappa'\mathscr{K},\\
&\partial^1_{\vec{\Lambda}} = \kappa'\mathscr{K} - \mathscr{K}\kappa' + \kappa'', \quad \partial^2_{\vec{\Lambda}} = -\kappa'\mathscr{K} - \kappa'',\\
&\qquad\qquad\qquad\qquad\qquad\quad\ \partial^3_{\vec{\Lambda}} = 0.
\end{aligned} \tag{3-3}$$

Again, this paper investigates only quadratic Poisson tensors and polynomial (or formal) LP-cochains. If the structure $\vec{\Lambda}$ is K-exact, that is, in view of notations due to the elimination of the module basis of multivector fields, $\vec{\Lambda} = \vec{\nabla} p$ for $p \in \mathscr{S}^3\mathbb{R}^{3*}$ if and only if $\vec{\nabla} \wedge \vec{\Lambda} = 0$, homology operator κ'' vanishes. If, moreover, p has an isolated singularity, the K^*-complex associated with $\mathscr{K}$ is exact up to injectivity of $\mathscr{K} = \vec{\nabla}(\cdot)$, and the K_*-complex associated with κ' is acyclic (see above) up to surjectivity of $\kappa' = (\cdot)\cdot\vec{\Lambda}$. Pichereau [2005] computed the LP-cohomology for a weight-homogeneous polynomial p with an isolated singularity.

Next we describe a generic cohomological technique for SRMI Poisson tensors in a finite-dimensional vector space. This approach extends (3-3) to dimension n and also reduces the LP-coboundary operator ∂_Λ to a single Koszul differential.

Poisson cohomology in dimension n.

Definition 9. Let $Y_i = \sum_r \ell_{ir}\partial_r$ be n linear vector fields in $\mathbb{R}^n$. Set

$$\mathcal{R} = \bigoplus_{p=0}^{n} \mathcal{R}^p = \bigoplus_{p=0}^{n} \mathbb{R}[\![x_1, \dots, x_n]\!] \otimes \bigwedge_n^p \langle \vec{\partial} \rangle,$$

$$\mathcal{P} = \bigoplus_{p=0}^{n} \mathcal{P}^p = D^{-1} \bigoplus_{p=0}^{n} \mathbb{R}[\![x_1, \dots, x_n]\!] \otimes \bigwedge_n^p \langle \vec{Y} \rangle,$$

where $D = \det \ell$ and $\bigwedge_n^p \langle \vec{\partial} \rangle$ and $\bigwedge_n^p \langle \vec{Y} \rangle$ are the terms of degree p of the Grassmann algebras on generators $\vec{\partial} = (\partial_1, \dots, \partial_n)$ and $\vec{Y} = (Y_1, \dots, Y_n)$, respectively. The spaces $\mathcal{R}$ and $\mathcal{P}$ are respectively the space of *real* and *potential* formal LP-cochains.

For $\boldsymbol{i} = (i_1, \dots, i_m) \in \{1, \dots, n\}^m$, with $i_1 < \cdots < i_m$ and $m \in \{1, \dots, n\}$, we denote by $\boldsymbol{I} = (I_1, \dots, I_{n-m})$ its complement in $\{1, \dots, n\}$. The definition of D gives $Y_1 \wedge \cdots \wedge Y_n = D\, \partial_1 \wedge \cdots \wedge \partial_n$. If we take the interior product of this equation with $dx_{\boldsymbol{I}} = dx_{I_1} \wedge \cdots \wedge dx_{I_{n-m}}$, we get

$$D\partial_{\boldsymbol{i}} = \sum_{\boldsymbol{k}} (-1)^{|\boldsymbol{i}|+|\boldsymbol{k}|} L_{\boldsymbol{k}\boldsymbol{i}} Y_{\boldsymbol{k}},$$

where $\boldsymbol{k}$ is a subscript analogous to $\boldsymbol{i}$, where $\partial_{\boldsymbol{i}}$ and $Y_{\boldsymbol{k}}$ are compact notations similar to $dx_{\boldsymbol{I}}$, where $|\cdot|$ is the sum of the components, and where $L_{\boldsymbol{k}\boldsymbol{i}}$ denotes some homogeneous polynomial. Setting $L^{\boldsymbol{k}\boldsymbol{i}} := L_{\boldsymbol{K}\boldsymbol{I}}$, we have a theorem:

Theorem 10. (i) *There is a canonical nonsurjective injection* $i : \mathcal{R} \to \mathcal{P}$.

(ii) *A homogeneous potential cochain* $D^{-1} \sum_{\boldsymbol{k}} P^{\boldsymbol{k}r} Y_{\boldsymbol{k}}$ (*of bidegree* (p, r), *where* p *is the exterior degree and* r *the polynomial degree*) *is real if and only if the* $n!/p!(n-p)!$ *homogeneous polynomials* $\sum_{\boldsymbol{k}} L^{\boldsymbol{k}\boldsymbol{i}} P^{\boldsymbol{k}r}$ (*of degree* $p + r$) *are divisible by* D; *in case* $p = 0$ *this condition means that* P^r *is divisible by* D.

Remark. The bigrading $\mathcal{P} = \bigoplus_{p=0}^{n} \bigoplus_{r=0}^{\infty} \mathcal{P}^{pr}$, defined on $\mathcal{P}$ by the exterior degree and the polynomial degree, induces a bigrading $\mathcal{R} = \bigoplus_{p=0}^{n} \bigoplus_{r=0}^{\infty} \mathcal{R}^{pr}$ on $\mathcal{R}$.

