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COUNTING CONJUGACY CLASSES IN THE UNIPOTENT RADICAL OF PARABOLIC SUBGROUPS OF $\mathrm{GL}_n(q)$

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Let q be a power of a prime p. Let P be a parabolic subgroup of the general linear group $\mathrm{GL}_n(q)$ that is the stabilizer of a flag in $\mathbb{F}_q^n$ of length at most 5, and let $U = O_p(P)$. We prove that, as a function of q, the number $k(U)$ of conjugacy classes of U is a polynomial in q with integer coefficients.

1. Introduction

Let $\mathrm{GL}_n(q)$ be the finite general linear group defined over the field $\mathbb{F}_q$ of q elements, where q is a power of a prime p. A longstanding conjecture attributed to G. Higman [1960] asserts that the number of conjugacy classes of a Sylow p-subgroup of $\mathrm{GL}_n(q)$ is given by a polynomial in q with integer coefficients. This has been verified by computer calculation by A. Vera-López and J. M. Arregi [2003] for $n \leq 13$. G. R. Robinson [1998] and J. Thompson [2004] have shown much interest in this conjecture. For recent related results, see [Alperin 2006; Evseev 2009; Goodwin and Röhrle 2008; 2009a; 2009b; 2009c].

The following question is precisely Higman's conjecture when $P = B$ is a Borel subgroup of $\mathrm{GL}_n(q)$.

Question 1.1. Let P be a parabolic subgroup of $\mathrm{GL}_n(q)$ and let $U = O_p(P)$. As a function of q, is the number $k(U)$ of conjugacy classes of U a polynomial in q?

Here we recall that $O_p(P)$ is by definition the largest normal p-subgroup of P. In this paper, we give an affirmative answer to Question 1.1 in the following cases.

Theorem 1.2. *Let P be a parabolic subgroup of* $\mathrm{GL}_n(q)$ *that is the stabilizer of a flag in $\mathbb{F}_q^n$ of length at most* 5, *and let $U = O_p(P)$. Then, as a function of q, the number $k(U)$ of conjugacy classes of U is a polynomial in q with integer coefficients.*

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We now explain the significance of the hypothesis imposed in Theorem 1.2. Let $\boldsymbol{P}$ be a parabolic subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$, and let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{P}$, where $\overline{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$. All instances when $\boldsymbol{P}$ acts on $\boldsymbol{U}$ with a finite number of orbits were determined in [Hille and Röhrle 1999]; this is precisely the case when $\boldsymbol{P}$ is the stabilizer of a flag in $\overline{\mathbb{F}}_q^n$ of length at most 5. So Theorem 1.2 deals with parabolic subgroups P of $\mathrm{GL}_n(q)$ that correspond to parabolic subgroups $\boldsymbol{P}$ of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ with a finite number of conjugacy classes in $\boldsymbol{U}$. In such cases, it is observed in [Hille and Röhrle 1999, Remark 4.13] that the parameterization of the $\boldsymbol{P}$-conjugacy classes in $\boldsymbol{U}$ is independent of q: This is the crucial point that we require for our proof of Theorem 1.2.

The proof involves a translation of the problem to a representation theoretic setting. More precisely, recall from [Hille and Röhrle 1999, Section 4] that the P-conjugacy classes in U correspond bijectively to the so-called Δ-filtered modules of a certain quasihereditary algebra $\mathscr{A}_t$. This allows us to see that the parameterization of the P-orbits in U is independent of q and that we can choose a set $\mathfrak{R}$ of representatives that are matrices with entries equal to 0 or 1. The other key point is that the structures of the centralizers $C_P(x)$ and $C_U(x)$ for $x \in \mathfrak{R}$ do not depend on q; this is covered in Propositions 2.2 and 2.4.

