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Let G be a finitely presented group. Scott and Swarup have constructed a canonical splitting of G that encloses all almost invariant sets over virtually polycyclic subgroups of a given length. We give an alternative construction of this regular neighborhood by showing that it is the tree of cylinders of a JSJ splitting.

1. Introduction

Scott and Swarup [2003] have constructed a canonical graph of groups decomposition (or splitting) of a finitely presented group G; this splitting encloses all almost invariant sets over virtually polycyclic subgroups of a given length n (the VPC_n groups), and in particular over virtually cyclic subgroups for $n = 1$.

Almost invariant sets generalize splittings: Whereas a splitting is analogous to an embedded codimension-one submanifold of a manifold M, an almost invariant set is analogous to an immersed codimension-one submanifold.

Two splittings are *compatible* if they have a common refinement, in that both can be obtained from the refinement by collapsing some edges. For example, two splittings induced by disjoint embedded codimension-one submanifolds are compatible.

Enclosing is a generalization of this notion to almost invariant sets: Take, in the analogy above, two codimension-one submanifolds F_1 and F_2 of M with F_1 immersed and F_2 embedded. Then F_1 is enclosed in a connected component of $M \setminus F_2$ if one can isotope F_1 into this component.

Scott and Swarup's construction is called the *regular neighborhood* of all almost invariant sets over VPC_n subgroups. This is analogous to the topological regular neighborhood of a finite union of (nondisjoint) immersed codimension-one submanifolds: It defines a splitting that encloses the initial submanifolds.

One main virtue of their splitting is that it is canonical: It is invariant under automorphisms of G. Because of this, it is often quite different from usual JSJ

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splittings, which are unique only up to deformation. There the canonical object is the JSJ deformation space [Forester 2003; Guirardel and Levitt 2009].

The main reason for this rigidity is that the regular neighborhood is defined in terms of enclosing. Enclosing, like compatibility of splittings, is more rigid than domination, which is the basis for usual JSJ theory. For instance, any two splittings in Culler–Vogtmann's outer space dominate each other, but they are compatible if and only if they lie in a common simplex.

We have shown a general construction producing a canonical splitting T_c from a canonical deformation space: the *tree of cylinders* [Guirardel and Levitt 2008]. It also enjoys strong compatibility properties. In this paper, we show that the splitting constructed by Scott and Swarup is a subdivision of the tree of cylinders of the usual JSJ deformation space.

More precisely, let T_J be the Bass–Serre tree of a JSJ splitting of G over VPC_n groups, as constructed for instance in [Dunwoody and Sageev 1999]. To construct the tree of cylinders, say that two edges are in the same cylinder if their stabilizers are commensurable. Cylinders are subtrees, and the tree T_c dual to the covering of T_J by cylinders is *the tree of cylinders* of T_J; see [Guirardel and Levitt 2008] or Section 2b below.

Theorem 4.1. *Let G be a finitely presented group, and let $n \geq 1$. Assume that G does not split over a VPC_{n-1} subgroup, and that G is not VPC_{n+1}. Let T_J be a JSJ tree of G over VPC_n subgroups, and let T_c be its tree of cylinders for the commensurability relation.*

Then the Bass–Serre tree of Scott and Swarup's regular neighborhood of all almost invariant subsets over VPC_n subgroups is equivariantly isomorphic to a subdivision of T_c.

This gives a new proof that this regular neighborhood is a tree. Deriving the regular neighborhood from a JSJ splitting, instead of building it from an abstract betweenness relation, seems to greatly simplify the construction, by completely avoiding the notion of good or good-enough position for almost invariant subsets.

There are two ingredients in our approach, to be found in Sections 3 and 4. (Section 2 recalls basic material about trees of cylinders, almost invariant sets, cross-connected components, and regular neighborhoods.)

The first ingredient is a general fact about almost invariant sets that are *based* on a given tree T. Consider any simplicial tree T with an action of G. Any edge e separates T into two half-trees, and this defines almost invariant sets Z_e and Z_e^* (see Section 3a). The collection $\mathcal{B}(T)$ of almost invariant subsets *based* on T is then defined by taking Boolean combinations of such sets Z_e.

Following Scott and Swarup, one defines cross-connected components of $\mathcal{B}(T)$ by using *crossing* of almost invariant sets. The set of cross-connected components is then endowed with a betweenness relation that allows one to construct a bipartite graph $RN(\mathcal{B}(T))$ associated to $\mathcal{B}(T)$. This is the *regular neighborhood* of $\mathcal{B}(T)$; see Definition 2.2.

Theorem 3.3. *Let G be a finitely generated group, and T a tree with a minimal action of G. Assume that no two groups commensurable to edge stabilizers are contained in each other with infinite index.*

Then the regular neighborhood $RN(\mathcal{B}(T))$ is equivariantly isomorphic to a subdivision of T_c, the tree of cylinders of T for the commensurability relation; in particular, $RN(\mathcal{B}(T))$ is a tree.

The hypothesis about edge stabilizers holds in particular if all edge stabilizers of T are VPC_n for a fixed n.

This theorem remains true if one enlarges $\mathcal{B}(T)$ to $\mathcal{B}(T) \cup \mathrm{QH}(T)$ by including almost invariant sets enclosed by quadratically hanging vertices of T. Geometrically, such a vertex is associated to a fiber bundle over a 2-dimensional orbifold $\mathbb{O}$. Any simple closed curve on $\mathbb{O}$ gives a way to blow up T by creating new edges and therefore new almost invariant sets. These sets are in $\mathrm{QH}(T)$, as well as those associated to immersed curves on $\mathbb{O}$. Under the same hypotheses as Theorem 3.3, we show in Theorem 3.11 that the regular neighborhood $RN(\mathcal{B}(T) \cup \mathrm{QH}(T))$ also is a subdivision of T_c.

The second ingredient, specific to the VPC_n case, is due to (but not explicitly stated by) Scott and Swarup [2003]. We believe it is worth emphasizing this statement, as it gives a very useful description of almost invariant sets over VPC_n subgroups in terms of a JSJ splitting T_J. In plain words, it says that any almost invariant set over a VPC_n subgroup is either dual to a curve in a QH subgroup, or is a Boolean combination of almost invariant sets dual to half-trees of T_J.

Theorem 4.2 [Dunwoody and Swenson 2000; Scott and Swarup 2003]. *Let G and T_J be as in Theorem 4.1.*

For any almost invariant subset X over a VPC_n *subgroup, the equivalence class* $[X]$ *belongs to* $\mathcal{B}(T_J) \cup \mathrm{QH}(T_J)$.

Theorem 4.2 is essentially another take on the proof of Scott and Swarup's [2003, Theorem 8.2], and makes a crucial use of algebraic torus theorems of [Dunwoody and Swenson 2000; Dunwoody and Roller 1993]. We give a proof in Section 4.

Theorem 4.1 is a direct consequence of Theorems 4.2 and 3.11.

2. Preliminaries

Let G be a fixed finitely generated group, which, in Section 4, is finitely presented.

2a. ***Trees.*** If Γ is a graph, we denote by $V(\Gamma)$ its set of vertices and by $E(\Gamma)$ its set of (closed) nonoriented edges.

A tree always means a simplicial tree T on which G acts without inversions. Given a family $\mathcal{E}$ of subgroups of G, an *$\mathcal{E}$-tree* is a tree whose edge stabilizers belong to $\mathcal{E}$. We denote by G_v or G_e the stabilizer of a vertex v or an edge e.

Given a subtree A, we denote by pr_A the projection onto A, mapping x to the point of A closest to x. If A and B are disjoint, or intersect in at most one point, then $\mathrm{pr}_A(B)$ is a single point, and we define the *bridge* between A and B as the segment joining $\mathrm{pr}_A(B)$ to $\mathrm{pr}_B(A)$.

A tree T is *nontrivial* if there is no global fixed point, and *minimal* if there is no proper G-invariant subtree.