Consider now a quadratic Poisson tensor Λ in $\mathbb{R}^n$. From now on, we assume that Λ is SRMI, and more precisely that there are n mutually commuting linear vector fields $Y_i = \sum_{r=1}^{n} \ell_{ir}\partial_r$ with $\ell \in \mathfrak{gl}(n, \mathbb{R}^{n*})$ such that $D = \det \ell \neq 0$ and

$$\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}, \quad \text{where } \alpha^{ij} \in \mathbb{R}.$$

Proposition 11. *The determinant* $D = \det \ell \in \mathcal{S}^n \mathbb{R}^{n*} \setminus \{0\}$ *is the unique joint eigenvector of the* Y_i *with eigenvalues* $\operatorname{div} Y_i \in \mathbb{R}$, *that is*, D *is up to multiplication by nonzero constants the unique nonzero polynomial of* $\mathbb{R}^n$ *that satisfies*

$$Y_i D = (\operatorname{div} Y_i) D \quad \textit{for all } i \in \{1, \dots, n\}.$$

Moreover, if $D = D_1 D_2$, where $D_1 \in \mathcal{S}^{n_1}\mathbb{R}^{n*}$ and $D_2 \in \mathcal{S}^{n_2}\mathbb{R}^{n*}$ with $n_1 + n_2 = n$ are two polynomials without common divisor, these factors D_1 and D_2 are also joint eigenvectors. Their eigenvalues λ_i and μ_i satisfy $\lambda_i + \mu_i = \operatorname{div} Y_i$.

Proof. For $i \in \{1, \ldots, n\}$,

$$\begin{aligned} 0 = [Y_i, Y_1 \wedge \cdots \wedge Y_n] &= [Y_i, D\partial_1 \wedge \cdots \wedge \partial_n] \\ &= (Y_i D)\partial_1 \wedge \cdots \wedge \partial_n - D(\operatorname{div} Y_i)\partial_1 \wedge \cdots \wedge \partial_n, \end{aligned}$$

so that $Y_i D = (\operatorname{div} Y_i)D$.

For uniqueness, let $P \in \mathcal{S}\mathbb{R}^{n*} \setminus \{0\}$ be another polynomial such that $Y_i P = (\operatorname{div} Y_i)P$ for all $i \in \{1, \ldots, n\}$. Then $Y_i\,(P/D) = 0$ in $Z = \{x \in \mathbb{R}^n,\ D(x) \neq 0\}$, and, reasoning as in the proof of Theorem 1, we conclude there exists $\alpha \in \mathbb{R}^*$ such that $P = \alpha D$.

Finally, because $((\operatorname{div} Y_i)D_1 - Y_i D_1)D_2 = D_1(Y_i D_2)$ and the polynomials D_1 and D_2 have no common divisor, $Y_i D_2 = P D_2$ and $(\operatorname{div} Y_i)D_1 - Y_i D_1 = Q D_1$, where $P = Q$ is a polynomial. Looking at degrees, we see $P = Q$ is constant. □

Remark. The eigenvalues $\operatorname{div} Y_i$ for $i \in \{1, \ldots, n\}$ cannot vanish simultaneously, for otherwise the polynomial $D \in \mathcal{S}^n\mathbb{R}^{n*} \setminus \{0\}$ vanishes everywhere.

Definition 12. The complex $0 \to \mathcal{R}^0 \to \mathcal{R}^1 \to \cdots \to \mathcal{R}^n \to 0$ with differential $\partial_\Lambda = [\Lambda, \cdot]_{\mathrm{SN}}$ is the formal LP-complex of Poisson tensor $\Lambda \in \mathcal{S}^2\mathbb{R}^{n*} \otimes \bigwedge^2 \mathbb{R}^n$. We denote the corresponding cohomology groups by $LH^*(\mathcal{R}, \Lambda)$.

The next theorem shows that if the cochains $C \in \mathcal{R}$ are read as $C = iC \in \mathcal{P}$, the LP-differential simplifies.

Theorem 13. *Set $\Lambda = \sum_{i<j} \alpha^{ij} Y_{ij}$, $\alpha^{ji} = -\alpha^{ij}$, and $X_i = \sum_{j \neq i} \alpha^{ij} Y_j$.*

(i) *Let $C = D^{-1} \sum_{\boldsymbol{k}} P^{\boldsymbol{k}r} Y_{\boldsymbol{k}} \in \mathcal{P}^{pr}$ be a homogeneous potential cochain. The LP-coboundary of C is given by*

$$\begin{aligned} \partial_\Lambda C &= \sum_{\boldsymbol{k}i} X_i(D^{-1}P^{\boldsymbol{k}r})Y_i \wedge Y_{\boldsymbol{k}} \\ &= D^{-1} \sum_{\boldsymbol{k}i} (X_i - \delta_i \operatorname{id})(P^{\boldsymbol{k}r})Y_i \wedge Y_{\boldsymbol{k}} \in \mathcal{P}^{p+1,r}, \end{aligned} \tag{3-4}$$

where $\delta_i = \operatorname{div} X_i \in \mathbb{R}$.

(ii) *The LP-coboundary operator ∂_Λ endows $\mathcal{P}$ with a differential complex structure and preserves the polynomial degree r. This LP-complex of Λ over $\mathcal{P}$ contains the LP-complex $(\mathcal{R}, \partial_\Lambda)$ of Λ over $\mathcal{R}$ as a differential subcomplex.*

Proof. If $C = f\boldsymbol{Y}$, with f a function and $\boldsymbol{Y}$ a wedge product of vector fields Y_k, we get

$$\partial_\Lambda(f\boldsymbol{Y}) = [\Lambda, f\boldsymbol{Y}]_{\mathrm{SN}} = [\Lambda, f]_{\mathrm{SN}} \wedge \boldsymbol{Y}, \tag{3-5}$$

since the Y_k commute. However,

$$\begin{aligned}[\Lambda, f]_{\mathrm{SN}} &= \sum_{i<j} \alpha^{ij}((Y_j f)Y_i - (Y_i f)Y_j) \\ &= \sum_i \Big(\sum_{j\neq i} \alpha^{ij}\, Y_j f\Big) Y_i = \sum_i (X_i f) Y_i. \end{aligned} \tag{3-6}$$

