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THREE CLASSES OF PSEUDOSYMMETRIC CONTACT METRIC 3-MANIFOLDS

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We study the class of pseudosymmetric contact metric 3-manifolds satisfying $Q\xi = \rho\xi$, where ρ is a smooth function constant along the characteristic flow. We classify the complete pseudosymmetric contact metric 3-manifolds of constant type satisfying $Q\xi = \rho\xi$, where ρ is a smooth function, and we also classify the complete (κ, μ, ν)-contact metric pseudosymmetric 3-manifolds of constant type.

1. Introduction

A Riemannian manifold (M^m, g) is said to be *semisymmetric* if its curvature tensor R satisfies the condition $R(X, Y) \cdot R = 0$ for all vector fields X, Y on M, where the dot means that $R(X, Y)$ acts as a derivation on R [Szabó 1982; 1985]. Semisymmetric Riemannian manifolds were first studied by E. Cartan. Obviously, locally symmetric spaces (those with $\nabla R = 0$) are semisymmetric, but the converse is not true, as was proved by H. Takagi [1972].

According to R. Deszcz [1992], a Riemannian manifold (M^m, g) is pseudosymmetric if its curvature tensor R satisfies $R(X, Y) \cdot R = L((X \wedge Y) \cdot R)$, where L is a smooth function and the endomorphism field $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y \tag{1-1}$$

for all vectors fields X, Y, Z on M, and $X \wedge Y$ similarly acts as a derivation on R.

The condition $R(X, Y) \cdot R = L((X \wedge Y) \cdot R)$ arose in the study of totally umbilical submanifolds of semisymmetric manifolds, as well as in the study of geodesic mappings of semisymmetric manifolds [Deszcz 1992]. If L is constant, M is called a pseudosymmetric manifold of constant type. Obviously, pseudosymmetric spaces generalize the semisymmetric ones where $L = 0$. In dimension 3, the pseudosymmetry condition of constant type is equivalent to the condition that the eigenvalues ρ_1, ρ_2, ρ_3 of the Ricci tensor satisfy $\rho_1 = \rho_2$ (up to numeration) and $\rho_3 =$ constant [Deprez et al. 1989; Kowalski and Sekizawa 1996b].

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Three-dimensional pseudosymmetric spaces of constant type have been studied by O. Kowalski and M. Sekizawa [1996b; 1996a; 1997; 1998]. N. Hashimoto and M. Sekizawa [2000] classified 3-dimensional conformally flat pseudosymmetric spaces of constant type, while G. Calvaruso [2006] gave the complete classification of conformally flat pseudosymmetric spaces of constant type for dimensions greater than two. J. T. Cho and J. Inoguchi [2005] studied pseudosymmetric contact homogeneous 3-manifolds. Finally, M. Belkhelfa, R. Deszcz and L. Verstraelen [Belkhelfa et al. 2005] studied pseudosymmetric Sasakian space forms in arbitrary dimension.

This article studies 3-dimensional pseudosymmetric contact metric manifolds, and is organized as follows. In Section 2, we give some preliminaries on pseudosymmetric manifolds and contact manifolds as well. In Section 3, we give the necessary conditions for a 3-dimensional contact metric manifold to be pseudosymmetric. In the remaining sections, we use the results of Section 3 to study 3-dimensional contact metric manifolds that satisfy one of the following:

- M is pseudosymmetric with $Q\xi = \rho\xi$, where ρ is a smooth function on M constant along the characteristic flow.
- M is pseudosymmetric of constant type with $Q\xi=\rho\xi$, where ρ a smooth function on M.
- M is pseudosymmetric of constant type and its curvature satisfies the (κ, μ, ν)-condition.

2. Preliminaries

Let (M^m, g) for $m \geq 3$ be a connected Riemannian smooth manifold. We denote by ∇ the Levi-Civita connection of M^m and by R the corresponding Riemannian curvature tensor with $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$.

A Riemannian manifold (M^m, g) for $m \geq 3$ was called *pseudosymmetric* by R. Deszcz [1992] if at every point of M the curvature tensor satisfies

$$(R(X, Y) \cdot R)(X_1, X_2, X_3) = L(((X \wedge Y) \cdot R)(X_1, X_2, X_3))$$

or equivalently

$$\begin{aligned} R(X, Y)(R(X_1, X_2)X_3) - R(R(X, Y)X_1, X_2)X_3 \\ - R(X_1, R(X, Y)X_2)X_3 - R(X_1, X_2)(R(X, Y)X_3) \\ = L\big((X \wedge Y)(R(X_1, X_2)X_3) - R((X \wedge Y)X_1, X_2)X_3 \\ - R(X_1, (X \wedge Y)X_2)X_3 - R(X_1, X_2)((X \wedge Y)X_3)\big) \end{aligned} \tag{2-1}$$

for all vectors fields X, Y, X_1, X_2, X_3 on M, where $X \wedge Y$ is given by (1-1) and L is a smooth function. For details and examples of pseudosymmetric manifolds, see [Belkhelfa et al. 2002; Deszcz 1992].

A contact manifold is a smooth manifold M^{2n+1} endowed with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Then there is an underlying contact metric structure (η, ξ, ϕ, g), where g is a Riemannian metric (the *associated metric*), ϕ is a global tensor of type $(1, 1)$, and ξ is a unique global vector field (the *characteristic* or *Reeb vector field*). These structure tensors satisfy

$$\begin{gathered} \phi^2 = -I + \eta \otimes \xi, \qquad \eta(X) = g(X, \xi), \qquad \eta(\xi) = 1, \\ d\eta(X, Y) = g(X, \phi Y), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \end{gathered} \tag{2-2}$$

The associated metrics can be constructed by the polarization of $d\eta$ on the contact subbundle defined by $\eta = 0$. Denoting by L the Lie differentiation, we define the tensors

$$h = \tfrac{1}{2} L_\xi \phi, \quad \tau = L_\xi g, \quad l = R(\,\cdot\,, \xi)\xi. \tag{2-3}$$

These tensors satisfy the formulas

$$\begin{aligned} &\phi\xi = h\xi = l\xi = 0, \qquad \eta \circ \phi = \eta \circ h = 0, \qquad d\eta(\xi, X) = 0, \\ &\operatorname{Tr} h = \operatorname{Tr} h\phi = 0, \qquad \nabla_X \xi = -\phi X - \phi h X, \qquad h\phi = -\phi h, \\ &hX = \lambda X \quad \text{implies} \quad h\phi X = -\lambda \phi X, \\ &\nabla_\xi h = \phi - \phi l - \phi h^2, \quad \phi l \phi - l = 2(\phi^2 + h^2), \\ &\nabla_\xi \phi = 0, \qquad \operatorname{Tr} l = g(Q\xi, \xi) = 2n - \operatorname{Tr} h^2. \end{aligned} \tag{2-4}$$

Now $\tau = 0$ (or equivalently $h = 0$) if and only if ξ is Killing, and then M is called K-contact. If the structure is normal, it is Sasakian. A K-contact structure is Sasakian only in dimension 3, and this fails in higher dimensions. For details about contact manifolds, see [Blair 2002].

