Pacific Journal of Mathematics

Volume 245 No. 2 April 2010

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by

E. F. Beckenbach (1906–1982) F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

Regular subscription rate for 2009: US$450 a year (10 issues). Special rate: US$225 a year to individual members of supporting institutions. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840 is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

VOLUME ENTROPY OF HILBERT GEOMETRIES

GAUTIER BERCK, ANDREAS BERNIG AND CONSTANTIN VERNICOS

We show that among all plane Hilbert geometries, the hyperbolic plane has maximal volume entropy. More precisely, we show that the volume entropy is bounded above by $2/(3-d) \leq 1$, where d is the Minkowski dimension of the extremal set of K, and we construct an explicit example of a plane Hilbert geometry with noninteger volume entropy. In arbitrary dimension, the hyperbolic space has maximal entropy among all Hilbert geometries satisfying some additional technical hypothesis. To achieve this result, we construct a new projective invariant of convex bodies, similar to the centro-affine area.

1. Introduction

In his famous Fourth Problem, Hilbert asked for a characterization of metric geometries whose geodesics are straight lines. He constructed a special class of examples, now called *Hilbert geometries* [Hilbert 1895; 1999], which have since attracted much interest; see, for example, [Nasu 1961; de la Harpe 1993; Karlsson and Noskov 2002; Socié-Méthou 2004; Foertsch and Karlsson 2005; Benoist 2006; Colbois and Vernicos 2007], and the two complementary surveys [Benoist 2008] and [Vernicos 2005].

A Hilbert geometry is a particularly simple metric space on the interior of a compact convex set K (see the definition below). This metric happens to be a complete Finsler metric whose set of geodesics contains the straight lines. Since the definition of the Hilbert geometry only uses cross-ratios, the Hilbert metric is a projective invariant. In the particular case where K is an ellipsoid, the Hilbert geometry is isometric to the usual hyperbolic space.

An important part of the above mentioned works, and of older ones, is to study how different or close to the hyperbolic geometry these geometries can be. For instance, if K is not an ellipsoid, Kay [1967, Corollary 1] showed that the metric is never Riemannian. This result is related to the fact that among all finite-dimensional normed vector spaces, many notions of curvatures are only satisfied

MSC2000: 51F99, 53A20, 53C60.
Keywords: metric geometry, Hilbert geometry, convex geometry.
Berck and Bernig were supported by the Schweizerischer Nationalfonds grants SNF PP002-114715/1 and 200020-121506/1.

by the Euclidean spaces [Kelly and Paige 1952; Kelly and Straus 1958; 1968]. However, if ∂K is sufficiently smooth, then the flag curvature, an analog of the sectional curvature, of the Hilbert metric is constant and equals -1; see [Shen 2001, Example 9.2.2]. Hence one can ask whether or not these geometries behave like negatively curved Riemannian manifolds. The example of the triangle geometry that is isometric to a two-dimensional normed vector space shows that things are a little more involved (see [de la Harpe 1993], and also theorems cited below). The present work is partially inspired by the feeling that Hilbert geometries might be thought of as geometries with Ricci curvature bounded from below, and focuses on the volume growth of balls.

Unlike the Riemannian case, where there is only one natural choice of volume, there are several good choices of volume on a Finsler manifold. We postpone this issue to Section 2 and fix just one volume (like the n-dimensional Hausdorff measure) for the moment.

Let $B(o,r)$ be the metric ball of radius r centered at o. The volume entropy of K is defined by the limit (provided it exists)

$$\operatorname{Ent} K := \lim_{r\to\infty} \frac{\log \operatorname{Vol} B(o,r)}{r}. \tag{1}$$

The entropy depends neither on the particular choice of the base point $o \in \operatorname{int} K$, nor on the particular choice of the volume. If $h = \operatorname{Ent} K$, then $\operatorname{Vol} B(o,r)$ behaves roughly as e^{hr}.

It is well known and easy to prove (see S. Gallot, D. Hulin and J. Lafontaine [Gallot et al. 2004, Section III.H]) that the volume of a ball of radius r in the n-dimensional hyperbolic space is given by

$$n\omega_n \int_0^r (\sinh s)^{n-1} ds = O(e^{(n-1)r}),$$

where ω_n is the volume of the Euclidean unit ball of dimension n. It follows that the entropy of an ellipsoid equals $n-1$.

In general, it is not known whether the limit above exists, although it does in several cases: It exists if the convex set K is divisible, which means that a discrete subgroup of the group of isometries of the Hilbert geometry acts cocompactly [Benoist 2004]. If the convex set is sufficiently smooth (for example, C^2 with positive curvature suffices), the entropy exists and equals $n-1$ (see the theorem of Colbois and Verovic below). In general, one may define lower and upper entropies $\underline{\operatorname{Ent}}$ and $\overline{\operatorname{Ent}}$ by replacing the limit in the definition (1) by lim inf or lim sup.

There is a well known conjecture (whose origin seems difficult to locate) saying that the hyperbolic space has maximal entropy among all Hilbert geometries of the same dimension:

Conjecture. *For any n-dimensional Hilbert geometry*, $\overline{\mathrm{Ent}}\, K \leq n-1$.

Notice that an analogous result in Riemannian geometry is a consequence of Bishop's volume comparison theorem for complete Riemannian manifolds of Ricci curvature bounded by $-(n-1)$ [Gallot et al. 2004, Theorem 3.101(i)].

Several particular cases of the conjecture were treated in the literature. The following one shows that the volume entropy does not characterize the hyperbolic geometry among all Hilbert geometries.

Theorem [Colbois and Verovic 2004]. *If K is C^2-smooth with strictly positive curvature*, *then the Hilbert metric of K is bi-Lipschitz to the hyperbolic metric and therefore* $\mathrm{Ent}\, K = n-1$.

The case of convex polytopes is rather well understood.

Theorem [Bernig 2009; Vernicos 2008b]. *The Hilbert metric associated to a convex body K is bi-Lipschitz to a normed space if and only if K is a polytope. In particular*, *the entropy of a polytope is* 0.

The two-dimensional case was earlier obtained by Colbois, Vernicos, and Verovic in [Colbois et al. 2008].

Instead of taking the volume of balls, a natural choice is to study the volume growth of the metric spheres $S(o,r)$. One may define a (spherical) entropy by

$$\mathrm{Ent}^s\, K := \lim_{r\to\infty} \frac{\log \mathrm{Vol}\, S(o,r)}{r}, \tag{2}$$

provided the limit exists. In general, one may define upper and lower spherical entropies $\overline{\mathrm{Ent}}^s\, K$ and $\underline{\mathrm{Ent}}^s\, K$ by replacing the limits in (2) by a lim sup or lim inf.

The next theorem is a spherical version of the theorem of Colbois and Verovic.

Theorem [Borisenko and Olin 2008]. *If K is an n-dimensional convex body of class C^3 with positive Gauss curvature*, *then* $\mathrm{Ent}^s = n-1$.

Our first main theorem treats the two-dimensional case. Recall that an extremal point of a convex body K is a point that is not a convex combination of two other points of K.

First main theorem. *Let K be a two-dimensional convex body. Let d be the upper Minkowski dimension of the set of extremal points of K. Then the entropy of K is bounded by*

$$\overline{\mathrm{Ent}}\, K \leq \frac{2}{3-d} \leq 1. \tag{3}$$

The inequality is sharp if K is smooth or contains some positively curved smooth part in the boundary. In this case the upper Minkowski dimension of $\mathrm{ex}\, K$ and the entropy are both 1. On the other hand, for polygons the upper Minkowski dimension of the set of extremal points and the entropy both vanish (see the theorem of [Colbois et al. 2008]), and the inequality is not sharp in this case.

It should be noted that the entropy behaves in a rather subtle way (see also [Vernicos 2008a] for a technical study of the entropy, complementary to this paper). As we have seen above, the entropy of a polygon vanishes. In contrast to this, we will construct a convex body with piecewise affine boundary whose entropy is between $1/4$ and $3/4$.

Our second main theorem applies in all dimensions. It weakens in a substantial way the assumptions in the theorem of Colbois and Verovic and strengthens its conclusions, for it gives not only the precise value of the entropy but also the *entropy coefficient*. To state it, we introduce a projective invariant of convex bodies, which is interesting in itself.

Let V be an n-dimensional vector space with origin o. Given a convex body K containing o in the interior, we define a positive function a on the boundary by the condition that for $p \in \partial K$ we have $-a(p)p \in \partial K$. The letter a stands for *antipodal*. If V is endowed with a Euclidean scalar product, we let $k(p)$ be the Gauss curvature and $n(p)$ be the outer normal vector at a boundary point p (whenever they are well-defined, which is almost everywhere the case following [Alexandroff 1939]).

Definition. The *centroprojective* area of K is

$$\mathcal{A}_p(K) := \int_{\partial K} \frac{\sqrt{k}}{\langle n, p\rangle^{(n-1)/2}} \left(\frac{2a}{1+a}\right)^{(n-1)/2} dA. \tag{4}$$

It is not quite obvious (but true, as we shall see) that this definition does not depend on the choice of the scalar product. In fact, the centroprojective area is invariant under *projective transformations* fixing the origin. The reader familiar with the theory of valuations may notice the similarity with the centroaffine surface area, whose definition is the same except that the second factor (containing the function a) does not appear. We refer to [Laugwitz 1965; Leichtweiß 1998] for more information on affine and centroaffine differential geometry.

Second main theorem. *If ∂K is $C^{1,1}$ or if $n = 2$, then*

$$\lim_{r\to\infty} \frac{\operatorname{Vol} B(o, r)}{\sinh^{n-1} r} = \frac{1}{n-1} \mathcal{A}_p(K). \tag{5}$$

In the first case, $\mathcal{A}_p(K) \neq 0$ and hence $\operatorname{Ent} K = n - 1$.

Our next theorem, together with the previous ones, shows that it suffices to assume K to be merely of class $C^{1,1}$ in the theorem of Borisenko and Olin.

Theorem. *For each convex body K,*

$$\underline{\operatorname{Ent}}^s K = \underline{\operatorname{Ent}}\, K \quad \textit{and} \quad \overline{\operatorname{Ent}}^s K = \overline{\operatorname{Ent}}\, K.$$

Plan of the paper. In the next section, we collect some well-known facts about convex bodies, Hilbert geometries and volumes on Finsler manifolds, and we prove a number of easy lemmas. Using some inequalities for volumes in normed spaces, we show that entropy and spherical entropy coincide for general convex bodies.

In Section 3, we use the lemmas to prove our main theorems. In Section 4, we give an intrinsic definition of the centroprojective surface area and study some of its properties. In particular, we show that it is upper semicontinuous with respect to Hausdorff topology.

2. Preliminaries on convex bodies and Hilbert geometries

2.1. ***Convex bodies.*** Let V be a finite-dimensional real vector space. By a *convex body*, we mean a compact convex set $K \subset V$ with nonempty interior (note that this last condition is sometimes not required in the literature). Most of the time, the convex bodies will be assumed to contain the origin in their interiors. In such a case, we will as usual call the *Minkowski functional* the positive, homogeneous function of degree 1 whose level set at height 1 is the boundary ∂K. It is a convex function, which by Alexandroff's theorem admits a quadratic approximation almost everywhere [Alexandroff 1939; Evans and Gariepy 1992, page 242]. In the following, boundary points where Alexandroff's theorem applies will be called *smooth*. If we assume the vector space to be equipped with an inner product, the principal curvatures of the boundary and its Gauss curvature k are well defined at every smooth point.

We will be concerned with generalizations and variations of *Blaschke's rolling theorem*, a proof of which may be found in [Leichtweiß 1993].

Theorem 2.1 [Blaschke 1956]. *Let K be a convex body in $\mathbb{R}^n$ whose boundary is C^2 with everywhere positive Gaussian curvature. Then there are two positive radii R_1 and R_2 such that for every boundary point p, there exists a ball of radius R_1 (respectively R_2) containing p on its boundary and contained in K (respectively containing K).*

We first remark that for the "inner part" of Blaschke's result, the regularity of the boundary may be lowered. Recall that the boundary of a convex body is $C^{1,1}$ provided it is C^1 and the Gauss map is Lipschitz continuous. Roughly speaking, the second condition says that the curvature of the boundary remains bounded, even if it is only almost everywhere defined. The following proposition then gives a geometrical characterization of such bodies [Hörmander 2007, Proposition 2.4.3; Bangert 1999; Hug 1999b].

Proposition 2.2. *The boundary of a convex body K is $C^{1,1}$ if and only if there exists some $R > 0$ such that K is the union of balls with radius R.*

Without assumption on the boundary, there is still an integral version of Blaschke's rolling theorem.

Theorem 2.3 [Schütt and Werner 1990]. *For a convex body K containing the unit ball of a Euclidean space and $p \in \partial K$, let $R(p) \in [0, \infty)$ be the radius of the biggest ball contained in K and containing p. Then for all $0 < \alpha < 1$,*

$$\int_{\partial K} R^{-\alpha} d\mathcal{H}^{n-1} < \infty. \tag{6}$$

We will need the following refinement of this theorem.

Proposition 2.4. *In the same situation as in Theorem 2.3, for each Borel subset $B \subset \partial K$ we have*

$$\int_B R^{-\alpha} d\mathcal{H}^{n-1} \le 2(n-1)^\alpha \Big(\frac{2^\alpha}{1-2^{\alpha-1}}\Big)^\alpha (\mathcal{H}^{n-1}(B))^{1-\alpha} (\mathcal{H}^{n-1}(\partial K))^\alpha. \tag{7}$$

In particular, for some constant C depending on K, we have

$$\int_B R^{-1/2} d\mathcal{H}^{n-1} \le C(\mathcal{H}^{n-1}(B))^{1/2}. \tag{8}$$

Proof. By [Schütt and Werner 1990, Lemma 4], we have for $0 \le t \le 1$

$$\mathcal{H}^{n-1}(\{p \in \partial K \mid R(p) \le t\}) \le (n-1)t\ \mathcal{H}^{n-1}(\partial K), \tag{9}$$

from which we deduce that, for each $0 < \epsilon < 1$,

$$\begin{aligned} \int_{\partial K \cap \{R<\epsilon\}} R^{-\alpha} d\mathcal{H}^{n-1} &= \sum_{i=0}^{\infty} \int_{\partial K \cap \{\epsilon 2^{-i-1} \le R < 2^{-i}\epsilon\}} R^{-\alpha} d\mathcal{H}^{n-1} \\ &\le \sum_{i=0}^{\infty} (\epsilon 2^{-i-1})^{-\alpha}\ \mathcal{H}^{n-1}(\partial K \cap \{\epsilon 2^{-i-1} \le R < 2^{-i}\epsilon\}) \\ &\le \sum_{i=0}^{\infty} (\epsilon 2^{-i-1})^{-\alpha} (n-1) 2^{-i}\epsilon\ \mathcal{H}^{n-1}(\partial K) \\ &= \epsilon^{1-\alpha}(n-1)\frac{2^\alpha}{1-2^{\alpha-1}}\ \mathcal{H}^{n-1}(\partial K). \end{aligned} \tag{10}$$

It follows that

$$\begin{aligned} \int_B R^{-\alpha} d\mathcal{H}^{n-1} &= \int_{B \cap \{R<\epsilon\}} R^{-\alpha} d\mathcal{H}^{n-1} + \int_{B \cap \{R \ge \epsilon\}} R^{-\alpha} d\mathcal{H}^{n-1} \\ &\le \epsilon^{1-\alpha}(n-1)\frac{2^\alpha}{1-2^{\alpha-1}}\ \mathcal{H}^{n-1}(\partial K) + \epsilon^{-\alpha}\ \mathcal{H}^{n-1}(B). \end{aligned}$$

We get the inequality of the lemma by choosing

$$\epsilon := \frac{1-2^{\alpha-1}}{2^\alpha(n-1)} \frac{\mathcal{H}^{n-1}(B)}{\mathcal{H}^{n-1}(\partial K)}.$$

□

2.2. ***Hilbert geometries.*** The *Hilbert distance* between two distinct points x and y in $\operatorname{int} K$ is defined by

$$d(x, y) := \tfrac{1}{2}\bigl|\log[a, b, x, y]\bigr|,$$

where a and b are the intersections of the line passing through x and y with the boundary ∂K, and $[a, b, x, y]$ denotes the cross-ratio (adopting the convention of [Bridson and Haefliger 1999]).

This distance is invariant under projective transformations. If K is an ellipsoid, the Hilbert geometry on $\operatorname{int} K$ is isometric to hyperbolic n-space.

Unbounded closed convex sets with nonempty interiors and not containing a straight line are projectively equivalent to convex bodies. Therefore, the definition of the distance naturally extends to the interiors of such convex sets. In particular, the convex sets bounded by parabolas are also isometric to the hyperbolic space.

Let us assume the origin o lies inside the interior of K. We will write $B(r)$ for the *metric ball* of radius r and centered at o. Its boundary, the *metric sphere*, will be denoted by $S(r)$. Let $a : \partial K \to \mathbb{R}_+$ be defined by the equation $-a(p)p \in \partial K$, so the letter a refers to the antipodal point. It is an easy exercise to check that metric spheres are parameterized by the boundary ∂K as

$$S(r) = \{\phi(p, r) : p \in \partial K\},$$

where

$$\phi : \partial K \times \mathbb{R}_+ \to \operatorname{int} K, \quad (p, r) \mapsto a\frac{e^{2r} - 1}{ae^{2r} + 1}\; p. \tag{11}$$

The Hilbert distance comes from a Finsler metric on the interior of K. Given $x \in \operatorname{int} K$ and $v \in T_x V$, the Finsler norm of v is given by

$$\|v\|_x = \frac{1}{2}\Bigl(\frac{1}{t_1} + \frac{1}{t_2}\Bigr), \tag{12}$$

where $t_1, t_2 > 0$ are such that $x \pm t_i v \in \partial K$. Again, we do not exclude that one of the t_i is infinite. Equivalently, if F_x is the Minkowski functional of $K - x$, then

$$\|v\|_x = \tfrac{1}{2}(F_x(v) + F_x(-v)).$$

The Finsler metric makes it possible to measure the length of a differentiable curve $c : I \to \operatorname{int} K$ by

$$l(c) := \int_I \bigl\|c'(t)\bigr\|_{c(t)} dt.$$

It is less trivial to measure the area (or volume) of higher dimensional subsets of $\operatorname{int} K$. In fact, different notions of volume are being used. The most important ones are the Busemann definition (which is equal to the Hausdorff n-dimensional measure) and the Holmes–Thompson definition. In the following, only properties of *volumes* in Finsler spaces (as defined in [Álvarez Paiva and Thompson 2004]) will be used:

- Vol is a Borel measure on $\operatorname{int} K$ that is absolutely continuous with respect to Lebesgue measure.
- If $A \subset K \subset L$, where K, L are compact convex sets, then the measure of A with respect to K is larger than the measure of A with respect to L.
- If K is an ellipsoid, then $\operatorname{Vol}(A)$ is the hyperbolic volume of A.

We will mainly investigate the following projective invariants of convex bodies.

Definition 2.5. The *upper and lower volume entropies* of K are

$$\overline{\operatorname{Ent}}(K) := \limsup_{r\to\infty} \frac{\log(\operatorname{Vol} B(r))}{r} \quad \text{and} \quad \underline{\operatorname{Ent}}(K) := \liminf_{r\to\infty} \frac{\log(\operatorname{Vol} B(r))}{r}.$$

If the upper and lower volume entropies of K coincide, their common value is called the volume entropy of K and is denoted by $\operatorname{Ent} K$.

Note that these invariants are independent of the choice of the center and of the choice of the volume definition.

2.3. ***Busemann's density.*** For simplicity, we restrict ourselves to Busemann's volume, although all results remain true for every other choice of volume. The reason is that the proofs of the crucial Propositions 2.7 and 2.8 below do not use any particular property of Busemann's volume, but only the axioms satisfied by every definition of volume.

The density of Busemann's volume (with respect to some Lebesgue measure $\mathscr{L}$) is given by $\sigma(x) = \omega_n/\mathscr{L}(B_x)$, where B_x is the tangent unit ball of the Finsler metric at x and ω_n is the (Euclidean) volume of the unit ball in $\mathbb{R}^n$. The volume of a Borel subset $A \subset \operatorname{int} K$ is thus given by $\operatorname{Vol}(A) = \int_A \sigma \, d\mathscr{L}$.

We now state and prove some propositions concerning upper bounds and asymptotic behaviors of Busemann's densities for points that are close to the boundaries of particular convex sets. We will make use of an auxiliary inner product, calling $\mathscr{L}$ and μ the corresponding Lebesgue measure and volume n-form. Busemann densities are defined with this particular choice of measure.

Proposition 2.6. *Let K and K' be closed convex sets not containing any straight line and let $\sigma : \operatorname{int} K \to \mathbb{R}$ and $\sigma' : \operatorname{int} K' \to \mathbb{R}$ be their corresponding Busemann densities. Let $p \in \partial K$, let E_0 be a support hyperplane of K at p, and let E_1 be a hyperplane parallel to E_0 intersecting K. Suppose that K and K' have the same intersection with the strip between E_0 and E_1 (in particular $p \in \partial K'$). Then*

$$\lim_{y\to p} \sigma(y)/\sigma'(y) = 1.$$

Proof. Let d be the distance between E_0 and E_1, and let (y_i) be a sequence of points of $\operatorname{int} K$ converging to p. We may suppose that the distance d_i between y_i and E_0 is strictly less than d. For every fixed point y_i and nonzero tangent vector $v \in T_{y_i}K$, let $t_1, t_2 \in \mathbb{R}_+ \cup \{\infty\}$ be such that $y_i \pm t_{1,2}v \in \partial K$; let t_1' and t_2' be the corresponding numbers for K'. Since at least one of $y_i + t_1 v$ and $y_i - t_2 v$ is inside the strip, say $y_i + t_1 v$, we must have $t_1 = t_1'$.

Either $t_2 = t_2'$ and $\|v\|_i = \|v\|_i'$, or $t_2 \neq t_2'$, in which case

$$\frac{t_1}{t_2}, \frac{t_1'}{t_2'} \leq \frac{d_i}{d - d_i}.$$

Therefore,

$$\frac{d - d_i}{d} \leq \frac{\|v\|_i}{\|v\|_i'} = \frac{1 + (t_1/t_2)}{1 + (t_1'/t_2')} \leq \frac{d}{d - d_i},$$

which shows that, as functions on $\mathbb{R}P^{n-1}$, the $\|\cdot\|_i / \|\cdot\|_i'$ uniformly converge to 1. Hence, for every ϵ and every i large enough, $(1-\epsilon)B_{y_i} \subset B'_{y_i} \subset (1+\epsilon)B_{y_i}$, which implies the convergence of σ/σ' to 1. □

Proposition 2.7. *Let $V = \mathbb{R}^n$ with its usual scalar product. Let P be the convex set bounded by the parabola $y = \sum_{i=1}^{n-1} (c_i/2)x_i^2$, with $c_1, \dots, c_{n-1} > 0$. Then*

$$\sigma(0, \dots, 0, 1-\lambda) = \frac{\sqrt{c}}{(2(1-\lambda))^{(n+1)/2}}, \quad \text{where } c = \prod_{i=1}^{n-1} c_i. \tag{13}$$

Proof. By the invariance of the Hilbert metric under projective transformations, the tangent unit sphere at any point of $\operatorname{int} P$ is an ellipse. At the point $(0, \dots, 0, 1-\lambda)$, the symmetry implies that the principal axes of this ellipse are parallel to the coordinate axes. Hence $\sigma = 1/\prod_{i=1}^{n} l_i$, where the l_i for $i = 1, \dots, n$ are the Euclidean lengths of the principal half-axes.

Now $l_i = \sqrt{2(1-\lambda)/c_i}$ for $i = 1, \dots, n-1$ and $l_n = 2(1-\lambda)$. □

Proposition 2.8. *Assume the origin o is inside* $\operatorname{int} K$. *For a smooth point p of ∂K, let $n(p)$ be the outward normal vector and let $k(p)$ be the Gauss curvature of ∂K at p. Then*

$$\lim_{\lambda \to 1} \sigma(\lambda p)(1-\lambda)^{(n+1)/2} = \frac{\sqrt{k(p)}}{(2\langle p, n(p)\rangle)^{(n+1)/2}}. \tag{14}$$

Proof. Let us choose a frame $(p; v_1, \dots, v_{n-1}, v_n)$, where $v_1, \dots, v_{n-1} \in T_p\partial K$ are unit vectors tangent to the principal curvature directions of ∂K at p and $v_n = -p$. In these coordinates, the boundary of K is locally the graph of a function

$$y = \sum_{i=1}^{n-1} (c_i/2)x_i^2 + R(|x|),$$

with $R(|x|) = o(|x|^2)$ and $c_1, \dots, c_{n-1} \geq 0$. We set $c := \prod_{i=1}^{n-1} c_i$. Then a short computation shows that $dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy = \mu/m$, where μ is the Euclidean n-form and $m := \mu(v_1, \dots, v_n) = \langle p, n(p) \rangle$. Also, the Gauss curvature at p is given by $k(p) = cm^{n-1}$.

Let us fix $\epsilon > 0$. Locally, the parabola defined by $y = \sum_{i=1}^{n-1} \frac{1}{2}(c_i + \epsilon)x_i^2$ lies inside K. Cutting it with some horizontal hyperplane, we obtain a convex body K' inside K. In particular, the metric of K' is greater than or equal to the metric of K; hence, $\sigma'(\lambda p) \geq \sigma(\lambda p)$ for λ near 1.

Then by Propositions 2.6 and 2.7,

$$\limsup_{\lambda \to 1} \sigma(\lambda p)(1-\lambda)^{(n+1)/2} \leq \lim_{\lambda \to 1} \sigma'(\lambda p)(1-\lambda)^{(n+1)/2} = \frac{\sqrt{\prod_{i=1}^{n-1}(c_i + \epsilon)}}{2^{(n+1)/2} m}. \tag{15}$$

Since $\sigma > 0$, this already settles the case $k = c = 0$, since ϵ was arbitrarily small.

If $c > 0$ and $0 < \epsilon < \min\{c_1, \dots, c_{n-1}\}$, the parabola P defined by

$$y = \sum_{i=1}^{n-1} \frac{c_i - \epsilon}{2} x_i^2$$

locally contains K. Cutting it with some horizontal hyperplane, we obtain a convex body K' inside P. Again by Propositions 2.6 and 2.7,

$$\liminf_{\lambda \to 1} \sigma(\lambda p)(1-\lambda)^{(n+1)/2} \geq \liminf_{\lambda \to 1} \sigma'(\lambda p)(1-\lambda)^{(n+1)/2} = \frac{\sqrt{\prod_{i=1}^{n-1}(c_i - \epsilon)}}{2^{(n+1)/2} m}. \tag{16}$$

From (15) and (16) (with $\epsilon \to 0$) we get

$$\lim_{\lambda \to 1} \sigma(\lambda p)(1-\lambda)^{(n+1)/2} = \frac{\sqrt{c}}{2^{(n+1)/2} m}. \qquad \square$$

Section 3 will start with the proof of a slight and somewhat technical refinement of our second main theorem. To state it precisely, we need to introduce the pseudo-Gauss curvature of the boundary of a convex set K in $\mathbb{R}^n$.

For a smooth point $p \in \partial K$, let $n(p)$ be the outward normal of ∂K at p. For each unit vector $e \in T_p \partial K$, let $H_e(p)$ be the affine plane containing p and directed by the vectors e and $n(p)$. We define R_e as the radius of the biggest disc containing p inside $K_e := K \cap H_e(p)$.

Definition 2.9. The *pseudo-Gauss curvature* $\bar{k}(p)$ of ∂K at p is the minimum of the numbers $\prod_{i=1}^{n-1} R_{e_i}(p)^{-1}$, where $e_1, \dots, e_{n-1}$ ranges over all orthonormal bases of $T_p \partial K$.

Proposition 2.10. *Let V be a Euclidean vector space of dimension n. Let K be a convex body containing the unit ball B. Then for $\frac{1}{2} \leq \lambda < 1$ and $p \in \partial K$,*

$$\sigma(\lambda p) \leq \frac{\omega_n n!}{2^n(1-\lambda)^{(n+1)/2}} \bar{k}(p)^{1/2}. \tag{17}$$

Proof. We use the same notation as in the definition of $\bar{k}$. We may suppose that $R_i := R_{e_i}(p) > 0$ for all i; otherwise the statement is trivial. By the definition of R_i, there is a 2-disc $B_i(p)$ of radius R_i inside K_{e_i} containing p. Let us denote by $B(e_i)$ the intersection of B with the affine plane $p + H_{e_i}$. Since $B(e_i)$ and $B_i(p) \subset K$,

$$\hat{C}_i := \operatorname{conv}(B(e_i) \times \{0\} \cup B_i(p) \times \{1\}) \subset K_{e_i} \times [0, 1].$$

Note that $\hat{C}_i$ is a truncated cone. Let E_i be the plane containing the line that is parallel to $T_p \partial K_{e_i}$ and that passes through the points $o \times \{0\}$ and $p \times \{1\}$. With $\pi : V \times [0, 1] \to V$ the projection on the first component, $C_i := \pi(E_i \cap \hat{C}_i) \subset K$ is bounded by a truncated conic.

In the nonorthogonal frame $(o; p, e_i)$, C_i is given by

$$(2R_i - 1)x^2 + 2(1 - R_i)x + y_1^2 \leq 1 \quad \text{for } 0 \leq x \leq 1.$$

Now let C be the convex hull of the union of the C_i. Then the polytope P with vertices

$$\big(\lambda, 0, \dots, \pm\sqrt{(1-\lambda)(2\lambda R_i - \lambda + 1)}, 0, \dots, 0\big), \quad (1, \vec{0}), \quad (2\lambda - 1, \vec{0})$$

lies inside C, with all but the last vertex being on the boundaries of the C_i.

Its volume is given by

$$\begin{aligned} \mathscr{L}(P) &= \frac{2^n \langle p, n(p)\rangle}{n!}(1-\lambda)^{(n+1)/2} \prod_{i=1}^{n-1} (2\lambda R_i - \lambda + 1)^{1/2} \\ &\geq \frac{2^n}{n!}(1-\lambda)^{(n+1)/2}(R_1 \cdot R_2 \cdots R_{n-1})^{1/2} = \frac{2^n}{n!}(1-\lambda)^{(n+1)/2} \bar{k}^{-1/2}(p). \end{aligned} \tag{18}$$

The factor $\langle p, n(p)\rangle$ in the first line appears because our coordinate system is not orthonormal. Since the unit ball is contained in K, this factor is at least 1.

From $P \subset C \subset K$ and the fact that P is centered at λp, we deduce that

$$\sigma(\lambda p) \leq \frac{\omega_n}{\mathscr{L}(P)} \leq \frac{\omega_n n!}{2^n}(1-\lambda)^{-(n+1)/2}\, \bar{k}^{1/2}(p). \qquad \square$$

The next proposition will be needed in the construction of a convex body with entropy between 0 and 1.

Proposition 2.11. *Let $K = oab$ be a triangle with $1 \leq oa$ and $ob \leq 2$, such that the distance from o to the line passing through a and b is at least 1. Let p be a* *point in the interior of the side ab and suppose that* $\min\{ap, bp\} \geq \epsilon > 0$. *Then for* $\lambda \geq 1/2$, *Busemann's density of K at* λp *is bounded above by*

$$\sigma(\lambda p) \leq 32\pi \max\left\{\frac{1}{\epsilon(1-\lambda)}, \frac{1}{\epsilon^2}\right\}.$$

Proof. The hypothesis on the triangle implies that $\sin(abo), \sin(bao) \geq 1/2$.

Let a' be the intersection with ob of the line passing through a and $z := \lambda p$, and define b' similarly.

The unit tangent ball at z is a hexagon centered at z. The length of one of its half-diagonals is the harmonic mean of za and za'; the length of the second half-diagonal is the harmonic mean of zb and zb'; and the third half-diagonal has length

$$\frac{2op}{\dfrac{1}{\lambda}+\dfrac{1}{1-\lambda}} \geq 1-\lambda.$$

An easy geometric argument shows that

$$za', zb \geq \tfrac{1}{2} pb \sin(abo) \geq \tfrac{1}{4}\epsilon \quad \text{and} \quad za, zb' \geq \tfrac{1}{2} pa \sin(bao) \geq \tfrac{1}{4}\epsilon.$$

The area A of the hexagon is at least half of the minimal product of two of its half-diagonals; hence, $A \geq \min\{\frac{1}{8}\epsilon(1-\lambda), \frac{1}{32}\epsilon^2\}$. ☐

2.4. ***Volume entropy of spheres.*** By definition, the entropy controls the volume growth of metric balls in Hilbert geometries. We show in this section that it coincides with the growth of areas of metric spheres. Again, there are several definitions of area of hypersurfaces in Finsler geometry. For simplicity, we consider Busemann's definition, which gives the Hausdorff $(n-1)$-measure of these hypersurfaces.

Lemma 2.12 (rough monotonicity of area). *There exist a monotone function f and a constant* $C_1 > 1$ *such that for all* $r > 0$,

$$C_1^{-1} f(r) \leq \operatorname{Area}(S(r)) \leq C_1 f(r). \tag{19}$$

Proof. Let $f(r)$ be the Holmes–Thompson area of $S(r)$. Since all area definitions agree up to some universal constant, inequality (19) is trivial. It remains to show that f is monotone.

If ∂K is C^2 with everywhere positive Gaussian curvature, then the tangent unit spheres of the Finsler metric are quadratically convex. According to [Álvarez Paiva and Fernandes 1998, Theorem 1.1 and Remark 2], there exists a Crofton formula for the Holmes–Thompson area, from which the monotonicity of f easily follows.

Such smooth convex bodies are dense in the set of all convex bodies for the Hausdorff topology; see for example [Hörmander 2007, Lemma 2.3.2]. By approximation, it follows that f is monotone for arbitrary K. ☐

Lemma 2.13 (coarea inequalities). *There exists a constant* $C_2 > 1$ *such that*

$$C_2^{-1}\,\mathrm{Area}(S(r)) \le \frac{\partial}{\partial r}\,\mathrm{Vol}(B(r)) \le C_2\,\mathrm{Area}(S(r)) \quad \textit{for all } r > 0.$$

Proof. Let $\mu := \sigma\, dx_1 \wedge \cdots \wedge dx_n$ be the volume form, and let α be the $(n-1)$-form on $S(r)$ whose integral equals the area.

Since

$$\mathrm{Vol}(B(r)) = \int_0^r \int_{S(s)} i_{\partial_r}\mu \; ds,$$

where ∂_r at $\lambda p \in S(s)$ is the tangent vector multiple of $\vec{op}$ with unit Finsler norm, we have to compare $i_{\partial_r}\mu$ and α.

We will assume that $S(r)$ is differentiable at λp. The section of the unit tangent ball by the tangent space $T_{\lambda p}S(r)$ will be called γ. By the definition of Busemann area, the area of γ measured with the form α is the constant $\alpha(\gamma) = \omega_{n-1}$.

In the same way, calling Γ the half unit ball containing ∂_r and bounded by γ, one has $\mu(\Gamma) = \frac{1}{2}\omega_n$.

Since Γ is convex, it contains the cone with base γ and vertex ∂_r. Therefore,

$$\tfrac{1}{n} i_{\partial_r}\mu(\gamma) \le \tfrac{1}{2}\omega_n. \tag{20}$$

By Brunn's theorem (see for example [Koldobsky 2005, Theorem 2.3]), the sections of the tangent unit ball with hyperplanes parallel to γ have an area less than or equal to the area of γ. Also the tangent unit ball has a supporting hyperplane at ∂_r which is parallel to γ. Therefore, by Fubini's theorem, the cylinder $\gamma \times ([0, 1]\cdot\partial_r)$ has a volume greater than or equal to the volume of Γ (even if it generally does not contain Γ). Hence,

$$\tfrac{1}{2}\omega_n \le i_{\partial_r}\mu(\gamma). \tag{21}$$

Inequalities (20) and (21) give

$$\frac{1}{2}\frac{\omega_n}{\omega_{n-1}}\alpha(\gamma) \le i_{\partial_r}\mu(\gamma) \le \frac{n}{2}\frac{\omega_n}{\omega_{n-1}}\alpha(\gamma),$$

from which the result easily follows. □

Theorem 2.14. *The spherical entropy coincides with the entropy. More precisely*,

$$\limsup_{r\to\infty} \frac{\log \mathrm{Area}(S(r))}{r} = \overline{\mathrm{Ent}}\, K \quad \textit{and} \quad \liminf_{r\to\infty} \frac{\log \mathrm{Area}(S(r))}{r} = \underline{\mathrm{Ent}}\, K.$$

Proof. For convenience, let $V(r) := \mathrm{Vol}\, B(r)$ and $A(r) := \mathrm{Area}\, S(r)$.

Using the previous two lemmas, one has, for all $r > 0$,

$$\begin{aligned} V(r) = \int_0^r V'(s)\, ds \le C_2 \int_0^r A(s)\, ds \le C_1C_2 \int_0^r f(s)\, ds \\ \le C_1C_2 f(r) r \le C_1^2 C_2 A(r) r. \end{aligned}$$

It follows that

$$\overline{\mathrm{Ent}}\, K = \limsup_{r\to\infty} \frac{\log V(r)}{r} \le \limsup_{r\to\infty} \frac{\log C_1^2 C_2 A(r) r}{r} = \limsup_{r\to\infty} \frac{\log \mathrm{Area}(S(r))}{r}.$$

Similarly, for each $\epsilon > 0$,

$$\begin{aligned} V(r(1+\epsilon)) &= \int_0^{r(1+\epsilon)} V'(s)\,ds \ge C_1^{-1} C_2^{-1} \int_0^{r(1+\epsilon)} f(s)\,ds \\ &\ge C_1^{-1} C_2^{-1} \int_r^{r(1+\epsilon)} f(s)\,ds \ge C_1^{-1} C_2^{-1} f(r) r\epsilon \ge C_1^{-2} C_2^{-1} A(r) r\epsilon, \end{aligned}$$

and hence

$$\begin{aligned} (1+\epsilon)\, \overline{\mathrm{Ent}}\, K &= (1+\epsilon) \limsup_{r\to\infty} \frac{\log V(r(1+\epsilon))}{r(1+\epsilon)} \\ &\ge \limsup_{r\to\infty} \frac{\log C_2^{-1} C_1^{-2} A(r) r\epsilon}{r} = \limsup_{r\to\infty} \frac{\log \mathrm{Area}(S(r))}{r}. \end{aligned}$$

Letting $\epsilon \to 0$ gives the first equality. The second one follows in a similar way. □

3. Entropy bounds

3.1. ***Upper entropy bound in arbitrary dimension.*** Our second main theorem will follow from the next result.

Theorem 3.1. *Let K be an n-dimensional convex body and $o \in \operatorname{int} K$. For a point $p \in \partial K$, we denote by $\bar{k}(p)$ its pseudo-Gauss curvature as in Definition 2.9. If*

$$\int_{\partial K} \bar{k}^{1/2}(p)dp < \infty, \tag{22}$$

then

$$\lim_{r\to\infty} \frac{\mathrm{Vol}\, B(o,r)}{\sinh^{n-1} r} = \frac{1}{n-1} \mathcal{A}_p(K). \tag{23}$$

In particular, $\overline{\mathrm{Ent}}\, K \le n-1$, *and if* $\mathcal{A}_p(K) \ne 0$, *then* $\overline{\mathrm{Ent}}\, K = n-1$.

Proof. Using the parameterization (11), the volume of metric balls is given by

$$\mathrm{Vol}(B(r)) = \int_0^r \int_{\partial K} F(p,r)\, d\mathcal{H}^{n-1},$$

where $F(p,r) := \sigma(\phi(p,r)) \operatorname{Jac}\phi(p,r)$.

The Jacobian may be explicitly computed:

$$\operatorname{Jac}\phi(p,r) = \frac{(e^{2r}-1)^{n-1} e^{2r}}{(ae^{2r}+1)^{n+1}} 2a^n (1+a)\langle p, n(p)\rangle.$$

In particular,

$$\lim_{r\to\infty} e^{2r} \operatorname{Jac}\phi(p,r) = 2(1+a)\langle p, n(p)\rangle / a. \tag{24}$$

On the other hand, for each smooth boundary point p we have, by Proposition 2.8,

$$\lim_{r\to\infty} \frac{\sigma(\phi(p,r))}{e^{(n+1)r}} = \frac{\sqrt{k(p)}}{(2\langle p, n(p)\rangle)^{(n+1)/2}} \frac{a^{(n+1)/2}}{(1+a)^{(n+1)/2}}. \tag{25}$$

Then, by Proposition 2.10 and the hypothesis (22),

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{e^{(n-1)r}} \int_{\partial K} F(p,r)\, d\mathcal{H}^{n-1} &= \int_{\partial K} \lim_{r\to\infty} \frac{F(p,r)}{e^{(n-1)r}}\, d\mathcal{H}^{n-1} \\ &= \int_{\partial K} \lim_{r\to\infty} \frac{\sigma(\phi(p,r))}{e^{(n+1)r}} \lim_{r\to\infty} e^{2r} \operatorname{Jac}\phi(p,r)\, d\mathcal{H}^{n-1} \\ &= \int_{\partial K} \frac{\sqrt{k(p)}}{(2\langle p, n(p)\rangle)^{(n-1)/2}} \left(\frac{a}{1+a}\right)^{n-1/2} d\mathcal{H}^{n-1} = \frac{1}{2^{n-1}} \mathcal{A}_p(K).\end{aligned} \tag{26}$$

By L'Hôpital's rule, we get

$$\lim_{r\to\infty} \frac{\operatorname{Vol}(B(r))}{e^{(n-1)r}} = \lim_{r\to\infty} \frac{\int_0^r \int_{\partial K} F(p,s) d\mathcal{H}^{n-1} ds}{(n-1)\int_0^r e^{(n-1)s} ds} = \frac{1}{2^{n-1}(n-1)} \mathcal{A}_p(K). \quad \square$$

Remark. The metric balls $B(r)$ are projective invariants of K. There is an affine version of the previous theorem using the affine balls $B_a(r) := \tanh(r)K$ (where multiplication is with respect to the center o). Under the same assumptions as in Theorem 3.1, we obtain that

$$\lim_{r\to\infty} \frac{\operatorname{Vol} B_a(r)}{e^{(n-1)r}} = \frac{1}{2^{n-1}(n-1)} \mathcal{A}_a(K),$$

where $\mathcal{A}_a(K)$ is the centroaffine area (see Section 4). The proof goes as the one above by replacing the function a by 1.

Corollary 3.2. *Suppose K is an n-dimensional convex body of class $C^{1,1}$. Then*

$$\operatorname{Ent} K = n-1.$$

Proof. For any $p \in \partial K$, $R(p)$ is the biggest radius of a ball in K containing p. By Proposition 2.2, there exists a constant $R > 0$ such that $R(p) \geq R$ for all $p \in \partial K$. It follows that the hypothesis (22) is satisfied and therefore $\operatorname{Ent} K \leq n-1$.

The Gauss map $\mathcal{G}\colon \partial K \to S^{n-1}$ is well defined and continuous. As a consequence of [Hug 1999a, Theorem 2.3] and [Hug 1998, Equation 2.7], the standard measure on the unit sphere is the push-forward of $k \cdot d\mathcal{H}^{n-1}$, that is,

$$\mathcal{G}_*(k \cdot d\mathcal{H}^{n-1}|_{\partial K}) = d\mathcal{H}^{n-1}|_{S^{n-1}},$$

and hence the curvature has a positive integral. Therefore, $\mathcal{A}_p(K) > 0$, and (23) implies that $\operatorname{Ent} K = n-1$. □

Corollary 3.3. *If K is an arbitrary n-dimensional convex body with $\mathcal{A}_p(K) \neq 0$, then $\underline{\operatorname{Ent}}\, K \geq n-1$.*

Proof. Arguing as in the proof of Theorem 3.1 and using Fatou's lemma instead of the dominated convergence theorem gives the result. □

3.2. ***The plane case.*** Let us now assume that $n = 2$. By Theorem 2.3, the hypothesis (22) is satisfied for each convex body K. Therefore

$$\overline{\operatorname{Ent}}\, K \leq 1 \tag{27}$$

and

$$\lim_{r\to\infty} \frac{\operatorname{Vol} B(o,r)}{\sinh r} = \mathcal{A}_p(K).$$

Next, we are going to prove a better bound for $\overline{\operatorname{Ent}}\, K$. To state our main result, we need to recall some basic notions of measure theory in a Euclidean space and refer to [Mattila 1995] for details. For a nonempty bounded set A, let $N(A,\epsilon)$ be the minimal number of ϵ-balls needed to cover A. Then the upper Minkowski dimension of A is defined as

$$\overline{\dim}\, A := \inf\bigl\{s : \limsup_{\epsilon\to 0} N(A,\epsilon)\epsilon^s = 0\bigr\}.$$

One should note that this dimension is invariant under bi-Lipschitz maps. In particular, it does not depend on a particular choice of inner product, and it is invariant under projective maps provided the considered subsets are bounded.

Recall that a point $p \in K$ is called *extremal* if it is not a convex combination of other points of K. The set of extremal points is a subset of ∂K, which we denote by $\operatorname{ex} K$.

First main theorem. *Let K be a plane convex body, and let d be the upper Minkowski dimension of* $\operatorname{ex} K$. *Then the entropy of K is bounded by*

$$\overline{\operatorname{Ent}}\, K \leq \frac{2}{3-d} \leq 1.$$

Proof. Since the entropy is independent of the choice of the center, we may suppose that the Euclidean unit ball around o is the maximum volume ellipsoid inside K. Then K is contained in the ball of radius 2 [Barvinok 2002].

Set $\epsilon := e^{-\alpha r}$, where $\alpha \leq 1$ will be fixed later. Divide the boundary of K into two parts:

$$\partial K = \mathcal{B} \cup \mathcal{G},$$

where $\mathcal{B}$ (the bad part) is the closed ϵ-neighborhood around the set of extremal points of K and $\mathcal{G}$ (the good part) is its complement.

Using Proposition 2.4 and equalities (24) and (25), we get an upper bound for large values of r:

$$\int_{r/2}^{r}\int_{\mathcal{B}} \sigma(\phi(p,s))\,\mathrm{Jac}\,\phi(p,s)\,d\mathcal{H}^1 ds \le O\big(e^r\sqrt{\mathcal{H}^1(\mathcal{B})}\big). \tag{28}$$

Next, let $p \in \mathcal{G}$. The endpoints of the maximal segment in ∂K containing p are extremal points of K and hence of distance at least ϵ from p. Therefore K contains a triangle as in Proposition 2.11, and if $s \ge r/2$, and r is sufficiently large,

$$\sigma(\phi(p,s)) = \sigma(\lambda\cdot p) \le 32\max\left\{\frac{1}{\epsilon(1-\lambda)}, \frac{1}{\epsilon^2}\right\} = \frac{32}{\epsilon(1-\lambda)}.$$

Integrating this from $r/2$ to r yields

$$\int_{r/2}^{r}\int_{\mathcal{G}} \sigma(\phi(p,s))\,\mathrm{Jac}\,\phi(p,s)\,d\mathcal{H}^1 ds = O(e^{\alpha r}). \tag{29}$$

Let d be the upper Minkowski dimension of the set of extremal points of K. Then, for each $\eta > 0$, $N(\mathrm{ex}\,K,\epsilon) = o(\epsilon^{-d-\eta})$ as $\epsilon \to 0$. By the definition of N, there is a covering of $\mathrm{ex}\,K$ by $N(\mathrm{ex}\,K,\epsilon)$ balls of radius ϵ. Hence there is a covering of $\mathcal{B}$ by $N(\mathrm{ex}\,K,\epsilon)$ balls with radius 2ϵ. The intersection of a 2ϵ-ball with ∂K has length less than $4\pi\epsilon$. It follows that $\mathcal{H}^1(\mathcal{B}) = o(\epsilon^{-d-\eta+1})$. Since the volume of $B(r/2)$ is bounded by $O(e^{r/2})$ (see (27)), the volume of $B(r)$ is bounded by

$$\begin{aligned}\mathrm{Vol}\,B(r) = \mathrm{Vol}\,B(r/2) &+ \int_{r/2}^{r}\int_{\mathcal{B}} \sigma(\phi(p,s))\,\mathrm{Jac}\,\phi(p,s)d\mathcal{H}^1 ds\\ &+ \int_{r/2}^{r}\int_{\mathcal{G}} \sigma(\phi(p,s))\,\mathrm{Jac}\,\phi(p,s)d\mathcal{H}^1 ds\\ = O(e^{r/2}) &+ O(e^{r(1-(\alpha(1-d-\eta))/2)}) + O(e^{\alpha r}).\end{aligned}$$

We fix α such that $1-\alpha(1-d-\eta)/2 = \alpha$, that is, $\alpha := 2/(3-d-\eta) > 2/3$. Then $\mathrm{Vol}\,B(r) = O(e^{\alpha r})$, which implies that the (upper) entropy of K is bounded by α. Since $\eta > 0$ was arbitrary, the result follows. □

3.3. ***An example of noninteger entropy.*** We will construct a plane convex body with piecewise affine boundary whose entropy is strictly between 0 and 1.

Let us choose a real number $s > 2$ and set $\alpha_i := C_s/i^s$, where $C_s > 0$ is sufficiently small, such that $3\sum_{i=1}^{\infty}\alpha_i < \pi$. We consider a centrally symmetric sequence E of points on S^1 such that the angles between consecutive points are $\alpha_1, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_2, \ldots$ (each angle appearing three times).

Theorem 3.4. *The entropy of $K = \mathrm{conv}(E)$ is bounded by*

$$0 < \frac{1}{s} \le \underline{\mathrm{Ent}}\,K \le \overline{\mathrm{Ent}}\,K \le \frac{2s-2}{3s-4} < 1.$$

Proof for lower bound. The unit sphere of radius r in the Hilbert geometry K is $\tanh r K$ and consists of an infinite number of segments.

An easy geometric computation shows that the middle segment $S_i(r)$ corresponding to $\alpha := \alpha_i$ has for each $r \geq 0$ length bounded from below by

$$l(S_i(r)) \geq \log\Big(\frac{2\tanh r}{1-\tanh r}\tan(\alpha/2)\sin(\alpha)+1\Big).$$

Set $i_0(r) := \lfloor (2C_s)^{1/s} e^{r/s} \rfloor$. Then, for sufficiently large r,

$$\frac{2\tanh r}{1-\tanh r}\tan(\alpha_i/2)\sin(\alpha_i) \leq 1 \quad \text{for all } i \geq i_0(r).$$

By the concavity of the log-function, we have $\log(1+x) \geq x\log 2 \geq x/2$ for $0 \leq x \leq 1$. Therefore

$$l(S(r)) \geq \sum_{i=i_0}^{\infty} \frac{\tanh r}{1-\tanh r}\tan(\alpha_i/2)\sin(\alpha_i).$$

For sufficiently large r, the first factor is bounded from below by $e^{2r}/4$, while the second is bounded from below by α_i^2. We thus get

$$l(S(r)) \geq \frac{e^{2r}}{4}\sum_{i=i_0}^{\infty}\alpha_i^2 = C_s^2\frac{e^{2r}}{4}\sum_{i=i_0}^{\infty}\frac{1}{i^{2s}} \geq C_s^2\frac{e^{2r}}{4}\int_{i_0}^{\infty}\frac{1}{x^{2s}}\,dx = C_s^2\frac{e^{2r}}{4(2s-1)i_0^{2s-1}}.$$

Replacing our explicit value for i_0 gives $l(S(r)) \geq Ce^{r/s}$ for sufficiently large r and some constant C (again depending on s). Hence $\underline{\mathrm{Ent}}\,K \geq 1/s$.

Proof for upper bound. For the upper bound in the statement, we apply our first main theorem. For this, we have to find an upper bound on the Minkowski dimension of $\mathrm{ex}\,K = E$.

Since the Minkowski dimension is invariant under bi-Lipschitz maps, we may replace distances on the unit circle by angular distances.

E has two accumulation points $\pm x_0$. For $\epsilon > 0$, let $N(\epsilon)$ be the number of ϵ-balls needed to cover E. We take one such ball around $\pm x_0$ and one further ball for each point in E not covered by these two balls.

The three points corresponding to the angle α_i are certainly in the ϵ-neighborhood of $\pm x_0$, provided that $3\sum_{j=i}^{\infty}\alpha_j \leq \epsilon$.

Now we compute

$$\sum_{j=i}^{\infty}\alpha_j = C_s\sum_{j=i}^{\infty}\frac{1}{j^s} \leq C_s\int_{i-1}^{\infty}\frac{1}{x^s}\,dx = \frac{C_s}{s-1}\frac{1}{(i-1)^{s-1}}.$$

It follows that all i satisfying $i \geq i_0 := (3C_s/(s-1))^{1/(s-1)}\,\epsilon^{1/(1-s)} + 1$ also satisfy the inequality above, and hence $N(\mathrm{ex}\,K, \epsilon) \leq 6i_0 + 2 \leq C\epsilon^{-1/(s-1)}$.

It follows that the upper Minkowski dimension is not larger than $1/(s-1)$. The upper bound of First main theorem gives

$$\overline{\mathrm{Ent}}\, K \leq \frac{2s-2}{3s-4}. \qquad \square$$

4. Centroprojective and centroaffine areas

In this section, we will take a closer look at the centroprojective area, which was introduced (in a nonintrinsic way) in the definition on page 204.

4.1. ***Basic definitions and properties.*** Geometrically speaking, both centroaffine and centroprojective areas are Riemannian volumes of the boundary ∂K.

We first give intrinsic definitions of the centroaffine metric and area. Let K be a convex body with a distinguished interior point, which we may suppose to be the origin o of V. The Minkowski functional of K is the unique positive function F that is homogeneous of degree 1 and whose level set at height 1 is the boundary ∂K. This function is convex and, according to Alexandroff's theorem, has almost everywhere a quadratic approximation.

Definition 4.1. Let v be a tangent vector to ∂K at a smooth point p. Then the *centroaffine seminorm* of v is $\|v\|_a := \sqrt{\mathrm{Hess}_p F(v, v)}$.

The square of the centroaffine seminorm is a quadratic function on the tangent, and hence we may define as usual a volume form, say ω_a (which vanishes if $\|\cdot\|_a$ is not definite).

Definition 4.2. The *centroaffine area* of K is $\mathcal{A}_a(K) := \int_{\partial K} |\omega_a|$.

It easily follows from the definitions that the centroaffine area is indeed an affine invariant of pointed convex bodies. Moreover, it is finite and vanishes on polytopes. The next proposition relates our definitions with the classical ones; its proof is a straightforward computation.

Proposition 4.3. *If the space is equipped with a Euclidean inner product*, *then the centroaffine area is given by*

$$\mathcal{A}_a(K) = \int_{\partial K} \frac{\sqrt{k}}{\langle n, p\rangle^{(n-1)/2}}\, dA,$$

with k the Gaussian curvature of ∂K at p, *where n is the unit vector normal to $T_p\partial K$*, *and where dA is the Euclidean area.*

To introduce the centroprojective area, we will consider a compact convex subset of the (real) n-dimensional projective space. Here the word "convex" means that each intersection with a projective line is connected.

The definitions of the centroprojective seminorm and area are merely the same as the centroaffine ones, but one has to replace the Minkowski functional by a projectively invariant function.

Definition 4.4. Let $K \subset \mathbb{P}^n$ be a convex body and $o \in \operatorname{int} K$. The *projective gauge function* is

$$G_K : \mathbb{P}^n \setminus \{o\} \to \mathbb{R} \cup \{\infty\}, \quad x \mapsto 2[q_1, o, x, q_2],$$

where q_1 and q_2 are the two intersections of ∂K with the line going through o and x.

Since the order of q_1 and q_2 is not fixed, this function is multivalued (in fact double-valued). Identifying $\mathbb{R} \cup \{\infty\}$ with $\mathbb{P}^1$, this function is continuous.

If p belongs to the boundary of K, then the two values of $G_K(p)$ are different, one of them being 2, the other being ∞. Hence there is some neighborhood U of p such that the restriction of G_K to U is the union of two continuous (in fact smooth) functions G_K^+ and G_K^- on U, where $G_K^+(p) = 2$ and $G_K^-(p) = \infty$.

Let v be a tangent vector to ∂K at a smooth point p. Since the restriction of G_K^+ to $\partial K \cap U$ is constant, the derivative of G_K^+ in the direction of v vanishes. Therefore, the Hessian of the restriction of G_K^+ to the tangent line is well defined.

Definition 4.5. The *centroprojective seminorm* of v is

$$\|v\|_p := \sqrt{\operatorname{Hess}_p G_K^+(v, v)}.$$

If we let ω_p be the induced volume form on ∂K, the *centroprojective area* of K is $\mathcal{A}_p(K) := \int_{\partial K} |\omega_p|$.

Proposition 4.6. *In a Euclidean space*,

$$\mathcal{A}_p(K) = \int_{\partial K} \frac{\sqrt{k}}{\langle n, p\rangle^{(n-1)/2}} \left(\frac{2a}{1+a}\right)^{(n-1)/2} dA.$$

In particular, *the intrinsic definition of $\mathcal{A}_p$ agrees with the definition given in the introduction.*

Proof. An easy computation shows that

$$[q_1, o, x, q_2] = \frac{1 + a(q_2)}{F(x) + a(q_2)} F(x).$$

Then, if p is a smooth point of ∂K and $v \in T_p \partial K$,

$$\operatorname{Hess}_p G_K(v, v) = \frac{2a(p)}{1 + a(p)} \operatorname{Hess}_p F(v, v). \qquad \square$$

4.2. ***Properties of the centroprojective area.*** Both centroaffine and centroprojective areas vanish on polytopes, and hence they are not continuous with respect to the Hausdorff topology on (pointed) bounded convex bodies. Nevertheless, the centroaffine area is upper-semicontinuous [Lutwak 1996]. The same holds true for the centroprojective area as shown in the next theorem.

Theorem 4.7. *The centroprojective area is finite, invariant under projective transformations, and upper-semicontinuous.*

Proof. From the above intrinsic definition, it follows that $\mathcal{A}_p$ is invariant under projective transformations. Also, since the function a on the boundary is bounded and positive, and since the centroaffine area is finite, it follows from Proposition 4.6 that the centroprojective area is also finite. It remains to show that it is upper-semicontinuous. Our proof is based on the fact that the centroaffine surface area $\mathcal{A}_a$ is semicontinuous [Lutwak 1996].

Let K be a bounded convex body containing the origin in its interior, and let (K_i) be a sequence of convex bodies with the same properties converging to K. Set

$$\tau(p) := \left(\frac{2a(p)}{1+a(p)}\right)^{(n-1)/2} \quad \text{for } p \in \partial K,$$

which is a continuous function on ∂K.

For each i, if a_i is the function corresponding to K_i and p_i is the radial projection of p on ∂K_i, define $\tau_i \in C(\partial K)$ by

$$\tau_i(p) := \left(\frac{2a_i(p_i)}{1+a_i(p_i)}\right)^{(n-1)/2}.$$

Since $K_i \to K$, τ_i converges uniformly to τ. Therefore $\|\tau_i - \tau\|_\infty < \epsilon$ for fixed $\epsilon > 0$ and all sufficiently large i.

Take a triangulation of the sphere and let $\partial K = \bigcup_{j=1}^m \Delta_j$ be its radial projection. Define $\partial K_i = \bigcup_{j=1}^m \Delta_{ij}$ similarly.

Choosing this triangulation sufficiently thin, there exist $t_1, \dots, t_m \in \mathbb{R}_+$ such that $|\tau(p) - t_j| < \epsilon$ on Δ_j. By the triangle inequality, $|\tau_i(p) - t_j| < 2\epsilon$ on Δ_{ij}.

We define

$$\mathcal{A}_p(K_i, \Delta_{ij}) := \int_{\Delta_{ij}} \frac{\sqrt{k(x)}}{\langle n(x), x\rangle^{(n-1)/2}} \tau_i \, d\mathcal{H}^{n-1}(x).$$

Clearly, $\mathcal{A}_p(K_i) = \sum_{j=1}^m \mathcal{A}_p(K_i, \Delta_{ij})$. We define $\mathcal{A}_p(K, \Delta_j)$, $\mathcal{A}_a(K_i, \Delta_{ij})$ and $\mathcal{A}_a(K, \Delta_j)$ similarly.

Fix p_j in the interior of Δ_j and consider the convex hulls $\hat{\Delta}_i$ of $\Delta_j \cup \{-p_j\}$ and $\hat{\Delta}_{ij}$ of $\Delta_{ij} \cup -p_j$. The boundary of $\hat{\Delta}_{ij}$ is the union of Δ_{ij} and flat simplices;

hence $\mathcal{A}_a(K_i, \Delta_{ij}) = \mathcal{A}_a(\hat{\Delta}_{ij})$. By the semicontinuity of $\mathcal{A}_a$, we obtain

$$\limsup_{i\to\infty} \mathcal{A}_a(K_i, \Delta_{ij}) = \limsup_{i\to\infty} \mathcal{A}_a(\hat{\Delta}_{ij}) \le \mathcal{A}_a(\hat{\Delta}_j) = \mathcal{A}_a(K, \Delta_j).$$

It follows that

$$\begin{aligned}\limsup_{i\to\infty} \mathcal{A}_p(K_i) &= \limsup_{i\to\infty} \sum_{j=1}^m \mathcal{A}_p(K_i, \Delta_{ij}) \\ &\le \limsup_{i\to\infty} \sum_{j=1}^m \mathcal{A}_a(K_i, \Delta_{ij})(t_j + 2\epsilon) \le \sum_{j=1}^m \mathcal{A}_a(K, \Delta_j)(t_j + 2\epsilon).\end{aligned}$$

On the other hand,

$$\mathcal{A}_p(K) = \sum_{j=1}^m \mathcal{A}_p(K, \Delta_j) \ge \sum_{j=1}^m \mathcal{A}_a(K, \Delta_j)(t_j - \epsilon),$$

from which we deduce that $\limsup_{i\to\infty} \mathcal{A}_p(K_i) \le \mathcal{A}_p(K) + 3\epsilon\mathcal{A}_a(K)$. □

The centroaffine surface area has the following important properties:

- $\mathcal{A}_a$ is a valuation on the space of compact convex subsets of V containing o in the interior. This means that whenever $K, L, K \cup L$ are such bodies, then
$$\mathcal{A}_a(K \cup L) = \mathcal{A}_a(K) + \mathcal{A}_a(L) - \mathcal{A}_a(K \cap L).$$
- $\mathcal{A}_a$ is upper semicontinuous with respect to the Hausdorff topology.
- $\mathcal{A}_a$ is invariant under $\mathrm{GL}(V)$.

A recent theorem by M. Ludwig and M. Reitzner [2007] states that the vector space of functionals with these three properties is generated by the constant valuation and $\mathcal{A}_a$. The centroprojective surface area satisfies the last two conditions, but is not a valuation.

Acknowledgments

We wish to thank Bruno Colbois and Daniel Hug for interesting discussions and Franz Schuster for useful remarks on an earlier version of this paper.