We now discuss some natural generalizations of Theorem 1.2. First consider the case of a normal subgroup $\boldsymbol{N}$ of $\boldsymbol{P}$ with $\boldsymbol{N} \subseteq \boldsymbol{U}$. Still assuming that there is only a finite number of $\boldsymbol{P}$-orbits in $\boldsymbol{U}$, we readily derive from the proof of Theorem 1.2 that $k(U, N)$, the number of U-conjugacy classes in $N = \boldsymbol{N} \cap U$, is given by a polynomial in q with integer coefficients. It should also be possible to prove that the number $k(U, N)$ is a polynomial in q with just the assumption that there are finitely many $\boldsymbol{P}$-orbits in $\boldsymbol{N}$. For example, for $\boldsymbol{N} = \boldsymbol{U}^{(l)}$ the l-th member of the descending central series of $\boldsymbol{U}$, there is a classification of all instances when $\boldsymbol{P}$ acts on $\boldsymbol{U}^{(l)}$ with a finite number of orbits; see [Brüstle and Hille 2000]. In such situations a generalization of the proof of Theorem 1.2 would require detailed knowledge of the $\boldsymbol{P}$-conjugacy classes in $\boldsymbol{N}$.

It is also natural to consider the generalization of Question 1.1, where $\mathrm{GL}_n(q)$ is replaced by any finite reductive group G, and also to consider the number $k(P, U)$ of P-conjugacy classes in U rather than $k(U)$. (To avoid degeneracies in the Chevalley commutator relations, it is sensible to only consider these generalizations when q is a power of a good prime for G.)

At present there are no known examples in which $k(U)$ is not given by a polynomial in q, and there are many cases not covered by Theorem 1.2, where $k(U)$ is given by a polynomial in q; see for example [Goodwin and Röhrle 2009b] and [Vera-López and Arregi 2003]. However, it is not necessarily the case that $k(P, U)$ is a polynomial in q. Indeed in [Goodwin 2007, Example 4.6], it shown that in case G is of type G_2, and $P = B$ is a Borel subgroup of G, the number $k(B, U)$ is given by two different polynomials depending on the residue of q modulo 3.

Let $\boldsymbol{P}$ be a parabolic subgroup of a reductive algebraic group $\boldsymbol{G}$ defined over $\mathbb{F}_q$, and suppose that $\boldsymbol{P}$ has finitely many conjugacy classes in $\boldsymbol{U}$; let P and U be the groups of $\mathbb{F}_q$-rational points of $\boldsymbol{P}$ and $\boldsymbol{U}$, respectively. Given the discussion after Theorem 1.2, a natural generalization to consider is whether the number $k(U)$ of conjugacy classes of U is a polynomial in q. Our proof of Theorem 1.2 is dependent on the detailed information about the P-conjugacy classes in U. For this reason the argument does not adapt to the case in which G is any finite reductive group. The main difficulty is that it is not clear whether the parameterization of $\boldsymbol{P}$-orbits in $\boldsymbol{U}$ and the structure of centralizers depends on the characteristic of the underlying ground field. Another problem is that centralizers $C_{\boldsymbol{P}}(u)$ for $u \in U$ need not be connected, so determining the P-classes in U from the $\boldsymbol{P}$-classes in $\boldsymbol{U}$ may be nontrivial.

2. Translation to representation theory

Here, we recall the relationship established in [Hille and Röhrle 1999, Section 4] between adjoint orbits of parabolic subgroups and modules for a certain quasi-hereditary algebra. This relationship is central to our proof of Theorem 1.2. In particular, it is crucial for Propositions 2.2 and 2.4, which describe the structure of certain centralizers. Throughout this section we work in generality over any field, before specializing to finite fields for the proof of Theorem 1.2 in Section 3.

Let K be any field, and let $n, t \in \mathbb{Z}_{\geq 1}$. Let $\boldsymbol{d} = (d_1, \ldots, d_t) \in \mathbb{Z}_{\geq 0}^t$ with $d_i \leq d_{i+1}$ and $d_t = n$. We define the parabolic subgroup $P(\boldsymbol{d}) = P_K(\boldsymbol{d})$ of $\mathrm{GL}_n(K)$ to be the stabilizer of the flag $0 \subseteq K^{d_1} \subseteq K^{d_2} \subseteq \ldots \subseteq K^{d_t}$ in K^n; any parabolic subgroup of $\mathrm{GL}_n(K)$ is conjugate to $P(\boldsymbol{d})$ for some $\boldsymbol{d}$. We write

$$U(\boldsymbol{d}) = U_K(\boldsymbol{d}) = \{u \in \mathrm{GL}_n(K) \mid (u-1)V_i \subseteq V_{i-1} \text{ for each } i\}$$