Let (M, ϕ, ξ, η, g) be a 3-dimensional contact metric manifold. Let U be the open subset of points $p \in M$ such that $h \neq 0$ in a neighborhood of p, and let U_0 be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. Because h is a smooth function on M, the set $U \cup U_0$ is an open and dense subset of M; thus a property that is satisfied in $U_0 \cup U$ is also satisfied in M. For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p (a ϕ-basis). On U, we put $he = \lambda e$, where λ is a nonvanishing smooth function that is supposed positive. From the third line of (2-4), we have $h\phi e = -\lambda \phi e$.

Lemma 2.1 [Gouli-Andreou and Xenos 1998a]. *On U we have*

$$\begin{aligned} &\nabla_\xi e = a\phi e, && \nabla_e e = b\phi e, && \nabla_{\phi e} e = -c\phi e + (\lambda-1)\xi, \\ &\nabla_\xi \phi e = -ae, && \nabla_e \phi e = -be + (1+\lambda)\xi, && \nabla_{\phi e}\phi e = ce, \\ &\nabla_\xi \xi = 0, && \nabla_e \xi = -(1+\lambda)\phi e, && \nabla_{\phi e}\xi = (1-\lambda)e, \end{aligned}$$

where a is a smooth function and

$$\begin{aligned} b &= \frac{1}{2\lambda}((\phi e \cdot \lambda) + A), \quad \text{with } A = S(\xi, e), \\ c &= \frac{1}{2\lambda}((e \cdot \lambda) + B), \quad \text{with } B = S(\xi, \phi e). \end{aligned} \tag{2-5}$$

From Lemma 2.1 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$ we can prove that

$$\begin{aligned} [e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi, \\ [e, \xi] &= \nabla_e \xi - \nabla_\xi e = -(a+\lambda+1)\phi e, \\ [\phi e, \xi] &= \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a-\lambda+1)e, \end{aligned} \tag{2-6}$$

and from (1-1) we estimate

$$\begin{aligned} &(e\wedge\phi e)e = -\phi e, && (e\wedge\xi)e = -\xi, && (\phi e\wedge\xi)\xi = \phi e, \\ &(e\wedge\phi e)\phi e = e, && (e\wedge\xi)\xi = e, && (\phi e\wedge\xi)\phi e = -\xi, \end{aligned} \tag{2-7}$$

while $(X \wedge Y)Z = 0$ whenever $X \neq Y \neq Z \neq X$ and $X, Y, Z \in \{e, \phi e, \xi\}$.

By direct computations we calculate the nonvanishing independent components of the Riemannian $(1, 3)$ curvature tensor field R to be

$$\begin{aligned} &R(\xi, e)\xi = -Ie - Z\phi e, && R(e, \phi e)e = -C\phi e - B\xi, \\ &R(\xi, \phi e)\xi = -Ze - D\phi e, && R(\xi, e)\phi e = -Ke + Z\xi, \\ &R(e, \phi e)\xi = Be - A\phi e, && R(\xi, \phi e)\phi e = He + D\xi, \\ &R(\xi, e)e = K\phi e + I\xi, && R(e, \phi e)\phi e = Ce + A\xi, \\ &R(\xi, \phi e)e = -H\phi e + Z\xi, \end{aligned} \tag{2-8}$$

where

$$\begin{aligned} C &= -b^2 - c^2 + \lambda^2 - 1 + 2a + (e\cdot c) + (\phi e \cdot b), \\ H &= b(\lambda - a - 1) + (\xi \cdot c) + (\phi e \cdot a), \\ K &= c(\lambda + a + 1) + (\xi \cdot b) - (e \cdot a), \\ I &= -2a\lambda - \lambda^2 + 1, \\ D &= 2a\lambda - \lambda^2 + 1, \\ Z &= \xi \cdot \lambda. \end{aligned} \tag{2-9}$$

Setting $X=e$, $Y=\phi e$ and $Z=\xi$ in the Jacobi identity $[[X, Y], Z]+[[Y, Z], X]+[[Z, X], Y]=0$ and using (2-6), we get

$$\begin{aligned} b(a+\lambda+1)-(\xi\cdot c)-(\phi e\cdot\lambda)-(\phi e\cdot a)&=0,\\ c(a-\lambda+1)+(\xi\cdot b)+(e\cdot\lambda)-(e\cdot a)&=0, \end{aligned}\tag{2-10}$$

or equivalently $A=H$ and $B=K$.

The components of the Ricci operator Q with respect to a ϕ-basis are

$$\begin{aligned} Qe&=(\tfrac{1}{2}r-1+\lambda^2-2a\lambda)e+Z\phi e+A\xi,\\ Q\phi e&=Ze+(\tfrac{1}{2}r-1+\lambda^2+2a\lambda)\phi e+B\xi,\\ Q\xi&=Ae+B\phi e+2(1-\lambda^2)\xi, \end{aligned}\tag{2-11}$$

where

$$r=\operatorname{Tr} Q=2(1-\lambda^2-b^2-c^2+2a+(e\cdot c)+(\phi e\cdot b)).\tag{2-12}$$

The relations (2-9) and (2-12) yield

$$C=-b^2-c^2+\lambda^2-1+2a+(e\cdot c)+(\phi e\cdot b)=2\lambda^2-2+\tfrac{1}{2}r,\tag{2-13}$$

and the relation on the last line of (2-4) gives $\operatorname{Tr} l=2(1-\lambda^2)$.

Definition 2.2 [Gouli-Andreou et al. 2008]. Let M^3 be a 3-dimensional contact metric manifold and $h=\lambda h^+-\lambda h^-$ the spectral decomposition of h on U. If

$$\nabla_{h^-X}h^-X=[\xi, h^+X]$$

for all vector fields X on M^3 and all points of an open subset W of U, and if $h=0$ on the points of M^3 that do not belong to W, then the manifold is said to be a *semi-K-contact* manifold.

