GRADIENT AND HARNACK INEQUALITIES ON NONCOMPACT MANIFOLDS WITH BOUNDARY

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By using the reflecting diffusion process and a conformal change of metric, a generalized maximum principle is established for (unbounded) time-space functions on a class of noncompact Riemannian manifolds with (nonconvex) boundary. As applications, Li–Yau-type gradient and Harnack inequalities are derived for the Neumann semigroup on a class of noncompact manifolds with (nonconvex) boundary. These generalize some previous ones obtained for the Neumann semigroup on compact manifolds with boundary. As a byproduct, the gradient inequality for the Neumann semigroup derived by Hsu on a compact manifold with boundary is confirmed on these noncompact manifolds.

1. Introduction

Suppose $M$ is a $d$-dimensional connected complete Riemannian manifold, and let $L = \Delta + Z$, where $Z$ is a $C^1$ vector field satisfying the curvature-dimension condition of Bakry and Émery [1984] given by

\begin{equation}
\Gamma_2(f, f) := \frac{1}{2} L \langle \nabla f, \nabla f \rangle - \langle \nabla L f, \nabla f \rangle \geq \frac{(Lf)^2}{m} - K |\nabla f|^2 \quad \text{for } f \in C^\infty(M)
\end{equation}

for some constants $K \geq 0$ and $m > d$. By [Qian 1998, page 138], this condition is equivalent to

\begin{equation}
\operatorname{Ric} - \nabla Z - \frac{Z \otimes Z}{m - d} \geq -K.
\end{equation}

When $Z = 0$ and $M$ is either without boundary or compact and with a convex boundary $\partial M$, Li and Yau [1986] found a now-famous gradient estimate for the (Neumann) semigroup $P_t$ generated by $L$:

\begin{equation}
|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \leq \frac{d\alpha^2}{2t} + \frac{d\alpha^2 K}{4(\alpha - 1)} \quad \text{for } t > 0 \text{ and } \alpha > 1
\end{equation}

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for all positive $f \in C_b(M)$. We note that in [Li and Yau 1986] the second term on the right side of (1-3) is $\alpha^2 K / (\sqrt{2}(\alpha - 1))$, but $\sqrt{2}$ here can be replaced by 4 according to a refined calculation; see for example [Davies 1989].

As an application, (1-3) implies a parabolic Harnack inequality for $P_t$:

$$ P_t f(x) \leq \left(\frac{t+s}{t}\right)^{\alpha/2} \left(P_{t+s} f(y)\right) \exp\left(\frac{ap(x, y)^2}{4s} + \frac{\alpha K ds}{4(\alpha - 1)}\right) $$

for $t > 0$ and $x, y \in M$,

where $\alpha > 1$ and $f \in C_b(M)$ is positive. From this Harnack inequality, one obtains Gaussian-type heat kernel bounds for $P_t$; see [Li and Yau 1986; Davies 1989].

The gradient estimate (1-3) has been extended and improved in several papers. See for example [Bakry and Qian 1999] for an improved version for $\alpha = 1$ with $Z \neq 0$ and $\partial M = \emptyset$, and see [Wang 1997] for an extension to a compact manifold with nonconvex boundary. The aim of this paper is to investigate the gradient and Harnack inequalities for $P_t$ on noncompact manifolds with (nonconvex) boundary.

Recall that the key step of Li and Yau’s argument for the gradient estimate (1-3) is to apply the maximum principle to the reference function

$$ G(t, x) := t(\nabla \log P_t f)^2 - \alpha \partial_t \log P_t f(x) \quad \text{for } t \in [0, T] \text{ and } x \in M. $$

When $M$ is compact without boundary, the maximum principle says that for any smooth function $G$ on $[0, T] \times M$ with $G(0, \cdot) \leq 0$ and $\sup G > 0$, there exists a maximal point of $G$ at which $\nabla G = 0$, $\partial_t G \geq 0$, and $\Delta G \leq 0$. When $M$ is compact with a convex boundary, the same assertion holds for the above specified function $G$, as observed in [Li and Yau 1986, proof of Theorem 1.1]. In [1997], J. Wang extended this maximum principle on a compact manifold with nonconvex boundary by taking

$$ G(t, x) = t(\phi \nabla \log P_t f)^2 - \alpha \partial_t \log P_t f(x) \quad \text{for } t \in [0, T] \text{ and } x \in M $$

for a nice function $\phi$ compensating the concavity of the boundary.

As for a noncompact manifold without boundary, Li and Yau [1986] established the gradient estimate by applying the maximal principle to a sequence of functions with compact support that approximate the original function $G$. An alternative is to apply directly the following generalized maximum principle:

**Lemma 1.1** [Yau 1975]. For any bounded smooth function $G$ on $[0, T] \times M$ with $G(0, \cdot) \leq 0$ and $\sup G > 0$, there exists a sequence $\{(t_n, x_n)\}_{n \geq 1} \subset [0, T] \times M$ such that

(i) $0 < G(t_n, x_n) \uparrow \sup G$ as $n \uparrow \infty$, and

(ii) for any $n \geq 1$,

$$ LG(t_n, x_n) \leq 1/n, \quad |\nabla G(t_n, \cdot)(x_n)| \leq 1/n, \quad \partial_t G(t_n, x_n) \geq 0. $$
To apply this generalized maximal principle for the gradient estimate, one has to first confirm the boundedness of \( G(t, \cdot) := t(|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f) \) on \([0, T] \times M\) for \( T > 0\).

Since the boundedness of this type of reference function is unknown when \( M\) is noncompact with a nonconvex boundary, we shall establish a generalized maximum principle on a class of noncompact manifolds with boundary for not necessarily bounded functions. Applying this principle to a suitable reference function \( G \), we derive the Li–Yau-type gradient and Harnack inequalities for Neumann semigroups. To establish such a maximum principle, we adopt a localization argument so that the classical maximum principle can be applied.