for the unipotent radical of $P(\boldsymbol{d})$, and

$$\mathfrak{u}(\boldsymbol{d}) = \mathfrak{u}_K(\boldsymbol{d}) = \{x \in \mathrm{M}_n(K) \mid xV_i \subseteq V_{i-1} \text{ for each } i\}$$

for the Lie algebra of $U(\boldsymbol{d})$. Then $P(\boldsymbol{d})$ acts on $\mathfrak{u}(\boldsymbol{d})$ via the adjoint action, that is, $g \cdot x = gxg^{-1}$ for $g \in P(\boldsymbol{d})$ and $x \in \mathfrak{u}(\boldsymbol{d})$. For $x \in \mathfrak{u}(\boldsymbol{d})$, we write $P \cdot x$ for the adjoint P-orbit of x and $C_P(x)$ for the centralizer of x in P; we define $U \cdot x$ and $C_U(x)$ analogously.

Though we are primarily interested in the conjugacy classes of $U(\boldsymbol{d})$ and the $P(\boldsymbol{d})$-conjugacy classes in $U(\boldsymbol{d})$, it is more convenient to consider the adjoint $P(\boldsymbol{d})$-orbits in $\mathfrak{u}(\boldsymbol{d})$. The map $x \mapsto 1 + x$ is a $P(\boldsymbol{d})$-equivariant isomorphism between $\mathfrak{u}(\boldsymbol{d})$ and $U(\boldsymbol{d})$, which means that the adjoint $P(\boldsymbol{d})$-orbits in $\mathfrak{u}(\boldsymbol{d})$ are in bijective correspondence with the $P(\boldsymbol{d})$-conjugacy classes in $U(\boldsymbol{d})$; this allows us to work with the adjoint orbits.

The quiver $\mathcal{Q}_t$ is defined to have vertex set $\{1, \dots, t\}$, and there are arrows $\alpha_i : i \to i+1$ and $\beta_i : i+1 \to i$ for $i = 1, \dots, t-1$. Here is an example of the quiver $\mathcal{Q}_t$ for $t = 5$:

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift right, "\alpha_1"'] & \underset{2}{\bullet} \arrow[l, shift right, "\beta_1"'] \arrow[r, shift right, "\alpha_2"'] & \underset{3}{\bullet} \arrow[l, shift right, "\beta_2"'] \arrow[r, shift right, "\alpha_3"'] & \underset{4}{\bullet} \arrow[l, shift right, "\beta_3"'] \arrow[r, shift right, "\alpha_4"'] & \underset{5}{\bullet} \arrow[l, shift right, "\beta_4"'] \end{tikzcd}$$

Let $I_t = I_{t,K}$ be the ideal of the path algebra $K\mathcal{Q}_t$ of $\mathcal{Q}_t$ generated by the relations

$$\beta_1\alpha_1 = 0 \quad \text{and} \quad \alpha_i\beta_i = \beta_{i+1}\alpha_{i+1} \quad \text{for } i = 1, \dots, t-2. \tag{2-1}$$

The algebra $\mathcal{A}_t = \mathcal{A}_{t,K}$ is defined to be the quotient $K\mathcal{Q}_t/I_t$.

Recall that an $\mathcal{A}_t$-module M is determined by a family of vector spaces $M(i)$ over K for $i = 1, \dots, t$ such that $M = \bigoplus_{i=1}^{t} M(i)$, and linear maps $M(\alpha_i) : M(i) \to M(i+1)$ and $M(\beta_i) : M(i+1) \to M(i)$ for $i = 1, \dots, t-1$ that satisfy the relations (2-1). The dimension vector $\dim M \in \mathbb{Z}^t_{\geq 0}$ of an $\mathcal{A}_t$-module is defined by $\dim M = (\dim M(1), \dots, \dim M(t))$.

Let $\mathcal{M}_t = \mathcal{M}_{t,K}$ be the category of $\mathcal{A}_t$-modules M such that $M(\alpha_i)$ is injective for all i. Write $\mathcal{M}_t(\boldsymbol{d}) = \mathcal{M}_{t,K}(\boldsymbol{d})$ for the class of modules in $\mathcal{M}_t$ with dimension vector $\boldsymbol{d}$. Hille and Röhrle show in [1999, Section 4] that the orbits of $P(\boldsymbol{d})$ in $\mathfrak{u}(\boldsymbol{d})$ are in bijection with the isoclasses in $\mathcal{M}_t(\boldsymbol{d})$ and moreover, using [Dlab and Ringel 1992, Sections 6 and 7],[1] that there is a unique structure of a quasihereditary algebra on $\mathcal{A}_t$ such that $\mathcal{M}_t$ is the category of Δ-*filtered* $\mathcal{A}_t$-modules.