From Lemma 2.1 and the relations (2-6), the condition above leads to $[\xi, e]=0$ when $X=e$ and to $\nabla_{\phi e}\phi e=0$ when $X=\phi e$. Hence on a semi-K-contact manifold, we have $a+\lambda+1=c=0$. If we apply the deformation

$$e\to\phi e,\quad \phi e\to e,\quad \xi\to-\xi,\quad \lambda\to-\lambda,\quad b\to c,\quad c\to b,$$

the contact metric structure remains the same. Hence a 3-dimensional contact metric manifold is semi-K-contact if $a-\lambda+1=b=0$.

Definition 2.3. In [Koufogiorgos et al. 2008], a (κ, μ, ν)-contact metric manifold is a contact metric manifold $(M^{2n+1}, \eta, \xi, \phi, g)$ on which the curvature tensor satisfies for every $X, Y\in X(M)$ the condition

$$\begin{aligned} R(X, Y)\xi=\kappa(\eta(Y)X-\eta(X)Y)&+\mu(\eta(Y)hX-\eta(X)hY)\\ &+\nu(\eta(Y)\phi hX-\eta(X)\phi hY), \end{aligned}\tag{2-14}$$

where κ, μ, ν are smooth functions on M. If $\nu = 0$, we have a generalized (κ, μ)-contact metric manifold [Koufogiorgos and Tsichlias 2000], and if also κ, μ are constants, then M is a contact metric (κ, μ)-space [Blair et al. 1995; Boeckx 2000].

In [Koufogiorgos et al. 2008], it was proved that for a (κ, μ, ν)-contact metric manifold M^{2n+1} of dimension greater than 3, the functions κ and μ are constants and ν is the zero function; in [Koufogiorgos and Tsichlias 2000], this was proved for generalized (κ, μ)-contact metric manifolds M^{2n+1} of dimension greater than 3.

Remark 2.4. If $M^3 = U_0$, the case treated in [Gouli-Andreou and Xenos 1998b], then Lemma 2.1 is expressed in a similar form with $\lambda = 0$, e is a unit vector field belonging to the contact distribution, and the functions A, B, D, H, I, K and Z satisfy $A = B = Z = H = K = 0$, $I = D = 1$ and $C = r/2 - 2$.

Proposition 2.5. *In a* 3*-dimensional contact metric manifold*, *we have*

$$Q\phi = \phi Q \quad \text{if and only if} \quad \xi \cdot \lambda = 2b\lambda - (\phi e \cdot \lambda) = 2c\lambda - (e \cdot \lambda) = a\lambda = 0. \tag{2-15}$$

Proof. The relations (2-11) by (2-2), (2-5), (2-9) and (2-13) yield

$$\begin{aligned}
(Q\phi - \phi Q)e &= 2Ze + 4a\lambda\phi e + B\xi, \\
(Q\phi - \phi Q)\phi e &= 4a\lambda e - 2Z\phi e - A\xi, \\
(Q\phi - \phi Q)\xi &= Be - A\phi e,
\end{aligned}$$

from which the proposition follows. □

3. Pseudosymmetric contact metric 3-manifolds

Let (M, η, g, ϕ, ξ) be a contact metric 3-manifold. In case $M = U_0$, that is, (ξ, η, ϕ, g) is a Sasakian structure, then M is a pseudosymmetric space of constant type [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis as in Lemma 2.1.

Lemma 3.1. *A contact metric* 3*-manifold* (M, η, g, ϕ, ξ) *is pseudosymmetric if and only if*

$$\begin{cases}
B(\xi\cdot\lambda) + (-2a\lambda - \lambda^2 + 1)A = LA, \\
\quad A(\xi\cdot\lambda) + (2a\lambda - \lambda^2 + 1)B = LB, \\
\quad (\xi\cdot\lambda)(\tfrac{1}{2}r + 2\lambda^2 - 2) + AB = L(\xi\cdot\lambda), \\
A^2 - |(\xi\cdot\lambda)|^2 + (2a\lambda - \lambda^2 + 1)(-2a\lambda - 3\lambda^2 + 3 - \tfrac{1}{2}r) \\
\qquad\qquad = L(-2a\lambda - 3\lambda^2 + 3 - \tfrac{1}{2}r), \\
B^2 - |(\xi\cdot\lambda)|^2 + (-2a\lambda - \lambda^2 + 1)(2a\lambda - 3\lambda^2 + 3 - \tfrac{1}{2}r) \\
\qquad\qquad = L(2a\lambda - 3\lambda^2 + 3 - \tfrac{1}{2}r),
\end{cases} \tag{3-1}$$

where L is the function in the pseudosymmetry definition (2-1).

Proof. Setting $X_1 = e,\ X_2 = \phi e$ and $X_3 = \xi$ in (2-1), we obtain

$$(R(X,Y)\cdot R)(e,\phi e,\xi) = L\big(((X\wedge Y)\cdot R)(e,\phi e,\xi)\big).$$

First we set $X = e$ and $Y = \phi e$. Then by virtue of (2-7) and (2-8), we obtain

$$(B(\xi\cdot\lambda)+(-2a\lambda-\lambda^2+1)A)e+(A(\xi\cdot\lambda)+(2a\lambda-\lambda^2+1)B)\phi e = L(Ae+B\phi e),$$

from which the first two equations of (3-1) follow at once.

Similarly, setting $X = \phi e$, $Y = \xi$ we obtain

$$\begin{aligned}&\big(A^2-|(\xi\cdot\lambda)|^2+(2a\lambda-\lambda^2+1)(-2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\big)e\\ &\quad+\big((\xi\cdot\lambda)(\tfrac{1}{2}r+2\lambda^2-2)+AB\big)\phi e = L\big((-2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)e+(\xi\cdot\lambda)\phi e\big),\end{aligned}$$

from which we get the next two equations of (3-1).

Finally, setting $X = e$ and $Y = \xi$, we have

$$\begin{aligned}&\big(B^2-|(\xi\cdot\lambda)|^2+(-2a\lambda-\lambda^2+1)(2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\big)\phi e\\ &\quad+\big((\xi\cdot\lambda)(\tfrac{1}{2}r+2\lambda^2-2)+AB\big)e = L\big((2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\phi e+(\xi\cdot\lambda)e\big),\end{aligned}$$

from which we obtain the last equation of (3-1). Using the equations (2-9) and (2-13), the system (3-1) takes the convenient form

$$\begin{aligned} ZB+IA &= LA,\\ ZA+DB &= LB,\\ ZC+AB &= LZ,\\ A^2-Z^2+D(I-C) &= L(I-C),\\ B^2-Z^2+I(D-C) &= L(D-C).\end{aligned} \tag{3-2}$$

□

Remark 3.2. If $L = 0$, the manifold is semisymmetric and the system (3-2) is in accordance with [Calvaruso and Perrone 2002, equations (3.1)–(3.5)].

