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**TRANSITIVE ACTIONS AND EQUIVARIANT COHOMOLOGY
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TRANSITIVE ACTIONS AND EQUIVARIANT COHOMOLOGY AS AN UNSTABLE \mathcal{A}^* -ALGEBRA

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A graded \mathbb{F}_p -algebra A with action of the Steenrod algebra \mathcal{A}^* is said to be Steenrod presentable if there is a polynomial ring $P = \mathbb{F}_p[u_1, \dots, u_n]$ with an action of \mathcal{A}^* and an \mathcal{A}^* -invariant ideal $I \subset P$ such that $A = P/I$ and the induced action of \mathcal{A}^* on P/I is the given one. It is shown that an action φ of a simple compact Lie group G on a homogeneous Kähler manifold $X = G/H$ has a Steenrod presentable equivariant cohomology for almost all primes p if and only if φ is conjugate to the standard action by left translation. Application to the case $H = T$ a maximal torus reproduces a former result of the author: namely, that every topological G -action on G/T is conjugate to the standard action by left translation with isotropy group a maximal torus.

1. Introduction

Suppose X to be a space, and let $A = H^*(X; \mathbb{F}_p)$ be its cohomology with coefficients in the prime field \mathbb{F}_p . Then on A there is an unstable action of the p -Steenrod algebra \mathcal{A}^* . On the other hand, given a presentation $A = P/I$, for an ideal $I \subset P$ where P is the polynomial algebra $P = \mathbb{F}_p[h_1, \dots, h_n]$, with $\deg h_i = d_i$, one might ask whether the given action of \mathcal{A}^* is induced by an action of \mathcal{A}^* on the polynomial algebra that leaves the defining ideal stable. In the case $p \neq 2$ and d_i prime to p for all i , a necessary condition is given by a theorem of Adams and Wilkerson [1980]; see also [Smith 1995, Theorem 10.5.1]. In particular it follows from this theorem that the polynomial ring P must be the invariant ring

$$P = \mathbb{F}_p[x_1, \dots, x_n]^W, \quad \deg x_i = 2,$$

where $W \subset \mathrm{GL}(n, \mathbb{F}_p)$ is a finite group of order $d_1 \dots d_n$ generated by pseudoreflections acting on $\mathbb{F}_p[x_1, \dots, x_n] = \mathbb{F}_p[V]$ in the standard way [Smith 1995; 1997].

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This implies that the degrees d_i can only assume certain precise values, which are exactly the Weyl–Coxeter degrees of the group W ; see for example [Smith 1995, p. 199].

In the following, we say that the \mathbb{F}_p -algebra A with an unstable action of \mathcal{A}^* is *Steenrod presentable* if there is a polynomial ring $P = \mathbb{F}_p[x_1, \dots, x_n]^W$ with the standard action of the Steenrod algebra \mathcal{A}^* and an \mathcal{A}^* -stable ideal $I \subset P$ such that $A \cong P/I$ with the induced \mathcal{A}^* -module structure.

As the main example of Steenrod presentable \mathbb{F}_p -algebras, we consider the cohomology of homogeneous spaces $X = G/H$, where $H \subset G$ is a closed connected maximal rank subgroup of a compact connected Lie group G . Then there is the standard fibration

$$G/H \longrightarrow B_H \longrightarrow B_G,$$

where B_K is the classifying space for the topological group K . If $H \subset G$ is a subgroup of maximal rank and if neither G nor H have p -torsion, the ring $H^*(G/H; \mathbb{F}_p)$ has a presentation

$$H^*(G/H; \mathbb{F}_p) \cong \frac{H^*(B_H; \mathbb{F}_p)}{H_+^*(B_G; \mathbb{F}_p) \cdot H^*(B_H; \mathbb{F}_p)}$$

such that the action of the Steenrod algebra \mathcal{A}^* on $H^*(G/H; \mathbb{F}_p)$ is induced by the standard action of \mathcal{A}^* on the ring $H^*(B_H; \mathbb{F}_p)$.