Suppose for this paragraph that K is infinite. Using the above bijection and the results from [DR], it was proved in [HR, Theorem 4.1] that there is a finite number of $P(\boldsymbol{d})$-orbits in $\mathfrak{u}(\boldsymbol{d})$ if and only if $t \leq 5$. This is deduced from the fact that $\mathcal{A}_t$ has finite Δ-representation type if and only if $t \leq 5$; see [DR, Proposition 7.2].

Let $t \leq 5$. Because the results in [HR, Section 4] are proved for an arbitrary field — see [HR, Remark 4.13] — the parametrization of indecomposable Δ-filtered $\mathcal{A}_t$-modules does not depend on the field K; we explain this more explicitly below. Let $\{I_1, \dots, I_m\}$ be a complete set of representatives of isoclasses of indecomposable Δ-filtered $\mathcal{A}_t$-modules, and write $\boldsymbol{d}_i$ for the dimension vector of I_i. Let $x_i \in \mathfrak{u}(\boldsymbol{d}_i)$ be such that the $P(\boldsymbol{d}_i)$-orbit of x_i corresponds to the isoclass of I_i. As discussed in [HR, Section 7] — see also [Brüstle et al. 1999, Figure 10] — one can choose x_i to be a matrix with entries 0 and 1, and these matrices do not depend on K. In particular, this implies that the modules I_i are absolutely indecomposable.

Another important consequence for us is the following lemma.

[1] These two references are henceforth abbreviated as [HR] and [DR].

Lemma 2.1. *Assume $t \le 5$. We may choose a set $\mathfrak{R}$ of representatives of the adjoint $P(\boldsymbol{d})$-orbits in $\mathfrak{u}(\boldsymbol{d})$ such that each element of $\mathfrak{R}$ is a matrix with all entries equal to* 0 *or* 1*. Moreover, the elements of $\mathfrak{R}$ do not depend on the field K, that is, the positions of entries equal to* 1 *do not depend on K.*

We still assume that $t \le 5$, and let $\boldsymbol{d} \in \mathbb{Z}^t_{\ge 0}$. Let $P = P(\boldsymbol{d})$, $U = U(\boldsymbol{d})$ and $x \in \mathfrak{u} = \mathfrak{u}(\boldsymbol{d})$. For the proof of Theorem 1.2 we need information about the structure of the centralizers $C_P(x)$ and $C_U(x)$; this is given by Propositions 2.2 and 2.4.

Let M be a Δ-filtered $\mathscr{A}_t$-module (with dimension vector $\boldsymbol{d}$) whose isoclass corresponds to the P-orbit of x. Extending the arguments of [HR, Section 4], one can show that the automorphism group $\mathrm{Aut}_{\mathscr{A}_t}(M)$ of M is isomorphic to $C_P(x)$. Below we explain the structure of $\mathrm{End}_{\mathscr{A}_t}(M)$ and $\mathrm{Aut}_{\mathscr{A}_t}(M)$; this uses standard arguments that we outline here for convenience. We proceed to explain how $C_U(x)$ is related to $\mathrm{End}_{\mathscr{A}_t}(M)$.

As above, let $\{I_1, \dots, I_m\}$ be a complete set of representatives of isoclasses of indecomposable Δ-filtered $\mathscr{A}_t$-modules. We may decompose M as a direct sum of indecomposable modules

$$M \cong \bigoplus_{i=1}^{m} n_i I_i, \quad \text{where } n_i \in \mathbb{Z}_{\ge 0}. \tag{2-2}$$

Then

$$\mathrm{End}_{\mathscr{A}_t}(M) \cong \bigoplus_{i,j=1}^{m} n_i n_j \,\mathrm{Hom}_{\mathscr{A}_t}(I_i, I_j)$$

as a vector space and composition is defined in the obvious way.