References

[Alexandroff 1939] A. D. Alexandroff, "Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it", *Leningrad State Univ. Annals Math. Ser.* **6** (1939), 3–35. MR 2,155a

[Álvarez Paiva and Fernandes 1998] J. C. Álvarez Paiva and E. Fernandes, "Crofton formulas in projective Finsler spaces", *Electron. Res. Announc. Amer. Math. Soc.* **4** (1998), 91–100. MR 99j:53097 Zbl 0910.53044

[Álvarez Paiva and Thompson 2004] J. C. Álvarez Paiva and A. C. Thompson, "Volumes on normed and Finsler spaces", pp. 1–48 in *A sampler of Riemann–Finsler geometry*, edited by D. Bao et al., Math. Sci. Res. Inst. Publ. **50**, Cambridge Univ. Press, 2004. MR 2006c:53079 Zbl 1078.53072

[Bangert 1999] V. Bangert, "Convex hypersurfaces with bounded first mean curvature measure", *Calc. Var. Partial Differential Equations* **8**:3 (1999), 259–278. MR 2000f:53043 Zbl 0960.53007

[Barvinok 2002] A. Barvinok, *A course in convexity*, Graduate Studies in Mathematics **54**, American Mathematical Society, Providence, RI, 2002. MR 2003j:52001 Zbl 1014.52001

[Benoist 2004] Y. Benoist, "Convexes divisibles, I", pp. 339–374 in *Algebraic groups and arithmetic*, edited by S. G. Dani and G. Prasad, Tata Inst. Fund. Res., Mumbai, 2004. MR 2005h:37073 Zbl 1084.37026

[Benoist 2006] Y. Benoist, "Convexes hyperboliques et quasiisométries", *Geometriae Dedicata* **122** (2006), 109–134. MR 2007k:20091 Zbl 1122.20020

[Benoist 2008] Y. Benoist, "A survey on divisible convex sets", pp. 1–18 in *Geometry, analysis and topology of discrete groups*, edited by L. Ji et al., Adv. Lect. Math. (ALM) **6**, Int. Press, Somerville, MA, 2008. MR 2464391 Zbl 1154.22016

[Bernig 2009] A. Bernig, "Hilbert geometry of polytopes", *Arch. Math.* (*Basel*) **92**:4 (2009), 314–324. MR 2010c:53106 Zbl 1171.53046

[Blaschke 1956] W. Blaschke, *Kreis und Kugel*, de Gruyter, Berlin, 1956. MR 17,1123d Zbl 0070.17501

[Borisenko and Olin 2008] A. A. Borisenko and E. A. Olin, "Asymptotic properties of Hilbert geometry", *Zh. Mat. Fiz. Anal. Geom.* **4**:3 (2008), 327–345. MR 2444106 Zbl 1167.53066

[Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Math. **319**, Springer, Berlin, 1999. MR 2000k:53038 Zbl 0988.53001

[Colbois and Vernicos 2007] B. Colbois and C. Vernicos, "Les géométries de Hilbert sont à géométrie locale bornée", *Ann. Inst. Fourier* (*Grenoble*) **57**:4 (2007), 1359–1375. MR 2008k:53160 Zbl 1123.53022

[Colbois and Verovic 2004] B. Colbois and P. Verovic, "Hilbert geometry for strictly convex domains", *Geometriae Dedicata* **105** (2004), 29–42. MR 2005e:53111 Zbl 1078.52002

[Colbois et al. 2008] B. Colbois, C. Vernicos, and P. Verovic, "Hilbert geometry for convex polygonal domains", preprint, 2008. arXiv 0804.1620

[Evans and Gariepy 1992] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992. MR 93f:28001 Zbl 0804.28001

[Foertsch and Karlsson 2005] T. Foertsch and A. Karlsson, "Hilbert metrics and Minkowski norms", *J. Geom.* **83**:1-2 (2005), 22–31. MR 2007e:51021 Zbl 1084.52008

[Gallot et al. 2004] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, 3rd ed., Springer, Berlin, 2004. MR 2005e:53001 Zbl 1068.53001

[de la Harpe 1993] P. de la Harpe, "On Hilbert's metric for simplices", pp. 97–119 in *Geometric group theory* (Sussex, 1991), vol. 1, edited by G. A. Niblo and M. A. Roller, London Math. Soc. Lecture Note Ser. **181**, Cambridge Univ. Press, 1993. MR 94i:52006 Zbl 0832.52002

[Hilbert 1895] D. Hilbert, "Ueber die gerade Linie als kürzeste Verbindung zweier Punkte", *Math. Ann.* **46** (1895), 91–96. JFM 26.0540.02

[Hilbert 1999] D. Hilbert, *Grundlagen der Geometrie*, 14th ed., Teubner-Archiv zur Mathematik. Supplement **6**, Teubner, Stuttgart, 1999. MR 2000j:01120 Zbl 0933.01031

[Hörmander 2007] L. Hörmander, *Notions of convexity*, Birkhäuser, Boston, MA, 2007. MR 2311920 Zbl 1108.32001

[Hug 1998] D. Hug, "Absolute continuity for curvature measures of convex sets, I", *Math. Nachr.* **195** (1998), 139–158. MR 99j:52002 Zbl 0938.52003

[Hug 1999a] D. Hug, "Absolute continuity for curvature measures of convex sets, II", *Math. Z.* **232**:3 (1999), 437–485. MR 2000m:52009 Zbl 0954.52006

[Hug 1999b] D. Hug, *Measures, curvatures and currents in convex geometry*, Habilitationsschrift, Universität Freiburg, 1999.

[Karlsson and Noskov 2002] A. Karlsson and G. A. Noskov, "The Hilbert metric and Gromov hyperbolicity", *Enseign. Math.* (2) **48**:1-2 (2002), 73–89. MR 2003f:53061 Zbl 1046.53026

[Kay 1967] D. C. Kay, "The ptolemaic inequality in Hilbert geometries", *Pacific J. Math.* **21** (1967), 293–301. MR 35 #4820 Zbl 0147.22406

[Kelly and Paige 1952] P. J. Kelly and L. J. Paige, "Symmetric perpendicularity in Hilbert geometries", *Pacific J. Math.* **2** (1952), 319–322. MR 14,308g Zbl 0048.13302

[Kelly and Straus 1958] P. Kelly and E. Straus, "Curvature in Hilbert geometries", *Pacific J. Math.* **8** (1958), 119–125. MR 20 #2748 Zbl 0081.16401

[Kelly and Straus 1968] P. Kelly and E. G. Straus, "Curvature in Hilbert geometries, II", *Pacific J. Math.* **25** (1968), 549–552. MR 38 #613

[Koldobsky 2005] A. Koldobsky, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs **116**, Amer. Math. Soc., Providence, RI, 2005. MR 2006a:42007 Zbl 1082.52002

[Laugwitz 1965] D. Laugwitz, *Differentialgeometrie in Vektorräumen, unter besonderer Berücksichtigung der unendlichdimensionalen Räume*, Vieweg, Braunschweig, 1965. MR 32 #406

[Leichtweiß 1993] K. Leichtweiß, "Convexity and differential geometry", pp. 1045–1080 in *Handbook of convex geometry*, vol. B, edited by P. M. Gruber and J. M. Wills, North-Holland, Amsterdam, 1993. MR 94j:52002 Zbl 0840.53038

[Leichtweiß 1998] K. Leichtweiß, *Affine geometry of convex bodies*, Barth, Heidelberg, 1998. MR 2000j:52005 Zbl 0899.52005

[Ludwig and Reitzner 2007] M. Ludwig and M. Reitzner, "A classification of SL(n) invariant valuations", preprint, 2007. To appear in Ann. Math.

[Lutwak 1996] E. Lutwak, "The Brunn–Minkowski–Firey theory, II: Affine and geominimal surface areas", *Adv. Math.* **118**:2 (1996), 244–294. MR 97f:52014

[Mattila 1995] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics **44**, Cambridge Univ. Press, 1995. MR 96h:28006 Zbl 0819.28004

[Nasu 1961] Y. Nasu, "On Hilbert geometry", *Math. J. Okayama Univ.* **10** (1961), 101–112. Zbl 0103.37901

[Schütt and Werner 1990] C. Schütt and E. Werner, "The convex floating body", *Math. Scand.* **66**:2 (1990), 275–290. MR 91i:52005

[Shen 2001] Z. Shen, *Lectures on Finsler geometry*, World Scientific, Singapore, 2001. MR 2002f:53032 Zbl 0974.53002

[Socié-Méthou 2004] E. Socié-Méthou, "Behaviour of distance functions in Hilbert–Finsler geometry", *Differential Geom. Appl.* **20**:1 (2004), 1–10. MR 2004i:53112 Zbl 1055.53057

[Vernicos 2005] C. Vernicos, "Introduction aux géométries de Hilbert", pp. 145–168 in *Actes de Séminaire de Théorie Spectrale et Géométrie*, Sémin. Théor. Spectr. Géom. **23**, Univ. Grenoble I, 2005. MR 2007m:51011 Zbl 1100.53031

[Vernicos 2008a] C. Vernicos, "Entropie volumique des géométries de Hilbert", preprint, 2008, Available at http://tinyurl.com/ydby7jt. To appear in Actes de Séminaire de Théorie Spectrale et Géométrie.

[Vernicos 2008b] C. Vernicos, "Lipschitz characterisation of polytopal Hilbert geometries", preprint, 2008. arXiv 0812.1032

Received March 12, 2009. Revised September 24, 2009.

GAUTIER BERCK
DÉPARTEMENT DE MATHÉMATIQUES
CHEMIN DU MUSÉE 23
1700 FRIBOURG
SWITZERLAND
gautier.berck@unifr.ch
http://perso.unifr.ch/gautier.berck

ANDREAS BERNIG
INSTITUT FÜR MATHEMATIK
JOHANN WOLFGANG GOETHE-UNIVERSITÄT FRANKFURT
ROBERT-MAYER-STR. 10
D-60054 FRANKFURT
GERMANY
bernig@math.uni-frankfurt.de
http://www.uni-frankfurt.de/fb/fb12/mathematik/an/bernig

CONSTANTIN VERNICOS
DEPARTMENT OF MATHEMATICS
LOGIC HOUSE, SOUTH CAMPUS
NATIONAL UNIVERSITY OF IRELAND, MAYNOOTH
CO. KILDARE
IRELAND
Constantin.Vernicos@maths.nuim.ie
http://costia.free.fr/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

ON ROUGH-ISOMETRY CLASSES OF HILBERT GEOMETRIES

THOMAS FOERTSCH

We prove that Hilbert geometries on uniformly convex Euclidean domains with C^2-boundaries are roughly isometric to the real hyperbolic spaces of corresponding dimension.

1. Introduction

Hilbert geometries generalize the Klein model of the real hyperbolic space from ellipsoids in $\mathbb{E}^n$, the n-dimensional Euclidean space, to arbitrary bounded convex subsets of $\mathbb{E}^n$. Karlsson and Noskov [2002] provide necessary conditions as well as sufficient conditions on the boundary of such a convex subset in order for its associated Hilbert geometry to be Gromov hyperbolic. Benoist [2003] even precisely determined such convex subsets, the associated Hilbert geometries of which are Gromov hyperbolic. Namely, such a bounded convex subset yields a Gromov hyperbolic Hilbert geometry if and only if its Euclidean boundary is locally the graph of a “quasisymmetrically convex” function.

Benoist [2006] proved that every two-dimensional Gromov hyperbolic Hilbert geometry is quasi-isometric to the real hyperbolic space of corresponding dimension. Here he also provides examples of Hilbert geometries in dimension ≥ 3 which are not quasi-isometric to real hyperbolic spaces.

For related discussions of non-Gromov hyperbolic Hilbert geometries, see also [Bernig 2009; Bletz-Siebert and Foertsch 2007; Colbois and Verovic 2008; Colbois et al. 2008].

Restricting their attention to so-called strictly (or, as one might prefer, uniformly) convex domains, Colbois and Verovic [2004] proved that the Hilbert geometries of such domains are bi-Lipschitz equivalent to the real hyperbolic space of corresponding dimension.

The purpose of this paper is to prove that such Hilbert geometries are even rough-isometric to the real hyperbolic spaces of corresponding dimension.

MSC2000: primary 53C60; secondary 51F99.
Keywords: Hilbert geometries, Gromov hyperbolicity, rough isometry.

Recall that a map $f : X \longrightarrow Y$ between metric spaces is called a *rough-isometric embedding* if there exists some $k \geq 0$ such that

$$|xx'| - k \leq |f(x)f(x')| \leq |xx'| + k \quad \text{for all} \quad x, x' \in X.$$

If, moreover, for all $y \in Y$ there exists an $x \in X$ such that $|yf(x)| \leq k$, then f is called a *rough isometry*.

Recall further that Gromov hyperbolicity is a rough-isometry invariant, and in the course setting of Gromov hyperbolic spaces, what one is generally interested in are the corresponding rough-isometry classes.

Theorem 1.1. *Let D be an open, bounded convex domain in $\mathbb{E}^n$. Suppose further that the boundary ∂D is of class C^2 and the curvature of ∂D is nonzero everywhere. Then the Hilbert geometry (D, h_κ^D) associated with D is rough-isometric to $\mathbb{H}_\kappa^n$.*

The proof relies on the equivalence of rough-isometry classes of visual, Gromov hyperbolic spaces and bi-Lipschitz classes of their boundaries at infinity. We recall in Section 2 the precise definition of Hilbert geometries and summarize such facts on Gromov hyperbolic spaces as will be needed in the proof of Theorem 1.1. In Section 3 we give proofs of some elementary geometric lemmata, which will also be quoted in the proof of Theorem 1.1 in Section 4.

2. Preliminaries

2.1. ***Hilbert geometries on uniformly convex domains with C^2-boundary.*** Let $\mathbb{E}^n = (\mathbb{R}^n, d_e) = (\mathbb{R}^n, |\cdot|)$ denote the n-dimensional Euclidean space. For the Euclidean distance of $x, y \in \mathbb{E}^n$ we write $|xy|$, and for the line segment between x and y we write $[x, y]$, while $L(x, y)$ denotes the whole straight line in $\mathbb{E}^n$ through x and y.

Given an open bounded convex domain $D \subset \mathbb{E}^n$ with boundary $\partial D \subset \mathbb{E}^n$ and some $\kappa < 0$ the Hilbert metric $h_\kappa^D : D \times D \longrightarrow \mathbb{R}_0^+$ is defined as follows. For $x, y \in D$ one defines

$$h_\kappa(x, y) := h_\kappa^D(x, y) := \begin{cases} \dfrac{1}{\sqrt{-\kappa}} \log \dfrac{|y\xi_{x,y}|\,|x\xi_{y,x}|}{|x\xi_{x,y}|\,|y\xi_{y,x}|} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

where $\xi_{x,y} \in L(x, y) \cap \partial D$ is uniquely determined by the condition $|\xi_{x,y}x| < |\xi_{x,y}y|$ ($\xi_{y,x} \in L(x, y) \cap \partial D$ by $|\xi_{y,x}x| > |\xi_{y,x}y|$, respectively). The expression

$$\frac{|y\xi_{x,y}|\,|x\xi_{y,x}|}{|x\xi_{x,y}|\,|y\xi_{y,x}|}$$

is called the cross ratio of the four collinear ordered points $\xi_{x,y}, x, y, \xi_{y,x}$ and is invariant under projective transformations. For the basic properties of the distance

h_κ see [Busemann 1955; de la Harpe 1993]; for example, the topology induced by h_κ on D coincides with the subspace topology inherited from $\mathbb{E}^n$. We shall refer to the metric space (D, h_κ) as a *Hilbert geometry*.

Note that if D is a ball or an ellipsoid, the associated Hilbert metric space (D, h_κ) is isometric to the real hyperbolic space of constant sectional curvature κ of corresponding dimension.

Now let $D \subset \mathbb{R}^n$ be an open bounded convex domain with boundary of class C^2. Let further $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^2-function satisfying $\rho|_D > 0$, $\rho|_{\partial D} = 0$, and $\rho|_{\mathbb{R}^n \setminus D} < 0$ such that its gradient $\nabla\rho$ is a unit vector field normal to ∂D and directed inside D. By $W_x : T_x\partial D \longrightarrow T_x\partial D$ we denote the curvature (or Weingarten) operator which assigns to each $v \in T_x\partial D$ the directional derivative of $\nabla\rho$ in direction v. From this curvature operator one obtains the second fundamental form II_x as the following bilinear form on $T_x\partial D$:

$$II_x(v, w) = \langle w, W_x(v)\rangle = \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial x^i \partial x^j} v_i w_j \quad \text{for } v, w \in T_x\partial D.$$

We call $k_x(u) := II_x(u, u)$ the normal curvature of ∂D at x in the direction of the unit tangent vector u.

In the case where the curvature of ∂D is nonzero everywhere, that is, where II is positive definite everywhere, there exists some constant $k_D > 0$ such that

$$k_D^{-1} \le k_x\left(\frac{u}{\|u\|}\right) \le k_D \quad \text{for } x \in \partial D,\ u \in T_x\partial D. \tag{1}$$

2.2. ***Gromov hyperbolic spaces and their boundaries at infinity.*** For X a metric space, the *Gromov product* of two points of X with respect to a third is defined by

$$(x \cdot y)_o := \tfrac{1}{2}(|xo| + |yo| - |xy|) \quad \text{for } o, x, y \in X.$$

The space X is called *Gromov hyperbolic* if there exists $\delta \ge 0$ such that

$$(x \cdot y)_o \ge \min\{(x \cdot z)_o,\ (z \cdot y)_o\} - \delta \quad \text{for } o, x, y, z \in X. \tag{2}$$

This notion of Gromov hyperbolicity is a rough-isometry invariant, and the objects of interest in this asymptotic theory are the corresponding rough-isometry classes rather than the spaces themselves.

To a Gromov hyperbolic metric space one associates a boundary at infinity, endowed with a certain quasimetric. For a broad class of Gromov hyperbolic spaces (those satisfying the visuality assumption — see below), the bi-Lipschitz class of this quasimetric canonically corresponds to the rough isometry class of the space.

Now let X be a Gromov hyperbolic metric space. A sequence $\{x_i\}$ of points $x_i \in X$ converges to infinity if $\lim_{i,j\to\infty}(x_i \cdot x_j)_o = \infty$. Two sequences $\{x_i\}$, $\{x_i'\}$

that converge to infinity are considered equivalent if $\lim_i (x_i \cdot x_i')_o = \infty$. Using the δ-inequality (2), one easily sees that this defines an equivalence relation for sequences in X converging to infinity. The boundary at infinity $\partial_\infty X$ of X is defined as the set of equivalence classes of sequences converging to infinity.

For points $\xi, \xi' \in \partial_\infty X$ one defines their Gromov product with respect to the basepoint $o \in X$ by

$$(\xi \cdot \xi')_o := \inf \liminf_{i \longrightarrow \infty} (x_i \cdot x_i')_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$ and $\{x_i'\} \in \xi'$.

It is a well-known fact (see for instance the remark following [Bridson and Haefliger 1999, Definition 1.19]) that in the geodesic setting the Gromov product $(\xi \cdot \xi')_o$ roughly measures the distance of o to the geodesic connecting ξ to ξ'. As we are going to use this fact later on, we formulate it as follows:

Lemma 2.1. *Fix $\delta > 0$. Then there exists a constant K such that if (X, d) is a proper geodesic Gromov hyperbolic space satisfying the δ-inequality* (2), *then $|d(x, im\{\gamma\} - (\xi \cdot \xi')_x)| < k$ for all $x \in X, \xi, \xi' \in \partial_\infty X$ and every geodesic line γ in (X, d) with $c(-\infty) = \xi$ and $c(\infty) = \xi'$.*

From the inequality (2) it immediately follows that $\rho_o : \partial_\infty X \times \partial_\infty X \longrightarrow \mathbb{R}_0^+$, given by $\rho_o(\xi, \xi') := e^{-(\xi \cdot \xi')_o}$, is a e^δ-quasimetric, that is,

$$\rho_o(\xi, \xi') \leq e^\delta \max\{\rho_o(\xi', \xi''), \rho_o(\xi'', \xi')\} \quad \text{for } \xi, \xi', \xi'' \in \partial_\infty X.$$

It is directly clear from the definition of the boundary quasimetrics that Gromov hyperbolic spaces X and X' which are rough-isometric to each other,

$$X \overset{\text{rough}}{\cong} X',$$

give rise to boundary quasimetric spaces $(\partial_\infty X, \rho_o)$ and $(\partial_\infty X', \rho_{o'})$ which are bi-Lipschitz equivalent,

$$(\partial_\infty X, \rho_o) \overset{\text{bi-Lip}}{\cong} (\partial_\infty X', \rho_{o'}).$$

For the converse statement to be true, it is clear that one has to ask the boundary somehow to represent the entire space. More precisely, recall that a metric space is called roughly geodesic if there exists some $k \geq 0$ such that any two points in the space can be joined by a k-rough geodesic, that is, a k-rough isometric embedding of a closed interval. A Gromov hyperbolic space X is called visual if for some $o \in X$ and some $k \geq 0$ every point $x \in X$ lies on a k-rough geodesic ray initiating in o. In particular, a visual Gromov hyperbolic space is roughly geodesic.

Bonk and Schramm [2000] described the morphism classes of the spaces on the one hand, and those of their boundaries, on the other hand, which correspond to each other under the assumption of visuality. The statement we will refer to can also be deduced as a corollary of [Buyalo and Schroeder 2007, Theorem 7.1.2].

Theorem 2.2 [Bonk and Schramm 2000; Buyalo and Schroeder 2007, Theorem 7.1.2]. *Let X and X' be visual Gromov hyperbolic spaces, and let $o \in X$ as well as $o' \in X'$. Then*

$$X \overset{\text{rough}}{\cong} X' \iff (\partial_\infty X, \rho_o) \overset{\text{bi-Lip}}{\cong} (\partial_\infty X', \rho_{o'}).$$

Note that in the case where the Gromov hyperbolic metric space is a CAT(-1)-space, the quasimetric ρ_o indeed satisfies the triangle inequality and hence is a metric. This was shown by Bourdon [1995]. In particular, consider the real hyperbolic space $\mathbb{H}^n$ in the Poincaré ball model. Then the Bourdon metric ρ_o with respect to the center of the ball o is precisely given by half the Euclidean metric on $\partial_\infty \mathbb{H}^n = S^{n-1} \subset \mathbb{E}^n$ [Buyalo and Schroeder 2007, p. 21].

Finally note that for a Gromov hyperbolic Hilbert geometry (D, h_D), the Gromov boundary can naturally be identified with ∂D, which follows from [Karlsson and Noskov 2002, Theorem 5.2] and [Foertsch and Karlsson 2005, Proposition 2].

Moreover, Hilbert geometries are visual. In fact, for any basepoint $o \in D$, every $x \in D$ lies on a geodesic ray initiating in o.

3. Four elementary geometric lemmata

This section contains the proofs of four elementary geometric lemmata, which will be referred to in the proof of Theorem 1.1 in Section 4. The complete section may be skipped at a first reading. The statements are not surprising, but we provide the proofs for the convenience of the reader.

Lemma 3.1. *Let $\gamma : [0, a] \longrightarrow \mathbb{E}^2$ be an arc-length parameterized straight line segment of length $0 < a \le 2\rho$ in a ball $B(r, \rho)$ around the origin $o \in \mathbb{E}^2$ with $\gamma(0), \gamma(a) \in \partial B(o, \rho)$, and denote by $l = l(\rho, a) > 0$ the distance of $\gamma(a/2)$ to the two-point set $L(o, \gamma(a/2)) \cap \partial B(o, \rho)$, for $a < 2\rho$, and $l = \rho$ otherwise. Then*

$$\frac{1}{\Lambda(\rho)} \sqrt{l(\rho, a)} \le a \le \Lambda(\rho) \sqrt{l(\rho, a)} \quad \textit{for } a \in [0, 2\rho],$$

with $\Lambda(\rho) := \max\{2\sqrt{2\rho}, 1/(2\sqrt{\rho})\}$.

Proof. This immediately follows from $a = 2\sqrt{2\rho - l(\rho, a)}\sqrt{l(\rho, a)}$ and $0 \le l(\rho, a) \le \rho$. □

Now let $R > r > 0$ and let S be a straight line segment in $\mathbb{E}^2$ of length x, the endpoints of which lie on $\partial B(o, R)$. $\overline{B(o, R)} \setminus S$ consists of two connected components $\tilde{B}$ and $\hat{B}$. For $x < r$, let $\tilde{B}(S)$ be the component disjoint from $B(o, R - r)$. Given $p \in B(o, R - r)$ and $q \in S$, define

$$w = w(p, q) := L(p, q) \cap \tilde{B}(S) \cap \partial B(o, R)$$

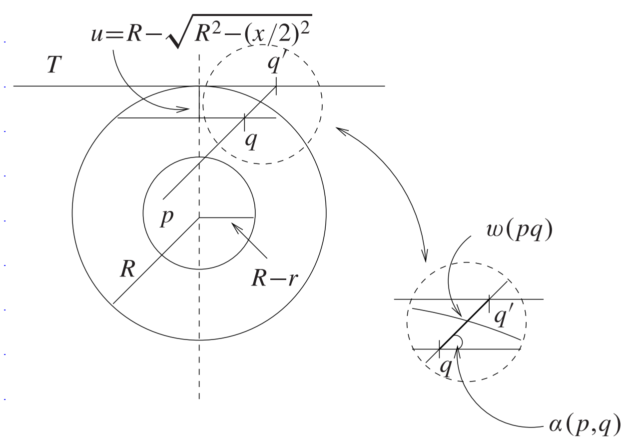

Figure 1. Notation in Lemma 3.2.

and set

$$m = m(x, R, r) := \max_{\substack{p \in B(o, R-r) \\ q \in S}} |qw(p, q)|.$$

Lemma 3.2. *Fix $R > r > 0$. Then*

$$m(x, R, r) \le \tilde{\Lambda} \left(R - \sqrt{R^2 - (x/2)^2} \right)$$

for $\tilde{\Lambda} = \tilde{\Lambda}(r, R) := \sin^{-1}(\arctan r/(4R))$.

Proof. For $p \in B(o, R-r)$ and $q \in S$, let $\alpha = \alpha(p, q)$ denote the angle $\alpha(p, q) := \angle_q(L(p, q), S) \in (0, \pi/2]$. Further, let T denote the tangential line to $\partial \tilde{B}(S) \setminus S$ parallel to S, and set $q' := T \cap L(p, q)$ and $v := |qq'|$. Then

$$|qw(p, q)| < v = \frac{u}{\sin \alpha(p, q)} \quad \text{with} \quad u := R - \sqrt{R^2 - (x/2)^2}.$$

Therefore it remains to prove that there exists $\alpha_0 > 0$ such that $\alpha(p, q) \ge \alpha_0$ for all $p \in B(o, R-r)$ and $q \in S$.

Since $x < r$, we deduce $u < r/2$ and therefore $\operatorname{dist}(S, \partial B(o, R-r)) > r/2$. It follows that we can choose

$$\alpha_0 := \arctan \frac{r/2}{2R} = \arctan \frac{r}{4R}. \qquad \square$$

Let $\rho_2 > \rho_1 > 0$ be fixed and C_{ρ_2}, C_{ρ_1} be circles in $\mathbb{E}^2$ of radius ρ_2 and ρ_1, respectively, such that $\#(C_{\rho_1} \cap C_{\rho_2}) = 1$ with the center o_{ρ_1} of C_{ρ_1} in the bounded component of $\mathbb{R}^2 \setminus C_2$. Let $q := C_{\rho_1} \cap C_{\rho_2}$, and denote by o_{ρ_2} the center of C_{ρ_2}. Further, let L_0 be the straight line through o_{ρ_2} orthogonal to $L(q, o_{\rho_2})$. By H we denote the half-space in $\mathbb{E}^2$ defined by L_0 such that H contains the center o_{ρ_1} of C_{ρ_1}. Now let $L_t \subset H$ be the parallel to L_0 in distance t of o_{ρ_2} for all $t \in [0, \rho_2)$

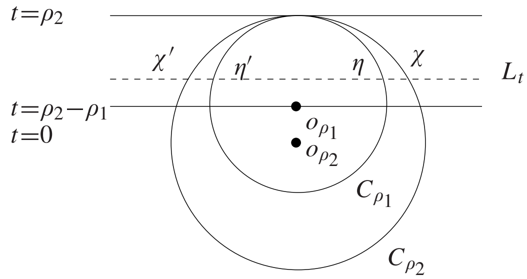

Figure 2. Illustration of the situation considered in Lemma 3.3.

and define $\chi_t, \chi_t', \eta_t, \eta_t' \in \mathbb{E}^2$ via $\{\chi_t, \chi_t'\} = L_t \cap C_{\rho_2}$ and $\{\eta_t, \eta_t'\} = L_t \cap C_{\rho_1}$ for all $t \in [\rho_2 - \rho_1, \rho_2)$.

Lemma 3.3. *Let $\rho_2 > \rho_1 > 0$. Then $|\chi_t \chi_t'| \le \hat{\Lambda}\, |\eta_t \eta_t'|$ for all $t \in [\rho_2 - \rho_1, \rho_2)$, with $\hat{\Lambda} = \hat{\Lambda}(\rho_1, \rho_2) := \sqrt{(2\rho_2 - \rho_1)/\rho_1}$.*

Proof. Consider the function $f : [\rho_2 - \rho_1, \rho_2) \longrightarrow \mathbb{R}^+$ given by

$$f(t) := \frac{|\chi_t \bar{\chi}_t|^2}{|\eta_t \bar{\eta}_t|^2} = \frac{{\rho_2}^2 - t^2}{{\rho_1}^2 - (t - (\rho_2 - \rho_1))^2} \quad \text{for all} \quad t \in [\rho_2 - \rho_1, \rho_2).$$

With $f'(t) \neq 0$ for all $t \in (\rho_2 - \rho_1, \rho_2)$, as well as

$$\lim_{t \to \rho_2} f(t) = \rho_2/\rho_1 \le (2\rho_2 - \rho_1)/\rho_1 = f(\rho_2 - \rho_1),$$

the claim follows. □

Lemma 3.4. *Let D be a bounded, convex domain in $\mathbb{E}^{n+1}$ with C^1-boundary ∂D. Then $(\partial D, d_e|_{\partial D \times \partial D})$ is bi-Lipschitz equivalent to $(S^n, d_e|S^n)$.*

Proof. Let $x \in D$ and let $r > 0$ be such that $B_r(x) \subset D$. Consider the map $\varphi : (\partial D, d_e|_{\partial D \times \partial D}) \longrightarrow (\partial B_r(x), d_e|_{\partial B_r(x) \times \partial B_r(x)})$, given by

$$\xi \mapsto \eta \in L(x, \xi) \cap \partial B_r(x) \quad \text{with} \quad |\eta\xi| = \operatorname{dist}(\xi, L(x, \xi) \cap \partial B_r(x)).$$

Obviously, $|\xi\xi'| \le |\varphi(\xi)\varphi(\xi')|$ for all $\xi, \xi' \in \partial D$. Moreover, for all $\alpha > 0$ there exists $\mu(\alpha)$ such that

$$|\xi\xi'| \ge \mu(\alpha)\, |\varphi(\xi)\varphi(\xi')| \quad \text{for } \xi, \xi' \in \partial D, \quad \text{with} \quad \angle_x(\xi, \xi') \ge \alpha.$$

Therefore we only have to consider angles approaching zero.

Let $R_{\xi,x} := |\xi x|$ and let $R_x := \{\max R_{\xi x} \mid \xi \in \partial D\}$. Let further T_ξ denote the tangent to ∂D at $\xi \in \partial D$ and set $\gamma_{x\xi} := \angle_\xi(T_\xi, L(x, \xi)) \in (0, \frac{\pi}{2})$. Then, since D is C^1 and convex and ∂D is compact, there exists $\gamma_0 > 0$ such that

$$\inf\{\gamma_{x\xi} \mid \xi \in \partial D\} = \min\{\gamma_{x\xi} \mid \xi \in \partial D\} \ge \gamma_0.$$

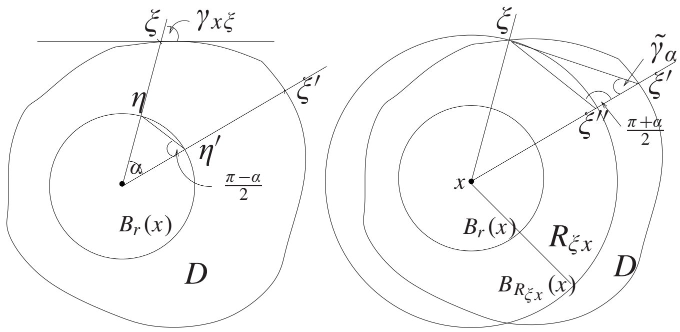

Figure 3. Notation used in the proof of Lemma 3.4.

Now consider $\xi, \xi' \in \partial D$ with $\angle_x(\xi, \xi') = \alpha$. Let $C_{x,\xi,\xi'}(R_{\xi,x})$ be the circle in $\operatorname{span}\{x, \xi, \xi'\}$ of radius $R_{\xi,x}$ and center x, and let $\xi'' := L(x, \xi') \cap C_{x,\xi,\xi'}(R_{\xi,x})$ with $\big||\xi'' x| - |\xi' x|\big| = |\xi' \xi''|$. Since $\angle_\xi(x, \xi'') = \frac{1}{2}(\pi - \alpha) = \angle_{\xi''}(x, \xi)$, we find

$$\frac{L_\alpha^\xi}{\sin \frac{1}{2}(\pi - \alpha)} = \frac{l_\alpha^\xi}{\sin \tilde{\gamma}_\alpha},$$

where $L_\alpha^\xi := |\xi' \xi|$, $l_\alpha^\xi := |\xi'' \xi|$ and $\tilde{\gamma}_\alpha := \angle_{\xi'}(\xi'', \xi)$.

Now, since $\sin \tilde{\gamma}_\alpha \to \sin \gamma_{x,\xi} \geq \sin \gamma_0$ as $\alpha \to 0$, it follows that for all $\xi \in \partial D$ there exists $\alpha_0(\xi)$ such that

$$L_\alpha^\xi \leq \frac{\sin \frac{1}{2}(\pi - \alpha)}{\sin \frac{1}{2}\gamma_0} l_\alpha^\xi$$

for all $\alpha \leq \alpha_0(\xi)$. Thus, since ∂D is compact, there also exist $\alpha_0 > 0$ as well as $\mu > 0$ such that $L_\alpha^\xi \leq \mu l_\alpha^\xi$ for all $\alpha < \alpha_0$, from which the claim follows. $\square$

4. Proof of Theorem 1.1

We prove that $(D, h_{-1}) \overset{\text{rough}}{\cong} \mathbb{H}^n_{-1}$. The rest of the claim follows as usual by merely rescaling the metric.

From [Karlsson and Noskov 2002, Theorem 5.2] and [Foertsch and Karlsson 2005, Proposition 2] it follows that the Gromov boundary at infinity of (D, h_{-1}) can naturally be identified with $\partial D \subset \mathbb{D}^n$. The main goal of this proof is to verify that for $x \in D$ the visual quasimetric ρ_x on ∂D is bi-Lipschitz equivalent to the restriction of the Euclidean metric $d_e = |\cdot|$ to ∂D.

Let k_D be as in (1) and set $\rho_1 := \sqrt{k_D^{-1}}$ and $\rho_2 := \sqrt{k_D}$. Fix $x \in D$ and let $R_x > r_x > 0$ be such that $B(x, r_x) \subset D \subset B(x, R_x)$. We want to show that

$$\rho_x \overset{\text{bi-Lip}}{\cong} d_e|_{\partial D} =: |\cdot|\,|_{\partial D}.$$

(i) In the first step we establish that

$$\text{there exists } \lambda > 0 \text{ such that } e^{-(\xi\cdot\xi')_x} \geq \frac{1}{\lambda}|\xi\xi'|, \quad \text{for all } \xi, \xi' \in \partial D.$$

Let therefore $\xi, \xi' \in \partial D$ and $y \in [\xi, \xi']$ satisfying $d(x, y) = \operatorname{dist}(x, [\xi, \xi'])$. Note that for $x \in [\xi, \xi']$ we have $e^{-(\xi\cdot\xi')_x} = 1$ and $e^{-(\xi\cdot\xi')_x} \geq \frac{1}{\lambda}|\xi\xi'|$ holds for $\lambda \geq \operatorname{diam} D$. Therefore we can assume in the following without loss of generality that $x \notin [\xi, \xi']$.

Now let $y' \in [\xi, \xi'] \cap D$ be arbitrary and $A, B \in L(x, y') \cap \partial D$ be defined via $|Ax| < |Ay|$ and $|By| < |Bx|$. Then, due to Lemma 2.1 and the inequalities $r_x \leq |xA|, |y'A|, |xB| \leq 2R_x$, we deduce the existence of $\tilde\lambda_1, \tilde\lambda_2 > 0$ only depending on (D, h_{-1}), r_x and R_x such that

$$e^{-(\xi\cdot\xi)_x} \;\geq\; \frac{1}{\tilde\lambda_1}e^{-h_1(x,y)} \geq \frac{1}{\tilde\lambda_1}e^{-h_1(x,y')} = \frac{1}{\tilde\lambda_1}\sqrt{\frac{|xA|\,|y'B|}{|xB|\,|y'A|}} \;\geq\; \frac{1}{\tilde\lambda_2}\sqrt{|y'B|}.$$

Thus it remains to show that there exists $\tilde\lambda_3 > 0$ only depending on (D, h_{-1}), r_x and R_x such that for all $\xi, \xi' \in \partial D$ there exists y' as above satisfying

$$\sqrt{|y'B|} \;\geq \frac{1}{\tilde\lambda_3}|\xi\xi'|. \tag{3}$$

To prove this, consider the two-dimensional plane Σ spanned by x, ξ, ξ'. The set $(\Sigma \cap D) \setminus [\xi, \xi']$ consists of two connected components. Denote by $\tilde\Sigma$ the connected component of this set not containing x. Since ∂D is C^2, there exists $B \in \partial\tilde\Sigma \setminus [\xi, \xi'] \subset \partial D$ such that the tangent $T(B)$ of $\partial\tilde\Sigma$ at B is parallel to $[\xi, \xi']$.

Let $T(B)^\perp \subset \Sigma$ denote the straight line through B orthogonal to $T(B)$. Let further C_{ρ_2} be the circle of radius ρ_2 in Σ through B, tangent to $T(B)$, which

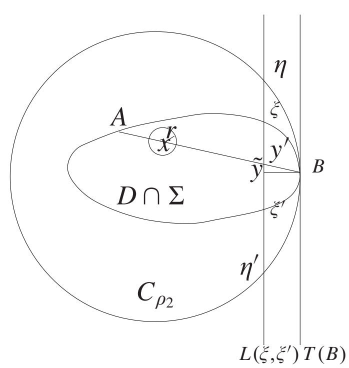

Figure 4. Situation in step (i) of the proof.

lies on the same side of $T(B)$ in Σ as D does. Now set $y' := [x, B] \cap [\xi, \xi']$, $\tilde{y} := T(B)^{\perp} \cap [\xi, \xi']$ as well as $\eta, \eta' \in L(\xi, \xi') \cap C_{\rho_2}$ such that $|\eta\xi| < |\eta\xi'|$ and $|\eta'\xi'| < |\eta'\xi|$.

Now we consider two cases:

- If $\mathrm{dist}([\xi, \xi'], T(B)) \geq \rho_2$, then (3) holds trivially for $|y'B|$ as above once $\lambda_3 \geq \mathrm{diam}(D)/\sqrt{\rho_2}$.
- If $\mathrm{dist}([\xi, \xi'], T(B)) < \rho_2$ we find with Lemma 3.1:

$$|\xi\xi'| \;\leq\; |\eta\eta'| \;\leq\; \Lambda(\rho_2)\sqrt{l(\rho_2, |\eta\eta'|)} \;=\; \Lambda(\rho_2)\sqrt{|\tilde{y}B|} \;\leq\; \Lambda(\rho_2)\sqrt{|y'B|}.$$

(ii) In the second step we establish that

$$\text{there exists } \lambda > 0 \text{ such that } e^{-(\xi\cdot\xi')_x} \leq \lambda|\xi\xi'|, \quad \text{for all } \xi, \xi' \in \partial D.$$

To do this, we choose x to be particularly nice: Let $E \in \partial D$, take the ball B_{ρ_1} of radius ρ_1 tangent to the tangent hyperplane $H(E)$ of ∂D at E such that ${B_{\rho_1}}^{\circ} \subset D$, and let x be the center of B_{ρ_1}. With x defined like this we have $|x\xi| \geq \rho_1$ for all $\xi \in \partial D$.

Now, for $\xi, \xi' \in \partial D$, $\xi \neq \xi'$, arbitrarily choose y as above and let $\bar{x} = \xi_{x,y}$, $\bar{y} = \xi_{y,x} \in \partial D$ be as in the definition of the Hilbert distance between x and y. Once again we can assume without loss of generality that $x \notin [\xi, \xi']$. Due to Lemma 2.1 and $r_x \leq |x\bar{x}|, |y\bar{x}|, |x\bar{y}| \leq 2R_x$ we deduce the existence of $\tilde{\lambda}_4, \tilde{\lambda}_5 > 0$ only depending on (D, h_{-1}), r_x and R_x such that

$$e^{-(\xi\cdot\xi')_x} \;\leq\; \tilde{\lambda}_4 e^{-h_1(x,y)} \;=\; \tilde{\lambda}_4\sqrt{\frac{|x\bar{x}|\cdot|y\bar{y}|}{|x\bar{y}|\cdot|y\bar{x}|}} \;\leq\; \tilde{\lambda}_5\sqrt{|y\bar{y}|}.$$

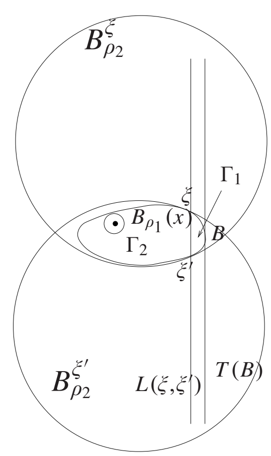

Figure 5. Notation in the proof of step (ii), with $i_0 = 1$.

Thus it remains to show that there exists $\tilde{\lambda}_6 > 0$ only depending on (D, h_1), r_x and R_x such that for all $\xi, \xi' \in \partial D$, the inequality $\sqrt{|y\bar{y}|} \le \tilde{\lambda}_6 |\xi\xi'|$ holds.

Since $|y\bar{y}| \le \operatorname{diam}(D)$, it suffices to restrict our attention to those $\xi, \xi' \in \partial D$ satisfying $|\xi\xi'| < 1/n$ for arbitrary but fixed $n \in \mathbb{N}$. We choose n as follows.

Let $\xi, \xi' \in \partial D$ and $\Sigma := \operatorname{span}\{x, \xi, \xi'\}$ as above. Let further $B^{\xi}_{\rho_2}$ and $B^{\xi'}_{\rho_2}$ denote the balls of radius ρ_2 through ξ and ξ' in Σ tangential to the tangents of $\partial D \cap \Sigma$ in ξ and ξ', respectively, such that $D \subset B^{\xi}_{\rho_2} \cap B^{\xi}_{\rho_2} =: \sigma$.

Then $\partial\sigma \setminus \{\xi, \xi'\}$ consists of two arcs γ_1 and γ_2 of length $l(\gamma_1)$ and $l(\gamma_2)$, respectively. Since ρ_1 and ρ_2 are fixed, it is immediate that there exists an $n_0 = n_0(\rho_1, \rho_2)$ such that from $|\xi\xi'| < \frac{1}{n_0}$, it follows that $\min\{l(\gamma_1), l(\gamma_2)\} < \rho_1$. Let us now assume without loss of generality (see above) that $|\xi\xi'| < \frac{1}{n_0}$.

We take $i_0 \in \{1, 2\}$ such that $l(\gamma_{i_0}) = \min\{l(\gamma_1), l(\gamma_2)\} < \rho_1$ and denote the connected components of $\sigma \setminus \{\xi, \xi'\}$ by Γ_1 and Γ_2 such that $\partial\Gamma_i = [\xi, \xi'] \cup \gamma_i$, $i = 1, 2$.

Since for each point $z \in \Gamma_{i_0}$ we have $\operatorname{dist}\{z, \partial D\} < \rho_1$, we deduce $x \notin \Gamma_{i_0}$ and thus $\bar{y} \in \Gamma_{i_0}$ for $\bar{y} = \xi_{y,x}$, as in the definition of the Hilbert distance between x and y.

Now let $B \in \Gamma_{i_0}$ and $T(B)$ be as in (i), and denote by B_{ρ_1} and B_{ρ_2} the balls in Σ of radii ρ_1 and ρ_2 through B, tangent to $T(B)$, which lie on the same side of $T(B)$ in Σ as D does. We denote the center of B_{ρ_1} by o_{ρ_1} and write T_{ρ_1} for the straight line through o_{ρ_1} parallel to $T(B)$. Further, let S be the strip bounded by $T(B)$ and T_{ρ_1}. Since $B \in \Gamma_{i_0}$ and thus $|\xi B|, |\xi' B| < \rho_1$, it follows that $\xi, \xi' \in (S \cap B_{\rho_2}) \setminus B^{\circ}_{\rho_1}$.

Thus we are exactly in the situation to apply Lemmata 3.1, 3.2 and 3.3. Let therefore $y' := T(B)^{\perp} \cap [\xi, \xi']$. Then we get

$$\begin{aligned}
\sqrt{|y\bar{y}|} &\le \tilde{\Lambda}(\rho_1, \rho_2) \cdot \sqrt{|y'B|} && \text{(by Lemma 3.2)} \\
&\le \tilde{\Lambda}(\rho_1, \rho_2) \cdot \Lambda(\rho_2) \cdot |\chi\chi'| && \text{(by Lemma 3.1)} \\
&\le \tilde{\Lambda}(\rho_1, \rho_2) \cdot \Lambda(\rho_2) \cdot \hat{\Lambda}(\rho_1, \rho_2) \cdot |\eta\eta'| && \text{(by Lemma 3.3)} \\
&\le \tilde{\Lambda}(\rho_1, \rho_2) \cdot \Lambda(\rho_2) \cdot \hat{\Lambda}(\rho_1, \rho_2) \cdot |\xi\xi'| =: \tilde{\lambda}_6 \cdot |\xi\xi'|,
\end{aligned}$$

where $\{\chi, \chi'\} := L(\xi, \xi') \cap C_{\rho_2}$ and $\{\eta, \eta'\} := L(\xi, \xi') \cap C_{\rho_1}$ and $C_{\rho_i} := \partial B_{\rho_i}$, $i = 1, 2$. Thus, applying Lemma 3.4, we have indeed established that the visual metric ρ_x on the boundary at infinity of (D, h_{-1}) is bi-Lipschitz equivalent to the angular boundary metric on $\partial\mathbb{H}^n_{-1}$. The claim therefore follows from Theorem 2.2 together with the obvious fact that (D, h_{-1}) is visual. $\square$

Acknowledgment

It is a pleasure to thank Mario Bonk for many discussions on Gromov hyperbolic spaces in general, and particularly on the subject of Gromov hyperbolic Hilbert geometries.

References

[Benoist 2003] Y. Benoist, "Convexes hyperboliques et fonctions quasisymétriques", *Publ. Math. Inst. Hautes Études Sci.* 97 (2003), 181–237. MR 2005g:53066 Zbl 1049.53027

[Benoist 2006] Y. Benoist, "Convexes hyperboliques et quasiisométries", *Geometriae Dedicata* **122** (2006), 109–134. MR 2007k:20091 Zbl 1122.20020

[Bernig 2009] A. Bernig, "Hilbert geometry of polytopes", *Arch. Math.* (*Basel*) **92**:4 (2009), 314–324. MR 2501287 Zbl 1171.53046

[Bletz-Siebert and Foertsch 2007] O. Bletz-Siebert and T. Foertsch, "The Euclidean rank of Hilbert geometries", *Pacific J. Math.* **231**:2 (2007), 257–278. MR 2008j:53073 Zbl 1155.53331

[Bonk and Schramm 2000] M. Bonk and O. Schramm, "Embeddings of Gromov hyperbolic spaces", *Geom. Funct. Anal.* **10**:2 (2000), 266–306. MR 2001g:53077 Zbl 0972.53021

[Bourdon 1995] M. Bourdon, "Structure conforme au bord et flot géodésique d'un CAT(−1)-espace", *Enseign. Math.* (2) **41**:1-2 (1995), 63–102. MR 96f:58120 Zbl 0871.58069

[Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **319**, Springer, Berlin, 1999. MR 2000k:53038 Zbl 0988.53001

[Busemann 1955] H. Busemann, *The geometry of geodesics*, Academic Press, New York, N. Y., 1955. MR 17,779a Zbl 0112.37002

[Buyalo and Schroeder 2007] S. Buyalo and V. Schroeder, *Elements of asymptotic geometry*, European Mathematical Society, Zürich, 2007. MR 2009a:53068 Zbl 1125.53036

[Colbois and Verovic 2004] B. Colbois and P. Verovic, "Hilbert geometry for strictly convex domains", *Geometriae Dedicata* **105** (2004), 29–42. MR 2005e:53111 Zbl 1078.52002

[Colbois and Verovic 2008] B. Colbois and P. Verovic, "Hilbert domains quasi-isometric to normed vector spaces", preprint, 2008. arXiv 0804.1619

[Colbois et al. 2008] B. Colbois, C. Vernicos, and P. Verovic, "Hilbert geometry for convex polygonal domains", preprint, 2008. arXiv 0804.1620

[Foertsch and Karlsson 2005] T. Foertsch and A. Karlsson, "Hilbert metrics and Minkowski norms", *J. Geom.* **83**:1-2 (2005), 22–31. MR 2007e:51021 Zbl 1084.52008

[de la Harpe 1993] P. de la Harpe, "On Hilbert's metric for simplices", pp. 97–119 in *Geometric group theory, Vol. 1* (Sussex, 1991), edited by G. A. Niblo and M. A. Roller, London Math. Soc. Lecture Note Ser. **181**, Cambridge Univ. Press, 1993. MR 94i:52006 Zbl 0832.52002

[Karlsson and Noskov 2002] A. Karlsson and G. A. Noskov, "The Hilbert metric and Gromov hyperbolicity", *Enseign. Math.* (2) **48**:1-2 (2002), 73–89. MR 2003f:53061 Zbl 1046.53026

Received July 30, 2009.

THOMAS FOERTSCH
MATHEMATISCHES INSTITUT
UNIVERSITÄT BONN
53115 BONN
GERMANY

foertsch@math.uni-bonn.de
http://www.math.uni-bonn.de/people/foertsch/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

ANALYTIC PROPERTIES OF DIRICHLET SERIES OBTAINED FROM THE ERROR TERM IN THE DIRICHLET DIVISOR PROBLEM

JUN FURUYA AND YOSHIO TANIGAWA

We discuss some analytic properties of Dirichlet series

$$Y(s) = \sum_{n=1}^{\infty} d(n)\Delta(n)n^{-s} \quad \text{for } \operatorname{Re} s > \tfrac{5}{4},$$

where $d(n)$ is the divisor function and $\Delta(x)$ is the error term in the Dirichlet divisor problem. In particular, we study an analytic continuation and an order of $Y(s)$. As applications, we study an analytic continuation and orders of several kinds of Dirichlet series related to $\Delta(x)$.

1. Introduction and statement of results

Let $d(n)$ be the divisor function, and let $\Delta(x)$ be the error term in the Dirichlet divisor problem, defined by

$$\Delta(x) = \sum_{n \le x} d(n) - x(\log x + 2\gamma - 1), \tag{1-1}$$

where γ is the Euler constant. A long history of research on $\Delta(x)$ has not settled the famous conjecture that $\Delta(x) = O(x^{1/4+\varepsilon})$, where ε is an arbitrarily small positive number. An efficient way to investigate $\Delta(x)$ is to consider the Dirichlet series whose coefficients involve $\Delta(x)$ or the related integrals.

In [Furuya et al. 2010], we considered properties of the Dirichlet series $D_j(s)$ defined by

$$D_j(s) = \sum_{n=1}^{\infty} \frac{\Delta(n)^j}{n^s},$$

for $j = 1$ and 2. It is easily seen that these functions are absolutely convergent for $\sigma > 5/4$ for $j = 1$ and $\sigma > 3/2$ for $j = 2$. Here, and in what follows, we denote the

MSC2000: 11L07, 11M41.
Keywords: analytic continuation, Dirichlet series, divisor problem, double zeta function, orders of Dirichlet series.
Furuya is supported by Grant-in-Aid for Scientific Research number 18740022.

complex number s as $s = \sigma + it$ with real numbers σ and t. We have established the analytic continuation and the locations of poles of these functions:

Theorem [Furuya et al. 2010, Theorems 1 and 2]. *The function $D_1(s)$ can be continued to the whole complex plane as a meromorphic function. This function has a double pole at $s = 1$ and a simple pole at $s = -2n$ with a nonnegative integer n. In particular, the Laurent expansion of $D_1(s)$ at $s = 1$ is given by*

$$D_1(s) = \frac{1}{2(s-1)^2} + \frac{\gamma + \frac{1}{4}}{s-1} + O(1).$$

The function $D_2(s)$ can be continued to the region $\operatorname{Re} s > 2/3$ as a meromorphic function. This function has a simple pole at $s = 3/2$ and a triple pole at $s = 1$.

One of the results in [Furuya et al. 2010] is the relationship between the Dirichlet series $D_2(s)$ and Lau and Tsang's conjecture [1995, Formula 1.3],

$$\int_1^x \Delta(u)^2 du = c_1 x^{3/2} - \frac{1}{4\pi^2} x \log^2 x + c_2 x \log x + O(x), \tag{1-2}$$

where c_1 and c_2 are certain constants. In particular, it was suggested that the second and third terms on the right side of (1-2) come from the residues of $D_2(s)$ at $s = 1$ [Furuya et al. 2010, Section 5].

In this paper, we first consider the Dirichlet series $Y(s)$ defined by

$$Y(s) = \sum_{n=1}^{\infty} \frac{d(n)\Delta(n)}{n^s},$$

which can be regarded as a modification of $D_1(s)$ and $D_2(s)$. We can easily see that the function $Y(s)$ is absolutely convergent in $\sigma > 5/4$, similarly to $D_1(s)$, since $d(n) = O(n^{\varepsilon})$ for an arbitrarily small positive number ε and

$$\sum_{n \le x} |\Delta(n)| = O(x^{5/4}).$$

As for the other analytic properties of $Y(s)$, we obtain this:

Theorem 1. *The Dirichlet series $Y(s)$ can be continued analytically to the region $\operatorname{Re} s > -1/3$ as a meromorphic function. In the region $\operatorname{Re} s \ge 1/2$, it has a simple pole at $s = 1/2$ with*

$$\operatorname*{Res}_{s=1/2} Y(s) = \frac{1}{16\pi^2} \sum_{n=1}^{\infty} \frac{d(n)^2}{n^{3/2}},$$

and it also has a pole of fourth order at $s = 1$, whose Laurent expansion at $s = 1$ is given by

$$Y(s) = \frac{3}{\pi^2 (s-1)^4} + \frac{12(\pi^2 \gamma - 3\zeta'(2))}{\pi^4 (s-1)^3} + \frac{a_{-2}}{(s-1)^2} + \frac{a_{-1}}{s-1} + \cdots,$$

*with some constants a_j, where $\zeta(s)$ denotes the Riemann zeta function. In the region $-1/3 < \operatorname{Re} s < 1/2$, the function $Y(s)$ has poles at $s = \rho/2$ if ρ satisfies the conditions $\zeta(\rho) = 0$ and $\zeta(\rho/2) \neq 0$.**

We shall give the proof of this theorem in two ways; see Sections 3 and 7.

As the first application of Theorem 1, we shall study the Dirichlet series related to the coefficient of $\tilde{\Delta}(n)$ defined by

$$\tilde{\Delta}(x) = {\sum_{n\le x}}' d(n) - x(\log x + 2\gamma - 1) - \tfrac{1}{4},$$

where $\sum_{n\le x}'$ indicates that the last term is to be halved if x is an integer. Commonly this definition is used as the error term instead of (1-1). Many properties of $\Delta(x)$ also hold in the case $\tilde{\Delta}(x)$; for example, these functions have same upper and lower bounds as $x \to \infty$. For the mean value theorem, we can see that

$$\int_1^x \Delta(u)\,du = \tfrac{1}{4}x + O(x^{3/4}) \quad \text{and} \quad \int_1^x \tilde{\Delta}(u)\,du = O(x^{3/4})$$

for $x \geq 1$, though the asymptotic behaviors, in particular the main terms of the higher power cases from 2 to 9, are the same. However, the difference between $\Delta(n)$ and $\tilde{\Delta}(n)$ for natural numbers n is essential in the study of the "discrete" mean values. Actually, these functions are connected by the relation

$$\tilde{\Delta}(n) = \Delta(n) - \tfrac{1}{2}d(n) - \tfrac{1}{4}, \tag{1-3}$$

for a natural number n; hence we have

$$\sum_{n\le x} \tilde{\Delta}(n)^k = \sum_{n\le x} \Delta(n)^k + \sum_{b=0}^{k-1}\sum_{a=0}^{k-b} \frac{k!(-1)^{b-k}2^{a+2b-2k}}{a!\,b!\,(k-a-b)!} \sum_{n\le x} d(n)^a \Delta(n)^b,$$

with a fixed natural number k [Furuya 2007, Formula 5.1]. In view of this formula, studying the discrete mean values of $\tilde{\Delta}(n)$ will require that we understand the function $\sum_{n\le x} d(n)^a \Delta(n)^b$. As noted in [Furuya 2007], it is very difficult to study this kind of sum in the case $a \geq 2$.

Now we consider the Dirichlet series

$$\widetilde{D}_j(s) = \sum_{n=1}^{\infty} \frac{\tilde{\Delta}(n)^j}{n^s} \quad \text{for } j = 1 \text{ and } 2.$$

It is easily seen that these functions are absolutely convergent for $\sigma > 5/4$ for $j = 1$ and $\sigma > 3/2$ for $j = 2$, similarly to the cases of $D_j(s)$. For the other properties, we have the following corollary.

*Needless to say, the last condition $\zeta(\rho/2) \neq 0$ holds if the Riemann Hypothesis is true.

Corollary 1. (1) *The function $\widetilde{D}_1(s)$ can be continued to the whole complex plane as a meromorphic function with a simple pole at $s=-2n$ for a nonnegative integer n; in particular, this function is holomorphic at $s=1$. The residue of $\widetilde{D}_1(s)$ at $s=-2n$ is the same as that of $D_1(s)$ and is given by*

$$\operatorname*{Res}_{s=-2n} \widetilde{D}_1(s) = -\frac{\zeta(-2n-1)}{2n+1}.$$

(2) *The function $\widetilde{D}_2(s)$ can be continued analytically as a meromorphic function to the region* $\operatorname{Re} s > 2/3$, *where it has a simple pole at $s=3/2$ and a pole of fourth order at $s=1$. The residue of $\widetilde{D}_2(s)$ at $s=3/2$ is given by*

$$\operatorname*{Res}_{s=3/2} \widetilde{D}_2(s) = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{d(n)^2}{n^{3/2}}.$$

The proof of this corollary is based on the relation (1-3), Theorem 1, and the known results concerning $D_j(s)$ and $\zeta(s)$. Actually, we have by (1-3) that

$$\begin{aligned}
\widetilde{D}_1(s) &= D_1(s) - \tfrac{1}{2}\zeta^2(s) - \tfrac{1}{4}\zeta(s),\\
\widetilde{D}_2(s) &= D_2(s) - \tfrac{1}{2}D_1(s) + \tfrac{1}{4}\zeta^2(s) + \tfrac{1}{16}\zeta(s) + \frac{\zeta^4(s)}{4\zeta(2s)} - Y(s).
\end{aligned}$$

The corollary follows immediately from these. (In fact, we need not use Theorem 1 to prove (1); we need only apply [Furuya et al. 2010, Theorem 1].)

Comparing this corollary with [Furuya et al. 2010, Theorems 1 and 2], we can see that the behaviors of $\widetilde{D}_j(s)$ and $D_j(s)$ are different. We also note that the residue of $\widetilde{D}_2(s)$ at $s=3/2$ is the same as that of $I_2(s)$, which is defined in the beginning of Section 2; see also Lemma 2 below.

We further study the properties of Dirichlet series related to $\Delta(x)$, especially the orders of $D_2(s)$ and $Y(s)$, whose analytic properties are poorly understood. Namely, their functional equations, approximate functional equations, and mean values are not known. It seems difficult to study the orders of these Dirichlet series in a satisfactory way. However:

Theorem 2. *Let $s=\sigma+it$ be a complex variable. For* $|t|\ge 2$, *we have*

$$Y(s) \ll \begin{cases} 1 & \text{for } \sigma > 5/4,\\ |t|^{(5-4\sigma)/3}\log^{5/2}|t| & \text{for } 1/2 \le \sigma \le 5/4. \end{cases}$$

Theorem 3. *Let $s=\sigma+it$ be a complex variable. For* $|t|\ge 2$, *we have*

$$D_2(s) \ll \begin{cases} 1 & \text{for } \sigma > 3/2,\\ \log|t| & \text{for } \sigma = 3/2,\\ |t|^{3-2\sigma}\log^4|t| & \text{for } 1 < \sigma < 3/2. \end{cases}$$

These are obtained by using mean value theorems of $\Delta(x)$ and the Phragmén–Lindelöf convexity theorem. The factor $\log^4|t|$ in Theorem 3 corresponds to the error estimate of the mean square of $\Delta(x)$ [Preissmann 1988]. We can improve it slightly by using the recent result of Lau and Tsang [2009, Theorem 2], but for simplicity we use the result of Preissmann here.

Finally, as an application of Theorem 1, we will study an analytic continuation of a certain kind of multiple zeta function. Such functions are of current interest, especially those of the Euler–Zagier type

$$\sum_{n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k}}.$$

As a generalization, one can consider two types of multiple series,

$$\sum_{n_1<n_2<\cdots<n_k} \frac{a_1(n_1)a_2(n_2)\cdots a_k(n_k)}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} \tag{1-4}$$

and

$$\sum_{n_1<n_2<\cdots<n_k} \frac{a_1(n_1)a_2(n_2-n_1)\cdots a_k(n_k-n_{k-1})}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}, \tag{1-5}$$

where $a_j(n)$ are certain arithmetical functions. Under suitable assumptions on the Dirichlet series $\sum_{n=1}^{\infty} a_j(n)n^{-s_j}$, the analytic properties for the multiple series of type (1-5) can be easily derived. Compared with (1-5), the series of type (1-4) is rather difficult, and it seems that [Akiyama and Ishikawa 2002] is the only character mod q_j is treated in the case $a_j(n)=\chi_j(n)$; that paper made use of the periodicity of χ_j to reduce the problem to the multiple Hurwitz zeta function

$$\sum_{n_1<n_2<\cdots<n_k} \frac{1}{(n_1+\alpha_1)^{s_1}(n_2+\alpha_2)^{s_2}\ldots(n_k+\alpha_k)^{s_k}}.$$

The multiple series that we consider here is of the form

$$D(s_1,s_2)=\sum_{m<n} \frac{d(m)d(n)}{m^{s_1}n^{s_2}}. \tag{1-6}$$

For $\operatorname{Re} s_2>1$ and $\operatorname{Re}(s_1+s_2)>2$, the series in (1-6) is absolutely convergent and represents a holomorphic function in s_1 and s_2. Since the divisor function $d(n)$ is not periodic, we should adopt a different approach than Akiyama and Ishikawa.

In the case $s_1=s_2=s$, it is easy to see that $D(s,s)$ has an analytic continuation to the whole plane $\mathbb{C}$, since trivially

$$D(s,s)=\tfrac{1}{2}\zeta(s)^4-\frac{\zeta(2s)^4}{2\zeta(4s)}.$$

For general s_j, the analytic continuation of (1-6) is as follows:

Theorem 4. *The multiple zeta function $D(s_1, s_2)$ can be continued analytically to a function meromorphic in the region in $\mathbb{C}^2$ given by*

$$\operatorname{Re} s_1 + \operatorname{Re} s_2 > \tfrac{1}{2}.$$

To prove Theorem 4, we employ previous results about the Dirichlet series $D_j(s)$, $Y(s)$, and $I_j(s)$ and their derivatives. More precisely, we will express $D(s_1, s_2)$ in terms of these functions and then use their analytic continuations. We can determine the singularities of $D(s_1, s_2)$ in the region $\operatorname{Re} s_1 + \operatorname{Re} s_2 > 1/2$ by using the explicit formula (6-3) for $D(s_1, s_2)$. However, we shall omit the details of these properties since we would like to state the properties of $D(s_1, s_2)$ as simply as possible.

2. Preliminaries

Here we prepare some lemmas. The first concerns the analytic properties of the integrals

$$I_j(s) = \int_1^\infty u^{-s} \Delta(u)^j \, du \quad \text{for } j = 1 \text{ and } 2.$$

We easily see that these integrals are absolutely convergent in the region $\sigma > 5/4$ for $j = 1$ and $\sigma > 3/2$ for $j = 2$.

Lemma 1 [Sitaramachandra Rao 1987]. *The function $I_1(s)$ can be continued to the whole complex plane as a function holomorphic except for a simple pole at $s = 1$,*[†] *and is expressed explicitly by*

$$I_1(s) = \frac{\zeta^2(s-1)}{s-1} - \frac{2\gamma - 1}{s-2} - \frac{1}{(s-2)^2}. \tag{2-1}$$

Lemma 2 [Furuya et al. 2010, Lemma 4]. *The function $I_2(s)$ can be continued analytically to the right half-plane $\sigma > 2/3$. It has a simple pole at $s = 3/2$ with residues*

$$\operatorname*{Res}_{s=3/2} I_2(s) = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{d(n)^2}{n^{3/2}},$$

while it has a triple pole at $s = 1$.

We will need several results about sums of $\Delta(n)$.

Lemma 3. *Let $\Delta(x)$ be the error term defined by* (1-1). *Then*

$$\sum_{n \le x} \Delta(n)^2 = c_1 x^{3/2} + F(x),$$

[†] The function $I_1(s)$ is holomorphic at $s = 2$, since the integral of $I_1(s)$ converges absolutely for $s = 2$. (This can also be checked using the Laurent expansion around $s = 2$ of the right side of (2-1).)

with $F(x) = O(x \log^4 x)$, *where* c_1 *is the constant defined in* (1-2).