So, throughout this note we shall assume that $p \neq 2$ and that B_G and B_H do not have p -torsion for all primes to be considered.

Suppose a compact connected Lie group K is acting in a reasonable way on $X = G/H$. Then X is totally nonhomologous to zero in the fibration

$$X \longrightarrow X_K \longrightarrow B_K,$$

where $X_K = E_K \times_K X$ is the Borel construction. Write $H^*(X; \mathbb{F}_p) = P/I_0$, $P = \mathbb{F}_p[h_1, \dots, h_n]$, where the ideal $I_0 \subset P$ is generated by a set g_1, \dots, g_n of multiplicative generators of the invariant ring $R_G = H^*(B_G; \mathbb{F}_p) \subset H^*(B_H; \mathbb{F}_p)$. As can be shown in the same way as in the proof of [Hauschild 1986, Theorem 1.1], the equivariant cohomology $H_K^*(X; \mathbb{F}_p) = H^*(X_K; \mathbb{F}_p)$ is a graded algebra over $R = H^*(B_K; \mathbb{F}_p)$, which can be written as $H_K^*(X; \mathbb{F}_p) = P_R/I$, where $P_R = R \otimes P$ and I is an ideal generated by homogeneous elements of the form $1 \otimes g_j - r_j$, where the r_j are elements of the ideal $R_+ P_R$ generated by the augmentation ideal of R . On the ring $P_R = H^*(B_K; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(B_H; \mathbb{F}_p)$ there is the natural unstable \mathcal{A}^* -module structure and the equivariant cohomology is *Steenrod presentable* if I is stable under this \mathcal{A}^* -action inducing the given \mathcal{A}^* -action on the quotient. Moreover, since the isomorphism $H^*(X; \mathbb{F}_p) \cong H_K^*(X; \mathbb{F}_p)/H_+^*(B_K; \mathbb{F}_p)H_K^*(X; \mathbb{F}_p)$ is induced by the inclusion $i: X \rightarrow X_K$ of the fiber, the Steenrod presentation of

$H_K^*(X; \mathbb{F}_p)$ induces the Steenrod presentation of $H^*(X; \mathbb{F}_p)$. For more information on Steenrod powers acting on equivariant cohomology, see [Allday and Puppe 1993; Quillen 1971].

2. Steenrod powers and rational cohomology

Observe that $X = G/H$ is now the fiber of two fibrations, and that in both fibrations it is totally nonhomologous to zero. Consequently there is the canonical epimorphism $j^*: H^*(B_H; \mathbb{F}_p) \rightarrow H^*(X; \mathbb{F}_p)$ induced by the inclusion $j: X \rightarrow B_H$ of the fiber. Moreover, let i^* be induced by the inclusion $i: X \rightarrow X_K$ of the fiber in the Borel fibration. Both maps commute of course with the respective \mathcal{A}^* -module structures.

Observation 1. *The equivariant cohomology $H_K^*(X; \mathbb{F}_p)$ is Steenrod presentable if and only if there is a homomorphism $J: H^*(B_H; \mathbb{F}_p) \rightarrow H_K^*(X; \mathbb{F}_p)$ making the following diagram commute:*

$$\begin{array}{ccc}
 & H_K^*(X; \mathbb{F}_p) & \\
 J \nearrow & & \downarrow i^* \\
 H^*(B_H; \mathbb{F}_p) & \xrightarrow{j^*} & H^*(X; \mathbb{F}_p)
 \end{array}$$

Proof. Let $\pi: X_K \rightarrow B_K$ be the projection in the Borel fibration, and then consider the homomorphism $\pi^* \otimes J: H^*(B_K; \mathbb{F}_p) \otimes H^*(B_H; \mathbb{F}_p) \rightarrow H_K^*(X; \mathbb{F}_p)$. This map is surjective and commutes with the respective \mathcal{A}^* -actions. Let $I = \text{Ker}(\pi^* \otimes J)$; then $H_K^*(X; \mathbb{F}_p) = (H^*(B_K; \mathbb{F}_p) \otimes H^*(B_H; \mathbb{F}_p))/I$ is a Steenrod presentation.