Remark 3.3. If the manifold M^3 is Sasakian and we work in a similar way, then (3-2) is reduced to the equation $(C-1)(L-1) = 0$. Cho and Inoguchi [2005] proved that M is a pseudosymmetric space of constant type. Hence, a Sasakian 3-manifold satisfying the condition $R(X,Y)\cdot R = L((X\wedge Y)\cdot R)$ with $L \neq 1$ is a space of constant scalar curvature $r = 6$, where L is some constant function on M^3.

Proposition 3.4. *Let M^3 be a 3-dimensional contact metric manifold satisfying $Q\phi = \phi Q$. Then M^3 is a pseudosymmetric space of constant type.*

Proof. Cho and Inoguchi [2005] have proved that contact metric 3-manifolds satisfying $Q\phi = \phi Q$ are pseudosymmetric. We know from [Blair et al. 1990] that in these manifolds the Ricci operator has the form $QX = \alpha X + \beta\eta(X)\xi$ or equivalently the Ricci tensor is given by the equation

$$S = \alpha g + \beta\eta \otimes \eta,$$

where $\alpha = \frac{1}{2}(r - \operatorname{Tr} l)$ and $\beta = \frac{1}{2}(3\operatorname{Tr} l - r)$, and the functions of the ϕ-sectional curvature and $\operatorname{Tr} l$ are constants. By [Koufogiorgos 1995], the ϕ-sectional curvature is given by $r/2 - \operatorname{Tr} l$. Hence in contact metric 3-manifolds with $Q\phi = \phi Q$, the function $r = \operatorname{Tr} Q$ is also constant; obviously the functions α and β in the equations above are constants as well. The manifold is quasi-Einstein and hence pseudosymmetric, and because β is constant it is pseudosymmetric of constant type, that is, L is constant. □

Remark 3.5. In dimension 3, the pseudosymmetry condition is equivalent to the Ricci-pseudosymmetry condition $R(X, Y)\cdot S = L((X \wedge Y)\cdot S)$, so (3-2) is also valid for the Ricci-pseudosymmetric contact metric 3-manifolds [Arslan et al. 1997].

4. Pseudosymmetric contact metric 3-manifolds with $Q\xi = \rho\xi$ and ρ constant in the direction of ξ

Theorem 4.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold such that $Q\xi = \rho\xi$, where ρ is a smooth function on M^3 constant along the characteristic direction ξ. Then there are at most six open subsets of M^3 for which their union is an open and dense subset inside of the closure of M^3 and each of them as an open submanifold of M^3 is either*

(a) *a Sasakian manifold*,

(b) *flat*,

(c) *locally isometric to one of the Lie groups* $\mathrm{SU}(2)$ *or* $\mathrm{SL}(2, \mathbb{R})$ *equipped with a left invariant metric*,

(d) *pseudosymmetric of constant type L and of constant scalar curvature r equal to* $2(1 - \lambda^2 + 2a)$,

(e) *semi-K contact with* $L = -3a^2 - 4a$, *or*

(f) *semi-K contact with* $L = a^2$.

Proof. We consider these next open subsets of M:

$$U_0 = \{p \in M : \lambda = 0 \quad \text{in a neighborhood of p}\},$$
$$U = \{p \in M : \lambda \neq 0 \quad \text{in a neighborhood of p}\},$$

where $U_0 \cup U$ is open and dense subset of M.

If $M = U_0$, then M is a pseudosymmetric space of constant type [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis.

The assumption $Q\xi = \rho\xi$ and (2-11) imply

$$\phi e \cdot \lambda = 2b\lambda, \tag{4-1}$$

$$e \cdot \lambda = 2c\lambda, \tag{4-2}$$

$$\rho = 2(1-\lambda^2), \tag{4-3}$$

where the smooth function ρ satisfies

$$\xi \cdot \rho = 0. \tag{4-4}$$

From (2-10), (4-1) and (4-2), we have

$$\xi \cdot c = -(\phi e \cdot a) + b(a - \lambda + 1), \tag{4-5}$$

$$\xi \cdot b = (e \cdot a) - c(\lambda + a + 1). \tag{4-6}$$

Under the conditions (4-1) and (4-2), the system (3-2) becomes

$$\begin{aligned} (C-L)Z &= 0, \\ -Z^2 + (D-L)(I-C) &= 0, \\ -Z^2 + (I-L)(D-C) &= 0, \end{aligned} \tag{4-7}$$

where Z, C, I, D are given by (2-9) and (2-13) and L is the smooth function of the pseudosymmetry condition.

From equations (4-3) and (4-4) we can deduce everywhere in U that

$$\xi \cdot \lambda = 0. \tag{4-8}$$

Differentiating the equations (4-1) and (4-2) with respect to e and ϕe respectively and subtracting, we get

$$[e, \phi e]\lambda = 2b(e \cdot \lambda) + 2\lambda(e \cdot b) - 2c(\phi e \cdot \lambda) - 2\lambda(\phi e \cdot c),$$

or because of (2-6), (4-1), (4-2) and (4-8), we obtain

$$e \cdot b = \phi e \cdot c. \tag{4-9}$$

Differentiating Equations (4-1) and (4-8) with respect to ξ and ϕe respectively and subtracting, we obtain $[\xi, \phi e]\lambda = 2\lambda(\xi \cdot b)$ or because of (2-6), (4-2) and (4-6)

$$\xi \cdot b = c(\lambda - a - 1), \tag{4-10}$$

$$e \cdot a = 2c\lambda. \tag{4-11}$$

Differentiating (4-2) and (4-8) with respect to ξ and e respectively and subtracting we obtain $[\xi, e]\lambda = 2\lambda(\xi \cdot c)$ or because of (2-6), (4-1) and (4-5)

$$\xi \cdot c = b(\lambda + a + 1), \tag{4-12}$$

$$\phi e \cdot a = -2b\lambda. \tag{4-13}$$

Differentiating (4-11) and (4-13) with respect to ϕe and e respectively and subtracting, we get

$$[\phi e, e]a = 2b(e \cdot \lambda) + 2\lambda(e \cdot b) + 2c(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot c)$$

or because of (2-6), (4-1), (4-2), (4-9), (4-11) and (4-13)

$$\xi \cdot a = -2\lambda(e \cdot b) - 2bc\lambda \tag{4-14}$$

Under the condition (4-8) everywhere in U the system (4-7) becomes

$$\begin{cases} (I - C)(D - L) = 0, \\ (D - C)(I - L) = 0. \end{cases}$$

or equivalently

$$\begin{cases} (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L) = 0, \\ (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(-2a\lambda - \lambda^2 + 1 - L) = 0. \end{cases}$$