For \( M\) noncompact without boundary, Li and Yau [1986] used such a localization argument to apply the maximal principle to functions with compact support; they then passed to the desired global estimate by taking a limit. To do this, they constructed cut-off functions using \( \rho_o \), the Riemannian distance function to a fixed point \( o \in M\). It turns out that this argument works also when \( \partial M\) is convex; see Section 2.1. For the nonconvex case, we will use the conformal change of metric introduced in [Wang 2007] to make a nonconvex boundary convex; see Section 2.2.

**Assumption A.** The manifold \( M\) is connected and complete with boundary \( \partial M\) and such that either

1. \( \partial M\) is convex, or
2. the second fundamental form of \( \partial M\) is bounded, the sectional curvature of \( M\) is bounded from above, and the injectivity radius \( i_{\partial M}\) of \( \partial M\) is positive.

Recall that the Riemannian distance function \( \rho_{\partial M}\) to the boundary is smooth on the set \( \{ \rho_{\partial M} < i_{\partial M}\}\).

Let \( N\) be the inward unit normal vector field on \( \partial M\). The second fundamental form of \( \partial M\) is

\[
\ll(X, Y) = -\langle \nabla_X N, Y \rangle \quad \text{for } X, Y \in T \partial M.
\]

The boundary \( \partial M\) is called convex if \( \ll \geq 0\). We are now ready to state our generalized maximal principle for possibly unbounded functions.

**Theorem 1.2.** Let \( M\) satisfy A, and let \( L\) satisfy (1-2). Let \( T > 0\), and let \( G\) be a smooth function on \([0, T] \times M\) such that \( NG|_{\partial M} \geq 0\), \( G(0, \cdot) \leq 0\) and \( \sup G > 0\). Then for any \( \varepsilon > 0\), there exists a sequence \( \{(t_n, x_n)\}_{n \geq 1} \subset (0, T] \times M\) such that Lemma 1.1(i) holds and for any \( n \geq 1\)

\[
LG(t_n, x_n) \leq \frac{G(t_n, x_n)^{1+\varepsilon}}{n}, \quad |\nabla G(t_n, x_n)| \leq \frac{G(t_n, x_n)^{1+\varepsilon}}{n}, \quad \partial_t G(t_n, x_n) \geq 0.
\]
Applying Theorem 1.2 to a proper choice of function $G$, we will derive the Li–Yau-type gradient estimate (1-5). We shall prove that the reflecting diffusion process $X_t$ generated by $L$ on $M$ is non explosive, so that the corresponding Neumann semigroup $P_t$ can be formulated as

$$P_t f(x) = E^x f(X_t) \quad \text{for } t \geq 0, \ x \in M, \ \text{and } f \in C_b(M),$$

where $E^x$ is the expectation taken for $X_0 = x$.

**Theorem 1.3.** Let $M$ satisfy A, and suppose $L$ satisfies (1-2) with $\|Z\|_\infty < \infty$. Then the reflecting $L$-diffusion process on $M$ is nonexplosive and the corresponding Neumann semigroup $P_t$ satisfies these assertions:

(i) If $\partial M$ is convex, then (1-3) holds with $m$ in place of $d$.

(ii) If $\partial M$ is nonconvex with $\frac{\|\xi\|}{\|\xi\|_\infty} \geq -\sigma$ for some $\sigma > 0$, then for any bounded $\phi \in C^\infty(M)$ with $\phi \geq 1$ and $N \log \phi|_{\partial M} \geq 2\sigma$, the gradient inequality

$$|\nabla \log P_t f|^2 - a \tilde{c}, \log P_t f \leq \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K(\phi, \varepsilon, \alpha)}{4(\alpha - \|\phi\|_\infty)}$$

holds for all positive $f \in C_b(M)$, $\alpha > \|\phi\|_\infty$, $t > 0$, $\varepsilon \in (0, 1)$ and $K(\phi, \varepsilon, \alpha) := \frac{1+\varepsilon}{1-\varepsilon} \left( K + \frac{1}{\varepsilon} \|\nabla \log \phi\|_\infty^2 + \frac{1}{2} \sup(-\phi^{-1}L\phi) + \frac{m\alpha^2 \|\nabla \log \phi\|_\infty^2(1+\varepsilon)}{8(\alpha - \|\phi\|_\infty)^2 \varepsilon (1-\varepsilon)} \right)$.

We emphasize that the results in Theorem 1.3 are new for noncompact manifolds with boundary. When $M$ is compact with a convex boundary, the first assertion was proved in [Li and Yau 1986] by using the classical maximum principle on compact manifolds, while when $M$ is compact with a nonconvex boundary, an inequality similar to (1-5) was proved in [Wang 1997] by using the “interior rolling $R$-ball” condition.

These two theorems will be proved in Sections 2 and 3. By a standard argument due to Li and Yau [1986], the gradient estimate (1-5) implies a Harnack inequality.

**Corollary 1.4.** In the situation of Theorem 1.3 the Neumann semigroup $P_t$ satisfies

$$P_t f(x) \leq \left( \frac{t+s}{t} \right)^{m(1+\varepsilon)\alpha/2(1-\varepsilon)} (P_{t+s} f(y)) \exp \left( \frac{\alpha \rho(x, y)^2}{4s} + \frac{amK(\phi, \varepsilon, \alpha)s}{4(\alpha - \|\phi\|_\infty)} \right)$$

for all positive $f \in C_b(M)$, $t, \varepsilon \in (0, 1)$, $\alpha > \|\phi\|_\infty$ and $x, y \in M$. In particular, if $\partial M$ is convex, then (1-4) holds with $m$ in place of $d$ and for all $\alpha > 1$. 
To derive explicit inequalities for the nonconvex case, we shall take a specific choice of \( \phi \) as in [Wang 2007]. Let \( i_{\partial M} \) be the injectivity radius of \( \partial M \), and let \( \rho_{\partial M} \) be the Riemannian distance to the boundary. We shall take \( \phi = \varphi \circ \rho_{\partial M} \) for a nice reference function \( \varphi \) on \([0, \infty)\). More precisely, let the sectional curvature satisfy \( \text{Sect}_M \leq k \) and \(-\sigma \leq \sigma \leq \gamma \) for some \( k, \sigma, \gamma > 0 \). Let

\[
h(s) = \cos(\sqrt{k}s) - (\gamma / \sqrt{k}) \sin(\sqrt{k}s) \quad \text{for } s \geq 0.
\]

