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By using the reflecting diffusion process and a conformal change of metric, a generalized maximum principle is established for (unbounded) time-space functions on a class of noncompact Riemannian manifolds with (nonconvex) boundary. As applications, Li-Yau-type gradient and Harnack inequalities are derived for the Neumann semigroup on a class of noncompact manifolds with (nonconvex) boundary. These generalize some previous ones obtained for the Neumann semigroup on compact manifolds with boundary. As a byproduct, the gradient inequality for the Neumann semigroup derived by Hsu on a compact manifold with boundary is confirmed on these noncompact manifolds.

1. Introduction

Suppose M is a d -dimensional connected complete Riemannian manifold, and let $L = \Delta + Z$, where Z is a C^1 vector field satisfying the curvature-dimension condition of Bakry and Émery [1984] given by

$$(1-1) \quad \Gamma_2(f, f) := \frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle \geq \frac{(Lf)^2}{m} - K|\nabla f|^2 \quad \text{for } f \in C^\infty(M)$$

for some constants $K \geq 0$ and $m > d$. By [Qian 1998, page 138], this condition is equivalent to

$$(1-2) \quad \text{Ric} - \nabla Z - \frac{Z \otimes Z}{m-d} \geq -K.$$

When $Z = 0$ and M is either without boundary or compact and with a convex boundary ∂M , Li and Yau [1986] found a now-famous gradient estimate for the (Neumann) semigroup P_t generated by L :

$$(1-3) \quad |\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \leq \frac{d\alpha^2}{2t} + \frac{d\alpha^2 K}{4(\alpha-1)} \quad \text{for } t > 0 \text{ and } \alpha > 1$$

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for all positive $f \in C_b(M)$. We note that in [Li and Yau 1986] the second term on the right side of (1-3) is $d\alpha^2 K/(\sqrt{2}(\alpha - 1))$, but $\sqrt{2}$ here can be replaced by 4 according to a refined calculation; see for example [Davies 1989].

As an application, (1-3) implies a parabolic Harnack inequality for P_t :

$$(1-4) \quad P_t f(x) \leq \left(\frac{t+s}{t}\right)^{d\alpha/2} (P_{t+s} f(y)) \exp\left(\frac{\alpha\rho(x,y)^2}{4s} + \frac{\alpha K ds}{4(\alpha-1)}\right)$$

for $t > 0$ and $x, y \in M$,

where $\alpha > 1$ and $f \in C_b(M)$ is positive. From this Harnack inequality, one obtains Gaussian-type heat kernel bounds for P_t ; see [Li and Yau 1986; Davies 1989].

The gradient estimate (1-3) has been extended and improved in several papers. See for example [Bakry and Qian 1999] for an improved version for $\alpha = 1$ with $Z \neq 0$ and $\partial M = \emptyset$, and see [Wang 1997] for an extension to a compact manifold with nonconvex boundary. The aim of this paper is to investigate the gradient and Harnack inequalities for P_t on noncompact manifolds with (nonconvex) boundary.

Recall that the key step of Li and Yau's argument for the gradient estimate (1-3) is to apply the maximum principle to the reference function

$$G(t, x) := t(|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f)(x) \quad \text{for } t \in [0, T] \text{ and } x \in M.$$

When M is compact without boundary, the maximum principle says that for any smooth function G on $[0, T] \times M$ with $G(0, \cdot) \leq 0$ and $\sup G > 0$, there exists a maximal point of G at which $\nabla G = 0$, $\partial_t G \geq 0$, and $\Delta G \leq 0$. When M is compact with a convex boundary, the same assertion holds for the above specified function G , as observed in [Li and Yau 1986, proof of Theorem 1.1]. In [1997], J. Wang extended this maximum principle on a compact manifold with nonconvex boundary by taking

$$G(t, x) = t(\phi|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f)(x) \quad \text{for } t \in [0, T] \text{ and } x \in M$$

for a nice function ϕ compensating the concavity of the boundary.

As for a noncompact manifold without boundary, Li and Yau [1986] established the gradient estimate by applying the maximal principle to a sequence of functions with compact support that approximate the original function G . An alternative is to apply directly the following generalized maximum principle:

Lemma 1.1 [Yau 1975]. *For any bounded smooth function G on $[0, T] \times M$ with $G(0, \cdot) \leq 0$ and $\sup G > 0$, there exists a sequence $\{(t_n, x_n)\}_{n \geq 1} \subset [0, T] \times M$ such that*

- (i) $0 < G(t_n, x_n) \uparrow \sup G$ as $n \uparrow \infty$, and
- (ii) for any $n \geq 1$,

$$LG(t_n, x_n) \leq 1/n, \quad |\nabla G(t_n, \cdot)(x_n)| \leq 1/n, \quad \partial_t G(t_n, x_n) \geq 0.$$

To apply this generalized maximal principle for the gradient estimate, one has to first confirm the boundedness of $G(t, \cdot) := t(|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f)$ on $[0, T] \times M$ for $T > 0$.

Since the boundedness of this type of reference function is unknown when M is noncompact with a nonconvex boundary, we shall establish a generalized maximum principle on a class of noncompact manifolds with boundary for not necessarily bounded functions. Applying this principle to a suitable reference function G , we derive the Li–Yau-type gradient and Harnack inequalities for Neumann semi-groups. To establish such a maximum principle, we adopt a localization argument so that the classical maximum principle can be applied.

For M noncompact without boundary, Li and Yau [1986] used such a localization argument to apply the maximal principle to functions with compact support; they then passed to the desired global estimate by taking a limit. To do this, they constructed cut-off functions using ρ_o , the Riemannian distance function to a fixed point $o \in M$. It turns out that this argument works also when ∂M is convex; see Section 2.1. For the nonconvex case, we will use the conformal change of metric introduced in [Wang 2007] to make a nonconvex boundary convex; see Section 2.2.

Assumption A. The manifold M is connected and complete with boundary ∂M and such that either

- (1) ∂M is convex, or
- (2) the second fundamental form of ∂M is bounded, the sectional curvature of M is bounded from above, and the injectivity radius $i_{\partial M}$ of ∂M is positive.