By combining (3-5) and (3-6), we get the first part of (3-4), whereas its second part is a consequence of Proposition 11. □

Corollary 14. *The LP-cohomology groups of* Λ *over* $\mathcal{R}$ *and* $\mathcal{P}$ *are bigraded, that is,*

$$LH(\mathcal{R}, \Lambda) = \bigoplus_{r=0}^{\infty} \bigoplus_{p=0}^{n} LH^{pr}(\mathcal{R}, \Lambda) \quad \text{and} \quad LH(\mathcal{P}, \Lambda) = \bigoplus_{r=0}^{\infty} \bigoplus_{p=0}^{n} LH^{pr}(\mathcal{P}, \Lambda),$$

where for instance $LH^{pr}(\mathcal{P}, \Lambda)$ *is defined by*

$$LH^{pr}(\mathcal{P}, \Lambda) = \ker(\partial_\Lambda : \mathcal{P}^{pr} \to \mathcal{P}^{p+1,r}) / \operatorname{im}(\partial_\Lambda : \mathcal{P}^{p-1,r} \to \mathcal{P}^{pr}).$$

In the following we deal with the terms $LP^{*r}(\mathcal{P}, \Lambda) = \bigoplus_{p=0}^{n} LP^{pr}(\mathcal{P}, \Lambda)$ of the LP-cohomology over $\mathcal{P}$ and with the corresponding part of the LP-cohomology of the subcomplex $\mathcal{R}$.

Theorem 15. *Let* E_r *be the real finite-dimensional vector space* $\mathcal{S}^r\mathbb{R}^{n*}$, *and let* $\vec{X}_\delta := (X_1 - \delta_1\,\mathrm{id}, \dots, X_n - \delta_n\,\mathrm{id})$, *where* $\delta_i = \operatorname{div} X_i$, *be the n-tuple of commuting linear operators* $X_i - \delta_i\,\mathrm{id}$ *on* E_r *defined in Theorem 13. The LP-cohomology space* $LH^{*r}(\mathcal{P}, \Lambda)$ *coincides with the Koszul cohomology space* $KH^*(\vec{X}_\delta, E_r)$.

Proof. This follows from $\partial_\Lambda = \sum_i (X_i - \delta_i\,\mathrm{id}) \otimes e_{Y_i}$, as proved in Theorem 13. □

Since $(\mathcal{R}, \partial_\Lambda)$ is a subcomplex of $(\mathcal{P}, \partial_\Lambda)$, we can use classical techniques (namely, the long exact cohomology sequence) to deduce the LP-cohomology of Λ from the Koszul cohomology associated with $\vec{X}_\delta$. More precisely, consider the relative cohomology $LH(\mathcal{P}, \mathcal{R}, \Lambda)$ of $(\mathcal{P}, \partial_\Lambda)$ with respect to $\mathcal{R}$, that is, the cohomology of the space $(\mathcal{P}/\mathcal{R}, \bar{\partial}_\Lambda)$, and let ϕ be the composition of ∂_Λ with the projection of $\mathcal{P}$ onto $\mathcal{R}$.

Theorem 16. *The LP-cohomology groups of a SRMI Poisson tensor* Λ *over the space* $\mathcal{R}$ *of cochains with coefficients in formal power series are given by*

$$LH^{pr}(\mathcal{R}, \Lambda) \simeq LH^{pr}(\mathcal{P}, \Lambda) / \ker^{pr} \phi_\sharp \oplus LH^{p-1,r}(\mathcal{P}, \mathcal{R}, \Lambda) / \ker^{p-1,r} \phi_\sharp.$$

Remark 1. This theorem reduces computing the groups $LH^{pr}(\mathcal{R}, \Lambda)$ to finding the groups $LH^{pr}(\mathcal{P}, \Lambda) \simeq KH^p(\vec{X}_\delta, E_r)$ associated to the operators $\vec{X}_\delta$ on $E_r = \mathcal{S}^r\mathbb{R}^{n*}$ induced by Λ, and to finding the relative cohomology groups $LH^{p-1,r}(\mathcal{P}, \mathcal{R}, \Lambda)$. It thus links Poisson and Koszul cohomology. In [Masmoudi and Poncin 2007], we showed via explicit computations in $\mathbb{R}^3$ that $\mathcal{P}$-cohomology (now identified as Koszul cohomology) and $\mathcal{S}$-cohomology (or relative cohomology) are less intricate than Poisson cohomology.

4. Koszul cohomology in a finite-dimensional vector space

In view of Remark 1, we now turn to the Koszul cohomology space $KH^*(\vec{X}_\lambda, E)$ associated to operators $\vec{X}_\lambda := (X_1 - \lambda_1 \,\mathrm{id}, \dots, X_n - \lambda_n \,\mathrm{id})$ made up of commuting linear transformations $\vec{X} := (X_1, \dots, X_n)$ of a finite-dimensional real vector space E and a point $\vec{\lambda} := (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Koszul cohomology is known to be closely connected with spectral theory: A fundamental principle of multivariate operator theory is that all essential spectral properties of operators $\vec{X}$ in a complex space should be understood in terms of properties of the Koszul complex induced by $\vec{X}_\lambda$ for $\vec{\lambda} \in \mathbb{C}^n$. Thus the complex setting is the natural one for investigating Koszul cohomology. To engage this point of view, it suffices to note that, if $\vec{X} \in \mathrm{End}_{\mathbb{R}}(E)$ are commuting $\mathbb{R}$-linear transformations of a real vector space E, and if $\vec{X}^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(E^{\mathbb{C}})$ are the corresponding commuting complexified $\mathbb{C}$-linear transformations of the complexification $E^{\mathbb{C}}$ of E, the cohomology $KH^*(\vec{X}^{\mathbb{C}}, E^{\mathbb{C}})$ of the complexification $K^*(\vec{X}^{\mathbb{C}}, E^{\mathbb{C}})$ of the complex $K^*(\vec{X}, E)$ is isomorphic to the complexification $KH^{*\mathbb{C}}(\vec{X}, E)$ of the cohomology of $K^*(\vec{X}, E)$.