Proof. This formula can be proved directly by using [Furuya 2005, Theorem 1] and the asymptotic formula

$$\int_1^x \Delta(u)^2 du = c_1 x^{3/2} + O(x \log^4 x) \tag{2-2}$$

due to Preissmann [1988]. □

Lemma 4.

$$\sum_{n \le x} d(n)\Delta(n) = \tfrac{1}{2}\sum_{n \le x} d(n)^2 + \tfrac{1}{2}\Delta(x)^2 - \tfrac{1}{2}(2\gamma - 1)^2 + \int_1^x (\log u + 2\gamma)\Delta(u)\,du.$$

We can write this sum explicitly as an asymptotic formula

$$\sum_{n \le x} d(n)\Delta(n) = \frac{1}{2\pi^2} x \log^3 x + c_3 x \log^2 x + c_4 x \log x + c_5 x + O(x^{3/4}\log x),$$

with suitable constants c_3, c_4 *and* c_5.

Proof. The first formula is derived from [Furuya 2007, Theorem 1] by putting $f(n) = d(n)$, which implies $g(x) = x(\log x + 2\gamma - 1)$ and $E(x) = \Delta(x)$. The second formula is [Furuya 2007, Corollary 1]. □

3. The function $Y(s)$

Let N be a sufficiently large positive number and let

$$Y_N(s) = \sum_{n \le N} d(n)\Delta(n)\, n^{-s} \tag{3-1}$$

for $\sigma > 5/4$. Also put $g(x) = x(\log x + 2\gamma - 1)$. Then by partial summation and the first formula of Lemma 4, we have

$$\begin{aligned} Y_N(s) &= N^{-s}\sum_{n \le N} d(n)\Delta(n) + s\int_1^N u^{-s-1}\sum_{n \le u} d(n)\Delta(n)\,du \\ &= s\int_1^N u^{-s-1}\left(\tfrac{1}{2}\sum_{n \le u} d(n)^2 + \tfrac{1}{2}\Delta(u)^2 - \tfrac{1}{2}(2\gamma - 1)^2 + \int_1^u g'(v)\Delta(v)\,dv\right)du \\ &\qquad + O(N^{1-\sigma}\log^3 N). \end{aligned}$$

For the double integral on the right side, we have

$$\begin{aligned} \int_1^N u^{-s-1}\int_1^u g'(v)\Delta(v)\,dv\,du &= \int_1^N g'(v)\Delta(v)\int_v^N u^{-s-1}\,du\,dv \\ &= \frac{1}{s}\int_1^N u^{-s} g'(u)\Delta(u)\,du + O(N^{1-\sigma}\log N). \end{aligned}$$

Furthermore, we have

$$\sum_{n\le x} d(n)^2 n^{-s} = x^{-s}\sum_{n\le x} d(n)^2 + s\int_1^x u^{-s-1}\sum_{n\le u} d(n)^2 du$$

by partial summation; hence,

$$s\int_1^N u^{-s-1}\sum_{n\le u} d(n)^2 du = \sum_{n\le N} d(n)^2 n^{-s} + O(N^{1-\sigma}\log^3 N).$$

Therefore

$$Y_N(s) = \tfrac{1}{2}\sum_{n\le N} d(n)^2 n^{-s} + \tfrac{1}{2}s\int_1^N u^{-s-1}\Delta(u)^2 du - \frac{(2\gamma-1)^2}{2} + \int_1^N u^{-s} g'(u)\Delta(u)du + O(N^{1-\sigma}\log^3 N).$$

In the above formula, we let $N\to\infty$ and get

$$Y(s) = \frac{\zeta(s)^4}{2\zeta(2s)} + \frac{sI_2(s+1)}{2} - \frac{(2\gamma-1)^2}{2} + 2\gamma I_1(s) - I_1'(s). \tag{3-2}$$

This expression holds for $\sigma > 5/4$. But we can easily see, by (3-2) and the analytic properties of $\zeta(s)$, $I_j(s)$ (for $j=1,2$) and $I_1'(s)$, that $Y(s)$ is continued analytically from $\sigma > 5/4$ to the region $\sigma > -1/3$.

Furthermore, we see that $Y(s)$ has poles at $s=1/2$ and $s=1$ in the region $\sigma\ge 1/2$. For $-1/3<\sigma<1/2$, the assertion in the theorem is easily derived from the right side of (3-2). The residue at $s=1/2$ is derived easily from Lemma 2, and the Laurent expansion of $Y(s)$ at $s=1$ is derived also by the right side of (3-2). This completes the proof of Theorem 1. □

4. The order of $Y(s)$

In this section, we prove Theorem 2. Specifically, we determine the order of $Y(s)$ on the vertical lines $\sigma=1/2$ and $\sigma=5/4$ and apply the Phragmén–Lindelöf convexity theorem for $1/2\le\sigma\le 5/4$.

First we consider the order of $Y(s)$ on the line $\sigma=1/2$. From (3-2), we have

$$Y(\tfrac{1}{2}+it) = \frac{\zeta(\frac{1}{2}+it)^4}{2\zeta(1+2it)} + \frac{\frac{1}{2}+it}{2}I_2(\tfrac{3}{2}+it) - \frac{(2\gamma-1)^2}{2} + 2\gamma I_1(\tfrac{1}{2}+it) - I_1'(\tfrac{1}{2}+it).$$

It is easily seen that

$$\frac{\zeta(\frac{1}{2}+it)^4}{\zeta(1+2it)} \ll |t|^{2/3}\log^7|t| \ll |t|.$$

We also have $I_1(\frac{1}{2}+it) \ll |t|^{-1}|\zeta(-\frac{1}{2}+it)|^2 \ll |t|$ from Lemma 1, and

$$I_1'(\tfrac{1}{2}+it) \ll |t|^{-1}|\zeta(-\tfrac{1}{2}+it)\zeta'(-\tfrac{1}{2}+it)| \ll |t|\log|t|$$

similarly. So it remains to consider $I_2(\frac{3}{2}+it)$:

Lemma 5. $I_2(\frac{3}{2}+it) \ll \log|t|$ *as* $|t| \to \infty$.

Proof. Assume $\sigma > 3/2$, and let X be a large parameter. Splitting the integral at X, we have

$$I_2(s) = \int_1^X u^{-s}\Delta(u)^2 du + \int_X^\infty u^{-s}\Delta(u)^2 du =: J_X^{(1)}(s) + J_X^{(2)}(s).$$

Using the mean value estimate (2-2) and integration by parts, we have

$$\begin{aligned} J_X^{(2)}(s) &= \Big[u^{-s}(c_1u^{3/2}+O(u\log^4 u))\Big]_X^\infty + s\int_X^\infty u^{-s-1}(c_1u^{3/2}+O(u\log^4 u))\,du \\ &= \frac{3c_1}{2s-3}X^{-s+3/2} + O(X^{1-\sigma}\log^4 X) + O\Big(|t|\int_X^\infty u^{-\sigma}\log^4 u\,du\Big). \end{aligned}$$

The integral in the last term converges absolutely in the region $\sigma > 1$ and is estimated as $O(|t|X^{1-\sigma}\log^4 X)$. Hence we have

$$J_X^{(2)}(\tfrac{3}{2}+it) \ll |t|^{-1} + X^{-1/2}\log^4 X + |t|X^{-1/2}\log^4 X.$$

Meanwhile,

$$J_X^{(1)}(\tfrac{3}{2}+it) \ll \int_1^X u^{-3/2}\Delta^2(u)\,du \ll \log X.$$

By taking, for example, $X = |t|^3$, we obtain the lemma. □

From these estimates, we obtain

$$Y(\tfrac{1}{2}+it) \ll |t|\log|t|. \tag{4-1}$$

Next we consider the order on the line $\sigma = 5/4$. Assuming first that $\sigma > 5/4$ as usual, we define

$$E_N(s) = Y(s) - Y_N(s) = \sum_{n>N}\frac{d(n)\Delta(n)}{n^s},$$

where $Y_N(s)$ is the function defined by (3-1).

Using partial summation and the second formula in Lemma 4, we have

$$\begin{aligned} E_N(s) = -N^{-s}&\Big(\frac{1}{2\pi^2}N\log^3 N + c_3N\log^2 N + c_4N\log N + c_5N\Big) \\ &+ s\int_N^\infty u^{-s-1}\Big(\frac{1}{2\pi^2}u\log^3 u + c_3u\log^2 u + c_4u\log u + c_5u\Big)\,du \\ &+ O(|t|N^{3/4-\sigma}\log N) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi^2(s-1)} N^{1-s} \log^3 N + \left(\frac{3s}{2\pi^2(s-1)^2} + \frac{c_3}{s-1} \right) N^{1-s} \log^2 N \\
&\quad + \left(\frac{6s}{2\pi^2(s-2)^3} + \frac{2c_3 s}{(s-1)^2} + \frac{c_4}{s-1} \right) N^{1-s} \log N \\
&\quad + \left(\frac{6s}{2\pi^2(s-1)^4} + \frac{2c_3 s}{(s-1)^3} + \frac{c_4 s}{(s-1)^2} + \frac{c_5}{s-1} \right) N^{1-s} \\
&\quad + O\big(|t| N^{3/4-\sigma} \log N\big).
\end{aligned}
$$

Hence, we get the estimate

$$
E_N(s) \ll \frac{N^{1-\sigma} \log^3 N}{|t|} + |t| N^{3/4-\sigma} \log N \tag{4-2}
$$

for $\sigma > 5/4$. Note that (4-2) holds true for $\sigma > 3/4$.

On the other hand, the first part of this division can be estimated as

$$
Y_N(\tfrac{5}{4} + it) \ll \left(\sum_{n \le N} \frac{d(n)^2}{n} \right)^{1/2} \left(\sum_{n \le N} \frac{\Delta(n)^2}{n^{3/2}} \right)^{1/2} \ll \log^{5/2} N.
$$

Taking $N = |t|^2$, we then get

$$
Y(\tfrac{5}{4} + it) \ll \log^{5/2} |t|. \tag{4-3}
$$

By (4-1), (4-3) and the Phragmén–Lindelöf principle, we obtain

$$
Y(\sigma + it) \ll |t|^{(5-4\sigma)/3} \log^{5/2} |t| \quad \text{for} \quad 1/2 \le \sigma \le 5/4,
$$

which completes the proof of Theorem 3. □

5. The order of $D_2(s)$

Let $\sigma > 3/2$. We divide the infinite series as

$$
\sum_{n=1}^{\infty} \frac{\Delta(n)^2}{n^s} = \left(\sum_{n \le N} + \sum_{n > N} \right) \frac{\Delta(n)^2}{n^s} = D_{2,N}^{(1)}(s) + D_{2,N}^{(2)}(s).
$$

By using partial summation and Lemma 2, we have

$$
\begin{aligned}
D_{2,N}^{(2)}(s) &= \frac{\frac{3}{2} c_1}{s - \frac{3}{2}} N^{-s+3/2} - N^{-s} F(N) + s \int_N^{\infty} u^{-s-1} F(u)\, du \\
&= \frac{\frac{3}{2} c_1}{s - \frac{3}{2}} N^{-s+3/2} + O\big(|s| N^{1-\sigma} \log^4 N\big).
\end{aligned} \tag{5-1}
$$

This estimate actually holds for $\sigma > 1$, and thus this formula gives the analytic continuation of $D_{2,N}^{(2)}(s)$ from $\sigma > 3/2$ into $\sigma > 1$.

We treat the case $s = \frac{3}{2} + it$. By (5-1), we have

$$D^{(2)}_{2,N}(\tfrac{3}{2} + it) \ll |t|^{-1} + |t| N^{-1/2} \log^4 N.$$

We have, by partial summation and Lemma 2 again,

$$D^{(1)}_{2,N}(\tfrac{3}{2} + it) \ll \sum_{n \le N} \frac{\Delta(n)^2}{n^{3/2}} \ll \log N.$$

Hence, by taking $N = |t|^3$, we have $D_2(\frac{3}{2} + it) \ll \log|t|$. This estimate gives the second assertion of Theorem 3.

To prove the third, we first consider the case $s = 1 + \varepsilon + it$, where ε is a fixed positive small number. By (5-1), we get

$$D^{(2)}_{2,N}(1 + \varepsilon + it) \ll |t|^{-1} N^{1/2-\varepsilon} + |t| N^{-\varepsilon} \log^4 N,$$

and

$$D^{(1)}_{2,N}(1 + \varepsilon + it) \ll N^{1/2-\varepsilon}.$$

Hence, by taking $N = |t|^2$, we get $D_2(1 + \varepsilon + it) \ll |t|^{1-2\varepsilon} \log^4|t|$.

Applying the Phragmén–Lindelöf convexity principle, we obtain

$$D_2(\sigma + it) \ll |t|^{3-2\sigma} \log^4|t|$$

for $1 < \sigma < 3/2$. This completes the proof of Theorem 3. $\square$

6. Proof of Theorem 4

Let

$$S(s_1, s_2) = \sum_{m \le n} \frac{d(m)d(n)}{m^{s_1} n^{s_2}}.$$

To prove Theorem 4, it is enough to consider the series $S(s_1, s_2)$, since

$$D(s_1, s_2) = S(s_1, s_2) - \frac{\zeta^4(s_1 + s_2)}{\zeta(2(s_1 + s_2))}. \tag{6-1}$$

Let $s_j = \sigma_j + it_j$ be complex variables. First assume that $\sigma_1 > 1$ and $\sigma_2 > 1$. For a large positive number N, we consider the finite sum

$$S_N(s_1, s_2) := \sum_{m \le n \le N} \frac{d(m)d(n)}{m^{s_1} n^{s_2}}.$$

By partial summation and (1-1), we have

$$S_N(s_1, s_2) = \sum_{n \le N} \frac{d(n)}{n^{s_2}} \sum_{m \le n} \frac{d(m)}{m^{s_1}}$$

$$
\begin{aligned}
&= \sum_{n\le N} \frac{d(n)}{n^{s_2}} \biggl(\frac{1}{n^{s_1}} \sum_{m\le n} d(m) + s_1 \int_1^n u^{-s_1-1} \Bigl(\sum_{m\le u} d(m) \Bigr) \, du \biggr) \\
&= \sum_{n\le N} \frac{d(n)(g(n)+\Delta(n))}{n^{s_1+s_2}} + s_1 \sum_{n\le N} \frac{d(n)}{n^{s_2}} \int_1^n u^{-s_1-1} g(u)\, du \\
&\qquad\qquad + s_1 \sum_{n\le N} \frac{d(n)}{n^{s_2}} \int_1^n u^{-s_1-1} \Delta(u)\, du,
\end{aligned}
$$

where $g(u) = u(\log u + 2\gamma - 1)$ as before. In the last term above, split the integral as $\int_1^n = \int_1^N - \int_n^N$. Then interchange the order of integral and summation and use partial summation and (1-1) again. We get eight terms:

$$
\begin{aligned}
S_N(s_1, s_2) &= \sum_{n\le N} \frac{d(n)g(n)}{n^{s_1+s_2}} + \sum_{n\le N} \frac{d(n)\Delta(n)}{n^{s_1+s_2}} + s_1 \sum_{n\le N} \frac{d(n)}{n^{s_2}} \int_1^n u^{-s_1-1} g(u)\, du \\
&\quad + s_1 \Bigl(\sum_{n\le N} \frac{d(n)}{n^{s_2}} \Bigr) \int_1^N u^{-s_1-1} \Delta(u)\, du \\
&\quad - s_1 \int_1^N u^{-s_1-s_2-1} \Delta(u) g(u)\, du - s_1 \int_1^N u^{-s_1-s_2-1} \Delta(u)^2 du \\
&\quad - s_1 s_2 \int_1^N u^{-s_1-1} \Delta(u) \int_1^u v^{-s_2-1} g(v)\, dv\, du \\
&\quad - s_1 s_2 \int_1^N u^{-s_1-1} \Delta(u) \int_1^u v^{-s_2-1} \Delta(v)\, dv\, du,
\end{aligned}
$$

which we define as $\sum_{j=1}^{8} I_{j,N}(s_1, s_2)$. We consider each $I_j(s_1, s_2)$ as $N \to \infty$. It is easy to see that

$$
\lim_{N\to\infty} I_{1,N}(s_1, s_2) = -(\zeta^2)'(s_1+s_2-1) + (2\gamma - 1)\zeta^2(s_1+s_2-1).
$$

By elementary calculations, we have

$$
\begin{aligned}
\lim_{N\to\infty} I_{3,N}(s_1, s_2) &= \frac{s_1}{s_1-1} (\zeta^2)'(s_1+s_2-1) \\
&\qquad - s_1 \biggl(\frac{1}{(s_1-1)^2} + \frac{2\gamma - 1}{s_1 - 1} \biggr) \bigl(\zeta^2(s_1+s_2-1) - \zeta^2(s_2) \bigr).
\end{aligned}
$$

The terms $I_{j,N}(s_1, s_2)$ for $j = 2, 4, 5, 6, 7$ can be written in terms of the functions $Y(s)$, $I_1(s)$, $I_1'(s)$ and $I_2(s)$. In fact, we have

$$
\begin{aligned}
&\lim_{N\to\infty} I_{2,N}(s_1, s_2) = Y(s_1+s_2), \\
&\lim_{N\to\infty} I_{4,N}(s_1, s_2) = s_1 \zeta^2(s_2) I_1(s_1+1),
\end{aligned}
$$

$$\lim_{N\to\infty} I_{5,N}(s_1,s_2) = -s_1\big(-I_1'(s_1+s_2)+(2\gamma-1)I_1(s_1+s_2)\big),$$
$$\lim_{N\to\infty} I_{6,N}(s_1,s_2) = -s_1 I_2(s_1+s_2+1),$$

and

$$\lim_{N\to\infty} I_{7,N}(s_1,s_2) = -\frac{s_1s_2}{s_2-1}I_1'(s_1+s_2) + s_1s_2\bigg(\frac{1}{(s_2-1)^2}+\frac{2\gamma-1}{s_2-1}\bigg)\big(I_1(s_1+s_2)-I_1(s_1+1)\big).$$

By Theorem 1 and Lemmas 1 and 2, the terms $\lim_{N\to\infty} I_{j,N}(s_1,s_2)$ for $j=1,\dots,7$ can be continued meromorphically to the region $\sigma_1+\sigma_2>-1/3$.

We treat the term $I_{8,N}(s_1,s_2)$ with the following lemma.

Lemma 6. *Let*

$$I^{(N)}(s_1,s_2) = \int_1^N u^{-s_1-1}\Delta(u)\int_u^N v^{-s_2-1}\Delta(v)\,dv\,du,$$

and

$$I(s_1,s_2) = \lim_{N\to\infty} I^{(N)}(s_1,s_2).$$

Then $I(s_1,s_2)$ defines a holomorphic function in the region $\sigma_2>\frac14$ and $\sigma_1+\sigma_2>\frac12$.

Proof. In the region $\sigma_2>\frac14$ and $\sigma_1+\sigma_2>\frac12$,

$$I^{(N)}(s_1,s_2) \ll \int_1^N u^{-\sigma_1-1}|\Delta(u)|u^{-\sigma_2+1/4}du \ll 1,$$

since

$$\int_a^b u^{\beta}|\Delta(u)|\,du \ll a^{\beta+5/4}$$

for $a\le b$ and $\beta<-5/4$. The lemma follows immediately. □

Now we consider the double integral

$$J_N(s_1,s_2) = \int_1^N u^{-s_1-1}\Delta(u)\int_1^u v^{-s_2-1}\Delta(v)\,dv\,du.$$

Splitting the innermost integral in $J_N(s_1,s_2)$ as $\int_1^u = \int_1^N - \int_u^N$, we have

$$J_N(s_1,s_2) = \int_1^N u^{-s_1-1}\Delta(u)\,du\int_1^N u^{-s_2-1}\Delta(u)\,du - I^{(N)}(s_1,s_2).$$

In $\sigma_2>\frac14$ and $\sigma_1+\sigma_2>\frac12$, we have

$$J(s_1,s_2) := \lim_{N\to\infty} J_N(s_1,s_2) = I_1(s_1+1)I_1(s_2+1) - I(s_1,s_2).$$

Hence, $J(s_1,s_2)$ is a meromorphic function there by Lemmas 1 and 6.

On the other hand, by the symmetric property

$$J_N(s_1, s_2) + J_N(s_2, s_1) = \int_1^N u^{-s_1-1}\Delta(u)du \int_1^N u^{-s_2-1}\Delta(u)\,du,$$

we obtain

$$J(s_1, s_2) = I_1(s_1+1)I_1(s_2+1) - J(s_2, s_1). \tag{6-2}$$

By applying the above argument on $I(s_2, s_1)$, we see that $J(s_1, s_2)$ is also defined in the region $\sigma_1 > \frac{1}{4}$ and $\sigma_1 + \sigma_2 > \frac{1}{2}$. Therefore we conclude that

$$\lim_{N\to\infty} I_{8,N}(s_1, s_2) = -s_1 s_2 J(s_1, s_2)$$

is meromorphic in $\sigma_1 + \sigma_2 > \frac{1}{2}$. This completes the proof of Theorem 4. $\square$

More concretely, the explicit form of the analytic continuation of $S(s_1, s_2)$ is given by

$$\begin{aligned} S(s_1, s_2) = Y(s_1+s_2) - s_1 I_2(s_1+s_2+1) - s_1 s_2 I_1(s_1+1)I_1(s_2+1) \\ - \frac{s_1}{s_2-1}I_1'(s_1+s_2) + s_1\left(\frac{s_2}{(s_2-1)^2} + \frac{2\gamma-1}{s_2-1}\right)I_1(s_1+s_2) \\ + s_1\zeta^2(s_2)I_1(s_1+1) - s_1 s_2\left(\frac{1}{(s_2-1)^2} + \frac{2\gamma-1}{s_2-1}\right)I_1(s_1+1) \\ + \frac{1}{s_1-1}(\zeta^2)'(s_1+s_2-1) - \left(\frac{s_1}{(s_1-1)^2} + \frac{2\gamma-1}{s_1-1}\right)\zeta^2(s_1+s_2-1) \\ + s_1\left(\frac{1}{(s_1-1)^2} + \frac{2\gamma-1}{s_1-1}\right)\zeta^2(s_2) + s_1 s_2 I(s_1, s_2). \end{aligned} \tag{6-3}$$

From this formula, we can determine the locations of singularities of $S(s_1, s_2)$, and thus $D(s_1, s_2)$ by (6-1), but we omit the details of this topic here.

7. An alternative approach to Theorem 1

We now give a proof of Theorem 1 by approaching (3-2) differently. In fact, we will not use the first result in Lemma 4, which is an identity for $\sum_{n\le x} d(n)\Delta(n)$.

Let $Y_N(s)$ and $g(x)$ be defined as above. By (1-1), we have

$$\begin{aligned} Y_N(s) &= \sum_{n\le N}\frac{d(n)}{n^s}\left(\sum_{m\le n} d(m) - g(n)\right) \\ &= \left(\sum_{n\le N} d(n)\right)\left(\sum_{n\le N}\frac{d(n)}{n^s}\right) - \sum_{m\le N}\left(d(m)\sum_{n\le m}\frac{d(n)}{n^s}\right) + \sum_{n\le N}\frac{d(n)^2}{n^s} - \sum_{n\le N}\frac{d(n)}{n^s}g(n). \end{aligned}$$

Further, since

$$\sum_{m\le N} d(m)\sum_{n\le m}\frac{d(n)}{n^s} = \sum_{m\le N}\frac{d(m)}{m^s}(g(m)+\Delta(m)) + s\sum_{m\le N} d(m)\int_1^m u^{-s-1}(g(u)+\Delta(u))du$$

by partial summation, we have, for $\sigma > 5/4$,

$$2Y_N(s) - \frac{\zeta^4(s)}{\zeta(2s)} = \Big(\sum_{n\le N} d(n)\Big)\Big(\sum_{n\le N}\frac{d(n)}{n^s}\Big) - 2\sum_{n\le N}\frac{d(n)}{n^s}g(n) - s\sum_{m\le N} d(m)\int_1^m u^{-s-1}(g(u)+\Delta(u))du + O(N^{1-\sigma}\log^3 N).$$

We now consider the transformation of $\int_1^N u^{-s}\Delta^2(u)\,du$. We have by (1-1)

$$\begin{aligned}\int_1^N u^{-s}\Delta^2(u)\,du &= \int_1^N u^{-s}\Delta(u)\Big(\sum_{n\le u} d(n) - g(u)\Big)\,du\\ &= \sum_{n\le N} d(n)\int_n^N u^{-s}\Delta(u)du - \int_1^N u^{-s}\Delta(u)g(u)\,du\\ &= \Big(\int_1^N u^{-s}\Delta(u)\,du\Big)\sum_{n\le N} d(n) - \sum_{n\le N} d(n)\int_1^n u^{-s}\Delta(u)\,du\\ &\qquad - \int_1^N u^{-s}\Delta(u)g(u)\,du.\end{aligned}$$

We obtain by this formula, and by applying partial summation to $\sum_{n\le N} d(n)n^{-s}$, $\sum_{n\le N} d(n)g(n)n^{-s}$, and $\sum_{n\le N} d(n)\int_1^n u^{-s-1}g(u)\,du$, that

$$\begin{aligned}&2Y_N(s) - \frac{\zeta^4(s)}{\zeta(2s)} - sI_2(s+1) + 2I_1'(s) - 4\gamma\, I_1(s)\\ &\quad = \Big(N^{-s}\sum_{n\le N} d(n) - 2N^{-s}g(N)\Big)\sum_{n\le N} d(n) + s\int_1^N u^{-s-1}g(u)^2du\\ &\qquad + 2\int_1^N g(u)\left(u^{-s}g'(u) - su^{-s-1}g(u)\right)du + O(N^{5/4-\sigma}\log N)\end{aligned}$$

for $\sigma > 5/4$. Furthermore, by applying the estimate $\Delta(x) = O(x^{1/3})$ and the formula

$$2\int_1^N u^{-s}g(u)g'(u)\,du = N^{-s}g(N)^2 - (2\gamma-1)^2 + s\int_1^N u^{-s-1}g(u)^2\,du,$$

which has been proved by integration by parts, we obtain

$$2Y_N(s)-\frac{\zeta^4(s)}{\zeta(2s)}-sI_2(s+1)+2I_1'(s)-4\gamma\,I_1(s)=-(2\gamma-1)^2+O(N^{5/4-\sigma}\log N)$$

for $\sigma > 5/4$. Thus, as N tends toward infinity, we obtain again (3-2). □

References

[Akiyama and Ishikawa 2002] S. Akiyama and H. Ishikawa, "On analytic continuation of multiple L-functions and related zeta-functions", pp. 1–16 in *Analytic number theory* (Beijing/Kyoto, 1999), edited by C. Jia and K. Matsumoto, Dev. Math. **6**, Kluwer, Dordrecht, 2002. MR 2003b:11093 Zbl 1028.11058

[Furuya 2005] J. Furuya, "On the average orders of the error term in the Dirichlet divisor problem", *J. Number Theory* **115**:1 (2005), 1–26. MR 2006h:11117 Zbl 1089.11055

[Furuya 2007] J. Furuya, "On the summatory function of a product of an arithmetical function and its relevant error term", *Ann. Sci. Math. Québec* **31**:2 (2007), 165–185. MR 2009k:11148 Zbl 05529025

[Furuya et al. 2010] J. Furuya, Y. Tanigawa, and W. Zhai, "Dirichlet series obtained from the error term in the Dirichlet divisor problem", preprint, 2010. To appear in *Monatshefte Math.*

[Lau and Tsang 1995] Y.-K. Lau and K.-M. Tsang, "Mean square of the remainder term in the Dirichlet divisor problem", *J. Théor. Nombres Bordeaux* **7**:1 (1995), 75–92. Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993). MR 98k:11126 Zbl 0844.11059

[Lau and Tsang 2009] Y.-K. Lau and K.-M. Tsang, "On the mean square formula of the error term in the Dirichlet divisor problem", *Math. Proc. Cambridge Philos. Soc.* **146**:2 (2009), 277–287. MR 2009k:11149 Zbl 05532374

[Preissmann 1988] E. Preissmann, "Sur la moyenne quadratique du terme de reste du problème du cercle", *C. R. Acad. Sci. Paris Sér. I Math.* **306**:4 (1988), 151–154. MR 89e:11056 Zbl 0654.10042

[Sitaramachandra Rao 1987] R. Sitaramachandra Rao, "An integral involving the remainder term in the Piltz divisor problem", *Acta Arith.* **48**:1 (1987), 89–92. MR 88h:11068

Received April 1, 2009. Revised September 14, 2009.

JUN FURUYA
DEPARTMENT OF INTEGRATED ARTS AND SCIENCE
OKINAWA NATIONAL COLLEGE OF TECHNOLOGY
NAGO, OKINAWA, 905-2192
JAPAN
jfuruya@okinawa-ct.ac.jp

YOSHIO TANIGAWA
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
NAGOYA, 464-8602
JAPAN
tanigawa@math.nagoya-u.ac.jp

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

A CONSTANT RANK THEOREM FOR LEVEL SETS OF IMMERSED HYPERSURFACES IN $\mathbb{R}^{n+1}$ WITH PRESCRIBED MEAN CURVATURE

CHANGQING HU, XI-NAN MA AND QIANZHONG OU

We prove a constant rank theorem on the second fundamental forms of level sets of immersed hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed mean curvature.

1. Introduction

Constant rank theorems have been a powerful tool in the study of convex solutions to partial differential equations. Caffarelli and Friedman [1985] first proved a constant rank theorem on solutions to a class of semilinear elliptic PDEs in two dimensions and hence proved the strict convexity of the solutions. Singer, Wong, Yau and Yau explored a similar idea [Singer et al. 1985]. Korevaar and Lewis [1987] extended Caffarelli and Friedman's results to the n-dimensional case. In the last decade, the constant rank theorem has been extended to fully nonlinear elliptic PDEs [Guan and Ma 2003; Caffarelli et al. 2007; Guan et al. 2006]; these authors found important applications for it in some geometric problems. For the convexity of level sets, Korevaar proved a constant rank theorem:

Theorem 1.1 [Korevaar 1990]. *Let Ω be a connected domain in $\mathbb{R}^n$. Let $u \in C^4(\Omega)$ solve*

$$Lu := A\Big(\Delta u - \frac{u_i u_j}{|\nabla u|^2} u_{ij}\Big) + B\Big(\frac{u_i u_j}{|\nabla u|^2} u_{ij}\Big) = f(u, |\nabla u|), \tag{1-1}$$

where A, B, f are C^2 functions of u, and $\mu := |\nabla u|$. These satisfy the structure conditions

(i) $(\sqrt{A/B}\,)_{\mu\mu} \geq 0$, *and*

(ii) $(f(u, \mu)/B\mu^2)_{\mu\mu} \leq 0$.

MSC2000: primary 35J15; secondary 53A10.
Keywords: constant rank theorem, level sets, mean curvature.
Hu was partially supported by NSFC numbers 10871138 and 10871139. Ma and Ou were supported by NSFC numbers 10671186 and 10871187.

Suppose that $|\nabla u| \neq 0$ *and that u has convex level sets* $\{x \in \Omega \mid u(x) \leq c\}$. *Then all the level sets of u have second fundamental forms with* (*the same*) *constant rank throughout* Ω.

The equations in Theorem 1.1 include p-Laplacian equations and mean curvature equations as special cases. In these cases we respectively take

$$A = \mu^{p-2}, \quad B = (p-1)\mu^{p-2} \quad \text{and} \quad A = \frac{1}{\sqrt{1+\mu^2}}, \quad B = \frac{1}{(1+\mu^2)^{3/2}}.$$

Korevaar [1990] used this theorem to prove some interesting results on the convexity of the level sets of solutions to elliptic PDEs. Recently, Xu [2008] generalized Theorem 1.1 to the case where the function f in (1-1) also depends on the coordinate variable x, and accordingly the structure condition (ii) turns into

$$\mu^3 \frac{f(x, u, 1/\mu)}{B(u, 1/\mu)} \quad \text{is convex in } (x, \mu).$$

In this paper, we will prove an analogous result on a class of immersed hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed mean curvature.

Let M^n be a smooth immersed hypersurface in $\mathbb{R}^{n+1}$, and let $X : M \to \mathbb{R}^{n+1}$ be the immersion satisfying

$$H = -f(X, N), \tag{1-2}$$

where H and N are respectively the mean curvature and unit normal vectors of M^n at X, and f is a smooth function in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let ξ be a fixed unit vector in $\mathbb{R}^{n+1}$. Then the height function of M^n corresponding to ξ can be expressed as $u(X) = \langle X, \xi\rangle$; here $\langle\cdot\,,\cdot\rangle$ means the usual Euclidean inner product in $\mathbb{R}^{n+1}$. Now, the level set of M^n corresponding to ξ with height c is defined as

$$\Sigma_c = \{X \in M^n \mid u(X) = c\}. \tag{1-3}$$

Suppose u has no critical point on M^n. Then Σ_c can be considered as a hypersurface in the hyperplane $\Pi = \{X \in \mathbb{R}^{n+1} \mid \langle X, \xi\rangle = c\}$.

With the above notations, our constant rank theorem on the level sets of an immersed hypersurface with prescribed mean curvature can be stated as follows:

Theorem 1.2. *Let* M^n *be an immersed hypersurface in* $\mathbb{R}^{n+1}$ *whose mean curvature satisfies* (1-2). *Assume that the height function u of* M^n *corresponding to* ξ *has no critical point, and that the level sets are all locally convex with respect to the normal direction* $-Du$, *that is, their second fundamental forms are positive semidefinite. Then the second fundamental forms of all the level sets have* (*the same*) *constant rank, provided* $f(X, N) = f(X) \geq 0$ *and the matrix*

$$2f\frac{\partial^2 f}{\partial X_A \partial X_B} - 3\frac{\partial f}{\partial X_A}\frac{\partial f}{\partial X_B} \tag{1-4}$$

is positive semidefinite, where $1 \le A, B \le n+1$. *In other words, when* f *is a positive function, the condition* (1-4) *simply means that* $f^{-1/2}$ *is a concave function in* $\mathbb{R}^{n+1}$.

Remark 1.3. For the more general case where $H = -f(X, N)$ as in (1-2), by (3-28) and (3-29) in Section 3, we still can choose the structure conditions on f to ensure the result of Theorem 1.2. For example, if $f(X, N) = \langle \xi, N\rangle^{\beta}$ with $\langle \xi, N\rangle > 0$ on M^n, then the structure condition is $\beta \ge 1$ or $\beta \le 0$.

Remark 1.4. Throughout, we adapt these conventions: The hypersurface M^n is orientable. We choose the unit normal vector field N so that it represents the orientation of M^n. The unit vector field normal to the level set Σ_c is obtained by projecting N onto the hyperplane $\Pi = \{X \in \mathbb{R}^{n+1} \mid \langle X, \xi\rangle = c\}$.

When do the solutions of elliptic PDEs have convex level sets? Gabriel [1957] proved that the level sets of the Green function on a 3-dimensional convex domain are strictly convex. Lewis [1977] extended Gabriel's result to p-harmonic functions in higher dimensions. Caffarelli and Spruck [1982] generalized Lewis's result to a class of semilinear elliptic PDEs. For recent progress, see [Colesanti and Salani 2003] and [Cuoghi and Salani 2006]. The constant rank theorem is an important step for the concrete convexity theorem, since one can use it to prove strict convexity results, as in, for example, [Korevaar 1990]. In practice, one always runs into difficulty at the critical points of the solution (or height functions in our case). In some sense our constant rank theorem is only a local and intermediate result.

In Section 2, we will give a formula for the curvatures of the level sets of an immersed hypersurface in $\mathbb{R}^{n+1}$. We prove it by the method of moving frames. We prove our main result, Theorem 1.2, in Section 3 using a calculation similar to the one in [Xu 2008].

2. Formulas of curvature of level sets

For a C^2 function u defined in a n-dimensional domain Ω in $\mathbb{R}^n$, let $\kappa_1, \dots, \kappa_{n-1}$ be the principal curvatures of the level sets of u with respect to the normal direction $-Du$. Then the k-th curvature of the level sets, denoted by L_k, is the k-th elementary symmetric function of $\kappa_1, \dots, \kappa_{n-1}$. Clearly, L_1 and L_{n-1} are respectively the mean curvature and Gauss curvature of the level sets. If u has no critical point, that is, $|\nabla u| \ne 0$, then Trudinger [1997] (see also [Gilbarg and Trudinger 1977]) expressed L_k as

$$L_k = \frac{\partial \sigma_{k+1}(D^2 u)}{\partial u_{ij}} u_i u_j |\nabla u|^{-k-2}, \tag{2-1}$$

where we use summation convention for repeated indices, and where $\sigma_k(D^2 u)$ is the k-th elementary symmetric function of the eigenvalues of the Hessian $(D^2 u)$.

There is an formula analogous to (2-1) on hypersurfaces in $\mathbb{R}^{n+1}$:

Proposition 2.1. *Let M^n be a smoothly immersed hypersurface in $\mathbb{R}^{n+1}$. Let u be its height function and Σ_c one of its level sets, with respect to a fixed unit vector ξ, as given in the last section. Then the k-th curvature of the level set Σ_c with respect to $-Du$ is*

$$L_k = \frac{\partial \sigma_{k+1}(\boldsymbol{B})}{\partial h_{ij}} u_i u_j |\nabla u|^{-(k+2)}. \tag{2-2}$$

Here $\boldsymbol{B} = (h_{ij})$ is the second fundamental form of M^n, $\sigma_k(\boldsymbol{B})$ is the k-th elementary symmetric function of the eigenvalues of $\boldsymbol{B}$, and u_i for $1 \le i \le n$ are the first order covariant derivatives of u computed in any orthonormal frame field on M^n.

Huang [1992] gave the formula (2-2) for $n = 2$. Here we give a complete proof by using moving frames. In this section, indices will run from 1 to $n-1$ when lower case and Greek; Latin indices will run from 1 to n when lower case and from 1 to $n+1$ when upper case.

For an orthonormal frame field $\{X; e_A\}$ in $\mathbb{R}^{n+1}$, we have

$$dX = \omega_A e_A \quad \text{and} \quad de_A = \omega_{A,B} e_B, \tag{2-3}$$

where $\{\omega_A\}$ is the dual frame of $\{e_A\}$, and $\{\omega_{A,B}\}$ are connection forms. Then the structure equations read as

$$d\omega_A = \omega_{A,B} \wedge \omega_B \quad \text{and} \quad d\omega_{A,B} = \omega_{A,C} \wedge \omega_{C,B}. \tag{2-4}$$

If we choose e_{n+1} to be the unit normal vector field N of M^n, then $\omega_{n+1} = 0$ on M^n, and hence by (2-4)

$$\omega_{n+1,i} \wedge \omega_i = 0. \tag{2-5}$$

Then Cartan's lemma implies $\omega_{n+1,i} = h_{ij}\omega_j$ and $h_{ij} = h_{ji}$, where $\boldsymbol{B} = (h_{ij})$ is the second fundamental form of M^n.

Proof of Proposition 2.1. First, we check that the right side of (2-2) is independent of the choice of the frame fields $\{X; e_i\}$ on M^n. Then we can just prove (2-2) in a special frame field.

Suppose $\{X; \bar{e}_i\}$ is another frame field on M^n. Then there is an orthogonal transformation T such that $(\bar{e}_1, \dots, \bar{e}_n) = (e_1, \dots, e_n)T$. Then

$$(\bar{u}_1, \dots, \bar{u}_n) = (u_1, \dots, u_n)T, \tag{2-6}$$

where $\nabla u = u_i e_i = \bar{u}_i \bar{e}_i$ is the gradient of u. Also, for the dual frame field and the connection forms we have

$$(\bar{\omega}_1, \dots, \bar{\omega}_n) = (\omega_1, \dots, \omega_n)T,$$
$$(\bar{\omega}_{1,n+1}, \dots, \bar{\omega}_{n,n+1}) = (\omega_{1,n+1}, \dots, \omega_{n,n+1})T.$$

Furthermore, for the second fundamental form we have

$$\bar{\boldsymbol{B}} = T^{-1}\boldsymbol{B}T. \tag{2-7}$$

Obviously $\sigma_k(\boldsymbol{B})$ and $|\nabla u|$ are invariant under the transformation T. Then the following equalities show that the right side of (2-2) is independent of the choice of $\{e_1, \dots, e_n\}$:

$$\begin{aligned}\frac{\partial\sigma_k(\boldsymbol{B})}{\partial h_{ij}}u_iu_j &= \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial \bar{h}_{ml}}\frac{\partial \bar{h}_{ml}}{\partial h_{ij}}u_iu_j = \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial \bar{h}_{ml}}\frac{\partial (T^{mp}h_{pq}T_{ql})}{\partial h_{ij}}u_iu_j \\ &= \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial \bar{h}_{ml}}T^{mi}T_{jl}u_iu_j = \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial \bar{h}_{ml}}T_{im}u_iT_{jl}u_j = \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial \bar{h}_{ml}}\bar{u}_m\bar{u}_l.\end{aligned} \tag{2-8}$$

Now we adapt the frame field above so that along the level set Σ_c, the e_α are its tangential vectors. Furthermore, we choose another frame field $\tilde{e}_A$ in $\mathbb{R}^{n+1}$ so that $\tilde{e}_{n+1} = \xi$ and $\tilde{e}_\alpha = e_\alpha$, and so that $\tilde{e}_n$ lies in the hyperplane Π and is normal to Σ_c with the same direction of the projection of $e_{n+1} = N$ on Π. With respect to this frame field, the structure equations of Σ_c are

$$d\tilde{\omega}_i = \tilde{\omega}_{i,j}\wedge\tilde{\omega}_j \quad \text{and} \quad d\tilde{\omega}_{ij} = \tilde{\omega}_{i,l}\wedge\tilde{\omega}_{l,j}. \tag{2-9}$$

On Σ_c, we have $\tilde{\omega}_n = 0$, which implies

$$\tilde{\omega}_{n,\alpha} = \tilde{h}_{\alpha\beta}\tilde{\omega}_\beta \quad \text{and} \quad \tilde{h}_{\alpha\beta} = \tilde{h}_{\beta\alpha}, \tag{2-10}$$

where $\tilde{h}_{\alpha\beta}$ is the second fundamental form of Σ_c in Π (with respect to the unit normal $\tilde{e}_n$).

Clearly e_n, e_{n+1} and $\tilde{e}_n, \tilde{e}_{n+1}$ are in the same 2-plane perpendicular to the e_α. Let ϕ be the angle between e_n and $\tilde{e}_n$. Then we have

$$\tilde{e}_n = e_n cos\phi + e_{n+1}\sin\phi \quad \text{and} \quad \tilde{e}_{n+1} = -\tilde{e}_n sin\phi + e_{n+1}\cos\phi. \tag{2-11}$$

Accordingly,

$$\tilde{\omega}_n = \omega_n\cos\phi + \omega_{n+1}\sin\phi, \quad \tilde{\omega}_{n+1} = -\omega_n\sin\phi + \omega_{n+1}\cos\phi, \quad \tilde{\omega}_\alpha = \omega_\alpha. \tag{2-12}$$

Taking the exterior derivative of (2-12), and using (2-4) and (2-12) again, we get

$$\begin{aligned} d\tilde{\omega}_n &= (d\phi + \omega_{n,n+1})\wedge\tilde{\omega}_{n+1} + ((\cos\phi)\omega_{n,\alpha} + (\sin\phi)\omega_{n+1,n})\wedge\omega_\alpha, \\ d\tilde{\omega}_{n+1} &= (-d\phi + \omega_{n+1,n})\wedge\tilde{\omega}_n + ((\cos\phi)\omega_{n+1,\alpha} - (\sin\phi)\omega_{n,\alpha})\wedge\omega_\alpha.\end{aligned} \tag{2-13}$$

Notice that $\tilde{\omega}_n = \tilde{\omega}_{n+1} = 0$ on Σ_c. Comparing (2-13) with (2-9), we have

$$\begin{aligned}\tilde{\omega}_{n,\alpha} &= (\cos\phi)\omega_{n,\alpha} + (\sin\phi)\omega_{n+1,\alpha}, \\ \tilde{\omega}_{n+1,\alpha} &= (-\sin\phi)\omega_{n,\alpha} + (\cos\phi)\omega_{n+1,\alpha}.\end{aligned} \tag{2-14}$$

On the other hand, $\langle \tilde{e}_\alpha, \xi\rangle = 0$ on Σ_c, and since $d(\langle \tilde{e}_\alpha, \xi\rangle) = \langle \tilde{\omega}_{\alpha,A}\tilde{e}_A, \xi\rangle$, we have $\tilde{\omega}_{\alpha,n+1} = 0$. This together with (2-14) implies

$$\begin{aligned}\tilde{\omega}_{n,\alpha} &= \frac{\cos^2\phi}{\sin\phi}\omega_{n+1,\alpha} + (\sin\phi)\omega_{n+1,\alpha}\\ &= \frac{1}{\sin\phi}\omega_{n+1,\alpha} = \frac{1}{\sin\phi}(h_{\alpha\beta}\omega_\beta + h_{\alpha n}\omega_n).\end{aligned} \tag{2-15}$$

Combining this with (2-10) gives

$$\tilde{h}_{\alpha\beta} = \frac{1}{\sin\phi}h_{\alpha\beta} \quad \text{and} \quad h_{\alpha,n} = 0. \tag{2-16}$$

From the definition of the height function u, we can see $u_i = e_i(\langle X, \xi\rangle) = \langle e_i, \xi\rangle$; in particular, $u_n = \langle e_n, \xi\rangle$. Note that $\tilde{e}_{n+1} = \xi$, hence the second equation of (2-11) implies $u_n = -\sin\phi$ and $\langle \xi, e_{n+1}\rangle = \cos\phi$. By the decomposition

$$\xi = \sum_1^n \langle \xi, e_i\rangle e_i + \langle \xi, e_{n+1}\rangle e_{n+1}$$

we deduce that $1 = |\nabla u|^2 + \cos^2\phi$ and therefore $|\nabla u| = \pm\sin\phi$. With e_n chosen suitably we may assume $\sin\phi > 0$. Then (2-16) becomes

$$\tilde{h}_{\alpha\beta} = \frac{1}{|\nabla u|}h_{\alpha\beta} \quad \text{and} \quad h_{\alpha n} = 0. \tag{2-17}$$

From this one can easily see that

$$\begin{aligned}L_k = \sigma_k(\tilde{h}_{\alpha\beta}) &= \frac{1}{|\nabla u|^k}\sigma_k(h_{\alpha\beta})\\ &= \frac{1}{|\nabla u|^{k+2}}\frac{\partial\sigma_{k+1}(\boldsymbol{B})}{\partial h_{nn}}u_n u_n = \frac{\partial\sigma_{k+1}(\boldsymbol{B})}{\partial h_{ij}}u_i u_j |\nabla u|^{-(k+2)},\end{aligned} \tag{2-18}$$

where we have used $|u_n| = |\nabla u|$. □

3. Proof of Theorem 1.2

We adapt the notations in Section 2, and collect these formulas for convenience:

$$\begin{aligned}
X_i &= e_i, &&\\
X_{ij} &= -h_{ij}e_{n+1} && \text{(Gauss formula)},\\
e_{n+1,i} &= h_{ij}e_j && \text{(Weingarten formula)},\\
h_{ijk} &= h_{ikj} && \text{(Codazzi equation)},\\
R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} && \text{(Gauss equation)},\\
h_{ijkl} &= h_{ijlk} + h_{im}R_{mjkl} + h_{jm}R_{mikl}, &&
\end{aligned} \tag{3-1}$$

and for the smooth function u on M^n we also have the Ricci identity

$$u_{ijk} = u_{ikj} + u_m R_{mijk},$$

where R_{ijkl} is the Riemann curvature tensor, and as for the rest of this section, repeated indices are summed from 1 to n, unless otherwise stated.

Proof of Theorem 1.2. Suppose the second fundamental forms of the level sets of M^n take the minimum rank k with $k \le n-2$ at a point $P \in M^n$. We will treat the case $k > 0$ first, and then show how to modify the argument for the case $k = 0$. With the assumption that the level sets are all locally convex, we find easily that

$$\begin{aligned} L_r(P) &= 0 \quad \text{for all } r > k, \\ L_r(P) &> 0 \quad \text{for all } r \le k, \end{aligned} \tag{3-2}$$

and moreover

$$\begin{aligned} Z := \{X \in M^n \mid{} & \text{the second fundamental form} \\ & \text{of the level sets of } M^n \text{ has rank } k \text{ at } X\} \\ = \{X \in M^n \mid{} & L_{k+1}(X) = 0\}. \end{aligned} \tag{3-3}$$

Obviously Z is a closed set in M^n. If we can show that Z is also open in M^n — that is, that there is a neighborhood U_P of P in M^n such that $L_{k+1} \equiv 0$ on U_P — then $Z = M^n$, which is the result in the theorem.

Now $L_{k+1}(P) = 0 = \min_{X \in M^n} L_{k+1}(X)$, so by the strong maximum principle, we need only to show that

$$\Delta L_{k+1}(X) \le 0 \quad \text{mod } \{L_{k+1}(X), \nabla L_{k+1}(X)\} \quad \text{in } U_P, \tag{3-4}$$

where we modify the terms of L_{k+1} and its first derivatives, coefficients are locally bounded, and Δ is the Beltrami–Laplace operator on M^n.

For the rest of this section, define

$$W := (h_{ij}) \quad \text{with } i, j \le n-1, \qquad L := L_{k+1}, \qquad F := \sigma_{k+2}(\boldsymbol{B}),$$

and

$$F^{ij} := \frac{\partial F}{\partial h_{ij}}, \qquad F^{ij,rs} := \frac{\partial^2 F}{\partial h_{ij} \partial h_{rs}}, \qquad F^{ij,rs,pq} := \frac{\partial^3 F}{\partial h_{ij} \partial h_{rs} \partial h_{pq}}.$$

Hence, by (2-2),

$$|\nabla u|^{k+3} L = F^{ij} u_i u_j. \tag{3-5}$$

Taking the covariant derivative of this, we get

$$\begin{aligned} (|\nabla u|^{k+3} L)_\alpha &= |\nabla u|^{k+3} L_\alpha + (|\nabla u|^{k+3})_\alpha L, \\ (F^{ij} u_i u_j)_\alpha &= F^{ij,rs} h_{rs\alpha} u_i u_j + 2 F^{ij} u_{i\alpha} u_j. \end{aligned} \tag{3-6}$$

Taking the covariant derivative again, we get

$$
\begin{aligned}
(|\nabla u|^{k+3}L)_{\alpha\alpha} &= |\nabla u|^{k+3}L_{\alpha\alpha} + 2(|\nabla u|^{k+3})_{\alpha}L_{\alpha} + (|\nabla u|^{k+3})_{\alpha\alpha}L,\\
(F^{ij}u_iu_j)_{\alpha\alpha} &= F^{ij,rs,pq}h_{pq\alpha}h_{rs\alpha}u_iu_j + F^{ij,rs}h_{rs\alpha\alpha}u_iu_j\\
&\quad + 4F^{ij,rs}h_{rs\alpha}u_{i\alpha}u_j + 2F^{ij}u_{i\alpha\alpha}u_j + 2F^{ij}u_{i\alpha}u_{j\alpha}.
\end{aligned}
\tag{3-7}
$$

For a fixed point X_0 in U_P, choose a frame $\{e_1, \dots, e_n\}$ such that u_i through u_{n-1} vanish, $|u_n| = |\nabla u| > 0$, the form W is diagonal, and $h_{11} \geq h_{22} \geq \cdots \geq h_{n-1,n-1}$. Then by (3-2) we see that with U_P small enough

$$
\begin{aligned}
&h_{rr}(X_0) = 0 && \text{mod } \{L(X_0), \nabla L(X_0)\} \quad \text{for all } r > k,\\
&h_{rr}(X_0) > \epsilon > 0 && \text{mod } \{L(X_0), \nabla L(X_0)\} \quad \text{for all } r \leq k,
\end{aligned}
\tag{3-8}
$$

where ϵ is a positive sufficiently small number (maybe depending on U_P).

In the following, all the calculations will be done at X_0, and the terms of $L(X_0)$ and $\nabla L(X_0)$ will be dropped, that is, all the equalities or inequalities should be understood mod$\{L(X_0), \nabla L(X_0)\}$.

Denote $G := \{h_{11}, \dots, h_{kk}\}$ and $B := \{h_{k+1,k+1}, \dots, h_{n-1,n-1}\}$. Use the same symbols for $G := \{1, \dots, k\}$ and $B := \{k+1, \dots, n-1\}$ (it won't cause confusion).

Now, by $L(P) = 0 = \min_{X \in M^n} L(X)$ we get

$$
\begin{aligned}
0 &= (|\nabla u|^{k+3}L)_{\alpha} = (F^{ij}u_iu_j)_{\alpha} = F^{ij,rs}h_{rs\alpha}u_iu_j + 2F^{ij}u_{i\alpha}u_j\\
&= u_n^2F^{nn,rr}h_{rr\alpha} + 2u_nF^{in}u_{i\alpha}\\
&= u_n^2\sigma_k(G)\textstyle\sum_{r\in B} h_{rr\alpha} + 2u_nF^{nn}u_{n\alpha} + 2u_n\sum_{i=1}^{n-1} F^{in}u_{i\alpha}\\
&= u_n^2\sigma_k(G)\textstyle\sum_{r\in B} h_{rr\alpha} - 2u_n\sigma_k(G)\sum_{i\in B} h_{ni}u_{i\alpha}.
\end{aligned}
\tag{3-9}
$$

Clearly

$$
\begin{aligned}
u_i &= \langle X, \xi\rangle_i = \langle X_i, \xi\rangle = \langle e_i, \xi\rangle,\\
u_{ij} &= \langle X_{ij}, \xi\rangle = -\langle h_{ij}N, \xi\rangle := h_{ij}w,
\end{aligned}
\tag{3-10}
$$

where $w = -\langle N, \xi\rangle = \pm\sqrt{1 - |\nabla u|^2}$.

Substituting (3-10) into (3-9), using (3-8), and noting that W is diagonal, we deduce

$$
\begin{aligned}
\textstyle\sum_{i\in B} h_{ii\alpha} &= 0 \quad \text{for all } \alpha < n,\\
u_n\textstyle\sum_{i\in B} h_{iin} &= 2\textstyle\sum_{i\in B} h_{ni}^2 w.
\end{aligned}
\tag{3-11}
$$

By (3-7) we have

$$
(|\nabla u|^{k+3}L)_{\alpha\alpha} = |\nabla u|^{k+3}L_{\alpha\alpha} + 2(|\nabla u|^{k+3})_{\alpha}L_{\alpha} + (|\nabla u|^{k+3})_{\alpha\alpha}L = (F^{ij}u_iu_j)_{\alpha\alpha}.
$$

That is,

$$
\begin{aligned}
|\nabla u|^{k+3} L_{\alpha\alpha} &= F^{ij,rs,pq} h_{pq\alpha} h_{rs\alpha} u_i u_j + F^{ij,rs} h_{rs\alpha\alpha} u_i u_j \\
&\qquad + 4F^{ij,rs} h_{rs\alpha} u_{i\alpha} u_j + 2F^{ij} u_{i\alpha\alpha} u_j + 2F^{ij} u_{i\alpha} u_{j\alpha} \\
&= u_n^2 F^{nn,rs,pq} h_{pq\alpha} h_{rs\alpha} + u_n^2 F^{nn,rs} h_{rs\alpha\alpha} \\
&\qquad + 4u_n F^{in,rs} h_{rs\alpha} u_{i\alpha} + 2u_n F^{in} u_{i\alpha\alpha} + 2F^{ij} u_{i\alpha} u_{j\alpha},
\end{aligned} \tag{3-12}
$$

which we decompose as $I + II + III + IV$, where

$$
\begin{aligned}
I &:= u_n^2 F^{nn,rs,pq} h_{pq\alpha} h_{rs\alpha}, & II &:= 4u_n F^{in,rs} h_{rs\alpha} u_{i\alpha}, \\
III &:= u_n^2 F^{nn,rs} h_{rs\alpha\alpha} + 2u_n F^{in} u_{i\alpha\alpha}, & IV &:= 2F^{ij} u_{i\alpha} u_{j\alpha}.
\end{aligned} \tag{3-13}
$$

Next we will compute the above terms step by step. First

$$
\begin{aligned}
I &:= u_n^2 F^{nn,rs,pq} h_{pq\alpha} h_{rs\alpha} \\
&= u_n^2 F^{nn,rr,ss} h_{rr\alpha} h_{ss\alpha} + u_n^2 F^{nn,rs,sr} h_{rs\alpha} h_{sr\alpha} =: I_1 + I_2,
\end{aligned} \tag{3-14}
$$

and

$$
\begin{aligned}
I_1 &:= u_n^2 F^{nn,rr,ss} h_{rr\alpha} h_{ss\alpha} \\
&= 2u_n^2 \sum_{r\in G, s\in B} F^{nn,rr,ss} h_{rr\alpha} h_{ss\alpha} + u_n^2 \sum_{r,s\in B} F^{nn,rr,ss} h_{rr\alpha} h_{ss\alpha} \\
&= 2u_n^2 \sum_{r\in G, s\in B} \sigma_{k-1}(G\,|\,r) h_{rr\alpha} h_{ss\alpha} + u_n^2 \sigma_{k-1}(G) \sum_{r,s\in B, r\neq s} h_{rr\alpha} h_{ss\alpha},
\end{aligned} \tag{3-15}
$$

where here and below we use the notation $\sigma_{k-1}(G\,|\,r) := \sigma_{k-1}(G\backslash\{h_{rr}\})$ and the convention $\sigma_0 = 1$. Substituting (3-11) into (3-15) yields

$$
\begin{aligned}
I_1 &= 2u_n^2 \sum_{r\in G, s\in B} \sigma_{k-1}(G\,|\,r) h_{rr\alpha} h_{ss\alpha} + u_n^2 \sigma_{k-1}(G) \sum_{r\in B} h_{rr\alpha} \Big(\sum_{s\in B} h_{ss\alpha} - h_{rr\alpha} \Big) \\
&= 4\omega u_n \sum_{s\in B} h_{sn}^2 \sum_{r\in G} \sigma_{k-1}(G\,|\,r) h_{rrn} - u_n^2 \sigma_{k-1}(G) \sum_{\alpha=1}^{n} \sum_{r\in B} h_{rr\alpha}^2 \\
&\qquad\qquad + 4\omega^2 \sigma_{k-1}(G) \Big(\sum_{s\in B} h_{sn}^2 \Big)^2.
\end{aligned}
$$

For the remaining term in (3-14), we have

$$
\begin{aligned}
I_2 &= 2u_n^2 \sum_{r\in G, s\in B} F^{nn,rs,sr} h_{rs\alpha} h_{sr\alpha} + u_n^2 \sum_{r,s\in B} F^{nn,rs,sr} h_{rs\alpha} h_{sr\alpha} \\
&= -2u_n^2 \sum_{\alpha=1}^{n} \sum_{r\in G, s\in B} \sigma_{k-1}(G\,|\,r) h_{rs\alpha}^2 - u_n^2 \sigma_{k-1}(G) \sum_{\alpha=1}^{n} \sum_{r,s\in B, r\neq s} h_{rs\alpha}^2.
\end{aligned}
$$

So for the first term in (3-13) we have

$$I = 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i)h_{iin}h_{jn}^2 - 2u_n^2 \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i)h_{ij\alpha}^2 + 4w^2\sigma_{k-1}(G)\Big(\sum_{j\in B} h_{jn}^2\Big)^2 - u_n^2\sigma_{k-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} h_{ij\alpha}^2. \tag{3-16}$$

To compute the second term in (3-13), first we have by using (3-10)

$$\begin{aligned} II &= 4wu_n F^{in,rs} h_{rs\alpha} h_{i\alpha} \\ &= 4wu_n F^{nn,rs} h_{rs\alpha} h_{n\alpha} + 4wu_n \sum_{i=1}^{n-1} F^{in,ni} h_{ni\alpha} h_{i\alpha} \\ &\quad + 4wu_n \sum_{i,j=1}^{n-1} F^{in,ji} h_{ji\alpha} h_{i\alpha} + 4wu_n \sum_{i=1}^{n-1} F^{in,rr} h_{rr\alpha} h_{i\alpha}. \end{aligned} \tag{3-17}$$

We decompose the last four terms as $II_1 + II_2 + II_3 + II_4$. By (3-11), the first can be treated as

$$\begin{aligned} II_1 = 4wu_n F^{nn,rr} h_{rr\alpha} h_{n\alpha} &= 4wu_n \sigma_k(G) \sum_{r\in B} h_{rr\alpha} h_{n\alpha} \\ &= 4wu_n \sigma_k(G) \sum_{r\in B} h_{rrn} h_{nn} = 8w^2 \sigma_k(G) h_{nn} \sum_{r\in B} h_{rn}^2. \end{aligned}$$

For the second and the third terms, straightforward calculations show that

$$II_2 = -4wu_n \sigma_k(G) \sum_{i\in B} h_{ni\alpha} h_{i\alpha} = -4wu_n \sigma_k(G) \sum_{i\in B} h_{nni} h_{in}, \tag{3-18}$$

and

$$\begin{aligned} II_3 &= 4wu_n \sum_{i,j\in B} F^{in,ji} h_{ji\alpha} h_{i\alpha} \\ &\quad + 4wu_n \sum_{i\in G, j\in B} F^{in,ji} h_{ji\alpha} h_{i\alpha} + 4wu_n \sum_{j\in G, i\in B} F^{in,ji} h_{ji\alpha} h_{i\alpha} \\ &= 4wu_n \sigma_{k-1}(G) \sum_{i,j\in B, i\neq j} h_{jn} h_{ijn} h_{in} + 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i) h_{nj} h_{ji\alpha} h_{i\alpha} \\ &\quad + 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i) h_{ni} h_{ijn} h_{jn}. \end{aligned} \tag{3-19}$$

Again by (3-11), the fourth term can be treated as

$$
\begin{aligned}
II_4 &= 4wu_n \sum_{i,r\in B} F^{in,rr} h_{rr\alpha} h_{i\alpha} + 4wu_n \sum_{i\in G, r\in B} F^{in,rr} h_{rr\alpha} h_{i\alpha} \\
&\qquad\qquad + 4wu_n \sum_{r\in G, i\in B} F^{in,rr} h_{rr\alpha} h_{i\alpha} \\
&= -4wu_n \sigma_{k-1}(G) \sum_{i,r\in B, i\neq r} h_{in} h_{rr\alpha} h_{i\alpha} - 4wu_n \sum_{i\in G, r\in B} \sigma_{k-1}(G\,|\,i) h_{ni} h_{rr\alpha} h_{i\alpha} \\
&\qquad\qquad - 4wu_n \sum_{r\in G, i\in B} \sigma_{k-1}(G\,|\,r) h_{ni} h_{rr\alpha} h_{i\alpha} \\
&= -4wu_n \sigma_{k-1}(G) \sum_{i,r\in B, i\neq r} h_{in}^2 h_{rrn} - 4wu_n \sum_{i\in G, r\in B} \sigma_{k-1}(G\,|\,i) h_{ni} h_{rri} h_{ii} \\
&\qquad - 4wu_n \sum_{i\in G, r\in B} \sigma_{k-1}(G\,|\,i) h_{ni}^2 h_{rrn} - 4wu_n \sum_{r\in G, i\in B} \sigma_{k-1}(G\,|\,r) h_{ni}^2 h_{rrn} \\
&= -4wu_n \sigma_{k-1}(G) \sum_{i\in B} h_{in}^2 \Big(\sum_{r\in B} h_{rrn} - h_{iin} \Big) \\
&\qquad - 4wu_n \sum_{i\in G} \sigma_{k-1}(G\,|\,i) h_{ii} h_{ni} \sum_{r\in B} h_{rri} - 4wu_n \sum_{i\in G} \sigma_{k-1}(G\,|\,i) h_{ni}^2 \sum_{r\in B} h_{rrn} \\
&\qquad\qquad - 4wu_n \sum_{r\in G, i\in B} \sigma_{k-1}(G\,|\,r) h_{ni}^2 h_{rrn} \\
&= 4wu_n \sigma_{k-1}(G) \sum_{i\in B} h_{in}^2 h_{iin} - 8w^2 \sigma_{k-1}(G) \Big(\sum_{i\in B} h_{in}^2 \Big)^2 \\
&\qquad - 8w^2 \sum_{i\in G, r\in B} \sigma_{k-1}(G\,|\,i) h_{in}^2 h_{rn}^2 - 4wu_n \sum_{i\in B} h_{ni}^2 \sum_{r\in G} \sigma_{k-1}(G\,|\,r) h_{rrn}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
II &= 8w^2 \sigma_k(G) h_{nn} \sum_{j\in B} h_{nj}^2 - 4wu_n \sigma_k(G) \sum_{j\in B} h_{nnj} h_{nj} \\
&\qquad - 8w^2 \sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i) h_{in}^2 h_{jn}^2 - 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i) h_{nj}^2 h_{iin} \\
&\quad + 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i) h_{i\alpha} h_{jn} h_{ij\alpha} + 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i) h_{ni} h_{nj} h_{ijn} \\
&+ 4wu_n \sigma_{k-1}(G) \sum_{i,j\in B, i\neq j} h_{ni} h_{nj} h_{ijn} + 4wu_n \sigma_{k-1}(G) \sum_{j\in B} h_{nj}^2 h_{jjn} \\
&\qquad\qquad - 8w^2 \sigma_{k-1}(G) \Big(\sum_{j\in B} h_{nj}^2 \Big)^2.
\end{aligned}
\tag{3-20}
$$