An element or a subgroup of G is *elliptic* in T if it has a global fixed point. An element that is not elliptic is *hyperbolic*. It has an axis on which it acts as a translation. If T is minimal, then it is the union of all translation axes of elements of G. In particular, if $Y \subset T$ is a subtree, then any connected component of $T \setminus Y$ is unbounded.

A subgroup A consisting only of elliptic elements fixes a point if it is finitely generated, a point or an end in general. If a finitely generated subgroup A is not elliptic, there is a unique minimal A-invariant subtree.

A tree T *dominates* a tree T' if there is an equivariant map $f : T \to T'$. Equivalently, any subgroup that is elliptic in T is also elliptic in T'. Having the same elliptic subgroups is an equivalence relation on the set of trees, and the equivalence classes are called *deformation spaces*; see [Forester 2002; Guirardel and Levitt 2007] for details.

2b. ***Trees of cylinders.*** Two subgroups A and B of G are *commensurable* if $A \cap B$ has finite index in both A and B.

Definition 2.1. We fix a conjugacy-invariant family $\mathcal{E}$ of subgroups of G such that

- any subgroup A commensurable with some $B \in \mathcal{E}$ lies in $\mathcal{E}$, and
- if $A, B \in \mathcal{E}$ are such that $A \subset B$, then $[B : A] < \infty$.

An $\mathcal{E}$-tree is a tree whose edge stabilizers belong to $\mathcal{E}$.

For instance, $\mathcal{E}$ may consist of all subgroups of G that are virtually $\mathbb{Z}^n$ for some fixed n, or all subgroups that are virtually polycyclic of Hirsch length exactly n.

In [Guirardel and Levitt 2008], we associated a tree of cylinders T_c to any $\mathcal{E}$-tree T, as follows. Two (nonoriented) edges of T are equivalent if their stabilizers are commensurable. A *cylinder* of T is an equivalence class Y. We identify Y with the union of its edges, which is a subtree of T.

Two distinct cylinders meet in at most one point. One can then define the tree of cylinders of T as the tree T_c dual to the covering of T by its cylinders, as in

[Guirardel 2004, Definition 4.8]. Formally, T_c is the bipartite tree with vertex set $V(T_c) = V_0(T_c) \sqcup V_1(T_c)$ defined as follows:

(1) $V_0(T_c)$ is the set of vertices x of T belonging to (at least) two distinct cylinders;

(2) $V_1(T_c)$ is the set of cylinders Y of T;

(3) there is an edge $\varepsilon = (x, Y)$ between $x \in V_0(T_c)$ and $Y \in V_1(T_c)$ if and only if x (viewed as a vertex of T) belongs to Y (viewed as a subtree of T).

Alternatively, one can define the *boundary* ∂Y of a cylinder Y as the set of vertices of Y belonging to another cylinder, and obtain T_c from T by replacing each cylinder by the cone on its boundary.

All edges of a cylinder Y have commensurable stabilizers, and we denote by $\mathcal{C} \subset \mathcal{E}$ the corresponding commensurability class. We sometimes view $V_1(T_c)$ as a set of commensurability classes.

2c. ***Almost invariant subsets.*** Given a subgroup $H \subset G$, consider the action of H on G by left multiplication. A subset $X \subset G$ is *H-finite* if it is contained in the union of finitely many H-orbits. Two subsets X and Y are *equivalent* if their symmetric difference $X + Y$ is H-finite. We denote by $[X]$ the equivalence class of X, and by X^* the complement of X.

An *H-almost invariant subset* (or an almost invariant subset over H) is a subset $X \subset G$ that is invariant under the (left) action of H and equivalent to the right-translate Xs for all $s \in G$. An H-almost invariant subset X is *nontrivial* if neither X nor its complement X^* is H-finite. Given $H < G$, the set of equivalence classes of H-almost invariant subsets is a *Boolean algebra* $\mathcal{B}_H$ for the usual operations.

If H contains H' with finite index, then any H-almost invariant subset X is also H'-almost invariant. Furthermore, two sets X and Y are equivalent over H' if and only if they are equivalent over H. It follows that, given a commensurability class $\mathcal{C}$ of subgroups of G, the set of equivalence classes of almost invariant subsets over subgroups in $\mathcal{C}$ is a *Boolean algebra* $\mathcal{B}_{\mathcal{C}}$.

Two almost invariant subsets X over H and Y over K are *equivalent* if their symmetric difference $X+Y$ is H-finite. By [Scott and Swarup 2003, Remark 2.9], this is a symmetric relation: $X + Y$ is H-finite if and only if it is K-finite. If X and Y are nontrivial, equivalence implies that H and K are commensurable.

The algebras $\mathcal{B}_{\mathcal{C}}$ are thus disjoint, except for the (trivial) equivalence classes of $\varnothing$ and G that belong to every $\mathcal{B}_{\mathcal{C}}$. We denote by $\mathcal{B}$ the union of the algebras $\mathcal{B}_{\mathcal{C}}$. It is the set of equivalence classes of all almost invariant sets, but it is not a Boolean algebra in general. There is a natural action of G on $\mathcal{B}$ induced by left translation (or conjugation).

2d. ***Cross-connected components and regular neighborhoods.*** Let X be an H-almost invariant subset, and Y a K-almost invariant subset. One says that X

crosses Y, or the pair $\{X, X^*\}$ crosses $\{Y, Y^*\}$, if none of the four sets $X^{(*)} \cap Y^{(*)}$ is H-finite (we denote by $X^{(*)} \cap Y^{(*)}$ the four possible intersections $X \cap Y$, $X^* \cap Y$, $X \cap Y^*$, and $X^* \cap Y^*$). By [Scott 1998], this is a symmetric relation. Note that X and Y do not cross if they are equivalent, and that crossing depends only on the equivalence classes of X and Y. Following [Scott and Swarup 2003], we will say that $X^{(*)} \cap Y^{(*)}$ is *small* if it is H-finite (or equivalently K-finite).

Now let $\mathscr{X}$ be a subset of $\mathscr{B}$. Let $\overline{\mathscr{X}}$ be the set of nontrivial unordered pairs $\{[X], [X^*]\}$ for $[X] \in \mathscr{X}$. A *cross-connected component* (CCC) of $\mathscr{X}$ is an equivalence class C for the equivalence relation generated on $\overline{\mathscr{X}}$ by crossing. We often say that X, rather than $\{[X], [X^*]\}$, belongs to C, or represents C. We denote by $\mathscr{H}$ the set of cross-connected components of $\mathscr{X}$.

Given three distinct cross-connected components C_1, C_2, C_3, we say that C_2 is *between* C_1 and C_3 if there are representatives X_i of C_i satisfying $X_1 \subset X_2 \subset X_3$.

A *star* is a subset $\Sigma \subset \mathscr{H}$ containing at least two elements, and maximal for the property that, given $C, C' \in \Sigma$, no $C'' \in \mathscr{H}$ is between C and C'. We denote by $\mathscr{S}$ the set of stars.

Definition 2.2. Let $\mathscr{X} \subset \mathscr{B}$ be a collection of almost invariant sets. Its *regular neighborhood* $RN(\mathscr{X})$ is the bipartite graph whose vertex set is $\mathscr{H} \sqcup \mathscr{S}$ (a vertex is either a cross-connected component or a star), and whose edges are pairs $(C, \Sigma) \in \mathscr{H} \times \mathscr{S}$ with $C \in \Sigma$. If $\mathscr{X}$ is G-invariant, then G acts on $RN(\mathscr{X})$.

This definition is motivated by the following remark, whose proof we leave to the reader.

Remark 2.3. Let T be any simplicial tree. Suppose that $\mathscr{H} \subset T$ meets any closed edge in a nonempty finite set. Define betweenness in $\mathscr{H}$ by $C_2 \in [C_1, C_3] \subset T$. Then the bipartite graph defined as above is isomorphic to a subdivision of T.

In the situation of Scott and Swarup [2003], a main result is that $RN(\mathscr{X})$ is a tree. We will reprove this fact by identifying $RN(\mathscr{X})$ with a subdivision of the tree of cylinders.