Then \( h \) is the unique solution to the differential equation \( h'' + k h = 0 \) with boundary conditions \( h(0) = 1 \) and \( h'(0) = -\gamma \). By the Laplacian comparison theorem for \( \rho_{\partial M} \) (see [Kasue 1984, Theorem 0.3] or [Wang 2007]),

\[
\Delta \rho_{\partial M} \geq \frac{(d-1)h'}{h}(\rho_{\partial M}) \quad \text{and} \quad \rho_{\partial M} < i_{\partial M} \wedge h^{-1}(0),
\]

where \( h^{-1}(0) = (1/\sqrt{k}) \arcsin(\sqrt{k}/\sqrt{k + \gamma^2}) \) is the first zero point of \( h \). Fix a positive number \( r_0 \leq i_{\partial M} \wedge h^{-1}(0) \), and let

\[
\delta = \frac{2\sigma}{\int_0^{r_0} h(s) - h(r_0)^{d-1} \, ds},
\]

\[
\varphi(r) = 1 + \delta \int_0^r (h(s) - h(r_0))^{1-d} \, ds \int_{s \wedge r_0}^r (h(u) - h(r_0))^{d-1} \, du.
\]

It is easy to see that \( \varphi \circ \rho_{\partial M} \) is differentiable with a Lipschitzian gradient. By a simple approximation argument, we may apply Theorem 1.3 and Corollary 1.4 to \( \phi = \varphi \circ \rho_{\partial M} \); see [Wang 2007, page 1436].

Obviously, (1-7) and \( N = \nabla \rho_{\partial M} \) imply

\[
\Delta \varphi \circ \rho_{\partial M} \geq -\delta \quad \text{and} \quad N \log \varphi \circ \rho_{\partial M} |_{\partial M} = \varphi'(0)/\varphi(0) = 2\sigma.
\]

Moreover, by [Wang 2007, (20)] we have

\[
\delta \leq 2\sigma dr_0^{-1} \quad \text{and} \quad \varphi(r_0) \leq 1 + \sigma dr_0.
\]

Thus, for \( \phi := \varphi \circ \rho_{\partial M} \) we have

\[
-\phi^{-1} L \phi \leq 2\sigma dr_0^{-1} + 2\sigma \|Z\|_{\infty}, \quad \|\nabla \log \phi\|_{\infty}^2 \leq 4\sigma^2,
\]

\[
\|\phi\|_{\infty} \leq \varphi(r_0) \leq 1 + \sigma dr_0.
\]

Combining these with Theorem 1.3 and Corollary 1.4, we obtain these explicit inequalities on a class of nonconvex and noncompact manifolds:

**Corollary 1.5.** Let \( i_{\partial M} > 0 \), and suppose \( \gamma \geq \sigma \geq 0 \) and \( \text{Sect}_M \leq k \) for some \( \gamma, \sigma, k > 0 \). If (1-2) holds and \( \|Z\|_{\infty} < \infty \), then for any positive number

\[
r_0 \leq \min \{ i_{\partial M}, (1/\sqrt{k}) \arcsin(\sqrt{k}/\sqrt{k + \gamma^2}) \},
\]
the inequalities
\[ |\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \leq \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K_{\varepsilon}}{4(\alpha - 1 - \sigma dr_0)} \]
and
\[ P_t f(x) \leq \left(\frac{t+s}{t}\right)^{m(1+\varepsilon)/2(1-\varepsilon)} (P_{t+s} f(y)) \exp\left(\frac{\alpha\rho(x, y)^2}{4s} + \frac{m\alpha K_{\varepsilon}s}{4(\alpha - 1 - \sigma dr_0)}\right) \]
hold for all positive \( f \in C_b(M) \), \( t > 0 \), \( \varepsilon \in (0, 1) \), \( \alpha > 1 + \sigma dr_0 \), and
\[ K_{\varepsilon} = \frac{1+\varepsilon}{1-\varepsilon} \left( K + \frac{4\sigma^2}{\varepsilon} + \frac{\sigma d}{r_0} + \sigma \|Z\|_\infty + \frac{m\alpha^2\sigma^2(1+\varepsilon)}{2(\alpha - 1 - \sigma dr_0)^2(1-\varepsilon)}\right). \]

Combining our gradient estimate with an approximation and a probabilistic argument, we can derive the gradient estimate (1-9) for a class of noncompact manifolds:

**Theorem 1.6.** Let \( M \) satisfy \( A \), and let \( L \) satisfy (1-2) with \( \|Z\|_\infty < \infty \). Let \( \kappa_1 \) and \( \kappa_2 \) be positive elements of \( C_b(M) \) such that
\[ (1-8) \quad \text{Ric} - \nabla Z \geq -\kappa_1 \quad \text{and} \quad \|L\| \geq -\kappa_2 \]
hold on \( M \) and \( \partial M \), respectively. Then
\[ (1-9) \quad |\nabla P_t f|(x) \leq E^x \left(|\nabla f|(X_t) \exp\left(\int_0^t \kappa_1(X_s) ds + \int_0^t \kappa_2(X_s) dl_s\right)\right) \]
holds for all \( f \in C^1_b(M) \), \( t > 0 \), and \( x \in M \).

Inequality (1-9) was first derived by Hsu [2002] on a compact manifold with boundary. In [2002, Theorem 3.7], Hsu applied the Itô formula to \( F(U_t, T-t) := U_t^{-1} \nabla P_{T-t} f(X_t) \), where \( U_t \) is the horizontal lift of \( X_t \) on the frame bundle \( O(M) \). Since \( M \) is compact, the (local) martingale part of this process is a real martingale (it may not be for noncompact \( M \)). Then the desired gradient estimate followed immediately from [2002, Corollary 3.6]. In Section 4, we will prove the boundedness of \( \nabla P_{T-t} f \) on \( [0, T] \times M \) for any \( T > 0 \) and \( f \in C^1_b(M) \), which leads to a simple proof of (1-9) for a class of noncompact manifolds.