Recall that the Riemannian distance function $\rho_{\partial M}$ to the boundary is smooth on the set $\{\rho_{\partial M} < i_{\partial M}\}$.

Let N be the inward unit normal vector field on ∂M . The second fundamental form of ∂M is

$$\text{II}(X, Y) = -\langle \nabla_X N, Y \rangle \quad \text{for } X, Y \in T\partial M.$$

The boundary ∂M is called convex if $\text{II} \geq 0$. We are now ready to state our generalized maximal principle for possibly unbounded functions.

Theorem 1.2. *Let M satisfy A, and let L satisfy (1-2). Let $T > 0$, and let G be a smooth function on $[0, T] \times M$ such that $NG|_{\partial M} \geq 0$, $G(0, \cdot) \leq 0$ and $\sup G > 0$. Then for any $\varepsilon > 0$, there exists a sequence $\{(t_n, x_n)\}_{n \geq 1} \subset (0, T] \times M$ such that Lemma 1.1(i) holds and for any $n \geq 1$*

$$LG(t_n, x_n) \leq \frac{G(t_n, x_n)^{1+\varepsilon}}{n}, \quad |\nabla G(t_n, x_n)| \leq \frac{G(t_n, x_n)^{1+\varepsilon}}{n},$$

$$\partial_t G(t_n, x_n) \geq 0.$$

Applying Theorem 1.2 to a proper choice of function G , we will derive the Li–Yau-type gradient estimate (1-5). We shall prove that the reflecting diffusion process X_t generated by L on M is non explosive, so that the corresponding Neumann semigroup P_t can be formulated as

$$P_t f(x) = E^x f(X_t) \quad \text{for } t \geq 0, x \in M, \text{ and } f \in C_b(M),$$

where E^x is the expectation taken for $X_0 = x$.

Theorem 1.3. *Let M satisfy A, and suppose L satisfies (1-2) with $\|Z\|_\infty < \infty$. Then the reflecting L -diffusion process on M is nonexplosive and the corresponding Neumann semigroup P_t satisfies these assertions:*

- (i) *If ∂M is convex, then (1-3) holds with m in place of d .*
- (ii) *If ∂M is nonconvex with $\Pi \geq -\sigma$ for some $\sigma > 0$, then for any bounded $\phi \in C^\infty(M)$ with $\phi \geq 1$ and $N \log \phi|_{\partial M} \geq 2\sigma$, the gradient inequality*

$$(1-5) \quad |\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \leq \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K(\phi, \varepsilon, \alpha)}{4(\alpha - \|\phi\|_\infty)}$$

holds for all positive $f \in C_b(M)$, $\alpha > \|\phi\|_\infty$, $t > 0$, $\varepsilon \in (0, 1)$ and

$K(\phi, \varepsilon, \alpha) :=$

$$\frac{1+\varepsilon}{1-\varepsilon} \left(K + \frac{1}{\varepsilon} \|\nabla \log \phi\|_\infty^2 + \frac{1}{2} \sup(-\phi^{-1} L\phi) + \frac{m\alpha^2 \|\nabla \log \phi\|_\infty^2 (1+\varepsilon)}{8(\alpha - \|\phi\|_\infty)^2 \varepsilon (1-\varepsilon)} \right).$$

We emphasize that the results in Theorem 1.3 are new for noncompact manifolds with boundary. When M is compact with a convex boundary, the first assertion was proved in [Li and Yau 1986] by using the classical maximum principle on compact manifolds, while when M is compact with a nonconvex boundary, an inequality similar to (1-5) was proved in [Wang 1997] by using the “interior rolling R -ball” condition.

These two theorems will be proved in Sections 2 and 3. By a standard argument due to Li and Yau [1986], the gradient estimate (1-5) implies a Harnack inequality. Let $\rho(x, y)$ be the Riemannian distance between $x, y \in M$, that is, the infimum of the length of all smooth curves in M that link x and y .

Corollary 1.4. *In the situation of Theorem 1.3 the Neumann semigroup P_t satisfies*

$$(1-6) \quad P_t f(x) \leq \left(\frac{t+s}{t} \right)^{m(1+\varepsilon)\alpha/2(1-\varepsilon)} (P_{t+s} f(y)) \exp \left(\frac{\alpha \rho(x, y)^2}{4s} + \frac{\alpha m K(\phi, \varepsilon, \alpha) s}{4(\alpha - \|\phi\|_\infty)} \right)$$

for all positive $f \in C_b(M)$, $t, \varepsilon \in (0, 1)$, $\alpha > \|\phi\|_\infty$ and $x, y \in M$. In particular, if ∂M is convex, then (1-4) holds with m in place of d and for all $\alpha > 1$.

To derive explicit inequalities for the nonconvex case, we shall take a specific choice of ϕ as in [Wang 2007]. Let $i_{\partial M}$ be the injectivity radius of ∂M , and let $\rho_{\partial M}$ be the Riemannian distance to the boundary. We shall take $\phi = \varphi \circ \rho_{\partial M}$ for a nice reference function φ on $[0, \infty)$. More precisely, let the sectional curvature satisfy $\text{Sect}_M \leq k$ and $-\sigma \leq \text{II} \leq \gamma$ for some $k, \sigma, \gamma > 0$. Let

$$h(s) = \cos(\sqrt{k} s) - (\gamma/\sqrt{k}) \sin(\sqrt{k} s) \quad \text{for } s \geq 0.$$