3. Regular neighborhoods as trees of cylinders

Now we fix a family $\mathscr{E}$ as in Definition 2.1. It is stable under commensurability, and a group of $\mathscr{E}$ cannot contain another with infinite index. Let T be an $\mathscr{E}$-tree.

In Section 3a, we define the set $\mathscr{B}(T)$ of almost invariant sets based on T, and we state the main result, Theorem 3.3: The regular neighborhood $RN(\mathscr{B}(T))$ of $\mathscr{B}(T)$ is up to subdivision the tree of cylinders T_c. In Section 3b, we represent elements of $\mathscr{B}(T)$ by special subforests of T. We then study the cross-connected components of $\mathscr{B}(T)$. We prove Theorem 3.3 in Section 3d by constructing a map Φ from the set of cross-connected components to T_c. In Section 3e we generalize Theorem 3.3 to

Theorem 3.11 by including almost invariant sets enclosed by quadratically hanging vertices of T.

3a. ***Almost invariant sets based on a tree.*** We fix a basepoint $v_0 \in V(T)$. If e is an edge of T, we denote by $\mathring{e}$ the open edge. Let T_e and T_e^* be the connected components of $T \setminus \mathring{e}$. The set of $g \in G$ such that $gv_0 \in T_e$ (respectively $gv_0 \in T_e^*$) is an almost invariant set Z_e (respectively Z_e^*) over G_e. Up to equivalence, it is independent of v_0. When we need to distinguish between Z_e and Z_e^*, we orient e and declare that the terminal vertex of e belongs to T_e.

Now consider a cylinder $Y \subset T$ and the corresponding commensurability class $\mathcal{C}$. Any Boolean combination of the Z_e for $e \in E(Y)$ is an almost invariant set over some subgroup $H \in \mathcal{C}$.

Definition 3.1. Given a cylinder Y, associated to a commensurability class $\mathcal{C}$, the *Boolean algebra of almost invariant subsets based on* Y is the subalgebra $\mathcal{B}_{\mathcal{C}}(T)$ of $\mathcal{B}_{\mathcal{C}}$ generated by the classes $[Z_e]$ for $e \in E(Y)$.

The set of *almost invariant subsets based on* T is the union $\mathcal{B}(T) = \bigcup_{\mathcal{C}} \mathcal{B}_{\mathcal{C}}(T)$, a subset of $\mathcal{B} = \bigcup_{\mathcal{C}} \mathcal{B}_{\mathcal{C}}$; just like $\mathcal{B}$, it is a union of Boolean algebras but not itself a Boolean algebra.

Proposition 3.2. *Let T and T' be minimal $\mathcal{E}$-trees. Then $\mathcal{B}(T) = \mathcal{B}(T')$ if and only if T and T' belong to the same deformation space.*

More precisely, T dominates T' if and only if $\mathcal{B}(T') \subset \mathcal{B}(T)$.

Proof. Suppose T dominates T'. After subdividing T (this does not change $\mathcal{B}(T)$), we may assume that there is an equivariant map $f : T \to T'$ sending every edge to a vertex or an edge. We claim that, given $e' \in E(T')$, there are only finitely many edges $e_i \in E(T)$ such that $f(e_i) = e'$. To see this, we may restrict to a G-orbit of edges of T, since there are finitely many such orbits. If e and ge both map onto e', then $g \in G_{e'}$. Because of the hypotheses on $\mathcal{E}$, the stabilizer G_e is contained in $G_{e'}$ *with finite index*. The claim follows.

Choose basepoints $v \in T$ and $v' = f(v) \in T'$. Then $Z_{e'}$ (defined using v') is a Boolean combination of the sets Z_{e_i} (defined using v), so $\mathcal{B}(T') \subset \mathcal{B}(T)$.

Conversely, assume $\mathcal{B}(T') \subset \mathcal{B}(T)$. Let $K \subset G$ be a subgroup elliptic in T. We show that it is also elliptic in T'.

If not, we can find an edge $e' = [v', w'] \subset T'$, and sequences $g_n \in G$ and $k_n \in K$, such that the sequences $g_n v'$ and $g_n k_n v'$ have no bounded subsequence, and $e' \subset [g_n v', g_n k_n v']$ for all n. (If K contains a hyperbolic element k, we choose e' on its axis, and we define $g_n = k^{-n}$ and $k_n = k^{2n}$; if K fixes an end ω, we want $g_n^{-1} e' \subset [v', k_n v']$, so we choose e' and g_n such that all edges $g_n^{-1} e'$ are contained on a ray ρ going out to ω, and then we choose k_n.) Defining $Z_{e'}$ using the vertex v' and a suitable orientation of e', we have $g_n \in Z_{e'}$ and $g_n k_n \notin Z_{e'}$.

Using a vertex of T fixed by K to define the almost invariant sets Z_e, we see that any element of $\mathcal{B}(T)$ is represented by an almost invariant set X satisfying $XK = X$. In particular, since $\mathcal{B}(T') \subset \mathcal{B}(T)$, there exist finite sets F_1 and F_2 such that $Z = (Z_{e'} \setminus G_{e'}F_1) \cup G_{e'}F_2$ is K-invariant on the right. For every n, one has $g_n k_n \in G_{e'}F_2$ (if $g_n, g_n k_n \in Z$) or $g_n \in G_{e'}F_1$ (if not), so one of the sequences $g_n k_n v'$ or $g_n v'$ has a bounded subsequence (because $G_{e'}$ is elliptic), a contradiction. □

Remark. The only fact used in the proof is that no edge stabilizer of T has infinite index in an edge stabilizer of T'.

Theorem 3.3. *Let T be a minimal $\mathcal{E}$-tree, with $\mathcal{E}$ as in Definition 2.1, and T_c its tree of cylinders for the commensurability relation. Let $\mathcal{X} = \mathcal{B}(T)$ be the set of almost invariant subsets based on T.*

Then $RN(\mathcal{X})$ is equivariantly isomorphic to a subdivision of T_c.

By Proposition 3.2, and [Guirardel and Levitt 2008, Theorem 1], $RN(\mathcal{X})$ and T_c only depend on the deformation space of T.

To prove the version of Theorem 3.3 stated in the introduction, one takes $\mathcal{E}$ to be the family of subgroups commensurable to an edge stabilizer of T.

The theorem will be proved in the next three subsections. We always fix a base vertex $v_0 \in T$.

3b. ***Special forests.*** Let S and S' be subsets of $V(T)$. We say that S and S' are *equivalent* if their symmetric difference is finite; we say S is trivial if it is equivalent to $\varnothing$ or $V(T)$.

The *coboundary* δS is the set of edges having one endpoint in S and one in S^* (the complement of S in $V(T)$). We shall be interested in sets S with finite coboundary. Since $\delta(S \cap S') \subset \delta S \cup \delta S'$, they form a Boolean algebra.

We also view such an S as a *subforest* of T, by including all edges whose endpoints are both in S; we can then consider the (connected) *components* of S. The set of edges of T is partitioned into edges in S, edges in S^*, and edges in $\delta S = \delta S^*$. Note that S is equivalent to the set of endpoints of its edges. In particular, S is finite (as a set of vertices) if and only if it contains finitely many edges.

We say that S is a *special forest* based on a cylinder Y if $\delta S = \{e_1, \dots, e_n\}$ is finite and contained in Y. If nonempty, S contains at least one vertex of Y. Each component of S (viewed as a subforest) is a component of $T \setminus \{\mathring{e}_1, \dots, \mathring{e}_n\}$, and S^* is the union of the other components of $T \setminus \{\mathring{e}_1, \dots, \mathring{e}_n\}$.

We define $\mathcal{B}_Y$ as the Boolean algebra of equivalence classes of special forests based on Y.