Below, we use the concept of joint spectrum $\sigma(\vec{X})$ of commuting bounded linear operators $\vec{X} = (X_1, \dots, X_n)$ on a complex vector space E. Such spectra are defined variously in the literature, where E may be a normed space, a Banach space, or a Hilbert space. Here we investigate Koszul cohomology in finite dimension and need the following characterizations of the elements of the joint spectrum $\sigma(\vec{X})$; for a proof, see [Bolotnikov and Rodman 2002].

Proposition 17. *Let $\vec{X} = (X_1, \dots, X_n)$ be an n-tuple of commuting operators on a finite-dimensional complex vector space E. Then these statements are equivalent for any fixed $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$:*

(a) $\vec{\lambda} \in \sigma(\vec{X})$.

(b) *There is a basis in E in which the matrices representing the X_j are all upper-triangular, and there is an index q in $1 \le q \le \dim E$ such that λ_j is the (q, q) entry of the matrix representing X_j for $j \in \{1, \dots, n\}$.*

(c) *For every basis in E in which matrices for the X_j are all upper-triangular, there is an index q as in* (b).

(d) *There is a nonzero vector x such that $X_j x = \lambda_j x$ for all $j \in \{1, \dots, n\}$*

(e) *There are no Y_j in the subalgebra of* $\mathrm{End}_{\mathbb{C}}(E)$ *generated by* id *and $\vec{X}$ that satisfy* $\sum_{j=1}^n Y_j (X_j - \lambda_j \,\mathrm{id}) = \mathrm{id}$.

We now supply some results about Koszul cohomology spaces, using the same notation as above. The first is obvious.

Proposition 18. *Let $\bigwedge = \bigwedge_n\langle\vec\eta\rangle$ be the exterior algebra on n with generators $\vec\eta$ over a field $\mathbb{F}$ of characteristic 0, and let $\vec h$ be the dual generators, that is, suppose $i_{h_k}\eta_\ell = \partial_{k\ell}$. We then have the homotopy formula $e_{\eta_\ell}i_{h_k} + i_{h_k}e_{\eta_\ell} = \delta_{k\ell}\,\mathrm{id}$, where i_{h_k} and e_{η_ℓ} are the creation and annihilation operators, respectively.*

Proposition 19. *Let $\vec{\mathcal{X}} \in \mathrm{End}_{\mathbb{F}}^{\times n}(E)$ and $\vec Y \in \mathrm{End}_{\mathbb{F}}^{\times n}(E)$ be n commuting linear operators $\vec{\mathcal{X}}$ and $\vec Y$, respectively, on a vector space E over $\mathbb{F}$. We denote by $\mathcal{K} = \sum_\ell \mathcal{X}_\ell \otimes e_{\eta_\ell}$ and $\kappa = \sum_k Y_k \otimes i_{h_k}$ the respective corresponding Koszul cohomology and homology operators. Then*

$$\mathcal{K}\kappa + \kappa\mathcal{K} = \Big(\sum_\ell Y_\ell\mathcal{X}_\ell\Big)\otimes\mathrm{id} + \sum_{k\ell}[\mathcal{X}_\ell, Y_k]\otimes e_{\eta_\ell}i_{h_k}.$$

Proof. This is a direct consequence of Proposition 18. □

Theorem 20. *Let $\vec X \in \mathrm{End}_{\mathbb{C}}^{\times n}(E)$ be n commuting endomorphisms of E, a finite-dimensional complex vector space, and let $\vec\lambda \in \mathbb{C}^n$. For splitting $E = E_1 \oplus E_2$, denote by $i_j : E_j \to E$ the injection of E_j into E and by $p_j : E \to E_j$ the projection of E onto E_j.*

If E_1 is stable under the operators X_ℓ, that is, $p_2X_\ell i_1 = 0$, and if $\vec\lambda$ is not in the joint spectrum $\sigma(\vec X')$ of the commuting operators $X'_\ell = p_2X_\ell i_2 \in \mathrm{End}_{\mathbb{C}}(E_2)$, then any cocycle $C \in E\otimes\bigwedge$ of the Koszul complex $K^*(\vec X_\lambda, E)$, where $\vec X_\lambda = \vec X - \vec\lambda\,\mathrm{id}_E$, is cohomologous to a cocycle $C_1 \in E_1\otimes\bigwedge$, with $\bigwedge = \bigwedge_n\langle\vec\eta\rangle$.

Proof. If $q(\vec X) \in \mathbb{C}[X_1, \dots, X_n] \subset \mathrm{End}_{\mathbb{C}}(E)$ denotes a polynomial in the X_ℓ, the map $q(\vec X)_2 = p_2q(\vec X)i_2$ coincides with the (same) polynomial $q(\vec X') \in \mathrm{End}_{\mathbb{C}}(E_2)$ in the X'_ℓ. Indeed, due to stability of E_1, we have

$$p_2X_\ell X_k i_2 = p_2X_\ell i_1p_1X_ki_2 + p_2X_\ell i_2p_2X_ki_2 = X'_\ell X'_2.$$

This implies that the X'_ℓ commute.

Since $\vec\lambda \notin \sigma(\vec X')$, Proposition 17(e) implies that there are n operators $\vec Y'$ in the subalgebra of $\mathrm{End}_{\mathbb{C}}(E_2)$ generated by id_{E_2} and $\vec X'$ such that

$$\sum_\ell Y'_\ell(X'_\ell - \lambda_\ell\,\mathrm{id}_{E_2}) = \mathrm{id}_{E_2}\,. \tag{4-1}$$

Hence $Y'_\ell = Q_\ell(\vec X')$ is a polynomial in the X'_k for any ℓ. Set $Y_\ell = Q_\ell(\vec X) \in \mathrm{End}_{\mathbb{C}}(E)$.