Now we deal with the third term in (3-13):

$$
\begin{aligned}
III &:= u_n^2 F^{nn,rs} h_{rs\alpha\alpha} + 2u_n F^{in} u_{i\alpha\alpha} \\
&= u_n^2 F^{nn,rr} h_{rr\alpha\alpha} + 2u_n F^{nn} u_{n\alpha\alpha} + 2u_n \sum_{i=1}^{n-1} F^{in} u_{i\alpha\alpha}.
\end{aligned} \tag{3-21}
$$

We decompose the last three terms as $III_1+III_2+III_3$. Using the exchange formula in (3-1), we can calculate

$$
\begin{aligned}
III_1 &= u_n^2 \sigma_k(G) \sum_{r\in B} h_{rr\alpha\alpha} \\
&= u_n^2 \sigma_k(G) \sum_{r\in B} (h_{r\alpha\alpha r} + h_{rm} R_{m\alpha r\alpha} + h_{\alpha m} R_{mrr\alpha}) \\
&= u_n^2 \sigma_k(G) \sum_{r\in B} h_{\alpha\alpha rr} \\
&\quad + u_n^2 \sigma_k(G) \sum_{r\in B} (h_{rm}(h_{mr}h_{\alpha\alpha} - h_{m\alpha}h_{\alpha r}) + h_{\alpha m}(h_{mr}h_{r\alpha} - h_{m\alpha}h_{rr})) \\
&= u_n^2 \sigma_k(G) \sum_{r\in B} H_{rr} + u_n^2 \sigma_k(G) \sum_{r\in B} (H h_{rm} h_{mr} - h_{rr} h_{m\alpha} h_{\alpha m}) \\
&= u_n^2 \sigma_k(G) \sum_{j\in B} H_{jj} + u_n^2 H \sigma_k(G) \sum_{j\in B} h_{jn}^2,
\end{aligned}
$$

and $III_2 = 2u_n \sigma_{k+1}(W) u_{n\alpha\alpha} = 0$. For the third term, we have

$$
\begin{aligned}
III_3 &= -2u_n \sigma_k(G) \sum_{i\in B} h_{in} u_{i\alpha\alpha} \\
&= -2u_n \sigma_k(G) \sum_{i\in B} h_{in} (u_{\alpha\alpha i} + u_m R_{\alpha m\alpha i}) \\
&= -2u_n \sigma_k(G) \sum_{i\in B} h_{in} (H\omega)_i - 2u_n \sigma_k(G) \sum_{i\in B} h_{in} u_m (h_{\alpha\alpha} h_{mi} - h_{\alpha i} h_{m\alpha}) \\
&= -2u_n \sigma_k(G) \sum_{i\in B} h_{in} (H_i \omega - H h_{ij} u_j) \\
&\qquad\qquad - 2u_n^2 \sigma_k(G) \sum_{i\in B} h_{in}^2 H + 2u_n^2 \sigma_k(G) \sum_{i\in B} h_{in}^2 h_{nn} \\
&= -2\omega u_n \sigma_k(G) \sum_{j\in B} h_{in} H_j + 2u_n^2 \sigma_k(G) h_{nn} \sum_{j\in B} h_{jn}^2.
\end{aligned}
$$

We have used in the calculations above that

$$
\omega_i = -\langle N, \xi\rangle_i = -\langle N_i, \xi\rangle = -\langle h_{ij} e_j, \xi\rangle = -h_{ij} u_j.
$$

Substituting our results for III_1, III_2, and III_3 into (3-21) yields

$$III = u_n^2\sigma_k(G)\sum_{j\in B} H_{jj} + u_n^2\sigma_k(G)H\sum_{j\in B} h_{jn}^2 - 2wu_n\sigma_k(G)\sum_{j\in B} h_{jn}H_j + 2u_n^2\sigma_k(G)h_{nn}\sum_{j\in B} h_{jn}^2. \tag{3-22}$$

We decompose the final term in (3-13) as $IV_1+IV_2+IV_3+IV_4$ by

$$\begin{aligned} IV &:= 2F^{ij}u_{i\alpha}u_{j\alpha} \\ &= 2F^{nn}u_{n\alpha}u_{n\alpha} + 4\sum_{i=1}^{n-1} F^{in}u_{i\alpha}u_{n\alpha} + 2\sum_{i=1}^{n-1} F^{ii}u_{i\alpha}u_{i\alpha} + 2\sum_{\substack{i,j=1\\ i\neq j}}^{n-1} F^{ij}u_{i\alpha}u_{j\alpha} \end{aligned}$$

It follows that $IV_1 = 2F^{nn}u_{n\alpha}u_{n\alpha} = 2\sigma_{k+1}(W)u_{n\alpha}u_{n\alpha} = 0$, and

$$\begin{aligned} IV_2 &= -4\sum_{i=1}^{n-1}\sigma_k(W\,|\,i)h_{in}u_{i\alpha}u_{n\alpha} = -4\sigma_k(G)\sum_{i\in B} h_{in}u_{i\alpha}u_{n\alpha} \\ &= -4w^2\sigma_k(G)\sum_{i\in B} h_{in}^2 h_{nn}. \end{aligned} \tag{3-23}$$

For the last two terms, we have

$$\begin{aligned} IV_3 &= 2\sum_{i\in G} F^{ii}u_{i\alpha}u_{i\alpha} + 2\sum_{i\in B} F^{ii}u_{i\alpha}u_{i\alpha} \\ &= -2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{jn}^2 u_{i\alpha}u_{i\alpha} + 2\sigma_k(G)\sum_{i\in B} h_{nn}u_{i\alpha}u_{i\alpha} \\ &\quad - 2\sum_{i,j\in B, i\neq j}\sigma_k(G)h_{jn}^2 u_{i\alpha}u_{i\alpha} - 2\sum_{j\in G, i\in B}\sigma_{k-1}(G\,|\,j)h_{jn}^2 u_{i\alpha}u_{i\alpha} \\ &= -2w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{jn}^2 h_{ii}^2 - 2w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{jn}^2 h_{in}^2 \\ &\quad + 2w^2\sigma_k(G)\sum_{i\in B} h_{nn}h_{in}^2 - 2w^2\sigma_{k-1}(G)\sum_{i,j\in B, i\neq j} h_{in}^2 h_{jn}^2 \\ &\quad - 2w^2\sum_{j\in G, i\in B}\sigma_{k-1}(G\,|\,j)h_{jn}^2 h_{in}^2 \\ &= -2w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{ii}^2 h_{jn}^2 - 4w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{in}^2 h_{jn}^2 \\ &\quad + 2w^2\sigma_k(G)\sum_{i\in B} h_{nn}h_{in}^2 - 2w^2\sigma_{k-1}(G)\sum_{i,j\in B, i\neq j} h_{in}^2 h_{jn}^2, \end{aligned}$$

and

$$
\begin{aligned}
IV_4 &= 2\sum_{i,j\in G, i\neq j} F^{ij}u_{i\alpha}u_{j\alpha} + 4\sum_{i\in G, j\in B} F^{ij}u_{i\alpha}u_{j\alpha} + 2\sum_{i,j\in B, i\neq j} F^{ij}u_{i\alpha}u_{j\alpha}\\
&= 0 + 4\sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i)h_{in}h_{jn}u_{i\alpha}u_{j\alpha} + 2\sigma_{k-1}(G)\sum_{i,j\in B, i\neq j} h_{in}h_{jn}u_{i\alpha}u_{j\alpha}\\
&= 4w^2\sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i)h_{in}^2h_{jn}^2 + 2w^2\sigma_{k-1}(G)\sum_{i,j\in B, i\neq j} h_{in}^2h_{jn}^2.
\end{aligned}
$$

Our final result for IV is then

$$
IV = -2w^2\sigma_k(G)h_{nn}\sum_{j\in B}h_{jn}^2 - 2w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{ii}^2h_{jn}^2. \tag{3-24}
$$

Combining (3-16), (3-20), (3-22) and (3-24) with (3-12) we have

$$
|\nabla u|^{k+3}L_{\alpha\alpha} := I + II + III + IV := A + B + C, \tag{3-25}
$$

where

$$
\begin{aligned}
C &:= \sigma_{k-1}(G)\Big(4wu_n\sum_{\substack{i,j\in B\\ i\neq j}} h_{ni}h_{nj}h_{ijn} + 4wu_n\sum_{j\in B}h_{nj}^2h_{jjn}\\
&\qquad\qquad - 4w^2\Big(\sum_{j\in B}h_{nj}^2\Big)^2 - u_n^2\sum_{\alpha=1}^n\sum_{i,j\in B}h_{ij\alpha}^2\Big)\\
&= -\sigma_{k-1}(G)\sum_{\alpha=1}^n\sum_{i,j\in B}(u_nh_{ij\alpha} - 2wh_{nj}h_{i\alpha})^2,
\end{aligned}
$$

and

$$
\begin{aligned}
A &:= \sigma_k(G)\Big(u_n^2\sum_{j\in B}H_{jj} - 2wu_n\sum_{j\in B}h_{jn}H_j - 4wu_n\sum_{j\in B}h_{nnj}h_{nj}\\
&\qquad + u_n^2H\sum_{j\in B}h_{jn}^2 + 2u_n^2h_{nn}\sum_{j\in B}h_{jn}^2 + 6w^2h_{nn}\sum_{j\in B}h_{nj}^2\Big)\\
&= \sigma_k(G)\Big(u_n^2\sum_{j\in B}H_{jj} - 6wu_n\sum_{j\in B}h_{jn}H_j + (3u_n^2 + 6w^2)H\sum_{j\in B}h_{jn}^2\Big)\\
&\quad + \sigma_k(G)\Big(-(6w^2 + 2u_n^2)\sum_{i\in G, j\in B}h_{ii}h_{jn}^2 + 4wu_n\sum_{i\in G, j\in B}h_{iij}h_{nj}\Big).
\end{aligned}
$$

The summand B is grouped in terms of $\sigma_{k-1}(G\,|\,i)$. We decompose the last two terms as A_1+A_2. It follows that

$$
\begin{aligned}
\text{(3-26)}\quad B+A_2 &= \sum_{\alpha=1}^{n}\sum_{i\in G,\,j\in B}\sigma_{k-1}(G\,|\,i)(-8w^2h_{i\alpha}^2h_{jn}^2+8wu_nh_{i\alpha}h_{jn}h_{ij\alpha}\\
&\qquad\qquad -2u_n^2h_{ij\alpha}^2-2u_n^2h_{ii}^2h_{jn}^2)\\
&= -2\sum_{\alpha=1}^{n}\sum_{i\in G,\,j\in B}\sigma_{k-1}(G\,|\,i)(u_nh_{ij\alpha}-2wh_{i\alpha}h_{jn})^2\\
&\qquad -2u_n^2\sigma_k(G)\sigma_1(G)\sum_{j\in B}h_{jn}^2.
\end{aligned}
$$

Combining (3-25) with (3-26), we finally get

$$
\begin{aligned}
\text{(3-27)}\quad |\nabla u|^{k+3}L_{\alpha\alpha} &= \sigma_k(G)\Big(u_n^2\sum_{j\in B}H_{jj}-6wu_n\sum_{j\in B}h_{jn}H_j\\
&\qquad\qquad +(3u_n^2+6w^2)H\sum_{j\in B}h_{jn}^2\Big)\\
&\quad -2\sum_{\alpha=1}^{n}\sum_{i\in G,\,j\in B}\sigma_{k-1}(G\,|\,i)(u_nh_{ij\alpha}\\
&\quad -2wh_{i\alpha}h_{jn})^2-2u_n^2\sigma_k(G)\sigma_1(G)\sum_{j\in B}h_{jn}^2\\
&\quad -\sigma_{k-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B}(u_nh_{ij\alpha}-2wh_{nj}h_{i\alpha})^2.
\end{aligned}
$$

Then, for $H=-f(X,N)$, the structure conditions on f is

$$\text{(3-28)}\qquad -u_n^2f_{jj}+6wu_nh_{nj}f_j-(6-3u_n^2)fh_{nj}^2\le 0\quad\text{for each } j\in B,$$

where we have used $w^2+u_n^2=1$. Now we can use the following formulas to get the structure condition on f. Following Guan, Lin, and Ma [Guan et al. 2006], we have for each $i\in\{1,2,\dots,n\}$

$$
\begin{aligned}
f_i &= \sum_{A=1}^{n+1}f_{X_A}e_i^A+f_{e_{n+1}}(e_{n+1})_i,\\
\text{(3-29)}\qquad f_{ii} &= \sum_{A,C=1}^{n+1}f_{X_AX_C}e_i^Ae_i^C+\sum_{A=1}^{n+1}f_{X_A}X_{ii}^A+2\sum_{A=1}^{n+1}f_{X_Ae_{n+1}}e_i^A(e_{n+1})_i\\
&\qquad +f_{e_{n+1},e_{n+1}}(e_{n+1})_i(e_{n+1})_i+f_{e_{n+1}}(e_{n+1})_{ii}.
\end{aligned}
$$

For example, if $f(X,N)=f(X)$, then f satisfies

$$\text{(3-30)}\qquad 3(1-u_n^2)f_j^2\le(2-u_n^2)ff_{jj}$$

and $f \geq 0$. Since $0 < u_n^2 \leq 1$, we reduce the structure conditions on f to

$$f \geq 0 \quad \text{and} \quad 3f_j^2 \leq 2ff_{jj} \quad \text{for all } j \in B. \tag{3-31}$$

So the structure conditions is $f \geq 0$ and the matrix

$$2f\frac{\partial^2 f}{\partial X_A \partial X_B} - 3\frac{\partial f}{\partial X_A}\frac{\partial f}{\partial X_B}$$

is positive semidefinite, where $1 \leq A, B \leq n+1$. Clearly (3-27) implies (3-4) under these conditions, which proves the case in which $k > 0$.

In case $k = 0$, only A_1 appears in (3-25), so this obviously finishes the proof of Theorem 1.2. □

References

[Caffarelli and Friedman 1985] L. A. Caffarelli and A. Friedman, "Convexity of solutions of semilinear elliptic equations", *Duke Math. J.* **52**:2 (1985), 431–456. MR 87a:35028 Zbl 0599.35065

[Caffarelli and Spruck 1982] L. A. Caffarelli and J. Spruck, "Convexity properties of solutions to some classical variational problems", *Comm. Partial Differential Equations* **7**:11 (1982), 1337–1379. MR 85f:49062 Zbl 0508.49013

[Caffarelli et al. 2007] L. Caffarelli, P. Guan, and X.-N. Ma, "A constant rank theorem for solutions of fully nonlinear elliptic equations", *Comm. Pure Appl. Math.* **60**:12 (2007), 1769–1791. MR 2008i:35067 Zbl 05223755

[Colesanti and Salani 2003] A. Colesanti and P. Salani, "Quasi-concave envelope of a function and convexity of level sets of solutions to elliptic equations", *Math. Nachr.* **258** (2003), 3–15. MR 2004f:35054 Zbl 1128.35332

[Cuoghi and Salani 2006] P. Cuoghi and P. Salani, "Convexity of level sets for solutions to nonlinear elliptic problems in convex rings", *Electron. J. Differential Equations* **2006** (2006), Article No. 124. MR 2007d:35083 Zbl 1128.35320

[Gabriel 1957] R. M. Gabriel, "A result concerning convex level surfaces of 3-dimensional harmonic functions", *J. London Math. Soc.* **32** (1957), 286–294. MR 19,848a Zbl 0087.09702

[Gilbarg and Trudinger 1977] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften **224**, Springer, Berlin, 1977. MR 57 #13109 Zbl 0361.35003

[Guan and Ma 2003] P. Guan and X.-N. Ma, "The Christoffel–Minkowski problem, I: Convexity of solutions of a Hessian equation", *Invent. Math.* **151**:3 (2003), 553–577. MR 2004a:35071

[Guan et al. 2006] P. Guan, C. Lin, and X. Ma, "The Christoffel–Minkowski problem, II: Weingarten curvature equations", *Chinese Ann. Math. Ser. B* **27**:6 (2006), 595–614. MR 2007k:35145

[Huang 1992] W. H. Huang, "Superharmonicity of curvatures for surfaces of constant mean curvature", *Pacific J. Math.* **152**:2 (1992), 291–318. MR 92k:53017 Zbl 0767.53040

[Korevaar 1990] N. J. Korevaar, "Convexity of level sets for solutions to elliptic ring problems", *Comm. Partial Differential Equations* **15**:4 (1990), 541–556. MR 91h:35118 Zbl 0725.35007

[Korevaar and Lewis 1987] N. J. Korevaar and J. L. Lewis, "Convex solutions of certain elliptic equations have constant rank Hessians", *Arch. Rational Mech. Anal.* **97**:1 (1987), 19–32. MR 88i:35054 Zbl 0624.35031

[Lewis 1977] J. L. Lewis, "Capacitary functions in convex rings", *Arch. Rational Mech. Anal.* **66**:3 (1977), 201–224. MR 57 #16638 Zbl 0393.46028

[Singer et al. 1985] I. M. Singer, B. Wong, S.-T. Yau, and S. S.-T. Yau, "An estimate of the gap of the first two eigenvalues in the Schrödinger operator", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **12**:2 (1985), 319–333. MR 87j:35280 Zbl 0603.35070

[Trudinger 1997] N. S. Trudinger, "On new isoperimetric inequalities and symmetrization", *J. Reine Angew. Math.* **488** (1997), 203–220. MR 99a:35076 Zbl 0883.52006

[Xu 2008] L. Xu, "A microscopic convexity theorem of level sets for solutions to elliptic equations", preprint, 2008.

Received August 17, 2008. Revised April 14, 2009.

CHANGQING HU
DEPARTMENT OF MATHEMATICS
SUZHOU UNIVERSITY
SUZHOU 215006
CHINA
huchangqing@suda.edu.cn

XI-NAN MA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI 230026
CHINA
xinan@ustc.edu.cn

QIANZHONG OU
DEPARTMENT OF MATHEMATICS
HEZHOU UNIVERSITY
HEZHOU 542800
CHINA
ouqzh@163.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

SYMPLECTIC SUPERCUSPIDAL REPRESENTATIONS OF GL(2n) OVER p-ADIC FIELDS

DIHUA JIANG, CHUFENG NIEN AND YUJUN QIN

This is part two of the authors' work on supercuspidal representations of GL($2n$) over p-adic fields. We consider the complete relations among the local theta correspondence, local Langlands transfer, and the local descent attached to a given irreducible symplectic supercuspidal representation of p-adic GL_{2n}. This is the natural extension of the work of Ginzburg, Rallis and Soudry and of Jiang and Soudry on the local descents and the local Langlands transfers. The approach undertaken in this paper is purely local. A mixed approach with both local and global methods, which works for more general classical groups, has been considered by Jiang and Soudry.

1. Introduction

Let $\mathcal{F}$ be a p-adic local field of characteristic zero. Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$. By the local Langlands conjecture for $\mathrm{GL}_{2n}(\mathcal{F})$, which is now a theorem of Harris and Taylor [2001] and of Henniart [2000], there exists an irreducible admissible $2n$-dimensional representation ϕ of the local Weil group $\mathcal{W}_{\mathcal{F}}$, that is, the local Langlands parameter

$$\phi : \mathcal{W}_{\mathcal{F}} \to \mathrm{GL}_{2n}(\mathbb{C}),$$

corresponding to τ with a set of required conditions. We say that τ is of symplectic type if the image $\phi(\mathcal{W}_{\mathcal{F}})$ is contained in the symplectic subgroup $\mathrm{Sp}_{2n}(\mathbb{C})$ of the complex dual group $\mathrm{GL}_{2n}(\mathbb{C})$ of $\mathrm{GL}_{2n}(\mathcal{F})$.

Because of their deep connection with Galois representations, symplectic supercuspidal representations (or more importantly cuspidal automorphic representations) have recently received much attention; see for instance [Ginzburg et al. 2004; Chenevier and Clozel 2009]. The symplectic irreducible unitary supercuspidal

MSC2000: primary 11F70, 22E50; secondary 11F85, 22E55.
Keywords: symplectic representation, Shalika models, local Langlands transfer, local descent, supercuspidal, representations of p-adic groups.
Jiang is supported in part by NSF (Unite States) grant DMS-0653742. Nien is supported by NSC 97-2115-M-006-007-, Taiwan. Qin is supported partly by the Program for Changjiang Scholars and Innovative Research Team in East China Normal University. All three authors are supported in part by NSFC 10701034, China.

representations of $\mathrm{GL}_{2n}(\mathcal{F})$ were characterized in [Shahidi 1990; 1992; Jacquet and Rallis 1996; Ginzburg et al. 1999; Jiang and Soudry 2003; 2004; Jiang and Qin 2007; Jiang et al. 2008] and were discussed in detail in [Jiang et al. 2008, Section 5]. We state these results as follows; the theorem's notation and terminology will explained in Section 2.

Theorem 1.1. *Suppose τ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$. Then the following are equivalent.*

(1) *τ is of symplectic type.*

(2) *The local exterior square L-factor $L(s, \tau, \Lambda^2)$ has a pole at $s = 0$.*

(3) *The local exterior square γ-factor $\gamma(s, \tau, \Lambda^2, \psi)$ has a pole at $s = 1$.*

(4) *τ has a nonzero Shalika model.*

(5) *The unitarily induced representation $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ is reducible at $s = 1$. In this case, $\mathrm{I}^{\mathrm{SO}_{4n}}(1, \tau)$ has the unique Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$, which has a nonzero generalized Shalika model.*

(6) *τ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(\mathcal{F})$.*

(7) *τ has a nonzero linear model, that is, a $\mathrm{GL}_n(\mathcal{F}) \times \mathrm{GL}_n(\mathcal{F})$-invariant functional.*

(8) *The unitarily induced representation $\mathrm{I}^{\mathrm{Sp}_{4n}}(s, \tau)$ of $\mathrm{Sp}_{4n}(\mathcal{F})$ is reducible at $s = 1/2$, and $\mathrm{I}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ has the unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$, which has a nonzero symplectic linear model, that is, a $\mathrm{Sp}_{2n}(\mathcal{F}) \times \mathrm{Sp}_{2n}(\mathcal{F})$-invariant functional.*

(9) *τ is a local Langlands functorial ψ-transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$.*

If one of the above holds for τ, then τ is self-dual.

The local Langlands functorial ψ-transfer from an irreducible ψ-generic supercuspidal representation $\tilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to the irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ is given by the [Ginzburg et al. 1999, corollary of Section 1.5]. The local exterior square L-function and gamma factor are given by the Shahidi method.

The equivalence of the characterizations in Theorem 1.1 can be explained by Figure 1. The complex dual groups of $\mathrm{SO}_{2n+1}(\mathcal{F})$ and the double metaplectic cover $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ of $\mathrm{Sp}_{2n}(\mathcal{F})$ are the same, namely, $\mathrm{Sp}_{2n}(\mathbb{C})$. In Figure 1, the map θc is the local theta correspondence for the reductive dual pairs $(\mathrm{SO}_{4n}, \mathrm{Sp}_{4n})$ and $(\mathrm{SO}_{2n+1}, \widetilde{\mathrm{Sp}}_{2n})$. The map G-G is the local Gelfand–Graev coefficient that takes representations from SO_{4n} to SO_{2n+1}. The map F-J is the local Fourier–Jacobi coefficient that takes representations from Sp_{4n} to $\widetilde{\mathrm{Sp}}_{2n}$. The map Lq is the composition of the parabolic induction from the standard parabolic subgroups with the Levi subgroup isomorphic to GL_{2n} in SO_{4n} and Sp_{4n}, and that takes the unique

$$
\begin{tikzcd}
\mathrm{SO}_{4n} \arrow[rr, leftrightarrow, "\theta\mathrm{c}"] \arrow[dd, "\text{G-G}"'] & \text{(A)} & \mathrm{Sp}_{4n} \arrow[dd, "\text{F-J}"] \\
\text{(B)} & \mathrm{GL}_{2n} \arrow[ul, "\mathrm{Lq}"] \arrow[ur, "\mathrm{Lq}"'] & \text{(C)} \\
\mathrm{SO}_{2n+1} \arrow[rr, leftrightarrow, "\theta\mathrm{c}"'] \arrow[ur, "\mathrm{Lt}"'] & \text{(D)} & \widetilde{\mathrm{Sp}}_{2n} \arrow[ul, "\mathrm{Lt}"]
\end{tikzcd}
$$

Diagram 1

Langlands quotient from the induced representations of SO_{4n} and Sp_{4n}, respectively. It is clear that G-G $\circ$ Lq and F-J $\circ$ Lq are the local descents from GL_{2n} to SO_{2n+1} and $\widetilde{\mathrm{Sp}}_{2n}$, respectively, in the sense of Ginzburg, Rallis and Soudry. Finally the map Lt is the local Langlands functorial transfer from SO_{2n+1} to GL_{2n} and from $\widetilde{\mathrm{Sp}}_{2n}$ to GL_{2n}.

For a given irreducible unitary symplectic supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, the maps in Figure 1 can be realized as in Figure 2, where notation is as follows. First, σ is an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, which lifts to τ by the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}, and $\tilde{\pi}$ is an irreducible ψ-generic supercuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$,

$$
\begin{tikzcd}
\mathcal{L}^{\mathrm{SO}_{4n}}(1,\tau) \arrow[rr, leftrightarrow, "\theta\mathrm{c}"] \arrow[dd, "\text{G-G}"'] & \text{(A)} & \mathcal{L}^{\mathrm{Sp}_{4n}}(1/2,\tau) \arrow[dd, "\text{F-J}"] \\
\text{(B)} & \tau \arrow[ul, "\mathrm{Lq}"] \arrow[ur, "\mathrm{Lq}"'] & \text{(C)} \\
\sigma \arrow[rr, leftrightarrow, "\theta\mathrm{c}"'] \arrow[ur, "\mathrm{Lt}"'] & \text{(D)} & \tilde{\pi} \arrow[ul, "\mathrm{Lt}"]
\end{tikzcd}
$$

Diagram 2

which lifts to τ by the local Langlands functorial ψ-transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to GL_{2n}. Consider the maximal parabolic subgroup P of SO_{4n} with Levi subgroup GL_{2n}. Then the unitarily parabolic induction $\mathrm{I}^{\mathrm{SO}_{4n}}(1, \tau)$ has a unique Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$, and similarly the unitarily parabolic induction $\mathrm{I}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ has a unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$. Finally, the local Gelfand–Graev coefficient takes $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ from $\mathrm{SO}_{4n}(\mathcal{F})$ back to $\mathrm{SO}_{2n+1}(\mathcal{F})$ and the local Fourier–Jacobi coefficient takes $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ from $\mathrm{Sp}_{4n}(\mathcal{F})$ back to $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$, respectively. Detailed discussion of these maps is found in Section 2.

Theorem 1.2. *For an irreducible unitary symplectic supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, Figure 2 is commutative.*

Now we explain the relation between Theorem 1.1 and Theorem 1.2, or the commutative diagrams Figure 1 and Figure 2.

Jiang and Soudry [2003] proved that for a given irreducible unitary symplectic supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, there exists uniquely an irreducible generic supercuspidal representation σ of $\mathrm{SO}_{2n+1}(\mathcal{F})$ and an irreducible ψ-generic supercuspidal representation $\tilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$, such that the subdiagram (D) is commutative. The local Langlands functorial transfer property for τ is equivalent to the existence of a pole at $s = 0$ of the local exterior square L-factor $L(s, \tau, \Lambda^2)$, or equivalently by definition a pole at $s = 1$ of the local exterior square γ-factor $\gamma(s, \tau, \Lambda^2, \psi)$. One very interesting point is the characterization in terms of the existence of a nonzero Shalika model (or functional) or of a nonzero linear model (or functional), following the idea of relative trace formula approach to the global Langlands functorial transfers. It was proved in [Jiang et al. 2008] that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, the existence of a nonzero Shalika model for τ is equivalent to the existence of a nonzero linear model for τ, although this result had been expected for a while. Jacquet and Rallis [1996] proved that the existence of a nonzero Shalika model for τ implies the existence of a nonzero linear model for τ.

For an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, why does the existence of a nonzero linear model for τ determine the local Langlands functorial transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to GL_{2n}, while the existence of a nonzero Shalika model for τ determines the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}? To answer this, Ginzburg, Rallis, and Soudry [Ginzburg et al. 1999] showed that if an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ has a nonzero linear model, that is, a nonzero $\mathrm{GL}_n(\mathcal{F}) \times \mathrm{GL}_n(\mathcal{F})$-invariant functional, then the unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ of the unitarily parabolic induction $\mathrm{I}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ (which is reducible) has a nonzero symplectic linear model, that is, a nonzero $\mathrm{Sp}_{2n}(\mathcal{F}) \times \mathrm{Sp}_{2n}(\mathcal{F})$-invariant functional. Based on the existence of a nonzero symplectic linear model for $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$, they show that the ψ-local descent (the

Fourier–Jacobi ψ-functor in this case) yields $\tilde{\pi}$ back to $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$. This proves that the subdiagram (C) is commutative.

The local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ was first obtained in [Jiang and Soudry 2003] by combining the subdiagrams (C) and (D) and by using the local converse theorem. More recently, Jiang and Soudry (see [Soudry 2008]) obtained the local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ via the global theory of the automorphic descent [Ginzburg et al. 2001]. Their method works for other classical groups as well.

In [Jiang and Qin 2007; Jiang et al. 2008], we began the task of establishing the local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ by using the existence of a nonzero Shalika model for τ of $\mathrm{GL}_{2n}(\mathcal{F})$ and of a nonzero generalized Shalika model for the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$. We proved by a purely local argument in [Jiang et al. 2008, Theorem 3.1] that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, the local Gelfand–Graev coefficient (a special type of twisted Jacquet functor) of the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$, which is a representation of $\mathrm{SO}_{2r+1}(\mathcal{F})$, vanishes for all $r < n$. Here, again using a purely local argument, we show that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, the local Gelfand–Graev coefficient of the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ is an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$; this, Theorem 2.5, is our main result. The proof idea was suggested by the global argument as in [Ginzburg et al. 2001]. Our proof goes similarly to the case of symplectic linear models in [Ginzburg et al. 1999], but is essentially based on the existence and uniqueness of a generalized Shalika model for the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$. The technical details are of independent interest, and are found in Sections 3, 4 and 5.

One fact that needs to be shown here is that the local Gelfand–Graev coefficient on $\mathrm{SO}_{2n+1}(\mathcal{F})$ from $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ lifts to τ via the local Langlands functorial transfer. In [Jiang and Soudry 2003; Soudry 2008], a global argument is used to show that this is the case. However, one would like to prove this by a purely local argument. One way to do this is to calculate explicitly the local Rankin–Selberg integral for the tensor product L-functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_r$ by using the supercuspidal representation constructed explicitly by the local Gelfand–Graev coefficient from $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$; however we do not do this here. Hence, the subdiagram (B) is commutative by Theorem 2.5 and the result in [Jiang and Soudry 2003; Soudry 2008].

Finally, we show that the subdiagram (A) is also commutative by using results of G. Muic [2006], which show that the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ and the Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ of $\mathrm{Sp}_{4n}(\mathcal{F})$ correspond to each other via the local theta correspondence. By combining this with Theorem 1.1, one deduces that the generalized Shalika model on $\mathrm{SO}_{4n}(\mathcal{F})$ and the symplectic linear model of $\mathrm{Sp}_{4n}(\mathcal{F})$ are related by the local theta correspondence. It would be interesting to check directly, without using Theorem 1.1, that the local theta correspondence relates the generalized Shalika model on $\mathrm{SO}_{4n}(\mathcal{F})$ and the symplectic linear model of $\mathrm{Sp}_{4n}(\mathcal{F})$.

In future work, we will study the explicit relations between Diagrams 1 and 2 and refined structures of the corresponding local Arthur packets.

2. Main result

We introduce definitions of various models and of the local descent in the case under consideration, and then state the main result for the local descent.

2.1. ***Shalika and generalized Shalika models.*** Let $\mathcal{F}$ be a finite extension of the p-adic number field $\mathbb{Q}_p$ for some rational prime p. Take the maximal parabolic subgroup $P_{n,n} = M_{n,n}N_{n,n}$ of GL_{2n} with

$$M_{n,n} = \mathrm{GL}_n \times \mathrm{GL}_n,$$
$$N_{n,n} = \left\{ n(X) = \begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix} \in \mathrm{GL}_{2n} \right\}.$$

Let ψ be a nontrivial character of $\mathcal{F}$. Define a character $\psi_{N_{n,n}}(n(X)) = \psi(\mathrm{tr}(X))$. The stabilizer of $\psi_{N_{n,n}}$ in $M_{n,n}$ is GL_n^{Δ}, the diagonal embedding of GL_n into $M_{n,n}$. Denote by

$$\mathcal{S}_n = \mathrm{GL}_n^{\Delta} \rtimes N_{n,n}$$

the Shalika subgroup. Denote by $\psi_{\mathcal{S}_n}$ the extension of $\psi_{N_{n,n}}$ from $N_{n,n}$ to the Shalika subgroup $\mathcal{S}_n$ such that $\psi_{\mathcal{S}_n}$ is trivial on GL_n^{Δ}. The Shalika functionals of an irreducible admissible representation (τ, V_τ) of $\mathrm{GL}_{2n}(\mathcal{F})$ are nonzero elements of the space $\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n})$. By the Frobenius reciprocity

$$\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n}) \cong \mathrm{Hom}_{\mathrm{GL}_{2n}(\mathcal{F})}(V_\tau, \mathrm{Ind}_{\mathcal{S}_n(\mathcal{F})}^{\mathrm{GL}_{2n}(\mathcal{F})}(\psi_{\mathcal{S}_n})),$$

any nonzero Shalika functional ℓ_ψ in $\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n})$ gives rise to an embedding

$$V_\tau \hookrightarrow \mathrm{Ind}_{\mathcal{S}_n(\mathcal{F})}^{\mathrm{GL}_{2n}(\mathcal{F})}(\psi_{\mathcal{S}_n}),$$

the image of which is called a local Shalika model of V_τ. Jacquet and Rallis [1996] (and also Nien [2009] by different argument) proved that the local Shalika model is unique for any irreducible admissible representation of $\mathrm{GL}_{2n}(\mathcal{F})$.

Jiang and Qin [2007] introduced the *generalized Shalika model* for $\mathrm{SO}_{4n}(\mathcal{F})$. Let $\nu_1 = 1$ and inductively define

$$\nu_n = \begin{pmatrix} & 1 \\ \nu_{n-1} & \end{pmatrix} \quad \text{for } n \geq 2 \text{ and } n \in \mathbb{N}. \tag{2-1}$$

Let SO_{4n} be the even special orthogonal group attached to the nondegenerate $4n$-dimensional quadratic vector space over $\mathcal{F}$ with respect to ν_{4n}. That is,

$$\mathrm{SO}_{4n} = \{g \in \mathrm{GL}_{4n} \mid {}^t g \cdot \nu_{4n} \cdot g = \nu_{4n}\}.$$

Let $P_{2n} = M_{2n}V_{2n}$ be the Siegel parabolic subgroup of SO_{4n}, consisting of elements of the form

$$(g, X) = \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \begin{pmatrix} \mathrm{I}_n & X \\ & \mathrm{I}_n \end{pmatrix}, \tag{2-2}$$

where $g \in \mathrm{GL}_{2n}$ and $g^* = \nu_{2n}\, {}^t g^{-1} \nu_{2n}$, and X satisfies ${}^t X = -\nu_{2n} X \nu_{2n}$.

The generalized Shalika group $\mathcal{H}_{2n}$ of SO_{4n} is the subgroup of P consisting of elements (g, X) with $g \in \mathrm{Sp}_{2n}$. Here the symplectic group is given by

$$\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid {}^t g \cdot J_{2n} \cdot g = J_{2n}\}, \quad \text{where } J_{2n} = \begin{pmatrix} & \nu_n \\ -\nu_n & \end{pmatrix} \quad \text{for } n \in \mathbb{N}.$$

Define a character $\psi_{\mathcal{H}}$ of $\mathcal{H}_{2n}(\mathcal{F})$ (we write $\mathcal{H} = \mathcal{H}_{2n}$ when n is understood) by letting

$$\begin{aligned} \psi_{\mathcal{H}}((g, X)) &= \psi(\mathrm{tr}(J_{2n} X \nu_{2n})) \\ &= \psi(\mathrm{tr}((\mathrm{diag}(-\mathrm{I}_n, \mathrm{I}_n))X)). \end{aligned}$$

It is well defined. The *generalized Shalika functional* or *$\psi_{\mathcal{H}}$-functional* of an irreducible admissible representation (σ, V_σ) of $\mathrm{SO}_{4n}(\mathcal{F})$ is a nonzero functional in the space

$$\mathrm{Hom}_{\mathrm{SO}_{4n}(\mathcal{F})}(V_\sigma, \mathrm{Ind}_{\mathcal{H}_{2n}(\mathcal{F})}^{\mathrm{SO}_{4n}(\mathcal{F})}(\psi_{\mathcal{H}})) = \mathrm{Hom}_{\mathcal{H}_{2n}(\mathcal{F})}(V_\sigma, \psi_{\mathcal{H}}).$$

Nien [2010] has shown the uniqueness of the generalized Shalika model. Hence one can use a nonzero generalized Shalika functional to define a generalized Shalika model for σ. To relate the Shalika model on GL_{2n} and the generalized Shalika model on SO_{4n}, we consider the following parabolic induction.

For an irreducible, unitary, supercuspidal representation (τ, V_τ) of $\mathrm{GL}_{2n}(\mathcal{F})$, we consider the unitary representation $\mathrm{I}(s, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ induced from the Siegel parabolic subgroup $P_{2n} = M_{2n}V_{2n}$, where the Levi part M_{2n} is isomorphic to GL_{2n}, via the bijection

$$a \in \mathrm{GL}_{2n} \mapsto m(a) := \mathrm{diag}(a, a^*) \in M_{2n}.$$

More precisely, a section $\phi_{\tau,s}$ in $\mathrm{I}(s, \tau)$ is a smooth function from $\mathrm{SO}_{4n}(\mathcal{F})$ to V_τ such that

$$\phi_{\tau,s}(m(a)ng) = |\det a|^{s/2+(2n-1)/2}\tau(a)\phi_{\tau,s}(g),$$

where $m(a) \in M_{2n}$ with $a \in \mathrm{GL}_{2n}(\mathcal{F})$ and $n \in V_{2n}$. In other words, one has

$$\mathrm{I}(s, \tau) = \mathrm{Ind}_{P_{2n}(\mathcal{F})}^{\mathrm{SO}_{4n}(\mathcal{F})}(|\det|^{s/2} \cdot \tau).$$

In the introduction, we used notation $\mathrm{I}^{\mathrm{SO}_{4n}}(s,\tau)$ for $\mathrm{I}(s,\tau)$ as a reminder that it is a representation of SO_{4n}. From now on, we simply use the notation $\mathrm{I}(s,\tau)$.

The relation between the Shalika model on GL_{2n} and the generalized Shalika model on SO_{4n} is given by the following theorem.

Theorem 2.2 [Jiang and Qin 2007, Theorem 3.1]. *The induced representation* $\mathrm{I}(s,\tau)$ *admits a nonzero generalized Shalika functional only when* $s=1$. *In that case*, $\mathrm{I}(1,\tau)$ *admits a nonzero generalized Shalika functional if and only if the supercuspidal datum* τ *admits a nonzero Shalika functional. The generalized Shalika functionals of* $\mathrm{I}(1,\tau)$ *are unique up to scalar, and if nonzero, they must factor through the unique Langlands quotient* $\mathscr{L}(1,\tau)$.

Again from now on we simply use $\mathscr{L}(1,\tau)$ rather than $\mathscr{L}^{\mathrm{SO}_{4n}}(1,\tau)$.

2.3. ***A family of degenerate Whittaker models.*** Degenerate Whittaker models for a reductive group G can be defined for any given nilpotent orbit in the Lie algebra $\mathfrak{g}$ of G; see [Mœglin and Waldspurger 1987]. Here, we consider a family of nilpotent orbits $\mathbb{O}_{2n,2n-k}$ of SO_{4n} corresponding to a family of partitions $[2(2n-k)+1, 1^{2k-1}]$ for $k=1,2,\dots,2n$. This family of degenerate Whittaker models on $\mathrm{SO}_{4n}(\mathscr{F})$ was considered in [Ginzburg et al. 1997] for construction of automorphic L-functions of orthogonal groups, and in [Ginzburg et al. 1999] for construction of the Ginzburg–Rallis–Soudry global descents. We take a family of unipotent subgroups N_k of SO_{4n} consisting of elements of type

$$n=n(u,b,z)=\begin{pmatrix} u & b & z\\ & \mathrm{I}_{4n-2k} & b'\\ & & u'\end{pmatrix}\in \mathrm{SO}_{4n}, \tag{2-3}$$

where $u=(u_{i,j})\in \mathrm{U}_k$, the maximal unipotent subgroup of GL_k consisting of all upper triangular unipotent matrices in GL_k, the block $b=(b_{i,j})$ is the implied size, and b' and u' are determined by b and u so that n belongs to SO_{4n}. We define a character ψ_k on N_k by

$$\psi_k(n):=\psi(u_{1,2}+\cdots+u_{k-1,k})\psi(b_{k,2n-k}+b_{k,2n-k+1}). \tag{2-4}$$

When $k=2n-1$, the subgroup N_k coincides with the unipotent radical N of the Borel subgroup of SO_{4n}, and ψ_k is the generic character of N. Let π be an irreducible admissible representation (π, V_π) of $\mathrm{SO}_{4n}(\mathscr{F})$. Then π has a nonzero ψ_k-functional if

$$\mathrm{Hom}_{\mathrm{SO}_{4n}(\mathscr{F})}(V_\pi, \mathrm{Ind}_{N_k(\mathscr{F})}^{\mathrm{SO}_{4n}(\mathscr{F})}(\psi_k))\cong \mathrm{Hom}_{N_k(\mathscr{F})}(V_\pi,\psi_k)\neq 0. \tag{2-5}$$

In this case, a nonzero element in $\mathrm{Hom}_{N_k(\mathscr{F})}(V_\pi,\psi_k)$ is called a ψ_k-functional of V_π, or more precisely, a ψ_k-degenerate Whittaker functional of V_π. For each

ψ_k-functional ℓ_{ψ_k}, we define

$$\mathcal{W}_{\psi_k,v}(g) := \ell_{\psi_k}(\pi(g)(v)) \quad \text{for } v \in V_\pi, \tag{2-6}$$

which yields a ψ_k-degenerate Whittaker model (also called an (N_k, ψ_k)-model) for V_π. In particular, when $k = 2n-1$, it produces a Whittaker model for V_π. Note that the different choices of the representatives in the $\mathcal{F}$-rational points of the unipotent orbit $\mathbb{O}_{2n,k}(\mathcal{F})$ produce different characters for $N_k(\mathcal{F})$, and hence different degenerate Whittaker models. However, the centralizers are all isomorphic, which is the $\mathcal{F}$-split $\mathrm{SO}_{4n-2k-1}(\mathcal{F})$. This is different from the case of odd orthogonal groups considered in [Jiang and Soudry 2007].

We recall the definition of Jacquet functor and module. Fix a closed subgroup $\widetilde{P} = \widetilde{N} \rtimes \widetilde{M}$ of SO_{4n} with unipotent radical $\widetilde{N}$ and a character χ on $\widetilde{N}$ normalized by $\widetilde{M}$. Then for a representation (V_π, π) of $\mathrm{SO}_{4n}(\mathcal{F})$, its Jacquet module with respect to $(\widetilde{N}, \chi)$ is defined by

$$\mathcal{J}\{\widetilde{N}, \chi\}(\pi) = V_\sigma / \operatorname{Span}\{\sigma(n)v - \chi(n)v \mid n \in \widetilde{N}, v \in V_\pi\},$$

viewed as a representation of $\widetilde{M}$. We call $\mathcal{J}\{\widetilde{N}, \chi\}$ the Jacquet functor with respect to $(\widetilde{N}, \chi)$. We write $\mathcal{J}\{\widetilde{N}\}$ for $\mathcal{J}\{\widetilde{N}, \chi\}$ when χ is trivial. For the family of ψ_k-degenerate Whittaker models, we abbreviate the corresponding family of ψ_k-twisted Jacquet modules by

$$\mathcal{J}\{\psi_k\}(V_\pi) := \mathcal{J}\{N_k, \psi_k\}(V_\pi), \tag{2-7}$$

viewed as a representation of $\mathrm{SO}_{4n-2k-1}(\mathcal{F})$.

Theorem 2.4 [Jiang et al. 2008, Theorem 3.1]. *Suppose (π, V_π) is an irreducible admissible representation of $\mathrm{SO}_{4n}(\mathcal{F})$. If π has a nonzero generalized Shalika model, then the ψ_k-twisted Jacquet modules $\mathcal{J}\{\psi_k\}(V_\pi)$ are all zero for $n \le k \le 2n$.*

For an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, we apply the family of the ψ_k-twisted Jacquet functors to the Langlands quotient $\mathcal{L}(1, \tau)$. By Theorem 2.4, the first interesting representation we get from $\mathcal{L}(1, \tau)$ is at $k = n-1$, that is,

$$\sigma_{n-1} = \sigma_{n-1}(\tau) := \mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1, \tau)), \tag{2-8}$$

which is an admissible representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. We call σ_{n-1} the local descent of τ from GL_{2n} to SO_{2n+1}. The main result of this paper is this:

Theorem 2.5. *Suppose τ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model. Then its local descent σ_{n-1} is irreducible, generic, and a supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.*

We prove Theorem 2.5 in Sections 3, 4, and 5. In Section 3, we prove that the local descent σ_{n-1} as defined in (2-8) is quasisupercuspidal, which means the (nontwisted) Jacquet module $\mathcal{J}\{N\}(\sigma_{n-1})$ is trivial for the unipotent radical N of every standard proper parabolic group of SO_{2n+1}; see Theorem 3.1 for details. Hence we can write the local descent σ_{n-1} as a direct sum

$$\sigma_{n-1} = \sigma_{n-1}^{1} \oplus \cdots \oplus \sigma_{n-1}^{r} \oplus \cdots,$$

where the σ_{n-1}^{i} are irreducible supercuspidal representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$. We show in Theorem 4.1(2) that the local descent σ_{n-1} has a nonzero Whittaker functional, which is unique up to a scalar. Hence among the summands σ_{n-1}^{i}, one and only one has a nonzero Whittaker functional, that is, it is generic. Finally, we prove in Theorem 5.1(2) that every irreducible supercuspidal summand in σ_{n-1} is generic. This implies that the local descent σ_{n-1} has only one irreducible summand, and therefore, σ_{n-1} is irreducible, generic, and supercuspidal, proving Theorem 2.5.

3. Supercuspidality of the local descent

We first prove the quasisupercuspidality of $\sigma_{n-1} = \sigma_{n-1}(\tau) = \mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1, \tau))$, as defined in (2-8) for any irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model.

We relate any standard Jacquet module of σ_{n-1} to further descent σ_k of $\mathcal{L}(1, \tau)$ with $k \geq n$ in the tower of the local Gelfand–Graev models for the Langlands quotient $\mathcal{L}(1, \tau)$. Because $\mathcal{L}(1, \tau)$ has a nonzero generalized Shalika model, all standard Jacquet modules of σ_{n-1} must be zero by Theorem 2.4. The same proof can be used to show that the local descents from $\mathcal{L}(1, \tau)$ satisfy the local tower property as in [Ginzburg et al. 1999], but we omit the details here.

First we have to fix notation. Consider the embedding of elements in SO_{2k-1} into SO_{2k}, so that the embedding of unipotent elements are described explicitly.

Let $n = n(u, b, c)$ be a unipotent element of SO_{2k-1} of type

$$n = n(u, b, c) = \begin{pmatrix} u & b & c \\ & 1 & b' \\ & & u^* \end{pmatrix} \in \mathrm{SO}_{2k-1} \tag{3-1}$$

where u is in U_{k-1}, the maximal upper triangular unipotent subgroup of GL_{k-1}. Then the embedding of n under the embedding from SO_{2k-1} into SO_{2k} is given by

$$n \mapsto \iota(n) = \begin{pmatrix} u & b & -b & c \\ & 1 & 0 & -b' \\ & & 1 & b' \\ & & & u^* \end{pmatrix} \in \mathrm{SO}_{2k}\,. \tag{3-2}$$

Theorem 3.1. *Let τ be an irreducible supercuspidal representation of* $\mathrm{GL}_{2n}(\mathscr{F})$ *with $n \geq 2$, such that $L(s, \tau, \Lambda^2)$ has a pole at $s = 0$. Then $\sigma_{n-1} = \mathscr{J}\{\psi_{n-1}\}(\mathscr{L}(1, \tau))$ is a quasisupercuspidal representation of* $\mathrm{SO}_{2n+1}(\mathscr{F})$.

Proof. For simplicity, we set $\sigma := \mathscr{L}(1, \tau)$, which is an admissible representation of $\mathrm{SO}_{2n+1}(\mathscr{F})$. Denote by U_{n-1} be the maximal (upper triangular) unipotent subgroup of $\mathrm{GL}_{n-1}(\mathscr{F})$. Recall that N_{2n} is the unipotent radical of Siegel parabolic groups of SO_{4n}. For $x \in \mathscr{F}$, denote by $u_{i,j}(x)$ the unipotent matrix in SO_{4n} corresponding to $x(e_i - e_j)$, the x-multiple of root $e_i - e_j$, and let $U_{i,j} = \{u_{i,j}(x) \mid x \in \mathscr{F}\}$.

There are n unipotent radicals Q_k for $1 \leq k \leq n$ corresponding to standard maximal parabolic subgroups of SO_{2n+1}, and given by

$$Q_k = \left\{ \begin{pmatrix} \mathrm{I}_k & C & D \\ & \mathrm{I}_{2n-2k+1} & C^* \\ & & \mathrm{I}_k \end{pmatrix} \right\} \subset \mathrm{SO}_{2n+1}\,.$$

Denote by $\imath$ the embedding of elements of SO_{2n+1} into SO_{2n+2} as in (3-2).

Let $H_1 = \imath(Q_k)N_{n-1}$, and denote its elements by

$$w(r, x, y, A, B) = \begin{pmatrix} r & x & y \\ & \begin{pmatrix} \mathrm{I}_k & A & B \\ & \mathrm{I}_{2n-2k+2} & A^* \\ & & \mathrm{I}_k \end{pmatrix} & x' \\ & & r^* \end{pmatrix} \quad \text{for } r \in \mathrm{U}_{n-1}.$$

Write $r = (r_{i,j})$ and $x = (x_{i,j})$ and so on. Let ψ_{H_1} be the trivial extension of ψ_{n-1} to H_1, that is,

$$\psi_{H_1}(w(r, x, y, A, B)) = \psi(r_{1,2} + \cdots + r_{n-2,n-1})\psi(x_{n-1,n+1} + x_{n-1,n+2}).$$

To show that $\mathscr{J}\{\psi_{n-1}\}(\sigma)$ is supercuspidal, it suffices to show that

$$\mathscr{J}\{\imath(Q_k)\}(\mathscr{J}\{\psi_{n-1}\}(\sigma)) = 0 \quad \text{for all } 1 \leq k \leq n.$$

We begin by assuming to the contrary that $\mathscr{J}\{\imath(Q_k)\}(\mathscr{J}\{\psi_{n-1}\}(\sigma)) \neq 0$ for some $1 \leq k \leq n$. Then there exists a nonzero functional Φ_1 on V_σ such that

$$\Phi_1(\sigma(g)v) = \psi_{H_1}(g)\Phi_1(v) \tag{3-3}$$

holds for $g \in H_1$ and $v \in V_\sigma$.

Let H_2 be the complement of $\prod_{i=1}^{n-1} U_{i,n}$ in H_1, and define a character ψ_{H_2} on H_2 by $\psi_{H_2} = \psi_{H_1}|_{H_2}$. Then $\Phi_1(\sigma(g)v) = \psi_{H_2}(g)\Phi_1(v)$ for $g \in H_2$ and $v \in V_\sigma$. Denote by η the permutation matrix in SO_{4n} corresponding to the permutation product $(1, \ldots, n-1, n)(3n+1, \ldots, 4n)$ of two cycles. Let $H_3 = \eta H_2 \eta^{-1}$ and $\psi_{H_3}(g) = \psi_{H_2}(\eta^{-1} g \eta)$ for $g \in H_3$. Now we have a nontrivial functional Φ_3 on V_σ such that

$\Phi_3(\sigma(g)v) = \psi_{H_3}(g)\Phi_3(v)$ for $g \in H_3$ and $v \in V_\sigma$. Note that the functional Φ_3 is given by $\Phi_3(v) = \Phi_2(\eta v)$ for $v \in V_\sigma$.

Let H_4 be a subgroup of $H_3 \bigcap N_n$, consisting of elements of the form of

$$h = (h_{i,j}) = \begin{pmatrix} \mathrm{I}_n & (0_{n\times(k-1)} \mid *) & * \\ & \mathrm{I}_{2n} & \dfrac{\overset{\frown}{*}}{\underset{\smile}{0}} \\ & & \mathrm{I}_n \end{pmatrix}, \quad \text{with } h_{1,2n} = -h_{1,2n+1}.$$

Let $\psi_{H_4} = \psi_{H_3}|_{H_4}$. That is, $\psi_{H_4}(h) = \psi(h_{n,2n} + h_{n,2n+1})$.

Let $H_5 = U_{1,2n}H_4$ and let ψ_{H_5} be the character of H_5 extending ψ_{H_4} with trivial value on $U_{1,2n}$. For $u_{1,2n}(x) \in U_{1,2n}$, the adjoint action $\mathrm{ad}(u_{1,2n}(x))$ preserves H_4 and ψ_{H_4}. Therefore there exists a character χ on $U_{1,2n}$ and a functional Φ_4 on V_σ such that

$$\Phi_4(\sigma(ug)v) = \chi(u)\psi_{H_4}(g)\Phi_4(v) \tag{3-4}$$

for $u \in U_{1,2n},\ g \in H_4$ and $v \in V_\sigma$.

Assume that $\chi(x) = \psi(ax)$ for some $a \in \mathscr{F}$. Note that

$$\mathrm{ad}(u_{n,1}(-a))u_{1,2n}(x) = u_{1,2n}(x)u_{n,2n}(-ax).$$

Also, $\mathrm{ad}(u_{n,1}(-a))$ preserves both H_4 and ψ_{H_4}. Define $\Phi_5(v) = \Phi_4(u_{n,1}(-a)v)$. Then

$$\Phi_5(\sigma(g)v) = \psi_{H_5}(g)\Phi_5(v) \tag{3-5}$$

for $g \in H_5$ and $v \in V_\sigma$.

Let $X_0 = H_5$ and $\psi^{(0)} = \psi_{H_5}$. For $1 \le m \le n$, let $X_m = U_{m,m+1} \cdots U_{m,n+k-1}$ and write its elements by

$$X_m(\vec{x}) = \mathrm{diag}(r, \mathrm{I}_2, r^*) \quad \text{for } r = (r_{i,j}) \in \mathrm{U}_{2n-1} \text{ and } \vec{x} \in \mathscr{F}^{n+k-m-1},$$

where the m-th row of r is $(0_{m-1}, 1, \vec{x}, 0_{n-k+1})$ and $r_{i,j} = \delta_{i,j}$ for $i \ne m$. Let $\psi^{(m)}$ be the restriction of the character ψ_n of N_n to the subgroup $X_m \cdots X_1 H_5$.

For each $0 \le m \le n$, we claim in general that there exists a nontrivial functional Φ_m on V_σ such that

$$\Phi_m(uv) = \psi^{(m)}(u)\Phi_m(v) \tag{3-6}$$

for $u \in X_m \cdots X_1 H_5$ and $v \in V_\sigma$.

We proceed by induction. For $m = 0$, the claim is true by Equation (3-5). Assume that the claim is true for $0 \le j-1 \le n-2$.

Note that X_j is abelian and that $\mathrm{ad}(X_j(\vec{x}))$ preserves $X_{j-1}\cdots X_1H_5$ and $\psi^{(j-1)}$. Hence there exists a character χ_j on X_j such that

$$\Phi_{j-1}(\sigma(ug)v)=\chi_j(u)\psi^{(j-1)}(g)\Phi_{j-1}(v) \tag{3-7}$$

holds for $u\in X_j,\ g\in X_{j-1}\cdots X_1H_5$ and $v\in V_\sigma$.

Assume that

$$\chi_j(X_j(t_1,\dots,t_{n+k-j-1}))=\psi(a_1t_1+\cdots+a_{n+k-j-1}t_{n+k-j-1})\quad \text{for } a_i\in\mathscr{F}.$$

If $a_i=0$ for all $1\le i\le n+k-j-1$, then Equation (3-7) induces a nontrivial functional on V_τ that is invariant under $\tau(u)$,

$$u\in\left\{\begin{pmatrix}\mathrm{I}_j & *\\ & \mathrm{I}_{2n-j}\end{pmatrix}\in\mathrm{GL}_{2n}\right\}.$$

This contradicts the supercuspidality of τ. Hence there exists a nonzero a_i. Let

$$m_0=\min\{1\le i\le n+k-j-1\mid a_i\neq 0\}.$$

Case 1. If $m_0=1$, define $\Phi_j(v)=\Phi_{j-1}(\tilde{\lambda}v)$, where $\tilde{\lambda}=\mathrm{diag}(\lambda,\lambda^*)\in\mathrm{SO}_{4n}$ and

$$\lambda=\begin{pmatrix}\mathrm{I}_j &&&&&\\ & a_1 &&&&\\ & a_2 & 1 &&&\\ & \vdots && \mathrm{I}_{n+k-j-3} &&\\ & a_{n+k-j-1} &&& 1 &\\ &&&&& \mathrm{I}_{n-k}\end{pmatrix}\in\mathrm{GL}_{2n}\,.$$

Note that

$$\begin{aligned}&\mathrm{ad}(\tilde{\lambda})X_j(t_1,\dots,t_{n+k-j-1})\\ &\qquad=X_j(-a_1^{-1}t_1-a_1^{-1}a_2t_2\cdots-a_1^{-1}a_{n+k-j-1}t_{n+k-j-1},t_2,\dots,t_{n+k-j-1}).\end{aligned}$$

Moreover, $\mathrm{ad}(\tilde{\lambda})$ preserves both $X_{j-1}\cdots X_1H_5$ and $\psi^{(j-1)}$. Hence

$$\Phi_j(\sigma(u)v)=\psi^{(j)}(u)\Phi_j(v)\quad\text{for } u\in X_j\cdots X_1H_5. \tag{3-8}$$

Case 2. If $m_0>1$, take $\theta=u_{j+1,m_0}(1)$ and $\tilde{\theta}=\mathrm{diag}(\theta,\theta^*)\in\mathrm{SO}_{4n}$, and then define $\Phi''_j(v)=\Phi_{j-1}(\tilde{\theta}v)$. Then, for $u\in X_j,\ g\in X_{j-1}\cdots X_1H_5$ and $v\in V_\sigma$,

$$\Phi''_j(\sigma(ug)v)=\chi'(u)\psi^{(j-1)}(g)\Phi''_j(v)$$

holds for some character χ' on X_j satisfying

$$\chi'(X_j(x_1,\dots,x_{n+k-j-1}))=\psi(b_1x_1+\cdots+b_{n+k-j-1}x_{n+k-j-1}),$$

with $b_1 \neq 0$. By repeating the same procedure as in the first case, we again reach the conclusion Equation (3-8).

By induction, we have shown that

$$\Phi_{n-1}(\sigma(u)v) = \psi^{(n-1)}(u)\Phi_{n-1}(v) \quad \text{for } u \in X_{n-1} \cdots X_1 H_5.$$

By similar argument, we also obtain that $\Phi'_n(\sigma(ug)v) = \chi''(u)\psi^{(n-1)}(g)\Phi'_n(v)$, where $u \in X_n$, $g \in X_{n-1} \cdots X_1 H_5$ and $v \in V_\sigma$ holds for some character χ'' on X_n satisfying $\chi''(X_n(x_1, \dots, x_{k-1})) = \psi(d_1 x_1 + \cdots + d_{k-1} x_{k-1})$.

Finally, we take $\Phi_n(v) = \Phi'_n(\mathrm{diag}(\gamma, \gamma^*)v)$ for $v \in V_\sigma$, where

$$\gamma = \begin{pmatrix} \mathrm{I}_n & & \\ & \mathrm{I}_{k-1} & \begin{pmatrix} 0, \dots, d_1 \\ \vdots \\ 0, \dots, d_{k-1} \end{pmatrix} \\ & & \mathrm{I}_{n-k+1} \end{pmatrix} \in \mathrm{GL}_{2n},$$

and obtain that $\Phi_n(\sigma(u)v) = \psi^{(n)}(u)\Phi_n(v)$ for $u \in X_n \cdots X_1 H_5$ and $v \in V_\sigma$. Since $N_n = X_n \cdots X_1 H_5$, this gives a nontrivial ψ_n-functional on V_σ, contradicting Theorem 2.4's conclusion that generalized Shalika models and (N_n, ψ_n)-models are disjoint. The initial assumption must be false, so

$$\mathscr{J}\{\imath(Q_k)\}(\mathscr{J}\{\psi_{n-1}\}(\sigma)) = 0 \quad \text{for all } 1 \leq k \leq n$$

and $\mathscr{J}\{\psi_{n-1}\}(\sigma)$ is quasisupercuspidal. □

4. Genericity of the local descent

By Theorem 3.1, the local descent $\sigma_{n-1} = \sigma_{n-1}(\tau) = \mathscr{J}\{\psi_{n-1}\}(\mathscr{L}(1, \tau))$ as defined in (2-8) is a quasisupercuspidal representation of $\mathrm{SO}_{2n+1}(\mathscr{F})$. We may write

$$\sigma_{n-1} = \sigma^1_{n-1} \oplus \cdots \oplus \sigma^r_{n-1} \oplus \cdots,$$

where the σ^i_{n-1} are irreducible supercuspidal representations of $\mathrm{SO}_{2n+1}(\mathscr{F})$. Note that τ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathscr{F})$ with a nonzero Shalika model.

With regard to the Whittaker functional of $\sigma_{n-1} = \mathscr{J}\{\psi_{n-1}\}(\mathscr{L}(1, \tau))$, recall from (2-3) and (2-4) that

$$N_{n-1} = \left\{ n(z, x, y) = \begin{pmatrix} z & x & y \\ & \mathrm{I}_{2n+2} & x' \\ & & z' \end{pmatrix} \;\middle|\; z \in \mathrm{U}_{n-1} \right\} \subset \mathrm{SO}_{4n} \tag{4-1}$$

and the character ψ_{n-1} of N_{n-1} is given by

$$\psi_{n-1}(n(z, x, y)) = \psi(z_{1,2} + \cdots + z_{n-2,n-1})\psi(x_{n-1,n+1} + x_{n-1,n+2}).$$

As in (2-7), the twisted Jacquet module $\sigma_{n-1} = \mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1,\tau))$ is a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. Let Z_k be the standard maximal unipotent subgroup of the split special orthogonal group SO_k consisting of upper-triangular matrices with 1 along the diagonals. That is,

$$Z_{2n+1} = \left\{ z(u,b,w) = \begin{pmatrix} u & b & w \\ & 1 & b' \\ & & u' \end{pmatrix} \in \mathrm{SO}_{2n+1} \;\middle|\; u = (u_{i,j}) \in \mathrm{U}_n \right\}. \tag{4-2}$$

We may write $b = (b_1, \dots, b_n)^t \in \mathcal{F}^n$. The Whittaker character $\psi_{Z_{2n+1}}$ of Z_{2n+1} is defined by

$$\psi_{Z_{2n+1}}(z(u,b,w)) = \psi(u_{1,2} + \cdots + u_{n-1,n} - b_n). \tag{4-3}$$

By the Frobenius reciprocity law, in order to show that σ_{n-1} has a nonzero Whittaker functional, it suffices to show that the twisted Jacquet module

$$\mathcal{J}\{Z_{2n+1}, \psi_{Z_{2n+1}}\}(\sigma_{n-1}) = \mathcal{J}\{Z_{2n+1}, \psi_{Z_{2n+1}}\}(\mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1,\tau)))$$

is nonzero.