### 2. Proof of Theorem 1.2

We consider the convex case and pass to the nonconvex case using the conformal change of metric constructed in [Wang 2007]. Without loss of generality, we may assume that \( \sup G := \sup_{[0,T]\times M} G > 1 \). (Otherwise, we simply replace \( G \) by \( mG \) for a sufficiently large \( m > 0 \).)
2.1. Convex $\partial M$. Fix $o \in M$, and let $\rho_o$ be the Riemannian distance to the point $o$. Since $\partial M$ is convex, there exists a minimal geodesic in $M$ of length $\rho(x, y)$ that links any $x$ and $y$ in $M$; see for example [Wang 2005a, Proposition 2.1.5]. So, by (1.2) and a comparison theorem (see [Qian 1998])

$$L \rho_o \leq \sqrt{K(m-1)} \coth(\sqrt{K/(m-1)} \rho_o)$$

holds outside $\{o\} \cup \text{cut}(o)$, where $\text{cut}(o)$ is the cut locus of $o$. In the sequel, we will set $L \rho_o = 0$ on $\text{cut}(o)$ so that this implies

(2.1) $L \sqrt{1 + \rho_o^2} \leq c_1$ on $M$

for some constant $c_1 > 0$.

Let $h \in C_0^\infty((0, \infty))$ be decreasing such that

$$h(r) = \begin{cases} 1 & \text{if } r \leq 1, \\ \exp(-(3-r)^{-1}) & \text{if } r \in [2, 3), \\ 0 & \text{if } r \geq 3. \end{cases}$$

Obviously, for any $\varepsilon > 0$ we have

(2.2) $\sup_{[0,\infty)} \{|h^{\varepsilon-1}h''| + |h^{\varepsilon-1}h'|\} < \infty$.

Let $W = \sqrt{1 + \rho_o^2}$, and take $\varphi_n = h(W/n)$ for $n \geq 1$. Then

(2.3) $\{\varphi_n = 1\} \uparrow M$ as $n \uparrow \infty$.

So, according to (2.1) and (2.2),

$$|\nabla \log \varphi_n| \leq \frac{c}{n \varphi_n^\varepsilon},$$

(2.4) $\varphi_n^{-1} L \varphi_n = \frac{h'(W/n)}{nh(W/n)} LW + \frac{h''(W/n)}{n^2 h(W/n)} |\nabla W|^2 \geq -\frac{c}{n \varphi_n^\varepsilon}$

holds for some constant $c > 0$ and all $n \geq 1$.

Let $G_n(t, x) = \varphi_n(x)G(t, x)$ for $t \in [0, T]$ and $x \in M$. Since $G_n$ is continuous with compact support, there exists $(t_n, x_n) \in [0, T] \times M$ such that

$$G_n(t_n, x_n) = \max_{[0, T] \times M} G_n.$$ 

By (2.3) and that $\sup G > 1$, we have $\lim_{n \to \infty} G(t_n, x_n) = \sup G > 1$. By renumbering from a sufficient large $n_0$, we may assume that $G_n(t_n, x_n)$ is greater than 1 and is increasing in $n$. In particular, Lemma 1.1(i) holds and

(2.5) $\varphi_n(x_n) \geq 1/G(t_n, x_n)$ for $n \geq 1$. 
Moreover, since \( G_n(0, \cdot) \leq 0 \), we have \( t_n > 0 \) and \( \partial_t G(t_n, x_n) \geq 0 \) for \( n \geq 1 \). Thus, it remains to confirm that
\[
\begin{align*}
|\nabla G(t_n, x_n)| &\leq c G(t_n, x_n)^{1+\varepsilon}/n \quad \text{and} \\
L G(t_n, x_n) &\leq c G(t_n, x_n)^{1+\varepsilon}/n \quad \text{for } n \geq 1
\end{align*}
\]
for some constant \( c > 0 \). Indeed, by using a subsequence \( \{(t_{mn}, x_{mn})\}_{n \geq 1} \) for \( m \geq c \) to replace \( \{(t_n, x_n)\}_{n \geq 1} \), one may reduce (2-6) with some \( c > 0 \) to that with \( c = 1 \).

Since \( x_n \) is the maximal point of \( G_n \), we have \( \nabla G_n(t_n, x_n) = 0 \) if \( x_n \in M \setminus \partial M \). If \( x_n \in \partial M \), we have \( \nabla G_n(t_n, x_n) \leq 0 \). Recall that \( \nabla G(t_n, \cdot) \geq 0 \) and \( G(t_n, x_n) > 0 \). Then, noting that \( N\rho_0 \leq 0 \) together with \( h' \leq 0 \) implies \( N\varphi_n \geq 0 \), we conclude that \( \nabla G_n(t_n, x_n) \geq 0 \). Hence, \( \nabla G_n(t_n, x_n) = 0 \). Moreover, since \( x_n \) is the maximal point of \( G_n(t_n, \cdot) \) on the closed manifold \( \partial M \), we have \( U G_n(t_n, x_n) = 0 \) for all \( U \in T \partial M \). Therefore, \( \nabla G_n(t_n, x_n) = 0 \) also holds for \( x_n \in \partial M \). Combining this with (2-4) and (2-5), we obtain
\[
|\nabla G(t_n, x_n)| \leq \frac{G(t_n, x_n)}{\varphi_n(x_n)} |\nabla \varphi_n| \leq \frac{c G(t_n, x_n)^{1+\varepsilon}}{n},
\]
which proves the first inequality in (2-6).

Finally, by (2-4), the inequality
\[
\varphi_n L_n G + GL_n \varphi_n + 2\langle \nabla G, \nabla \varphi_n \rangle \geq \varphi_n L_n G - \frac{c \varphi_n^{1-\varepsilon}}{n} G - \frac{2c \varphi_n^{1-\varepsilon}}{n} |\nabla G| =: \Phi
\]
holds on \( \{G_n > 0\} \setminus \text{cut}(\partial) \). By Lemma 2.1 below we obtain at the point \((t_n, x_n)\) that
\[
L G \leq \frac{c}{n \varphi_n} G + \frac{2c}{n \varphi_n} |\nabla G|.
\]
Combining this with (2-5) and the first inequality in (2-6), we get
\[
L G(t_n, x_n) \leq \frac{c}{n} G^{1+2\varepsilon}(t_n, x_n)
\]
for some constant \( c > 0 \) and all \( n \geq 1 \). Since \( \varepsilon > 0 \) is arbitrary, we may replace \( \varepsilon \) by \( \varepsilon/2 \) (recall that \( G(t_n, x_n) \geq 1 \)). This proves the second inequality in (2-6).