On the other hand, given a Steenrod presentation

$$H_K^*(X; \mathbb{F}_p) = (H^*(B_K; \mathbb{F}_p) \otimes H^*(B_H; \mathbb{F}_p))/I,$$

and $J: H^*(B_H; \mathbb{F}_p) \rightarrow H_K^*(X; \mathbb{F}_p)$ given by

$$H^*(B_H; \mathbb{F}_p) \ni \zeta_H \mapsto 1 \otimes \zeta_H \pmod I,$$

then J commutes with the \mathcal{A}^* -actions and $i^* \circ J = j^*$. □

Let X, X' be spaces such that the rational cohomology rings $H^*(X; \mathbb{Q})$ and $H^*(X'; \mathbb{Q})$ are finitely generated as graded \mathbb{Q} -algebras. Then we have to define what it means for a homomorphism $\theta: H^*(X; \mathbb{Q}) \leftarrow H^*(X'; \mathbb{Q})$ to commute with Steenrod powers for almost all primes p . Let $y_1, \dots, y_m \in H^*(X'; \mathbb{Q})$ be a set of multiplicative generators; similarly, let $x_1, \dots, x_n \in H^*(X; \mathbb{Q})$ be a set of multiplicative generators. Then $\theta(y_i) = p_i(x_1, \dots, x_n) \in H^*(X; \mathbb{Q})$ are polynomials. Let Prime_θ be the (finite) subset of primes which appear as divisors of the denominators of the coefficients of the p_i . Then for all $p \notin \text{Prime}_\theta$ there are unique

homomorphisms $\theta_p, \bar{\theta}_p$ which make the following diagram commute [Adams and Mahmud 1976]:

$$\begin{array}{ccc}
 H^*(X; \mathbb{Q}) & \xleftarrow{\theta} & H^*(X'; \mathbb{Q}) \\
 \uparrow & & \uparrow \\
 H^*(X; \mathbb{Z}_{(p)}) & \xleftarrow{\theta_p} & H^*(X'; \mathbb{Z}_{(p)}) \\
 \downarrow & & \downarrow \\
 H^*(X; \mathbb{F}_p) & \xleftarrow{\bar{\theta}_p} & H^*(X'; \mathbb{F}_p)
 \end{array}$$

Here the vertical maps are induced by the canonical maps $\mathbb{Z}_{(p)} \rightarrow \mathbb{Q}$ and $\mathbb{Z}_{(p)} \rightarrow \mathbb{F}_p$ respectively. We say that θ commutes with the Steenrod powers for almost all primes p if the $\bar{\theta}_p$ commute with Steenrod powers for $p \notin \text{Prime}_\theta$.

Definition 2. Let K be a compact Lie group acting on $X = G/H$. Then we say that the rational equivariant cohomology $H_K^*(X; \mathbb{Q})$ is Steenrod presentable if there is a lifting J of the edge homomorphism j^*

$$\begin{array}{ccc}
 & & H_K^*(X; \mathbb{Q}) \\
 & \nearrow J & \downarrow i^* \\
 H^*(B_H; \mathbb{Q}) & \xrightarrow{j^*} & H^*(X; \mathbb{Q})
 \end{array}$$

such that $\bar{J}_p: H^*(B_H; \mathbb{F}_p) \rightarrow H_K^*(X; \mathbb{F}_p)$ commutes with Steenrod powers for almost all p .

A homogeneous space G/H such that $\text{rank } G = \text{rank } H$ is Kähler if and only if $H = Z(K)$ is the centralizer of a (not necessarily maximal) torus K , or, equivalently, if H is conjugate to an isotropy group of the adjoint representation [Besse 1987, Chapter 8].