Given a special forest S based on Y, we define $X_S = \{g \mid gv_0 \in S\}$. It is an almost invariant set over $H = \bigcap_{e \in \delta S} G_e$, a subgroup of G belonging to the commensurability class $\mathcal{C}$ associated to Y; we denote its equivalence class by $[X_S]$. Every element of $\mathcal{B}(T)$ may be represented in this form. More precisely:

Lemma 3.4. *Let Y be a cylinder associated to a commensurability class $\mathscr{C}$. Then the map $S \mapsto [X_S]$ induces an isomorphism of Boolean algebras between $\mathscr{B}_Y$ and $\mathscr{B}_{\mathscr{C}}(T)$.*

Proof. It is easy to check that $S \mapsto [X_S]$ is a morphism of Boolean algebras. It is onto because the set T_e used to define the almost invariant set Z_e is a special forest (based on the cylinder containing e). It remains to determine the "kernel", namely to show that X_S is H-finite if and only if S is finite (where H denotes any group in $\mathscr{C}$).

First suppose that S is finite. Then S is contained in Y since it contains any connected component of $T \setminus Y$ that it intersects. Since δS is finite, no vertex x of S has infinite valence in T. In particular, for each vertex $x \in S$, the group G_x is commensurable with H. It follows that $\{g \in G \mid g.v_0 = x\}$ is H-finite, and X_S is H-finite.

If S is infinite, one of its components is infinite, and by minimality of T there exists a hyperbolic element $g \in G$ such that $g^n v_0 \in S$ for all $n \geq 0$. Thus $g^n \in X_S$ for $n \geq 0$. If X_S is H-finite, one can find a sequence n_i going to infinity, and $h_i \in H$, such that $g^{n_i} = h_i g^{n_0}$. Since H is elliptic in T, the sequence $h_i g^{n_0} v_0$ is bounded, a contradiction. □

Lemma 3.5. *Let S and S' be special forests.*

(1) *If S and S' are infinite and based on distinct cylinders, and if $S \cap S'$ is finite, then $S \cap S' = \varnothing$.*

(2) *If X_S crosses $X_{S'}$, then S and S' are based on the same cylinder.*

(3) *$X_S \cap X_{S'}$ is small if and only if $S \cap S'$ is finite.*

Proof. For part (1), assume that S and S' are infinite and based on $Y \neq Y'$, and that $S \cap S'$ is finite. Let $[u, u']$ be the bridge between Y and Y' (with $u = u'$ if Y and Y' intersect in a point). Since u and u' lie in more than one cylinder, they have infinite valence in T.

Assume first that $u \in S$. Then S contains all components of $T \setminus \{u\}$, except finitely many of them (which intersect Y). In particular, S contains Y'. If S' contains u', it contains u by the same argument, and $S \cap S'$ contains infinitely many edges incident on u, a contradiction. If S' does not contain u', it is contained in S, also a contradiction.

We may therefore assume $u \notin S$ and $u' \notin S'$. It follows that S (respectively S') is contained in the union of the components of $T \setminus \{u\}$ (respectively $T \setminus \{u'\}$) which intersect Y (respectively Y'), so S and S' are disjoint.

Part (2) is a consequence of [Scott and Swarup 2003, Proposition 13.5], but here is a direct argument. Assume that S and S' are based on $Y \neq Y'$, and let $[u, u']$

be as above. Up to replacing S and S' by their complements, we have $u \notin S$ and $u' \notin S'$. The argument above shows that $S \cap S' = \varnothing$, so X_S does not cross $X_{S'}$.

For part (3), first suppose that $S \cap S'$ is finite. If, say, S is finite, then X_S is H-finite by Lemma 3.4, so $X_S \cap X_{S'}$ is small. Assume therefore that S and S' are infinite. If they are based on distinct cylinders, then $X_S \cap X_{S'} = \varnothing$ by part (1). If they are based on the same cylinder, then $S \cap S'$ is itself a finite special forest, so $X_S \cap X_{S'} = X_{S \cap S'}$ is small by Lemma 3.4. Conversely, if $S \cap S'$ is infinite, one shows that $X_S \cap X_{S'}$ is not H-finite as in the proof of Lemma 3.4, using g such that $g^n v \in S \cap S'$ for all $n \geq 0$. □

Remark 3.6. If S and S' are infinite and $X_S \cap X_{S'}$ is small, then S and S' are equivalent to disjoint special forests. This follows from the lemma if they are based on distinct cylinders. If not, one replaces S' by $S' \cap S^*$.

3c. *Peripheral cross-connected components.* Theorem 3.3 is trivial if T is a line, so we can assume that each vertex of T has valence at least 3 (we now allow G to act with inversions). We need to understand cross-connected components. By Lemma 3.5(2), every such component is based on a cylinder, so we focus on a given Y. We first define peripheral special forests and almost invariant sets.

Recall that ∂Y is the set of vertices of Y that belong to another cylinder. Let $v \in \partial Y$ be a vertex whose valence in Y is finite. Let $e_1, \dots, e_n$ be the edges of Y containing v, oriented towards v. Let $S_{v,Y} = T_{e_1} \cap \cdots \cap T_{e_n}$ (recall that T_e denotes the component of $T \setminus \mathring{e}$ containing the terminal point of e). It is a subtree satisfying $S_{v,Y} \cap Y = \{v\}$, with coboundary $\delta S_{v,Y} = \{e_1, \dots, e_n\}$. We say that $S_{v,Y}$, and any special forest equivalent to it, is *peripheral* (but $S^*_{v,Y}$ is not peripheral in general).

We denote by $X_{v,Y}$ the almost invariant set corresponding to $S_{v,Y}$, and we say that X is peripheral if it is equivalent to some $X_{v,Y}$. Both $S_{v,Y}$ and $S^*_{v,Y}$ are infinite, so $X_{v,Y}$ is nontrivial by Lemma 3.4.

We claim that $C_{v,Y} = \{\{[X_{v,Y}], [X^*_{v,Y}]\}\}$ is a complete cross-connected component of $\mathcal{B}(T)$, called a peripheral CCC. Indeed, assume that $X_{v,Y}$ crosses some X_S. Then S is based on Y by Lemma 3.5, but since $S_{v,Y}$ contains no edge of Y, it is contained in S_X or S^*_X, which prevents crossing.

Note that if $C_{v,Y} = C_{v',Y'}$, then $Y = Y'$ (because an H-almost invariant subset determines the commensurability class of H), and $v = v'$ except when Y is a single edge vv', in which case $X_{v,Y} = X^*_{v',Y}$.

Lemma 3.7. *Let Y be a cylinder. There is at most one nonperipheral cross-connected component C_Y based on Y. There is exactly one if and only if $|\partial Y| \neq 2, 3$.*

Proof. The proof is in three parts.

We first claim that, given any infinite connected nonperipheral special forest S based on Y, there is an edge $e \subset S \cap Y$ such that both connected components of $S \setminus \{\mathring{e}\}$ are infinite.

Assume there is no such e. Then $S \cap Y$ is locally finite: Given $v \in S$, all but finitely many components of $S \setminus \{v\}$ are infinite, so infinitely many edges incident on v satisfy the claim if v has infinite valence in $S \cap Y$.

Since S is infinite and nonperipheral, $S \cap Y$ is not reduced to a single point. We orient every edge e of $S \cap Y$ so that $S \cap T_e$ is infinite and $S \cap T_e^*$ is finite. If a vertex v of $S \cap Y$ is terminal in every edge of $S \cap Y$ that contains it, S is peripheral. We may therefore find an infinite ray $\rho \subset S \cap Y$ consisting of positively oriented edges. Since every vertex of T has valence ≥ 3, every vertex of ρ is the projection onto ρ of an edge of δS, contradicting the finiteness of δS. This proves the claim.

Secondly, to show that there is at most one nonperipheral cross-connected component, we fix two nontrivial forests S and S' based on Y, and we show that X_S and $X_{S'}$ are in the same CCC if they do not belong to peripheral CCCs. We can assume that $X_S \cap X_{S'}$ is small, and by Remark 3.6 that $S \cap S'$ is empty. We may also assume that every component of S and S' is infinite.