We observed above that I_i is absolutely indecomposable, which means that $\mathrm{End}_{\mathscr{A}_t}(I_i)$ is a local ring, and that we have the decomposition $\mathrm{End}_{\mathscr{A}_t}(I_i) = K \oplus \mathfrak{m}_i$, where K is acting by scalars and $\mathfrak{m}_i$ is the maximal ideal. Therefore,

$$n_i^2 \,\mathrm{End}_{\mathscr{A}_t}(I_i) \cong \mathrm{M}_{n_i}(K) \oplus \mathrm{M}_{n_i}(\mathfrak{m}_i),$$

where $\mathrm{M}_{n_i}(K)$ is a subalgebra and $\mathrm{M}_{n_i}(\mathfrak{m}_i)$ is an ideal. In fact, $\mathrm{M}_{n_i}(\mathfrak{m}_i)$ is the Jacobson radical of $n_i^2 \,\mathrm{End}_{\mathscr{A}_t}(I_i)$.

Now one can see that the Jacobson radical of $\mathrm{End}_{\mathscr{A}_t}(M)$ is

$$J(\mathrm{End}_{\mathscr{A}_t}(M)) \cong \bigoplus_{i=1}^{m} \mathrm{M}_{n_i}(\mathfrak{m}_i) \oplus \bigoplus_{i \ne j} n_i n_j \,\mathrm{Hom}_{\mathscr{A}_t}(I_i, I_j).$$

There is a complement to $J(\mathrm{End}_{\mathscr{A}_t}(M))$ in $\mathrm{End}_{\mathscr{A}_t}(M)$ denoted by $C(\mathrm{End}_{\mathscr{A}_t}(M))$ with

$$C(\mathrm{End}_{\mathscr{A}_t}(M)) \cong \bigoplus_{i=1}^{m} \mathrm{M}_{n_i}(K).$$

We can now describe the automorphism group $\mathrm{Aut}_{\mathscr{A}_t}(M)$. We have

$$\mathrm{Aut}_{\mathscr{A}_t}(M) \cong U(C(\mathrm{End}_{\mathscr{A}_t}(M))) \ltimes (1_M + J(\mathrm{End}_{\mathscr{A}_t}(M))),$$

with $U(C(\mathrm{End}_{\mathscr{A}_t}(M)))$ the group of units of $C(\mathrm{End}_{\mathscr{A}_t}(M))$ and $1_M + J(\mathrm{End}_{\mathscr{A}_t}(M))$ the unipotent group $\{1_M + \phi \mid \phi \in J(\mathrm{End}_{\mathscr{A}_t}(M))\}$. We have $U(C(\mathrm{End}_{\mathscr{A}_t}(M))) \cong \prod_{i=1}^m \mathrm{GL}_{n_i}(K)$, and therefore

$$\mathrm{Aut}_{\mathscr{A}_t}(M) \cong \prod_{i=1}^m \mathrm{GL}_{n_i}(K) \ltimes N,$$

where N is a split unipotent group over K. By saying N is a *split unipotent group*, we mean that N has a normal series with all quotients isomorphic to the additive group K. The dimension of N is

$$\delta := \sum_{i=1}^m n_i^2(\dim \mathrm{End}_{\mathscr{A}_t}(I_i) - 1) + \sum_{i \neq j} n_i n_j \dim \mathrm{Hom}_{\mathscr{A}_t}(I_i, I_j). \tag{2-3}$$

One can compute all Hom-groups $\mathrm{Hom}_{\mathscr{A}_t}(I_i, I_j)$ from the underlying Auslander–Reiten quivers of $\mathscr{A}_t$ in [DR, pages 221 and 222]; see also [Brüstle et al. 1999, Appendix A]. The dimensions $\dim \mathrm{Hom}_{\mathscr{A}_t}(I_i, I_j)$ are independent of K. Therefore, the positive integer δ is also independent of K.

We said above that $\mathrm{Aut}_{\mathscr{A}_t}(M)$ is isomorphic to $C_P(x)$, so we have the following proposition.

Proposition 2.2. *The Levi decomposition of $C_P(x)$ is given by*

$$C_P(x) \cong \prod_{i=1}^m \mathrm{GL}_{n_i}(K) \ltimes N,$$

where N, the unipotent radical of $C_P(x)$, is a split unipotent group over K of dimension δ.

Remark 2.3. It is natural to ask whether Proposition 2.2 still holds if $t > 5$. The arguments above do apply in case K is assumed to be algebraically closed. It would be interesting to know what happens in general, and also if Corollary 3.1 holds for $t > 5$.