To compose the two twisted Jacquet functors $\mathcal{J}\{Z_{2n+1}, \psi_{Z_{2n+1}}\}$ and $\mathcal{J}\{\psi_{n-1}\}$, we set $E_1 = \tilde{\imath}(Z_{2n+1})N_{n-1}$ and let ψ_{E_1} be the character of E_1 defined by

$$\psi_{E_1}(vn) = \psi_{Z_{2n+1}}(v)\psi_{n-1}(n) \quad \text{for } v \in Z_{2n+1} \text{ and } n \in N_{n-1},$$

where $\tilde{\imath} : \mathrm{SO}_{2k+1} \hookrightarrow \mathrm{SO}_{4n}$ is given by

$$g \in \mathrm{SO}_{2k+1} \mapsto \tilde{\imath}(g) = \mathrm{diag}(\mathrm{I}_{2n-k-1}, \imath(g), \mathrm{I}_{2n-k-1})$$

for any $k = 0, 1, \dots, 2n-1$, and the embedding $\imath$ is defined in (3-2). Hence

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) = \mathcal{J}\{Z_{2n+1}, \psi_{Z_{2n+1}}\} \circ \mathcal{J}\{\psi_{n-1}\}(V_\pi)$$

for any irreducible admissible representation (π, V_π) of $\mathrm{SO}_{4n}(\mathcal{F})$.

We put $k = 2n$ in the maximal unipotent subgroup of SO_{4n} defined in (2-3), so that

$$N_{2n} = \left\{ n(z,y) = \begin{pmatrix} z & y \\ & z' \end{pmatrix} \;\middle|\; z \in \mathrm{U}_{2n} \right\}. \tag{4-4}$$

Define a degenerate character $\tilde{\psi}$ of N_{2n} by

$$\tilde{\psi}(n(z,y)) = \psi(z_{1,2} + \cdots + z_{2n-1,2n}).$$

We define the twisted Jacquet module $\mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi)$ for any irreducible admissible representation (π, V_π) of $\mathrm{SO}_{4n}(\mathcal{F})$.

Theorem 4.1. *Let π be an irreducible smooth representation of* SO_{4n} *that admits a nonzero generalized Shalika model.*

(1) *There exists a vector space isomorphism between the two twisted Jacquet modules*, *that is*,

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi).$$

(2) *The local descent σ_{n-1} has a nonzero Whittaker functional*, *which is unique up to a scalar.*

Proof. The proof of (1) needs to use the local version of the Fourier expansion for representations, in particular, the [Ginzburg et al. 1999, General Lemma]. We treat the various cases in Sections 4.2–4.12.

We show here that (2) follows from (1). Take π to be $\mathcal{L}(1, \tau)$ and consider $\mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi) = \mathcal{J}\{N_{2n}, \tilde{\psi}\}(\mathcal{L}(1, \tau))$. We may write $N_{2n} = \mathrm{U}_{2n} \ltimes V_{2n}$, where V_{2n} is the unipotent radical of the Siegel parabolic subgroup P_{2n} of SO_{4n} as defined in (2-2). Then we decompose the twisted Jacquet functor as

$$\mathcal{J}\{N_{2n}, \tilde{\psi}\} = \mathcal{J}\{\mathrm{U}_{2n}, \psi_{\mathrm{U}_{2n}}\}^{\mathrm{GL}_{2n}} \circ \mathcal{J}\{V_{2n}\}$$

where the left part of the composition is the Whittaker functor of GL_{2n} and the right is the nontwisted Jacquet functor (that is, the constant term functor along V_{2n}).

Consider first $\mathcal{J}\{V_{2n}\}(\mathcal{L}(1, \tau))$. By [Bernstein and Zelevinsky 1977, Geometric Lemma], we obtain that

$$\mathcal{J}\{V_{2n}\}(\mathcal{L}(1, \tau)) \simeq \tau \otimes |\det|^{-1/2}$$

as representations of $\mathrm{GL}_{2n}(\mathcal{F})$. By the local uniqueness of Whittaker model of τ, we see that the space

$$\mathcal{J}\{\mathrm{U}_{2n}, \psi_{\mathrm{U}_{2n}}\}^{\mathrm{GL}_{2n}} \circ \mathcal{J}\{V_{2n}\}(\mathcal{L}(1, \tau))$$

is one-dimensional. Therefore, $\mathcal{J}\{E_1, \psi_{E_1}\}(\mathcal{L}(1, \tau))$ is one-dimensional by (1); in particular, the local descent σ_{n-1} has a unique Whittaker functional. □

4.2. We start to prove (1) of Theorem 4.1 by constructing a few intermediate twisted Jacquet modules relating $\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi)$ and $\mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi)$. The relations are explained in terms of the local versions of Fourier expansions for representations; this is called the General Lemma in [Ginzburg et al. 1999], and also here.

In this subsection and Section 4.3, we consider the general case when (π, V_π) is any smooth representation of $\mathrm{SO}_{4n}(\mathcal{F})$.

Let

$$C_1 = \{\tilde{\imath}(v)n \mid v \in Z_{2n+1},\ n = n(z, x, y) \text{ such that } x_{n-1,1} = 0\}.$$

Let $\psi_{C_1} = \psi_{E_1}|_{C_1}$. For $i = 1, \dots, n$, let

$$X_i = \left\{ \begin{pmatrix} \mathrm{I}_{n-1} & x & 0 \\ & \mathrm{I}_{2n+2} & x' \\ & & \mathrm{I}_{n-1} \end{pmatrix} \in N_{n-1} \;\middle|\; x_{s,t} \in \delta_{s,n-1}\delta_{t,i} \cdot \mathscr{F} \right\},$$

where $\delta_{i,j}$ is defined by that $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$. For $i = 1, \dots, n-1$, set

$$Y_i = \{\mathrm{I}_{4n} + \lambda E_{n+i-1,2n+1} - \lambda E_{2n,3n+2-i} \mid \lambda \in \mathscr{F}\} \subset \mathrm{SO}_{4n},$$

where $E_{i,j} = (e_{k,l})$, $e_{k,l} = \delta_{k,i}\delta_{l,j}$, and set

$$Y_n = \left\{ \begin{pmatrix} \mathrm{I}_{2n-2} & & \\ & h & \\ & & \mathrm{I}_{2n-2} \end{pmatrix} \;\middle|\; h = \begin{pmatrix} 1 & x & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \right\} \subset \mathrm{SO}_{4n}\,.$$

Note that X_1 is the complement of C_1 in E_1, that is, $E_1 = C_1 \rtimes X_1$. Let $D_1 = C_1 \rtimes Y_1$, and letψ_{D_1} be the trivial extension of ψ_{C_1} to D_1. This forms a setting which for which the General Lemma applies. Hence we have

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{D_1, \psi_{D_1}\}(V_\pi).$$

For $i = 2, \dots, n$, define a series of subgroups C_i of $Z_{2n+2}N_{n-1}$ by

$$C_i = \left\{ vn \;\middle|\; v = \begin{pmatrix} u & t & w \\ & \iota(h) & t' \\ & & u' \end{pmatrix} \in Z_{2n+2}, \quad \begin{array}{c} u \in \mathrm{U}_{i-1},\ h \in Z_{2n+3-2i}, \\ n = n(z, x, y) \in N_{n-1}, \\ x_{n-1,1} = x_{n-1,2} = \cdots = x_{n-1,i} = 0 \end{array} \right\},$$

where Z_{2n+2} is identified with its embedding in the middle diagonal part of SO_{4n}. Let ψ^i be the character of C_i defined by

$$\psi^i(vn) = \psi_{n-1}(n)\psi(u_{1,2} + \cdots + u_{i-2,i-1} + t_{i-1,1})\psi_{Z_{2n+3-2i}}(h).$$

Then X_i and Y_i both normalize C_i and ψ^i. The trivial extensions of ψ^i to $C_i \rtimes X_i$ and $C_i \rtimes Y_i$ are still denoted by ψ^i. Let $D_i := C_i \rtimes Y_i$. Then $D_{i-1} \simeq C_i \rtimes X_i$ for $i = 2, \dots n$ and the characters ψ^{i-1} and ψ^i of D_{i-1} are equal. Again, this is the setting of the General Lemma, and we obtain

$$\mathcal{J}\{D_{i-1}, \psi^{i-1}\}(V_\pi) \simeq \mathcal{J}\{D_i, \psi^i\}(V_\pi) \quad \text{for } i = 2, \dots, n.$$

Hence we obtain a vector space isomorphism of twisted Jacquet modules:

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{D_n, \psi^n\}(V_\pi).$$

Note that

$$D_n = \left\{ \begin{pmatrix} z & y & w \\ & h & y' \\ & & z' \end{pmatrix} \;\middle|\; \begin{array}{l} h = \begin{pmatrix} 1 & \tilde{f} & -f & w \\ & 1 & 0 & f \\ & & 1 & -\tilde{f} \\ & & & 1 \end{pmatrix} \in Z_4, \\ z \in \mathrm{U}_{2n-2} \text{ with } z_{n-1,i} = 0 \text{ for } i \geq n \end{array} \right\} \subset Z_{4n}.$$

Then we also have the isomorphism $\mathcal{J}\{D_n, \psi^n\}(V_\pi) \simeq \mathcal{J}\{D_n, \psi_{D_n}\}(V_\pi)$ of vector spaces, where the character ψ_{D_n} of D_n is given by

$$\psi_{D_n}(v) = \psi(z_{1,2} + z_{2,3} + \cdots + z_{2n-3,2n-2} + y_{n-1,2} + y_{n-1,3} - f).$$

4.3. Let ν be the permutation matrix in GL_{2n} given by

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & \dots & 2n-1 & 2n \\ 2 & 4 & \dots & 2(n-1) & 1 & 3 & \dots & 2n-1 & 2n \end{pmatrix},$$

and identify it with its embedding $m(\nu)$, where $m : g \in \mathrm{GL}_{2n} \mapsto \mathrm{diag}(g, g^*) \in \mathrm{SO}_{4n}$. Let $E = \nu D_n \nu^{-1}$, and define a character ψ_E of E by

$$\psi_E(n) := \psi_{D_n}(\nu^{-1} n \nu) \quad \text{for } n \in E.$$

Let $T(n)$ be the subgroup of GL_{2n} consisting of certain elements $t = (t_{i,j})$, as follows: Let $\bar{t}_j = (t_{j+1,j}, \dots, t_{2n,j})^t$ and $t_i = (t_{i,i+1}, \dots, t_{i,2n})$ for $i, j \leq 2n - 1$.

- For $1 \leq i \leq 2n$, require $t_{i,i} = 1$.
- For $j \leq n-2$, require that the (single-element) rows of $\bar{t}_{2j-1}$ alternate between arbitrary and zero, except for the last 4, which are all zero; require that $\bar{t}_{2n-3}$ and $\bar{t}_{2n-1}$ vanish.
- For $j \leq n$, require that $\bar{t}_{2j}$ vanishes.
- For $i \leq n$, require that $t_{2i-1} = (0 \;\; * \;\; 0 \;\; * \;\; \dots \;\; * \;\; 0 \;\; * \;\; *)$.
- Require $t_{2(n-1)} = (0, *)$.

Then

$$E = \left\{ n = \begin{pmatrix} t & X \\ & t' \end{pmatrix} \;\middle|\; t \in T(n) \right\}$$

and the character ψ_E is given by

$$\psi_E(n) = \psi(t_{1,3} + t_{2,4} + \cdots + t_{2n-3,2n-1} + t_{2n-2,2n} + x_{2n-2,1} + x_{2n-1,1}). \tag{4-5}$$

Example 4.4. In the case of $n = 4$,

$$T(4) = \left\{ \begin{pmatrix} 1 & 0 & * & 0 & * & 0 & * & * \\ * & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & 0 & * & * \\ * & 0 & * & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_8 .$$

Since $\psi_E(n) = \psi_{D_n}(\nu^{-1} n \nu)$ for all $n \in E$, we have the vector space isomorphism

$$\mathcal{J}\{D_n, \psi_{D_n}\}(V_\pi) \simeq \mathcal{J}\{E, \psi_E\}(V_\pi).$$

Next, we will apply the General Lemma to fill the zeros of t_{2i-1} from right to left, using $\bar{t}_{2i-1}$.

Let

$$\begin{aligned} Y^{i,1} &= \{m(\mathrm{I}_{2n} + yE_{2i,1}) \mid y \in \mathcal{F}\} \quad \text{for } i = 1, \dots, n-2, \\ X^{1,j} &= \{m(\mathrm{I}_{2n} + xE_{1,2j}) \mid x \in \mathcal{F}\} \quad \text{for } j = 2, \dots, n-1, \\ E^{i,1} &= \{n \in E \mid n_{j,1} = 0, \forall j > 2i\} \cdot \textstyle\prod_{j=i+2}^{n} X^{1,j} \quad \text{for } i \le n-3, \\ E^{n-2,1} &= E, \\ C^{i,1} &= \{n \in E^{i,1} \mid n_{2i,1} = 0\}, \quad D^{1,i+1} = C^{i,1} X^{1,i+1}, \quad A^{1,i+1} = D^{1,i+1} Y^{i,1}. \end{aligned}$$

Define a series of characters $\psi^{i,1} = \psi_E|_{C^{i,1}}$. Extend $\psi^{i,1}$ trivially to $D^{1,i+1}$ as $\psi^{i,1}_{D^{1,i+1}}$ and to $E^{i,1}$ as $\psi^{i,1}_{E^{i,1}}$. Note that

$$D^{1,i+1} = E^{i-1,1} \quad \text{and} \quad \psi^{i,1}_{D^{1,i+1}}|_{C^{i-1,1}} = \psi^{i-1,1}.$$

By the General Lemma, we have vector space isomorphisms

$$\mathcal{J}\{E^{i,1}, \psi^{i,1}_{E^{i,1}}\}(V_\pi) \simeq \mathcal{J}\{D^{1,i+1}, \psi^{i,1}_{D^{1,i+1}}\}(V_\pi) \simeq \mathcal{J}\{E^{i-1,1}, \psi^{i-1,1}_{E^{i-1,1}}\}(V_\pi)$$

for $i = n-2, \dots, 2$. In particular, we have

$$\mathcal{J}\{E, \psi_E\}(V_\pi) \simeq \mathcal{J}\{D^{1,2}, \psi^{1,1}_{D^{1,2}}\}(V_\pi).$$

Note that the GL_{2n} part of $D^{1,2}$ looks like $\left(\begin{smallmatrix} \mathrm{I}_2 & * \\ 0 & T' \end{smallmatrix}\right)$ with $T' \in T(n-1)$. Now let

$$\begin{aligned} Y^{r,s} &= \{m(\mathrm{I}_{2n} + yE_{2r,2s-1}) \mid y \in \mathcal{F}\} \quad \text{for } 1 \le r, s \le n-2, \\ X^{r,s} &= \{m(\mathrm{I}_{2n} + xE_{2r-1,2s}) \mid x \in \mathcal{F}\} \quad \text{for } 1 \le r \le n-2 \text{ and } 1 \le s \le n-1. \end{aligned}$$

For $1 \le j \le i \le n-2$, we define

$$E^{i,j} = \tilde{E}^{i,j} \prod_{s=i+2}^{n-1} X^{j,s}, \quad \text{where } \tilde{E}^{i,j} = \left\{ \begin{pmatrix} t & X \\ & t' \end{pmatrix} \in \mathrm{SO}_{4n} \right\},$$

where $t_{\ell,2j-1} = 0$ for all $\ell > 2i$ and otherwise is of the form

$$t = \begin{pmatrix} \mathrm{I}_2 & & * & * \\ & \ddots & & \\ & & \mathrm{I}_2 & * \\ & & & Z \end{pmatrix}, \quad \text{with } Z \in T(n-j+1),$$

We further define

$$C^{i,j} = \left\{ n = \begin{pmatrix} t & X \\ & t' \end{pmatrix} \in E^{i,j} \;\middle|\; t_{2i,2j-1} = 0 \right\},$$
$$D^{j,i+1} = C^{i,j} X^{j,i+1}, \quad A^{j,i+1} = D^{j,i+1} Y^{i,j}.$$

We also define $\psi^{i,j} = \psi_E|_{C^{i,j}}$. Note that $D^{j,i+1} \simeq A^{i-1,j}$ for $i \ge j+1$ and that $D^{j,j+1} \simeq A^{n-1,j+1}$. The relations among those $\psi^{i,j}$ and their trivial extensions $\psi^{i,j}_{D^{j,i+1}}$ and $\psi^{i,j}_{A^{i,j}}$ to $D^{j,i+1}$ and $A^{i,j}$, respectively, are compatible in the sense of the General Lemma. We then have vector space isomorphisms

$$\begin{aligned} \mathcal{J}\{E, \psi\}(V_\pi) \simeq \mathcal{J}\{D^{1,2}, \psi^{1,1}_{D^{1,2}}\}(V_\pi) \simeq \cdots \simeq \mathcal{J}\{D^{j,j+1}, \psi^{j,j}_{D^{j,j+1}}\}(V_\pi) \\ \simeq \cdots \simeq \mathcal{J}\{D^{n-2,n-1}, \psi^{n-2,n-2}_{D^{n-2,n-1}}\}(V_\pi). \end{aligned}$$

Denote by B_n the standard Borel subgroup of GL_n. The subgroup $D^{n-2,n-1}$ consists of elements of the form

$$\begin{pmatrix} t & X \\ & t' \end{pmatrix} \in \mathrm{SO}_{4n}, \quad \text{with } t = \begin{pmatrix} \mathrm{I}_2 & y_1 & * & \cdots & * \\ & \mathrm{I}_2 & y_2 & \cdots & * \\ & & \ddots & & \\ & & & \mathrm{I}_2 & y_{n-1} \\ & & & & z \end{pmatrix},$$

where $y_1, \ldots, y_{n-2} \in \mathrm{Mat}_2$, $y_{n-1} \in \mathrm{B}_2$ and $z \in \mathrm{U}_2$. The character $\psi^{n-2,n-1}_{D^{n-2,n-1}}$ is given by

$$\psi^{n-2,n-2}_{D^{n-2,n-1}}(n) = \psi(\mathrm{tr}(y_1 + \cdots + y_{n-1}))\psi(x_{2n-2,1} + x_{2n-1,1}). \tag{4-6}$$

Proposition 4.5. *Let π be a smooth representation of* SO_{4n}. *Then there exists a vector space isomorphism between two twisted Jacquet modules given by*

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{D^{n-2,n-1}, \psi^{n-2,n-1}_{D^{n-2,n-2}}\}(V_\pi).$$

So far we have only assumed π to be a smooth representation of $\mathrm{SO}_{4n}(\mathcal{F})$.

4.6. The next step is to eliminate the character place $x_{2n-2,1}$ in (4-6). We need two auxiliary results, Propositions 4.7 and 4.11. We assume that V_π is an irreducible admissible representation of $\mathrm{SO}_{4n}(\mathcal{F})$ with a nonzero generalized Shalika model.

We define

$$D = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \;\middle|\; T = \begin{pmatrix} t_1 & z_1 & \dots & \dots & * \\ & t_2 & z_2 & \dots & * \\ & & \dots & & \dots \\ & & & t_{n-1} & z_{n-1} \\ & & & & t_n \end{pmatrix}, t_i \in \mathrm{U}_2,\ z_i \in \mathrm{B}_2 \right\}, \tag{4-7}$$

and a character $\psi_D(n) = \psi(\mathrm{tr}(z_1 + \dots + z_{n-1}) + x_{2n-2,1} + x_{2n-1,1})$ of D.

Proposition 4.7. *Let π be an irreducible smooth representation of SO_{4n} admitting a nonzero generalized Shalika model. Then there exists a vector space isomorphism*

$$\mathcal{J}\{D^{n-2,n-1}, \psi^{n-1,n-1}_{D^{n-2,n-1}}\}(V_\pi) \simeq \mathcal{J}\{D, \psi_D\}(V_\pi).$$

Proof. After applying the General Lemma $n-2$ times, we have the vector space isomorphisms

$$\mathcal{J}\{D^{n-2,n-1}, \psi^{n-1,n-1}_{D^{n-2,n-1}}\}(V_\pi) \simeq \mathcal{J}\{H_1, \psi_{H_1}\}(V_\pi),$$

where

$$H_1 = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \;\middle|\; T = \begin{pmatrix} \mathrm{I}_2 & z_1 & \dots & \dots & * \\ & t_2 & z_2 & \dots & * \\ & & \dots & \dots & \\ & & & t_{n-1} & z_{n-1} \\ & & & & t_n \end{pmatrix}, t_i \in \mathrm{U}_2,\ z_i \in \mathrm{B}_2 \right\},$$

$$\psi_{H_1}(n) = \psi(\mathrm{tr}(z_1 + \dots + z_{n-1}) + x_{2n-2,1} + x_{2n-1,1}) \quad \text{for } n \in H_1.$$

Note that the group

$$m\left(\left\{\left(\begin{array}{cc:c} 1 & * & 0 \\ & 1 & 0 \\ \hdashline & & \mathrm{I}_{2n-2} \end{array}\right)\right\}\right) \subset m(\mathrm{GL}_{2n}) \subset \mathrm{SO}_{4n}$$

normalizes H_1 and ψ_{H_1}.

For $\lambda \in \mathcal{F}^*$, define a character $\psi'_{D,\lambda}$ of D by

$$\psi'_{D,\lambda}(n) = \psi(\mathrm{tr}(z_1 + \dots + z_{n-1}) + x_{n-2,1} + x_{n-1,1})\psi(\lambda t),$$

where $t_1 = \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)$ as in H_1. By the conclusion of the next lemma, Lemma 4.8, the only twisted Jacquet module that remains is the one corresponding to $\lambda = 0$. In this case we have $\psi'_{D,0} = \psi_D$, and therefore $\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{D, \psi_D\}(V_\pi)$. □

Lemma 4.8. *Assume that π is an irreducible representation of* SO_{4n} *admitting a nonzero generalized Shalika model. Then*

$$\mathcal{J}\{D, \psi'_{D,\lambda}\}(V_\pi) = 0 \quad \textit{for all } \lambda \in \mathcal{F}^*.$$

Proof. First we consider the case of $\lambda = 1$. Let $\psi'_D := \psi'_{D,1}$. Then for $n = \left(\begin{smallmatrix} T & X \\ & T' \end{smallmatrix}\right) \in D$ we have

$$\psi'_D(n) = \psi(T_{1,2} + T_{1,3} + \textstyle\sum_{i=2}^{n-2} T_{i,i+2})\psi(x_{2n-2,1} + x_{2n-1,1}).$$

Let

$$z_1 = \mathrm{diag}(Z, \mathrm{I}_{2n-3}) \in \mathrm{GL}_{2n}, \qquad \text{with } Z = \begin{pmatrix} 1 & & \\ & 1 & 0 \\ & 1 & 1 \end{pmatrix}.$$

Then z_1 normalizes D. Let $\psi_{D,1}$ be the character of D defined by

$$\psi_{D,1}(n) = \psi'_D(z_1 n z_1^{-1}) = \psi(T_{1,2} + T_{2,4} + \textstyle\sum_{i=2}^{n-2} T_{i,i+2})\psi(x_{2n-2,1} + x_{2n-1,1}). \tag{4-8}$$

Clearly there exists a vector space isomorphism

$$\mathcal{J}\{D, \psi'_D\}(V_\pi) \simeq \mathcal{J}\{D, \psi_{D,1}\}(V_\pi). \tag{4-9}$$

For $i = 2, \ldots, n-1$, let $z_i = \mathrm{I}_{2n} + E_{2i+1,2i} \in \mathrm{GL}_{2n}$, and let $\psi_{D,i}$ be the character of D defined by $\psi_{D,i}(n) := \psi_{D,i-1}(z_i n z_i^{-1})$. Then we have

$$\psi_{D,i}(n) = \begin{cases} \psi(T_{1,2} + T_{2i,2i+3} + \sum_{j=2}^{n-2} T_{j,j+2})\psi(x_{2n-2,1} + x_{2n-1,1}) & \\ & \text{if } 2 \le i \le n-2, \\ \psi(T_{1,2} + \sum_{j=2}^{n-2} T_{j,j+2})\psi(x_{2n-1,1} + 2x_{2n-2,1}) & \text{if } i = n-1. \end{cases}$$

It is clear that

$$\mathcal{J}\{D, \psi_{D,i}\}(V_\pi) \simeq \mathcal{J}\{D, \psi_{D,i+1}\}(V_\pi) \quad \text{for } i = 2, \ldots, n-2. \tag{4-10}$$

From (4-9) and (4-10), we have the vector space isomorphism

$$\mathcal{J}\{D, \psi'_D\}(V_\pi) \simeq \mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi).$$

Now we assume to the contrary that

$$\mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi) \neq 0. \tag{4-11}$$

Then by the Frobenius reciprocity law, there exists a nonzero functional ℓ on V_π such that

$$\ell(\pi(n)v) = \psi_{D,n-1}(n)\ell(v) \quad \text{for } n \in D \text{ and } v \in V_\pi. \tag{4-12}$$

Such a functional ℓ on V_π factors through $\mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi)$. Hence the nonvanishing of $\mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi)$ is equivalent to the nonvanishing of such ℓ.

Let μ be the permutation matrix in GL_{2n} given by

$$\begin{array}{lll} \mu(1)=1, & \mu(2i-2)=i & \text{for } i=2,\dots,n, \\ \mu(2n)=2n, & \mu(2i-1)=n+i-1 & \text{for } i=2,\dots,n, \end{array}$$

which can be identified with its embedding $m(\mu)$ in SO_{4n}. Denote by Ni_k the set of nilpotent elements in GL_k. Then

$$F := \mu D\mu^{-1} = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \,\middle|\, T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \begin{array}{l} \alpha,\delta,\gamma \in \mathrm{B}_n \cap \mathrm{Ni}_n,\ \beta \in \mathrm{B}_n, \\ \gamma_{i,i+1} = 0 \ \text{ for } i=1,\dots,n-1 \end{array} \right\}.$$

Example 4.9. When $n=4$, the T in F are of the form

$$\left(\begin{array}{cccc:cccc} 1 & * & * & * & * & * & * & * \\ 0 & 1 & * & * & 0 & * & * & * \\ 0 & 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ \hdashline 0 & 0 & * & * & 1 & * & * & * \\ 0 & 0 & 0 & * & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Let ψ_F be the character of F defined by

$$\psi_F(n) = \psi_{D,n-1}(\mu^{-1}n\mu) = \psi\big(\textstyle\sum_{i=1,i\neq n}^{2n-2} T_{i,i+1} + T_{n,2n} + 2X_{n,1} + X_{2n-1,1}\big).$$

Define a linear functional on V_π by $\ell_F(v) = \ell(\pi(\mu^{-1})v)$ for $v \in V_\pi$. Then ℓ_F is a nonzero functional on V_π satisfying $\ell_F(\pi(n)v) = \psi_F(n)\ell_F(v)$ for $n \in F$. Since ℓ_F factors through $\mathcal{J}\{F, \psi_F\}(V_\pi)$, the latter must be nonzero.

Again, by the General Lemma, we get $\mathcal{J}\{F, \psi_F\}(V_\pi) \simeq \mathcal{J}\{F', \psi_{F'}\}(V_\pi)$, where

$$F' = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \,\middle|\, T \in \mathrm{U}_{2n},\ T_{n,n+i} = 0 \ \text{ for } i=1,\dots,n-1 \right\}$$

and the character $\psi_{F'}$ is given by

$$\psi_{F'}(n) = \psi\big(\textstyle\sum_{i=1,i\neq n}^{2n-2} T_{i,i+1} + T_{n,2n} + 2X_{n,1} + X_{2n-1,1}\big). \tag{4-13}$$

Example 4.10. The T in F' are of the form

$$\left(\begin{array}{cccc:cccc} 1 & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ \hdashline 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

(Compare this form with the one in Example 4.9 to see how the General Lemma works.)

Since $\mathcal{J}\{F, \psi_F\}(V_\pi) \simeq \mathcal{J}\{F', \psi_{F'}\}(V_\pi) \neq 0$, there is a nonzero linear functional $\ell_{F'}$ on V_π such that $\ell_{F'}(\pi(n)\, v) = \psi_{F'}(n)\ell_{F'}(v)$ for $n \in F'$.

Next, we consider the intersection $F'_n := F' \cap N_n$. Then

$$F'_n = \left\{ \begin{pmatrix} \alpha & \beta & x & y \\ & \mathrm{I}_n & 0 & x' \\ & & \mathrm{I}_n & \beta' \\ & & & \alpha' \end{pmatrix} \,\middle|\, \begin{array}{l} \alpha \in \mathrm{U}_n,\ \beta \in \mathrm{B}_n, \\ \beta_{n,i} = 0 \ \text{ for } i = 1, \dots, n-1 \end{array} \right\}. \tag{4-14}$$

and $\ell_{F'}$ is a nonzero linear functional on V_π such that

$$\ell_{F'}(\pi(n)\, v) = \psi_{F'}(n)\ell_{F'}(v) \quad \text{for } n \in F'_n.$$

Note that F'_n differs from N_n by the requirements on their elements at the β entries of (4-14). Now we will apply the local version of Fourier expansion to "fill the zeros of β".

Define a series of subgroups $F'_n \subset F'_{n-1} \subset \cdots \subset F'_1 = N_n$ as follows. Let

$$F'_i = \left\{ \begin{pmatrix} \alpha & \beta & x & y \\ & \mathrm{I}_n & 0 & x' \\ & & \mathrm{I}_n & \beta' \\ & & & \alpha' \end{pmatrix} \in N_n \,\middle|\, \alpha \in \mathrm{U}_n,\ \beta_{n,j} = 0 \ \text{ for } j = 1, \dots, i-1 \right\}. \tag{4-15}$$

Let $\psi_{F'_i}$ be the character of F'_i defined by the same formula of (4-13), that is, by

$$\psi_{F'_i}(n) = \psi(\alpha_{1,2} + \cdots + \alpha_{n-1,n} + \beta_{n,2n} + 2x_{n,1}).$$

Now we use induction in reversed order. The case of $i = n$ is shown in (4-14). Assume for some $2 \le i \le n$ that we have a nonzero linear functional ℓ_i on V_π

satisfying the quasiinvariance property

$$\ell_i(\pi(n)\,v) = \psi_{F'_i}(n)\ell_i(v) \quad \text{for } n \in F'_i. \tag{4-16}$$

We show that the functional ℓ_{i-1} is an extension of ℓ_i such that (4-16) holds with i replaced by $i-1$.

Note that the root group of $e_n - e_{i-1}$ normalizes the character $\psi_{F'_i}$. There are two possibilities:

(i) The ℓ_i having the $(F'_i, \psi_{F'_i})$-quasiinvariance property can be trivially extended to ℓ_{i-1} with the $(F'_{i-1}, \psi_{F'_{i-1}})$-quasiinvariance property, and we are done.

(ii) The ℓ_i can be nontrivially extended to a nonzero linear functional ℓ'_{i-1} with the $(F'_{i-1}, \psi_{F'_{i-1}})$-quasiinvariance property, such that

$$\ell'_{i-1}(\pi(n)\,v) = \tilde{\psi}_{F'_{i-1}}(n)\ell'_{i-1}(v) \quad \text{for } n \in F'_{i-1}.$$

Then

$$\tilde{\psi}_{F'_i}(n) = \psi(\alpha_{1,2} + \cdots + \alpha_{n-1,n} + \beta_{n,2n} + 2x_{n,1})\psi(c\,\beta_{n,i}) \quad \text{for some } c \in \mathcal{F}^*.$$

Let $z = \mathrm{I}_{2n} + \alpha\; E_{n+i,2n} \in \mathrm{GL}_{2n}$. Then we can choose a certain $\alpha \in \mathcal{F}^*$ such that z normalizes F'_i and changes $\tilde{\psi}_{F'_{i-1}}$ back to the character $\psi_{F'_{i-1}}$. Hence we get (4-16) for ℓ_{i-1}.

By induction, we get a nonzero linear functional ℓ_1 on V_π that factors through $\mathcal{J}\{N_n, \psi_n\}(V_\pi)$.

By assumption, V_π has a nonzero generalized Shalika model. It follows from Theorem 2.4 that such a V_π has no nonzero twisted Jacquet module $\mathcal{J}\{N_n, \psi_n\}(V_\pi)$. Hence ℓ_1 must be zero.

Therefore, the assumption (4-11) must be wrong and $\mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi)$ must be zero. The proves the case when $\lambda = 1$.

If $\lambda \neq 1$, conjugation by $m(a)$ with $a = \mathrm{diag}(\lambda^{-1}, 1, \lambda^{-1}, 1, \ldots, \lambda^{-1}, 1) \in \mathrm{GL}_{2n}$ will give a vector space isomorphism $\mathcal{J}\{D, \psi'_{D,\lambda}\}(V_\pi) \simeq \mathcal{J}\{D, \psi_{D,\lambda}\}(V_\pi)$, where $\psi_{D,\lambda}$ is almost the same with the character of D defined in (4-8) except that the coefficient of $x_{2n-1,1}$ is λ^{-1}. In the proof of the case when $\lambda = 1$, we see that the coefficients of $x_{2n-1,1}$ and $x_{2n-2,1}$ play no role and a similar argument applies. □

Proposition 4.11. *Let π be a smooth representation of* SO_{4n}. *Then*

$$\mathcal{J}\{D, \psi_D\}(V_\pi) \simeq \mathcal{J}\{D, \tilde{\psi}_D\}(V_\pi),$$

where $\tilde{\psi}$ is the character of D defined (*in the notation of* (4-7)) *by*

$$\tilde{\psi}_D(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{2n-1,1}).$$

Proof. The proof is almost the same as that of Lemma 4.8. We give only a sketch.

First, let $\overline{\mathrm{B}}_n$ denote the opposite standard Borel subgroup of GL_n. By the General Lemma, we have the vector space isomorphism

$$\mathcal{J}\{D, \tilde{\psi}_D\}(V_\pi) \simeq \mathcal{J}\{\tilde{D}, \tilde{\psi}_{\tilde{D}}\}(V_\pi),$$

where

$$\tilde{D} = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \middle| \; T = \begin{pmatrix} 1 & * & * & \cdots & & \cdots & * \\ & t_1 & z_1 & * & \cdots & & * \\ & & t_2 & z_2 & * & \cdots & * \\ & & & \ddots & & & \vdots \\ & & & & t_{n-2} & z_{n-2} & * \\ & & & & & \mathrm{I}_2 & * \\ & & & & & & 1 \end{pmatrix}, \; t_i \in \mathrm{U}_2, \; z_i \in \overline{\mathrm{B}}_2 \right\} \tag{4-17}$$

and $\tilde{\psi}_{\tilde{D}}(n) = \psi(\sum_{i_1}^{2n-2} T_{i,i+2})\psi(x_{2n-2,1} + x_{2n-1,1})$ is the character of $\tilde{D}$.

Second, let

$$z = \begin{pmatrix} \mathrm{I}_{2n-3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which normalizes $\tilde{D}$ and changes $\tilde{\psi}_{\tilde{D}}$ to $\tilde{\psi}'_{\tilde{D}}$, defined in the notation of (4-17) by

$$\tilde{\psi}'_{\tilde{D}}(n) = \psi(\textstyle\sum_{i_1}^{2n-2} T_{i,i+2})\psi(x_{2n-1,1}).$$

Finally, use the General Lemma to transform the $\overline{\mathrm{B}}_2$ of the first part into B_2. $\square$

4.12. We are ready to prove Theorem 4.1(1). The proof is similar to that of [Ginzburg et al. 1999, Theorem 4.2.1], employing the local version of the Fourier expansion of representations. Let ν be the permutation matrix in GL_{4n} such that $\nu_{i,2i-1} = 1$ and $\nu_{2n+i,2i} = 1$ for $i = 1, \dots, 2n$, and $\nu_{i,j} = 0$ otherwise. Let $B = \nu D \nu^{-1}$, and define a character ψ_B of B by $\psi_B(e) = \tilde{\psi}_D(\nu^{-1} e \nu)$ for $e \in B$. Then we have the vector space isomorphism $\mathcal{J}\{D, \tilde{\psi}_D\}(V_\pi) \simeq \mathcal{J}\{B, \psi_B\}(V_\pi)$. Note that

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \; \begin{array}{l} \alpha, \delta \in \mathrm{U}_{2n}, \; \beta \in \mathrm{B}_{2n}, \\ \gamma \in \mathrm{B}_{2n} \cap \mathrm{Ni}_{2n} \text{ and } \gamma_{i,i+1} = 0 \text{ for } i = 1, \dots, 2n \end{array} \right\}, \tag{4-18}$$

and the character ψ_B is $\psi_B(e) = \psi(\alpha_{1,2} + \cdots + \alpha_{n,n+1} - \alpha_{n+1,n+2} - \cdots - \alpha_{2n-1,2n})$.

Example 4.13. For $n = 4$, elements in B are of the form

$$
\left(\begin{array}{cccccccc:cccccccc}
1 & \boxed{*} & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
 & 1 & \boxed{*} & * & * & * & * & * & 0 & * & * & * & * & * & * & * \\
 & & 1 & \boxed{*} & * & * & * & * & 0 & 0 & * & * & * & * & * & * \\
 & & & 1 & \boxed{*} & * & * & * & 0 & 0 & 0 & * & * & * & * & * \\
 & & & & 1 & \boxed{*} & * & * & 0 & 0 & 0 & 0 & * & * & * & * \\
 & & & & & 1 & \boxed{*} & * & 0 & 0 & 0 & 0 & 0 & * & * & * \\
 & & & & & & 1 & \boxed{*} & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
 & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
\hdashline
0 & 0 & * & * & * & * & * & * & 1 & * & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & 0 & 1 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 1 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 1 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 1 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right),
$$

where the boxes indicate the nontrivial character positions of ψ_B.

Our goal is to "fatten" β in (4-18), using the entries of γ, by successive applications of the General Lemma, until we transform $\mathcal{J}\{B, \psi_B\}$ into $\mathcal{J}\{V_{2n}, \tilde{\psi}\}$. Let

$$\mathscr{X} = \left\{ x \in \mathrm{Mat}_{2n}(\mathscr{F}) \;\middle|\; \begin{pmatrix} \mathrm{I}_{2n} & x \\ & \mathrm{I}_{2n} \end{pmatrix} \in \mathrm{SO}_{4n} \right\}.$$

For $x \in \mathscr{X}$, write

$$\epsilon(x) = \begin{pmatrix} \mathrm{I}_{2n} & x \\ & \mathrm{I}_{2n} \end{pmatrix} \quad \text{and} \quad \bar{\epsilon}(x) = \begin{pmatrix} \mathrm{I}_{2n} & 0 \\ x & \mathrm{I}_{2n} \end{pmatrix}.$$

For a subspace $S \subset \mathscr{X}$, define

$$\epsilon(S) = \{\epsilon(x) \mid x \in S\} \quad \text{and} \quad \bar{\epsilon}(S) = \{\bar{\epsilon}(x) \mid x \in S\}.$$

Put

$$\mathscr{X}_0 = \{x \in \mathscr{X} \mid x \in \mathrm{B}_{2n}\},$$
$$\mathscr{Y}_0 = \{x \in \mathscr{X} \mid x \in \mathrm{B}_{2n} \cap \mathrm{Ni}_{2n} \text{ and } x_{i,i+1} = 0 \text{ for } i = 1, \dots, n-1\}.$$

For $1 \le i < j-1$, define

$$\mathcal{Y}_{i,j} = \{x \in \mathcal{X}_0 \mid x_{r,l} = 0 \text{ for } r, l < j-1 \text{ and } x_{r,j} = 0 \text{ for } r \ge i\},$$
$$\mathcal{Y}^{i,j} = \mathrm{I} + \mathcal{F}(E_{i,j} - E_{2n+1-j,2n+1-i}).$$

Then elements in B can be written in the form

$$v = \epsilon(x)m(z)\bar{\epsilon}(y), \tag{4-19}$$

with $x \in \mathcal{X}_0$, $y \in \mathcal{Y}_0$ and $z \in \mathrm{U}_{2n}$. Let $\mathcal{Y}_{1,3} = \{x \in \mathcal{X}_0 \mid x_{1,3} = 0\}$. Let $C^{1,3}$ be the subgroup of the form (4-19) such that $y \in \mathcal{Y}_{1,3}$. Then $C^{1,3} = \epsilon(\mathcal{X}_0)m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{13})$. Let $Y^{1,3} = \bar{\epsilon}(\mathcal{Y}^{1,3})$. Denote by $X^{2,1} = \epsilon(\mathcal{X}^{2,1})$, where $\mathcal{X}^{2,1} = \mathcal{F}(e_{2,1} - e_{2n,2n-1})$. Let $\psi_B^{1,3} = \psi_B|_{C^{1,3}}$, $B^{1,3} = B$, and $D^{1,3} = C^{1,3}X^{2,1}$. Put $\mathcal{X}_{2,1} = \mathcal{X}_0 \oplus \mathcal{X}^{2,1}$. Then $D^{1,3} = \epsilon(\mathcal{X}_{2,1})m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{1,3})$. By the General Lemma, we conclude that

$$\mathcal{J}\{B^{1,3}, \psi^{1,3}_{B^{1,3}}\}(V_\pi) \simeq \mathcal{J}\{D^{1,3}, \psi^{1,3}_{D^{1,3}}\}(V_\pi),$$

where $\psi^{1,3}_{D^{1,3}}$ is the character of $D^{1,3}$, which is trivial on $\epsilon(\mathcal{X}_{2,1}) \cdot \overline{\mathcal{Y}}_{1,3}$.

Define $\mathcal{X}^{r,s} = \mathrm{I} + \mathcal{F}(E_{r,s} - E_{2n+1-s,2n+1-r})$ for $1 \le s < r \le 2n$. Let

$$\mathcal{X}_{r,s} = \mathcal{X}_0 \oplus \Big(\bigoplus_{q<l\le r-1} \mathcal{X}^{l,q}\Big) \oplus \Big(\bigoplus_{q=s}^{r-1} \mathcal{X}^{r,q}\Big) \quad \text{for } 1 \le s < r \le n.$$

For $1 \le i < j-1$ and $j \le n+1$, let $C^{i,j} = \epsilon(\mathcal{X}_{j-1,i+1})m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{i,j})$ if $i+1 \le j-1$. For $1 \le i < j \le n+1$, we define $Y^{i,j} = \bar{\epsilon}(\mathcal{Y}^{i,j})$ and $X^{j,i} = \epsilon(\mathcal{X}^{j,i})$, and also define

$$B^{i,j} = C^{i,j}Y^{i,j}, \quad D^{i,j} = C^{i,j}X^{j-1,i}, \quad A^{i,j} = D^{i,j}Y^{i,j}.$$

Let $\psi^{i,j}$ be the character of $C^{i,j}$, which is trivial on $\epsilon(\mathcal{X}_{j-1,i+1}) \cdot \bar{\epsilon}(\mathcal{Y}_{i,j})$. Then by the General Lemma, we have the vector space isomorphism

$$\mathcal{J}\{B^{i,j}, \psi^{i,j}_{B^{i,j}}\}(V_\pi) \simeq \mathcal{J}\{D^{i,j}, \psi^{i,j}_{D^{i,j}}\}(V_\pi)$$

for all $1 \le i < j-1$, $j \le n+1$.

Note that for $2 \le i < j-1$, $j \le n+1$, we have

$$D^{i,j} = B^{i-1,j} \quad \text{and} \quad \psi^{i,j}_{D^{i,j}} = \psi^{i-1,j}_{B^{i-1,j}},$$

and for $j = 3, \dots, n+1$, we have

$$D^{1,j} = B^{j-1,j+1} \quad \text{and} \quad \psi^{1,j}_{D^{1,j}} = \psi^{j-1,j+1}_{B^{j-1,j+1}}.$$

We conclude by the General Lemma again that

$$\mathcal{J}\{B^{1,3}, \psi^{1,3}_{B^{1,3}}\}(V_\pi) \simeq \mathcal{J}\{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi) \tag{4-20}$$

as vector spaces. Note that $D^{1,n+1} = \epsilon(\mathcal{X}_{n,1})m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{1,n+1})$.

So far in this proof, we have not used any particular property of V_π. We are now going to use the property that V_π has a nonzero generalized Shalika model.

For $n+1 \le r \le 2n-1$ and $1 \le s \le 2n-r$, define

$$\mathscr{X}_{r,s} = \mathscr{X}_{n,1} \oplus \Big(\bigoplus_{\substack{n+1 \le l \le r-1 \\ 1 \le q \le 2n-l}} \mathscr{X}_{l,q} \Big) \oplus \Big(\bigoplus_{q=s}^{2n-r} \mathscr{X}^{r,q} \Big).$$

Then $X^{n+1,n-1}$ normalizes $D^{1,n+1}$ and $\psi^{1,n+1}_{D^{1,n+1}}$. Considering its action on the right side of (4-20), we claim that for any nontrivial character ξ of $X^{n+1,n-1}$,

$$\mathscr{J}\{X^{n+1,n-1}, \xi\}(\mathscr{J}\{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi)) = 0,$$

and hence we must have the trivial character for this action. We assume to the contrary that, by the Frobenius reciprocity law, there exists ℓ a nonzero linear functional on V_π such that

$$\ell(\pi(xn)v) = \psi^{1,n+1}_{1,n+1}(n)\xi(x)\ell(v) \quad \text{for all } x \in X^{n+1,n-1},\ n \in D^{1,n+1} \text{ and } v \in V_\pi.$$

We may assume that there is a $\lambda \in \mathscr{F}^*$ such that $\xi(x(t)) = \psi(\lambda\, t)$, where $x(t) = \mathrm{I}_{4n} + t(E_{n+1,3n-1} - E_{n+2,3n})$. Then ℓ is a nonzero linear functional on V_π such that

$$\ell(\pi(n)v) = \psi^{1,n+1}_{D^{1,n+1}}(n)\ell(v) \quad \text{for } n \in X^{n+1,n-1}D^{1,n+1} \cap N_{n+1}.$$

Note that $X^{n+1,n-1}D^{1,n+1} \cap N_{n+1}$ consists of elements of the form

$$\begin{pmatrix} z & y & w \\ & \mathrm{I}_{2n-2} & y' \\ & & z' \end{pmatrix} \in \mathrm{SO}_{4n}, \tag{4-21}$$

with $z \in \mathrm{U}_{n+1}$ and $y \in \mathrm{Mat}_{n+1,2n-2}$ such that $y_{n+1,n+i} = 0$ for $i = 1, \dots, n-1$.

Now the situation is similar to that of (4-14). The same argument shows that ℓ can be extended trivially to N_{n+1} so that

$$\ell(\pi(n)v) = \psi^{1,n+1}_{N_{n+1}}(n)\ell(v) \quad \text{for } n \in N_{n+1},$$

where $\psi^{1,n+1}_{N_{n+1}}$ is the trivial extension of restriction of $\psi^{1,n+1}_{D^{1,n+1}}$ to $D^{1,n+1} \cap N_{n+1}$.

Note that for an element $n \in N_{n+1}$ of the form (4-21),

$$\psi^{1,n+1}_{D^{1,n+1}}(n) = \psi(z_{1,2} + \cdots + z_{n,n+1})\psi(y_{n+1,1} + y_{n+1,2n-2}).$$

Let ν' be the permutation matrix in GL_{2n} defined by

$$\nu'(i) = \begin{cases} i & \text{if } i = 1, \dots, n+1, \\ 2n & \text{if } i = n+2, \\ i-1 & \text{if } i = n+3, \dots, 2n, \end{cases}$$

which is identified with its embedding $m(\nu')$ in SO_{4n}. Then ν' normalizes N_{n+1} and transforms $\psi^{1,n+1}_{N_{n+1}}$ into ψ_{n+1}. Hence we obtain a nonzero linear functional that factors through $\mathcal{J}\{\psi_{n+1}\}(V_\pi)$. In particular, we have $\mathcal{J}\{\psi_{n+1}\}(V_\pi) \neq 0$.

On the other hand, V_π has a nonzero generalized Shalika model by assumption. Following Theorem 2.4, $\mathcal{J}\{\psi_{n+1}\}(V_\pi)$ must be zero. We get a contradiction. Hence $X^{n+1,n-1}$ must act trivially on $\mathcal{J}\{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi)$.

Next we continue this process. Define

$$B^{n-2,n+2} = D^{1,n+1} X^{n+1,n-1},$$

and extend $\psi^{1,n+1}_{D^{1,n+1}}$ to a character $\psi^{n-2,n+2}_{B^{n-2,n+2}}$ on $B^{n-2,n+2}$ by making it trivial on $X^{n+1,n-1}$. Thus we have

$$\mathcal{J}\{B^{n-2,n+2}, \psi^{n-2,n+2}_{B^{n-2,n+2}}\}(V_\pi) \simeq \mathcal{J}\{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi).$$

Now we can repeat the argument as before, by replacing the $n-2$ coordinates of $\bigoplus_{i=1}^{n-2} \mathcal{Y}_{i,n+2}$ with $\bigoplus_{i=1}^{n-2} \mathcal{X}_{n+1,i}$. For $1 \le i \le n-2$ and $j \ge n+2$, define $C^{i,j} = \epsilon(\mathcal{X}_{j-1,i+1}) m(\mathrm{U}_{2n}) \bar{\epsilon}(\mathcal{Y}_{i,j})$ and

$$B^{i,j} = C^{i,j} Y^{i,j}, \quad D^{i,j} = C^{i,j} X^{j-1,i}, \quad A^{i,j} = D^{i,j} Y^{i,j}.$$

Let $\psi^{i,n+2}$ be the character of $C^{i,n+2}$, which is trivial on $\ell(C_{n+1,i+1})\bar{\ell}(Y_{i,n+2})$. By the General Lemma, we conclude that

$$\mathcal{J}\{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi) \simeq \mathcal{J}\{D^{1,n+2}, \psi^{1,n+2}_{D^{1,n+2}}\}(V_\pi) \tag{4-22}$$

as vector spaces. Then, by using the property that V_π has a nonzero generalized Shalika model, we show that $X^{n+2,n-2}$ acts trivially on the right side of (4-22). As before, we get

$$\mathcal{J}\{D^{1,n+2}, \psi^{1,n+2}_{D^{1,n+2}}\}(V_\pi) \simeq \cdots \simeq \mathcal{J}\{D^{1,2n-1}, \psi^{1,2n-1}_{D^{1,2n-1}}\}(V_\pi)$$

as vector spaces. Note that $D^{1,2n-1} = N_{2n}$ and $\psi^{1,2n-1}_{1,2n-1} = \tilde{\psi}$. We conclude that

$$\mathcal{J}\{D^{1,2n-1}, \psi^{1,2n-1}_{D^{1,2n-1}}\}(V_\pi) = \mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi).$$

This concludes the proof of part (1) of Theorem 4.1.

5. Irreducibility of the local descent

To finish the proof of Theorem 2.5, it remains to show that σ_{n-1} is irreducible. In Sections 3 and 4, we proved that, as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, the local descent $\sigma_{n-1} = \mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1,\tau))$, as defined in (2-8), is quasisupercuspidal and has a unique nonzero Whittaker functional. Hence it is enough to show that any irreducible summand of σ_{n-1} is generic, that is, has a nonzero Whittaker functional. This is proved in Theorem 5.1(2). Theorem 5.1, whose proof is standard, may be viewed as a generalization of the geometric lemma of Bernstein and Zelevinsky [1977] for the twisted Jacquet functor $\mathcal{J}\{\psi_{n-1}\}$ applied to $\mathcal{L}(1,\tau)$. For a similar discussion for the metaplectic and symplectic groups, see [Ginzburg et al. 1999]

For a given irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, recall that $\mathrm{I}(s,\tau)$ is the induced representation of $\mathrm{SO}_{4n}(\mathcal{F})$ from the supercuspidal datum (P_{2n},τ) as defined in Section 2.1. The unique Langlands quotient of $\mathrm{I}(s,\tau)$ at $s=1$ is $\mathcal{L}(1,\tau)$.

Theorem 5.1. *Suppose (V_σ,σ) is an irreducible supercuspidal representation of* $\mathrm{SO}_{2n+1}(\mathcal{F})$.

(1) *If* $\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\tau)),V_\sigma)$ *is nonzero for any* $s\in\mathbb{C}$, *then* σ *is generic.*

(2) *If* $\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1,\tau)),V_\sigma)$ *is nonzero, then* σ *is generic.*

Clearly part (2) follows from part (1) by the exactness of the twisted Jacquet functors. Part (1) is proved in Section 5.7.

We start by investigating the structure of $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\tau))$ to determine the genericity of σ. We realize the irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ by its Whittaker model $\mathcal{W}(\tau,\psi)$, and realize the induced representation $\mathrm{I}(s,\tau)$ as $\mathrm{I}(s,\mathcal{W}(\tau,\psi))$. Then we consider $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$.

5.2. ***The twisted Jacquet module $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$.*** We consider first the orbital structure of the closed subgroup $\mathrm{SO}_{2n+2}\cdot N_{n-1}$ acting on the generalized flag variety $P_{2n}\backslash\mathrm{SO}_{4n}$ over the p-adic field $\mathcal{F}$, and then consider the semisimplification of $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

For $j=1,\dots,2n$, let

$$P_j=\left\{\begin{pmatrix} h & * & * \\ & g & * \\ & & h^* \end{pmatrix}\;\middle|\; h\in\mathrm{GL}_j,\ g\in\mathrm{SO}_{4n-2j}\right\}$$

be the standard maximal parabolic subgroup of SO_{4n}. Then the generalized Bruhat decomposition $P_{2n}\backslash\mathrm{SO}_{4n}/P_{n-1}$ has a complete set of representatives given by $\{\gamma_i\mid i\in 2\mathbb{N},\ n\le i\le 2n\}$, where for $i\in 2\mathbb{N}$ with $n\le i\le 2n$,

$$\gamma_i=\begin{pmatrix} & & \nu_{2n-i} \\ & \mathrm{I}_{2i} & \\ \nu_{2n-i} & & \end{pmatrix}$$

and ν_j is as defined in (2-1). For $k=0,1,\dots,n-1$, let M_k be the standard maximal parabolic subgroup of GL_{n-1} corresponding to the partition $(k,n-k-1)$

of $n-1$ such that the Levi part is $\mathrm{GL}_k \times \mathrm{GL}_{n-k-1}$ and the unipotent radical is

$$L_k = \left\{ \begin{pmatrix} \mathrm{I}_k & \\ A & \mathrm{I}_{n-1-k} \end{pmatrix} \in \mathrm{GL}_{n-1} \;\middle|\; A \in \mathrm{Mat}_{n-1-k,k} \right\}.$$

Lemma 5.3. *The orbits of the closed subgroup $\mathrm{SO}_{2n+2} \cdot N_{n-1}$ acting on the generalized flag variety $P_{2n} \setminus \mathrm{SO}_{4n}$ are represented by elements of the form $\gamma_i w$, where $n \le i \le 2n$ is even and the w are elements of $W(\mathrm{GL}_{n-1})$ given by*

$$\begin{cases} w \in [W(\mathrm{GL}_{2n-i}) \times W(\mathrm{GL}_{i-n-1})] \backslash W(\mathrm{GL}_{n-1}) & \text{if } i \ne n, \\ w = \mathrm{id} & \text{if } i = n. \end{cases}$$

Here $W(\mathrm{GL}_m)$ denotes the Weyl group of GL_m.

Proof. Clearly, we have $\mathrm{SO}_{2n+2}\, N_{n-1} \subset P_{n-1}$. Hence we can choose $\gamma_i w$ to be the representative of any double cosets in $P_{2n} \backslash \mathrm{SO}_{4n} / [\mathrm{SO}_{2n+2}\, N_{n-1}]$, for some

$$w \in [\gamma_i^{-1} P_{2n} \gamma_i \cap P_{n-1}] \backslash P_{n-1} / [\mathrm{SO}_{2n+2}\, N_{n-1}].$$

Since $M_{2n-i} \subset \gamma_i^{-1} P_{2n} \gamma_i \cap P_{n-1}$, we may choose a set of representatives for $[\gamma_i^{-1} P_{2n} \gamma_i \cap P_{n-1}] \backslash P_{n-1} / [\mathrm{SO}_{2n+2}\, N_{n-1}]$ from $M_{2n-i} \backslash \mathrm{GL}_{n-1} / N_{n-1}$. Then a complete set of representatives for $M_{2n-i} \backslash \mathrm{GL}_{n-1} / N_{n-1}$ can be chosen from

$$[W(\mathrm{GL}_{2n-i}) \times W(\mathrm{GL}_{i-n-1})] \backslash W(\mathrm{GL}_{n-1}). \qquad \square$$

Let $\alpha_1, \ldots, \alpha_{n-2}$ denote the simple roots of GL_{n-1} with respect to N_{n-1}. Let

$$\{x_{\alpha_j}(t) = \mathrm{I}_{n-1} + t E_{j,j+1} \mid t \in \mathcal{F}\}$$

denote the one parameter unipotent subgroup of N_{n-1} corresponding to the root α. We will take $w = \mathrm{id}$ to be the representative of the coset $W(\mathrm{GL}_k) \times W(\mathrm{GL}_{n-1-k})$ in $W(\mathrm{GL}_{n-1})$.

Lemma 5.4 [Ginzburg et al. 1999, Lemma 4.3]. *If a Weyl group element w belongs to $[W(\mathrm{GL}_k) \times W(\mathrm{GL}_{n-1-k})] \backslash W(\mathrm{GL}_{n-1})$ and is the identity, then there exists a simple root α_j such that $w x_{\alpha_j}(t) w^{-1} \in L_k$ for all $t \in \mathcal{F}$.*

Next we consider the semisimplification of the module $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s, \mathcal{W}(\tau, \psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. It is a standard process to decompose the representation by spaces of functions on $\mathrm{SO}_{4n}(\mathcal{F})$ according the orbital decomposition obtained in Lemma 5.3.

It is clear that among the orbits

$$\mathbb{O}_{i,w} = [P_{2n}]\gamma_i w [\mathrm{SO}_{2n+2}\, N_{n-1}] \quad \text{for } i \in 2\mathbb{N} \text{ with } n \le i \le 2n,$$

the orbit $\mathbb{O}_{2[(n+1)/2],\mathrm{id}}$ is the unique open orbit. Let E be a union of orbits $\mathbb{O}_{i,\omega}$. We denote by $\mathcal{S}(E, \tau_s)$ the space of smooth functions ϕ on E that are compactly supported modulo P_{2n}, have values in the Whittaker model $\mathcal{W}(\tau, \psi)$ and are such that

$$\phi\left(\begin{pmatrix} a & * \\ & a^* \end{pmatrix} g, r\right) = |\det a|^{s/2+n-1/2}\phi(g, ra) \quad \text{for } g \in \mathrm{SO}_{4n} \text{ and } a, r \in \mathrm{GL}_{2n}.$$

We may arrange the orbits in a sequence

$$P_{2n}\,\mathrm{SO}_{2n+2}\,N_{n-1} = \Omega_1, \ \ldots, \ \Omega_l = \mathbb{O}_{2[(n+1)/2],\mathrm{id}}$$

such that $F_i = \bigcup_{j=1}^{i} \Omega_j$ is closed in SO_{4n}. It is clear that Ω_i is open in F_i and F_{i-1} is closed in F_i. We obtain the exact sequence

$$0 \to \mathcal{S}(\Omega_{i+1}, \tau_s) \xrightarrow{e} \mathcal{S}(F_{i+1}, \tau_s) \xrightarrow{r} \mathcal{S}(F_i, \tau_s) \to 0, \tag{5-1}$$

where the map e is the natural embedding and r is the restriction to F_i. Apply the twisted Jacquet functor $\mathcal{J}\{\psi_{n-1}\}$ to the exact sequence (5-1). Since the Jacquet functors are exact, we obtain another exact sequence

$$0 \to \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\Omega_{i+1}, \tau_s)) \to \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(F_{i+1}, \tau_s)) \to \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(F_i, \tau_s)) \to 0.$$

We obtain the semisimplification of $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s, \mathcal{W}(\tau, \psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$ as $\bigoplus_{i=1}^{l} \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\Omega_i, \tau_s))$.

Next, we study the space $\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\Omega_i, \tau_s))$ for $i = 1, 2, \ldots, l$. We assume for the rest of this section unless stated otherwise that all inductions are unnormalized.

As $\mathrm{SO}_{2n+2}\,N_{n-1}$ module, we have

$$\mathcal{S}(\mathbb{O}_{i,w}, \tau_s) \simeq \text{c-Ind}_{P_{2n}^{\gamma_{i,w}}}^{\mathrm{SO}_{2n+2}\,N_{n-1}} (\delta_{P_{2n}}^{1/2}\tau_s)^{\gamma_{i,w}},$$

where c-Ind denotes the compact induction and

$$R_{i,w} = P_{2n}^{\gamma_{i,w}} := (\gamma_i w)^{-1} P_{2n} \gamma_i w \cap \mathrm{SO}_{2n+1}\,N_{n-1}.$$

Lemma 5.5. *With notation above, the following vanishing properties hold.*

(1) *For* $w \neq \mathrm{id}$,

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{R_{i,w}}^{\mathrm{SO}_{2n+2}\,N_{n-1}} (\delta_{P_{2n}}^{1/2}\tau_s)^{\gamma_{i,w}}) = 0 \quad \textit{for } i \geq 2[(n+1)/2].$$

(2) *For* $w = \mathrm{id}$,

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{R_{i,\mathrm{id}}}^{\mathrm{SO}_{2n+2}\,N_{n-1}} (\delta_{P_{2n}}^{1/2}\tau_s)^{\gamma_{i,\mathrm{id}}}) = 0 \quad \textit{for } i > 2[(n+1)/2].$$

Proof. When $w \neq \mathrm{id}$, by Lemma 5.4, there is a simple root subgroup $x(t)$ inside N_{n-1} such that $\gamma_i w x(t)(\gamma_i w)^{-1}$ lies in the unipotent radical of P_{2n}. This shows that

$$x(t) \in R_{i,w} \cap N_{n-1} \quad \text{and} \quad (\delta_{P_{2n}}^{1/2}\tau_s)^{\gamma_{i,w}}(x(t)) = \mathrm{id},$$

while $\psi_{n-1}(x(t)) = \psi(t)$.

When $w = \mathrm{id}$ and $i > 2[(n+1)/2]$, the root subgroup $x_\alpha(t)$ of SO_{4n} is for $\alpha = e_{n-1} + e_{2n}$ invariant under the conjugation by $\gamma_i'^{-1}$. Hence

$$x(t) \in R_{i,w} \cap N_{n-1} \quad \text{and} \quad (\delta_{P_{2n}}^{1/2} \tau_s)^{\gamma_{i,w}}(x(t)) = \mathrm{id},$$

while $\psi_{n-1}(x(t)) = \psi(t)$. □

Therefore, we are left with $\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\Omega_l, \tau_s))$ for the Zariski open orbit $\Omega_l = \mathbb{O}_{2[(n+1)/2],\mathrm{id}}$. To summarize:

Proposition 5.6. *We have*

$$\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s, \mathcal{W}(\tau, \psi))) \simeq \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\mathbb{O}_{2[(n+1)/2],\mathrm{id}}, \tau_s))$$

for all $s \in \mathbb{C}$ as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

5.7. ***Proof of Theorem 5.1(1).*** Keep the previous notation. By Proposition 5.6,

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s, \tau)), V_\sigma) \\ &\qquad\simeq \mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\mathbb{O}_{2[(n+1)/2],\mathrm{id}}, \tau_s)), V_\sigma),\end{aligned}$$

reducing the proof to understanding the structure of $\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\mathbb{O}_{2[(n+1)/2],\mathrm{id}}, \tau_s))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

It is more convenient to choose ν_{4n} as representative of the orbit $\mathbb{O} = \mathbb{O}_{2[(n+1)/2],\mathrm{id}}$ than the original $\gamma_{2[(n+1)/2],\mathrm{id}}$. Then

$$\mathcal{S}(\mathbb{O}, \tau_s) \simeq \text{c-Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2}\, N_{n-1}} (\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}},$$

where $P_{2n}^{\nu_{4n}} = \nu_{4n}^{-1} P_{2n} \nu_{4n} \cap \mathrm{SO}_{2n+2}\, N_{n-1}$. Let $\mathfrak{Q}_{n+1}$ be the maximal standard parabolic subgroup of SO_{2n+2} whose Levi component is isomorphic to GL_{n+1}, and let $\mathfrak{Q}_{n+1}^-$ be the opposite parabolic subgroup. Then we have

$$\begin{aligned}P_{2n}^{\nu_{4n}} &= \left\{ m\left(\begin{pmatrix} z & c \\ & \mathrm{I}_{n+1} \end{pmatrix}\right) \in \mathrm{SO}_{4n} \;\middle|\; z \in \mathrm{U}_{n-1} \right\} \cdot \overline{\mathfrak{Q}}_{n+1} \\ &:= m(\mathrm{U}_{2n,n-1}) \cdot \mathfrak{Q}_{n+1}^-,\end{aligned}$$

where $\mathrm{U}_{2n,j}$ is the subgroup of the unipotent radical U_{2n} of the standard Borel subgroup of GL_{2n} consisting of elements of type

$$\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{2n-j} \end{pmatrix} \in \mathrm{U}_{2n} \quad \text{with } z \in \mathrm{U}_j.$$

For

$$\phi \in \text{c-Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2}\, N_{n-1}} (\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}} \quad \text{and} \quad q = \begin{pmatrix} a & 0 \\ * & a^* \end{pmatrix} \in \mathfrak{Q}_{n+1}^-,$$

we have

$$(5\text{-}2)\quad (\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}}\,(\mathrm{diag}(\mathrm{I}_{n-1}, q, \mathrm{I}_{n-1}))\,(\phi)(g,r) = |\det a|^{-(s/2+n-1/2)}\phi(g, r(\mathrm{diag}\,\mathrm{I}_{n-1}, a)),$$

and for $\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{n+1}\end{pmatrix} \in \mathrm{U}_{2n,n-1}$, we have

$$(5\text{-}3)\qquad (\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}}\Bigl(m\Bigl(\begin{pmatrix} z & c \\ & \mathrm{I}_{n+1}\end{pmatrix}\Bigr)\Bigr)(\phi)(g,r) = \phi\Bigl(g, r\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{n+1}\end{pmatrix}\Bigr).$$

To understand $\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\mathbb{O}, \tau_s))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, we consider the double coset decomposition $P_{2n}^{\nu_{4n}} \backslash \mathrm{SO}_{2n+2} \cdot N_{n-1} / \mathrm{SO}_{2n+1} \cdot N_{n-1}$, which reduces the proof to the computation of the double cosets

$$\mathcal{Q}_{n+1}^{-} \backslash \mathrm{SO}_{2n+2} / \mathrm{SO}_{2n+1}\,.$$

Next proposition shows that it has only one orbit.