**Lemma 2.1.** The reflecting \( L \)-diffusion process is nonexplosive, and for any \( \Phi \) in \( C_b(M) \) such that
\[
\Phi \leq L G_n = GL \varphi_n + \varphi_n L G + 2\langle \nabla \varphi_n, \nabla G \rangle \quad \text{on } \{G_n > 0\} \setminus \text{cut}(\partial),
\]
we have \( \Phi(t_n, x_n) \leq 0 \) for all \( n \geq 1 \).

**Proof.** Let \( X_t \) be the reflecting \( L \)-diffusion process generated by \( L \), and let \( U_t \) be its horizontal lift on the frame bundle \( O(M) \). By the Itô formula for \( \rho_o(X_t) \)
found by Kendall [1987] for \( \partial M = \emptyset \) and by the fact that \( N_{\rho_0}|_{\partial M} \leq 0 \) when \( \partial M \) is nonempty but convex, we have

\[
(2-7) \quad d\rho_0(X_t) = \sqrt{2}(\nabla \rho_0(X_t), U_t dB_t + L\rho_0(X_t) dt - dl_t + dl'_t,
\]

where \( B_t \) is the \( d \)-dimensional Brownian motion, where \( L\rho_0 \) is taken to be zero on \( \{o\} \cup \text{cut}(o) \), and where \( l_t \) and \( l'_t \) are two increasing processes such that \( l'_t \) increases only when \( X_t = o \), while \( l_t \) increases only when \( X_t \in \text{cut}(o) \cup \partial M \) (note that \( l'_t = 0 \) for \( d \geq 2 \)).

Combining this with (2-1) we obtain

\[
d\sqrt{1 + \rho_0^2(X_t)} \leq dM_t + L\sqrt{1 + \rho_0^2(X_t)} dt \leq dM_t + c_1 dt
\]

for some martingale \( M_t \). This implies immediately that \( X_t \) does not explode.

Now, let us take \( X_0 = x_0 \). Since \( h' \leq 0 \), it follows from (2-7) that

\[
(2-8) \quad d\varphi_n(X_t) \geq \sqrt{2}(\nabla \varphi_n(X_t), U_t dB_t + L\varphi_n(X_t) dt,
\]

where we set \( L\varphi_n = 0 \) on \( \text{cut}(o) \) as above.

On the other hand, since \( NG(t, \cdot) \geq 0 \), we may apply the Itô to \( G(t, X_t) \) to obtain

\[
(2-9) \quad dG(t, X_t) \geq \sqrt{2}(\nabla G(t, X_t), U_t dB_t + LG(t, X_t) dt.
\]

Because \( G_n(t_n, x_n) > 0 \), there exists an \( r > 0 \) such that \( G_n > 0 \) on \( B(x_n, r) \), the geodesic ball in \( M \) centered at \( x_n \) with radius \( r \). Let

\[
\tau = \inf\{t \geq 0 : X_t \notin B(x_n, r)\}.
\]

Then (2-8) and (2-9) imply

\[
dG_n(t_n, X_t) \geq dM_t + LG_n(t_n, \cdot)(X_t) dt \geq dM_t + \Phi(t_n, X_t) dt \quad \text{for } t \leq \tau
\]

for some martingale \( M_t \). Since \( G_n(t_n, X_t) \leq G_n(t_n, x_n) \) and \( X_0 = x_n \), this implies that

\[
0 \geq \mathbb{E}G_n(t_n, X_{t \wedge \tau}) - G_n(t_n, x_n) \geq \mathbb{E} \int_0^{t \wedge \tau} \Phi(t_n, X_s) ds.
\]

Therefore, the continuity of \( \Phi \) implies that

\[
\Phi(t_n, x_n) = \lim_{t \to 0} \frac{1}{E(t \wedge \tau)} \mathbb{E} \int_0^{t \wedge \tau} \Phi(t_n, X_s) ds \leq 0.
\]

2.2. Nonconvex \( \partial M \). Under our assumptions on \( M \), there exists a constant \( R > 1 \) and a function \( \phi \in C^\infty(M) \) such that

\[
1 \leq \phi \leq R, \quad |\nabla \phi| \leq R, \quad N \log \phi|_{\partial M} \geq \sigma.
\]

By [Wang 2007, Lemma 2.1], the boundary \( \partial M \) is convex under the new metric \( \langle \cdot, \cdot \rangle' := \phi^{-2} \langle \cdot, \cdot \rangle \). Let \( L' = \phi^2 L \). By [Wang 2007, Equation (9)], the vector
$U' := \phi U$ is unit under the new metric for any unit vector $U \in TM$, and the corresponding Ricci curvature satisfies

\begin{equation}
\text{Ric}'(U', U') \geq \phi^2 \text{Ric}(U, U) + \phi \Delta \phi - (d - 3)|\nabla \phi|^2 - 2(U \phi)^2 + (d - 2)\phi \text{Hess}_{\phi}(U, U).
\end{equation}

Let $\Delta'$ be the Laplacian induced by the new metric. By [Wang 2007, Lemma 2.2], we have

\begin{equation}
L' := \phi^2 L = \Delta' + (d - 2)\phi \nabla \phi + \phi^2 Z =: \Delta' + Z'.
\end{equation}

Noting that

\begin{equation}
\nabla_X' Y = \nabla_X Y - (X, \nabla \log \phi) Y - (Y, \nabla \log \phi) X + (X, Y) \nabla \log \phi 
\end{equation}

for $X, Y \in TM$, we have

\begin{equation}
\langle \nabla_U' Z', U' \rangle' = \langle \nabla_U Z', U \rangle - \langle Z', \nabla \log \phi \rangle = \phi^2 \langle \nabla_U Z, U \rangle + (U \phi^2)(Z, U) + (d - 2)(U \phi)^2 + (d - 2)\phi \text{Hess}_{\phi}(U, U) - \langle Z', \nabla \log \phi \rangle.
\end{equation}