We now wish to give the structure of the centralizer $C_U(x)$. By further extending the arguments in [HR, Section 4], one sees that there is an isomorphism

$$C_U(x) \cong 1_M + \mathrm{End}'_{\mathscr{A}_t}(M),$$

where

$$\mathrm{End}'_{\mathscr{A}_t}(M) := \{\phi \in \mathrm{End}_{\mathscr{A}_t}(M) \mid \phi M(l) \subseteq M(l-1) \text{ for all } l\};$$

here we are identifying $M(l-1)$ with its image in $M(l)$ under $M(\alpha_{l-1})$. We have that $\mathrm{End}'_{\mathcal{A}_t}(M)$ is a nilpotent ideal of $\mathrm{End}_{\mathcal{A}_t}(M)$. We define

$$\mathrm{Hom}'_{\mathcal{A}_t}(I_i, I_j) := \{\phi \in \mathrm{Hom}_{\mathcal{A}_t}(I_i, I_j) \mid \phi I_i(l) \subseteq I_j(l-1) \text{ for all } l\}.$$

Then we have the isomorphism

$$\mathrm{End}'_{\mathcal{A}_t}(M) \cong \bigoplus_{i,j=1}^{m} n_i n_j \,\mathrm{Hom}'_{\mathcal{A}_t}(I_i, I_j).$$

We write

$$\delta' := \dim \mathrm{End}'_{\mathcal{A}_t}(M) = \sum_{i,j=1}^{m} n_i n_j \dim \mathrm{Hom}'_{\mathcal{A}_t}(I_i, I_j). \tag{2-4}$$

From the Auslander–Reiten quivers of $\mathcal{A}_t$ exhibited in [DR, pages 221 and 222], one can compute the dimensions $\dim \mathrm{Hom}'_{\mathcal{A}_t}(I_i, I_j)$. These integers are independent of K, so that δ' is also independent of K. The discussion above proves the following proposition.

Proposition 2.4. *The centralizer $C_U(x)$ is a δ'-dimensional split unipotent group over K.*

3. Proof of Theorem 1.2

Let q be a prime power and let $K = \mathbb{F}_q$ be the field of q elements. Let $t \leq 5$ and let $\boldsymbol{d} \in \mathbb{Z}^t_{\geq 0}$. Let $P = P(\boldsymbol{d})$, $U = U(\boldsymbol{d})$ and $\mathfrak{u} = \mathfrak{u}(\boldsymbol{d})$ be as in the previous section, so that P is a parabolic subgroup of $\mathrm{GL}_n(q)$.

The following corollary is a key step in our proof of Theorem 1.2. It follows immediately from Propositions 2.2 and 2.4 along with the elementary fact that the order of a general linear group over $\mathbb{F}_q$ is given by a polynomial in q. The positive integers in the statement are determined in (2-2), (2-3) and (2-4).

Corollary 3.1. *Let $x \in \mathfrak{u}$. Then there are positive integers $n_1, \dots, n_m$, δ and δ' independent of q such that*

$$|C_P(x)| = \prod_{i=1}^{m} |\mathrm{GL}_{n_i}(q)| \cdot q^{\delta} \quad \text{and} \quad |C_U(x)| = q^{\delta'}.$$

In particular, $|C_P(x)|$ and $|C_U(x)|$ are polynomials in q with integer coefficients.

Proof of Theorem 1.2. We must prove that $k(U)$ is given by a polynomial in q. As discussed in the previous section $k(U)$ is equal to $k(U, \mathfrak{u})$, the number of adjoint U-orbits in $\mathfrak{u}$. We will prove that $k(U, \mathfrak{u})$ is a polynomial in q with integer coefficients.