Then h is the unique solution to the differential equation $h'' + kh = 0$ with boundary conditions $h(0) = 1$ and $h'(0) = -\gamma$. By the Laplacian comparison theorem for $\rho_{\partial M}$ (see [Kasue 1984, Theorem 0.3] or [Wang 2007]),

$$(1-7) \quad \Delta \rho_{\partial M} \geq \frac{(d-1)h'}{h}(\rho_{\partial M}) \quad \text{and} \quad \rho_{\partial M} < i_{\partial M} \wedge h^{(-1)}(0),$$

where $h^{(-1)}(0) = (1/\sqrt{k}) \arcsin(\sqrt{k}/\sqrt{k+\gamma^2})$ is the first zero point of h . Fix a positive number $r_0 \leq i_{\partial M} \wedge h^{(-1)}(0)$, and let

$$\delta = \frac{2\sigma(1-h(r_0))^{d-1}}{\int_0^{r_0} (h(s)-h(r_0))^{d-1} ds},$$

$$\varphi(r) = 1 + \delta \int_0^r (h(s)-h(r_0))^{1-d} ds \int_{s \wedge r_0}^{r_0} (h(u)-h(r_0))^{d-1} du.$$

It is easy to see that $\varphi \circ \rho_{\partial M}$ is differentiable with a Lipschitzian gradient. By a simple approximation argument, we may apply Theorem 1.3 and Corollary 1.4 to $\phi = \varphi \circ \rho_{\partial M}$; see [Wang 2007, page 1436].

Obviously, (1-7) and $N = \nabla \rho_{\partial M}$ imply

$$\Delta \varphi \circ \rho_{\partial M} \geq -\delta \quad \text{and} \quad N \log \varphi \circ \rho_{\partial M}|_{\partial M} = \varphi'(0)/\varphi(0) = 2\sigma.$$

Moreover, by [Wang 2007, (20)] we have

$$\delta \leq 2\sigma dr_0^{-1} \quad \text{and} \quad \varphi(r_0) \leq 1 + \sigma dr_0.$$

Thus, for $\phi := \varphi \circ \rho_{\partial M}$ we have

$$-\phi^{-1} L\phi \leq 2\sigma dr_0^{-1} + 2\sigma \|Z\|_\infty, \quad \|\nabla \log \phi\|_\infty^2 \leq 4\sigma^2,$$

$$\|\phi\|_\infty \leq \varphi(r_0) \leq 1 + \sigma dr_0.$$

Combining these with Theorem 1.3 and Corollary 1.4, we obtain these explicit inequalities on a class of nonconvex and noncompact manifolds:

Corollary 1.5. *Let $i_{\partial M} > 0$, and suppose $\gamma \geq \text{II} \geq -\sigma$ and $\text{Sect}_M \leq k$ for some $\gamma, \sigma, k > 0$. If (1-2) holds and $\|Z\|_\infty < \infty$, then for any positive number*

$$r_0 \leq \min\{i_{\partial M}, (1/\sqrt{k}) \arcsin(\sqrt{k}/\sqrt{k+\gamma^2})\},$$

the inequalities

$$|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \leq \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K_\varepsilon}{4(\alpha-1-\sigma dr_0)}$$

and

$$P_t f(x) \leq \left(\frac{t+s}{t}\right)^{m(1+\varepsilon)\alpha/2(1-\varepsilon)} (P_{t+s} f(y)) \exp\left(\frac{\alpha\rho(x,y)^2}{4s} + \frac{m\alpha K_\varepsilon s}{4(\alpha-1-\sigma dr_0)}\right)$$

for $x, y \in M$

hold for all positive $f \in C_b(M)$, $t > 0$, $\varepsilon \in (0, 1)$, $\alpha > 1 + \sigma dr_0$, and

$$K_\varepsilon = \frac{1+\varepsilon}{1-\varepsilon} \left(K + \frac{4\sigma^2}{\varepsilon} + \frac{\sigma d}{r_0} + \sigma \|Z\|_\infty + \frac{m\alpha^2\sigma^2(1+\varepsilon)}{2(\alpha-1-\sigma dr_0)^2\varepsilon(1-\varepsilon)} \right).$$

Combining our gradient estimate with an approximation and a probabilistic argument, we can derive the gradient estimate (1-9) for a class of noncompact manifolds:

Theorem 1.6. *Let M satisfy A, and let L satisfy (1-2) with $\|Z\|_\infty < \infty$. Let κ_1 and κ_2 be positive elements of $C_b(M)$ such that*

$$(1-8) \quad \text{Ric} - \nabla Z \geq -\kappa_1 \quad \text{and} \quad \text{II} \geq -\kappa_2$$

hold on M and ∂M , respectively. Then

$$(1-9) \quad |\nabla P_t f|(x) \leq \mathbb{E}^x \left(|\nabla f|(X_t) \exp\left(\int_0^t \kappa_1(X_s) ds + \int_0^t \kappa_2(X_s) dl_s\right) \right)$$

holds for all $f \in C_b^1(M)$, $t > 0$, and $x \in M$.

Inequality (1-9) was first derived by Hsu [2002] on a compact manifold with boundary. In [2002, Theorem 3.7], Hsu applied the Itô formula to $F(U_t, T-t) := U_t^{-1} \nabla P_{T-t} f(X_t)$, where U_t is the horizontal lift of X_t on the frame bundle $O(M)$. Since M is compact, the (local) martingale part of this process is a real martingale (it may not be for noncompact M). Then the desired gradient estimate followed immediately from [2002, Corollary 3.6]. In Section 4, we will prove the boundedness of $\nabla P_{(\cdot)} f$ on $[0, T] \times M$ for any $T > 0$ and $f \in C_b^1(M)$, which leads to a simple proof of (1-9) for a class of noncompact manifolds.

2. Proof of Theorem 1.2

We consider the convex case and pass to the nonconvex case using the conformal change of metric constructed in [Wang 2007]. Without loss of generality, we may assume that $\sup G := \sup_{[0, T] \times M} G > 1$. (Otherwise, we simply replace G by mG for a sufficiently large $m > 0$.)

2.1. Convex ∂M . Fix $o \in M$, and let ρ_o be the Riemannian distance to the point o . Since ∂M is convex, there exists a minimal geodesic in M of length $\rho(x, y)$ that links any x and y in M ; see for example [Wang 2005a, Proposition 2.1.5]. So, by (1-2) and a comparison theorem (see [Qian 1998])

$$L\rho_o \leq \sqrt{K(m-1)} \coth(\sqrt{K/(m-1)}\rho_o)$$

holds outside $\{o\} \cup \text{cut}(o)$, where $\text{cut}(o)$ is the cut locus of o . In the sequel, we will set $L\rho_o = 0$ on $\text{cut}(o)$ so that this implies

$$(2-1) \quad L\sqrt{1 + \rho_o^2} \leq c_1 \quad \text{on } M$$

for some constant $c_1 > 0$.