Combining this with (2-10) and the properties of $\phi$ mentioned above, we find a constant $c_1 > 0$ such that

\begin{equation}
\text{Ric}'(U', U') - \langle \nabla_U' Z', U' \rangle' \geq \phi^2 (\text{Ric} - \nabla Z)(U, U) - c_1 \quad \text{for } |U| = 1.
\end{equation}

Moreover, since

\begin{equation}
(Z' \otimes' Z')(U', U') := ((Z', U')')^2 = \phi^{-2}(Z', U)^2 \leq 2(d - 2)^2 \langle \nabla \phi, U \rangle^2 + 2\phi^2 \langle Z, U \rangle^2 \leq 2(d - 2)^2 R^2 + 2\phi^2 (Z \otimes Z)(U, U),
\end{equation}

it follows from (1-2) and (2-11) that

\begin{equation}
\text{Ric}' - \nabla' Z' - \frac{Z' \otimes' Z'}{2(m - d)} \geq -\phi^2 K - c_2 \geq -K'
\end{equation}

holds for the metric $\langle \cdot, \cdot \rangle'$ and some constants $c_2, K' > 0$. Therefore, we may apply Lemma 2.1 to $L'$ on the convex Riemannian manifold $(M, \langle \cdot, \cdot \rangle')$ to conclude that the desired sequence $\{(t_n, x_n)\}$ exists.

3. Proofs of Theorem 1.3 and Corollary 1.4

**Proof of Theorem 1.3.** When $\partial M$ is convex, Lemma 2.1 ensures that $X_t$ does not explode. If $\partial M$ is nonconvex, this can be confirmed by reparametrizing the time of the process. More precisely, let $X'_t$ be the reflecting diffusion process on $M$ generated by $L' := \phi^2 L$ constructed in Section 2.2. Since $L' = \Delta' + Z'$
satisfies (1-2) for some $K > 0$ on the convex manifold $(M, \langle \cdot, \cdot \rangle')$, the process $X'_t$ generated by $L'$ is nonexplosive by Lemma 2.1. Since $X_t = X'_{\hat{t}^{-1}(t)}$, where $\hat{t}^{-1}$ is the inverse of

$$ t \mapsto \hat{t}(t) = \int_0^t \phi^2(X'_s) \, ds, $$

we have $t\|\phi\|_\infty^2 \leq \hat{t}^{-1}(t) \leq t$, and the process $X_t$ is nonexplosive as well.

Let $f \in C^1_0(M)$ be strictly positive, and let $u(t, x) = \log P_t f(x)$. For a fixed number $T > 0$, we will apply Theorem 1.2 to the reference function

$$ G(t, x) = t\{\phi(x)|\nabla u|^2(t, x) - au_i(t, x)\} \quad \text{for } t \in [0, T] \text{ and } x \in M. $$

Note that $\|N\phi \| \geq -\sigma$ and $N\log \phi \geq 2\sigma$ imply

$$ N\phi \geq 2\sigma \phi,$$

$$ N|\nabla P_t f|^2 = 2\text{Hess}_{P_t f}(N, \nabla P_t f) = 2\|\nabla P_t f, \nabla P_t f\| \geq -2\sigma \|\nabla P_t f\|^2. $$

Since $P_t f$ and hence $u_t$ satisfy the Neumann boundary condition, this implies that

$$ NG = t\{N(\phi)|\nabla u|^2 + \frac{\phi}{(P_t f)^2} N|\nabla P_t f|^2\} \geq t\{2\sigma \phi |\nabla u|^2 - 2\sigma \phi |\nabla u|^2\} = 0 $$

on $\partial M$.

According to [Ledoux 2000, (1.14)], inequality (1-2) implies

$$(3-1) \quad L|\nabla u|^2 - 2\langle \nabla Lu, \nabla u \rangle \geq -2K|\nabla u|^2 + \frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}. $$

By multiplying this inequality by $\varepsilon$ and (1-1) by $2(1 - \varepsilon)$ and by combining the results, we obtain

$$ L|\nabla u|^2 \geq 2\langle \nabla Lu, \nabla u \rangle - 2K|\nabla u|^2 + \frac{2(1 - \varepsilon)(L)^2}{m} + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}. $$

It is also easy to check that $Lu = u_t - |\nabla u|^2$ and $\hat{t}^{-1}|\nabla u|^2 = 2\langle \nabla u, \nabla u_t \rangle$. Then we arrive at

$$(3-2) \quad (L - \hat{t}^{-1})|\nabla u|^2 \geq \frac{2(1 - \varepsilon)}{m}(\langle \nabla u|^2 - u_t \rangle^2 + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2} - 2\langle \nabla u, \nabla|\nabla u|^2 \rangle - 2K|\nabla u|^2. $$

On the other hand,

$$ -\alpha(L - \hat{t}^{-1})u_t = 2\alpha \langle \nabla u, \nabla u_t \rangle = 2\langle \nabla u, \nabla (\phi|\nabla u|^2 - t^{-1}G) \rangle $$

$$ = 2\phi(\nabla u, \nabla |\nabla u|^2) + 2|\nabla u|^2(\nabla u, \nabla \phi - 2t^{-1}(\nabla u, \nabla G). $$
Combining this with (3-2), we obtain

\[(L - \partial_t)G = -\frac{G}{t} + \frac{2(1 - \varepsilon)}{m} \left( (\nabla u)^2 - u_t \right)^2 - 2K \phi t |\nabla u|^2 + 2t \phi (\nabla u, \nabla \phi) \right) \]

\[\geq -\frac{G}{t} + \frac{2(1 - \varepsilon)}{m} \left( (\nabla u)^2 - u_t \right)^2 - 2K \phi t |\nabla u|^2 + 2t |\nabla \phi| \cdot |\nabla u|^2 + t |\nabla u|^2 L \phi.\]