To study this system we consider the open subsets

$$V = \{p \in U : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \text{ in a neighborhood of } p\},$$

$$V' = \{p \in U : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0 \text{ in a neighborhood of } p\},$$

where $V \cup V'$ is open and dense in the closure of U. We also have the equation

$$(-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L) = 0.$$

Hence we consider the open subsets

$$V_1 = \{p \in V : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \text{ in a neighborhood of } p\},$$

$$V_2 = \{p \in V : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0 \text{ in a neighborhood of } p\},$$

where the set $V_1 \cup V_2$ is open and dense in the closure of V. For V', in which $-2a\lambda - \lambda^2 + 1 - L = 0$, we consider the open subsets

$$V_3 = \{p \in V' : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \text{ in a neighborhood of } p\},$$
$$V_4 = \{p \in V' : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0 \text{ in a neighborhood of } p\},$$

where $V_3 \cup V_4$ is open and dense in the closure of V'. We describe the previous sets more precisely as

$$\begin{aligned} V_1 = \{p \in V \subseteq U :\ & -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \\ & \text{in a neighborhood of } p\}, \\ V_2 = \{p \in V \subseteq U :\ & 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & 2a\lambda - \lambda^2 + 1 - L = 0 \\ & \text{in a neighborhood of } p\}, \\ V_3 = \{p \in V' \subseteq U :\ & -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & -2a\lambda - \lambda^2 + 1 - L = 0 \\ & \text{in a neighborhood of } p\}, \\ V_4 = \{p \in V' \subseteq U :\ & -2a\lambda - \lambda^2 + 1 - L = 0, \\ & 2a\lambda - \lambda^2 + 1 - L = 0 \quad \text{in a neighborhood of } p\}, \end{aligned}$$

and the set $\bigcup V_i$ is open and dense in the closure of U.

In V_1, we have

$$-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0,$$
$$2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0.$$

Subtracting these two equations we find that $a = 0$ in $V_1 \subset U$. Hence we conclude that the structure has the property $Q\phi = \phi Q$ (Proposition 2.5), that L is constant (Proposition 3.4) and the classification results from [Blair et al. 1990] and [Blair and Chen 1992] hold.

In V_2, we have

$$2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0,$$
$$-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0,$$

(hence $a \neq 0$) or equivalently

$$2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \tag{4-15}$$

$$2a\lambda - \lambda^2 + 1 - L = 0. \tag{4-16}$$

Differentiating (4-15) with respect to ξ and using (4-8), (4-10), (4-12) and (4-14), we obtain

$$\xi \cdot e \cdot c + \xi \cdot \varphi e \cdot b = -4bc\lambda^2 + 8bc\lambda - 4\lambda^2(e \cdot b) + 4\lambda(e \cdot b). \tag{4-17}$$

Differentiating (4-10) and (4-12) with respect to ϕe and e respectively, we use (4-1), (4-2), (4-9), (4-11), (4-13), and adding we obtain

$$\phi e \cdot \xi \cdot b + e \cdot \xi \cdot c = 2\lambda(e \cdot b) + 8bc\lambda. \tag{4-18}$$

Subtract (4-17) and (4-18) and using (2-6), (4-9) and (4-14), we obtain

$$e \cdot b = \phi e \cdot c = -bc, \tag{4-19}$$

$$\xi \cdot a = 0. \tag{4-20}$$

Differentiating (4-20) and (4-13) with respect to ϕe and ξ respectively and subtracting, we obtain $[\phi e, \xi]a = 2\lambda(\xi \cdot b)$, or because of (2-6), (4-10), (4-11) and since $\lambda \neq 0$ in U, we have

$$c(a - \lambda + 1) = 0. \tag{4-21}$$

Differentiating (4-20) and (4-11) with respect to e and ξ respectively and subtracting, we obtain $[\xi, e]a = 2\lambda(\xi \cdot c)$, or because of (2-6), (4-12), (4-13) and since $\lambda \neq 0$ in U, we have

$$b(a + \lambda + 1) = 0. \tag{4-22}$$

Differentiating (4-16) with respect to ξ, ϕe and e and using (4-1), (4-2), (4-8), (4-11), (4-13) and (4-20) we obtain respectively

$$\xi \cdot L = 0, \tag{4-23}$$

$$\phi e \cdot L = 4ab\lambda - 8b\lambda^2, \tag{4-24}$$

$$e \cdot L = 4ac\lambda. \tag{4-25}$$

To study the system (4-21) and (4-22), we consider the open subsets

$$G = \{p \in V_2 : b = 0 \text{ in a neighborhood of } p\},$$

$$G' = \{p \in V_2 : b \neq 0 \text{ in a neighborhood of } p\},$$

where $G \cup G'$ is open and dense in the closure of V_2. Having also $c(\lambda - a - 1) = 0$ we consider the open subsets

$$G_1 = \{p \in G : c = 0 \text{ in a neighborhood of } p\},$$
$$G_2 = \{p \in G : c \neq 0 \text{ in a neighborhood of } p\},$$

where $G_1 \cup G_2$ is open and dense in the closure of G. The set G' (where $b \neq 0$ or equivalently $\lambda + a + 1 = 0$) is decomposed similarly as

$$G_3 = \{p \in G' : c = 0 \text{ in a neighborhood of } p\},$$
$$G_4 = \{p \in G' : c \neq 0 \text{ in a neighborhood of } p\},$$

where $G_3 \cup G_4$ is open and dense in the closure of G'. The sets G_1, G_2, G_3 and G_4 are described more specifically as

$$G_1 = \{p \in G \subset V_2 : b = c = 0 \text{ in a neighborhood of } p\},$$
$$G_2 = \{p \in G \subset V_2 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_3 = \{p \in G' \subset V_2 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_4 = \{p \in G' \subset V_2 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$

The set $\bigcup G_i$ is open and dense subset of V_2. We have $V_2 \subset U$, where $\lambda \neq 0$; hence $G_4 = \varnothing$.