Since S is not peripheral, it contains two disjoint infinite special forests S_1 and S_2 based on Y: This is clear if S has several components, and follows from the claim otherwise. Construct S_1' and S_2' similarly. Then $X_{S_1} \cup X_{S_1'}$ crosses both X_S and $X_{S'}$, so X_S and $X_{S'}$ are in the same cross-connected component.

Finally, we discuss the existence of C_Y. If $|\partial Y| \geq 4$, choose $v_1, \dots, v_4 \in \partial Y$, and consider edges e_1, e_2, e_3 of Y such that each v_i belongs to a different component S_i of $T \setminus \{\mathring{e}_1, \mathring{e}_2, \mathring{e}_3\}$. These components are infinite because $v_i \in \partial Y$, and $X_{S_1 \cup S_2}$ belongs to a nonperipheral CCC.

If ∂Y is empty, then $Y = T$ and existence is clear. If ∂Y is nonempty, minimality of T implies that Y is the convex hull of ∂Y (replacing every cylinder by the convex hull of its boundary yields an invariant subtree). From this we deduce that $|\partial Y| \neq 1$, and every CCC based on Y is peripheral if $|\partial Y|$ equals 2 or 3. There is one peripheral CCC if $|\partial Y| = 2$ (that is, Y is a single edge) and three if $|\partial Y| = 3$. □

Remark 3.8. The proof shows that, if $|\partial Y| \geq 4$, then for all $u \neq v$ in ∂Y, the nonperipheral CCC is represented by a special forest S such that $u \in S$ and $v \in S^*$.

3d. ***Proof of Theorem 3.3.*** From now on we assume that T has more than one cylinder; otherwise there is exactly one cross-connected component, and both $RN(\mathcal{X})$ and T_c are points.

It will be helpful to distinguish between a cylinder $Y \subset T$ or a point $\eta \in \partial Y$, and the corresponding vertex of T_c. We therefore denote by Y_c or η_c the vertex of T_c corresponding to Y or η.

Recall that $\mathcal{H}$ denotes the set of cross-connected components of $\mathcal{X} = \mathcal{B}(T)$. Consider the map $\Phi : \mathcal{H} \to T_c$ defined as follows:

- If $C = C_Y$ is a nonperipheral CCC, then $\Phi(C) = Y_c \in V_1(T_c)$.

- If $C = C_{v,Y}$ is peripheral, and $\#\partial Y \geq 3$, then $\Phi(C)$ is the midpoint of the edge $\varepsilon = (v_c, Y_c)$ of T_c.
- If $\#\partial Y = 2$, and C is the peripheral CCC based on Y, then $\Phi(C) = Y_c$.

In all cases, the distance between $\Phi(C)$ and Y_c is at most $1/2$. If C is peripheral, $\Phi(C)$ has valence 2 in T_c.

Clearly, Φ is one-to-one. By Remark 2.3, it now suffices to show that the image of Φ meets every closed edge, and Φ preserves betweenness: For $C_1, C_2, C_3 \in \mathcal{H}$, C_2 is between C_1 and C_3 if and only if $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$.

The first fact is clear because $\Phi(\mathcal{H})$ contains all vertices $Y_c \in V_1(T_c)$ with $|\partial Y| \neq 3$ and the three points at distance $1/2$ from Y_c if $|\partial Y| = 3$. To control betweenness, we need a couple of technical lemmas.

If S is a nontrivial special forest, we denote by $[\![S]\!]$ the cross-connected component represented by the almost invariant set X_S.

Let $Y \subset T$ be a cylinder. We denote by $\mathrm{pr}_Y : T \to Y$ the projection. If Y' is another cylinder, then $\mathrm{pr}_Y(Y')$ is a single point. This point belongs to two cylinders and hence defines a vertex of $V_0(T_c)$ that is at distance 1 from Y_c on the segment of T_c joining Y_c to Y'_c.

Let Y be a cylinder with $|\partial Y| \geq 4$. For each nontrivial special forest S' that is either based on some $Y' \neq Y$, or based on Y and peripheral, we define a point $\eta_Y(S') \in Y \subset T$ as follows. If S' is based on $Y' \neq Y$, we define $\eta_Y(S')$ to be $\mathrm{pr}_Y(Y')$. If S' is equivalent to some $S_{v,Y}$, we define $\eta_Y(S') = v$; note that in this case $\eta_Y(S'^*)$ is not defined.

Lemma 3.9. *Let Y be a cylinder with $|\partial Y| \geq 4$. Consider two nontrivial special forests S, S' with $[\![S']\!] \neq C_Y$ and $[\![S]\!] = C_Y$, and assume $S' \subset S$.*

Then $\eta = \eta_Y(S') \in Y$ is defined, $\eta \in S$, and S' contains an equivalent subforest S'' with $S'' \subset \mathrm{pr}_Y^{-1}(\{\eta\}) \subset S$.

Moreover, $\Phi([\![S']\!])$ lies in the connected component of $T_c \setminus \{Y_c\}$ containing η_c.

Proof. Let Y' be the cylinder on which S' is based.

If $Y' = Y$, then S'^* is not peripheral, so S' is peripheral. Thus η is defined, and S' is equivalent to its subforest $S'' = S_{Y,\eta}$. Then $S'' = \mathrm{pr}_Y^{-1}(\{\eta\}) \subset S$. In this case $\Phi([\![S']\!])$ is the midpoint of the edge (η_c, Y_c) of T_c.

Assume that $Y' \neq Y$. Then $\eta = \mathrm{pr}_Y(Y') \in S$; otherwise Y' would be disjoint from S and hence from S', a contradiction. It follows that $\mathrm{pr}_Y^{-1}(\{\eta\}) \subset S$. If $\eta \in S'$, then S' contains the complement of $\mathrm{pr}_Y^{-1}(\{\eta\})$, so $S = T$, a contradiction. Thus $\eta \notin S'$ and therefore $S' \subset \mathrm{pr}_Y^{-1}(\{\eta\})$. The "moreover" is clear in this case since η_c is between Y_c and Y'_c, and $\Phi([\![S']\!])$ is at distance $\leq 1/2$ from Y'_c. □

Lemma 3.10. *Let $S = S_{Y,u}$ be peripheral, and let S' be a nontrivial special forest with $[\![S']\!] \neq [\![S]\!]$. Recall that u_c is the vertex of T_c associated to u.*

(1) *If $S' \subset S$, then $\Phi([\![S']\!])$ belongs to the component of $T_c \setminus \{\Phi([\![S]\!])\}$ that contains u_c.*

(2) *If $S \subset S'$, then $\Phi([\![S']\!])$ belongs to the component of $T_c \setminus \{\Phi([\![S]\!])\}$ that does not contain u_c.*

Proof. If $S' \subset S$, then S' is based on some $Y' \neq Y$. Since $S' \subset S = \mathrm{pr}_Y^{-1}(\{u\})$, we have $Y' \subset \mathrm{pr}_Y^{-1}(\{u\})$ and u_c is between Y_c and Y'_c in T_c. The result follows since $\Phi([\![S]\!])$ is 1/2-close to Y_c and $\Phi([\![S']\!])$ is 1/2-close to Y'_c.

If $S \subset S'$ and $Y \neq Y'$, we have $\mathrm{pr}_Y(Y') \neq u$ because $S' \neq T$, and the lemma follows. If $Y = Y'$, the lemma is immediate. □

We can now show that Φ preserves betweenness. Consider three distinct cross-connected components $C_1, C_2, C_3 \in \mathcal{H}$. Let Y_2 be the cylinder on which C_2 is based. Note that $|\partial Y_2| \geq 4$ if C_2 is nonperipheral.

First assume that C_2 is between C_1 and C_3. By definition, there exist almost invariant subsets X_i representing C_i such that $X_1 \subset X_2 \subset X_3$. By Lemma 3.4, one can find special forests S_i with $[X_{S_i}] = [X_i]$. By Remark 3.6, since the C_i are distinct, one can assume $S_1 \subset S_2 \subset S_3$ (if necessary, replace S_2 by $S_2 \cap S_3$, and then S_1 by $S_1 \cap S_2 \cap S_3$).