Noting that

\[\frac{\varepsilon \phi t |\nabla u|^2}{2|\nabla u|^2} - 2t |\nabla \phi| \cdot |\nabla u|^2 \geq -\frac{2t |\nabla \phi|^2 |\nabla u|^2}{\varepsilon \phi},\]

we get

\[(3-3) \quad (L - \partial_t)G \geq -\frac{G}{t} + \frac{2(1 - \varepsilon)}{m} \left( (\nabla u)^2 - u_t \right)^2 - 2K \phi t |\nabla u|^2 - 2t |\nabla \phi| \cdot |\nabla u|^2 + t |\nabla u|^2 L \phi.\]

We assume that \(G > 0\), otherwise the proof is done. Since \(G(0, \cdot) = 0\) and \(NG|_{\partial M} \geq 0\), we can apply Theorem 1.2. Let \(((t_n, x_n))\) be fixed in Theorem 1.2 with, for example, \(\varepsilon = 1/2\). Then,

\[(3-4) \quad (L - \partial_t)G(t_n, x_n) \leq \frac{G^{3/2}(t_n, x_n)}{n} \quad \text{and} \quad |\nabla G|(t_n, x_n) \leq \frac{G^{3/2}(t_n, x_n)}{n}.\]

From now on, we evaluate functions at the point \((t_n, x_n)\), so that \(t = t_n\).

Let \(\mu = |\nabla u|^2 / G\). We have

\[|\nabla u|^2 - u_t = \left( \mu - \frac{(\mu t - 1) \phi}{\alpha t} \right) G = \frac{\mu t (\alpha - \phi) + \phi}{\alpha t} G.\]

Combining this with (3-3) and (3-4), we arrive at

\[(3-5) \quad \frac{2(1 - \varepsilon)}{m} \frac{\mu t (\alpha - \phi) + \phi}{\alpha t} G^2 \]

\[\leq \frac{G^{3/2}}{n} + \frac{G}{t} + \frac{2\sqrt{\mu} G^2}{n} + 2t |\nabla \phi| (\mu G)^{3/2} + (2k \phi + 2e^{-1} \phi^{-1} |\nabla \phi|^2 - L \phi) \mu t G.\]

Since it is easy to see that

\[(\mu t (\alpha - \phi) + \phi)^2 \geq \max \{ \phi^2, 4 \mu t (\alpha - \phi) \phi, (2t (\alpha - \phi))^3/2 \sqrt{\phi} \mu^{3/2} \},\]
we may multiply both sides of (3-5) by \( t(\mu t(\alpha - \phi) + \phi)^{-2}G^{-2} \) to obtain

\[
\frac{2(1-\varepsilon)\phi}{ma^2(1+\varepsilon)} \leq \frac{c't}{n(1 + \sqrt{G})} + \frac{1}{\phi^2G} + \frac{2K + 2e^{-1}|\nabla \log \phi|^2 - \phi^{-1}L\phi}{4(\alpha - \phi)G} + \frac{|\nabla \log \phi|\sqrt{t\phi}}{(\alpha - \phi)^{3/2}\sqrt{2G}}
\]

\[
\leq \frac{c't}{n(1 + \sqrt{G})} + \frac{1}{\phi^2G} + \frac{2K + 2e^{-1}|\nabla \log \phi|^2 - \phi^{-1}L\phi}{4(\alpha - \phi)G} + \frac{|\nabla \log \phi|\sqrt{t\phi}}{(\alpha - \phi)^{3/2}\sqrt{2G}} + \frac{2(1-\varepsilon)c\phi}{16(\alpha - \phi)^3(1-\varepsilon)G}
\]

for some constant \( c' > 0 \). Taking \( n \to \infty \) and noting that \( \phi \geq 1 \), we conclude that \( \theta := \sup G \) satisfies

\[
\frac{2(1-\varepsilon)}{ma^2(1+\varepsilon)} \leq \frac{1}{\theta} \left( 1 + \frac{2K + 2e^{-1}|\nabla \log \phi|^2_\infty + \sup(\phi^{-1}L\phi)}{4(\alpha - \|\phi\|_\infty)} \right) T + \frac{|\nabla \log \phi|^2_\infty ma^2(1+\varepsilon)T}{(\alpha - \|\phi\|_\infty)^3(1-\varepsilon)}
\]

Combining this with \( \theta \geq G(T, x) = T(\phi(x)|\nabla u|^2(T, x) - au_1(T, x)) \) for \( x \in M \), we obtain

\[
\phi(x)|\nabla u|^2(T, x) - au_1(T, x) \leq \frac{ma^2(1+\varepsilon)}{2(1-\varepsilon)} \left( \frac{1}{T} + \frac{2K + 2e^{-1}|\nabla \log \phi|^2_\infty + \sup(\phi^{-1}L\phi)}{4(\alpha - \|\phi\|_\infty)} \right) T + \frac{|\nabla \log \phi|^2_\infty ma^2(1+\varepsilon)}{(\alpha - \|\phi\|_\infty)^3(1-\varepsilon)}
\]

for all \( x \in M \). Then the proof is completed since \( T > 0 \) is arbitrary. \( \square \)

**Proof of Corollary 1.4.** By Theorem 1.3, the proof is standard according to [Li and Yau 1986]. For \( x, y \in M \), let \( \gamma : [0, 1] \to M \) be the shortest curve in \( M \) linking \( x \) and \( y \) such that \( |\dot{\gamma}| = \rho(x, y) \). Then, for any \( s, t > 0 \) and \( f \in C^\infty_b(M) \), it follows from (1-5) that

\[
\frac{d}{dr} \log P_{t+rs}f(\gamma_r) = s\partial_u \log P_u f(\gamma_r)|_{u=t+rs} + (\dot{\gamma}_r, \nabla P_{t+rs}f(\gamma_r))
\]

\[
\geq \frac{s}{\alpha} |\nabla \log P_{t+rs}f|^2(\gamma_r) - \rho(x, y)|\nabla \log f| (\gamma_r) + \frac{s}{\alpha} \left( \frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)} + \frac{maK(\phi, e, \alpha)}{4(\alpha - 1\|\phi\|_\infty)} \right)
\]

\[
\geq -\frac{\alpha}{4s} - \frac{s}{\alpha} \left( \frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)} + \frac{maK(\phi, e, \alpha)}{4(\alpha - 1\|\phi\|_\infty)} \right).
\]
We complete the proof by integrating with respect to $dr$ over $[0, 1]$.

\[ \square \]

4. Proof of Theorem 1.6

We first provide a simple proof of (1-9) under an extra assumption that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$ for any $T > 0$; we then drop this assumption by an approximation argument.