Here is the main theorem of this article.

Theorem 3. Let G be a simple compact connected Lie group and $H \subset G$ be a closed connected subgroup of maximal rank such that $X = G/H$ is Kähler and let G act topologically on $X = G/H$. Then the following statements are equivalent.

- (i) The equivariant cohomology $H_G^*(X; \mathbb{Q})$ is Steenrod presentable.
- (ii) The group G acts transitively on X with an isotropy group conjugate to K , where K is a maximal rank subgroup of G isomorphic to H by an automorphism of G which is inner with the possible exception of the even Spin groups.
- (iii) There is an isomorphism $H_G^*(X; \mathbb{Q}) \cong R_H$ as R_G -algebras.

As a corollary, we recover an earlier result from [Hauschild 1985]. (See also [Hauschild 1986] and the introduction of [Hauschild 2006], where the uniqueness problem for locally smooth $SU(n + 1)$ -actions on $SU(n + 1)/S(U(n - 1) \times U(2))$ is considered.)

Theorem 4. *Let G be a simple compact connected Lie group, and let $T \subset G$ be a maximal torus. Let G act nontrivially on $X = G/T$ via φ . Then up to conjugacy, φ is the standard transitive G -action on X with isotropy group conjugate to T .*

Proof (for a proof using obstruction theory, see the Appendix). Write $H^*(B_T; \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_n]$, $\deg x_i = 2$. Let $R_G = H^*(B_G; \mathbb{Q})$ and write

$$H_G^*(X; \mathbb{Q}) = \frac{R_G[X_1, \dots, X_n]}{I}, \quad \deg X_i = 2.$$

Define $J(x_i) = \bar{X}_i$, where the \bar{X}_i is the class of X_i . Let p be a prime such that J_p and \bar{J}_p are defined.

The values of the Steenrod powers $\mathcal{P}^k(x_i)$ and $\mathcal{P}^k(\bar{X}_i)$ are completely determined by the instability conditions, that is, we have $\mathcal{P}^k(x_i) = x_i^p$ for $k = 1$ and $\mathcal{P}^k(x_i) = 0$ for $k > 1$. The same holds in $H_G^*(X; \mathbb{F}_p)$; that is, $\mathcal{P}^k(\bar{X}_i) = \bar{X}_i^p$ for $k = 1$ and $\mathcal{P}^k(\bar{X}_i) = 0$ for $k > 1$. It follows that $\mathcal{P}^k \bar{J}_p(x_i) = \bar{J}_p \mathcal{P}^k(x_i)$ for all i . By simple induction using the Cartan rule, one gets the relation $\mathcal{P}^k \circ \bar{J}_p = \bar{J}_p \circ \mathcal{P}^k$ for all $k \geq 0$ and almost all primes p . So, the equivariant cohomology is Steenrod presentable and the result follows from Theorem 3. \square

3. A proof of the main theorem

The following definitions synthesize certain cohomological properties of symplectic manifolds and are taken from the paper [Allday 1998]. We consider cohomology with coefficients in a field \mathbb{Q} , with $\text{char } \mathbb{Q} = 0$. As a coefficient field of cohomology, the symbol \mathbb{Q} will be omitted in this paragraph.

Definition 5. Let X be a Poincaré duality space over \mathbb{Q} with formal dimension $2n$.

- (i) The space X is said to be c -symplectic (that is, cohomologically symplectic) if there is $w \in H^2(X)$ such that $w^n \neq 0$.
- (ii) If X is c -symplectic, for $0 \leq j \leq n$, consider the map $w^j : H^{n-j}(X) \rightarrow H^{n+j}(X)$, defined as $a \mapsto w^j a$, for all $a \in H^{n-j}(X)$. Then X is said to satisfy the hard Lefschetz condition if w^j is an isomorphism for all j . In this case X is also said to be c -Kähler.