Let $h \in C_0^\infty([0, \infty))$ be decreasing such that

$$h(r) = \begin{cases} 1 & \text{if } r \leq 1, \\ \exp(-(3-r)^{-1}) & \text{if } r \in [2, 3), \\ 0 & \text{if } r \geq 3. \end{cases}$$

Obviously, for any $\varepsilon > 0$ we have

$$(2-2) \quad \sup_{[0, \infty)} \{|h^{\varepsilon-1}h''| + |h^{\varepsilon-1}h'|\} < \infty.$$

Let $W = \sqrt{1 + \rho_o^2}$, and take $\varphi_n = h(W/n)$ for $n \geq 1$. Then

$$(2-3) \quad \{\varphi_n = 1\} \uparrow M \quad \text{as } n \uparrow \infty.$$

So, according to (2-1) and (2-2),

$$(2-4) \quad \begin{aligned} |\nabla \log \varphi_n| &\leq \frac{c}{n\varphi_n^\varepsilon}, \\ \varphi_n^{-1}L\varphi_n &= \frac{h'(W/n)}{nh(W/n)}LW + \frac{h''(W/n)}{n^2h(W/n)}|\nabla W|^2 \geq -\frac{c}{n\varphi_n^\varepsilon} \end{aligned}$$

holds for some constant $c > 0$ and all $n \geq 1$.

Let $G_n(t, x) = \varphi_n(x)G(t, x)$ for $t \in [0, T]$ and $x \in M$. Since G_n is continuous with compact support, there exists $(t_n, x_n) \in [0, T] \times M$ such that

$$G_n(t_n, x_n) = \max_{[0, T] \times M} G_n.$$

By (2-3) and that $\sup G > 1$, we have $\lim_{n \rightarrow \infty} G(t_n, x_n) = \sup G > 1$. By renumbering from a sufficient large n_0 , we may assume that $G_n(t_n, x_n)$ is greater than 1 and is increasing in n . In particular, Lemma 1.1(i) holds and

$$(2-5) \quad \varphi_n(x_n) \geq 1/G(t_n, x_n) \quad \text{for } n \geq 1.$$

Moreover, since $G_n(0, \cdot) \leq 0$, we have $t_n > 0$ and $\partial_t G(t_n, x_n) \geq 0$ for $n \geq 1$. Thus, it remains to confirm that

$$(2-6) \quad \begin{aligned} |\nabla G(t_n, x_n)| &\leq cG(t_n, x_n)^{1+\varepsilon}/n \quad \text{and} \\ LG(t_n, x_n) &\leq cG(t_n, x_n)^{1+\varepsilon}/n \quad \text{for } n \geq 1 \end{aligned}$$

for some constant $c > 0$. Indeed, by using a subsequence $\{(t_{mn}, x_{mn})\}_{n \geq 1}$ for $m \geq c$ to replace $\{(t_n, x_n)\}_{n \geq 1}$, one may reduce (2-6) with some $c > 0$ to that with $c = 1$.

Since x_n is the maximal point of G_n , we have $\nabla G_n(t_n, x_n) = 0$ if $x_n \in M \setminus \partial M$. If $x_n \in \partial M$, we have $NG_n(t_n, x_n) \leq 0$. Recall that $NG(t_n, \cdot) \geq 0$ and $G(t_n, x_n) > 0$. Then, noting that $N\rho_0 \leq 0$ together with $h' \leq 0$ implies $N\varphi_n \geq 0$, we conclude that $NG_n(t_n, x_n) \geq 0$. Hence, $NG_n(t_n, x_n) = 0$. Moreover, since x_n is the maximal point of $G_n(t_n, \cdot)$ on the closed manifold ∂M , we have $UG_n(t_n, x_n) = 0$ for all $U \in T\partial M$. Therefore, $\nabla G_n(t_n, x_n) = 0$ also holds for $x_n \in \partial M$. Combining this with (2-4) and (2-5), we obtain

$$|\nabla G(t_n, x_n)| \leq \frac{G(t_n, x_n)}{\varphi_n(x_n)} |\nabla \varphi_n| \leq \frac{cG(t_n, x_n)^{1+\varepsilon}}{n},$$

which proves the first inequality in (2-6).

Finally, by (2-4), the inequality

$$\varphi_n L_n G + GL_n \varphi_n + 2\langle \nabla G, \nabla \varphi_n \rangle \geq \varphi_n L_n G - \frac{c\varphi_n^{1-\varepsilon}}{n} G - \frac{2c\varphi_n^{1-\varepsilon}}{n} |\nabla G| =: \Phi$$

holds on $\{G_n > 0\} \setminus \text{cut}(o)$. By Lemma 2.1 below we obtain at the point (t_n, x_n) that

$$LG \leq \frac{c}{n\varphi_n^\varepsilon} G + \frac{2c}{n\varphi_n} |\nabla G|.$$

Combining this with (2-5) and the first inequality in (2-6), we get

$$LG(t_n, x_n) \leq \frac{c}{n} G^{1+2\varepsilon}(t_n, x_n)$$

for some constant $c > 0$ and all $n \geq 1$. Since $\varepsilon > 0$ is arbitrary, we may replace ε by $\varepsilon/2$ (recall that $G(t_n, x_n) \geq 1$). This proves the second inequality in (2-6).

Lemma 2.1. *The reflecting L -diffusion process is nonexplosive, and for any Φ in $C_b(M)$ such that*

$$\Phi \leq LG_n = GL\varphi_n + \varphi_n LG + 2\langle \nabla \varphi_n, \nabla G \rangle \quad \text{on } \{G_n > 0\} \setminus \text{cut}(o),$$

we have $\Phi(t_n, x_n) \leq 0$ for all $n \geq 1$.