**Lemma 4.1.** If that $f \in C_b^1(M)$ is such that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$ for any $T > 0$, then (1-9) holds.

**Proof:** For any $\varepsilon > 0$, let $\eta_s = \sqrt{\varepsilon + |\nabla P_{t-s} f|^2}(X_s)$ for $s \leq t$. By the Itô formula, we have

\[
d\eta_s = dM_s + \frac{L|\nabla P_{t-s} f|^2 - 2(\nabla L P_{t-s} f, \nabla P_{t-s} f)}{2\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s) ds
\]

\[
- \frac{|\nabla |\nabla P_{t-s} f|^2|}{4(\varepsilon + |\nabla P_{t-s} f|^2)^{3/2}}(X_s) ds + \frac{N|\nabla P_{t-s} f|^2}{2\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s) ds
\]

for $s \leq t$, where $M_s$ is a local martingale. Combining this with (1-8) and (3-1), with $\kappa_1$ in place of $K_0$, we obtain

\[
d\eta_s \geq dM_s - \frac{\kappa_1|\nabla P_{t-s} f|^2}{\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s) ds - \frac{\kappa_2|\nabla P_{t-s} f|^2}{\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s) ds
\]

\[
\geq dM_s - \kappa_1(X_s) \eta_s ds - \kappa_2(X_s) \eta_s ds \quad \text{for } s \leq t.
\]

Now $\eta_s$ is bounded on $[0, t]$, and by the proof of [Wang 2005b, Lemma 2.1] we have $\mathbb{E}e^{\lambda \eta_s} < \infty$ for all $\lambda > 0$. This implies that

\[
[0, t] \ni s \mapsto \sqrt{\varepsilon + |\nabla P_{t-s} f|^2}(X_s) \exp\left(\int_0^s \kappa_1(X_s) ds + \int_0^s \kappa_2(X_s) ds\right)
\]

is a submartingale for any $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we conclude that

\[
[0, t] \ni s \mapsto |\nabla P_{t-s} f|(X_s) \exp\left(\int_0^s \kappa_1(X_s) ds + \int_0^s \kappa_2(X_s) ds\right)
\]

is a submartingale as well.

According to Lemma 4.1, it suffices to confirm the boundedness of $|\nabla P_{(\cdot)} f|$ on $[0, T] \times M$ for any $T > 0$ and $f \in C_b^1(M)$. We shall start from $f \in C_0^\infty(M)$ with $\partial M = 0$, then pass to $f \in C_b^1(M)$ by combining an approximation argument and Lemma 4.1.

**Case a.** Let $f \in C_0^\infty(M)$ with $\partial M = 0$. We have

\[
P_t f = f + \int_0^t P_s L f ds.
\]

(4-1)
Since $Lf$ is bounded, there is a $c > 0$ such that $Lf + c \geq 1$. Applying Corollary 1.5 with for example $\alpha = 2 + \sigma dr_0$ and $\epsilon = 1/2$, but using $Lf + c$ in place of $f$, we obtain
\[
|\nabla P_s L f| = |\nabla P_s (Lf + c)| \\
\leq \|Lf + c\|_\infty \left(\alpha \|P_s L^2 f\|_\infty + \frac{m(1 + \epsilon)\alpha^2}{2(1 - \epsilon)s} + \frac{ma^2 K_\epsilon}{4(\alpha - 1 - \sigma dr_0)}\right)^{1/2} \\
\leq c_1/\sqrt{s} \quad \text{for } s \leq T
\]
for some constant $c_1 > 0$. Combining this with (4-1) we conclude that, for some constant $c_2 > 0$,
\[
|\nabla P_t f| \leq |\nabla f| + \int_0^t \frac{c_1}{\sqrt{s}} \, ds \leq c_2 \quad \text{for } t \leq T.
\]

**Case b.** Let $f \in C_0^\infty(M)$. There exists a sequence of functions $\{f_n\}_{n \geq 1} \subset C_0^\infty(M)$ such that $Nf_n|_{\partial M} = 0$, $f_n \to f$ uniformly as $n \to \infty$, and $\|\nabla f_n\|_\infty \leq 1 + \|\nabla f\|_\infty$ holds for any $n \geq 1$; see for example [Wang 1994]. By Case a and Lemma 4.1, (1-9) holds with $f_n$ in place of $f$ such that
\[
|P_t f_n(x) - P_t f_n(y)| \leq C \quad \text{for } t \leq T, \ n \geq 1, \ x \neq y
\]
for some constant $C > 0$. Letting first $n \to 0$ and then $y \to x$, we conclude that $|\nabla (\cdot f)|$ is bounded on $[0, T] \times M$.

**Case c.** Let $f \in C_b^\infty(M)$. Let $\{g_n\}_{n \geq 1} \subset C_0^\infty(M)$ be such that $0 \leq g_n \leq 1$, $|\nabla g_n| \leq 2$ and $g_n \uparrow 1$ as $n \uparrow \infty$. By Case b and Lemma 4.1, we may apply (1-9) to $g_n f$ in place of $f$ such that
\[
|P_t (g_n f)(x) - P_t (g_n f)(y)| \leq C \quad \text{for } t \leq T, \ n \geq 1, \ x \neq y
\]
for some constant $C > 0$. By the same reason as in Case b, we conclude that $|\nabla P_t (\cdot f)|$ is bounded on $[0, T] \times M$.

**Case d.** Finally, for $f \in C_b^1(M)$, there exist $\{f_n\}_{n \geq 1} \subset C_b^\infty(M)$ such that $f_n \to f$ uniformly as $n \to \infty$ and $\|\nabla f_n\|_\infty \leq \|\nabla f\|_\infty + 1$ for any $n \geq 1$. The proof is completed by the same reason as in Cases b and c. \qed

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