Let X be a c -symplectic space with $w \in H^2(X)$ as in the definition above. Let G be a compact connected Lie group acting on X . Then $g^*(w) = w$ for all $g \in G$. In this way any action of a compact connected Lie group on a c -symplectic space is considered to be a cohomologically symplectic action.

Definition 6. Let X be a c -symplectic space with c -symplectic class $w \in H^2(X)$. Let a torus G act on X . Then the action is said to be cohomologically Hamiltonian (c -Hamiltonian) if $w \in \text{Im}\{i^*: H_G^2(X) \rightarrow H^2(X)\}$, where $i: X \rightarrow X_G$ is the inclusion of the fiber in the bundle $X_G \rightarrow B_G$.

The main reason we have restricted ourselves to homogeneous spaces G/H with the Kähler property is the following result, which can be considered a generalization of a theorem of Atiyah [1983] (see also [Guillemin and Sternberg 1982; Audin 1991, Corollary 4.2.3]). For the definition of uniformity see [Allday and Puppe 1993, Definition 3.6.17]. For other consequences of the Kähler property, see [Allday et al. 2002].

Theorem 7 [Allday 1998]. *Let the r -torus $G = T^r$ act on a closed c -symplectic manifold X in an effective, uniform, c -Hamiltonian way. Then X^G has at least $r + 1$ connected components.*

The conditions of the theorem are always satisfied if X is totally nonhomologous to zero in the Borel fibration [Allday and Puppe 1993]. Let G be a torus and suppose G is acting on a c -symplectic manifold X with vanishing odd cohomology. As we have seen before, the equivariant cohomology can be written as $H_G^*(X) = R_G[h_1, \dots, h_n]/I$ where $R_G = H^*(B_G)$ and the h_1, \dots, h_n is a system of homogeneous multiplicative generators, I the defining ideal. Let $X^G = F_1 + F_2 + \dots + F_s$ be the decomposition of the fixed space X^G into its connected components. Then for every α , $1 \leq \alpha \leq s$, we choose a point $p_\alpha \in F_\alpha$ and define a prime ideal P_α as the kernel of the composed homomorphism $R_G[h_1, \dots, h_n] \rightarrow H_G^*(X) \rightarrow H_G^*(p_\alpha) \cong R_G$. Here the first homomorphism is the natural projection and the second is given by restricting equivariant cohomology classes to $E_G \times_G \{p_\alpha\}$. Then the radical of I is given by $\sqrt{I} = \bigcap_\alpha P_\alpha$. Moreover there is a natural bijection between the primary components of the ideal I and the connected components of X^G . For more details on these standard facts on equivariant cohomology see [Allday and Puppe 1993; Hsiang 1975]. The following lemma is an immediate consequence of the result of Allday.

Lemma 8. *Let the r -torus $G = T^r$ act on a closed c -symplectic manifold X with vanishing odd cohomology. Suppose G is acting on X in an effective, uniform, c -Hamiltonian way. Then there exists a connected component F of X^G such that the prime ideal $P \subset R[h_1, \dots, h_n]$ belonging to F is of the kind $P = (h_1 - \beta_1, \dots, h_n - \beta_n)$ with $\beta_i \in R^{\deg h_i}$ and some $\beta_i \neq 0$.*

Proof of the main theorem. (i) \Rightarrow (ii): Let $R_G = H^*(B_G)$ and let $R_H = H^*(B_H) \cong \mathbb{Q}[h_1, \dots, h_n]$. Suppose $H_G^*(X) = (R_G \otimes_{\mathbb{Q}} R_H)/I_G$ to be a Steenrod presentation. Let $T \subset G$ be a maximal torus; then the equivariant cohomology of the induced T -action is given by $H_T^*(X) \cong H_G^*(X) \otimes_{R_G} R_T$. Let $I_T \subset R_G \otimes_{\mathbb{Q}} R_T$ be the ideal