If applied to operators $\vec X_\lambda$ and $\vec Y$, Proposition 19 implies that

$$\Big(\sum_\ell Y_\ell(X_\ell - \lambda_\ell\,\mathrm{id}_E)\Big)\otimes\mathrm{id}_\wedge + \sum_{k\ell}[X_\ell - \lambda_\ell\,\mathrm{id}_E, Y_k]\otimes e_{\eta_\ell}i_{h_k} = \mathcal{K}\kappa + \kappa\mathcal{K}, \tag{4-2}$$

where $\mathcal{K}$ and κ are respectively the Koszul cohomology and homology operators associated with $\vec{X}_\lambda$ and $\vec{Y}$ on E. Since Y_k is a polynomial in the commuting endomorphisms X_ℓ, the second term on the left side of (4-2) vanishes. Hence, when evaluating both sides on a cocycle $C = e \otimes w$ of cochain complex $K^*(\vec{X}_\lambda, E)$, we get

$$(Q(\vec{X})(e))w = \mathcal{K}\kappa(e \otimes w),$$

where $Q(\vec{X}) = \sum_\ell Y_\ell(X_\ell - \lambda_\ell \operatorname{id}_E) = \sum_\ell Q_\ell(\vec{X})(X_\ell - \lambda_\ell \operatorname{id}_E)$ is a polynomial in the X_ℓ. Absent the factor w, the left side reads

$$Q(\vec{X})(e) = p_1 Q(\vec{X}) i_1 p_1(e) + p_2 Q(\vec{X}) i_1 p_1(e) + p_1 Q(\vec{X}) i_2 p_2(e) + p_2 Q(\vec{X}) i_2 p_2(e),$$

where the second term on the right vanishes in view of the stability of E_1, and the last term equals $p_2(e)$, in view of the first sentence of the proof of Theorem 20 and (4-1). Finally, the cocycle $C = e \otimes w$ is cohomologous to

$$C_1 = C - \mathcal{K}\kappa C = \big(p_1(e) - p_1 Q(\vec{X}) i_1 p_1(e) - p_1 Q(\vec{X}) i_2 p_2(e)\big) \otimes w \in E_1 \otimes \textstyle\bigwedge. \quad \square$$

This theorem has a number of new and partially practical consequences.

First, if $\vec{X} \in \operatorname{End}_{\mathbb{C}}^{\times n}(E)$ are n commuting $\mathbb{C}$-linear endomorphisms, the complex finite-dimensional vector space E on which these operators act has a direct sum decomposition $E = \bigoplus_{\vec{\mu} \in \mathbb{C}^n} E^\mu$, where the primary subspace of E associated with the weight $\vec{\mu}$, namely

$$E^\mu = \bigcap_i E^{\mu_i} = \bigcap_i \bigcup_{n \in \mathbb{N}} \ker(X_i - \mu_i \operatorname{id})^n,$$

is stable under the action of the operators $\vec{X}$ [Bourbaki 1975, théorème 1]. Let us also mention that

$$E^{\mu_i} = \bigcup_{n \in \mathbb{N}} \ker(X_i - \mu_i \operatorname{id})^n = \ker(X_i - \mu_i \operatorname{id})^{m_i},$$

where m_i denotes the multiplicity of the solution μ_i of the characteristic polynomial of X_i, and that $\dim E^{\mu_i} = m_i$. Since the multiplicity $\mathfrak{m}$ of $\vec{0}$ in the joint spectrum of the commuting operators $X_i - \mu_i \operatorname{id}$ coincides with its multiplicity in the joint spectrum of the operators $(X_i - \mu_i \operatorname{id})^m$, where $m = \sup\{m_i\}$, we easily see that the dimension of $E^\mu = \bigcap_i \ker(X_i - \mu_i \operatorname{id})^m$ cannot exceed $\mathfrak{m}$.

Another consequence of Theorem 20 is that Koszul cohomology $KH^*(\vec{X}_\lambda, E)$, roughly speaking, is made up of weak joint eigenvectors with eigenvalues λ_ℓ.

Corollary 21. *Let $\vec{\lambda} \in \mathbb{C}^n$, and let $\vec{X} \in \operatorname{End}_{\mathbb{C}}^{\times n}(E)$ be n commuting endomorphisms of a finite-dimensional complex vector space E. Denote by $\bigwedge = \bigwedge_n \langle \vec{\eta} \rangle$ the Grassmann algebra with n generators $\vec{\eta}$. Any cocycle $C \in E \otimes \bigwedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle $C_1 \in E^\lambda \otimes \bigwedge$.*

This corollary has a useful variant. Choose a supplementary subspace $E^{(2)}$ in E of the stable subspace $\ker \vec{X}_\lambda := \bigcap_\ell \ker(X_\ell - \lambda_\ell \,\mathrm{id})$ of joint eigenvectors, and denote by $\vec{X}^{(2)}$ the restrictions of the operators $\vec{X}$ to $E^{(2)}$; see Theorem 20. We can iterate this procedure, choosing a supplementary subspace $E^{(3)}$ in $E^{(2)}$ of $\ker \vec{X}^{(2)}_\lambda$, introducing the restrictions $\vec{X}^{(3)}$ of the operators $\vec{X}^{(2)}$ to $E^{(3)}$, and so on, until we get $\ker \vec{X}^{(s+1)}_\lambda = 0$. We will show that the direct sum of these kernels coincides with the space E^λ above, so that Corollary 21 means that any cocycle $C \in E \otimes \bigwedge$ of $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle $C_1 \in \bigoplus_{a=1}^s \ker \vec{X}^{(a)}_\lambda \otimes \bigwedge$, where of course $\ker \vec{X}^{(1)}_\lambda = \ker \vec{X}_\lambda$.