Proposition 5.8. *Over any field k of characteristic zero, the generalized flag variety $\mathcal{Q}_{n+1}^{-}(k) \backslash \mathrm{SO}_{2n+2}(k)$ has only one orbit under the action of* $\mathrm{SO}_{2n+1}(k)$.

Proof. Let $X = k^{2n+2}$ be a k-vector space, written with its elements as column vector, with a quadratic form q defined by $\frac{1}{2}\nu_{2n+2}$. Then $\mathrm{SO}(X) \simeq \mathrm{SO}_{2n+2}$ Let $e_1, \dots, e_{2n+2}$ be the standard basis of X, $v_0 = e_{n+1} + e_{n+2}$. Let $Y = (k \cdot v_0)^{\perp}$. Then $\dim Y = 2n+1$ and $\mathrm{SO}(Y) = \mathrm{SO}_{2n+1}$. Note that Y has a basis

$$(5\text{-}4)\qquad e_{n+1} - e_{n+2}, e_1, e_2 \dots, e_n, e_{n+3}, \dots, e_{2n+1}, e_{2n+2}.$$

Then a basis of X can be chosen to be

$$(5\text{-}5)\qquad e_{n+1} + e_{n+2}, e_{n+1} - e_{n+2}, e_1, e_2, \dots, e_n, e_{n+3}, \dots, e_{2n+1}, e_{2n+2}.$$

Let $g \in \mathrm{SO}(X)$ such that $g(v_0) = v_0$. Then $g(Y) = Y$. Assume that the matrix of $g|_Y$ in the basis (5-4) is A_g. Then g in the basis (5-5) is $\mathrm{diag}(1, A_g)$. As $\det g = 1$, we must have $\det(A_g) = 1$; hence $g \in \mathrm{SO}(Y)$, so the stabilizer of v_0 is $\mathrm{SO}(Y)$.

Note that $q(v_0) = 1$. Let $Z = \{v \in X \mid q(v) = 1\}$. Then SO_{2n+2} acts transitively on Z. To show the proposition, we only need to show that $\mathcal{Q}_{n+1}^{-}$ acts on Z transitively. In fact, if $\mathcal{Q}_{n+1}^{-}$ acts transitively on Z, then, letting $h \in \mathrm{SO}(X)$, there exists $t \in \mathcal{Q}_{n+1}^{-}$ such that $h \cdot v_0 = t \cdot v_0$. Hence $(t^{-1}h) \cdot v_0 = v_0$, and then $t^{-1}h \in \mathrm{SO}_{2n+1}$ and $h \in \mathcal{Q}_{n+1}^{-}\,\mathrm{SO}_{2n+1}$. This means that $\mathrm{SO}_{2n+2} = \mathcal{Q}_{n+1}^{-}\,\mathrm{SO}_{2n+1}$.

Now we show that $\mathcal{Q}_{n+1}^{-}$ acts transitively on Z. We only need to show that any element of Z can be moved to v_0 under the action of some element in $\mathcal{Q}_{n+1}^{-}$. Let $v = (v_1, v_2) \in X$ with $v_1, v_2 \in k^{n+1}$. Take $g \in \mathcal{Q}_{n+1}^{-}$ to be

$$g = \begin{pmatrix} a & 0 \\ b & a^{*}\end{pmatrix} \quad \text{with } a \in \mathrm{GL}_{n+1}\,.$$

Then the action of g on v is given by $g \cdot v = (av_1, bv_1 + a^* v_2)^t$.

Assume now $q(v) = 1$. Then $v_1 \neq 0$, otherwise $q(v) = 0$. Then there is $a \in \mathrm{GL}_{n+1}$ such that $av_1 = (0, \dots, 0, 1)^t$. For this a, there exists $b \in \mathrm{Mat}_{n+1}(k)$ such that $bv_1 = (1, 0, \dots, 0)^t - a^* v_2$, since $v_1 \neq 0$. Now

$$g = \begin{pmatrix} a & 0 \\ b & a^* \end{pmatrix} \in \mathcal{Q}_{n+1}^- \quad \text{and} \quad g \cdot v = v_0. \qquad \square$$

It follows that $P_{2n}^{\nu_{4n}} \backslash \mathrm{SO}_{2n+2} N_{n-1} = [P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}] \backslash \mathrm{SO}_{2n+1} N_{n-1}$. By restriction to $\mathrm{SO}_{2n+1} N_{n-1}$, we have

$$\text{c-Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}}) \simeq \text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F}) \ltimes N_{n-1}(\mathcal{F})$. Hence

$$\begin{aligned} &\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})) \\ &\qquad \simeq \mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})) \end{aligned}$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

Define $\psi_{\mathrm{U}_{2n,n-1}}(u(z, c)) := \psi_{n-1}|_{\mathrm{U}_{2n,n-1}}(u(z, c))$.

Proposition 5.9. *With notation above*,

$$\begin{aligned} &\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})) \\ &\qquad \simeq \text{c-Ind}_{\mathcal{P}_n^-}^{\mathrm{SO}_{2n+1}} (\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau') |\det|^{-s/2-1/2}) \end{aligned}$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$, *where* $\mathcal{P}_n^- := \mathcal{Q}_{n+1}^- \cap \mathrm{SO}_{2n+1}$, *the representation* τ' *is obtained by restriction to* $\mathcal{P}_n^-(\mathcal{F})$ *of the representation of* $\mathcal{Q}_{n+1}^-(\mathcal{F})$ *given by* (5-2) *and* (5-3), *and* $\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')$ *denotes the twisted Jacquet module of* τ' *along* $(\mathrm{U}_{2n,n-1}, \psi_{\mathrm{U}_{2n,n-1}})$.

Proof. Let f be a section in

$$\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}}).$$

Consider the restriction of f to $\mathrm{SO}_{2n+1}(\mathcal{F})$. It is clear that this restriction map factors through

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})),$$

so we still denote the restriction by $f \mapsto f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$. By (5-2) and (5-3), the restriction $f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$ belongs to the space

$$\text{c-Ind}_{\mathcal{P}_n^-}^{\mathrm{SO}_{2n+1}} (\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau') |\det|^{-s/2-1/2}).$$

By using the orbital decomposition in Proposition 5.8 and (5-3), it is not hard to check that $f \mapsto f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$ is in fact injective. The argument is the same as in the proof of [Ginzburg et al. 1999, formula (6.5)] and similar to that of [Kudla 1986, Lemma 5.3]. We omit the details.

The surjectivity can be verified as follows. Assume that we have a smooth $\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(V_{\tau'})$-valued function g on SO_{2n+1}, compactly supported modulo $\mathcal{P}_n^-$, satisfying

$$g(qy) = \mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')|\det|^{-s/2-1/2}(q)g(y) \quad \text{for } q \in \mathcal{P}_n^- \text{ and } y \in \mathrm{SO}_{2n+1}.$$

Since g is locally constant, we may pull back g to a smooth $V_{\tau'}$-valued function g' on SO_{2n+1}, compactly supported modulo $\mathcal{P}_n^-$, satisfying

$$g'(qy) = \tau'(q)|\det|^{-(s/2+n-1/2)}g'(y) \quad \text{for } q \in \mathcal{P}_n^- \text{ and } y \in \mathrm{SO}_{2n+1}.$$

The unipotent subgroup N_{n-1} can be written as $N_{n-1} = m(\mathrm{U}_{2n,n-1}) \ltimes N''$, where N'' is the intersection of N_{n-1} with the unipotent radical V_{2n} of P_{2n}. Then

$$\mathrm{SO}_{2n+1}\, N_{n-1} = \mathrm{SO}_{2n+1}\, \mathrm{U}''_{2n,n-1},$$

which is in fact a homeomorphism. Indeed, let $z'y'x' = zyx$ with $x, x' \in \mathrm{SO}_{2n+1}$, $z, z' \in B_{n-1}$ and $y, y' \in N''$. Then $y = (z^{-1}z')y'(x'x^{-1}) \in N''$. Hence $x = x'$, $z = z'$ and $y = y'$.

Then we can pull back g to a section f in

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1}\, N_{n-1}}^{\mathrm{SO}_{2n+1}\, N_{n-1}}((\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}})),$$

which is defined as follows. Choose a compactly supported smooth function ϕ on N'' that has a nonzero projection under the twisted Jacquet functor with respect to $(N'', \psi_{n-1}|_{N''})$, and define $f'(uyx, r) := \phi(y)g'(x, ru)$, for all $x \in \mathrm{SO}_{2n+1}$, $u \in \mathrm{U}_{2n,n-1}$, $y \in N''$, and $r \in \mathrm{GL}_{2n}$. It is clear that f' is a nonzero section in

$$\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1}\, N_{n-1}}^{\mathrm{SO}_{2n+1}\, N_{n-1}}((\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}}).$$

By checking the action of N_{n-1} on f', it is clear that f' factors through

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1}\, N_{n-1}}^{\mathrm{SO}_{2n+1}\, N_{n-1}}((\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}})),$$

whose image f has the restriction to $\mathrm{SO}_{2n+1}(\mathcal{F})$ equal to g. □

The elements of $\mathcal{P}_n^-$ have the form

$$\begin{pmatrix} b & & & \\ x & 1 & 0 & \\ -x & 0 & 1 & \\ y' & -x' & x' & b^* \end{pmatrix} \in \mathrm{SO}_{2n+2}(\mathcal{F}),$$

which is identified (following the embedding we assumed as before) with

$$\begin{pmatrix} b & & \\ x & 1 & \\ y & x' & b^* \end{pmatrix} \in \mathrm{SO}_{2n+1}(\mathcal{F}).$$

Following the discussions above, we deduce that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}&(\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\tau)), V_\sigma)\\ &\simeq \mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\text{c-Ind}_{\mathcal{P}_n^-(\mathcal{F})}^{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')|\det|^{-s/2-1/2}), V_\sigma).\end{aligned}$$

By the Frobenius reciprocity law, the last space is isomorphic to

$$\mathrm{Hom}_{\mathcal{P}_n^-(\mathcal{F})}(\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')|\det|^{-s/2-1/2}, V_\sigma).$$

By the assumption of Theorem 5.1, the last space is nonzero. Since the argument below only uses the genericity of τ' and the supercuspidality of σ and does not depend on the value s, we may consider, for simplicity, only the nonzero space $\mathrm{Hom}_{\mathcal{P}_n^-(\mathcal{F})}(\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau'), V_\sigma)$. Any nonzero element ξ in it is a $\mathcal{P}_n^-(\mathcal{F})$-equivariant, linear map from $\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')$ to V_σ. In particular, for any $v \in V_{\tau'}$, we have

$$\sigma\left(\begin{pmatrix} a & & \\ x & 1 & \\ y & x' & a^* \end{pmatrix}\right)(\xi(v)) = \xi(\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')\left(\begin{pmatrix} \mathrm{I}_{n-1} & & \\ & a & \\ & x & 1 \end{pmatrix}\right)(v)). \tag{5-6}$$

Take $a = \mathrm{I}_n$ and consider the action of the unipotent radical of $\mathcal{P}_n^-(\mathcal{F})$, which is denoted by $\mathcal{V}_n^-(\mathcal{F})$ and consists of elements of the form

$$v^-(x,y) := \begin{pmatrix} \mathrm{I}_n & & \\ x & 1 & \\ y & x' & \mathrm{I}_n \end{pmatrix}.$$

Then (5-6) implies that the center (the elements of type $v^-(0,y)$) of $\mathcal{V}_n^-(\mathcal{F})$ acts on V_σ trivially. Since V_σ is supercuspidal, there is a nonzero vector $v \in \mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(V_{\tilde{\tau}})$ such that the unipotent radical of $\mathcal{V}_n^-(\mathcal{F})$ acts on $\xi(v)$ by a nontrivial character. Since the $\mathrm{GL}_n(\mathcal{F})$ acts on the x-part (more precisely, the quotient of $\mathcal{V}_n^-(\mathcal{F})$ modulo the center) with two orbits, we may assume that

$$\sigma(v^-(x,y))(\xi(v)) = \psi_{\mathcal{V}_n^-}(v^-(x,y))\xi(v) = \psi(x_n)\xi(v) \quad \text{for } x = (x_1, \cdots, x_n)$$

where $\psi_{\mathcal{V}_n^-}$ is a nonzero character of $\mathcal{V}_n^-(\mathcal{F})$. In other words, the map ξ descends to a map from $\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')$ to $\mathcal{J}\{\psi_{\mathcal{V}_n^-}\}(V_\sigma)$.

By (5-6), we have

$$\xi(\mathcal{J}\{\psi_{n-1}\}(\tau')\left(\begin{pmatrix} \mathrm{I}_{n-1} & & \\ & a & \\ & x & 1 \end{pmatrix}\right)(v)) = \psi(x_n)\xi(v).$$

Now consider the subgroup $B_{2n,n}$ of GL_{2n} consisting of elements of the form

$$b(z, c, e, y, d) := \begin{pmatrix} z & c & e \\ 0 & 1 & y \\ 0 & 0 & d \end{pmatrix} \quad \text{with } d \in \mathrm{GL}_n(\mathcal{F}) \text{ and } z \in \mathrm{U}_{n-1}.$$

Let μ be the Weyl element of GL_{2n} that corresponds to the elementary matrix $\mathrm{diag}(\mathrm{I}_{n-1}, \nu_{n+1})$. Then it is easy to see that

$$\xi(\mathcal{J}\{\psi_{n-1}\}(\tau')(b(z, c, e, y, \mathrm{I}_n))(\mu v)) = \psi_{U_{n-1}}(z)\psi(c_{n-1})\psi(y_1)\xi(\mu v).$$

This means that the map ξ factors through the n-th derivative $\tilde{\tau}^{(n)}$ in the sense of [Bernstein and Zelevinsky 1976]. Therefore, we can view ξ as a map from the n-th derivative $\tilde{\tau}^{(n)}$ to $\mathcal{J}\{\psi_{\mathcal{V}_n^-}\}(V_\sigma)$, which has the equivalence property, for $a \in \mathrm{GL}_{n-1}$, that

$$\mathcal{J}\{\psi_{\mathcal{V}_n^-}\}(\sigma)\left(\begin{pmatrix} a & 0 \\ x & 1 \end{pmatrix}\right)\xi(v) = \xi\left((\tau')^{(n)}\left(\begin{pmatrix} \mathrm{I}_n & & \\ & 1 & x^* \\ & 0 & \nu_{n-1}a\nu_{n-1} \end{pmatrix}\right)\right)(\mu v),$$

where $x^* = (x_{n-1}, x_{n-2}, \dots, x_1)$ if $x = (x_1, \dots, x_{n-1})$.

Now we come back to the situation of (5-6) with $a \in \mathrm{GL}_{n-1}$. We repeat the same process with the supercuspidality of σ and the genericity of τ. Eventually, we arrive at the $2n$-th derivative of τ', which is the twisted Jacquet module of Whittaker type. The equivalence property in this last case shows that V_σ has a nonzero Whittaker functional. Hence it is generic. This finishes the proof of Theorem 5.1(1).

Acknowledgments

Jiang thanks David Soudry for sharing ideas and thoughts during their collaboration on the local descents for classical groups (as announced in [Soudry 2008]), which inspired the idea of establishing the local descent in this case by using the theory of generalized Shalika models for SO_{4n}. We thank the referee for valuable comments.

References

[Bernstein and Zelevinsky 1976] I. N. Bernšteĭn and A. V. Zelevinskiĭ, "Representations of the group GL(n, F), where F is a local non-Archimedean field", *Uspehi Mat. Nauk* **31**:3(189) (1976), 5–70. In Russian; translated in *Russ. Math. Surveys* **31**:3 (1976), 1–68. MR 54 #12988 Zbl 0348.43007

[Bernstein and Zelevinsky 1977] I. N. Bernstein and A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, I", *Ann. Sci. École Norm. Sup.* (4) **10**:4 (1977), 441–472. MR 58 #28310 Zbl 0412.22015

[Chenevier and Clozel 2009] G. Chenevier and L. Clozel, "Corps de nombres peu ramifiés et formes automorphes autoduales", *J. Amer. Math. Soc.* **22**:2 (2009), 467–519. MR 2476781

[Ginzburg et al. 1997] D. Ginzburg, I. Piatetski-Shapiro, and S. Rallis, *L functions for the orthogonal group*, Mem. Amer. Math. Soc. **128**:611, 1997. MR 98m:11041 Zbl 0884.11022

[Ginzburg et al. 1999] D. Ginzburg, S. Rallis, and D. Soudry, "On a correspondence between cuspidal representations of GL_{2n} and $\widetilde{\mathrm{Sp}}_{2n}$", *J. Amer. Math. Soc.* **12** (1999), 849–907. MR 2000b:22018 Zbl 0928.11027

[Ginzburg et al. 2001] D. Ginzburg, S. Rallis, and D. Soudry, "Generic automorphic forms on $\mathrm{SO}(2n+1)$: functorial lift to $\mathrm{GL}(2n)$, endoscopy, and base change", *Internat. Math. Res. Notices* 14 (2001), 729–764. MR 2002g:11065 Zbl 1060.11031

[Ginzburg et al. 2004] D. Ginzburg, D. Jiang, and S. Rallis, "On the nonvanishing of the central value of the Rankin–Selberg L-functions", *J. Amer. Math. Soc.* **17**:3 (2004), 679–722. MR 2005g:11078 Zbl 1057.11029

[Harris and Taylor 2001] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies **151**, Princeton University Press, Princeton, NJ, 2001. MR 2002m:11050 Zbl 1036.11027

[Henniart 2000] G. Henniart, "Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique", *Invent. Math.* **139**:2 (2000), 439–455. MR 2001e:11052 Zbl 1048.11092

[Jacquet and Rallis 1996] H. Jacquet and S. Rallis, "Uniqueness of linear periods", *Compositio Math.* **102**:1 (1996), 65–123. MR 97k:22025 Zbl 0855.22018

[Jiang and Qin 2007] D. Jiang and Y. Qin, "Residues of Eisenstein series and generalized Shalika models for SO_{4n}", *J. Ramanujan Math. Soc.* **22**:2 (2007), 101–133. MR 2008c:11077 Zbl 1175.11025

[Jiang and Soudry 2003] D. Jiang and D. Soudry, "The local converse theorem for $\mathrm{SO}(2n+1)$ and applications", *Ann. of Math.* (2) **157**:3 (2003), 743–806. MR 2005b:11193 Zbl 1049.11055

[Jiang and Soudry 2004] D. Jiang and D. Soudry, "Generic representations and local Langlands reciprocity law for p-adic SO_{2n+1}", pp. 457–519 in *Contributions to automorphic forms, geometry, and number theory*, edited by H. Hida et al., Johns Hopkins Univ. Press, Baltimore, MD, 2004. MR 2005f:11272 Zbl 1062.11077

[Jiang and Soudry 2007] D. Jiang and D. Soudry, "On the genericity of cuspidal automorphic forms of SO_{2n+1}", *J. Reine Angew. Math.* **604** (2007), 187–209. MR 2008g:22027a Zbl 1139.11027

[Jiang et al. 2008] D. Jiang, C. Nien, and Y. Qin, "Local Shalika models and functoriality", *Manuscripta Math.* **127**:2 (2008), 187–217. MR 2442895 Zbl 1167.11021

[Kudla 1986] S. S. Kudla, "On the local theta-correspondence", *Invent. Math.* **83**:2 (1986), 229–255. MR 87e:22037 Zbl 0583.22010

[Mœglin and Waldspurger 1987] C. Mœglin and J.-L. Waldspurger, "Modèles de Whittaker dégénérés pour des groupes p-adiques", *Math. Z.* **196**:3 (1987), 427–452. MR 89f:22024 Zbl 0612.22008

[Muić 2006] G. Muić, "On the structure of the full lift for the Howe correspondence of $(\mathrm{Sp}(n), \mathrm{O}(V))$ for rank-one reducibilities", *Canad. Math. Bull.* **49**:4 (2006), 578–591. MR 2007j:22021 Zbl 1123.22011

[Nien 2009] C. Nien, "Uniqueness of Shalika models", *Canad. J. Math.* **61**:6 (2009), 1325–1340. MR MR2488457

[Nien 2010] C. Nien, "Local uniqueness of generalized Shalika models for SO($4n$)", *J. Algebra* **323**:2 (2010), 437–457.

[Shahidi 1990] F. Shahidi, "A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups", *Ann. of Math.* (2) **132**:2 (1990), 273–330. MR 91m:11095

[Shahidi 1992] F. Shahidi, "Twisted endoscopy and reducibility of induced representations for p-adic groups", *Duke Math. J.* **66**:1 (1992), 1–41. MR 93b:22034 Zbl 0785.22022

[Soudry 2008] D. Soudry, "Local descent from GL(n) to classical groups", *Oberwolfach Reports* **5** (2008), 247–250.

Received January 7, 2009.

DIHUA JIANG
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN 55455
UNITED STATES
dhjiang@math.umn.edu
http://www.math.umn.edu/~dhjiang

CHUFENG NIEN
DEPARTMENT OF MATHEMATICS
NATIONAL CHENG KUNG UNIVERSITY
TAINAN 701
TAIWAN
nienpig@mail.ncku.edu.tw
http://www.math.ncku.edu.tw/~cfnien/

YUJUN QIN
DEPARTMENT OF MATHEMATICS
EAST CHINA NORMAL UNIVERSITY
DONGCHUAN ROAD 500
SHANGHAI 200241
CHINA
yjqin@math.ecnu.edu.cn
http://math.ecnu.edu.cn/~yjqin/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

CONFORMALLY OSSERMAN MANIFOLDS

YURI NIKOLAYEVSKY

To the memory of Novica Blažić (1959–2005),
a remarkable mathematician and a wonderful person.

An algebraic curvature tensor is called Osserman if the eigenvalues of the associated Jacobi operator are constant on the unit sphere. A Riemannian manifold is called conformally Osserman if its Weyl conformal curvature tensor at every point is Osserman. We prove that a conformally Osserman manifold of dimension $n \neq 3, 4, 16$ is locally conformally equivalent either to a Euclidean space or to a rank-one symmetric space.

1. Introduction

An *algebraic curvature tensor* $\mathcal{R}$ on a Euclidean space $\mathbb{R}^n$ is a (3, 1) tensor having the same symmetries as the curvature tensor of a Riemannian manifold. For $X \in \mathbb{R}^n$, the *Jacobi operator* $\mathcal{R}_X : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\mathcal{R}_X Y = \mathcal{R}(X, Y)X$. The Jacobi operator is symmetric, and $\mathcal{R}_X X = 0$ for all $X \in \mathbb{R}^n$.

Definition 1.1. An algebraic curvature tensor $\mathcal{R}$ is *Osserman* if the eigenvalues of the Jacobi operator $\mathcal{R}_X$ do not depend on the choice of a unit vector $X \in \mathbb{R}^n$.

One of the algebraic curvature tensors naturally associated to a Riemannian manifold (apart from the curvature tensor itself) is the Weyl conformal curvature tensor.

Definition 1.2. A Riemannian manifold is (*pointwise*) *Osserman* if its curvature tensor at every point is Osserman. It is *conformally Osserman* if its Weyl tensor everywhere at every point is Osserman.

It is well known (and easy to check directly) that a Riemannian space locally isometric to a Euclidean space or to a rank-one symmetric space is Osserman. The question of whether the converse is true ("every pointwise Osserman manifold is flat or locally rank-one symmetric") is known as the Osserman conjecture [1990]. The first result on the Osserman conjecture, the affirmative answer for manifolds of

MSC2000: 53B20, 53A30.
Keywords: Osserman manifold, Weyl tensor, Jacobi operator, Clifford structure.
Supported by the FSTE grant.

dimension not divisible by 4, was published before the conjecture itself [Chi 1988]. In the following two decades, substantial progress was made in understanding Osserman and related classes of manifolds, both in the Riemannian and pseudo-Riemannian settings; see the books [Gilkey 2001; 2007; García-Río et al. 2002].

The Osserman conjecture is proved in the most cases, exception being when the dimension of an Osserman manifold is 16 and one of the eigenvalues of the Jacobi operator has multiplicity 7 or 8 [N 2003; 2004; 2005; 2006]. The main difficulty in proving the conjecture in these remaining cases lies in the fact that the Cayley projective plane (and its hyperbolic dual) are Osserman, with the multiplicities of the nonzero eigenvalues of the Jacobi operator being exactly 7 and 8; moreover, the curvature tensor of the Cayley projective plane is essentially different from that of the other rank-one symmetric spaces, as it does not admit a Clifford structure (see Section 2 for details). This is the only known Osserman curvature tensor without a Clifford structure, and to prove the Osserman conjecture in full, it would be very desirable to show that there are no other exceptions.

The study of conformally Osserman manifolds was started by Blažić and Gilkey [2004] and was continued in [Blažić and Gilkey 2005; Blažić et al. 2005; Gilkey 2007; Blažić et al. 2008]. Every Osserman manifold is conformally Osserman (which easily follows from the formula for the Weyl tensor and the fact that every Osserman manifold is Einstein), since also every manifold is locally conformally equivalent to an Osserman manifold.

Theorem 1.3 (main result). *A connected C^∞ Riemannian conformally Osserman manifold of dimension $n \neq 3, 4, 16$ is locally conformally equivalent to a Euclidean space or to a rank-one symmetric space.*

This theorem answers the conjecture made in [Blažić et al. 2005], with three exceptions. (For conformally Osserman manifolds of dimension $n > 6$, not divisible by 4, this conjecture is proved in [Blažić and Gilkey 2004, Theorem 1.4].)

Note that the nature of the three excepted dimensions in Theorem 1.3 is different. In dimension three, the Weyl tensor vanishes, hence giving no information about the manifold at all. In dimension four, even a "genuine" pointwise Osserman manifold may not be locally symmetric (see the examples of "generalized complex space forms" in [Gilkey et al. 1995, Corollary 2.7] and [Olszak 1989]). As proved in [Chi 1988], the Osserman conjecture is still true in dimension four, but in a more restrictive version: One requires the eigenvalues of the Jacobi operator to be constant on the whole unit tangent bundle (a Riemannian manifold having this property is called *globally Osserman*). One might wonder whether the conformal counterpart of this result is true. Blažić and Gilkey [2005] found the elegant characterization that a four-dimensional Riemannian manifold is conformally Osserman if and only if it is either self-dual or anti-self-dual.

In dimension 16, both the conformal and the original Osserman conjecture remain open; for partial results, see [N 2005; 2006] in the Riemannian case and Theorem 3.1 in the conformal case.

As a rather particular case of Theorem 1.3, we obtain an analogue of the Weyl–Schouten theorem for rank-one symmetric spaces: A Riemannian manifold of dimension greater than four having "the same" Weyl tensor as that of one of the complex/quaternionic projective spaces or their noncompact duals is locally conformally equivalent to that space. More precisely:

Theorem 1.4. *Let M_0^n denote one of the spaces $\mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$, $\mathbb{H}P^{n/4}$ or $\mathbb{H}H^{n/4}$, and let W_0 be the Weyl tensor of M_0^n at some point $x_0 \in M_0^n$. Suppose that for every point x of a Riemannian manifold M^n with $n > 4$ there exists a linear isometry $\iota : T_x M^n \to T_{x_0} M_0^n$ that maps the Weyl tensor of M^n at x on a positive multiple of W_0. Then M^n is locally conformally equivalent to M_0^n.*

The claim follows from [Blažić and Gilkey 2004, Theorem 1.4] for $M_0^n = \mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$ and $n > 6$. The fact that the dimension $n = 16$ is not excluded (as compared to Theorem 1.3) follows from Theorem 3.1.

We assume all the object (manifolds, metrics, vector and tensor fields) to be smooth (of class C^∞), although all the results remain valid for class C^k, with sufficiently large k.

The paper is organized as follows. In Section 2, we give some background on Osserman algebraic curvature tensors and on Clifford structures and prove some technical lemmas. The proof of Theorem 1.3 is given in Section 3. Theorem 1.3 is deduced from a more general Theorem 3.1. We first prove the local version using the second Bianchi identity, and then the global version by showing that the "algebraic type" of the Weyl tensor is the same at all points of a connected conformally Osserman Riemannian manifold (in particular, a nonzero Osserman Weyl tensor cannot degenerate to zero).

2. Algebraic curvature tensors with a Clifford structure

2.1. ***Clifford structure.*** The requirement that an algebraic curvature tensor $\mathcal{R}$ be Osserman is algebraically quite restrictive. In most cases, such a tensor can be obtained by the following construction, suggested in [Gilkey et al. 1995], which generalizes the curvature tensor of complex and quaternionic projective space.

Definition 2.2. A *Clifford structure* $\mathrm{Cliff}(\nu; J_1, \dots, J_\nu; \lambda_0, \eta_1, \dots, \eta_\nu)$ on $\mathbb{R}^n$ is a set of $\nu \geq 0$ anticommuting almost Hermitian structures J_i and $\nu + 1$ real numbers $\lambda_0, \eta_1, \dots \eta_\nu$, with $\eta_i \neq 0$. An algebraic curvature tensor $\mathcal{R}$ on $\mathbb{R}^n$ *has a Clifford* *structure* $\mathrm{Cliff}(\nu;\, J_1, \dots, J_\nu;\, \lambda_0, \eta_1, \dots, \eta_\nu)$ if

$$\mathcal{R}(X,Y)Z = \lambda_0(\langle X,Z\rangle Y - \langle Y,Z\rangle X) + \sum_{i=1}^{\nu} \eta_i (2\langle J_i X, Y\rangle J_i Z + \langle J_i Z, Y\rangle J_i X - \langle J_i Z, X\rangle J_i Y). \tag{2-1}$$

When it does not create ambiguity, we write $\mathrm{Cliff}(\nu;\, J_1, \dots, J_\nu;\, \lambda_0, \eta_1, \dots, \eta_\nu)$ simply as $\mathrm{Cliff}(\nu)$.

Remark 2.3. Definition 2.2 implies that the operators J_i are skew-symmetric and orthogonal and satisfy the equations

$$\langle J_i X, J_j X\rangle = \delta_{ij}\|X\|^2 \quad \text{and} \quad J_i J_j + J_j J_i = -2\delta_{ij}\,\mathrm{id}$$

for all $i, j = 1, \dots, \nu$ and all $X \in \mathbb{R}^n$. This implies that every algebraic curvature tensor with a Clifford structure is Osserman, as by (2-1) the Jacobi operator has the form $\mathcal{R}_X Y = \lambda_0(\|X\|^2 Y - \langle Y, X\rangle X) + \sum_{i=1}^{\nu} 3\eta_i \langle J_i X, Y\rangle J_i X$. So for a unit vector X, the eigenvalues of $\mathcal{R}_X$ are λ_0 (of multiplicity $n-1-\nu$ if $\nu < n-1$), 0, and $\lambda_0 + 3\eta_i$ for $i = 1, \dots, \nu$.

The converse — every Osserman algebraic curvature tensor has a Clifford structure — is true in all dimensions but $n = 16$ and also in many cases when $n = 16$, as follows from [N 2005, Proposition 1 and the penultimate paragraph of the proof of Theorems 1 and 2], [N 2004, Proposition 1] and [N 2006, Proposition 2.1]. The only known counterexample is the curvature tensor $R^{\mathbb{O}P^2}$ of the Cayley projective plane (more precisely, any algebraic curvature tensor of the form $\mathcal{R} = aR^{\mathbb{O}P^2} + bR^1$, where R^1 is the curvature tensor of the unit sphere $S^{16}(1)$ and $a \neq 0$).

A Clifford structure $\mathrm{Cliff}(\nu)$ on Euclidean $\mathbb{R}^n$ turns it into a Clifford module; see [Atiyah et al. 1964, Part 1], [Husemoller 1975, Chapter 11], and [Lawson and Michelsohn 1989, Chapter 1] for standard facts on Clifford algebras and Clifford modules). A *Clifford algebra* $\mathrm{Cl}(\nu)$ on ν generators $x_1, \dots, x_\nu$ is an associative unital algebra over $\mathbb{R}$ defined by the relations $x_i x_j + x_j x_i = -2\delta_{ij}$. The homomorphism $\sigma : \mathrm{Cl}(\nu) \to \mathrm{End}(\mathbb{R}^n)$ of associative algebras defined on generators by $\sigma(x_i) = J_i$ and $\sigma(1) = \mathrm{id}$ is a representation of $\mathrm{Cl}(\nu)$ on $\mathbb{R}^n$. Since all the J_i are orthogonal and skew-symmetric, σ gives rise to an *orthogonal multiplication* defined as follows. In the Euclidean space $\mathbb{R}^\nu$, fix an orthonormal basis $e_1, \dots, e_\nu$. For every $u = \sum_{i=1}^{\nu} u_i e_i \in \mathbb{R}^\nu$ and every $X \in \mathbb{R}^n$, define

$$J_u X = \sum\nolimits_{i=1}^{\nu} u_i J_i X \tag{2-2}$$

(when $u = e_i$, we abbreviate J_{e_i} to J_i). The map $J : \mathbb{R}^\nu \times \mathbb{R}^n \to \mathbb{R}^n$ defined by (2-2) is an orthogonal multiplication: $\|J_u X\|^2 = \|u\|^2 \|X\|^2$ (similarly, we can define an orthogonal multiplication $J : \mathbb{R}^{\nu+1} \times \mathbb{R}^n \to \mathbb{R}^n$ by $J_u X = u_0 X + \sum_{i=1}^{\nu} u_i J_i X$ for $u =$

$\sum_{i=0}^{\nu} u_i e_i \in \mathbb{R}^{\nu+1}$, where $e_0, \dots, e_\nu$ is an orthonormal basis for Euclidean $\mathbb{R}^{\nu+1}$). For $X \in \mathbb{R}^n$, denote

$$\mathcal{J}X = \operatorname{Span}(J_1X, \dots, J_\nu X) \quad \text{and} \quad \mathcal{I}X = \operatorname{Span}(X, J_1X, \dots, J_\nu X).$$

We also use the complexified versions of these subspaces, which we denote by $\mathcal{J}_{\mathbb{C}}X$ and $\mathcal{I}_{\mathbb{C}}X$ respectively for $X \in \mathbb{C}^n$.

If $\mathbb{R}^n$ is a $\mathrm{Cl}(\nu)$-module (equivalently, if there exists an algebraic curvature tensor with a Clifford structure $\mathrm{Cliff}(\nu)$ on $\mathbb{R}^n$), then

$$\nu \le 2^b + 8a - 1, \quad \text{where } n = 2^{4a+b}c,\ c \text{ is odd, and } 0 \le b \le 3; \tag{2-3}$$

see, for instance, [Husemoller 1975, Theorem 11.8.2].

As a direct consequence of (2-3), we have the following inequalities.

Lemma 2.4. *Let $\mathcal{R}$ be an algebraic curvature tensor with a Clifford structure* $\mathrm{Cliff}(\nu)$ *on $\mathbb{R}^n$. Suppose that $n > 4$ and $n \ne 8, 16$. Then*

(i) $n \ge 3\nu + 3$, *with equality only when $n = 6$ and $\nu = 1$, or $n = 12$ and $\nu = 3$, or $n = 24$ and $\nu = 7$;*

(ii) $n > 4\nu - 2$, *except when $n = 24$ and $\nu = 7$ or $n = 32$ and $\nu = 9$;*

(iii) *there exists an integer l such that $\nu < 2^l < n$.*

Proof. Let $\rho(n) = 2^b + 8a - 1$, the right side of (2-3). Then $\nu \le \rho(n)$. First suppose that $n = 2^m c$, with $m = 4a + b \ge 6$, where $0 \le b \le 3$ and c is odd. We claim that $n > 4\rho(n)$. Indeed, $n \ge 2^m = 2^{4a+b}$, so it suffices to show that $2^{4a-2} > 1 + 2^{3-b}a - 2^{-b}$. The latter inequality follows from $2^{4a-2} > 1 + 8a$, when $a \ge 2$, and is also true when $a = 1$ and $b = 2, 3$. Since $n > 4\rho(n)$, (ii) is obvious, (i) is satisfied (since $\rho(n) > 3$), and (iii) is satisfied with $l = m - 1$.

In each of the remaining cases ($n = 2^m c$, with an odd c and $m = 0, \dots, 5$), $\rho(n)$ can be computed explicitly and the claim follows by a routine check. $\square$

2.5. *Clifford structures on $\mathbb{R}^8$ and the octonions.* The proof of Theorem 1.3 in the generic case uses that ν is small relative to n (with the required estimates given in Lemma 2.4). However, in the case $n = 8$, the number ν can be as large as 7, according to (2-3). Consider this case in more detail. In [N 2004], it is shown that every Osserman algebraic curvature tensor $\mathcal{R}$ on $\mathbb{R}^8$ has a Clifford structure, and that either $\mathcal{R}$ has a $\mathrm{Cliff}(3)$ structure with $J_1J_2 = \pm J_3$, or an existing $\mathrm{Cliff}(\nu)$ structure can be complemented to a $\mathrm{Cliff}(7)$ structure. More precisely:

Lemma 2.6. (1) *Suppose $\mathcal{R}$ is an algebraic curvature tensor on $\mathbb{R}^8$ with Clifford structure* $\mathrm{Cliff}(\nu; J_1, \dots, J_\nu; \lambda_0, \eta_1, \dots, \eta_\nu)$. *Then exactly one of two possibilities may occur: either $\mathcal{R}$ has a Clifford structure* $\mathrm{Cliff}(3)$ *with $J_1J_2 = J_3$, or there exist $7 - \nu$ operators $J_{\nu+1}, \dots, J_7$ such that $J_1, \dots, J_7$ are anticommuting almost Hermitian structures with $J_1J_2 \dots J_7 = \mathrm{id}_{\mathbb{R}^8}$ and $\mathcal{R}$ has a Clifford* *structure* $\mathrm{Cliff}(7;\, J_1, \dots, J_7;\, \lambda_0 - 3\xi, \eta_1 + \xi, \dots, \eta_\nu + \xi, \xi, \dots, \xi)$ *for any* $\xi \neq -\eta_i, 0$.

(2) *Let* $\mathbb{O}$ *be the octonion algebra with inner product defined by* $\|u\|^2 = uu^*$, *where* * *is the octonion conjugation, and let* $\mathbb{O}' = 1^\perp$, *the space of imaginary octonions. Then, in the second case in part* (1), *there exist linear isometries* $\imath_1 : \mathbb{R}^8 \to \mathbb{O}$ *and* $\imath_2 : \mathbb{R}^7 \to \mathbb{O}'$ *such that the orthogonal multiplication* (2-2) *is given by* $J_u X = \imath_1(X)\imath_2(u)$.

Proof. (1) This claim is proved in [N 2004, Lemma 5]. The proof is based on the fact that every representation σ of $\mathrm{Cl}(\nu)$ on $\mathbb{R}^8$, except for the representations of $\mathrm{Cl}(3)$ with $J_1 J_2 = \pm J_3$, is a restriction of a representation of $\mathrm{Cl}(7)$ on $\mathbb{R}^8$ to $\mathrm{Cl}(\nu) \subset \mathrm{Cl}(7)$. It follows that the almost Hermitian structures $J_1, \dots, J_\nu$ defined by σ can be complemented by almost Hermitian structures $J_{\nu+1}, \dots, J_7$ such that $J_1, \dots, J_7$ anticommute, and so $\mathcal{R}$ can be written in the form (2-1), with a formal summation up to 7 on the right side (but with $\eta_i = 0$ when $i = \nu + 1, \dots, 7$). To obtain a Cliff(7) structure for $\mathcal{R}$, according to Definition 2.2, we only need to make all the η_i nonzero. This can be done using the identity

$$\langle X, Z\rangle Y - \langle Y, Z\rangle X = \sum_{i=1}^{7} \tfrac{1}{3}\bigl(2\langle J_i X, Y\rangle J_i Z + \langle J_i Z, Y\rangle J_i X - \langle J_i Z, X\rangle J_i Y\bigr), \tag{2-4}$$

which is gotten from the polarized identity

$$\|X\|^2 Y - \langle X, Y\rangle X = \textstyle\sum_{i=1}^{7} \langle J_i X, Y\rangle J_i X,$$

which is true because for $X \neq 0$ the vectors $\|X\|^{-1}X, \|X\|^{-1}J_1X, \dots, \|X\|^{-1}J_7X$ form an orthonormal basis for $\mathbb{R}^8$. Then by (2-1), $\mathcal{R}$ has a Clifford structure $\mathrm{Cliff}(7;\, J_1, \dots, J_7;\, \lambda_0 - 3\xi, \eta_1 + \xi, \dots, \eta_\nu + \xi, \xi, \dots, \xi)$ for any $\xi \neq -\eta_i, 0$.

(2) This claim is proved in [N 2004, the beginning of Section 5.1]. The proof is based on the following. There are two nonisomorphic representations of $\mathrm{Cl}(7)$ on $\mathbb{R}^8$. By identifying $\mathbb{R}^8$ with the octonion algebra $\mathbb{O}$ via a linear isometry, these representations are given by the orthogonal multiplications $J_u X = uX$ and $J_u X = Xu$ respectively [Lawson and Michelsohn 1989, Section I.8]. Since $(uX)^* = X^* u^* = -X^* u$ for all $u, X \in \mathbb{O}$ with $u \perp 1$, the first representation is orthogonally equivalent to the second one, with the operators J_i replaced by $-J_i$. Since changing the signs of the J_i does not affect the form of the algebraic curvature tensor (2-1), we can always assume that a Cliff(7) structure for an algebraic curvature tensor on $\mathbb{R}^8$ is given by the orthogonal multiplication $J_u X = \imath_1(X)\imath_2(u)$. □

In the proof of Theorem 1.3 for $n = 8$, we will usually identify $\mathbb{R}^8$ with $\mathbb{O}$ and identify $\mathbb{R}^7$ with $\mathbb{O}'$ via some fixed linear isometries $\imath_1$ and $\imath_2$, and we will simply write the orthogonal multiplication in the form

$$J_u X = Xu, \tag{2-5}$$

where $X \in \mathbb{R}^8 = \mathbb{O}$ and $u \in \mathbb{O}'$. The proof of Theorem 1.3 for $n = 8$ extensively uses computations in the octonion algebra $\mathbb{O}$, in particular, the standard identities

$$\begin{aligned} a^* &= 2\langle a, 1\rangle 1 - a, & \langle a, b\rangle &= \langle a^*, b^*\rangle = \tfrac{1}{2}(a^*b + b^*a),\\ a(ab) &= a^2 b, & \langle a, bc\rangle &= \langle b^*a, c\rangle = \langle ac^*, b\rangle,\\ (ab^*)c + (ac^*)b &= 2\langle b, c\rangle a, & \langle ab, ac\rangle &= \langle ba, ca\rangle = \|a\|^2\langle b, c\rangle \end{aligned}$$

for any $a, b, c \in \mathbb{O}$, and the like; see for example [Harvey and Lawson 1982, Section IV]. It also uses the fact that $\mathbb{O}$ is a division algebra; in particular, any nonzero octonion is invertible: $a^{-1} = \|a\|^{-2}a^*$. We will also use the *bioctonions* $\mathbb{O}\otimes\mathbb{C}$, the algebra over the $\mathbb{C}$ that has same multiplication table as $\mathbb{O}$. Since all the identities above are polynomial, they still hold for bioctonions, with the complex inner product on $\mathbb{C}^8$, the underlying linear space of $\mathbb{O}\otimes\mathbb{C}$. However, the bioctonion algebra is not a division algebra (and has zero-divisors: $(\mathrm{i}1 + e_1)(\mathrm{i}1 - e_1) = 0$).

The proof of Theorem 1.3 will require a technical lemma.

Lemma 2.7. (1) *Let $J_1, \dots, J_\nu$ be anticommuting almost Hermitian structures on $\mathbb{R}^n$, and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a homogeneous polynomial map of degree m such that $F(X) \in \mathcal{J}X$ for all $X \in \mathbb{R}^n$. Suppose that $n > 4$, and also $\nu \le 3$ if $n = 8$ and $\nu \le 7$ if $n = 16$. Then there exist homogeneous polynomials c_i for $i = 1, \dots, \nu$ of degree $m - 1$ such that $F(X) = \sum_{i=1}^{\nu} c_i(X) J_i X$.*

With the same assumption, but with $\mathcal{J}$ replaced by $\mathcal{I}$, an additional homogeneous degree $m - 1$ polynomial c_0 appears, and $c_0(X)X$ is added to $F(X)$.

(2) *Let $J_1, \dots, J_\nu$ be anticommuting almost Hermitian structures on $\mathbb{R}^n$. Suppose that $n > 4$ and that $\nu \le 3$ if $n = 8$. Let $1 \le k \le \nu$ and let a_j for $1 \le j \le \nu$ with $j \ne k$ be $\nu - 1$ vectors in $\mathbb{R}^n$ such that*

$$\sum_{j\ne k}(\langle a_j, J_k Y\rangle J_j Y + \langle a_j, Y\rangle J_k J_j Y) = 0 \quad \textit{for all } Y \in \mathbb{R}^n. \tag{2-6}$$

Then either $a_j = 0$ for all $j \ne k$, or $\nu = 1$, or $\nu = 3$, $J_1 J_2 = \varepsilon J_3$, $\varepsilon = \pm 1$, and $a_j = J_i v$, where $\{i, j, k\} = \{1, 2, 3\}$ and $v \ne 0$.

(3) *Let N^n be a smooth Riemannian manifold and let $J_1, \dots, J_\nu$ be anticommuting almost Hermitian structures on N^n. Suppose that for every nowhere vanishing smooth vector field X on N^n, the distribution $\mathcal{J}X = \mathrm{Span}(J_1 X, \dots, J_\nu X)$ is smooth (that is, the ν-form $J_1 X \wedge \cdots \wedge J_\nu X$ is smooth). Then for every $x \in N^n$, there exists a neighborhood $\mathcal{U} = \mathcal{U}(x)$ and smooth anticommuting almost Hermitian structures $\tilde{J}_1, \dots, \tilde{J}_\nu$ on $\mathcal{U}$ such that $\mathrm{Span}(\tilde{J}_1 X, \dots, \tilde{J}_\nu X) = \mathrm{Span}(J_1 X, \dots, J_\nu X)$ for any vector field X on $\mathcal{U}$.*

Proof. (1) It is sufficient to prove the assertion for the case $F(X) \in \mathcal{J}X$.

Since for every $X \neq 0$, the vectors $X, J_1X, \dots, J_\nu X$ are orthogonal and have the same length $\|X\|$, we have

$$\|X\|^2 F(X) = f_0(X)X + \sum_{i=1}^{\nu} f_i(X) J_i X,$$

where $f_0(X) = \langle F(X), X\rangle$ and $f_i(X) = \langle F(X), J_i X\rangle$ are homogeneous polynomials of degree $m+1$ of X (or possibly zeros). Taking the squared lengths of the both sides we get

$$\|X\|^2 \|F(X)\|^2 = f_0^2(X) + \sum_{i=1}^{\nu} f_i^2(X),$$

so the sum of squares of the $\nu+1$ polynomials $f_0(X), f_1(X), \dots, f_\nu(X)$ is divisible by $\|X\|^2$. For $X = (x_1, \dots, x_n)$, let $(\|X\|^2)$ be the ideal of $\mathbb{R}[X]$ generated by $\|X\|^2 = \sum_j x_j^2$, and let $\mathbf{R}$ be the quotient of $\mathbb{R}[X]$ by this ideal. Let π be the natural projection from $\mathbb{R}[X]$ to $\mathbf{R}$. We have $\sum_{i=0}^{\nu} \hat{f}_i^2 = 0$, where $\hat{f}_i = \pi f_i$. If at least one of the $\hat{f}_i$ is nonzero (say the ν-th one), then $\sum_{i=0}^{\nu-1} (\hat{f}_i/\hat{f}_\nu)^2 = -1$ in $\mathbb{F}$, the field of fractions of the ring $\mathbf{R}$. The field $\mathbb{F}$ is isomorphic to the field $\mathbb{L}_{n-1} = \mathbb{R}(x_1, \dots, x_{n-1}, \sqrt{-d})$, where $d = x_1^2 + \dots + x_{n-1}^2$ (an isomorphism from $\mathbb{L}_{n-1}$ to $\mathbb{F}$ is induced by the map $(a + b\sqrt{-d})/c \to (a + bx_n)/c$, with $a, b, c \in \mathbb{R}[x_1, \dots, x_{n-1}]$ and $c \neq 0$). By [Pfister 1995, Theorem 3.1.4], the level of the field $\mathbb{L}_{n-1}$, the minimal number of elements whose sum of squares is -1, is 2^l, where $2^l < n \le 2^{l+1}$. It follows that we arrive at a contradiction in all the cases when $\nu < 2^l < n$. This means that $\hat{f}_i = 0$ for all $i = 0, \dots, \nu$, so each of the f_i is divisible by $\|X\|^2$ in $\mathbb{R}[X]$, so

$$F(X) = (\|X\|^{-2} f_0(X))X + \sum_{i=1}^{\nu} (\|X\|^{-2} f_i(X)) J_i X,$$

with all the nonzero coefficients on the right side being homogeneous polynomials of degree $m-1$. The claim now follows from Lemma 2.4(iii).

(2) If $\nu = 1$, Equation (2-6) is trivially satisfied. If $\nu = 2$, the claim follows immediately by taking the inner product of (2-6) with $J_1 J_2 Y$. Suppose $\nu = 3$. Taking the inner product of (2-6) with $J_i Y$ and $i \neq k$, we obtain

$$\langle a_i, J_k Y\rangle \|Y\|^2 = \langle a_j, Y\rangle \langle J_i J_k J_j, Y\rangle,$$

where $\{i, j, k\} = \{1, 2, 3\}$. It follows that the polynomial $\langle J_i J_k J_j Y, Y\rangle$ is divisible by $\|Y\|^2$. Since the operator $J_i J_k J_j$ is symmetric and orthogonal, it equals $\tilde{\varepsilon}\,\mathrm{id}$, with $\tilde{\varepsilon} = \pm 1$; hence $J_1 J_2 = \varepsilon J_3$ with $\varepsilon = \pm 1$. Then $-J_k a_i = \tilde{\varepsilon} a_j$, so $J_i a_j = -\tilde{\varepsilon} J_i J_k a_i = -\tilde{\varepsilon} J_i J_k a_i = J_j a_i$. Therefore for all i, j such that $\{i, j, k\} = \{1, 2, 3\}$, we have $a_j = J_i v$ and $a_i = J_j v$, and we can assume that $v \neq 0$, since otherwise $a_i = a_j = 0$.

Now suppose $\nu > 3$ and let $L = \mathrm{Span}(a_j)$. It follows from (2-6) that if $Y \perp L$, then $J_k Y \perp L$, so L is J_k-invariant. Polarizing (2-6) we obtain

$$\sum_{j\neq k}(\langle a_j, J_k X\rangle J_j Y + \langle a_j, X\rangle J_k J_j Y) + \sum_{j\neq k}(\langle a_j, J_k Y\rangle J_j X + \langle a_j, Y\rangle J_k J_j X) = 0.$$

It follows that, for all $X \perp L$ and all $Y \in \mathbb{R}^n$,

$$\sum_{j\neq k}(\langle a_j, J_k Y\rangle J_j X + \langle a_j, Y\rangle J_k J_j X) = 0,$$

that is, with $u(Y) = \sum_{j\neq k}\langle a_j, J_k Y\rangle e_j$ and $v(Y) = \sum_{j\neq k}\langle a_j, Y\rangle e_j$, we have that $J_{u(Y)}X = -J_k J_{v(Y)}X$. Note that $u(Y)$ and $v(Y)$ are perpendicular to e_k. Now, fix an arbitrary $Y \in \mathbb{R}^n$ and choose a unit vector w perpendicular to $u(Y)$, $v(Y)$ and e_k in $\mathbb{R}^\nu$ (this is possible since $\nu > 3$). Then $J_w J_{u(Y)}X = -J_w J_k J_{v(Y)}X$, so $\langle J_w J_k J_{v(Y)}X, X\rangle = 0$ for all $X \in L^\perp$. If $v(Y) \neq 0$, the operator $\|v(Y)\|^{-1} J_w J_k J_{v(Y)}$ is symmetric and orthogonal, so the maximal dimension of its isotropic subspace is $n/2 < n - (\nu - 1) = \dim L^\perp$ (the inequality follows from Lemma 2.4(ii)), which is a contradiction. Hence $v(Y) = 0$ for all $Y \in \mathbb{R}^n$, so all the a_j are zeros.

(3) We first prove the lemma assuming $2\nu \leq n$. In this case, the proof closely follows the arguments in the proof of [N 2003, Lemma 3.1].

Let $Y_0 \in T_x N^n$ be a unit vector. Since $2\nu \leq n$, there exists a unit vector $E \in T_x N^n$ that is not in the range of the map $\Phi : S^{\nu-1} \times S^{\nu-1} \to S^{n-1}$, $\Phi(u, v) \mapsto J_u J_v Y_0$. Then $\mathcal{J}E \cap \mathcal{J}Y_0 = 0$. It follows that on some neighborhood $\mathcal{U}'$ of x, there exist smooth unit vector fields Y and E_n such that $E_n(x) = E$, $Y(x) = Y_0$ and $\mathcal{J}E_n \cap \mathcal{J}Y = 0$ at every point $y \in \mathcal{U}'$. By assumption, the ν-dimensional distribution $\mathcal{J}E_n$ is smooth, so we can choose ν smooth orthonormal sections $E_1, \dots, E_\nu$ of it, and then define anticommuting almost Hermitian structures $\tilde{J}_\alpha$ on $\mathcal{U}'$ satisfying $\tilde{J}_\alpha E_n = E_\alpha$ by setting $\tilde{J}_\alpha = \sum_{\beta=1}^{\nu} a_{\alpha\beta} J_\beta$, where $(a_{\alpha\beta})$ is the $\nu \times \nu$ orthogonal matrix given by $a_{\alpha\beta} = \langle E_\alpha, J_\beta E_n\rangle$.

Let $E_{\nu+1}, \dots, E_{n-1}$ be orthonormal vector fields on $\mathcal{U}'$ such that $E_1, \dots, E_n$ is an orthonormal frame, and for a vector field X on $\mathcal{U}'$, let $\tilde{J}X$ denote the $n \times \nu$ matrix whose column vectors are $\tilde{J}_1 X, \dots, \tilde{J}_\nu X$ relative to the frame $E_1, \dots, E_n$. Then $(\tilde{J}X)^t \tilde{J}X = \|X\|^2 I_\nu$ and all the $\nu \times \nu$ minors of the matrix $\tilde{J}X$ are smooth functions on $\mathcal{U}'$. Moreover, the entries of the matrices $\tilde{J}E_i$ for $i = 1, \dots, n$ are the rearranged entries of the matrices $\tilde{J}_\alpha$ for $\alpha = 1, \dots, \nu$ relative to the basis $\{E_i\}$, so to prove that the $\tilde{J}_\alpha$ are smooth it suffices to show that all the entries of the matrices $\tilde{J}E_i$ are smooth (on a possibly smaller neighborhood). Write $\tilde{J}E_i = \binom{K_i}{P_i}$, where K_i and P_i are respectively $\nu \times \nu$ and $(n - \nu) \times \nu$ matrix-valued functions on $\mathcal{U}'$;

note that $\tilde{J}E_n = \binom{I_\nu}{0}$. For an arbitrary $t \in \mathbb{R}$, all the $\nu \times \nu$ minors of the matrix

$$\tilde{J}(E_i + tE_n) = \begin{pmatrix} K_i + tI_\nu \\ P_i \end{pmatrix}$$

are smooth. For every entry $(P_i)_{k\alpha}$, where $k = \nu+1, \dots, n$ and $\alpha = 1, \dots, \nu$, the coefficient of $t^{\nu-1}$ in the $\nu \times \nu$ minor of $\tilde{J}(E_i + tE_n)$ consisting of $\nu - 1$ out of the first ν rows (omitting the α-th row) and the k-th row is $\pm(P_i)_{k\alpha}$, so all the entries of all the P_i are smooth.

For the vector field Y defined above, write $\tilde{J}Y = \binom{K}{P}$. Since $P = \sum_{i=1}^n \langle Y, E_i \rangle P_i$, all the entries of P are smooth on $\mathcal{U}'$. Moreover, since $\mathcal{J}Y \cap \mathcal{J}E_n = 0$, the spans of the vector columns of the matrices $\tilde{J}Y$ and $\tilde{J}E_n = \binom{I_\nu}{0}$ have trivial intersection, so $\operatorname{rk} P = \nu$ at every point $y \in \mathcal{U}'$. Therefore we can choose the rows $\nu + 1 \le b_1 < \cdots < b_\nu \le n$ of the matrix P at the point x so that the corresponding minor $P_{(b)} = P_{b_1 \dots b_\nu}$ is nonzero. Then the same minor $P_{(b)}$ is nonzero on a (possibly smaller) neighborhood $\mathcal{U} \subset \mathcal{U}'$ of x. Taking all the $\nu \times \nu$ minors of $\tilde{J}Y$ consisting of $\nu - 1$ out of ν rows of $P_{(b)}$ and one row of K, we obtain that all the entries of K are smooth on $\mathcal{U}$. Moreover, for an arbitrary $t \in \mathbb{R}$, all the $\nu \times \nu$ minors of the matrix

$$\tilde{J}(tE_i + Y) = \begin{pmatrix} tK_i + K \\ tP_i + P \end{pmatrix}$$

are smooth. Computing the coefficient of t in all the $\nu \times \nu$ minors of $\tilde{J}(tE_i + Y)$ consisting of $\nu - 1$ out of ν rows of $(tP_i + P)_{(b)}$ and one row of $tK_i + K$, and using the fact that all the entries of K, P and P_i are smooth on $\mathcal{U}$, we obtain that all the entries of K_i are also smooth on $\mathcal{U}$. Therefore all the entries of all the matrices $\tilde{J}E_i$ are smooth on $\mathcal{U}$; hence the anticommuting almost Hermitian structures $\tilde{J}_\alpha$ are also smooth on $\mathcal{U}$.

Since ν and n must satisfy inequality (2-3) (and hence those of Lemma 2.4), the above proof works in all the cases except when $n = 4$ and $\nu = 3$ and when $n = 8$ and $\nu = 5, 6, 7$. The former case is easy: Taking any smooth orthonormal frame E_i on a neighborhood of x and defining $\tilde{J}_\alpha = \sum_{\beta=1}^3 a_{\alpha\beta} J_\beta$ with the orthogonal 3×3 matrix $(a_{\alpha\beta})$ given by $a_{\alpha\beta} = \langle E_\alpha, J_\beta E_4 \rangle$, we see that all the entries of the $\tilde{J}_\alpha$ relative to the basis E_i are ± 1 and 0.

The proof in the cases that $n = 8$ and $\nu = 5, 6, 7$ is based on the fact that, except when $\nu = 3$ and $J_1 J_2 = \pm J_3$, any set of anticommuting almost Hermitian structures $J_1, \dots, J_\nu$ on $\mathbb{R}^8$ can be complemented by almost Hermitian structures $J_{\nu+1}, \dots, J_7$ to a set $J_1, \dots, J_7$ of anticommuting almost Hermitian structures on $\mathbb{R}^8$ (this is Lemma 2.6(1)).

If $n = 8$ and $\nu = 7$, choose an arbitrary smooth almost Hermitian structure J_7 on some neighborhood $\mathcal{U}$ of x and complement it by anticommuting almost Hermitian structures $J_1, \dots, J_6$ at every point of $\mathcal{U}$. Then for every smooth nowhere vanishing vector field X on $\mathcal{U}$, $\mathrm{Span}(J_1X, \dots, J_6X) = (\mathrm{Span}(X, J_7X))^\perp$ is a smooth distribution. This reduces the case $n = 8$ and $\nu = 7$ to the case $n = 8$ and $\nu = 6$.

Let $n = 8$ and $\nu = 6$, and let J_7 be an almost Hermitian structure complementing $J_1, \dots, J_6$ at every point $x \in N^n$. Using the first part of the proof (or the fact that J_7X spans the one-dimensional smooth distribution $(\mathrm{Span}(J_1X, \dots, J_6X) \oplus \mathbb{R}X)^\perp$ for every nonvanishing smooth vector field X) we can assume that J_7 is smooth on a neighborhood $\mathcal{U}$ of $x \in N^n$. Choose a smooth orthonormal frame $E_1, \dots, E_8$ on (a possibly smaller neighborhood) $\mathcal{U}$ such that the matrix of J_7 relative to E_i is $\left(\begin{smallmatrix} 0 & I_4 \\ -I_4 & 0 \end{smallmatrix}\right)$ and define the almost Hermitian structure $\tilde{J}_6$ on $\mathcal{U}$ by

$$\tilde{J}_6E_2 = E_1, \quad \tilde{J}_6E_4 = E_3, \quad \tilde{J}_6E_6 = -E_5, \quad \tilde{J}_6E_8 = -E_7.$$

Then J_7 and $\tilde{J}_6$ anticommute; hence we can complement them by almost Hermitian structures $J'_1, \dots, J'_5$ on $\mathcal{U}$ so that $J'_1, \dots, J'_5, \tilde{J}_6, J_7$ are anticommuting almost Hermitian structures. Moreover, since both J_7 and $\tilde{J}_6$ are smooth on $\mathcal{U}$, the five-dimensional distribution $\mathrm{Span}(J'_1X, \dots, J'_5X) = (\mathrm{Span}(X, J_7X, \tilde{J}_6X))^\perp$ is smooth for every smooth nowhere vanishing vector field X on $\mathcal{U}$. This reduces the case $n = 8$ and $\nu = 6$ to the case $n = 8$ and $\nu = 5$. Indeed, if $\tilde{J}_1, \dots, \tilde{J}_5$ are smooth anticommuting almost Hermitian structures on $\mathcal{U}$ such that $\mathrm{Span}(\tilde{J}_1X, \dots, \tilde{J}_5X) = \mathrm{Span}(J'_1X, \dots, J'_5X)$ for every vector field X, then $\tilde{J}_1, \dots, \tilde{J}_5, \tilde{J}_6$ are the required almost Hermitian structures, since

$$\begin{aligned}\mathrm{Span}(\tilde{J}_1X, \dots, \tilde{J}_6X) &= \mathrm{Span}(J'_1X, \dots, J'_5X, \tilde{J}_6X) \\ &= (\mathrm{Span}(X, J_7X))^\perp = \mathrm{Span}(J_1X, \dots, J_6X),\end{aligned}$$

for every vector field X on $\mathcal{U}$, and $\tilde{J}_6$ anticommutes with every $\tilde{J}_\alpha$ for $\alpha = 1, \dots, 5$, since it anticommutes with every J'_α for $\alpha = 1, \dots, 5$.