If S_2 is peripheral, $\Phi(C_1)$ and $\Phi(C_3)$ are in distinct components of $T_c \setminus \{\Phi(C_2)\}$ by Lemma 3.10, so $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$. If S_2^* is peripheral, we apply the same argument using $S_3^* \subset S_2^* \subset S_1^*$.

Assume therefore that C_2 is nonperipheral. Lemma 3.9 implies that the points $\eta_1 = \eta_{Y_2}(S_1)$ and $\eta_3 = \eta_{Y_2}(S_3^*)$ are defined, and $\eta_1 \in S_2$ and $\eta_3 \in S_2^*$. In particular, we have $\eta_1 \neq \eta_3$. By the "moreover", we get $\Phi(C_2) \in [\Phi(C_1), \Phi(C_3)]$ since $\Phi(C_2) = (Y_2)_c$.

Now assume that C_2 is not between C_1 and C_3, and choose S_i with $[\![S_i]\!] = C_i$. By Remark 3.6, we may assume that for each $i \in \{1, 3\}$ some inclusion $S_i^{(*)} \subset S_2^{(*)}$ holds. Since C_2 is not between C_1 and C_3, we may assume after changing S_i to S_i^* if needed that $S_1 \subset S_2$ and $S_3 \subset S_2$.

If S_2 or S_2^* is peripheral, Lemma 3.10 implies that $\Phi(C_1)$ and $\Phi(C_3)$ lie in the same connected component of $T_c \setminus \{\Phi(C_2)\}$, so $\Phi(C_2)$ is not between $\Phi(C_1)$ and $\Phi(C_3)$.

Assume therefore that C_2 is nonperipheral. Lemma 3.9 says that the points $\eta_1 = \eta_{Y_2}(S_1)$ and $\eta_3 = \eta_{Y_2}(S_3)$ are defined, and we may assume $S_i \subset \mathrm{pr}_{Y_2}^{-1}(\{\eta_i\})$. If $\eta_1 = \eta_3$, then $\Phi(C_2)$ does not lie between $\Phi(C_1)$ and $\Phi(C_3)$ by the "moreover" of Lemma 3.9. If $\eta_1 \neq \eta_3$, consider $\tilde{S}_2$ with $[\![\tilde{S}_2]\!] = C_2$ such that $\eta_1 \in \tilde{S}_2$ and $\eta_3 \in \tilde{S}_2^*$ (it exists by Remark 3.8). Then $S_1 \subset \mathrm{pr}_{Y_2}^{-1}(\eta_1) \subset \tilde{S}_2$ and $S_3 \subset \mathrm{pr}_{Y_2}^{-1}(\eta_3) \subset \tilde{S}_2^*$, so C_2 lies between C_1 and C_3, a contradiction.

This ends the proof of Theorem 3.3. □

3e. ***Quadratically hanging vertices.*** We say that a vertex stabilizer G_v of T is a *QH-subgroup* if there is an exact sequence $1 \to F \to G_v \xrightarrow{\pi} \Sigma \to 1$, where $\Sigma = \pi_1(\mathbb{O})$ is a hyperbolic 2-orbifold group and every incident edge group G_e is peripheral: It is contained with finite index in the preimage by π of a boundary subgroup $B = \pi_1(C)$, with C a boundary component of $\mathbb{O}$. We say that v is a *QH-vertex* of T.

We now define almost invariant sets based on v. They will be included in our description of the regular neighborhood.

We view Σ as a convex cocompact Fuchsian group acting on $\mathbb{H}^2$. Let $\overline{H}$ be any nonperipheral maximal two-ended subgroup of Σ (represented by an immersed curve or 1-suborbifold). Let γ be the geodesic invariant by $\overline{H}$. It separates $\mathbb{H}^2$ into two half-spaces $P^{\pm}$, which may be interchanged by certain elements of $\overline{H}$.

Let $\overline{H}_0$ be the stabilizer of P^+, which has index at most 2 in $\overline{H}$, and let x_0 be a basepoint. We define an $\overline{H}_0$-almost invariant set $\overline{X} \subset \Sigma$ as the set of $g \in \Sigma$ such that $gx_0 \in P^+$. (If $\overline{H}$ is the fundamental group of a two-sided simple closed curve on $\mathbb{O}$, there is a one-edge splitting of Σ over $\overline{H}$, and $\overline{X}$ is a Z_e as in Section 3a.)

The preimage of $\overline{X}$ in G_v is an almost invariant set X_v over the preimage H_0 of $\overline{H}_0$. We extend it to an almost invariant set X of G as follows. Let S' be the set of vertices $u \neq v$ of T such that, denoting by e the initial edge of the segment $[v, u]$, the geodesic of $\mathbb{H}^2$ invariant under $G_e \subset G_v$ lies in P^+; note that it lies in either P^+ or P^-. Then X is the union of X_v with the set of $g \notin G_v$ such that $gv \in S'$.

Starting from $\overline{H}$, we have thus constructed an almost invariant set X, which is well defined up to equivalence and complementation (because of the choices of x_0 and $P^{\pm}$). We say that X is a *QH-almost invariant subset* based on v. We let $\mathrm{QH}_v(T)$ be the set of equivalence classes of QH-almost invariant subsets obtained from v as above (varying $\overline{H}$), and we let $\mathrm{QH}(T)$ be the union of all $\mathrm{QH}_v(T)$ when v ranges over all QH-vertices of T.

Theorem 3.11. *With $\mathscr{E}$ and T as in Theorem 3.3, let $\hat{\mathscr{X}} = \mathscr{B}(T) \cup \mathrm{QH}(T)$. Then $RN(\hat{\mathscr{X}})$ is isomorphic to a subdivision of T_c.*

Proof. The proof is similar to that of Theorem 3.3.

If X is a QH-almost invariant subset as constructed above, we call $S = S' \cup \{v\}$ the *QH-forest* associated to X. We say that it is based on v. The coboundary of S is infinite, but all its edges contain v. We may therefore view S as a subtree of T (the union of v with certain components of $T \setminus \{v\}$). It is a union of cylinders. We let $S^* = (T \setminus S) \cup \{v\}$, so that $S \cap S^* = \{v\}$.

Note that S cannot contain a peripheral special forest $S_{v,Y}$, with Y a cylinder containing v (this is because the subgroup $\overline{H} \subset \Sigma$ was chosen nonperipheral).

Conversely, given a QH-forest S, one can recover H_0, which is the stabilizer of S, and the equivalence class of X. In other words, there is a bijection between $\mathrm{QH}_v(T)$

and the set of QH-forests based on v. We denote by X_S the almost invariant set X corresponding to S (it is well defined up to equivalence). Note that X_S is not a subset of $\{g \in G \mid gv \in S\}$, and these sets have the same intersection with $G \setminus G_v$.

The following fact is analogous to Lemma 3.5.

Lemma 3.12. *Let S be a QH-forest based on v. Let S' be a nontrivial special forest, or a QH-forest based on $v' \neq v$.*

(1) *$X_S \cap X_{S'}$ is small if and only if $S \cap S' = \varnothing$.*

(2) *X_S and $X_{S'}$ do not cross.*

Proof. When S' is a special forest, we use v as a basepoint to define $X_{S'}$ as the set $\{g \mid gv \in S'\}$. Beware that X_S is properly contained in $\{g \mid gv \in S\}$.

We claim that if S' is a special forest with $v \notin S'$ and $S \cap S' \neq \varnothing$, then $X_{S'} \subset X_S$. Let Y' be the cylinder on which S' is based. Since each connected component of S' contains a point in Y', there is a point $w \neq v$ in $S \cap Y'$. Since S is a union of cylinders, S contains Y'. All connected components of S' therefore contain a point of S and so are contained in $S \setminus \{v\}$ since $v \notin S'$. We deduce $X_{S'} \subset X_S$.