Proof. Let X_t be the reflecting L -diffusion process generated by L , and let U_t be its horizontal lift on the frame bundle $O(M)$. By the Itô formula for $\rho_o(X_t)$

found by Kendall [1987] for $\partial M = \emptyset$ and by the fact that $N\rho_o|_{\partial M} \leq 0$ when ∂M is nonempty but convex, we have

$$(2-7) \quad d\rho_o(X_t) = \sqrt{2}\langle \nabla \rho_o(X_t), U_t dB_t \rangle + L\rho_o(X_t)dt - dl_t + dl'_t,$$

where B_t is the d -dimensional Brownian motion, where $L\rho_o$ is taken to be zero on $\{o\} \cup \text{cut}(o)$, and where l_t and l'_t are two increasing processes such that l'_t increases only when $X_t = o$, while l_t increases only when $X_t \in \text{cut}(o) \cup \partial M$ (note that $l'_t = 0$ for $d \geq 2$). Combining this with (2-1) we obtain

$$d\sqrt{1 + \rho_o^2(X_t)} \leq dM_t + L\sqrt{1 + \rho_o^2(X_t)} dt \leq dM_t + c_1 dt$$

for some martingale M_t . This implies immediately that X_t does not explode.

Now, let us take $X_0 = x_n$. Since $h' \leq 0$, it follows from (2-7) that

$$(2-8) \quad d\varphi_n(X_t) \geq \sqrt{2}\langle \nabla \varphi_n(X_t), U_t dB_t \rangle + L\varphi_n(X_t)dt,$$

where we set $L\varphi_n = 0$ on $\text{cut}(o)$ as above.

On the other hand, since $NG(t_n, \cdot) \geq 0$, we may apply the Itô to $G(t_n, X_t)$ to obtain

$$(2-9) \quad dG(t_n, X_t) \geq \sqrt{2}\langle \nabla G(t_n, X_t), U_t dB_t \rangle + LG(t_n, X_t)dt.$$

Because $G_n(t_n, x_n) > 0$, there exists an $r > 0$ such that $G_n > 0$ on $B(x_n, r)$, the geodesic ball in M centered at x_n with radius r . Let

$$\tau = \inf\{t \geq 0 : X_t \notin B(x_n, r)\}.$$

Then (2-8) and (2-9) imply

$$dG_n(t_n, X_t) \geq dM_t + LG_n(t_n, \cdot)(X_t)dt \geq dM_t + \Phi(t_n, X_t)dt \quad \text{for } t \leq \tau$$

for some martingale M_t . Since $G_n(t_n, X_t) \leq G_n(t_n, x_n)$ and $X_0 = x_n$, this implies that

$$0 \geq EG_n(t_n, X_{t \wedge \tau}) - G_n(t_n, x_n) \geq E \int_0^{t \wedge \tau} \Phi(t_n, X_s) ds.$$

Therefore, the continuity of Φ implies that

$$\Phi(t_n, x_n) = \lim_{t \rightarrow 0} \frac{1}{E(t \wedge \tau)} E \int_0^{t \wedge \tau} \Phi(t_n, X_s) ds \leq 0. \quad \square$$

2.2. Nonconvex ∂M . Under our assumptions on M , there exists a constant $R > 1$ and a function $\phi \in C^\infty(M)$ such that

$$1 \leq \phi \leq R, \quad |\nabla \phi| \leq R, \quad N \log \phi|_{\partial M} \geq \sigma.$$

By [Wang 2007, Lemma 2.1], the boundary ∂M is convex under the new metric $\langle \cdot, \cdot \rangle' := \phi^{-2} \langle \cdot, \cdot \rangle$. Let $L' = \phi^2 L$. By [Wang 2007, Equation (9)], the vector

$U' := \phi U$ is unit under the new metric for any unit vector $U \in TM$, and the corresponding Ricci curvature satisfies

$$(2-10) \quad \text{Ric}'(U', U') \geq \phi^2 \text{Ric}(U, U) + \phi \Delta \phi - (d-3)|\nabla \phi|^2 \\ - 2(U\phi)^2 + (d-2)\phi \text{Hess}_\phi(U, U).$$

Let Δ' be the Laplacian induced by the new metric. By [Wang 2007, Lemma 2.2], we have

$$L' := \phi^2 L = \Delta' + (d-2)\phi \nabla \phi + \phi^2 Z =: \Delta' + Z'.$$

Noting that

$$\nabla'_X Y = \nabla_X Y - \langle X, \nabla \log \phi \rangle Y - \langle Y, \nabla \log \phi \rangle X + \langle X, Y \rangle \nabla \log \phi \quad \text{for } X, Y \in TM,$$

we have

$$\langle \nabla_{U'} Z', U' \rangle' = \langle \nabla_U Z', U \rangle - \langle Z', \nabla \log \phi \rangle \\ = \phi^2 \langle \nabla_U Z, U \rangle + (U\phi)^2 \langle Z, U \rangle + (d-2)(U\phi)^2 \\ + (d-2)\phi \text{Hess}_\phi(U, U) - \langle Z', \nabla \log \phi \rangle.$$

Combining this with (2-10) and the properties of ϕ mentioned above, we find a constant $c_1 > 0$ such that

$$(2-11) \quad \text{Ric}'(U', U') - \langle \nabla_{U'} Z', U' \rangle' \geq \phi^2 (\text{Ric} - \nabla Z)(U, U) - c_1 \quad \text{for } |U| = 1.$$

Moreover, since

$$(Z' \otimes Z')(U', U') := (\langle Z', U' \rangle')^2 = \phi^{-2} \langle Z', U \rangle^2 \\ \leq 2(d-2)^2 \langle \nabla \phi, U \rangle^2 + 2\phi^2 \langle Z, U \rangle^2 \\ \leq 2(d-2)^2 R^2 + 2\phi^2 \langle Z \otimes Z \rangle(U, U),$$

it follows from (1-2) and (2-11) that

$$\text{Ric}' - \nabla' Z' - \frac{Z' \otimes Z'}{2(m-d)} \geq -\phi^2 K - c_2 \geq -K'$$

holds for the metric $\langle \cdot, \cdot \rangle'$ and some constants $c_2, K' > 0$. Therefore, we may apply Lemma 2.1 to L' on the convex Riemannian manifold $(M, \langle \cdot, \cdot \rangle')$ to conclude that the desired sequence $\{(t_n, x_n)\}$ exists.