Let $n = 8$ and $\nu = 5$. Let J_6 and J_7 be anticommuting almost Hermitian structures complementing $J_1, \dots, J_5$ at every point $x \in N^n$. Since $\mathrm{Span}(J_6X, J_7X) = (\mathrm{Span}(J_1X, \dots, J_5X))^\perp$, we can choose such J_6 and J_7 to be smooth on a neighborhood $\mathcal{U}$ of $x \in N^n$, by the first part of the proof. Choose a smooth orthonormal frame $E_1, \dots, E_8$ on (a possibly smaller neighborhood) $\mathcal{U}$ as follows. First choose an arbitrary smooth unit vector field E_1 on $\mathcal{U}$. The vector fields J_6E_1 and J_7E_1 are orthonormal; set $E_2 = -J_6E_1$, $E_3 = -J_7E_1$. The unit vector field $J_6J_7E_1$ is orthogonal to E_1, J_6E_1 and J_7E_1; set $E_4 = -J_6J_7E_1$. Choose an arbitrary smooth unit section E_5 of the smooth distribution $(\mathrm{Span}(E_1, E_2, E_3, E_4))^\perp$ on $\mathcal{U}$. That distribution is both J_6- and J_7-invariant, so we can set, similar to above, $E_6 = J_6E_5$, $E_7 = J_7E_5$ and $E_8 = -J_6J_7E_5$. Now define the almost Hermitian structure $\tilde{J}_5$ on $\mathcal{U}$ whose matrix in the frame E_i is $\left(\begin{smallmatrix} 0 & I_4 \\ -I_4 & 0 \end{smallmatrix}\right)$. Then $\tilde{J}_5$, J_6 and J_7 are anticommuting almost Hermitian structures on $\mathcal{U}$, with $\tilde{J}_5J_6 \neq \pm J_7$; hence we can complement them by almost Hermitian structures $J_1', \dots, J_4'$ on $\mathcal{U}$ in such a way that $J_1', \dots, J_4', \tilde{J}_5, J_6, J_7$ are anticommuting almost Hermitian structures. Moreover, since $\tilde{J}_5$, J_6 and J_7 are smooth on $\mathcal{U}$, the four-dimensional distribution $\mathrm{Span}(J_1'X, \dots, J_4'X) = (\mathrm{Span}(X, \tilde{J}_5X, J_6X, J_7X))^\perp$ is smooth for every smooth nowhere vanishing vector field X on $\mathcal{U}$. By the first part of the proof, we can find smooth anticommuting almost Hermitian structures $\tilde{J}_1, \dots, \tilde{J}_4$ on (a possibly smaller) neighborhood $\mathcal{U}$ such that $\mathrm{Span}(\tilde{J}_1X, \dots, \tilde{J}_4X) = \mathrm{Span}(J_1'X, \dots, J_4'X)$ for every vector field X. Then $\tilde{J}_1, \dots, \tilde{J}_4, \tilde{J}_5$ are the required almost Hermitian structures, since

$$\begin{aligned}\mathrm{Span}(\tilde{J}_1X, \dots, \tilde{J}_5X) &= \mathrm{Span}(J_1'X, \dots, J_4'X, \tilde{J}_5X)\\ &= (\mathrm{Span}(X, J_6X, J_7X))^\perp = \mathrm{Span}(J_1X, \dots, J_5X)\end{aligned}$$

for every vector field X on $\mathcal{U}$, and $\tilde{J}_5$ anticommutes with every $\tilde{J}_\alpha$ for $\alpha = 1, 2, 3, 4$, since it anticommutes with every J_α' for $\alpha = 1, 2, 3, 4$. □

3. Conformally Osserman manifolds: Proof of Theorem 1.3

Let M^n be a smooth conformally Osserman Riemannian manifold with $n \neq 3, 4$. If $n = 2$, the manifold is locally conformally flat, so we can assume that $n > 4$. Combining [N 2005, Proposition 1 and the penultimate paragraph of the proof of Theorems 1 and 2] with [N 2004, Proposition 1] and [N 2006, Proposition 2.1], we obtain that the Weyl tensor of M^n has a Clifford structure for all $n \neq 16$, and also for $n = 16$ provided the Jacobi operator W_X has an eigenvalue of multiplicity at least 9 (note that the Jacobi operator of any Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ has an eigenvalue of multiplicity at least 7, for topological reasons). In the latter case, W has a Clifford structure $\mathrm{Cliff}(\nu)$, with $\nu \leq 6$, at every point on M^n.

To prove Theorem 1.3 it therefore suffices to prove the following theorem.

Theorem 3.1. *Let M^n be a connected smooth Riemannian manifold whose Weyl tensor at every point $x \in M^n$ has a Clifford structure* $\mathrm{Cliff}(\nu(x))$. *Suppose that $n > 4$, and additionally that $\nu(x) \leq 4$ if $n = 16$. Then there exists a space M_0^n from the list $\mathbb{R}^n, \mathbb{C}P^{n/2}, \mathbb{C}H^{n/2}, \mathbb{H}P^{n/4}, \mathbb{H}H^{n/4}$ (Euclidean space and the rank-one symmetric spaces with their standard metrics) such that M^n is locally conformally equivalent to M_0^n.*

Note that by Theorem 3.1, every point of M^n has a neighborhood conformally equivalent to a domain of the same "model space". Also note that the theorem says something also in the case $n = 16$, whereas Theorem 3.1 does not.

We start with a sketch of the proof of Theorem 3.1. First, we show that the Clifford structure for the Weyl tensor can be chosen locally smooth on an open, dense subset $M' \subset M^n$ (see Lemma 3.2 for the precise statement). To simplify the form of the curvature tensor R of M^n, we combine the λ_0-part of W (from (2-1)) with the difference $R - W$, so that R has the form (3-1) for some smooth symmetric operator field ρ at every point of M'. The technical core of the proof is Lemmas 3.5 and 3.6, which establish various identities for the covariant derivatives of ρ, the J_i and the η_i, using the second Bianchi identity for the curvature tensor of the form (3-1). Lemma 3.6 treats the case $(n, \nu) = (8, 7)$ and uses the octonion arithmetic; Lemma 3.5 treats all the other cases, and uses the fact that ν is small compared to n — see Lemma 2.4. It follows from the identities of Lemma 3.5 and Lemma 3.6 that, unless the Weyl tensor vanishes, the metric on M' can be locally changed to a conformal one whose curvature tensor again has the form (3-1), but with the two additional features: First, all the η_i are locally constant, and second, ρ is a Codazzi tensor, that is, $(\nabla_X \rho)Y = (\nabla_Y \rho)X$. By the result of [Derdziński and Shen 1983], exterior products of the eigenspaces of a symmetric Codazzi tensor are invariant under the curvature operator on the two-forms. Using that, we prove in Lemma 3.7 that ρ must be a multiple of the identity, so, by (3-1), M' is locally conformally equivalent to an Osserman manifold. The affirmative answer to the Osserman conjecture in the cases for n and ν considered in Theorem 3.1, given by [N 2003, Theorem 1.2], implies that M' is locally conformally equivalent to one of the spaces listed in Theorem 3.1. This proves Theorem 3.1 at the generic points. To prove Theorem 3.1 globally, we first show, using Lemma 3.9, that M splits into a disjoint union of a closed subset M_0, on which the Weyl tensor vanishes, and nonempty open connected subsets M_α, each of which is locally conformal to one of the rank-one symmetric spaces $\mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{H}H^{n/4}$. On every M_α, the conformal factor f is a well-defined positive smooth function. Assuming that there exists at least one M_α and that $M_0 \neq \varnothing$, we show in Lemma 3.10 that there exists a point $x_0 \in M_0$ on the boundary of a geodesic ball $B \subset M_\alpha$ such that both $f(x)$ and $\nabla f(x)$ tend to zero when $x \to x_0$ for $x \in B$. Then the positive function $u = f^{(n-2)/4}$ satisfies the elliptic equation (3-31) in B, with $\lim_{x \to x_0, x \in B} u(x) = 0$; hence by the boundary point theorem, the limiting value of the inner derivative of u at x_0 must be positive. This contradiction implies that either $M = M_0$ or $M = M_\alpha$.

Proof of Theorem 3.1. For $n > 4$, let M^n be a connected smooth Riemannian manifold whose Weyl tensor at every point has a Clifford structure. Define the function $N : M^n \to \mathbb{N}$ so that $N(x)$ is the number of distinct eigenvalues of the Jacobi operator W_X associated to the Weyl tensor, where X is an arbitrary nonzero vector from $T_x M^n$. Since the Weyl tensor is Osserman, $N(x)$ is well defined. Moreover, since the set of symmetric operators having no more than N_0 distinct eigenvalues is closed in the linear space of symmetric operators on $\mathbb{R}^n$, the function $N(x)$ is lower semicontinuous, that is, every subset $\{x : N(x) \leq N_0\}$ is closed in M^n. Let M' be the set of points where the function $N(x)$ is continuous. It is easy to see that M' is an open and dense (but possibly disconnected) subset of M^n. The following lemma shows that the Clifford structure for the Weyl tensor is locally smooth on every connected component of M'.

Lemma 3.2. *For $n > 4$, let M^n be a smooth Riemannian manifold whose Weyl tensor has a Clifford structure at every point. If $n = 16$, we additionally require that at every point $x \in M^{16}$, the Weyl tensor has a Clifford structure* $\mathrm{Cliff}(\nu(x))$ *with $\nu(x) \neq 8$.*

Let M' be the (open, dense) subset of M^n, at the points of which the number of distinct eigenvalues of the Jacobi operator associated to the Weyl tensor of M^n is locally constant. Then for every $x \in M'$, there exists a neighborhood $\mathcal{U} = \mathcal{U}(x)$, a number $\nu \geq 0$, smooth functions $\eta_1, \dots, \eta_\nu : \mathcal{U} \to \mathbb{R} \setminus \{0\}$, a smooth symmetric linear operator field ρ, and smooth anticommuting almost Hermitian structures J_i for $i = 1, \dots, \nu$, on $\mathcal{U}$ such that the curvature tensor of M^n has the form

$$R(X,Y)Z = \langle X,Z\rangle \rho Y + \langle \rho X, Z\rangle Y - \langle Y,Z\rangle \rho X - \langle \rho Y, Z\rangle X + \sum_{i=1}^{\nu} \eta_i (2\langle J_i X, Y\rangle J_i Z + \langle J_i Z, Y\rangle J_i X - \langle J_i Z, X\rangle J_i Y), \tag{3-1}$$

for all $y \in \mathcal{U}$ and $X, Y, Z \in T_y M^n$. Moreover, if $n = 8$, then the curvature tensor has the form (3-1) *either with $\nu = 3$ and $J_1 J_2 = \pm J_3$, or with $\nu = 7$ for all $y \in \mathcal{U}$.*

Proof. Let X be a smooth unit vector field on M^n. Since the Weyl tensor W is a smooth Osserman algebraic curvature tensor, the characteristic polynomial of $W_{X|X^\perp}$ (of the restriction of the Jacobi operator W_X to the subspace $X^\perp$) does not depend on X and is a well-defined smooth map $p : M^n \to \mathbb{R}_{n-1}[t]$, $y \mapsto p_y(t)$, where $\mathbb{R}_{n-1}[t]$ is the $(n-1)$-dimensional affine space of polynomials of degree $n-1$ with leading term $(-t)^{n-1}$. Since all the roots of $p_y(t)$ are real and the number of different roots is constant on every connected component of M', the eigenvalues $\mu_0, \mu_1, \dots, \mu_l$ of $W_{X|X^\perp}$ are smooth functions and their multiplicities $m_0, m_1, \dots, m_l$ are constant on every connected component of M' (we chose the labeling so that $m_0 = \max\{m_0, m_1, \dots, m_l\}$.

First consider the case $n \neq 8$. The Weyl tensor has a Clifford structure given by (2-1) at every point of M'. By Lemma 2.4, for $n > 4$ with $n \neq 8, 16$, we have $n - 1 - \nu > \nu$ for any Clifford structure on $\mathbb{R}^n$. By (2-3), we have $\nu \leq 8$ for $n = 16$, so by assumption, the inequality $n - 1 - \nu > \nu$ also holds for $n = 16$. Then the biggest multiplicity of an eigenvalue of $W_{X|X^\perp}$ is $n - 1 - \nu$; see Remark 2.3. So $\nu = n - 1 - m_0$ is constant and the function $\lambda_0 = \mu_0$ is smooth on every connected component of M'. Moreover, for every smooth unit vector field X on M' and every $i = 1, \dots, l$, the μ_i-eigendistribution of $W_{X|X^\perp}$ is $\mathrm{Span}_{j:\lambda_0+3\eta_j=\mu_i}(J_j X)$. Since λ_0 and μ_i are smooth functions on every connected component of M', so is η_j. Moreover, on every connected component of M', every distribution $\mathrm{Span}_{j:\lambda_0+3\eta_j=\mu_i}(J_j X)$ is smooth and has a constant dimension m_i for any nowhere vanishing smooth vector field X. By Lemma 2.7(3), there exists a neighborhood $\mathcal{U}_i(x)$ and smooth anticommuting almost Hermitian structures $\tilde{J}_j$ (for j such that $\lambda_0 + 3\eta_j = \mu_i$) on $\mathcal{U}_i(x)$ such that

$$\operatorname{Span}_{j:\lambda_0+3\eta_j=\mu_i}(J_j X) = \operatorname{Span}_{j:\lambda_0+3\eta_j=\mu_i}(\tilde{J}_j X).$$

Let $\widetilde{W}$ be the algebraic curvature tensor on $\mathcal{U} = \bigcap_{i=1}^{l} \mathcal{U}_i(x)$ with the Clifford structure $\mathrm{Cliff}(\nu;\, \tilde{J}_1, \dots, \tilde{J}_\nu;\, \lambda_0, \eta_1, \dots, \eta_\nu)$. Then $\nu = n-1-m_0$ is constant and all the $\tilde{J}_i$, η_i and λ_0 are smooth on $\mathcal{U}$. Moreover, for every unit vector field X on $\mathcal{U}$, the Jacobi operators $\widetilde{W}_X$ and W_X have the same eigenvalues and the same eigenspaces by construction; hence $\widetilde{W}_X = W_X$, which implies $\widetilde{W} = W$.

Now consider the case $n = 8$. By Lemma 2.6, at every point $x \in M'$, the Weyl tensor either has a Cliff(3) structure with $J_1 J_2 = J_3$ or a Cliff(7) structure (but not both). Since on every connected component M_α of M' the eigenvalues of the operator $W_{X|X^\perp}$ with $X \neq 0$ have constant number and multiplicity, Remark 2.3 implies that the only case when M_α may potentially contain points of both kinds is when one of the eigenvalues of $W_{X|X^\perp}$ with $X \neq 0$ on M_α has multiplicity 4 and the Clifford structure at every point $x \in M_\alpha$ is either

$$\mathrm{Cliff}(3;\, J_1, J_2, J_3;\, \lambda_0, \eta_1, \eta_2, \eta_3)$$

with $J_1 J_2 = J_3$, or

$$\mathrm{Cliff}(7;\, J_1, \dots, J_7;\, \lambda_0 - 3\xi, \eta_1 + \xi, \eta_2 + \xi, \eta_3 + \xi, \xi, \xi, \xi, \xi),$$

where $\eta_1, \eta_2, \eta_3 \neq 0$ (some of them can be equal) and $\xi \neq -\eta_i, 0$. The eigenvalues of $W_{X|X^\perp}$ with $\|X\| = 1$ at every point $x \in M_\alpha$ are λ_0, of multiplicity 4, and $\lambda_0 + 3\eta_i$. Let X be an arbitrary nowhere vanishing smooth vector field on a neighborhood $\mathcal{U} \subset M_\alpha$ of a point $x \in M_\alpha$. Then the four-dimensional eigendistribution of $W_{X|X^\perp}$ corresponding to the eigenvalue of multiplicity 4 is smooth; hence its orthogonal complement, the distribution $\operatorname{Span}(J_1 X, J_2 X, J_3 X)$, is also smooth. By Lemma 2.7(3), there are smooth anticommuting almost Hermitian structures $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$ such that $\operatorname{Span}(\tilde{J}_1 X, \tilde{J}_2 X, \tilde{J}_3 X) = \operatorname{Span}(J_1 X, J_2 X, J_3 X)$ on (a possibly smaller) neighborhood $\mathcal{U}$. By Lemma 2.7(1) with $F(X) = \tilde{J}_i X$, every $\tilde{J}_i$ is a linear combination of the $J_j : \tilde{J}_i = \sum_{j=1}^{3} a_{ij} J_j$, and moreover, the matrix (a_{ij}) must be orthogonal, since the $\tilde{J}_i$ are anticommuting almost Hermitian structures. It follows that $\tilde{J}_1 \tilde{J}_2 \tilde{J}_3 = \pm J_1 J_2 J_3$. The operator on the left side is smooth on $\mathcal{U}$, the one on the right side is $\pm \mathrm{id}_{\mathbb{R}^8}$ at the points where the Clifford structure is Cliff(3) with $J_1 J_2 = J_3$, and is symmetric with trace zero at the points where the Clifford structure is Cliff(7), which follows from the identity $J_4 (J_1 J_2 J_3) J_4 = J_1 J_2 J_3$. Therefore all the points of $\mathcal{U}$ either have a Cliff(3) structure with $J_1 J_2 = J_3$ or a Cliff(7) structure. In both cases, the Clifford structure for W can be taken to be smooth:

In the first, this follows from the arguments similar to those in the first part of the proof, since $\nu < n-1-\nu$; in the second, we apply Lemma 2.7(3) to every eigendistribution of $W_{X|X^\perp}$.

Thus for any $x \in M'$, the Weyl tensor on a neighborhood $\mathcal{U} = \mathcal{U}(x)$ has the form (2-1), with a constant ν and smooth λ_0, η_i and J_i. Then the curvature tensor has the form (3-1) with the operator ρ given by

$$\rho = \frac{1}{n-2}\,\mathrm{Ric} + \Big(\frac{\lambda_0}{2} - \frac{\mathrm{scal}}{2(n-1)(n-2)}\Big)\,\mathrm{id},$$

where Ric is the Ricci operator and scal is the scalar curvature. Since λ_0 is a smooth function, the operator field ρ is also smooth. $\square$

Remark 3.3. In fact, the proof shows that if an algebraic curvature tensor field $\mathcal{R}$ has a Clifford structure at every point of a Riemannian manifold (and $\nu \neq 8$ when $n = 16$), then it has a Clifford structure of the same class of differentiability as $\mathcal{R}$ on a neighborhood of every generic point of the manifold.

Remark 3.4. It follows from Lemma 2.6(1) (in fact, from Equation (2-4)) that, in the case $n = 8$ and $\nu = 7$ we can replace ρ by $\rho - \frac{3}{2} f\,\mathrm{id}$ and η_i by $\eta_i + f$ in (3-1) without changing R, where f is an arbitrary smooth function on $\mathcal{U}$. If we want the resulting Clifford structure to be Cliff(7), we additionally require that $\eta_i + f$ is nowhere zero.

Let $x \in M'$, and let $\mathcal{U} = \mathcal{U}(x)$ be its neighborhood defined in Lemma 3.2. By the second Bianchi identity, $(\nabla_U R)(X, Y)Y + (\nabla_Y R)(U, X)Y + (\nabla_X R)(Y, U)Y = 0$. Substituting R from (3-1) and using the fact that the operators J_i and their covariant derivatives are skew-symmetric and the operator ρ and its covariant derivatives are symmetric we get

$$\begin{aligned}
\text{(3-2)}\quad & \langle X, Y\rangle((\nabla_U\rho)Y - (\nabla_Y\rho)U) + \|Y\|^2((\nabla_X\rho)U - (\nabla_U\rho)X) \\
&+ \langle U, Y\rangle((\nabla_Y\rho)X - (\nabla_X\rho)Y) + \langle(\nabla_Y\rho)U - (\nabla_U\rho)Y, Y\rangle X \\
&+ \langle(\nabla_X\rho)Y - (\nabla_Y\rho)X, Y\rangle U + \langle(\nabla_U\rho)X - (\nabla_X\rho)U, Y\rangle Y \\
&+ \sum\nolimits_{i=1}^{\nu} 3(X(\eta_i)\langle J_iY, U\rangle - U(\eta_i)\langle J_iY, X\rangle)J_iY \\
&+ \sum\nolimits_{i=1}^{\nu} Y(\eta_i)(2\langle J_iU, X\rangle J_iY + \langle J_iY, X\rangle J_iU - \langle J_iY, U\rangle J_iX) \\
&+ \sum\nolimits_{i=1}^{\nu} \eta_i\big((3\langle(\nabla_U J_i)X, Y\rangle + 3\langle(\nabla_X J_i)Y, U\rangle + 2\langle(\nabla_Y J_i)U, X\rangle)J_iY \\
&\qquad + 3\langle J_iX, Y\rangle(\nabla_U J_i)Y + 3\langle J_iY, U\rangle(\nabla_X J_i)Y + 2\langle J_iU, X\rangle(\nabla_Y J_i)Y \\
&\qquad + \langle(\nabla_Y J_i)Y, X\rangle J_iU + \langle J_iY, X\rangle(\nabla_Y J_i)U \\
&\qquad - \langle(\nabla_Y J_i)Y, U\rangle J_iX - \langle J_iY, U\rangle(\nabla_Y J_i)X\big) = 0.
\end{aligned}$$

Taking the inner product of (3-2) with X and assuming X, Y and U to be orthogonal, we obtain

$$\begin{aligned}(3\text{-}3)\quad & \|X\|^2\langle Q(Y),U\rangle+\|Y\|^2\langle Q(X),U\rangle\\ &+\sum\nolimits_{i=1}^{\nu}3(X(\eta_i)\langle J_iY,U\rangle-Y(\eta_i)\langle J_iX,U\rangle-U(\eta_i)\langle J_iY,X\rangle)\langle J_iY,X\rangle\\ &+\sum\nolimits_{i=1}^{\nu}3\eta_i\big((2\langle(\nabla_UJ_i)X,Y\rangle+\langle(\nabla_XJ_i)Y,U\rangle+\langle(\nabla_YJ_i)U,X\rangle)\langle J_iY,X\rangle\\ &\qquad-\langle J_iY,U\rangle\langle(\nabla_XJ_i)X,Y\rangle-\langle J_iX,U\rangle\langle(\nabla_YJ_i)Y,X\rangle\big)=0,\end{aligned}$$

where $Q:\mathbb{R}^n\to\mathbb{R}^n$ is the quadratic map defined by

$$\langle Q(X),U\rangle=\langle(\nabla_X\rho)U-(\nabla_U\rho)X,X\rangle. \tag{3-4}$$

Note that $\langle Q(X),X\rangle=0$.

Lemma 3.5. *Under the assumptions of Lemma 3.2, let $x\in M'$ and let $\mathcal{U}$ be the corresponding neighborhood of x. Suppose that if $n=8$, then $\nu=3$ and $J_1J_2=J_3$ on $\mathcal{U}$, and if $n=16$, then $\nu\le 4$. For every point $y\in\mathcal{U}$, identify T_yM^n with Euclidean $\mathbb{R}^n$ via a linear isometry.*

(i) *There exist $m_i, b_{ij}\in\mathbb{R}^n$ with $i,j=1,\dots,\nu$ such that for all $X,Y,U\in\mathbb{R}^n$ and all $i,j=1,\dots,\nu$,*

$$Q(Y)=3\sum\nolimits_{k=1}^{\nu}\langle m_k,Y\rangle J_kY, \tag{3-5a}$$

$$(\nabla_XJ_i)X=\eta_i^{-1}(\|X\|^2m_i-\langle m_i,X\rangle X)+\sum\nolimits_{j=1}^{\nu}\langle b_{ij},X\rangle J_jX, \tag{3-5b}$$

$$b_{ij}+b_{ji}=\eta_i^{-1}J_jm_i+\eta_j^{-1}J_im_j, \tag{3-5c}$$

$$\nabla\eta_i=2J_im_i, \tag{3-5d}$$

$$\sum_{j\ne i}(\langle\eta_ib_{ij}+\eta_jb_{ji},J_iY\rangle J_jY+\langle\eta_ib_{ij}+\eta_jb_{ji},Y\rangle J_iJ_jY)=0. \tag{3-5e}$$

(ii) *These equations hold*:

$$(\nabla_Y\rho)U-(\nabla_U\rho)Y=\sum_{i=1}^{\nu}(2\langle J_iY,U\rangle m_i-\langle m_i,Y\rangle J_iU+\langle m_i,U\rangle J_iY), \tag{3-6a}$$

$$b_{ij}(3-\eta_i\eta_j^{-1})+b_{ji}(3-\eta_j\eta_i^{-1})=0\quad \textit{for } i\ne j, \tag{3-6b}$$

$$J_im_i=\eta_ip\quad \textit{for } i=1,\dots,\nu \textit{ and some } p\in\mathbb{R}^n. \tag{3-6c}$$

Proof. (i) We split the proof of these assertions into two cases: the *exceptional case*, when either $n=6$ and $\nu=1$, or $n=12$, $\nu=3$ and $J_1J_2=\pm J_3$, or $n=8$, $\nu=3$ and $J_1J_2=J_3$, and the *generic case*, consisting of all the other Clifford structures considered in the lemma.

Generic case. From (3-3) we obtain

$$\|X\|^{-2}\langle Q(X), U\rangle + \|Y\|^{-2}\langle Q(Y), U\rangle = 0 \quad \text{for all } X \perp \mathcal{J}Y,\ \ X, Y \perp \mathcal{J}U, \text{ and } X, Y, U \neq 0. \tag{3-7}$$

We want to show that $\langle Q(X), U\rangle = 0$ for all $X \perp \mathcal{J}U$. This is immediate when $n > 3\nu+3$. Indeed, $\operatorname{codim}(\mathcal{J}U+\mathcal{J}X) > \nu+1$ for any $U \neq 0$ and any unit $X \perp \mathcal{J}U$, so we can choose unit vectors $Y_1, Y_2 \perp \mathcal{J}U + \mathcal{J}X$ such that $Y_1 \perp \mathcal{J}Y_2$. Then (3-7) implies that $\langle Q(X), U\rangle = -\langle Q(Y_1), U\rangle = \langle Q(Y_2), U\rangle = -\langle Q(X), U\rangle$.

Consider the case $n \leq 3\nu + 3$. By Lemma 2.4(i), this could only happen when $n = 12$ and $\nu = 3$ or $n = 24$ and $\nu = 7$ (for the pairs (n, ν) belonging to the generic case), and in both cases, $n = 3\nu+3$. Choose and fix an arbitrary $U \neq 0$ and consider the quadratic form $q(X) = \langle Q(X), U\rangle$ defined on the $(2\nu + 2)$-dimensional space $L = (\mathcal{J}U)^{\perp}$. Suppose $q \neq 0$. By (3-7), the restriction of q to the unit sphere of L is not a constant, so it attains its maximum (respectively minimum) on a great sphere S_1 (respectively S_2). The subspaces L_1 and L_2 defined by S_1 and S_2 are orthogonal. Moreover by (3-7), we have $L_2 \supset (\mathcal{J}X)^{\perp} \cap L$ for any nonzero $X \in L_1$, which implies that $\dim L_2 \geq \nu + 1$. Similarly $\dim L_1 \geq \nu + 1$, so $\dim L_1 = \dim L_2 = \nu + 1$ since $L_1 \perp L_2$, and $L = L_1 \oplus L_2$. It follows that $q(X) = c(\|\pi_1 X\|^2 - \|\pi_2 X\|^2)$ for some $c > 0$, where $\pi_i : L \to L_i$ is the orthogonal projection. Also, $L_2 = (\mathcal{J}X)^{\perp} \cap L$ for all nonzero $X \in L_1$, which means that the subspace $L_1 = L_2^{\perp} \cap L$ (and similarly L_2) is $\pi\mathcal{J}$-invariant, where $\pi : \mathbb{R}^n \to L$ is the orthogonal projection, and even furthermore $\pi\mathcal{J}X = L_\alpha$ for every nonzero $X \in L_\alpha$ for $\alpha = 1, 2$, by dimension count. Let $X = X_1 + X_2$ and $Y = Y_1 + Y_2 \in L$, where $X_\alpha = \pi_\alpha X$ and $Y_\alpha = \pi_\alpha Y$. The condition $Y \perp \mathcal{J}X$ is equivalent to

$$\langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle = \langle \pi J_i X_1, Y_1\rangle + \langle \pi J_i X_2, Y_2\rangle = 0 \quad \text{for all } i = 1, \dots \nu.$$

Take arbitrary orthonormal bases for L_1 and for L_2 and let $M_\alpha(X_\alpha)$ for $\alpha = 1, 2$ be the $(\nu+1) \times (\nu+1)$ matrix whose columns relative to the chosen basis for L_α are $X_\alpha, \pi J_1 X_\alpha, \dots, \pi J_\nu X_\alpha$. Then $Y \perp \mathcal{J}X$ if and only if $M_1(X_1)^t Y_1 = -M_2(X_2)^t Y_2$. Since for $\alpha = 1, 2$, and any nonzero $X_\alpha \in L_\alpha$, the columns of $M_\alpha(X_\alpha)$ span L_α, we obtain $Y_2 = -(M_2(X_2)^t)^{-1} M_1(X_1)^t Y_1$ for any $X_2 \neq 0$. Then, since

$$q(X) = c(\|X_1\|^2 - \|X_2\|^2) \quad \text{and} \quad q(Y) = c(\|Y_1\|^2 - \|Y_2\|^2),$$

Equation (3-7) implies $\|Y_1\|^2\|X_1\|^2 - \|Y_2\|^2\|X_2\|^2 = 0$, so

$$\|Y_1\|^2\|X_1\|^2 - \|(M_2(X_2)^t)^{-1} M_1(X_1)^t Y_1\|^2 \|X_2\|^2 = 0$$

for any $X_1, Y_1 \in L_1$ and any nonzero $X_2 \in L_2$. It follows that

$$\|X_1\|^2 (M_1(X_1)^t M_1(X_1))^{-1} = \|X_2\|^2 (M_2(X_2)^t M_2(X_2))^{-1}$$

for any nonzero $X_\alpha \in L_\alpha$. Thus for some positive definite symmetric $(\nu+1)\times(\nu+1)$ matrix T, we have

$$M_\alpha(X_\alpha)^t M_\alpha(X_\alpha) = \|X_\alpha\|^2 T$$

for all $X_\alpha \in L_\alpha$ with $\alpha = 1, 2$. Then for any $X = X_1 + X_2 \in L$ with $X_\alpha \in L_\alpha$, and any $i = 1, \dots, \nu$,

$$\begin{aligned}\|\pi J_i X\|^2 = \|\pi J_i X_1\|^2 + \|\pi J_i X_2\|^2 &= (M_1(X_1)^t M_1(X_1) + M_2(X_2)^t M_2(X_2))_{ii}\\ &= T_{ii}(\|X_1\|^2 + \|X_2\|^2) = T_{ii}\|X\|^2.\end{aligned}$$

On the other hand, $\pi J_i X = J_i X - \|U\|^{-2}\sum_{j=1}^{\nu}\langle J_i X, J_j U\rangle J_j U$ for any $X \in L$, so $\|\pi J_i X\|^2 = \|X\|^2 - \|U\|^{-2}\sum_{j=1}^{\nu}\langle J_i X, J_j U\rangle^2$. It follows that

$$\|X\|^2\|U\|^2(1 - T_{ii}) = \sum_{j=1}^{\nu}\langle J_i X, J_j U\rangle^2 = \sum_{j=1}^{\nu}\langle X, J_i J_j U\rangle^2$$

for an arbitrary $X \in L$. Since $\dim L = 2\nu + 2 > \nu$, we can choose a nonzero $X \in L$ orthogonal to the ν vectors $J_i J_j U$, for $j = 1, \dots, \nu$. This implies $T_{ii} = 1$, and so $X \perp J_i J_j U$, for all $i, j = 1, \dots, \nu$ and all $X \in L = (\mathcal{J}U)^\perp$. Therefore $J_i J_j U \in \mathcal{J}U$ for all $i, j = 1, \dots, \nu$ and all $U \in \mathbb{R}^n$ for which the quadratic form $q(X) = \langle Q(X), U\rangle$ defined on $(\mathcal{J}U)^\perp$ is nonzero. If this is true for at least one U, then this is true for a dense subset of $\mathbb{R}^n$, which implies that $J_i J_j U \in \mathcal{J}U$ for all $i, j = 1, \dots, \nu$ and all $U \in \mathbb{R}^n$. Then by Lemma 2.7(1), $J_i J_j U = \sum_{k=1}^{\nu} a_{ijk} J_k U$ for $i \neq j$ for some constants a_{ijk}, which implies that $\langle J_k J_i J_j U, U\rangle = a_{ijk}\|U\|^2$, so for all triples of pairwise distinct i, j, k, the symmetric operator $J_k J_i J_j$ on $\mathbb{R}^n$ is a multiple of the identity. This is impossible when $\nu > 3$ (since for $l \neq i, j, k$, the operator $J_l J_k J_i J_j$ must be orthogonal and symmetric). The only remaining cases are $n = 12$ and $\nu = 3$, with $J_1 J_2 J_3 = \pm\,\mathrm{id}$, and $n = 6$ and $\nu = 1$, which are considered under the exceptional case below.

Therefore $\langle Q(X), U\rangle = 0$ for $X \perp \mathcal{J}U$, so $Q(X) \in \mathcal{J}X$ for all $X \in \mathbb{R}^n$. By Lemma 2.7(1) (and the fact that $\langle Q(X), X\rangle = 0$), this implies (3-5a) for some vectors $m_i \in \mathbb{R}^n$.

To prove (3-5b) and (3-5c), we first show that for an arbitrary $X \neq 0$, there is a dense subset of the Y in $(\mathcal{J}X)^\perp$ such that $\mathfrak{J}X \cap \mathfrak{J}Y = 0$. This follows from the dimension count (compare to [N 2003, Lemma 3.2(1)]). For $X \neq 0$, define the cone $\mathscr{C}X = \{J_u J_v X : u, v \in \mathbb{R}^\nu\}$; see (2-2). Since

$$\dim \mathscr{C}X \leq 2\nu - 1 < n - (\nu + 1) = \dim(\mathcal{J}X)^\perp,$$

where the inequality in the middle follows from Lemma 2.4(i), the complement to $\mathscr{C}X$ is dense in $(\mathcal{J}X)^\perp$. This complement is the required subset, since the condition $Y \notin \mathscr{C}X$ is equivalent to $\mathfrak{J}X \cap \mathfrak{J}Y = 0$. Substituting such X, Y into (3-3) we obtain by (3-5a)

$$\sum_{i=1}^{\nu}(\|X\|^2\langle m_i, Y\rangle - \eta_i\langle(\nabla_X J_i)X, Y\rangle)J_iY \\ + \sum_{i=1}^{\nu}(\|Y\|^2\langle m_i, X\rangle - \eta_i\langle(\nabla_Y J_i)Y, X\rangle)J_iX = 0.$$

Since $\mathcal{J}X \cap \mathcal{J}Y = 0$, all the coefficients vanish, so

$$\|X\|^2\langle m_i, Y\rangle - \eta_i\langle(\nabla_X J_i)X, Y\rangle = 0$$

for all $X \in \mathbb{R}^n$, all $i = 1, \dots, \nu$, and all Y from a dense subset of $(\mathcal{J}X)^\perp$, which implies that $(\nabla_X J_i)X - \eta_i^{-1}\|X\|^2 m_i \in \mathcal{J}X$ for all $X \in \mathbb{R}^n$. Equation (3-5b) then follows from Lemma 2.7(1). Equation (3-5c) follows from (3-5b) and the fact that $\langle(\nabla_X J_i)X, J_jX\rangle + \langle(\nabla_X J_j)X, J_iX\rangle = 0$.

To prove (3-5d) and (3-5e), substitute $X = J_kY$ and $U \perp X, Y$ into (3-3). Since $\langle J_iY, X\rangle = \|Y\|^2\delta_{ik}$, the first term in the second sum equals

$$3\eta_k(2\langle(\nabla_U J_k)X, Y\rangle + \langle(\nabla_X J_k)Y, U\rangle + \langle(\nabla_Y J_k)U, X\rangle)\|Y\|^2.$$

Since J_k is orthogonal and skew-symmetric,

$$\langle(\nabla_U J_k)X, Y\rangle = \langle(\nabla_U J_k)J_kY, Y\rangle = -\langle J_k(\nabla_U J_k)Y, Y\rangle = \langle(\nabla_U J_k)Y, J_kY\rangle = 0.$$

Next,

$$\begin{aligned}\langle(\nabla_Y J_k)U, X\rangle &= -\langle(\nabla_Y J_k)J_kY, U\rangle = \langle J_k(\nabla_Y J_k)Y, U\rangle \\ &= \langle(\eta_k^{-1}\|Y\|^2 J_k m_k + \textstyle\sum_{j=1}^{\nu}\langle b_{kj}, Y\rangle J_kJ_jY, U\rangle\end{aligned}$$

by (3-5b). Similarly, since $Y = -J_kX$, it follows from (3-5b) that

$$\begin{aligned}\langle(\nabla_X J_k)Y, U\rangle &= \langle J_k(\nabla_X J_k)X, U\rangle \\ &= \langle J_k(\eta_k^{-1}(\|X\|^2 m_k - \langle m_k, X\rangle X) + \textstyle\sum_{j=1}^{\nu}\langle b_{kj}, X\rangle J_jX), U\rangle \\ &= \langle\eta_k^{-1}\|Y\|^2 J_k m_k + \textstyle\sum_{j\neq k}\langle b_{kj}, J_kY\rangle J_jY - \langle b_{kk}, J_kY\rangle J_kY, U\rangle.\end{aligned}$$

Substituting this into (3-3) and using (3-5a) and (3-5b), we obtain after simplification

$$\text{(3-8)}\quad \|Y\|^2(\langle 2J_k m_k, U\rangle - U(\eta_k)) \\ + \sum_{j=1}^{\nu}\langle\eta_k b_{kj} + \eta_j b_{jk}, \langle J_jY, U\rangle J_kY + \langle J_kJ_jY, U\rangle Y\rangle = 0.$$

By [N 2003, Lemma 3.2(3)] for all $U \in \mathbb{R}^n$, we can find a nonzero Y such that $U \perp \mathcal{J}Y + \mathcal{J}J_kY$. Substituting such a Y into (3-8) proves (3-5d). Then (3-8) simplifies to (3-5e).

Exceptional case. Here either $n = 6$ and $\nu = 1$, or $n = 12$, $\nu = 3$ and $J_1 J_2 = \pm J_3$, or $n = 8$ and $\nu = 3$ and $J_1 J_2 = J_3$.

In all these cases, the Clifford structure has the "J^2 property" that $\mathcal{J}\mathcal{J}X = \mathcal{J}\mathcal{J}X = \mathcal{J}X$ for every $X \in \mathbb{R}^n$. In particular, if $Y \perp \mathcal{J}X$, then $\mathcal{J}Y \perp \mathcal{J}X$.

Substitute $X = J_k U$ and $Y \perp \mathcal{J}X = \mathcal{J}U$ into (3-2) and take the inner product of the resulting equation with $J_k Y$. Using the J^2 property and the fact that $\langle(\nabla_Y J_k)U, J_k U\rangle = \langle(\nabla_Y J_k)Y, J_k Y\rangle = 0$, we get

$$-J_k((\nabla_{J_k U}\rho)U-(\nabla_U\rho)J_k U)+2\|U\|^2\nabla\eta_k+3\eta_k((\nabla_U J_k)J_k U-(\nabla_{J_k U}J_k)U)\in\mathcal{J}U.$$

The expression $F(U)$ on the left side is a quadratic map from $\mathbb{R}^n$ to itself. By Lemma 2.7(1), $F(U)$ is a linear combination of $U, J_1 U, \dots, J_\nu U$ whose coefficients are linear forms of U. In particular, the cubic polynomial $\langle F(U), J_k U\rangle$ must be divisible by $\|U\|^2$. Since J_k is orthogonal and skew-symmetric,

$$\langle(\nabla_U J_k)J_k U - (\nabla_{J_k U} J_k)U, J_k U\rangle = 0,$$

so there exists a vector $m_k \in \mathbb{R}^n$ such that

$$\langle(\nabla_{J_k U}\rho)U - (\nabla_U\rho)J_k U, U\rangle = -3\|U\|^2\langle m_k, U\rangle.$$

It follows that the quadratic map Q defined by (3-4) satisfies

$$\langle Q(U), J_k U\rangle = 3\|U\|^2\langle m_k, U\rangle \quad \text{for all } U \in \mathbb{R}^n \text{ and all } k = 1, \dots, \nu.$$

Since $\langle Q(U), U\rangle = 0$, we can define a quadratic map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $U \in \mathbb{R}^n$,

$$Q(U) = T(U) + 3\textstyle\sum_{k=1}^{\nu}\langle m_k, U\rangle J_k U \quad \text{and} \quad T(U) \perp \mathcal{J}U. \tag{3-9}$$

Taking $U = J_k X$, $X, U \perp \mathcal{J}Y$ in (3-3) and using (3-9) we obtain

$$-J_k T(Y) + 3\|Y\|^2 m_k - 3\eta_k(\nabla_Y J_k)Y \in \mathcal{J}Y.$$

From Lemma 2.7(1) it follows that the expression on the left side is a linear combination of $Y, J_1 Y, \dots, J_\nu Y$ whose coefficients are linear forms of Y, so for some vectors $b_{ij} \in \mathbb{R}^n$,

$$(\nabla_Y J_i)Y = \eta_i^{-1}(m_i\|Y\|^2 - \langle m_i, Y\rangle Y) - (3\eta_i)^{-1}J_i T(Y) + \sum_{j=1}^{\nu}\langle b_{ij}, Y\rangle J_j Y. \tag{3-10}$$

Since $\langle(\nabla_Y J_i)Y, J_j Y\rangle$ is antisymmetric in i and j and $J_i T(Y) \perp \mathcal{J}Y$ by (3-9) and the J^2 property, the b_{ij} satisfy (3-5c).

Take $X = J_k Y$ and $U \perp \mathcal{J}Y = \mathcal{J}X$ in (3-3). Since $\langle(\nabla_U J_k)J_k Y, Y\rangle = 0$,

$$\langle(\nabla_Y J_k)U, X\rangle = -\langle(\nabla_Y J_k)J_k Y, U\rangle = \langle J_k(\nabla_Y J_k)Y, U\rangle,$$

and similarly $\langle(\nabla_X J_k)Y, U\rangle = -\langle(\nabla_X J_k)J_k X, U\rangle = \langle J_k(\nabla_X J_k)X, U\rangle$, we obtain from (3-9) and (3-10) after simplification that

$$2T(Y) + 2T(J_k Y) - 3\|Y\|^2(\nabla\eta_k - 2J_k m_k) \in \mathcal{J}Y. \tag{3-11}$$

In case $n = 6$ and $\nu = 1$, we can prove the remaining identities (3-5a), (3-5b), (3-5d) and (3-5e) of Lemma 3.5(i) as follows. Taking in (3-3) nonzero X, Y, U such that the subspaces $\mathcal{J}X$, $\mathcal{J}Y$ and $\mathcal{J}U$ are mutually orthogonal we obtain by (3-9)

$$\|X\|^{-2}\langle T(X), U\rangle + \|Y\|^{-2}\langle T(Y), U\rangle = 0,$$

which is, essentially, (3-7). Replacing Y by $J_1 Y$ and using (3-11) we get

$$2T(X) + 3\|X\|^2(\nabla\eta_1 - 2J_1 m_1) \in \mathcal{J}X.$$

The same is true with X replaced by $J_1 X$. Then by (3-11), $\nabla\eta_1 - 2J_1 m_1 \in \mathcal{J}X$ for all $X \in \mathbb{R}^6$, so $\nabla\eta_1 - 2J_1 m_1 = 0$, which is (3-5d). Then $T(X) \in \mathcal{J}X$; hence $T(X) = 0$, since $T(X) \perp \mathcal{J}X$ by (3-9). Now (3-5a) follows from (3-9), (3-5b) follows from (3-10), and (3-5e) is trivially satisfied, as $\nu = 1$.

In the cases $n = 8, 12$, $\nu = 3$ and $J_1 J_2 = J_3$ (if $J_1 J_2 = -J_3$, we replace J_3 by $-J_3$ without changing the curvature tensor (3-1)), we argue as follows. Adding (3-11) with $k = 1$ and with $k = 2$ and then subtracting (3-11) with $k = 3$ and Y replaced by $J_1 Y$ we get

$$4T(Y) - 3\|Y\|^2((\nabla\eta_1 - 2J_1 m_1) + (\nabla\eta_2 - 2J_2 m_2) - (\nabla\eta_3 - 2J_3 m_3)) \in \mathcal{J}Y.$$

This remains true under a cyclic permutation of the indices $1, 2, 3$, which implies $(\nabla\eta_k - 2J_k m_k) - (\nabla\eta_i - 2J_i m_i) \in \mathcal{J}Y$ for all $i, k = 1, 2, 3$ and all $Y \in \mathbb{R}^n$. Then $\nabla\eta_k - 2J_k m_k = \nabla\eta_i - 2J_i m_i = 4V/3$ for some vector $V \in \mathbb{R}^n$, and $T(Y) - \|Y\|^2 V$ belongs to $\mathcal{J}Y$ by the above. Since $T(Y) \perp \mathcal{J}Y$ by (3-9), we obtain

$$T(Y) = \|Y\|^2 V - \langle Y, V\rangle Y - \textstyle\sum_{i=1}^3 \langle J_i Y, V\rangle J_i Y,$$

so

$$\begin{aligned}
\nabla\eta_i &= 2J_i m_i + \tfrac{4}{3}V,\\
Q(Y) &= \|Y\|^2 V - \langle Y, V\rangle Y + \textstyle\sum_{j=1}^3 \langle 3m_j + J_j V, Y\rangle J_j Y,\\
(\nabla_Y J_i)Y &= (3\eta_i)^{-1}\big(\|Y\|^2(3m_i - J_i V) - \langle 3m_i - J_i V, Y\rangle Y\\
&\qquad + \textstyle\sum_{j=1}^3 \langle 3\eta_i b_{ij} - J_j J_i V, Y\rangle J_j Y\big),
\end{aligned} \tag{3-12}$$

where the second equation follows from (3-9) and the third from (3-10) and the fact that $J_1 J_2 = J_3$.

Substitute $X = J_k Y$ into (3-3) again, with an arbitrary $U \perp X, Y$. Using (3-12) and that the J_i are skew-symmetric, orthogonal and anticommute, we obtain after simplification that

$$\sum_{i=1}^{3}\langle 3a_{ik}-2J_iJ_kV,\, J_kY\rangle J_iY+\sum_{i=1}^{3}\langle 3a_{ik}-2J_iJ_kV,\, Y\rangle J_kJ_iY\in \operatorname{Span}(Y,\, J_kY),$$

where $a_{ik}=\eta_k b_{ki}+\eta_i b_{ik}$. Taking $k=1$ and using that $J_1J_2=J_3$, we get from the coefficient of J_2Y that $3J_1a_{12}-4J_2V+3a_{13}=0$, so $4V=-3J_2a_{13}+3J_3a_{12}$. Cyclically permuting the indices $1, 2, 3$ and using that $a_{ik}=a_{ki}$, we get $V=0$, which implies (3-5e). Since $V=0$, equations (3-5a), (3-5d) and (3-5b) follow from (3-12).

(ii) By (3-4) and (3-5a),

$$\langle(\nabla_X\rho)U-(\nabla_U\rho)X,\, X\rangle=3\sum_{i=1}^{\nu}\langle m_i,\, X\rangle\langle J_iX,\, U\rangle\quad\text{for all } X,\, U\in\mathbb{R}^n.$$

Polarizing this equation and using the fact that the covariant derivative of ρ is symmetric, we obtain

$$\begin{aligned}\langle(\nabla_X\rho)U,\, Y\rangle+\langle(\nabla_Y\rho)U,\, X\rangle-2\langle(\nabla_U\rho)Y,\, X\rangle&\\ =3\sum_{i=1}^{\nu}(\langle m_i,\, Y\rangle\langle J_iX,\, U\rangle+\langle m_i,\, X\rangle\langle J_iY,\, U\rangle).&\end{aligned}$$

Subtracting the same equation with Y and U interchanged, we get

$$\begin{aligned}\langle(\nabla_Y\rho)U-(\nabla_U\rho)Y,\, X\rangle=\sum_{i=1}^{\nu}(2\langle m_i,\, X\rangle\langle J_iY,\, U\rangle&\\ +\langle m_i,\, Y\rangle\langle J_iX,\, U\rangle-\langle m_i,\, U\rangle\langle J_iX,\, Y\rangle),&\end{aligned}$$

which proves (3-6a).

To establish (3-6b), substitute $X\perp\mathcal{J}Y,\ U=J_kY$ into (3-2). Using the equations of part (i) and (3-6a) we obtain after simplification that

$$\begin{aligned}3(\nabla_XJ_k)Y-(\nabla_YJ_k)X&\\ =-3\eta_k^{-1}\langle m_k,\, Y\rangle X+\sum_{i=1}^{\nu}\eta_k^{-1}\langle\eta_ib_{ik}+2\delta_{ik}J_km_k,\, Y\rangle J_iX\quad&\bmod\,(\mathcal{J}Y).\end{aligned}$$

Subtracting thrice polarized Equation (3-5b) (with $i=k$) and solving for $(\nabla_YJ_k)X$, we get, for all $X\perp\mathcal{J}Y$,

$$(\nabla_YJ_k)X=\sum_{i=1}^{\nu}\tfrac{1}{4}\eta_k^{-1}\langle 3\eta_kb_{ki}-\eta_ib_{ik}-2\delta_{ik}J_km_k,\, Y\rangle J_iX\quad\bmod\,(\mathcal{J}Y).\tag{3-13}$$

Choose $s\neq k$ and define the subset $S_{ks}\subset\mathbb{R}^n\oplus\mathbb{R}^n$ by

$$S_{ks}=\{(X,\, Y):X,\, Y\neq 0\text{ and }X,\, J_kX,\, J_sX\perp\mathcal{J}Y\}.$$

It is easy to see that $(X, Y) \in S_{ks}$ if and only if $(Y, X) \in S_{ks}$ and that replacing $\mathcal{J}Y$ by $\mathcal{I}Y$ in the definition of S_{ks} gives the same set S_{ks}. Moreover, $\{X : (X, Y) \in S_{ks}\}$ (and hence $\{Y : (X, Y) \in S_{ks}\}$) spans $\mathbb{R}^n$. If $n = 8$, $\nu = 3$ and $J_1 J_2 = J_3$, this follows from the J^2-property; in all other cases, it follows from [N 2003, Lemma 3.2(4)]. For $(X, Y) \in S_{ks}$, take the inner product of (3-13) with $J_s X$. Since $\langle(\nabla_Y J_k)X, J_s X\rangle$ is antisymmetric in k and s, we get $\langle(3 - \eta_k\eta_s^{-1})b_{ks} + (3 - \eta_s\eta_k^{-1})b_{sk}, Y\rangle = 0$ for a set of Y spanning $\mathbb{R}^n$. This proves (3-6b).

To prove (3-6c), we apply of Lemma 2.7(2) to (3-5e). If $\nu = 1$, there is nothing to prove; in fact, if $\nu = 1$ and $n \geq 8$, Theorem 3.1 follows from [Blažić and Gilkey 2004, Theorem 1.1]. If $\eta_i b_{ij} + \eta_j b_{ji} = 0$ for all $i \neq j$, then by (3-6b), $b_{ij} + b_{ji} = 0$ for all $i \neq j$, so $\eta_i^{-1} J_j m_i = -\eta_j^{-1} J_i m_j$ by (3-5c). Acting by $J_i J_j$ we obtain that the vector $\eta_i^{-1} J_i m_i$ is the same for all $i = 1, \dots, \nu$.

The only remaining possibility is $\nu = 3$, $J_1 J_2 = J_3$ (if $J_1 J_2 = -J_3$ we can replace J_3 by $-J_3$ without changing the curvature tensor (3-1)), and $\eta_k b_{ki} + \eta_i b_{ik} = J_j \upsilon$ for all the triples $\{i, j, k\} = \{1, 2, 3\}$, where $\upsilon \neq 0$. We will show that this leads to a contradiction. Note that by (2-3), the existence of a Cliff(3) structure implies that n is divisible by 4, so $n \geq 8$ by the assumption of the lemma.

If $\eta_i = \eta_k$ for some $i \neq k$, then from (3-6b) and $\eta_k b_{ki} + \eta_i b_{ik} = J_j \upsilon$ it follows that $\upsilon = 0$, a contradiction. Otherwise, if the η_i are pairwise distinct, we get

$$b_{ik} = (3\eta_i - \eta_k)(4\eta_i(\eta_i - \eta_k))^{-1} J_j \upsilon \quad \text{for } \{i, j, k\} = \{1, 2, 3\}.$$

Substituting this into (3-5c) and acting by J_j on both sides, we get

$$\eta_i^{-1} J_i m_i - \eta_k^{-1} J_k m_k = \tfrac{1}{4}\varepsilon_{ik}(\eta_i^{-1} + \eta_k^{-1})\upsilon \quad \text{for } \{i, j, k\} = \{1, 2, 3\},$$

where for $i \neq k$ we define $\varepsilon_{ik} = \pm 1$ by $J_i J_k = \varepsilon_{ik} J_j$. It is easy to see that $\varepsilon_{jk} = -\varepsilon_{jk}$ and $\varepsilon_{jk} = \varepsilon_{ij}$, where $\{i, j, k\} = \{1, 2, 3\}$. Then

$$\textstyle\sum_{i=1}^{3} \eta_i^{-1} = 0 \quad \text{and} \quad \eta_i^{-1} J_i m_i = \tfrac{1}{12}\varepsilon_{jk}(\eta_j^{-1} - \eta_k^{-1})\upsilon + w \quad \text{for some } w \in \mathbb{R}^n.$$

It then follows from (3-5d) that $\nabla\eta_i = (1/6)\varepsilon_{jk}\eta_i(\eta_j^{-1} - \eta_k^{-1})\upsilon + 2\eta_i w$, which implies

$$\nabla \ln|\eta_1\eta_2\eta_3| = 6w \quad \text{and} \quad \nabla \ln|\eta_i\eta_j^{-1}| = -\tfrac{1}{2}\varepsilon_{ij}\eta_k^{-1}\upsilon.$$

Let $\mathcal{U}' \subset \mathcal{U}$ be a neighborhood of x on which $\nabla \ln|\eta_1\eta_2^{-1}| \neq 0$. Then υ is a nowhere vanishing smooth vector field on $\mathcal{U}'$. Multiplying the metric on $\mathcal{U}$ by a function e^f changes neither the Weil tensor nor the J_i, and multiplies every η_i by e^{-f} and ∇ acting on functions by e^{-f}. Taking $f = (1/3)\ln|\eta_1\eta_2\eta_3|$ we can assume that $w = 0$ on $\mathcal{U}'$, so that $C = \eta_1\eta_2\eta_3$ is a constant. Then, since $\sum_{i=1}^{3} \eta_i^{-1} = 0$, we get

$$\nabla\eta_i = \pm\tfrac{1}{6}\upsilon\sqrt{1 - 4C^{-1}\eta_i^3}.$$

It follows that $\upsilon = \nabla t$ for some smooth function $t : \mathcal{U}' \to \mathbb{R}$ such that $\eta_i = -36C\wp(t + c_i)$, where $\wp$ is the Weierstrass function satisfying

$$(\tfrac{d}{dt}\wp(t))^2 = 4\wp(t)^3 + 6^{-6}C^{-2}$$

and $c_i \in \mathbb{R}$. Summarizing these identities, we have pointwise pairwise nonequal functions $\eta_i : \mathcal{U}' \to \mathbb{R} \setminus \{0\}$ satisfying

$$\begin{aligned}
&\upsilon = \nabla t \neq 0, && \nabla\eta_i = \tfrac{1}{6}\varepsilon_{jk}\eta_i(\eta_j^{-1} - \eta_k^{-1})\upsilon,\\
&0 = \textstyle\sum_{i=1}^{3} \eta_i^{-1}, && C = \text{const} = \textstyle\prod_{i=1}^{3} \eta_i,\\
&m_i = -\tfrac{1}{12}\varepsilon_{jk}\eta_i(\eta_j^{-1} - \eta_k^{-1})J_i\upsilon, && b_{ii} = \tfrac{1}{12}\varepsilon_{jk}(\eta_j^{-1} - \eta_k^{-1})\upsilon,\\
&b_{ij} = (3\eta_i - \eta_j)(4\eta_i(\eta_i - \eta_j))^{-1}J_k\upsilon,
\end{aligned} \tag{3-14}$$

for $\{i, j, k\} = \{1, 2, 3\}$, where we used (3-5c) to compute b_{ii}. Then Equation (3-13) simplifies to

$$(\nabla_Y J_k)X = \sum_{i \neq k} \tfrac{1}{2}(\eta_k - \eta_i)^{-1}\langle J_j\upsilon, Y\rangle J_i X \mod (\mathcal{J}Y) \quad \text{for all } X \perp \mathcal{J}Y.$$

By the J^2-property, $\mathcal{J}Y \perp \mathcal{J}X$, so to find the "mod$(\mathcal{J}Y)$" part, we have to compute the inner products of $(\nabla_Y J_k)X$ with Y, J_1Y, J_2Y and J_3Y. Since

$$\begin{aligned}
\langle(\nabla_Y J_k)X, Y\rangle &= -\langle(\nabla_Y J_k)Y, X\rangle,\\
\langle(\nabla_Y J_k)X, J_kY\rangle &= -\langle(\nabla_Y J_k)J_kY, X\rangle = \langle J_k(\nabla_Y J_k)Y, X\rangle,\\
\langle(\nabla_Y J_k)X, J_iY\rangle &= -\langle(\nabla_Y J_k)J_iY, X\rangle = -\langle(\varepsilon_{ki}(\nabla_Y J_j) - J_k(\nabla_Y J_i))Y, X\rangle
\end{aligned}$$

(from $J_kJ_i = \varepsilon_{ki}J_j$), these products can be found using (3-5b). Simplifying by (3-14) we get

$$\begin{aligned}
(\nabla_Y J_k)X = \tfrac{1}{12}\varepsilon_{ij}(\eta_i^{-1} - \eta_j^{-1})&(\langle J_k\upsilon, X\rangle Y + \langle\upsilon, X\rangle J_kY)\\
&+ \tfrac{1}{4}\eta_k^{-1}\sum_{i\neq k}\langle J_j\upsilon, X\rangle J_iY + \sum_{i\neq k}\tfrac{1}{2}(\eta_k - \eta_i)^{-1}\langle J_j\upsilon, Y\rangle J_iX,
\end{aligned}$$

for all $X \perp \mathcal{J}Y$, where $\{i, j, k\} = \{1, 2, 3\}$. To compute $(\nabla_Y J_k)X$ when $X \in \mathcal{J}Y$, we again use (3-5b) and the fact that, for $\{i, j, k\} = \{1, 2, 3\}$,

$$(\nabla_Y J_k)J_k = -J_k(\nabla_Y J_k) \quad \text{and} \quad (\nabla_Y J_k)J_i = \varepsilon_{ki}(\nabla_Y J_j) - J_k(\nabla_Y J_i).$$

Simplifying by (3-14) and using the above equation we get after some calculation

$$\begin{aligned}
(\nabla_Y J_k)X = \tfrac{1}{12}\varepsilon_{ij}(\eta_i^{-1} - \eta_j^{-1})&(\langle J_k\upsilon, X\rangle Y + \langle\upsilon, X\rangle J_kY - \langle X, Y\rangle J_k\upsilon - \langle X, J_kY\rangle\upsilon)\\
+ \tfrac{1}{4}\eta_k^{-1}\sum_{i\neq k}&(\langle J_j\upsilon, X\rangle J_iY - \langle J_iY, X\rangle J_j\upsilon) + \sum_{i\neq k}\tfrac{1}{2}(\eta_k - \eta_i)^{-1}\langle J_j\upsilon, Y\rangle J_iX,
\end{aligned}$$

for all $X, Y \in \mathbb{R}^n$, where $\{i, j, k\} = \{1, 2, 3\}$. For $a, b \in \mathbb{R}^n$, let $a \wedge b$ be the skew-symmetric operator defined by $(a \wedge b)X = \langle a, X\rangle b - \langle b, X\rangle a$. Then the above equation can be written in the form

$$\nabla_Y J_k = \tfrac{1}{12}\varepsilon_{ij}(\eta_i^{-1} - \eta_j^{-1})(J_k v \wedge Y + v \wedge J_k Y) + \tfrac{1}{4}\eta_k^{-1}\sum_{i\neq k} J_j v \wedge J_i Y + \sum_{i\neq k}\tfrac{1}{2}(\eta_k - \eta_i)^{-1}\langle J_j v, Y\rangle J_i,$$

that is, for $\{i, j, k\} = \{1, 2, 3\}$,

$$\begin{aligned} &\nabla_Y J_k = [J_k, AY], \qquad AY = \sum_{i=1}^{3}(\tfrac{1}{2}\lambda_i J_i Y \wedge J_i v + \omega_i \langle J_i v, Y\rangle J_i), \\ &\lambda_i = \tfrac{1}{6}\varepsilon_{jk}(\eta_j^{-1} - \eta_k^{-1}), \qquad \omega_i = \tfrac{1}{4}\varepsilon_{jk}(\eta_k - \eta_j)^{-1} \end{aligned} \tag{3-15}$$

where we used that, for $\{i, j, k\} = \{1, 2, 3\}$,

$$[J_k, a \wedge b] = J_k a \wedge b + a \wedge J_k b \quad \text{and} \quad [J_k, J_i] = 2\varepsilon_{ki} J_j.$$

By the Ricci formula, $\nabla^2_{Z,Y} J_k - \nabla^2_{Y,Z} J_k = [J_k, R(Y, Z)]$, where the tensor field $\nabla^2 J_k$ is defined by

$$\nabla^2_{Z,Y} J_k = \nabla_Z(\nabla_Y J_k) - \nabla_{\nabla_Z Y} J_k \quad \text{for vector fields } Y, Z \text{ on } \mathcal{U}'.$$

Since $\nabla_Y J_k = [J_k, AY]$ by (3-15), this is equivalent to the fact that the operator $F(Y, Z) = (\nabla_Z A)Y - (\nabla_Y A)Z - [AY, AZ] - R(Y, Z)$ commutes with all the J_s for all $Y, Z \in \mathbb{R}^n$ and all $s = 1, 2, 3$. By (3-1), we have

$$R(Y, Z) = Y \wedge \rho Z + \rho Y \wedge Z + \sum_{i=1}^{3}\eta_i(J_i Y \wedge J_i Z + 2\langle J_i Y, Z\rangle J_i),$$

so using (3-15) and the identities

$$\begin{aligned} &[a \wedge b, c \wedge d] = \langle a, d\rangle c \wedge b - \langle a, c\rangle d \wedge b - \langle b, d\rangle c \wedge a + \langle b, c\rangle d \wedge a, \\ &[J_s, a \wedge b] = J_s a \wedge b + a \wedge J_s b, \end{aligned}$$

we obtain

$$F(Y, Z) = V(Y, Z) + \sum_{i=1}^{3}\langle K_i Y, Z\rangle J_i + S(Y, Z), \tag{3-16}$$

where

$$\begin{aligned} &S(Y, Z) \in (\mathcal{J}Y + \mathcal{J}Z) \wedge \mathbb{R}^n, \\ &V(Y, Z) = -\tfrac{1}{2}\textstyle\sum_{i=1}^{3}\langle J_i Z, Y\rangle(\lambda_i^2 v \wedge J_i v + \varepsilon_{jk}(\lambda_j\lambda_k - \lambda_i\lambda_k - \lambda_j\lambda_i) J_j v \wedge J_k v) \\ &\qquad \in \mathcal{J}v \wedge \mathcal{J}v, \end{aligned}$$

and for subspaces $L_1, L_2 \subset \mathbb{R}^n$, we denote by $L_1 \wedge L_2$ the subspace of the space $\mathfrak{o}(n)$ of skew-symmetric operators on $\mathbb{R}^n$ defined by

$$L_1 \wedge L_2 = \mathrm{Span}(a \wedge b : a \in L_1,\ b \in L_2).$$

Note that if L_1 and L_2 are $\mathcal{J}$-invariant (that is, $\mathcal{J} L_\alpha \subset L_\alpha$), then $L_1 \wedge L_2$ is $\mathrm{ad}_{\mathcal{J}}$-invariant, that is, $[J_s, L_1 \wedge L_2] \subset L_1 \wedge L_2$.

From (3-15) and the facts that

$$\omega_i \lambda_i = (24C)^{-1} \eta_i, \quad \tfrac{d}{dt}\omega_i = 4\omega_i^2 + (12C)^{-1}\eta_i, \quad \textstyle\sum_i \omega_i^{-1} = 0,$$

which follow from (3-14) and (3-15), we obtain

$$\text{(3-17)} \quad K_i = -\omega_i((4\omega_i + \lambda_i) v \wedge J_i v + 4\varepsilon_{jk}(\omega_j + \omega_k) J_j v \wedge J_k v \\ + \lambda_i(48C + \|v\|^2) J_i + (J_i H + H J_i)),$$

where $\{i, j, k\} = \{1, 2, 3\}$ and H is the symmetric operator associated to the Hessian of t, that is, $\langle HY, Z\rangle = Y(Zt) - (\nabla_Y Z)t$ for vector fields Y and Z on $\mathcal{U}'$.