We now prove (1). If $S \cap S' = \varnothing$, then $X_S \cap X_{S'} = \varnothing$. We assume $S \cap S' \neq \varnothing$, and we show that $X_S \cap X_{S'}$ is not small. If S' is a QH-forest, then $v \in S'$ or $v' \in S$. If for instance $v \in S'$, then $X_S \cap X_{S'}$ is not small because it contains $X_S \cap G_v$. Now assume that S' is a special forest. If $v \in S'$, the same argument applies, so assume that $v \notin S'$. The claim implies $X_{S'} \subset X_S$, so $X_S \cap X_{S'}$ is not small.

To prove (2), first consider the case where S' is a QH-forest. Up to changing S and S' to S^* or S'^*, one can assume $S \cap S' = \varnothing$, so X_S does not cross $X_{S'}$. If S' is a special forest, we can assume $v \notin S'$ by changing S' to S'^*. By the claim, X_S does not cross $X_{S'}$. □

The lemma implies that no element of $\mathrm{QH}(T)$ crosses an element of $\mathcal{B}(T)$, and elements of $\mathrm{QH}_v(T)$ do not cross elements of $\mathrm{QH}_{v'}(T)$ for $v \neq v'$.

Since $\mathrm{QH}_v(T)$ is a cross-connected component, the set $\hat{\mathcal{H}}$ of cross-connected components of $\mathcal{B}(T) \cup \mathrm{QH}(T)$ is therefore the set of cross-connected components of $\mathcal{B}(T)$, together with one new cross-connected component $\mathrm{QH}_v(T)$ for each QH-vertex v.

One extends the map Φ defined in the proof of Theorem 3.3 to a map $\hat{\Phi} : \hat{\mathcal{H}} \to T_c$ by sending $\mathrm{QH}_v(T)$ to v (viewed as a vertex of $V_0(T_c)$ since a QH-vertex belongs to infinitely many cylinders). We need to prove that $\hat{\Phi}$ preserves betweenness.

Lemmas 3.9 and 3.10 extend immediately to the case where S' is a QH-forest: one just needs to define $\eta_Y(S') = \mathrm{pr}_Y(v')$ for S' based on v', so that v' plays the role of Y' in the proofs. (In the proof of Lemma 3.9, the assertion that $\eta \notin S'$ should be replaced by the fact that $S' \cap Y$ contains no edge; this holds since otherwise S' would contain Y.) This allows to prove that, if C_2 is not a component $\mathrm{QH}_v(T)$, then $\Phi(C_2)$ is between $\Phi(C_1)$ and $\Phi(C_3)$ if and only if C_2 lies between C_1 and C_3.

To treat the case when $C_2 = \mathrm{QH}_v(T)$, we need a cylinder-valued projection η_v. Let Y be a cylinder or a QH-vertex distinct from v. We define $\eta_v(Y)$ as the cylinder of T containing the initial edge of $[v, x]$ for any $x \in Y$ different from v. Equivalently, $\eta_v(Y)$ is Y if $v \in Y$, the cylinder containing the initial edge of the bridge joining x to Y otherwise.

If v lies in a cylinder Y^0, denote by $\eta_v^{-1}(Y^0)$ the union of cylinders Y such that $\eta_v(Y) = Y^0$. Equivalently, this is the set of points $x \in T$ such that $x = v$ or $[x, v]$ contains an edge of Y^0.

As before, $[\![S]\!]$ denotes the cross-connected component represented by X_S.

Lemma 3.13. *Let S be a QH-forest based on v. Let S' be a nontrivial special forest, or a QH-forest based on $v' \neq v$. Let Y' be the cylinder or QH-vertex on which S' is based, and let $Y'^0 = \eta_v(Y')$.*

If $S' \subset S$, then $S' \subset \eta_v^{-1}(Y'^0) \subset S$.

Moreover, $\Phi([\![S']\!])$ and $Y_c'^0$ lie in the same component of $T_c \setminus \{\Phi([\![S]\!])\}$.

We leave the proof of this lemma to the reader.

Assume now that $S_1 \subset S_2 \subset S_3$ with $[\![S_i]\!] = C_i$ and S_2 based on v. For $i = 1, 3$, let $Y_i^0 = \eta_v(Y_i)$. Then $S_1 \subset \eta_v^{-1}(Y_1^0) \subset S_2$ and $S_3^* \subset \eta_v^{-1}(Y_3^0) \subset S_2^*$. In particular, $Y_1^0 \neq Y_3^0$. Since $(Y_1^0)_c$ and $(Y_3^0)_c$ are neighbors of v_c, they lie in distinct components of $T_c \setminus \{\Phi(C_2)\}$. By Lemma 3.13, so do $\Phi([\![S_1]\!])$ and $\Phi([\![S_3]\!])$.

Conversely, assume that C_2 does not lie between C_1 and C_3, and consider $S_1 \subset S_2$ and $S_3 \subset S_2$ with $[\![S_i]\!] = C_i$. For $i = 1, 3$, let Y_i^0 be as above. If $Y_1^0 = Y_3^0$, then $\Phi(C_2)$ is not between $\Phi(C_1)$ and $\Phi(C_3)$ by Lemma 3.13, and we are done. If $Y_1^0 \neq Y_3^0$, these cylinders correspond to distinct peripheral subgroups of G_v, with invariant geodesics $\gamma_1 \neq \gamma_3$. There exists a nonperipheral group $\overline{H} \subset \Sigma$, as in the beginning of this subsection, whose invariant geodesic separates γ_1 and γ_3. Let S_2' be the associated QH-forest. Then $[\![S_2']\!] = C_2$ and, up to complementation, $\eta_v^{-1}(Y_1^0) \subset S_2'$ and $\eta_v^{-1}(Y_3^0) \subset S_2'^*$. It follows that $S_1 \subset S_2'$ and $S_3^* \subset S_2'^*$, so C_2 lies between C_1 and C_3, contradicting our assumptions. □

4. The regular neighborhood of Scott and Swarup

A group is VPC_n if some finite index subgroup is polycyclic of Hirsch length n. For instance, VPC_0 groups are finite groups, VPC_1 groups are virtually cyclic groups, and VPC_2 groups are virtually $\mathbb{Z}^2$ (but not all VPC_n groups are virtually abelian for $n \geq 3$).

Fix $n \geq 1$. We assume that G is finitely presented and does not split over a VPC_{n-1} subgroup. We also assume that G itself is not VPC_{n+1}. All trees we consider here are assumed to have VPC_n edge stabilizers.

A subgroup $H \subset G$ is *universally elliptic* if it is elliptic in every tree. A tree is universally elliptic if all its edge stabilizers are.

A tree is a *JSJ tree* (over VPC_n subgroups) if it is universally elliptic, and maximal for this property: it dominates every universally elliptic tree. JSJ trees exist (because G is finitely presented) and belong to the same deformation space, called the JSJ deformation space; see [Guirardel and Levitt 2009].

A vertex stabilizer G_v of a JSJ tree is *flexible* if it is not VPC_n and is not universally elliptic. It follows from [Dunwoody and Sageev 1999] that a flexible vertex stabilizer is a QH-subgroup, as defined in Section 3e: There is an exact sequence $1 \to F \to G_v \to \Sigma \to 1$, where $\Sigma = \pi_1(\mathbb{O})$ is the fundamental group of a hyperbolic 2-orbifold, F is VPC_{n-1}, and every incident edge group G_e is peripheral. Note that the QH-almost invariant subsets X constructed in Section 3e are over VPC_n subgroups.

We can now describe the regular neighborhood of all almost invariant subsets of G over VPC_n subgroups as a tree of cylinders.

Theorem 4.1. *Let G be a finitely presented group, and let $n \geq 1$. Assume that G does not split over a* VPC_{n-1} *subgroup and that G is not* VPC_{n+1}. *Let T be a JSJ tree over* VPC_n *subgroups, and let T_c be its tree of cylinders for the commensurability relation.*

Then Scott and Swarup's regular neighborhood of all almost invariant subsets over VPC_n *subgroups is equivariantly isomorphic to a subdivision of T_c.*

This is immediate from Theorem 3.11 and Theorem 4.2, which says one can read any almost invariant set over a VPC_n subgroup in a JSJ tree T, and which follows from [Dunwoody and Swenson 2000] and [Scott and Swarup 2003, Theorem 8.2].