In G_1, we have $b = 0$ and $c = 0$. From (4-1), (4-2), (4-8), (4-11), (4-13), (4-14), (4-23), (4-24) and (4-25), we find that λ, a and L are constant in G_1 with $\lambda, a \neq 0$; hence from (2-12) the scalar curvature $r = 2(1 - \lambda^2 + 2a)$ is also constant.

In G_2, we have $b = 0$ and $\lambda - a - 1 = 0$. Hence we have a semi-K contact structure. Then (4-16) and $a = \lambda - 1$ give $L = (\lambda - 1)^2 = a^2 \neq 0$.

In G_3, we have $c = 0$ and $\lambda + a + 1 = 0$. Similarly, we have a semi-K contact structure with $L = -3\lambda^2 - 2\lambda + 1 = -3a^2 - 4a$, with $a \neq 0$.

In V_3,

$$-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \tag{4-26}$$
$$-2a\lambda - \lambda^2 + 1 - L = 0. \tag{4-27}$$

We similarly obtain the system of (4-21) and (4-22) with $a \neq 0$, while for the function L, we have (4-23) as well as $\phi e \cdot L = -4ab\lambda$ and $e \cdot L = -4ac\lambda - 8c\lambda^2$.

We consider the open subsets

$$G'_1 = \{p \in V_3 : b = c = 0 \text{ in a neighborhood of p}\},$$
$$G'_2 = \{p \in V_3 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of p}\},$$
$$G'_3 = \{p \in V_3 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of p}\},$$
$$G'_4 = \{p \in V_3 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of p}\}.$$

The set $\bigcup G_i'$ is open and dense subset of V_3. We have $V_3 \subset U$, where $\lambda \neq 0$; hence G_4' is empty.

In G_1', we have $b = 0$ and $c = 0$. As in case of G_1, the functions λ, a, L and r are constants.

In G_2', we have $b = 0$ and $\lambda - a - 1 = 0$. Hence we have a semi-K contact structure with $L = -3\lambda^2 + 2\lambda + 1 = -3a^2 - 4a$, with $a \neq 0$.

In G_3', we have $c = 0$ and $\lambda + a + 1 = 0$. We have a semi-K contact structure with $L = (\lambda + 1)^2 = a^2 \neq 0$.

In V_4 we have $-2a\lambda - \lambda^2 + 1 - L = 0$ and $2a\lambda - \lambda^2 + 1 - L = 0$. Subtracting these two equations we obtain $a = 0$ in $V_4 \subset U$, and hence as in case of V_1 we have the structure $Q\phi = \phi Q$.

Finally, the sets U_0, V_1 and V_4, G_1 and G_1', G_3 and G_2', G_2 and G_3' satisfy the structures a, b and c, d, e and f respectively of Theorem 4.1. □

5. Pseudosymmetric contact metric 3-manifolds of constant type with $Q\xi = \rho\xi$

Theorem 5.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold of constant type such that $Q\xi = \rho\xi$, where ρ is a smooth function on M^3. Then ρ is constant. If M^3 is also complete then it is either a Sasakian manifold (meaning* $\operatorname{Tr} l = 2$) *or locally isometric to one of the following Lie groups equipped with a left invariant metric*: SU(2); SO(3); SL(2, $\mathbb{R}$); $E(2)$, *the rigid motions of Euclidean 2-space*; $E(1, 1)$, *the rigid motions of Minkowski 2-space*; *or* $O(1, 2)$, *the Lorentz group of linear maps preserving the quadratic form $t^2 - x^2 - y^2$.*

Proof. We consider open subsets

$$U_0 = \{p \in M : \lambda = 0 \text{ in a neighborhood of } p\},$$

$$U = \{p \in M : \lambda \neq 0 \text{ in a neighborhood of } p\},$$

where $U_0 \cup U$ is open and dense subset of M.

If $M = U_0$, then it is a pseudosymmetric space of constant type; see [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis. The assumption $Q\xi = \rho\xi$ and (2-11) imply

$$\phi e \cdot \lambda = 2b\lambda, \tag{5-1}$$

$$e \cdot \lambda = 2c\lambda, \tag{5-2}$$

$$\rho = 2(1 - \lambda^2), \tag{5-3}$$

where ρ is a smooth function on M. From (2-10), (5-1) and (5-2) we have

$$\xi \cdot c = -(\phi e \cdot a) + b(a - \lambda + 1), \tag{5-4}$$

$$\xi \cdot b = (e \cdot a) - c(\lambda + a + 1). \tag{5-5}$$

Under the conditions (5-1) and (5-2) the system (3-2) becomes

$$\begin{aligned}(C-L)Z&=0,\\ -Z^2+(D-L)(I-C)&=0,\\ -Z^2+(I-L)(D-C)&=0,\end{aligned}\tag{5-6}$$

where Z, C, I and D are given by (2-9) and (2-13) and L is the constant of the pseudosymmetry condition.

We work in the open subset U and suppose that there is a point p in U where $Z=\xi\cdot\lambda\neq 0$. The function Z is smooth, so because of its continuity there is an open neighborhood U_1 of p such that $U_1\subset U$ and $Z=\xi\cdot\lambda\neq 0$ everywhere in U_1. From the first equation of (5-6), we get $C=L$ in U_1, or equivalently

$$(e\cdot c)+(\phi e\cdot b)=L+b^2+c^2-\lambda^2+1-2a.\tag{5-7}$$

Differentiating (5-7) with respect to ξ, we get

$$\xi\cdot e\cdot c+\xi\cdot\phi e\cdot b=2b(\xi\cdot b)+2c(\xi\cdot c)-2\lambda(\xi\cdot\lambda)-2(\xi\cdot a),$$

which because of (5-4) and (5-5) becomes

$$\xi\cdot e\cdot c+\xi\cdot\phi e\cdot b=2b(e\cdot a)-2c(\phi e\cdot a)-2\lambda(\xi\cdot\lambda)-2(\xi\cdot a)-4bc\lambda.\tag{5-8}$$

Next, we differentiate (5-4) and (5-5) with respect to e and ϕe, respectively. Adding the results, we have

$$\begin{aligned}e\cdot\xi\cdot c+\phi e\cdot\xi\cdot b=-[e,\phi e]a-(a+\lambda+1)(\phi e\cdot c)+(a-\lambda+1)(e\cdot b)\\ -c(\phi e\cdot a)+b(e\cdot a)-4bc\lambda.\end{aligned}$$