3. Proofs of Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3. When ∂M is convex, Lemma 2.1 ensures that X_t does not explode. If ∂M is nonconvex, this can be confirmed by reparametrizing the time of the process. More precisely, let X'_t be the reflecting diffusion process on M generated by $L' := \phi^2 L$ constructed in Section 2.2. Since $L' = \Delta' + Z'$

satisfies (1-2) for some $K > 0$ on the convex manifold $(M, \langle \cdot, \cdot \rangle')$, the process X'_t generated by L' is nonexplosive by Lemma 2.1. Since $X_t = X'_{\xi^{-1}(t)}$, where ξ^{-1} is the inverse of

$$t \mapsto \xi(t) = \int_0^t \phi^2(X'_s) ds,$$

we have $t \|\phi\|_\infty^{-2} \leq \xi^{-1}(t) \leq t$, and the process X_t is nonexplosive as well.

Let $f \in C_b^1(M)$ be strictly positive, and let $u(t, x) = \log P_t f(x)$. For a fixed number $T > 0$, we will apply Theorem 1.2 to the reference function

$$G(t, x) = t\{\phi(x)|\nabla u|^2(t, x) - \alpha u_t(t, x)\} \quad \text{for } t \in [0, T] \text{ and } x \in M.$$

Note that $\Pi \geq -\sigma$ and $N \log \phi \geq 2\sigma$ imply

$$N\phi \geq 2\sigma\phi,$$

$$N|\nabla P_t f|^2 = 2 \text{Hess}_{P_t f}(N, \nabla P_t f) = 2\Pi(\nabla P_t f, \nabla P_t f) \geq -2\sigma|\nabla P_t f|^2.$$

Since $P_t f$ and hence u_t satisfy the Neumann boundary condition, this implies that

$$NG = t\left\{(N\phi)|\nabla u|^2 + \frac{\phi}{(P_t f)^2} N|\nabla P_t f|^2\right\} \geq t\{2\sigma\phi|\nabla u|^2 - 2\sigma\phi|\nabla u|^2\} = 0 \quad \text{on } \partial M.$$

According to [Ledoux 2000, (1.14)], inequality (1-2) implies

$$(3-1) \quad L|\nabla u|^2 - 2\langle \nabla Lu, \nabla u \rangle \geq -2K|\nabla u|^2 + \frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}.$$

By multiplying this inequality by ε and (1-1) by $2(1 - \varepsilon)$ and by combining the results, we obtain

$$L|\nabla u|^2 \geq 2\langle \nabla Lu, \nabla u \rangle - 2K|\nabla u|^2 + \frac{2(1 - \varepsilon)(Lu)^2}{m} + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}.$$

It is also easy to check that $Lu = u_t - |\nabla u|^2$ and $\partial_t|\nabla u|^2 = 2\langle \nabla u, \nabla u_t \rangle$. Then we arrive at

$$(3-2) \quad \begin{aligned} (L - \partial_t)|\nabla u|^2 &\geq \frac{2(1 - \varepsilon)}{m} (|\nabla u|^2 - u_t)^2 + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2} - 2\langle \nabla u, \nabla|\nabla u|^2 \rangle - 2K|\nabla u|^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} -\alpha(L - \partial_t)u_t &= 2\alpha\langle \nabla u, \nabla u_t \rangle = 2\langle \nabla u, \nabla(\phi|\nabla u|^2 - t^{-1}G) \rangle \\ &= 2\phi\langle \nabla u, \nabla|\nabla u|^2 \rangle + 2|\nabla u|^2\langle \nabla u, \nabla\phi \rangle - 2t^{-1}\langle \nabla u, \nabla G \rangle. \end{aligned}$$

Combining this with (3-2), we obtain

$$\begin{aligned}
(L - \partial_t)G &= -\frac{G}{t} + t(\phi(L - \partial_t)|\nabla u|^2 + |\nabla u|^2 L\phi + 2\langle \nabla \phi, \nabla |\nabla u|^2 \rangle) \\
&\quad + t(2\phi \langle \nabla u, \nabla |\nabla u|^2 \rangle + 2|\nabla u|^2 \langle \nabla u, \nabla \phi \rangle - 2t^{-1} \langle \nabla u, \nabla G \rangle) \\
&\geq -\frac{G}{t} + \frac{2(1-\varepsilon)\phi t}{m} (|\nabla u|^2 - u_t)^2 + \frac{\varepsilon\phi t |\nabla |\nabla u|^2|^2}{2|\nabla u|^2} - 2K\phi t |\nabla u|^2 \\
&\quad - 2|\nabla u| \cdot |\nabla G| - 2t|\nabla u|^3 |\nabla \phi| - 2t|\nabla \phi| \cdot |\nabla |\nabla u|^2| + t|\nabla u|^2 L\phi.
\end{aligned}$$

Noting that

$$\frac{\varepsilon\phi t |\nabla |\nabla u|^2|^2}{2|\nabla u|^2} - 2t|\nabla \phi| \cdot |\nabla |\nabla u|^2| \geq -\frac{2t|\nabla \phi|^2 |\nabla u|^2}{\varepsilon\phi},$$

we get

$$\begin{aligned}
(3-3) \quad (L - \partial_t)G &\geq -\frac{G}{t} + \frac{2(1-\varepsilon)\phi t}{m} (|\nabla u|^2 - u_t)^2 - 2K\phi t |\nabla u|^2 - 2|\nabla u| \cdot |\nabla G| \\
&\quad - 2t|\nabla u|^3 |\nabla \phi| + t|\nabla u|^2 L\phi - \frac{2t|\nabla \phi|^2 |\nabla u|^2}{\varepsilon\phi}.
\end{aligned}$$

We assume that $\sup G > 0$, otherwise the proof is done. Since $G(0, \cdot) = 0$ and $NG|_{\partial M} \geq 0$, we can apply Theorem 1.2. Let $\{(t_n, x_n)\}$ be fixed in Theorem 1.2 with, for example, $\varepsilon = 1/2$. Then,

$$(3-4) \quad (L - \partial_t)G(t_n, x_n) \leq \frac{G^{3/2}(t_n, x_n)}{n} \quad \text{and} \quad |\nabla G|(t_n, x_n) \leq \frac{G^{3/2}(t_n, x_n)}{n}.$$

From now on, we evaluate functions at the point (t_n, x_n) , so that $t = t_n$.