Proof of Corollary 21. Let $E_\lambda := \ker \vec{X}_\lambda \subset E^\lambda$ and denote by $E(\lambda)$ a supplementary subspace of E_λ in E^λ. Set

$$E = E^\lambda \oplus \bigoplus\nolimits_{\vec{\mu} \neq \vec{\lambda}} E^\mu =: E_1 \oplus E_2$$

and use the notation of Theorem 20. Assume that $\vec{\lambda} \in \sigma(\vec{X}')$, that is, that $\ker \vec{X}'_\lambda \neq 0$. If $e \in E_2$ is a nonvanishing joint eigenvector of the $X'_\ell = p_2 X_\ell i_2$ with eigenvalues λ_ℓ, we have $X_\ell e = p_1 X_\ell e + \lambda_\ell e$ for any ℓ. Since E_2 is fixed by the X_ℓ, we get $p_1 X_\ell e = 0$, so that $e \in E_1 \cap E_2 = 0$, a contradiction; finally $\vec{\lambda} \notin \sigma(\vec{X}')$ and we finish using Theorem 20.

On the other hand, set

$$E = E_\lambda \oplus \Big(E(\lambda) \oplus \bigoplus\nolimits_{\vec{\mu} \neq \vec{\lambda}} E^\mu \Big) =: E_1 \oplus E_2.$$

Note that the operators $\vec{X}'$ in this decomposition coincide with the $\vec{X}^{(2)}$ above. Now $(X_\ell - \lambda_\ell \,\mathrm{id})e = p_1 X_\ell e \in E_1 = \ker \vec{X}_\lambda$. Any $e \in \ker \vec{X}'_\lambda = \ker \vec{X}^{(2)}_\lambda$ belongs to E^λ. Also $\ker \vec{X}_\lambda \oplus \ker \vec{X}^{(2)}_\lambda \subset E^\lambda$; more generally, $\bigoplus_{a=1}^s \ker \vec{X}^{(a)}_\lambda \subset E^\lambda$. Since, as is easily checked, the dimension of this direct sum is equal to the multiplicity $\mathfrak{m}$ (which is no less than $\dim E^\lambda$) of $\vec{0}$ in the joint spectrum $\sigma(\vec{X}_\lambda)$, this direct sum coincides with E^λ. □

We next recover a well-known result, and then an important special case.

Corollary 22. *Let $\vec{X} \in \mathrm{End}^{\times n}_{\mathbb{C}}(E)$ be n commuting endomorphisms, and let $\vec{\lambda} \in \mathbb{C}^n$. Then $KH^*(\vec{X}_\lambda, E)$ is trivial if and only if* $\dim(\ker \vec{X}_\lambda) = 0$.

Corollary 23. *Assume the conditions of Corollary 21. If for any $\ell \in \{1, \dots, n\}$ the kernel and image of $X_\ell - \lambda_\ell$* id *are supplementary in E, then any cocycle $C \in E \otimes \bigwedge$ of the Koszul complex $K^*(\vec{X}_\lambda, E)$ is cohomologous to a cocycle $C_1 \in \ker \vec{X}_\lambda \otimes \bigwedge$.*

Proofs. First the forward implication in Corollary 22. If there exists $e \in \ker \vec{X}_\lambda \setminus \{0\}$, then

$$\mathcal{K}_{\vec{X}_\lambda} e = \sum_{\ell=1}^n (X_\ell - \lambda_\ell \,\mathrm{id}) e \eta_\ell = 0,$$

so that e is a nonbounding 0-cocycle. As for Corollary 23, it follows from the proof of Corollary 21 that if there is a nonzero vector $e \in \ker \vec{X}_{\lambda}^{(2)} \subset E^{(2)}$, then for every ℓ we have $(X_\ell - \lambda_\ell \,\mathrm{id})e \in \ker \vec{X}_\lambda \cap \mathrm{im}(X_\ell - \lambda_\ell \,\mathrm{id}) = 0$, so that $e \in \ker \vec{X}_\lambda \cap E^{(2)} = 0$, which is a contradiction. □

5. Koszul cohomology associated with Poisson cohomology

We now return to the Koszul cohomology implemented by a SRMI tensor of $\mathbb{R}^n$. Recall the QPT tensor Λ from (1-1) and the conditions under which it is SRMI. Theorems 13 and 15 identify the main building block of the LP-cohomology of Λ as the Koszul cohomology space $KH^*(\vec{X}_\delta, E_r)$. We noted that this cohomology can be deduced from its complex counterpart $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$, which, according to Corollaries 21–23, is closely related to the joint eigenvectors and spectrum of $\vec{X}^{\mathbb{C}}$ or $\vec{X}_\delta^{\mathbb{C}}$. We now further investigate $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$. In particular, we reduce its computation to a problem of linear algebra, and describe the spectrum of the commuting transformations $\vec{X}_\delta^{\mathbb{C}}$.

Proposition 24. *Let $a_j = J^{-1}Y_j \in \mathfrak{gl}(n, \mathbb{R})$ for $j \in \{1, \dots, n\}$. Any basis of $\mathbb{C}^n$ in which the $\vec{a}$ are upper-triangular naturally induces a basis of $E_r^{\mathbb{C}} = \mathcal{S}^r\mathbb{C}^{n*}$ in which the $\vec{X}_\delta^{\mathbb{C}}$ are upper-triangular.*

In what follows, the use of super- and subscripts is dictated by aesthetics and not by contra- or covariance.

Proof. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $z = (z_1, \dots, z_n) \in C^n$. As usual, we set $Y_k = \sum_m \ell_{km}\partial_{x_m} = \sum_{mp} a_k^{mp} x_p \partial_{x_m}$ and use notations as $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ for $\beta \in \mathbb{N}^n$.