Theorem 4.2. *Let G and T be as above.*

For any almost invariant subset X over a VPC_n *subgroup, the equivalence class* $[X]$ *belongs to* $\mathcal{B}(T) \cup \mathrm{QH}(T)$.

Proof. We essentially follow the proof by Scott and Swarup [2003],[1] and we adopt their definitions. All trees considered here have VPC_n edge stabilizers.

Let X be a nontrivial almost invariant subset over a VPC_n subgroup H. We first consider the case where X crosses strongly some other almost invariant subset. Then by [Dunwoody and Swenson 2000, Proposition 4.11], H is contained as a nonperipheral subgroup in a QH-vertex stabilizer W of some tree T'. When acting on T, the group W fixes a QH-vertex $v \in T$; see [Guirardel and Levitt 2009, Remark 7.20].

Note that H is not peripheral in G_v, because it is not peripheral in W. Since (G, H) only has 2 coends [2003, Proposition 13.8], there are (up to equivalence) only two almost invariant subsets over subgroups commensurable with H (namely X and X^*), and therefore $[X] \in \mathrm{QH}_v(T)$.

[1] From here on, [2003] refers to [Scott and Swarup 2003].

From now on, we assume that X crosses strongly no other almost invariant subset over a VPC_n subgroup. Then, by [Dunwoody and Roller 1993] and [Dunwoody and Swenson 2000, Section 3], there is a nontrivial tree T_0 with one orbit of edges and an edge stabilizer H_0 commensurable with H.

Since X crosses strongly no other almost invariant set, H and H_0 are universally elliptic; see [Guirardel 2005, Lemme 11.3]. In particular, T dominates T_0. It follows that there is an edge of T with stabilizer contained in H_0 (necessarily with finite index). This edge is contained in a cylinder Y associated to the commensurability class of H.

The main case is when T has no edge e such that Z_e crosses X. (See Section 3a for the definition of Z_e.) The following lemma implies that X is enclosed in some vertex v of T.

Lemma 4.3. *Suppose G is finitely generated. Let $X \subset G$ be a nontrivial almost invariant set over a finitely generated subgroup H. Let T be a tree with an action of G. If X crosses no Z_e, then X is enclosed in some vertex $v \in T$.*

Proof. The argument follows a part of the proof of [2003, Proposition 5.7].

Given two almost invariant subsets, we use the notation $X \geq Y$ when $Y \cap X^*$ is small. The noncrossing hypothesis says that each edge e of T may be oriented so that $Z_e \geq X$ or $Z_e \geq X^*$. If one can choose both orientations for some e, then X is equivalent to Z_e, so X is enclosed in both endpoints of e and we are done.

We orient each edge of T in this manner. We color the edge blue or red according to whether $Z_e \geq X$ or $Z_e \geq X^*$. No edge can have both colors. If e is an oriented edge, and if e' lies in T_e^*, then e' is oriented towards e, so that $Z_e \subset Z_{e'}$, and e' has the same color as e. In particular, given a vertex v, either all edges containing v are oriented towards v, or there exists exactly one edge containing v and oriented away from v, and all edges containing v have the same color.

If v is as in the first case, X is enclosed in v by definition. If there is no such v, then all edges have the same color and are oriented towards an end of T. By [2003, Lemma 2.31], G is contained in the R-neighborhood of X for some $R > 0$, so X is trivial, a contradiction. □

Let v be a vertex of T enclosing X. In particular, $H \subset G_v$. The set $X_v = X \cap G_v$ is an H-almost invariant subset of G_v (note that G_v is finitely generated). By [2003, Lemma 4.14], there is a subtree $S \subset T$ containing v, with $S \setminus \{v\}$ a union of components of $T \setminus \{v\}$, such that X is equivalent to $X_v \cup \{g \mid g.v \in S \setminus \{v\}\}$.

Lemma 4.4. *The H-almost invariant subset X_v of G_v is trivial.*

Proof. Otherwise, by [Dunwoody and Roller 1993; Dunwoody and Swenson 2000], there is a G_v-tree T_1 with one orbit of edges and an edge stabilizer H_1 commensurable with H, and an edge $e_1 \subset T_1$, such that Z_{e_1} lies up to equivalence in the Boolean algebra generated by the orbit of X_v under the commensurator of H in G_v.

Note that G_e is elliptic in T_1 for each edge e of T incident to v: By symmetry of strong crossing [2003, Proposition 13.3], G_e does not cross strongly any translate of X, and thus does not cross strongly Z_{e_1}, so G_e is elliptic in T_1 [Guirardel 2005, lemme 11.3]. This ellipticity allows us to refine T by creating new edges with stabilizer conjugate to H_1. Since H_1 is universally elliptic, this contradicts the maximality of the JSJ tree T. □

After replacing X by an equivalent almost invariant subset or its complement, and possibly changing S to $(T \setminus S) \cup \{v\}$, we can assume that $X = \{g \mid g.v \notin S\}$. Recall that Y is the cylinder defined by the commensurability class of H.

Lemma 4.5. *The coboundary δS, consisting of edges vw with $w \notin S$, is a finite set of edges of Y.*

This implies that $[X] \in \mathcal{B}(T)$, ending the proof when X crosses no Z_e.

Proof of Lemma 4.5. Let E be the set of edges of δS, oriented so that $X = \bigsqcup_{e \in E} Z_e$ (we use v as a basepoint to define Z_e). Let A be a finite generating system of G such that, for all $a \in A$, the open segment (av, v) does not meet the orbit of v. One can construct such a generating system from any finite generating system by iteratively replacing $\{a\}$ by the pair $\{g, g^{-1}a\}$ if (av, v) contains some $g.v$.

Let Γ be the Cayley graph of (G, A). For any subset $Z \subset G$, denote by δZ the set of edges of Γ having one endpoint in Z and the other endpoint in $G \setminus Z$. By our choice of A, no edge joins a vertex of Z_e to a vertex of $Z_{e'}$ for $e \neq e'$. It follows that $\delta X = \bigsqcup_{e \in E} \delta Z_e$.

Since δX is H-finite, the set δZ_e is H-finite for each $e \in E$, and E is contained in a finite union of H-orbits. Let $e \in E$. Since δZ_e is G_e-invariant and H-finite, $G_e \cap H$ has finite index in G_e. Since G_e and H are both VPC_n, they are commensurable, so the H-orbit of e is finite. It follows that $E \subset Y$ and that E is finite. □

We now turn to the case when X crosses some of the Z_e. For each $e \in E(T)$, the intersection number $i(Z_e, X)$ is finite [Scott 1998], which means that there are only finitely many edges e' in the orbit of e such that $Z_{e'}$ crosses X. Since T/G is finite, let $e_1, e_1^{-1}, e_2, e_2^{-1}, \ldots, e_n, e_n^{-1}$ be the finite set of oriented edges e such that Z_e crosses X, where we denote by $e \mapsto e^{-1}$ the orientation-reversing involution. Note that $e_i \subset Y$ by [2003, Proposition 13.5]. Now X is a finite union of sets of the form $X' = X \cap Z_{e_1^{\pm 1}} \cap \cdots \cap Z_{e_n^{\pm 1}}$. Since X' does not cross any Z_e, its equivalence class lies in $\mathcal{B}(T)$ by the argument above and so does $[X]$. □

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VINCENT GUIRARDEL
INSTITUT DE MATHÉMATIQUES DE TOULOUSE
UNIVERSITÉ DE TOULOUSE ET CNRS (UMR 5219)
118 ROUTE DE NARBONNE
F-31062 TOULOUSE CEDEX 9
FRANCE

guirardel@math.univ-toulouse.fr
www.math.univ-toulouse.fr/~guirardel

GILBERT LEVITT
LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME
UNIVERSITÉ DE CAEN ET CNRS (UMR 6139)
BP 5186
F-14032 CAEN CEDEX
FRANCE

levitt@math.unicaen.fr