Let $\mu = |\nabla u|^2/G$. We have

$$|\nabla u|^2 - u_t = \left(\mu - \frac{(\mu t - 1)\phi}{\alpha t} \right) G = \frac{\mu t(\alpha - \phi) + \phi}{\alpha t} G.$$

Combining this with (3-3) and (3-4), we arrive at

$$\begin{aligned}
(3-5) \quad &\frac{2(1-\varepsilon)\phi(\mu t(\alpha - \phi) + \phi)^2}{m\alpha^2 t} G^2 \\
&\leq \frac{G^{3/2}}{n} + \frac{G}{t} + \frac{2\sqrt{\mu}G^2}{n} + 2t|\nabla \phi|(\mu G)^{3/2} + (2k\phi + 2\varepsilon^{-1}\phi^{-1}|\nabla \phi|^2 - L\phi)\mu t G.
\end{aligned}$$

Since it is easy to see that

$$(\mu t(\alpha - \phi) + \phi)^2 \geq \max\{\phi^2, 4\mu t(\alpha - \phi)\phi, (2t(\alpha - \phi))^{3/2}\sqrt{\phi\mu^{3/2}}\},$$

we may multiply both sides of (3-5) by $t(\mu t(\alpha - \phi) + \phi)^{-2}G^{-2}$ to obtain

$$\begin{aligned} \frac{2(1-\varepsilon)\phi}{m\alpha^2} &\leq \frac{c't}{n(1\wedge\sqrt{G})} + \frac{1}{\phi^2G} + \frac{2K + 2\varepsilon^{-1}|\nabla\log\phi|^2 - \phi^{-1}L\phi}{4(\alpha - \phi)G}t \\ &\quad + \frac{|\nabla\log\phi|\sqrt{t\phi}}{(\alpha - \phi)^{3/2}\sqrt{2G}} \\ &\leq \frac{c't}{n(1\wedge\sqrt{G})} + \frac{1}{\phi^2G} + \frac{2K + 2\varepsilon^{-1}|\nabla\log\phi|^2 - \phi^{-1}L\phi}{4(\alpha - \phi)G} \\ &\quad + \frac{|\nabla\log\phi|^2m\alpha^2(1+\varepsilon)t}{16(\alpha - \phi)^3\varepsilon(1-\varepsilon)G} + \frac{2(1-\varepsilon)\varepsilon\phi}{m\alpha^2(1+\varepsilon)} \end{aligned}$$

for some constant $c' > 0$. Taking $n \rightarrow \infty$ and noting that $\phi \geq 1$, we conclude that $\theta := \sup G$ satisfies

$$\begin{aligned} \frac{2(1-\varepsilon)}{m\alpha^2(1+\varepsilon)} &\leq \frac{1}{\theta} \left(1 + \frac{2K + 2\varepsilon^{-1}\|\nabla\log\phi\|_\infty^2 + \sup(-\phi^{-1}L\phi)}{4(\alpha - \|\phi\|_\infty)}T \right. \\ &\quad \left. + \frac{\|\nabla\log\phi\|_\infty^2m\alpha^2(1+\varepsilon)T}{16(\alpha - \|\phi\|_\infty)^3\varepsilon(1-\varepsilon)} \right). \end{aligned}$$

Combining this with $\theta \geq G(T, x) = T(\phi(x)|\nabla u|^2(T, x) - \alpha u_t(T, x))$ for $x \in M$, we obtain

$$\begin{aligned} \phi(x)|\nabla u|^2(T, x) - \alpha u_t(T, x) &\leq \frac{m\alpha^2(1+\varepsilon)}{2(1-\varepsilon)} \left(\frac{1}{T} + \frac{2K + 2\varepsilon^{-1}\|\nabla\log\phi\|_\infty^2 + \sup(-\phi^{-1}L\phi)}{4(\alpha - \|\phi\|_\infty)} \right. \\ &\quad \left. + \frac{\|\nabla\log\phi\|_\infty^2m\alpha^2(1+\varepsilon)}{16(\alpha - \|\phi\|_\infty)^3\varepsilon(1-\varepsilon)} \right) \end{aligned}$$

for all $x \in M$. Then the proof is completed since $T > 0$ is arbitrary. \square

Proof of Corollary 1.4. By Theorem 1.3, the proof is standard according to [Li and Yau 1986]. For $x, y \in M$, let $\gamma : [0, 1] \rightarrow M$ be the shortest curve in M linking x and y such that $|\dot{\gamma}| = \rho(x, y)$. Then, for any $s, t > 0$ and $f \in C_b^\infty(M)$, it follows from (1-5) that

$$\begin{aligned} \frac{d}{dr} \log P_{t+rs} f(\gamma_r) &= s\partial_u \log P_u f(\gamma_r)|_{u=t+rs} + \langle \dot{\gamma}_r, \nabla P_{t+rs} f(\gamma_r) \rangle \\ &\geq \frac{s}{\alpha} |\nabla \log P_{t+rs} f|^2(\gamma_r) - \rho(x, y) |\nabla \log f|(\gamma_r) \\ &\quad - s \left(\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)} + \frac{m\alpha K(\phi, \varepsilon, \alpha)}{4(\alpha - 1\|\phi\|_\infty)} \right) \\ &\geq -\frac{\alpha}{4s} - s \left(\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)} + \frac{m\alpha K(\phi, \varepsilon, \alpha)}{4(\alpha - \|\phi\|_\infty)} \right). \end{aligned}$$

We complete the proof by integrating with respect to dr over $[0, 1]$. \square

4. Proof of Theorem 1.6

We first provide a simple proof of (1-9) under an extra assumption that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$ for any $T > 0$; we then drop this assumption by an approximation argument.