We complexify

$$E_r = \mathcal{S}^r\mathbb{R}^{n*} = \left\{P \in C^\infty(\mathbb{R}^n) : P(x) = \textstyle\sum_{|\beta|=r} r_\beta x^\beta,\ x \in \mathbb{R}^n,\ r_\beta \in \mathbb{R}\right\}$$

to $E_r^{\mathbb{C}} \simeq \mathcal{S}^r\mathbb{C}^{n*}$ by replacing $\mathbb{R}$ with $\mathbb{C}$. The complexification $Y_k^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(E_r^{\mathbb{C}})$ of $Y_k \in \mathrm{End}_{\mathbb{R}}(E_r)$ is the holomorphic vector field

$$Y_k^{\mathbb{C}} = \sum_{mp} a_k^{mp} z_p \partial_{z_m} \in \mathrm{Vect}^{10}(\mathbb{C}^n) \quad \text{of } \mathbb{C}^n.$$

There is a unitary matrix $U \in \mathrm{U}(n, \mathbb{C})$ such that $b_j = U^{-1}a_jU$ is upper-triangular, and there is a corresponding basis $(e'_1, \dots, e'_n)$ of $\mathbb{C}^n$ such that the a_j themselves are upper-triangular. Let $(\varepsilon'_1, \dots, \varepsilon'_n)$ be the corresponding dual basis.

Express z in this basis as $z = \sum_j \mathfrak{z}_j e'_j \in \mathbb{C}^n$. If viewed as a basis of the space $E_r^{\mathbb{C}}$ of degree r homogeneous polynomials of $\mathbb{C}^n$, the induced basis of the space $\mathcal{S}^r\mathbb{C}^{n*}$ of symmetric covariant r-tensors of $\mathbb{C}^n$ is the set $\mathfrak{z}^\beta$ with $\beta \in \mathbb{N}^n$ and $|\beta| = r$.

To find the matrices of the operators $\vec{X}_\delta^{\mathbb{C}}$ in the basis $\mathfrak{z}^\beta$, we arrange its elements by the lexicographic order $\prec$ and perform the coordinate change $z = U\mathfrak{z}$ and put

$\partial_z = \widetilde{\partial_{\mathfrak{z}} z}^{-1} \partial_{\mathfrak{z}}$ in the first order differential operators $(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}}$. We get

$$\begin{aligned}(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}} &= \sum_k \alpha^{jk} \sum_{m\le p} b_k^{mp} \mathfrak{z}_p \partial_{\mathfrak{z}_m} - \delta_j \,\mathrm{id}^{\mathbb{C}} \\ &= \sum_{km} \alpha^{jk} b_k^{mm} (\mathfrak{z}_m \partial_{\mathfrak{z}_m} - \mathrm{id}^{\mathbb{C}}) + \sum_k \sum_{m<p} \alpha^{jk} b_k^{mp} \mathfrak{z}_p \partial_{\mathfrak{z}_m},\end{aligned}$$

since $\delta_j = \operatorname{div} X_j = \sum_{km} \alpha^{jk} a_k^{mm} = \sum_{km} \alpha^{jk} b_k^{mm}$. These operators are then upper-triangular in the $\mathfrak{z}^\beta$ basis, since

$$(X_j - \delta_j \,\mathrm{id})^{\mathbb{C}} \mathfrak{z}^\beta = \sum_{km} \alpha^{jk} b_k^{mm} \;(\beta_m - 1)\, \mathfrak{z}^\beta + \sum_k \sum_{m<p} \alpha^{jk} b_k^{mp} \beta_m \,\mathfrak{z}^{\beta - e_m + e_p}, \tag{5-1}$$

where $\mathfrak{z}^{\beta - e_m + e_p} \prec \mathfrak{z}^\beta$. □

With matrices b_j as above, let $B \in \mathfrak{gl}(n, \mathbb{C})$ be the matrix $B_{jk} = b_j^{kk}$.

Theorem 25. *The joint spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ of the $\vec{X}_\delta^{\mathbb{C}}$ is given by*

$$\sigma_r(\vec{X}_\delta^{\mathbb{C}}) = \big\{\alpha B I : I \in (\mathbb{N} \cup \{-1\})^n,\, |I| = r - n\big\} \subset \mathbb{C}^n, \quad \textit{where } |I| = \textstyle\sum_j I_j.$$

Proof. This follows directly from Proposition 17 and (5-1). □

Remark. In Proposition 11, we showed $Y_k D = (\operatorname{div} Y_k) D$ that for all k, where $D = \det \ell \in E_n \subset E_n^{\mathbb{C}}$. Then $X_j^{\mathbb{C}} D = X_j D = (\operatorname{div} X_j) D = \delta_j \,\mathrm{id}^{\mathbb{C}} D$ for all j, so that $\vec{0} = (0, \dots, 0) \in \sigma_n(\vec{X}_\delta^{\mathbb{C}})$. This is immediately recovered from Theorem 25.

Set $K_r(\vec{X}_\delta^{\mathbb{C}}) = \{I \in \ker(\alpha B) : I \in (\mathbb{N} \cup \{-1\})^n, |I| = r - n\}$. Corollary 22 can then be reformulated as follows.

Corollary 26. *$KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ is acyclic if and only if $K_r(\vec{X}_\delta^{\mathbb{C}}) = \varnothing$.*

Proof. $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_r^{\mathbb{C}})$ is trivial if and only if $\dim(\ker \vec{X}_\delta^{\mathbb{C}}) = 0$, which is true if and only if $\vec{0} \notin \sigma_r(\vec{X}_\delta^{\mathbb{C}})$, that is, if and only if $K_r(\vec{X}_\delta^{\mathbb{C}}) = \varnothing$. □