Subtracting this from (5-8), we get

$$\begin{aligned}[\xi,e]c+[\xi,\phi e]b=b(e\cdot a)-c(\phi e\cdot a)-2(\xi\cdot a)-2\lambda(\xi\cdot\lambda)+[e,\phi e]a\\ +(a+\lambda+1)(\phi e\cdot c)-(a-\lambda+1)(e\cdot b),\end{aligned}$$

or because of (2-6),

$$\begin{aligned}&(a+\lambda+1)(\phi e\cdot c)+(\lambda-a-1)(e\cdot b)\\ &\quad=b(e\cdot a)-c(\phi e\cdot a)-2(\xi\cdot a)-2\lambda(\xi\cdot\lambda)-b(e\cdot a)\\ &\qquad+c(\phi e\cdot a)+2(\xi\cdot a)+(\lambda+a+1)(\phi e\cdot c)+(\lambda-a-1)(e\cdot b).\end{aligned}$$

Equivalently, $\lambda(\xi\cdot\lambda)=0$, and because we work in $U_1\subset U$, we have $\xi\cdot\lambda=0$, which is a contradiction. Hence, we can deduce everywhere in U that

$$\xi\cdot\lambda=0.\tag{5-9}$$

Working as previously, we obtain the equations

$$e \cdot b = \phi e \cdot c, \tag{5-10}$$

$$\xi \cdot b = c(\lambda - a - 1), \tag{5-11}$$

$$e \cdot a = 2c\lambda, \tag{5-12}$$

$$\xi \cdot c = b(\lambda + a + 1), \tag{5-13}$$

$$\phi e \cdot a = -2b\lambda. \tag{5-14}$$

Under the condition (5-9) everywhere in U the system (5-6) becomes

$$\begin{cases} (I - C)(D - L) = 0, \\ (D - C)(I - L) = 0, \end{cases}$$

or equivalently

$$\begin{cases} (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L) = 0, \\ (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(-2a\lambda - \lambda^2 + 1 - L) = 0. \end{cases}$$

To study this system, we consider (as previously) the open subsets

$$\begin{aligned} V_1 = \{p \in U :\ & -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{in a neighborhood of } p\}, \\ V_2 = \{p \in U :\ & 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & 2a\lambda - \lambda^2 + 1 - L = 0 \qquad\qquad \text{in a neighborhood of } p\}, \\ V_3 = \{p \in U :\ & -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ & -2a\lambda - \lambda^2 + 1 - L = 0 \qquad\qquad \text{in a neighborhood of } p\}, \\ V_4 = \{p \in U :\ & -2a\lambda - \lambda^2 + 1 - L = 0, \quad 2a\lambda - \lambda^2 + 1 - L = 0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{in a neighborhood of } p\}. \end{aligned}$$

The set $\bigcup V_i$ is open and dense in the closure of U. We shall prove that the functions λ and a are constants at V_i for $i = 1, 2, 3, 4$.

In V_1, we have

$$-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0,$$
$$2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0.$$

Subtracting these two equations we can deduce that $a = 0$ in $V_1 \subset U$. Hence from (5-12) and (5-14), we have $c = b = 0$, and from (5-1) and (5-2), we have $\phi e \cdot \lambda = e \cdot \lambda = 0$, which together with (5-9) give $\lambda =$ constant in V_1. Moreover, if we put $a = b = c = 0$ in one of the equations of the set V_1, we finally get $\lambda^2 = 1$.

In V_2,

$$2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \tag{5-15}$$

$$2a\lambda - \lambda^2 + 1 - L = 0. \tag{5-16}$$

Differentiating (5-16) with respect to ξ, ϕe and e and using (5-9), (5-12) and (5-14), we obtain respectively

$$\xi \cdot a = 0, \tag{5-17}$$

$$b(a - 2\lambda) = 0, \quad ac = 0.$$

Differentiating (5-12) and (5-17) with respect to ξ and e respectively and subtracting, we obtain $[\xi, e]a = 2\lambda(\xi \cdot c)$ or because of (2-6), (5-13) and (5-14)

$$b(\lambda + a + 1) = 0. \tag{5-18}$$

Similarly, differentiating (5-14) with respect to ξ and (5-17) with respect to ϕe and subtracting, we have $[\xi, \phi e]a = -2\lambda(\xi \cdot b)$ or because of (2-6), (5-11) and (5-12)

$$c(\lambda - a - 1) = 0. \tag{5-19}$$

We study the system of (5-18) and (5-19). As in the previous section, we consider open subsets

$$G_1 = \{p \in V_2 : b = c = 0 \text{ in a neighborhood of } p\},$$
$$G_2 = \{p \in V_2 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_3 = \{p \in V_2 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_4 = \{p \in V_2 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$

The set $\bigcup G_i$ is open and dense subset of V_2. We have $V_2 \subset U$ where $\lambda \neq 0$; hence G_4 is empty.

In G_1, we have $b = 0$ and $c = 0$. From (5-1) and (5-2) we can conclude $\phi e \cdot \lambda = e \cdot \lambda = 0$, which together with (5-9) implies λ is constant in G_1. Similarly from (5-12), (5-14) and (5-17), a is constant.

In G_2, we have $b = 0$ and $\lambda - a - 1 = 0$. The second of these together with (5-16) gives $\lambda^2 - 2\lambda + 1 - L = 0$. If we assume $e \cdot \lambda \neq 0$, we differentiate this equation twice with respect to e, and we obtain $e \cdot \lambda = 0$, which contradicts our assumption. Hence, $e \cdot \lambda = 0$ (and $c = 0$) and (5-1) gives $\phi e \cdot \lambda = 0$, or finally λ is constant in G_2 and $a = \lambda - 1$ is also constant.

In G_3, we have $c = 0$ and $\lambda + a + 1 = 0$. The first equation gives $e \cdot \lambda = 0$ by (5-2), while the second together with (5-16) gives $-3\lambda^2 - 2\lambda + 1 - L = 0$. Differentiating this equation with respect to ϕe, we get $(3\lambda + 1)(\phi e \cdot \lambda) = 0$. Suppose there is a point $p \in G_3$ at which $\phi e \cdot \lambda \neq 0$. Then, there is a neighborhood F of p in which

$\phi e \cdot \lambda \neq 0$. In that neighborhood we must have $\lambda = -1/3$ by the last equation; hence $\phi e \cdot \lambda = 0$, a contradiction. Thus $\phi e \cdot \lambda = 0$ everywhere in G_3, which gives $b = 0$. In G_3, we note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$, so λ is constant in G_3. Obviously a is also constant because $a = -\lambda - 1$. Moreover, if we put $b = c = 0$ and $a = -\lambda - 1$ in (5-15), we get $\lambda^2 = 1$.