Lemma 4.1. *If that $f \in C_b^1(M)$ is such that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$ for any $T > 0$, then (1-9) holds.*

Proof. For any $\varepsilon > 0$, let $\eta_s = \sqrt{\varepsilon + |\nabla P_{t-s} f|^2}(X_s)$ for $s \leq t$. By the Itô formula, we have

$$\begin{aligned} d\eta_s = dM_s + & \frac{L|\nabla P_{t-s} f|^2 - 2\langle \nabla L P_{t-s} f, \nabla P_{t-s} f \rangle}{2\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}^2}(X_s) ds \\ & - \frac{|\nabla |\nabla P_{t-s} f|^2|^2}{4(\varepsilon + |\nabla P_{t-s} f|^2)^{3/2}}(X_s) ds + \frac{N|\nabla P_{t-s} f|^2}{2\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s) dL_s \end{aligned}$$

for $s \leq t$, where M_s is a local martingale. Combining this with (1-8) and (3-1), with κ_1 in place of K_0 , we obtain

$$\begin{aligned} d\eta_s & \geq dM_s - \frac{\kappa_1 |\nabla P_{t-s} f|^2}{\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s) ds - \frac{\kappa_2 |\nabla P_{t-s} f|^2}{\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s) dL_s \\ & \geq dM_s - \kappa_1(X_s)\eta_s ds - \kappa_2(X_s)\eta_s dL_s \quad \text{for } s \leq t. \end{aligned}$$

Now η_s is bounded on $[0, t]$, and by the proof of [Wang 2005b, Lemma 2.1] we have $Ee^{\lambda t} < \infty$ for all $\lambda > 0$. This implies that

$$[0, t] \ni s \mapsto \sqrt{\varepsilon + |\nabla P_{t-s} f|^2}(X_s) \exp\left(\int_0^s \kappa_1(X_s) ds + \int_0^s \kappa_2(X_s) dL_s\right)$$

is a submartingale for any $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we conclude that

$$[0, t] \ni s \mapsto |\nabla P_{t-s} f|(X_s) \exp\left(\int_0^s \kappa_1(X_s) ds + \int_0^s \kappa_2(X_s) dL_s\right)$$

is a submartingale as well. \square

According to Lemma 4.1, it suffices to confirm the boundedness of $|\nabla P_{(\cdot)} f|$ on $[0, T] \times M$ for any $T > 0$ and $f \in C_b^1(M)$. We shall start from $f \in C_0^\infty(M)$ with $Nf|_{\partial M} = 0$, then pass to $f \in C_b^1(M)$ by combining an approximation argument and Lemma 4.1.

Case a. Let $f \in C_0^\infty(M)$ with $Nf|_{\partial M} = 0$. We have

$$(4-1) \quad P_t f = f + \int_0^t P_s L f ds.$$

Since Lf is bounded, there is a $c > 0$ such that $Lf + c \geq 1$. Applying Corollary 1.5 with for example $\alpha = 2 + \sigma dr_0$ and $\varepsilon = 1/2$, but using $Lf + c$ in place of f , we obtain

$$\begin{aligned} |\nabla P_s Lf| &= |\nabla P_s(Lf + c)| \\ &\leq \|Lf + c\|_\infty \left(\alpha \|P_s L^2 f\|_\infty + \frac{m(1 + \varepsilon)\alpha^2}{2(1 - \varepsilon)s} + \frac{m\alpha^2 K_\varepsilon}{4(\alpha - 1 - \sigma dr_0)} \right)^{1/2} \\ &\leq c_1/\sqrt{s} \quad \text{for } s \leq T \end{aligned}$$

for some constant $c_1 > 0$. Combining this with (4-1) we conclude that, for some constant $c_2 > 0$,

$$|\nabla P_t f| \leq |\nabla f| + \int_0^t \frac{c_1}{\sqrt{s}} ds \leq c_2 \quad \text{for } t \leq T.$$

Case b. Let $f \in C_0^\infty(M)$. There exists a sequence of functions $\{f_n\}_{n \geq 1} \subset C_0^\infty(M)$ such that $Nf_n|_{\partial M} = 0$, $f_n \rightarrow f$ uniformly as $n \rightarrow \infty$, and $\|\nabla f_n\|_\infty \leq 1 + \|\nabla f\|_\infty$ holds for any $n \geq 1$; see for example [Wang 1994]. By Case a and Lemma 4.1, (1-9) holds with f_n in place of f , so that

$$\frac{|P_t f_n(x) - P_t f_n(y)|}{\rho(x, y)} \leq C \quad \text{for } t \leq T, n \geq 1, x \neq y$$

for some constant $C > 0$. Letting first $n \rightarrow \infty$ and then $y \rightarrow x$, we conclude that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$.

Case c. Let $f \in C_b^\infty(M)$. Let $\{g_n\}_{n \geq 1} \subset C_0^\infty$ be such that $0 \leq g_n \leq 1$, $|\nabla g_n| \leq 2$ and $g_n \uparrow 1$ as $n \uparrow \infty$. By Case b and Lemma 4.1, we may apply (1-9) to $g_n f$ in place of f such that

$$\frac{|P_t(g_n f)(x) - P_t(g_n f)(y)|}{\rho(x, y)} \leq C \quad \text{for } t \leq T, n \geq 1, x \neq y$$

for some constant $C > 0$. By the same reason as in Case b, we conclude that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$.

Case d. Finally, for $f \in C_b^1(M)$, there exist $\{f_n\}_{n \geq 1} \subset C_b^\infty(M)$ such that $f_n \rightarrow f$ uniformly as $n \rightarrow \infty$ and $\|\nabla f_n\|_\infty \leq \|\nabla f\|_\infty + 1$ for any $n \geq 1$. The proof is completed by the same reason as in Cases b and c. \square

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