generated by I_G , that is, $I_T = I_G \cdot (R_T \otimes_{\mathbb{Q}} R_H)$; then $H_T^*(X) \cong R_T[h_1, \dots, h_n]/I_T$. By the previous lemma there is a connected component $F \subset X^T$ of the fixed set X^T such that the corresponding prime ideal has the form $P = (h_1 - \beta_1, \dots, h_n - \beta_n)$ with $(\beta_1, \dots, \beta_n) \neq 0$. In particular, the restriction homomorphism $H_T^*(X) \rightarrow H_T^*({p}) \cong R_T$, $p \in F$ is nontrivial. Let $G_p \subset G$ be the isotropy group of p . It follows from the commutativity of the diagram

$$\begin{array}{ccc} H_T^*(X) & \longrightarrow & H_T^*({p}) \cong R_T \\ \cup \uparrow & & \cup \uparrow \\ H_G^*(X) & \xrightarrow{\text{res}_p} & H_G^*(G(p)) \cong R_{G_p} \end{array}$$

that the restriction homomorphism

$$\text{res}_p: H_G^*(X) \rightarrow H_G^*(G(p)) \cong R_{G_p}$$

must also be nontrivial. Let $U = G_p^o$ be the connected component of the unit element in G_p , and let $\eta: H^*(B_{G_p}) \rightarrow H^*(B_U)$ be the homomorphism induced by the inclusion $U \subset G_p$. Then consider the composition $\theta = \eta \circ \text{res}_p \circ J$

$$\theta: H^*(B_H) \xrightarrow{J} H_G^*(X) \xrightarrow{\text{res}_p} H^*(B_{G_p}) \xrightarrow{\eta} H^*(B_U).$$

It follows from the construction and the hypothesis that θ commutes with the Steenrod powers in \mathcal{A}^* for almost all primes p . Let LT be the Lie algebra of the maximal torus T . Let $\Sigma \subset LT$ be the kernel of the projection $LT \rightarrow T$. After [Adams and Mahmud 1976, Theorem 1.5] there is an \mathbb{R} -linear map $\phi: LT \rightarrow LT$ carrying $\Sigma \otimes \mathbb{Q}$ into $\Sigma \otimes \mathbb{Q}$ such that the following diagram is commutative.

$$\begin{array}{ccc} H^*(B_H) & \xrightarrow{\theta} & H^*(B_U) \\ \downarrow & & \downarrow \\ H^*(B_T) & \xrightarrow{\phi^*} & H^*(B_T) \end{array}$$

Here ϕ^* is the graded ring homomorphism induced by the linear map ϕ . The existence of this map is a consequence of [Adams and Mahmud 1976, Lemma 1.2]. The vertical maps are the homomorphisms induced by the standard fibrations $B_T \rightarrow B_H$ and $B_T \rightarrow B_U$. It follows from our assumption that θ is nontrivial, which implies that ϕ^* is also nontrivial. Observe that the map θ induces exactly the homomorphism $\bar{\theta}: H^*(G/H) \rightarrow H^*(G/U)$ induced by the map $G/U \rightarrow G/G_p \cong$

$G(p) \subset X = G/H$. This means that we have a commutative diagram

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\bar{\theta}} & H^*(G/U) \\ \uparrow & & \uparrow \\ H^*(B_H) & \xrightarrow{\theta} & H^*(B_U) \end{array}$$

where the vertical maps are the edge homomorphisms for the fibrations $B_H \rightarrow B_G$ and $B_U \rightarrow B_G$, respectively. It follows that θ sends the ideal

$$H_+^*(B_G) \cdot H^*(B_H) \subset H^*(B_H)$$

generated by the invariants of the Weyl group in $H^*(B_H)$ into the ideal

$$H_+^*(B_G) \cdot H^*(B_U) \subset H^*(B_U)$$

generated by the same invariants in $H^*(B_U)$. Then ϕ^* sends the ideal

$$H_+^*(B_G) \cdot H^*(B_T) \subset H^*(B_T)$$

into the ideal

$$H_+^*(B_G) \cdot H^*(B_T) \subset H^*(B_T),$$

therefore inducing a graded and nontrivial homomorphism

$$\bar{\phi}^* : H^*(G/T) \longrightarrow H^*(G/T).$$

Since G is a simple Lie group we can apply [Hauschild 1985, Lemma 4.1]. Therefore $\bar{\phi}^*$ must be a surjective map and consequently must be an isomorphism. Now the commutative diagrams above induce a commutative diagram