We have proved that λ is constant at every G_i for $i = 1, 2, 3$, while the set $G_1 \cup G_2 \cup G_3$ is an open and dense subset of V_2; hence λ is constant in V_2 and the equations $b(a - 2\lambda) = 0$ and $ac = 0$ are satisfied because $b = c = 0$.

In V_3,

$$-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \tag{5-20}$$

$$-2a\lambda - \lambda^2 + 1 - L = 0. \tag{5-21}$$

Working as we did for the set V_2, we get again the first equation of (5-17), and

$$ab = 0 \quad \text{and} \quad c(a + 2\lambda) = 0$$

and the system of (5-18) and (5-19). We similarly consider the open subsets

$$G'_1 = \{p \in V_3 : b = c = 0 \text{ in a neighborhood of } p\},$$
$$G'_2 = \{p \in V_3 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$
$$G'_3 = \{p \in V_3 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$
$$G'_4 = \{p \in V_3 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\},$$

The set $\bigcup G'_i$ is open and dense subset of V_3. We have $V_3 \subset U$ where $\lambda \neq 0$; hence G'_4 is empty.

In G'_1, we have $b = 0$ and $c = 0$. From (5-1) and (5-2), we can conclude $\phi e \cdot \lambda = e \cdot \lambda = 0$, which together with (5-9) implies λ is constant in G'_1. From (5-12), (5-14) and (5-17) we obtain that a constant in G'_1.

In G'_2, we have $b = 0$ and $\lambda - a - 1 = 0$. The first equation gives $\phi e \cdot \lambda = 0$ from (5-1), while the second together with (5-21) gives $-3\lambda^2 + 2\lambda + 1 - L = 0$. Differentiating this equation with respect to e, we get $(-3\lambda + 1)(e \cdot \lambda) = 0$. Suppose that there is a point $p \in G'_2$ at which $e \cdot \lambda \neq 0$. Then, there is a neighborhood F' of p in which $e \cdot \lambda \neq 0$. In that neighborhood we must have from the last equation that $\lambda = 1/3$ and $e \cdot \lambda = 0$, a contradiction. Hence $e \cdot \lambda = 0$ everywhere in G'_2, which gives $c = 0$. In G'_2, we note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$, so λ is constant in G'_2. Obviously a is also constant because $a = \lambda - 1$. Moreover, if we put $b = c = 0$ and $a = \lambda - 1$ in (5-20) we get $\lambda^2 = 1$.

In G'_3, we have $c = 0$ and $\lambda + a + 1 = 0$. The second equation together with (5-21) gives $\lambda^2 + 2\lambda + 1 - L = 0$. Assuming $\phi e \cdot \lambda \neq 0$, we differentiate this equation twice with respect to ϕe and obtain $\phi e \cdot \lambda = 0$, a contradiction. Thus, $\phi e \cdot \lambda = 0$

everywhere in G_3', which gives $b = 0$. From (5-2), we get $e \cdot \lambda = 0$. We note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$, so λ is constant in G_3' and obviously so is $a = -\lambda - 1$.

We have proved that λ is constant in every G_i' for $i = 1, 2, 3$ while the set $G_1' \cup G_2' \cup G_3'$ is open and dense in the closure of V_3; hence λ is constant at V_3 and the equations $ab = 0$ and $c(a + 2\lambda) = 0$ are satisfied because $b = c = 0$.

In V_4, we have $2a\lambda - \lambda^2 + 1 - L = 0$ and $-2a\lambda - \lambda^2 + 1 - L = 0$. Subtracting these two equations, we can deduce that $a = 0$ in $V_4 \subset U$. Hence from (5-12) and (5-14), we have $c = b = 0$, and from (5-1) and (5-2), we have $\phi e \cdot \lambda = e \cdot \lambda = 0$, which together with (5-9) implies λ is constant in V_4. Moreover, if we put $a = 0$ in one of the equations of the set V_4, we finally obtain $\lambda^2 = 1 - L \geq 0$.

We have proved that λ is constant in every V_i for $i = 1, 2, 3, 4$. The set $V_1 \cup V_2 \cup V_3 \cup V_4$ is open and dense inside of the closure of U; hence λ is constant at U and because of (5-3) the function ρ is constant at U. Finally if the manifold M^3 is complete, we may use the main theorem of [Koufogiorgos 1995] to complete the proof. □

6. Pseudosymmetric (κ, μ, ν)-contact metric 3-manifolds of constant type

Theorem 6.1. *A* 3*-dimensional* (κ, μ, ν)*-contact metric pseudosymmetric manifold of constant type is either a Sasakian manifold or a* (κ, μ)*-contact metric manifold. In the second case, if* M^3 *is also complete, then it is locally isometric to one of the following Lie groups equipped with a left invariant metric*: SU(2); SO(3); SL(2, $\mathbb{R}$); $E(2)$, *the rigid motions of Euclidean* 2*-space*; $E(1, 1)$, *the rigid motions of Minkowski* 2*-space*; *or* $O(1, 2)$, *the Lorentz group consisting of linear transformations preserving the quadratic form* $t^2 - x^2 - y^2$).

Proof. We work as in the previous section. If $M = U_0$, then (ξ, η, ϕ, g) is a Sasakian structure that is a pseudosymmetric space of constant type with $\kappa = 1$, $\mu \in \mathbb{R}$ and $h = 0$. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis. From (2-14) we can calculate these components of the Riemannian curvature tensor:

$$R(\xi, e)\xi = -(\kappa + \lambda\mu)e - \lambda\nu\phi e, \qquad R(e, \phi e)\xi = 0,$$
$$R(\xi, \phi e)\xi = -\lambda\nu e - (\kappa - \lambda\mu)\phi e.$$

By virtue of (2-8), we can conclude that

$$A = B = 0, \quad Z = \lambda\nu, \quad D = \kappa - \lambda\mu, \quad I = \kappa + \lambda\mu, \tag{6-1}$$

and hence the system (3-2) gives again the system (5-6). First we get $Z = \xi \cdot \lambda = 0$ or equivalently $\nu = 0$ and then that λ, a are constants. Finally from (2-9) and (6-1) we have $\kappa = 1 - \lambda^2$ and $\mu = -2a$, and from the main theorem of [Koufogiorgos 1995] and [Boeckx 2000, Theorem 3], we can complete the proof. □

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