Example 27. Consider the structure Λ_2 of the Dufour–Haraki classification as in Theorem 4, and assume that $a \neq 0$ and $b = 0$. It is easily checked that the matrix

$$U = \begin{pmatrix} 0 & i/\sqrt{2} & -i/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix}$$

transforms the matrices a_ℓ into upper-triangular matrices b_ℓ. A short computation shows that $K_{3t}(\vec{X}_\delta^{\mathbb{C}})$ for $t \in \mathbb{N}$ contains a single point $I_t = (t-1, t-1, t-1)$, so that the multiplicity μ of $\vec{0}$ in the joint spectrum $\sigma_{3t}(\vec{X}_\delta^{\mathbb{C}})$ equals 1; see the proof of Theorem 25. It follows that the $KH^*(\vec{X}_\delta^{\mathbb{C}}, E_{3t}^{\mathbb{C}})$ are not trivial; see Corollary 26. Furthermore, since the matrices b_ℓ are in fact diagonal in this case, (5-1) implies that $\mathfrak{z}_1^t \mathfrak{z}_2^t \mathfrak{z}_3^t$ belongs to the kernel of the $\vec{X}_\delta^{\mathbb{C}}$ in $E_{3t}^{\mathbb{C}}$. Looking at dimensions, we see that this kernel is $\mathbb{C}\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t$ and that the reduced operators $\vec{X}_\delta^{\mathbb{C}(j)}$ for $j \in \{2, \ldots, s\}$ do not exist, that is, that $s = 1$. Then, since the change to canonical coordinates is $z = U\mathfrak{z}$ by the proof of Proposition 24, the space $KH^p(\vec{X}_\delta^{\mathbb{C}}, E_{3t}^{\mathbb{C}})$ for $p \in \{0, 1, 2, 3\}$ and $t \in \mathbb{N}$ is contained in

$$\mathfrak{z}_1^t\mathfrak{z}_2^t\mathfrak{z}_3^t \bigoplus_{j_1<\ldots<j_p} \mathbb{C}Y_{j_1\ldots j_p} = (z_1^2 + z_2^2)^t z_3^t \bigoplus_{j_1<\ldots<j_p} \mathbb{C}Y_{j_1\ldots j_p}.$$

This easy consequence agrees with the results of [Masmoudi and Poncin 2007] modulo slight changes in notation — showing that the new approach is more efficient, though the same results could also be obtained via complexification.

Example 28. For Λ_3 of the Dufour–Haraki classification with parameter $a = 0$, the multiplicity of $\vec{0}$ in the spectrum $\sigma_r(\vec{X}_\delta^{\mathbb{C}})$ equals 0 or 1, depending on the value of r. The computations are similar to those of Example 27 except in the case $r = 3$, which generates multiplicity 3. Since for Λ_3 the matrices a_ℓ are already lower-triangular, a coordinate change is not necessary, and it is easily seen that $s = 3$ and

$$\ker_3 \vec{X}_\delta^{\mathbb{C}} = \mathbb{C}z_1^2 z_3, \quad \ker_3 \vec{X}_\delta^{\mathbb{C}(2)} = \mathbb{C}z_1 z_2 z_3, \quad \ker_3 \vec{X}_\delta^{\mathbb{C}(3)} = \mathbb{C}z_2^2 z_3.$$

The next two theorems follow from similar computations; no proofs are given. In both, the Y_i are those defined in Theorem 4,

Theorem 29. *If $a \neq 0$, the cohomology spaces of the structure Λ_3 are*

$$\begin{aligned}
LH^{0*}(\mathcal{R}, \Lambda_3) &= \mathbb{R}, \\
LH^{1*}(\mathcal{R}, \Lambda_3) &= \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Y_3, \\
LH^{2*}(\mathcal{R}, \Lambda_3) &= \mathbb{R}Y_{23} \oplus \mathbb{R}Y_{31} \oplus \mathbb{R}(2yz\partial_{31} + y^2\partial_{12}), \\
LH^{3*}(\mathcal{R}, \Lambda_3) &= \mathbb{R}\partial_{123} \oplus \mathbb{R}y^2 z\partial_{123}.
\end{aligned}$$

Theorem 30. *If $a \neq 0$, the cohomology spaces of the structure Λ_9 are*

$$LH^{0*}(\mathcal{R}, \Lambda_9) = \mathbb{R}, \qquad LH^{2*}(\mathcal{R}, \Lambda_9) = \textstyle\bigoplus_{r\in\mathbb{N}} H_r^2,$$

$$LH^{1*}(\mathcal{R}, \Lambda_9) = \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Y_3, \qquad LH^{3*}(\mathcal{R}, \Lambda_9) = \textstyle\bigoplus_{r\in\mathbb{N}} \mathbb{R}z^r\partial_{123},$$

where

$$H_0^2 = \mathbb{R}\partial_{23}, \qquad H_1^2 = \mathbb{R}C_1^0, \qquad H_3^2 = \mathbb{R}C_1^2,$$

$$\begin{aligned}
H_2^2 = \mathbb{R}x^2\partial_{23} + \mathbb{R}xz(\partial_{23} - 2^{-1}\partial_{31}) + \mathbb{R}(xz\partial_{12} - z^2\partial_{23}) \\
+ \mathbb{R}(yz\partial_{12} + (-27a^2x^2 - 9axz + 3ay^2 - z^2)\partial_{31}),
\end{aligned}$$

$$H_{r+1}^2 = \mathbb{R}C_1^r + \mathbb{R}C_2^r \quad \textit{for } r \geq 3,$$

with

$$C_1^r = -a(xz^r + ry^2z^{r-1})\partial_{12} + (9a^2xy^r + a(3r-1)(r+1)^{-1}z^{r+1})\partial_{23} + ayz^r\partial_{31}$$

and

$$C_2^r = \big(9a^2xy^2z^{r-2} - 9ar^{-1}xz^r + 3a(r-3)(r-1)^{-1}y^2z^{r-1} - 3(r-1)r^{-1}(r+1)^{-1}z^{r+1}\big)\partial_{23} + (-a(r-2)y^4z^{r-3} + y^2z^{r-1})\partial_{12} + (6a(r-1)^{-1}xyz^{r-1} - ay^3z^{r-2} - r^{-1}yz^r)\partial_{31},$$

and where the terms that contain a negative power of x, y, or z are ignored.

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