$$\begin{array}{ccc} H^*(G/H) & \xrightarrow{\bar{\theta}} & H^*(G/U) \\ \downarrow & & \downarrow \\ H^*(G/T) & \xrightarrow{\bar{\phi}^*} & H^*(G/T) \end{array}$$

where the vertical maps are the respective inclusions of invariants under the Weyl groups WH , WU respectively. It follows that the homomorphism $\bar{\theta}$ must be injective which implies dimensions $cd_{\mathbb{Q}}(X) \leq cd_{\mathbb{Q}}(G/U)$ for the respective rational cohomology. But G/H and G/U are closed oriented manifolds and therefore $\dim X \leq \dim G/U$, which implies $\dim X = \dim G(p)$. It follows that $X = G/G_p$, that is, the action is transitive. Now X is 1-connected and therefore G_p must be connected, that is, $G_p = G_p^o = U$. It follows that $G/H = G/U$ and $\bar{\theta}$ is an isomorphism. By a theorem of Papadima [1986], the isomorphism $\bar{\phi}^*$ is induced by an automorphism of the root system of G . This implies that the root systems

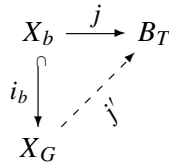
of the maximal rank subgroups H and U are conjugate by such an automorphism, and consequently, the groups H and U are conjugate by an automorphism which is inner with the possible exception of the Spin groups.

(ii) \Rightarrow (i): We have $X_G = E_G \times_G G/U \cong E_G/U = B_U$. But $H \cong U$ and so $X_G \cong B_H$ and therefore $H_G^*(X) \cong R_H$ as R_G -algebras.

(iii) \Rightarrow (i): Take $J = \text{Id}: R_H \rightarrow R_H$. □

Appendix

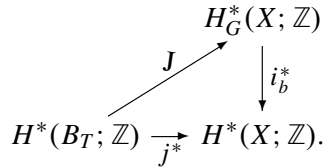
Proof of Theorem 4 using obstruction theory. Let $\pi : X_G \rightarrow B_G$ be the projection, let $b \in B_G$, and let $X_b = \pi^{-1}(b) \subset X_G$ be the fiber over b . Let $i_b : X_b \rightarrow X_G$ be the corresponding inclusion. Then consider the extension problem



The obstruction to extend the inclusion $j : X_b \rightarrow B_T$ to a map $j' : X_G \rightarrow B_T$ is to be found in the group $H^3(X_G, X_b; \pi_2(B_T))$. Consider the following piece of the long exact cohomology sequence of the pair (X_G, X_b) .

$$H^2(X_G; \mathbb{Z}) \rightarrow H^2(X_b; \mathbb{Z}) \rightarrow H^3(X_G, X_b; \mathbb{Z}) \rightarrow H^3(X_G, \mathbb{Z}) \rightarrow \dots$$

Now the first arrow, induced by the inclusion of the fiber, is surjective whereas $H^3(X_G; \mathbb{Z}) = 0$. It follows $H^3(X_G, X_b; \mathbb{Z}) = 0$ and so $H^3(X_G, X_b; \mathbb{Z}^n) = 0$. We thus have a lifting $J = j'^*$ which gives rise to the commutative diagram



By the definition of J as a map induced geometrically, we conclude that $H_G^*(X; \mathbb{Q})$ is Steenrod presentable. Using the equivalence between (i) and (ii) in Theorem 3 and the standard fact that two maximal tori are conjugate, the result follows. □

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