We may choose a set of representatives $\mathcal{R}$ of the adjoint P-orbits in $\mathfrak{u}$, as in Lemma 2.1, and consider $\mathcal{R}$ to be independent of q. We have

$$k(U, \mathfrak{u}) = \sum_{x \in \mathcal{R}} k(U, P \cdot x),$$

where $k(U, P \cdot x)$ is the number of U-orbits contained in $P \cdot x$. For $x \in \mathfrak{u}$ and $g \in P$, we have $C_U(g \cdot x) = gC_U(x)g^{-1}$. Therefore, we get $|U \cdot x| = |U \cdot (g \cdot x)|$ and $k(U, P \cdot x) = |P \cdot x|/|U \cdot x|$. It follows that

$$k(U, \mathfrak{u}) = \sum_{x \in \mathcal{R}} k(U, P \cdot x) = \sum_{x \in \mathcal{R}} \frac{|P \cdot x|}{|U \cdot x|} = \frac{|P|}{|U|} \sum_{x \in \mathcal{R}} \frac{|C_U(x)|}{|C_P(x)|} = |L| \sum_{x \in \mathcal{R}} \frac{|C_U(x)|}{|C_P(x)|},$$

where L is a Levi subgroup of P. Since $|L|$ is a polynomial in q, Corollary 3.1 and the fact that $\mathcal{R}$ is independent of q imply $k(U, \mathfrak{u}) = k(U)$ is a rational function in q. Since $k(U)$ takes integer values for all prime powers, standard arguments show that $k(U)$ is in fact a polynomial in q with rational coefficients; see for example [Goodwin and Röhrle 2009a, Lemma 2.11].

Let $\boldsymbol{P}$ be the subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ corresponding to P and let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{P}$. The *commuting variety of* $\boldsymbol{U}$ is the closed subvariety of $\boldsymbol{U} \times \boldsymbol{U}$ defined by

$$\mathcal{C}(\boldsymbol{U}) = \{(u, u') \in \boldsymbol{U} \times \boldsymbol{U} \mid uu' = u'u\}.$$

Setting $\mathcal{C}(U) = \mathcal{C}(\boldsymbol{U}) \cap (U \times U)$ and using the Burnside counting formula, we get

$$|\mathcal{C}(U)| = \sum_{x \in U} |C_U(x)| = |U| \cdot k(U).$$

Since $|U| = q^{\dim \boldsymbol{U}}$ and $k(U)$ is a polynomial in q with rational coefficients, so is $|\mathcal{C}(U)|$. Now using the Grothendieck trace formula applied to $\mathcal{C}(\boldsymbol{U})$ (see [Digne and Michel 1991, Theorem 10.4]), standard arguments prove that the coefficients of this polynomial are integers; see for example [Reineke 2006, Propopsition 6.1]. Thus, it follows that $k(U)$ is a polynomial function in q with integer coefficients, as claimed. □

Remark 3.2. Let $t \leq 5$ and $\boldsymbol{d}, \boldsymbol{d}' \in \mathbb{Z}_{\geq 0}^t$ with $d_t = d'_t = n$. Suppose that $P = P(\boldsymbol{d})$ and $Q = P(\boldsymbol{d}')$ are *associated parabolic subgroups* of $\mathrm{GL}_n(\mathbb{F}_q)$, that is, P and Q have Levi subgroups that are conjugate in $\mathrm{GL}_n(q)$. This means that there is a $\sigma \in S_n$ such that $d_i - d_{i-1} = d'_{\sigma(i)} - d'_{\sigma(i)-1}$ for all $i = 1, \ldots, t$, with the convention that $d_0 = d'_0 = 0$. Let $U = U(\boldsymbol{d})$ and $V = U(\boldsymbol{d}')$. A consequence of [HR, Corollary 4.7] is that the number $k(P, U)$ of P-conjugacy classes in U is the same as $k(Q, V)$; see [Goodwin and Röhrle 2009a, Corollary 4.8] for similar phenomena. However, it is not always the case that the number of conjugacy classes of U is the same as the number of conjugacy classes of V. For example, take $t = 3$ and consider the dimension vectors $\boldsymbol{d} = (2, 3, 4)$ and $\boldsymbol{d}' = (1, 3, 4)$. Then $P(\boldsymbol{d})$ and $P(\boldsymbol{d}')$ are associated parabolic subgroups of $\mathrm{GL}_4(q)$. Let $U = U(\boldsymbol{d})$ and $V = U(\boldsymbol{d}')$. Then by direct calculation one can check that

$$\begin{aligned} k(U) &= (q-1)^3 + 6(q-1)^2 + 5(q-1) + 1 \\ &\neq (q-1)^4 + 4(q-1)^3 + 6(q-1)^2 + 5(q-1) + 1 = k(V). \end{aligned}$$

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