Since $[F(Y, Z), J_s] = 0$ and the subspace $\mathcal{J}Y + \mathcal{J}Z$ is $\mathcal{J}$-invariant (and hence $(\mathcal{J}Y + \mathcal{J}Z) \wedge \mathbb{R}^n$ is $\mathrm{ad}_{\mathcal{J}}$-invariant), it follows from (3-16) that for all $Y, Z \in \mathbb{R}^n$ and all $s = 1, 2, 3$,

$$\text{(3-18)} \qquad [V(Y, Z), J_s] + \sum_{i=1}^{3} \langle K_i Y, Z\rangle [J_i, J_s] \in (\mathcal{J}Y + \mathcal{J}Z) \wedge \mathbb{R}^n.$$

Take $Y, Z \in \mathcal{J}v$ in (3-18). Then $\mathcal{J}Y + \mathcal{J}Z = \mathcal{J}v$ and $[V(Y, Z), J_s] \in \mathcal{J}v \wedge \mathcal{J}v$ by the J^2 property, so (3-18) simplifies to $\sum_{i \neq s} \varepsilon_{is} \langle K_i Y, Z\rangle J_j \in \mathcal{J}v \wedge \mathbb{R}^n$, where $\{i, j, s\} = \{1, 2, 3\}$. Project this onto the subspace $(\mathcal{J}v)^\perp \wedge (\mathcal{J}v)^\perp \subset \mathfrak{o}(n)$ by the standard inner product on $\mathfrak{o}(n)$, and use that $(\mathcal{J}v)^\perp$ is $\mathcal{J}$-invariant and $n \geq 8$. Then we get $\langle K_i Y, Z\rangle = 0$ for all $i = 1, 2, 3$ and all $Y, Z \in \mathcal{J}v$. Introduce the operators

$$\hat{J}_i = \pi_{\mathcal{J}v} J_i \pi_{\mathcal{J}v} \quad \text{and} \quad \hat{H} = \pi_{\mathcal{J}v} H \pi_{\mathcal{J}v} \quad \text{on } \mathcal{J}v.$$

Since $\mathcal{J}v$ is $\mathcal{J}$-invariant, the $\hat{J}_i$ are anticommuting almost Hermitian structures on $\mathcal{J}v$. Then the condition $\langle K_i Y, Z\rangle = 0$ for $Y, Z \in \mathcal{J}v$ and (3-17) imply

$$(4\omega_i + \lambda_i) v \wedge \hat{J}_i v + 4\varepsilon_{jk}(\omega_j + \omega_k)\hat{J}_j v \wedge \hat{J}_k v + \lambda_i(48C + \|v\|^2)\hat{J}_i + \hat{J}_i \hat{H} + \hat{H}\hat{J}_i = 0.$$

Multiplying by $\hat{J}_i$ and taking the trace we obtain for $\{i, j, k\} = \{1, 2, 3\}$

$$4\|v\|^2(\omega_i + \omega_j + \omega_k) + \lambda_i(96C + 3\|v\|^2) + \mathrm{Tr}\,\hat{H} = 0,$$

so $\lambda_i(96C + 3\|v\|^2)$ does not depend on $i = 1, 2, 3$. Since the λ_i are pairwise distinct (otherwise the condition $\sum_{i=1}^{3} \eta_i^{-1} = 0$ from (3-14) is violated), we get $\|v\|^2 = -32C$.

Now take $Y, Z \perp \mathcal{J}v$ in (3-18). Projecting to $\mathcal{J}v \wedge \mathcal{J}v$ and using the fact that $\mathcal{J}v \wedge \mathcal{J}v$ is $\mathrm{ad}_{\mathcal{J}}$-invariant we obtain that the operator $V(Y, Z) + \sum_{i=1}^{3} \langle K_i Y, Z \rangle \hat{J}_i$ on $\mathcal{J}v$ commutes with every $\hat{J}_s$. The centralizer of the set $\{\hat{J}_1, \hat{J}_2, \hat{J}_3\}$ in the Lie algebra $\mathfrak{o}(4) = \mathfrak{o}(\mathcal{J}v)$ is the three-dimensional subalgebra spanned by

$$v \wedge \hat{J}_i v - \varepsilon_{jk} \hat{J}_j v \wedge \hat{J}_k v \quad \text{for } \{i, j, k\} = \{1, 2, 3\}$$

(right multiplication by the imaginary quaternions). Substituting $V(Y, Z)$ from (3-16) and using that

$$\hat{J}_i = \|v\|^{-2}(v \wedge \hat{J}_i v + \varepsilon_{jk} \hat{J}_j v \wedge \hat{J}_k v),$$

we obtain that the operator $V(Y, Z) + \sum_{i=1}^{3} \langle K_i Y, Z \rangle \hat{J}_i$ commutes with all the $\hat{J}_s$ for $Y, Z \perp \mathcal{J}v$ if and only if

$$-\tfrac{1}{2} \langle J_i Z, Y \rangle (\lambda_i^2 + \lambda_j \lambda_k - \lambda_i \lambda_k - \lambda_j \lambda_i) + 2\|v\|^{-2} \langle K_i Y, Z \rangle = 0 \quad \text{for all } i = 1, 2, 3.$$

Substituting the λ_i from (3-15) and $\langle K_i Y, Z \rangle$ from (3-17) and taking into account that $\|v\|^2 = -32C$, which is shown above, we obtain

$$\langle (J_i H + H J_i - 32C\lambda_i J_i) Y, Z \rangle = 0 \quad \text{for all } Y, Z \perp \mathcal{J}v \text{ and all } i = 1, 2, 3.$$

Then

$$\pi (J_i H + H J_i) \pi = 32C \lambda_i \pi J_i \pi,$$

where $\pi = \pi_{(\mathcal{J}v)^\perp}$. Multiplying both sides by $\pi J_i \pi$ from the right and using that $[\pi, J_i] = 0$ (as $(\mathcal{J}v)^\perp$ is $\mathcal{J}$-invariant), we get $\pi (J_i H J_i - H) \pi = -32C \lambda_i \pi$. Taking the traces of the both sides we obtain $-2\,\mathrm{Tr}(\pi H \pi) = -32C \lambda_i (n-4)$, which is a contradiction since $n > 4$ and the λ_i are pairwise distinct, which follows from the equation $\sum_{i=1}^{3} \eta_i^{-1} = 0$ of (3-14). $\square$

The next lemma shows that the relations similar to (3-5) and (3-6) of Lemma 3.5 also hold in all the remaining cases when $n = 8$, that is, when $\nu \neq 3$ and when $\nu = 3$ and $J_1 J_2 \neq \pm J_3$. As shown in Lemma 3.2, in all these cases the Weyl tensor has a smooth Cliff(7) structure in a neighborhood $\mathcal{U}$ of every point $x \in M'$. Moreover, Lemma 2.6(2), that Cliff(7) structure is an almost Hermitian octonion structure, in the following sense. For every $y \in \mathcal{U}$, we can identify $\mathbb{R}^8 = T_y M^8$ with $\mathbb{O}$ and of $\mathbb{R}^7$ with $\mathbb{O}' = 1^\perp$ via linear isometries $\imath_1$ and $\imath_2$ respectively so that the orthogonal multiplication (2-2) defined by Cliff(7) has the form (2-5): $J_u X = Xu$ for every $X \in \mathbb{R}^8 = \mathbb{O}$ and $u \in \mathbb{O}'$.

Lemma 3.6. *Let $x \in M' \subset M^8$, and let $\mathcal{U}$ be the neighborhood of x defined in Lemma 3.2. For every point $y \in \mathcal{U}$, identify $\mathbb{R}^8 = T_y M^8$ with $\mathbb{O}$ via a linear isometry so that the Clifford structure* Cliff(7) *on $\mathbb{R}^8$ is given by* (2-5). *Then there* *exist* $m, t, b_{ij} \in \mathbb{R}^8 = \mathbb{O}$ *with* $i, j = 1, \dots, 7$ *such that for all* $X, U \in \mathbb{R}^8 = \mathbb{O}$,

$$(\nabla_U J_i)X = \sum_{j=1}^{7} \langle b_{ij}, U\rangle X e_j + (X(U^* m) - \langle m, U\rangle X)e_i + \langle m, U e_i\rangle X, \tag{3-19a}$$

$$b_{ij} + b_{ji} = 0, \tag{3-19b}$$

$$(\nabla_X \rho)U - (\nabla_U \rho)X = 2\sum_{i=1}^{7} \eta_i (\langle m e_i, U\rangle X e_i - \langle m e_i, X\rangle U e_i + 2\langle X e_i, U\rangle m e_i) + \tfrac{3}{4}(X \wedge U)t, \tag{3-19c}$$

$$\nabla \eta_i = -4\eta_i m - \tfrac{1}{2} t. \tag{3-19d}$$

Proof. In the proof we use standard identities of the octonion arithmetic (some of them are given in Section 2.5).

By [N 2004, Lemma 7], for the Clifford structure Cliff(7) given by (2-5), there exist $b_{ij} \in \mathbb{R}^8$, with $i, j = 1, \dots, 7$, satisfying (3-19b) and an ($\mathbb{R}$-)linear operator $A : \mathbb{O} \to \mathbb{O}'$ such that for all $X, U \in \mathbb{R}^8 = \mathbb{O}$,

$$(\nabla_U J_i)X = \sum_{j=1}^{7} \langle b_{ij}, U\rangle X e_j + (X \cdot AU)e_i + \langle AU, e_i\rangle X. \tag{3-20}$$

Equation (3-2) is a polynomial equation in 24 real variables, the coordinates of the vectors $X, Y, U \in \mathbb{R}^8$. It still holds if we allow X, Y, U to be complex and extend the tensors J_i, ∇J_i and $\langle \cdot, \cdot \rangle$ to $\mathbb{C}^8$ by complex linearity. The complexified inner product $\langle \cdot, \cdot \rangle$ takes values in $\mathbb{C}$ and is a nonsingular quadratic form on $\mathbb{C}^8$. Moreover, Equation (2-5) is still true if we identify $\mathbb{C}^8$ with the bioctonion algebra $\mathbb{O} \otimes \mathbb{C}$, and $\mathbb{C}^7$ with $1^\perp = \mathbb{O}' \otimes \mathbb{C}$, the orthogonal complement to 1 in $\mathbb{O} \otimes \mathbb{C}$.

Let $Y \in \mathbb{O} \otimes \mathbb{C}$ be a nonzero isotropic vector (that is, $Y^* Y = 0$) and let

$$\mathcal{J}_{\mathbb{C}} Y = \mathrm{Span}_{\mathbb{C}}(J_1 Y, \dots, J_7 Y).$$

Then $Y \in \mathcal{J}_{\mathbb{C}} Y$ and the space $\mathcal{J}_{\mathbb{C}} Y$ is isotropic: The inner product of any two vectors from $\mathcal{J}_{\mathbb{C}} Y$ vanishes. Choose $X, U \in \mathcal{J}_{\mathbb{C}} Y$ and take the inner product of the complexified (3-2) with X. Since X, Y and U are mutually orthogonal, we get (3-3), which further simplifies to

$$\sum_{i=1}^{7} \eta_i \langle J_i X, U\rangle \langle (\nabla_Y J_i)Y, X\rangle = 0,$$

since

$$\|X\|^2 = \|Y\|^2 = \langle J_i Y, X\rangle = \langle J_i Y, U\rangle = 0.$$

Using (3-20) we obtain

$$\sum_{i=1}^{7} \eta_i \langle J_i X, U\rangle \langle (Y \cdot AY)e_i, X\rangle = 0$$

for all isotropic vectors Y and for all $X, U \in \mathcal{J}_{\mathbb{C}} Y$. It follows that $Y \cdot AY$ is perpendicular to $\sum_{i=1}^{7} \eta_i \langle J_i X, U\rangle X e_i$ for all $X, U \in \mathcal{J}_{\mathbb{C}} Y$. Since $Y \cdot AY = J_{AY} Y \in \mathcal{J}_{\mathbb{C}} Y$ and $\mathcal{J}_{\mathbb{C}} Y$ is isotropic, we get $Y \cdot AY \perp \mathcal{J}_{\mathbb{C}} Y$, so $Y \cdot AY$ is perpendicular to

$$\mathcal{J}_{\mathbb{C}} Y + \operatorname{Span}_{\mathbb{C}}(\{\textstyle\sum_{i=1}^{7} \eta_i \langle J_i X, U\rangle J_i X \mid X, U \in \mathcal{J}_{\mathbb{C}} Y\}).$$

Following the arguments in the proof of [N 2004, Lemma 8] starting with formula (29), we obtain that $AU = U^* m - \langle U, m\rangle 1$ for some $m \in \mathbb{O}$. Then (3-19a) follows from (3-20).

To prove (3-19c) and (3-19d), introduce the vectors $f_{ij} \in \mathbb{R}^8$ for $i, j = 1, \dots, 8$ and the quadratic map $T : \mathbb{R}^8 \to \mathbb{R}^8$ (similar to the map Q of (3-4)) by

$$f_{ij} = (\eta_i - \eta_j) b_{ij} + \delta_{ij}(\nabla \eta_i - 2\eta_i m), \tag{3-21}$$

$$\langle T(X), U\rangle = \tfrac{1}{3}\langle (\nabla_X \rho)U - (\nabla_U \rho)X, X\rangle - \textstyle\sum_{i=1}^{7} \eta_i \langle m e_i, X\rangle \langle X e_i, U\rangle. \tag{3-22}$$

Note that $f_{ij} = f_{ji}$ and $\langle T(X), X\rangle = 0$. Take X, Y, U to be mutually orthogonal vectors in $\mathbb{R}^8$. By (3-19a) and (3-19b),

$$\begin{aligned}\langle (\nabla_U J_i)X, Y\rangle &= \sum_{j=1}^{7} \langle b_{ij}, U\rangle \langle X e_j, Y\rangle - \langle m, U\rangle \langle X e_i, Y\rangle + \langle (X(U^* m))e_i, Y\rangle \\ &= \sum_{j=1}^{7} \langle b_{ij} - \delta_{ij} m, U\rangle \langle X e_j, Y\rangle + \langle m((e_i Y^*)X), U\rangle,\end{aligned}$$

so every term on the left side of (3-3) can be written as the inner product of U with a vector depending on X and Y. Since U is arbitrary other than being perpendicular to X and Y, we find after substituting (2-5) and (3-19a) into (3-3) and rearranging the terms that

$$\begin{aligned}&\|X\|^2 T(Y) + \|Y\|^2 T(X) \\ &\quad + \sum_{i=1}^{7} \big(2\eta_i \langle Y e_i, X\rangle (m((e_i Y^*)X) + (Y(X^* m))e_i) \\ &\qquad\qquad + \langle Y e_j, X\rangle (\langle f_{ij}, X\rangle Y e_i - \langle f_{ij}, Y\rangle X e_i) - \langle Y e_i, X\rangle \langle Y e_j, X\rangle f_{ij}\big) \\ &\hspace{20em} \in \operatorname{Span}(X, Y),\end{aligned}$$

for all $X \perp Y$, where we used the fact that $(X(Y^* m))e_i = -(Y(X^* m))e_i$, since $X \perp Y$. Taking the inner products with X and with Y, we obtain that the left side of the above (before the "$\in$") is equal to $\langle T(Y), X\rangle X + \langle T(X), Y\rangle Y$ for all $X \perp Y$. Taking $X = Yu$ with $u = \sum_{i=1}^{7} u_i e_i \in \mathbb{O}'$ and regrouping the terms, we obtain

$$
\begin{aligned}
\text{(3-23)}\quad & \|u\|^2 T(Y) + T(Yu) \\
&+ 2\sum_{i=1}^{7} \eta_i u_i (2\langle Y, me_i\rangle Yu - 2\langle Yu, me_i\rangle Y + 2\|Y\|^2 (mu)e_i) \\
&+ \sum_{i,j=1}^{7} u_j (\langle f_{ij} + 8\delta_{ij}\eta_i m, Yu\rangle Ye_i - \langle f_{ij} + 8\delta_{ij}\eta_i m, Y\rangle (Yu)e_i) \\
&- \sum_{i,j=1}^{7} \|Y\|^2 u_i u_j f_{ij} = \|Y\|^{-2}\langle T(Y), Yu\rangle Yu + \|Y\|^{-2}\langle T(Yu), Y\rangle Y,
\end{aligned}
$$

where we used

$$
\begin{aligned}
& m((e_i Y^*)X) + (Y(X^* m))e_i \\
&= 2\langle Y, me_i\rangle Yu - 2\langle Yu, me_i\rangle Y + 4\langle Yu, m\rangle Ye_i - 4\langle Y, m\rangle (Yu)e_i + 2\|Y\|^2 (mu)e_i,
\end{aligned}
$$

which follows from

$$m((e_i Y^*)X) = (Y(X^* m))e_i - 2\langle m, Ye_i\rangle X - 2\langle X, me_i\rangle Y$$

for all X, Y, and

$$(Y(X^* m))e_i = -2\langle Y, m\rangle (Yu)e_i - 2\langle Y, mu\rangle Ye_i + \|Y\|^2 (mu)e_i$$

for $X = Yu$ and $u \perp 1$. By Lemma 2.7(1) (with $\nu = 1$ and $\mathcal{J}Y = \mathrm{Span}(Y, Yu)$) we obtain that both coefficients on the right side of (3-23), $\|Y\|^{-2}\langle T(Y), Yu\rangle$ and $\|Y\|^{-2}\langle T(Yu), Y\rangle$, are linear forms of $Y \in \mathbb{R}^8$ for every $u \in \mathbb{O}'$. Since $\langle T(Y), Y\rangle$ is zero, this implies that there exists an ($\mathbb{R}$-)linear operator $C : \mathbb{O} \to \mathbb{O}'$ such that $\|Y\|^{-2} Y^* T(Y) = CY$, so $T(Y) = Y \cdot CY$ for all $Y \in \mathbb{O}$. Substituting this to (3-23) and rearranging the terms, we obtain

$$
\begin{aligned}
\text{(3-24)}\quad & (Yu)\Big(C(Yu) - \sum_{i,j=1}^{7} u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Y\rangle e_i\Big) \\
&+ Y\Big(\|u\|^2 CY + \sum_{i=1}^{7} \big(4\eta_i u_i (\langle Y, me_i\rangle u - \langle Yu, me_i\rangle 1 + Y^*((mu)e_i)) \\
&\qquad + u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Yu\rangle e_i \\
&\qquad - u_i u_j Y^* f_{ij} - \langle CY, u\rangle u + \langle C(Yu), u\rangle 1\big)\Big) = 0,
\end{aligned}
$$

The left side of (3-24) has the form $(Yu)L(Y, u) + YF(Y, u)$, where $L(Y, u)$ and $F(Y, u)$ are ($\mathbb{R}$-) linear operators on $\mathbb{O}$ for every $u \in \mathbb{O}'$. By [N 2004, Lemma 6], for every unit octonion $u \in \mathbb{O}'$, $L(Y, u) = \langle a(u), Y\rangle 1 + \langle t(u), Y\rangle u + Y^* p(u)$ for some functions $a, t, p : S^6 \subset \mathbb{O}' \to \mathbb{O}$. Extending a, t, p by homogeneity (of degree 1, 0, 1 respectively) to $\mathbb{O}'$ we obtain

$$C(Yu) - \sum_{i,j=1}^{7} u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Y\rangle e_i = \langle a(u), Y\rangle 1 + \langle t(u), Y\rangle u + Y^* p(u)$$

for all $u \in \mathbb{O}'$. Moreover, $p(u) = -a(u)$, since $C(Y) \perp 1$. By the linearity of the left side in u, we get

$$\begin{aligned}\langle a(u_1+u_2) - a(u_1) - a(u_2), Y\rangle 1 + \langle t(u_1+u_2) - t(u_1), Y\rangle u_1 \\ + \langle t(u_1+u_2) - t(u_2), Y\rangle u_2 + Y^*(a(u_1+u_2) - a(u_1) - a(u_2)) = 0\end{aligned}$$

for all $u_1, u_2 \in \mathbb{O}'$. Then $Y^*(a(u_1+u_2) - a(u_1) - a(u_2)) \in \mathrm{Span}(1, u_1, u_2)$ for all $Y \in \mathbb{O}$, which is only possible when $a(u)$ is linear, that is, $a(u) = Bu$ for some ($\mathbb{R}$-)linear operator $B : \mathbb{O}' \to \mathbb{O}$. It follows that $t(u_1+u_2) = t(u_1) = t(u_2)$, that is, $t \in \mathbb{O}$ is a constant. So

$$C(Yu) = \sum_{i,j=1}^{7} u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Y\rangle e_i + \langle Bu, Y\rangle 1 + \langle t, Y\rangle u - Y^* Bu.$$

Taking the inner product of the both sides with $v \in \mathbb{O}'$ and subtracting from the resulting equation the same equation with u and v interchanged, we obtain $\langle C(Yu), v\rangle - \langle C(Yv), u\rangle = \langle Bv, Yu\rangle - \langle Bu, Yv\rangle$, since $f_{ij} = f_{ji}$ by (3-21). It follows that $\langle C^t v - Bv, Yu\rangle = \langle C^t u - Bu, Yv\rangle$, where C^t is the operator adjoint to C. Now taking $u \perp v$ and $Y = uv$, we get

$$\|u\|^2 \langle C^t v - Bv, v\rangle = -\|v\|^2 \langle C^t u - Bu, u\rangle,$$

which implies $C = B^t$. Then from the above,

$$\langle C(Yu), e_i\rangle = \sum_{j=1}^{7} u_j \langle f_{ij} + 8\delta_{ij}\eta_i m, Y\rangle + \langle t, Y\rangle u_i - \langle Bu, Ye_i\rangle = \langle Be_i, Yu\rangle,$$

so $\sum_{j=1}^{7} u_j (f_{ij} + \delta_{ij}(8\eta_i m + t)) + (Bu)e_i + (Be_i)u = 0$. Therefore

(3-25) $T(Y) = Y \cdot CY = Y \cdot B^t Y$ and $f_{ij} = -\delta_{ij}(8\eta_i m + t) - (Be_i)e_j - (Be_j)e_i$.

Substituting (3-25) to (3-24) and simplifying, we obtain

$$-\langle Lu \cdot u, Y\rangle Y - \langle Lu, Y\rangle Yu + \|Y\|^2 Lu \cdot u = 0,$$

where $Lu = 4Bu - tu - 4\sum_{i=1}^{7} \eta_i u_i m e_i$. Taking $Y \perp Lu, Lu \cdot u$ we get $Lu = 0$, so

(3-26) $$Bu = \tfrac{1}{4} tu + \textstyle\sum_{i=1}^{7} \eta_i u_i m e_i.$$

Substituting (3-26) into the first equation of (3-25) and then into (3-22) and simplifying, we obtain that for arbitrary $X, U \in \mathbb{O}$,

$$\langle(\nabla_X\rho)U - (\nabla_U\rho)X, X\rangle = \tfrac{3}{4}(\langle t, X\rangle\langle X, U\rangle - \|X\|^2\langle t, U\rangle) + 6\textstyle\sum_{i=1}^{7}\eta_i\langle Xe_i, U\rangle\langle me_i, X\rangle.$$

Polarizing this equation we get

$$\begin{aligned}\langle(\nabla_Y\rho)U - (\nabla_U\rho)Y, X\rangle + \langle(\nabla_X\rho)U - (\nabla_U\rho)X, Y\rangle &= \tfrac{3}{4}(\langle t, X\rangle\langle Y, U\rangle + \langle t, Y\rangle\langle X, U\rangle - 2\langle X, Y\rangle\langle t, U\rangle)\\ &\quad + 6\sum_{i=1}^{7}\eta_i(\langle Xe_i, U\rangle\langle me_i, Y\rangle + \langle Ye_i, U\rangle\langle me_i, X\rangle).\end{aligned}$$

Subtracting the same equation with X and U interchanged and using the fact that ρ is symmetric, we get (3-19c). The second equation of (3-25) and (3-26) give $f_{ii} = -6\eta_i m - t/2$, which by (3-21) implies (3-19d). □

Lemma 3.7. *In the assumptions of Theorem 3.1, let $x \in M'$, where $M' \subset M^n$ is defined in Lemma 3.2. Then there exists a neighborhood $\mathcal{U} = \mathcal{U}(x)$ and a smooth metric on $\mathcal{U}$ conformally equivalent to the original metric whose curvature tensor has the form* (3-1), *with ρ a multiple of the identity.*

Proof. Let $x \in M'$ and let $\mathcal{U}$ be the neighborhood of x on which the Weyl tensor has the smooth Clifford structure defined in Lemma 3.2. We can assume that $\nu > 0$, since in the case of a Cliff(0) structure, the curvature tensor given by (3-1) has the form $R(X, Y)Z = \langle X, Z\rangle\rho Y + \langle\rho X, Z\rangle Y - \langle Y, Z\rangle\rho X - \langle\rho Y, Z\rangle X$, so the Weyl tensor vanishes. Then the metric on $\mathcal{U}$ is locally conformally flat, that is, is conformally equivalent to a one with $\rho = 0$.

If $n = 8$ and $\nu = 7$, and all the η_i at x are equal, then they are equal at some neighborhood of x (by the definition of M'). By Remark 3.4, we can replace ρ by $\rho + 3\eta_1/2\,\mathrm{id}$ and η_i by $0 = \eta_i - \eta_1$ in (3-1), thus arriving at the case $\nu = 0$ considered above.

For the remaining part of the proof, we will assume that in the case $n = 8$ and $\nu = 7$, at least two of the η_i at x are different; up to relabeling, let $\eta_1 \neq \eta_2$ at x and also on a neighborhood of x (replace $\mathcal{U}$ by a smaller neighborhood if necessary). Let f be a smooth function on $\mathcal{U}$ and let $\langle\cdot,\cdot\rangle' = e^f\langle\cdot,\cdot\rangle$. Then $W' = W$, $J_i' = J_i$, $\eta_i' = e^{-f}\eta_i$ and on functions, $\nabla' = e^{-f}\nabla$, where we use the prime for objects associated to the metric $\langle\cdot,\cdot\rangle'$. Moreover, the curvature tensor R' still has the form (3-1), and all the identities of Lemmas 3.5 and 3.6 remain valid.

In the cases considered in Lemma 3.5, the ratios η_i/η_1 are constant, which follows from (3-5d) and (3-6c). In particular, taking $f = \ln|\eta_1|$ we obtain that η_1' is a constant, so all the η_i' are constant; $m_i' = 0$ by (3-5d), so $(\nabla_Y'\rho')U - (\nabla_U'\rho')Y = 0$

by (3-6a). In the case $n = 8$ and $\nu = 7$ (Lemma 3.6), take $f = \ln|\eta_1 - \eta_2|$. Then by (3-19d), we have $\nabla f = -4m$ and $\nabla' \eta_i' = -(1/2)e^{-2f} t$, which imply $m' = -(1/4)\nabla' \ln|\eta_1' - \eta_2'| = 0$ and $t' = e^{-2f} t$, again by (3-19d) for the metric $\langle\,\cdot\,,\cdot\,\rangle'$. Then by (3-19c), we have $(\nabla_X' \rho')U - (\nabla_U' \rho')X = (3/4)(X \wedge' U)t'$. By Remark 3.4, we can replace ρ' by $\tilde\rho = \rho' + (3/2)(\eta_1' + C)\,\mathrm{id}$ and η_i' by $\tilde\eta_i = \eta_i' - (\eta_1' + C)$ without changing the curvature tensor R' given by (3-1). (Here C is a constant chosen in such a way that $\tilde\eta_i \neq 0$ anywhere on $\mathcal{U}$.) Then by (3-19c) and (3-19d), $(\nabla_X' \tilde\rho)U - (\nabla_U' \tilde\rho)X = 0$ for the metric $\langle\,\cdot\,,\cdot\,\rangle'$.

Dropping the primes and the tildes, we obtain that, up to a conformal smooth change of the metric on $\mathcal{U}$, the curvature tensor has the form (3-1), with ρ satisfying the identity

$$(\nabla_Y \rho)X = (\nabla_X \rho)Y \quad \text{for all } X, Y,$$

that is, with ρ being a symmetric *Codazzi tensor*.

Then by [Derdziński and Shen 1983, Theorem 1], at every point of $\mathcal{U}$ for any three eigenspaces $E_\beta, E_\gamma, E_\alpha$ of ρ with $\alpha \notin \{\beta, \gamma\}$, the curvature tensor satisfies $R(X, Y)Z = 0$ for all $X \in E_\beta,\ Y \in E_\gamma$ and $Z \in E_\alpha$. It then follows from (3-1) that

$$\text{(3-27)}\quad \sum_{i=1}^{\nu} \eta_i \left(2\langle J_i X, Y\rangle J_i Z + \langle J_i Z, Y\rangle J_i X - \langle J_i Z, X\rangle J_i Y\right) = 0$$
$$\text{for all } X \in E_\beta,\ Y \in E_\gamma,\ Z \in E_\alpha, \quad \text{with } \alpha \notin \{\beta, \gamma\}.$$

Suppose ρ is not a multiple of the identity. Let $E_1, \dots, E_p$ for $p \geq 2$ be the eigenspaces of ρ. If $p > 2$, write $E_1' = E_1$ and $E_2' = E_2 \oplus \cdots \oplus E_p$. Then by linearity, (3-27) holds for any $X, Y \in E_\alpha'$ and $Z \in E_\beta'$ such that $\{\alpha, \beta\} = \{1, 2\}$. Hence to prove the lemma it suffices to show that (3-27) leads to a contradiction in the assumption $p = 2$. For the rest of the proof, suppose that $p = 2$. Let $d_\alpha = \dim E_\alpha$ with $d_1 \leq d_2$.

Choose $Z \in E_\alpha$, $\ X, Y \in E_\beta$, $\ \alpha \neq \beta$, and take the inner product of (3-27) with X. We get $\sum_{i=1}^{\nu} \eta_i \langle J_i X, Y\rangle\langle J_i X, Z\rangle = 0$. It follows that for every $X \in E_\alpha$, the subspaces E_1 and E_2 are invariant subspaces of the symmetric operator $\hat R_X \in \mathrm{End}(\mathbb{R}^n)$ defined by $\hat R_X Y = \sum_{i=1}^{\nu} \eta_i \langle J_i X, Y\rangle J_i X$. So $\hat R_X$ commutes with the orthogonal projections $\pi_\beta : \mathbb{R}^n \to E_\beta$ for $\beta = 1, 2$. Then for all $\alpha, \beta = 1, 2$ (α and β can be equal), all $X \in E_\alpha$ and all $Y \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{\nu} \eta_i \langle J_i X, \pi_\beta Y\rangle J_i X = \sum_{i=1}^{\nu} \eta_i \langle J_i X, Y\rangle \pi_\beta J_i X.$$

Taking $Y = J_j X$ we get that $\pi_\beta J_j X \subset \mathcal{J}X$; that is, $\pi_\beta \mathcal{J}X \subset \mathcal{J}X$ for all $X \in E_\alpha$ with $\alpha, \beta = 1, 2$. Since $\pi_1 + \pi_2 = \mathrm{id}$, we obtain $\mathcal{J}X \subset \pi_1 \mathcal{J}X \oplus \pi_2 \mathcal{J}X \subset \mathcal{J}X$; hence $\mathcal{J}X = \pi_1 \mathcal{J}X \oplus \pi_2 \mathcal{J}X$. Since every function $f_{\alpha\beta} : E_\alpha \to \mathbb{Z}$, $X \mapsto \dim \pi_\beta \mathcal{J}X$ with $\alpha, \beta = 1, 2$ is lower semicontinuous, and $f_{\alpha 1}(X) + f_{\alpha 2}(X) = \nu$ for all nonzero

$X \in E_\alpha$, there exist constants $c_{\alpha\beta}$ with $c_{\alpha 1} + c_{\alpha 2} = \nu$ such that $\dim \pi_\beta \mathcal{J} X = c_{\alpha\beta}$ for all $\alpha, \beta = 1, 2$ and all nonzero $X \in E_\alpha$.

Let $X, Y \in E_\alpha$, $Z \in E_\beta$ and $\beta \neq \alpha$. Taking the inner product of (3-27) with $J_j Z$ for $j = 1, \dots, \nu$, we get

$$2\eta_j \langle J_j X, Y\rangle \|Z\|^2 = \sum_{i \neq j} \eta_i (\langle J_i Z, X\rangle\langle J_i Y, J_j Z\rangle - \langle J_i Z, Y\rangle\langle J_i X, J_j Z\rangle).$$

Since $\langle J_i Z, X\rangle = \langle J_i \pi_\beta Z, X\rangle = -\langle Z, \pi_\beta J_i X\rangle$ (and similarly for $\langle J_i Z, Y\rangle$), the right side, viewed as a quadratic form of $Z \in E_\beta$, vanishes for all Z in the intersection of $\pi_\beta \mathcal{J} X)^\perp$ and $(\pi_\beta \mathcal{J} Y)^\perp$, that is, on a subspace of dimension at least $d_\beta - 2c_{\alpha\beta}$. So for $\alpha \neq \beta$, either $2c_{\alpha\beta} \geq d_\beta$, or $\mathcal{J} E_\alpha \perp E_\alpha$, that is, $\pi_\beta \mathcal{J} X = \mathcal{J} X$ for all $X \in E_\alpha$, so $c_{\alpha\beta} = \nu$.

Similarly, if $Z \in E_\alpha$, $X, Y \in E_\beta$ and $\beta \neq \alpha$, the inner product of (3-27) with $J_j X$ for $j = 1, \dots, \nu$ gives

$$\eta_j \langle J_j Z, Y\rangle \|X\|^2 = \sum_{i=1}^{\nu} \eta_i (-2\langle J_i X, Y\rangle\langle J_i Z, J_j X\rangle + \langle J_i Z, X\rangle\langle J_i Y, J_j X\rangle).$$

Because

$$\langle J_i X, Y\rangle = -\langle X, \pi_\beta J_i Y\rangle \quad \text{and} \quad \langle J_i Z, X\rangle = -\langle X, \pi_\beta J_i Z\rangle,$$

the right side, viewed as a quadratic form of $X \in E_\beta$, vanishes on the intersection of $(\pi_\beta \mathcal{J} Y)^\perp$ and $(\pi_\beta \mathcal{J} Z)^\perp$, whose dimension is at least $d_\beta - c_{\alpha\beta} - c_{\beta\beta}$. We obtain that for $\alpha \neq \beta$, either $c_{\alpha\beta} + c_{\beta\beta} \geq d_\beta$, or $\mathcal{J} E_\alpha \perp E_\beta$, that is, $\pi_\beta \mathcal{J} Z = 0$ for all $Z \in E_\alpha$, so $c_{\alpha\beta} = 0$. Since $c_{\alpha\beta} = 0$ contradicts both $2c_{\alpha\beta} \geq d_\beta$ and $c_{\alpha\beta} = \nu$ (since $\nu > 0$), we must have $c_{\alpha\beta} + c_{\beta\beta} \geq d_\beta$. Then $2\nu = \sum_{\alpha\beta} c_{\alpha\beta} \geq d_1 + d_2 = n$.

This proves the lemma in all the cases when $2\nu < n$, that is, in all the cases except for $n = 8$ and $\nu \geq 4$ (which follows from Lemma 2.4).

Consider the case $n = 8$. We identify $\mathbb{R}^8$ with $\mathbb{O}$ and assume that the J_i act as in (2-5). Let $D : \mathbb{O} \to \mathbb{O}$ be the symmetric operator defined by $D1 = 0$ and $De_i = \eta_i e_i$. By (2-4), condition (3-27) still holds if we replace D by $D + c\mathrm{Im}$, where Im is the operator of taking the imaginary part of an octonion. So we can assume that the eigenvalue of the maximal multiplicity of $D|_{\mathbb{O}'}$ is zero (one of them, if there are more than one). Then in (3-27), $\nu = \operatorname{rk} D$. By construction, we have $\nu \leq 6$, and we only need to consider the cases when $\nu \geq 4$, as shown above.

By (2-5),

$$\langle J_i X, Y\rangle J_i Z = \langle X e_i, Y\rangle Z e_i = \langle e_i, X^* Y\rangle Z e_i,$$

so

$$\sum_{i=1}^{\nu} \eta_i \langle J_i X, Y\rangle J_i Z = \sum_{i=1}^{\nu} \eta_i \langle e_i, X^* Y\rangle Z e_i = \sum_{i=1}^{7} \langle De_i, X^* Y\rangle Z e_i = Z D(X^* Y),$$

since D is symmetric and $D1 = 0$. Then (3-27) can be rewritten as

$$2ZD(X^*Y) + XD(Z^*Y) - YD(Z^*X) = 0 \tag{3-28}$$
$$\text{for all } X, Y \in E_\beta,\ Z \in E_\alpha \text{ an } \alpha \neq \beta.$$

Taking the inner product of (3-28) with X (and using the fact that D is symmetric, $D1 = 0$ and $Y^*X = 2\langle X, Y\rangle 1 - X^*Y$), we obtain $\langle D(X^*Y), X^*Z\rangle = 0$. It follows that for every $X \in E_\beta$, the subspaces E_1 and E_2 are invariant subspaces of the symmetric operator $L_X D L_X^t$, where $L_X : \mathbb{O} \to \mathbb{O}$ is the left multiplication by X (note that $L_{X^*} = L_X^t$ and that $L_X D L_X^t$ coincides with the operator $\hat{R}_X$ introduced above). So $L_X D L_X^t$ commutes with both orthogonal projections $\pi_\alpha : \mathbb{R}^8 \to E_\alpha$ for $\alpha = 1, 2$. It follows that for every α, β (not necessarily distinct) and every $X \in E_\beta$, the operator D commutes with $L_X^t \pi_\alpha L_X = \|X\|^2 \pi_{X^* E_\alpha}$, that is,

(3-29) $X^* E_\alpha$ is an invariant subspace of D for all α, β, and all $X \in E_\beta$.

Consider all the possible cases for the dimensions d_α of the subspaces E_α.

Let $(d_1, d_2) = (1, 7)$, and let u be a nonzero vector in E_1. Then by (3-29), every line spanned by X^*u with $X \perp u$ (that is, every line in $\mathbb{O}'$) is an invariant subspace of D. It follows that $D|_{\mathbb{O}'}$ is a multiple of the identity, which is a contradiction since $\operatorname{rk} D = \nu$ for $4 \le \nu \le 6$.

Let $(d_1, d_2) = (2, 6)$, and let $E_1 = \operatorname{Span}(u, ue)$ for $e \in \mathbb{O}'$, and let $\|e\| = \|u\| = 1$. Then $E_2 = uL$, where $L = \operatorname{Span}(1, e)^\perp$. Let U be any element of L. By (3-29) with $E_\alpha = E_1$ and $X = uU^* = -uU \in E_2$, every two-plane $\operatorname{Span}(U, (Uu^*)(ue))$ is an invariant subspace of D. Note that $(Uu^*)(ue) \in L$, and that the operator J defined by $JU = (Uu^*)(ue)$ is an almost Hermitian structure on L. Then L is an invariant subspace of D since it is as the sum of invariant subspaces $\operatorname{Span}(U, JU)$ and $JD|_L U \in \operatorname{Span}(U, JU)$ (since $\operatorname{Span}(U, JU)$ is both J- and $D|_L$-invariant). From Lemma 2.7(1), it follows that the operator $JD|_L$ is a linear combination of $\mathrm{id}|_L$ and J. Since D is symmetric and its eigenvalue of maximal multiplicity is zero, we have $D|_L = 0$. Then $\nu = \operatorname{rk} D \le 1$, which is a contradiction.

For the cases $(d_1, d_2) = (3, 5)$, $(4, 4)$, we use the notion of Cayley plane. A four-dimensional subspace $\mathscr{C} \subset \mathbb{O}$ is called a *Cayley plane* if $X(Y^*Z) \in \mathscr{C}$ for orthonormal octonions $X, Y, Z \in \mathscr{C}$. This definition coincides with [Harvey and Lawson 1982, Definition IV.1.23], if we disregard the orientation. We will need the following properties of the Cayley plane (they can be found in [ibid., Section IV] or proved directly):

(i) A Cayley plane is well defined; moreover, if $X(Y^*Z) \in \mathscr{C}$ for some triple X, Y, Z of orthonormal octonions in $\mathscr{C}$, then the same is true for any (possibly nonorthonormal) triple $X, Y, Z \in \mathscr{C}$.

(ii) If $\mathscr{C}$ is a Cayley plane, then the subspace $X^*\mathscr{C}$ is the same for all nonzero $X \in \mathscr{C}$; we call this subspace $\mathscr{C}^*\mathscr{C}$.

(iii) If $\mathscr{C}$ is a Cayley plane, then $\mathscr{C}^\perp$ is also a Cayley plane and $\mathscr{C}^{\perp *}\mathscr{C}^\perp = \mathscr{C}^*\mathscr{C}$. Moreover, for all nonzero $X \in \mathscr{C}^\perp$, the subspace $X^*\mathscr{C}$ is the same and is equal to $(\mathscr{C}^*\mathscr{C})^\perp$.

(iv) For every nonzero $e \in \mathbb{O}$ and every pair of orthonormal imaginary octonions u, v, the subspace $\mathscr{C} = \mathrm{Span}(e, eu, ev, (eu)v)$ is a Cayley plane; every Cayley plane can be obtained in this way.

Let $(d_1, d_2) = (3, 5)$. Then E_1 is contained in a Cayley plane $\mathscr{C}$ (spanned by E_1 and $X(Y^*Z)$ for some orthonormal vectors $X, Y, Z \in E_1$), so $\mathscr{C}^\perp \subset E_2$. Let U be a unit vector in the orthogonal complement to $\mathscr{C}^\perp$ in E_2. Then $X^*E_2 = \mathscr{C}^*\mathscr{C} \oplus \mathbb{R}(X^*U)$ for every nonzero $X \in \mathscr{C}^\perp$ by properties (ii) and (iii). Since for any two invariant subspaces of a symmetric operator, their intersection and the orthogonal complement to it in each of them are also invariant, it follows from (3-29) that both $\mathscr{C}^*\mathscr{C}$ and every line $\mathbb{R}(X^*U)$ with $X \in \mathscr{C}^\perp$ are invariant subspaces of D. Then D restricts to a multiple of the identity on the four-dimensional space $(\mathscr{C}^\perp)^*U$. Since the eigenvalue of maximal multiplicity of D is zero, $\mathbb{R}1 \oplus (\mathscr{C}^\perp)^*U \subset \operatorname{Ker} D$. Then $\nu = \operatorname{rk} D \le 3$, which is again a contradiction.

Let now $d_1 = d_2 = 4$. First suppose E_1 is not a Cayley plane. Let X_1 and X_2 be orthonormal vectors in E_1. Then $X_1^*E_1 \cap X_2^*E_1$ contains $\mathrm{Span}(1, X_1^*X_2)$, since $X_2^*X_1 = -X_1^*X_2$. Also, for any unit vector $Y \in X_1^*E_1 \cap X_2^*E_1$ orthogonal to $\mathrm{Span}(1, X_1^*X_2)$, we have $Y = X_1^*X_3 = X_2^*X_4$ for some $X_3, X_4 \in E_1$ such that $X_3, X_4 \perp X_1, X_2$, which implies $X_2(X_1^*X_3) = X_4 \in E_1$, so E_1 is a Cayley plane by property (i). It follows that $X_1^*E_1 \cap X_2^*E_1 = \mathrm{Span}(1, X_1^*X_2)$. Since by (3-29) both subspaces on the left side are invariant under D and since $\mathbb{R}1$ is an invariant subspace of D, we obtain that every line $\mathbb{R}(X_1^*X_2)$ for $X_1, X_2 \in E_1$ is an invariant subspace of D (that is, $X_1^*X_2$ is an eigenvector of D). Then the space $L = \mathrm{Span}(E_1^*E_1)$ lies in an eigenspace of D, so $D|_L$ is a multiple of $\mathrm{id}|_L$. If $X_1, X_2, X_3 \in E_1$ are orthonormal, then $X_2^*X_3 \notin X_1^*E_1$, since E_1 is not a Cayley plane. So $\dim L \ge 5$. Since the eigenvalue of maximal multiplicity of D is zero, $\nu = \operatorname{rk} D \le 3$, a contradiction.

Let again $d_1 = d_2 = 4$, and let E_1 be a Cayley plane. Then $E_2 = (E_1)^\perp$ is also a Cayley plane by property (iii). Also, by the same property, $E_1^*E_1 = E_2^*E_2 = V_1$ and $E_1^*E_2 = E_1^*E_2 = V_2$, where V_1 and V_2 are mutually orthogonal four-dimensional subspaces of $\mathbb{O}$, and $1 \in V_1$. From (3-29), both V_1 and V_2 are invariant under D. Let $X, Y \in E_1$ and $Z, W \in E_2$, with $X, Z \ne 0$, and let $u = X^{-1}Y$ and $v = Z^{-1}W$. Since $X^{-1} = \|X\|^{-2}X^*$, we have $L_{X^{-1}}E_1 = V_1$ by property (ii). Similarly, $L_{Z^{-1}}E_2 = V_1$. Taking the inner product of (3-28) with W we obtain

$$2\|Z\|^2\|X\|^2\langle Du, v\rangle - \langle D(Z^*(Xu)), Z^*(Xv)\rangle = -\langle D(Z^*X), Z^*((Xu)v)\rangle$$

for all $X \in E_1$, $Z \in E_2$ and $u, v \in V_1$. The left side is symmetric in u, v. Since $(Xu)v = -(Xv)u$ for any $u \perp v$ with $u, v \perp 1$, we obtain $\langle D(Z^*X), Z^*((Xu)v)\rangle = 0$ for all $u, v \in V_1$ with $u \perp v$ and $u, v \perp 1$, and all $X \in E_1$ and $Z \in E_2$. Given any nonzero orthogonal $X, X' \in E_1$, we can find $u, v \in V_1$ with $u \perp v$ and $u, v \perp 1$ such that $X' = (Xu)v$. To see this, note that $Xu \in E_1$ for every $u \in V_1 = E_1^* E_1$ by property (i). Since L_X is nonsingular, $L_X(V_1 \cap 1^\perp)$ is a three-dimensional subspace of E_1. The same is true with X replaced by X'. Therefore $Xu = X'v$ for some $u, v \in V_1 \cap 1^\perp$; hence $X' = -\|v\|^{-2}(Xu)v$. Since $X' \perp X$, we get $\langle X, (Xu)v\rangle = 0$, so $u \perp v$. Thus $\langle D(Z^*X), Z^*X'\rangle = 0$ for any $Z \in E_2$ and any orthogonal $X, X' \in E_1$. Since $Z^*E_1 = V_2$ for any nonzero $Z \in E_2$ by properties (ii) and (ii), and since the operator L_{Z^*} is orthogonal when $\|Z\| = 1$, we get $\langle Dv_1, v_2\rangle = 0$ for any two orthogonal vectors $v_1, v_2 \in V_2$. It follows that the restriction of D to its invariant subspace V_2 is a multiple of the identity. Since $V_2 \subset \mathbb{O}'$ and the eigenvalue of $D|_{\mathbb{O}'}$ of maximal multiplicity is zero, we obtain $\mathbb{R}1 \oplus V_2 \subset \operatorname{Ker} D$. Then $\nu = \operatorname{rk} D \le 3$, which is a contradiction. □

Remark 3.8. It follows from the proof of Lemma 3.7 that the algebraic statement "a symmetric operator satisfying (3-27) is a multiple of the identity" is valid when $2\nu < n$. In particular, when $n = 16$, it remains true if we relax the restrictions $\nu \le 4$ of Theorem 3.1 to $\nu \ne 8$ (as for $n = 16$ and $\nu \le 8$ by (2-3)).

Lemma 3.7 implies Theorem 3.1 at the generic points. Indeed, by Lemma 3.7, every $x \in M'$ has a neighborhood $\mathcal{U}$ that is either conformally flat or is conformally equivalent to a Riemannian manifold whose curvature tensor has the form (3-1), with ρ being a multiple of the identity, that is, whose curvature tensor has a Clifford structure. It follows from [N 2003, Theorem 1.2] and [N 2004, Proposition 2] that $\mathcal{U}$ is conformally equivalent to an open subset of one of five model spaces: the rank-one symmetric spaces $\mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$, $\mathbb{H}P^{n/4}$ or $\mathbb{H}H^{n/4}$, or Euclidean space.

To prove Theorem 3.1 in full, we show first that the same is true for any $x \in M^n$, and second that the model space, to a domain of which $\mathcal{U}$ is conformally equivalent, is the same for all $x \in M^n$.

We normalize the standard metric $\tilde{g}$ on each of the spaces $\mathbb{C}P^{n/2}$, $\mathbb{C}H^{n/2}$, $\mathbb{H}P^{n/4}$ and $\mathbb{H}H^{n/4}$ so that the sectional curvature K_σ satisfies $|K_\sigma| \in [1, 4]$. Then the curvature tensor of each has a Clifford structure $\operatorname{Cliff}(\nu; J_1, \dots, J_\nu; \varepsilon, \varepsilon, \dots, \varepsilon)$. Here $\nu = 1, 3$ and $\varepsilon = \pm 1$. The J_i are smooth anticommuting almost Hermitian structures with $J_1 J_2 = \pm J_3$ when $\nu = 3$, and satisfy

$$\tilde{\nabla}_Z J_i = \sum_{j=1}^{m} \omega_i^j(Z) J_j,$$

where ω_i^j are smooth 1-forms with $\omega_i^j + \omega_j^i = 0$ and $\tilde{\nabla}$ is the Levi-Civita connection for $\tilde{g}$. Denote the corresponding spaces by $M_{\nu,\varepsilon}$ and their Weyl tensors by $W_{\nu,\varepsilon}$, so that

$$M_{1,1} = (\mathbb{C}P^{n/2}, \tilde{g}), \qquad M_{1,-1} = (\mathbb{C}H^{n/2}, \tilde{g}),$$
$$M_{3,1} = (\mathbb{H}P^{n/4}, \tilde{g}), \qquad M_{3,-1} = (\mathbb{H}H^{n/4}, \tilde{g}).$$

We start with a technical lemma:

Lemma 3.9. *Let $(N^n, \langle\cdot,\cdot\rangle)$ be a smooth Riemannian space locally conformally equivalent to one of the $M_{\nu,\varepsilon}$, so that $\tilde{g} = f\langle\cdot,\cdot\rangle$ for a positive smooth function $f = e^{2\phi} : N^n \to \mathbb{R}$. Then the curvature tensor R and the Weyl tensor W of $(N^n, \langle\cdot,\cdot\rangle)$ satisfy*

$$\text{(3-30a)} \quad R(X,Y) = (X \wedge KY + KX \wedge Y) + \varepsilon f(X \wedge Y + T(X,Y)), \quad \textit{where}$$
$$T(X,Y) = \textstyle\sum_{i=1}^{\nu} (J_i X \wedge J_i Y + 2\langle J_i X, Y\rangle J_i),$$
$$K = H(\phi) - \nabla\phi \otimes \nabla\phi + \tfrac{1}{2}\|\nabla\phi\|^2 \,\mathrm{id},$$

$$\text{(3-30b)} \quad W(X,Y) = W_{\nu,\varepsilon}(X,Y) = \varepsilon f\bigl(-\tfrac{3\nu}{n-1} X \wedge Y + T(X,Y)\bigr),$$

$$\text{(3-30c)} \quad \|W\|^2 = C_{\nu n} f^2, \quad \textit{where } C_{\nu n} = 6\nu n(n+2)(n-\nu-1)(n-1)^{-1},$$

$$\text{(3-30d)} \quad (\nabla_Z W)(X,Y) = \varepsilon Z f\bigl(-\tfrac{3\nu}{n-1} X \wedge Y + T(X,Y)\bigr) + \tfrac{1}{2}\varepsilon\bigl([T(X,Y), \nabla f \wedge Z] + T((\nabla f \wedge Z)X, Y) + T(X, (\nabla f \wedge Z)Y)\bigr),$$

where $X \wedge Y$ is the linear operator defined by $(X \wedge Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X$, $H(\phi)$ is the symmetric operator associated to the Hessian of ϕ, and both ∇ and the norm are computed with respect to $\langle\cdot,\cdot\rangle$.

Proof. The curvature tensor of $M_{\nu,\varepsilon}$ has the form

$$\tilde{R}(X,Y) = \varepsilon\bigl(X \tilde{\wedge} Y + \textstyle\sum_{i=1}^{\nu} (J_i X \tilde{\wedge} J_i Y + 2\tilde{g}(J_i X, Y) J_i)\bigr),$$

where $(X \tilde{\wedge} Y)Z = \tilde{g}(X,Z)Y - \tilde{g}(Y,Z)X$. Under the conformal change of metric, the curvature tensor transforms as $\tilde{R}(X,Y) = R(X,Y) - (X \wedge KY + KX \wedge Y)$. Since $\tilde{g}(X,Y) = f\langle X, Y\rangle$ and $X \tilde{\wedge} Y = f(X \wedge Y)$ and because the J_i remain anticommuting almost Hermitian structures for $\langle\cdot,\cdot\rangle$, Equation (3-30a) follows.

The fact that the Weyl tensor has the form (3-30b) follows from the definition of W; the norm of W can be computed directly using that the J_i are orthogonal and that $J_1 J_2 = \pm J_3$ when $\nu = 3$.

From

$$\tilde{\nabla}_Z J_i = \textstyle\sum_{j=1}^{\nu} \omega_i^j(Z) J_j \quad \text{and} \quad \tilde{\nabla}_Z X = \nabla_Z X + Z\phi X + X\phi Z - \langle X, Z\rangle \nabla\phi,$$

where $\tilde{\nabla}$ is the Levi-Civita connection for $\tilde{g}$, we get

$$\nabla_Z J_i = \textstyle\sum_{j=1}^{\nu} \omega_i^j(Z) J_j + [J_i, \nabla\phi \wedge Z]$$

(where we used the fact that $[J_i, X \wedge Y] = J_i X \wedge Y + X \wedge J_i Y$). Then

$$(\nabla_Z T)(X, Y) = [T(X, Y), \nabla\phi \wedge Z] + T((\nabla\phi \wedge Z)X, Y) + T(X, (\nabla\phi \wedge Z)Y),$$

which, together with (3-30b), proves (3-30d). □

For every point $x \in M'$, there exists a neighborhood $\mathcal{U}$ of x and a positive smooth function $f : \mathcal{U} \to \mathbb{R}$ such that the Riemannian space $(\mathcal{U}(x), f\langle\cdot,\cdot\rangle)$ is isometric to an open subset of one of the five model spaces ($M_{\nu,\varepsilon}$ or $\mathbb{R}^n$), so at every point $x \in M'$, the Weyl tensor W of M^n either vanishes or has the form given in (3-30b). The Jacobi operators associated to the different Weyl tensors $W_{\nu,\varepsilon}$ in (3-30b) differ by their multiplicities and the signs of their eigenvalues, so every point $x \in M'$ has a neighborhood conformally equivalent to a domain of exactly one of the model spaces. Moreover, the function $f > 0$ is well defined at all the points where $W \neq 0$, since $\|W\|^2 = C_{\nu n} f^2$ by (3-30c).

By continuity, the Weyl tensor W of M^n either has the form $W_{\nu,\varepsilon}$ or vanishes at every point $x \in M^n$ (since M' is open and dense in M^n — see Lemma 3.2). Moreover, every point $x \in M^n$ at which the Weyl tensor has the form $W_{\nu,\varepsilon}$ has a neighborhood in which the Weyl tensor has the same form. Hence $M^n = M_0 \cup \bigcup_\alpha M_\alpha$, where $M_0 = \{x : W(x) = 0\}$ is closed, and every M_α is a nonempty open connected subset with $\partial M_\alpha \subset M_0$ such that the Weyl tensor has the same form $W_{\nu,\varepsilon} = W_{\nu(\alpha),\varepsilon(\alpha)}$ at every point $x \in M_\alpha$. In particular, since $M_\alpha \subset M'$, each M_α is locally conformal to one of the model spaces $M_{\nu,\varepsilon}$.

If $M = M_0$ or if $M_0 = \varnothing$, the theorem is proved. Otherwise, suppose that $M_0 \neq \varnothing$ and that there exists at least one component M_α. Let $y \in \partial M_\alpha \subset M_0$ and let $B_\delta(y)$ be a small geodesic ball of M centered at y that is strictly geodesically convex (any two points from $B(y)$ can be connected by a unique geodesic segment lying in $B_\delta(y)$, and that segment realizes the distance between them). Let $x \in B_{\delta/3}(y) \cap M_\alpha$ and let $r = \operatorname{dist}(x, M_0)$. Then the geodesic ball $B = B_r(x)$ lies in M_α and is strictly convex. Moreover, ∂B contains a point $x_0 \in M_0$. Replacing x by the midpoint of the segment $[x x_0]$ and r by $r/2$, if necessary, we can assume that all the points of ∂B, except for x_0, lie in M_α.

The function f is positive and smooth on $\overline{B} \setminus \{x_0\}$ (that is, on an open subset containing $\overline{B} \setminus \{x_0\}$, but not containing x_0). We are interested in the behavior of $f(x)$ when $x \in B$ approaches x_0.

Lemma 3.10. *When $x \to x_0$ while staying in B, both f and ∇f have a finite limit. Moreover,* $\lim_{x \to x_0, x \in B} f(x) = 0$.

Proof. The fact that $\lim_{x \to x_0, x \in B} f(x) = 0$ follows from (3-30c) and the fact that $W|_{x_0} = 0$ (since $x_0 \in M_0$).

Since the Riemannian space $(B, f\langle\cdot,\cdot\rangle)$ is locally isometric to a rank-one symmetric space $M_{\nu,\varepsilon}$ and is also simply connected, there exists a smooth isometric immersion $\iota : (B, f\langle\cdot,\cdot\rangle) \to M_{\nu,\varepsilon}$. Since f is smooth on $\overline{B}\setminus\{x_0\}$ and $\lim_{x\to x_0, x\in B} f(x)=0$, the range of ι is a bounded domain in $M_{\nu,\varepsilon}$. Moreover, since $\lim_{x\to x_0, x\in B} f(x)=0$, every sequence of points in B converging to x_0 in the metric $\langle\cdot,\cdot\rangle$ is a Cauchy sequence for the metric $f\langle\cdot,\cdot\rangle$. It follows that there exists a limit $\lim_{x\to x_0, x\in B}\iota(x)\in M_{\nu,\varepsilon}$. Defining for every $x\in B$ the point $\mathcal{J}|_x = \operatorname{Span}(J_1,\dots,J_\nu)$ in the Grassmanian $G(\nu, \bigwedge^2 T_xM^n)$, we find that there exists a limit

$$\lim_{x\to x_0, x\in B} \mathcal{J}|_x =: \mathcal{J}|_{x_0} \in G(\nu, \textstyle\bigwedge^2 T_{x_0}M^n).$$

In particular, if Z is a continuous vector field on $\overline{B}$, then there exists a unit continuous vector field Y on $\overline{B}$ such that $Y \perp Z, \mathcal{J}Z$ on B. For such two vector fields, the function

$$\theta(Y,Z) = \langle \textstyle\sum_{j=1}^n (\nabla_{E_j} W)(E_j, Y)Y, Z\rangle$$

(where E_j is an orthonormal frame on $\overline{B}$) is well defined and continuous on $\overline{B}$. Using (3-30d), we obtain by a direct computation that at the points of B,

$$\theta(Y,Z) = \frac{\varepsilon(n-3)}{2(n-1)}\langle(3\nu\nabla f\wedge Y-(n-1)T(\nabla f,Y))Y,Z\rangle = \frac{-3\varepsilon\nu(n-3)}{2(n-1)}\langle\nabla f, Z\rangle$$

(where we used that $\|Y\|=1$ and $Y\perp Z, \mathcal{J}Z$). Since $\theta(Y,Z)$ is continuous on $\overline{B}$, there exists a limit $\lim_{x\to x_0, x\in B} Zf$. Since Z is an arbitrary continuous vector field on $\overline{B}$, ∇f has a finite limit when $x\to x_0$ while staying in B. □

Since $\lim_{x\to x_0, x\in B} f(x)=0$ and the J_i are orthogonal, the second term on the right side of (3-30a) tends to 0 when $x\to x_0$ in B. Therefore the (3,1) tensor field defined by $(X,Y)\to(X\wedge KY + KX\wedge Y)$ has a finite limit (namely $R|_{x_0}$) when $x\to x_0$ in B. It follows that the symmetric operator K has a finite limit at x_0. Computing the trace of K and using the fact that $\phi=\frac12 \ln f$, we get

$$\text{(3-31)}\qquad \triangle u = Fu \text{ on } B, \quad \text{where } u = f^{(n-2)/4} \text{ and } F=\tfrac12(n-2)\operatorname{Tr}K.$$

Both functions F and u are smooth on $\overline{B}\setminus\{x_0\}$ and have a finite limit at x_0. Moreover, $\lim_{x\to x_0, x\in B} u(x)=0$ by Lemma 3.10 and $u(x)>0$ for $x\in\overline{B}\setminus\{x_0\}$. The domain B is a small geodesic ball, so it satisfies the inner sphere condition (the radii of curvature of the sphere ∂B are uniformly bounded). By the boundary point theorem [Fraenkel 2000, Section 2.3], the inner directional derivative of u at x_0 (which exists by Lemma 3.10 if we define $u(x_0)=0$ by continuity) is positive.

Since $\nabla u = (1/4)(n-2)f^{(n-6)/4}\nabla f$ in B, we arrive at a contradiction with Lemma 3.10 in all cases except for $n=6$. To finish the proof in that case, we show that the limit $\lim_{x\to x_0, x\in B}\nabla f(x)$, which exists by Lemma 3.10, is zero. When $n=6$, we have $\nu=1$ by (2-3), so $T(X,Y)=JX\wedge JY+2\langle JX,Y\rangle J$, where

$J = J(x)$ is smooth on $\overline{B} \setminus \{x_0\}$ and has a limit when $x \to x_0$ while in B (see the proof of Lemma 3.10). Using the covariant derivative of T computed in Lemma 3.9 and (3-30d), we obtain that on B,

$$\begin{aligned}
&(\nabla_U \nabla_Z W)(X, Y)\\
&= \varepsilon\langle H(f)U, Z\rangle(-\tfrac{3}{5}X \wedge Y + T(X, Y))\\
&\quad + \tfrac{1}{2}\varepsilon([T(X, Y), H(f)U \wedge Z] + T((H(f)U \wedge Z)X, Y) + T(X, (H(f)U \wedge Z)Y))\\
&\quad + \tfrac{1}{2}\varepsilon f^{-1} Zf([T(X, Y), \nabla f \wedge U] + T((\nabla f \wedge U)X, Y) + T(X, (\nabla f \wedge U)Y))\\
&\quad + \tfrac{1}{4}\varepsilon f^{-1}[[T(X, Y), \nabla f \wedge U] + T((\nabla f \wedge U)X, Y)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + T(X, (\nabla f \wedge U)Y), \nabla f \wedge Z]\\
&\quad + \tfrac{1}{4}\varepsilon f^{-1}([T((\nabla f \wedge Z)X, Y), \nabla f \wedge U] + T((\nabla f \wedge U)(\nabla f \wedge Z)X, Y)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + T((\nabla f \wedge Z)X, (\nabla f \wedge U)Y))\\
&\quad + \tfrac{1}{4}\varepsilon f^{-1}([T(X, (\nabla f \wedge Z)Y), \nabla f \wedge U] + T((\nabla f \wedge U)X, (\nabla f \wedge Z)Y)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + T(X, (\nabla f \wedge U)(\nabla f \wedge Z)Y)),
\end{aligned}$$

where $H(f)$ is the symmetric operator associated to the Hessian of f. Taking $U = Z = E_j$, where $\{E_j\}$ is an orthonormal basis, and summing up by j we find after some computation

$$\begin{aligned}
\sum_{j=1}^{6}(\nabla_{E_j}\nabla_{E_j} W)(X, Y)&\\
= \varepsilon \triangle f(-\tfrac{3}{5}X \wedge Y &+ T(X, Y)) - \varepsilon f^{-1}\|\nabla f\|^2 T(X, Y)\\
&+ \varepsilon f^{-1}(T(X, Y)\nabla f \wedge \nabla f + T((X \wedge Y)\nabla f, \nabla f))\\
&+ \tfrac{3}{2}\varepsilon f^{-1}(\nabla f \wedge (X \wedge Y)\nabla f + J\nabla f \wedge (X \wedge Y)J\nabla f).
\end{aligned}$$

Since both ∇f and J are smooth on $\overline{B} \setminus \{x_0\}$ and have limits when $x \to x_0$ while in B, there exist unit vector fields X and Y that are continuous on $\overline{B}$ and satisfy $\mathcal{J}X, \mathcal{J}Y \perp \nabla f$ and $\mathcal{J}X \perp \mathcal{J}Y$. For such X and Y,

$$\sum_{j=1}^{6}(\nabla_{E_j}\nabla_{E_j} W)(X, Y) = \varepsilon \triangle f(-\tfrac{3}{5}X \wedge Y + JX \wedge JY) - \varepsilon f^{-1}\|\nabla f\|^2 JX \wedge JY.$$

Since the left side is continuous on $\overline{B}$ and $\lim_{x \to x_0, x \in B} \triangle f = 0$ by (3-31) and Lemma 3.10, we obtain that the field $f^{-1}\|\nabla f\|^2 JX \wedge JY$ of skew-symmetric operators has a limit at x_0. Taking the trace of its square, we find that there exists a limit $\lim_{x \to x_0, x \in B} f^{-2}\|\nabla f\|^4$, which implies $\lim_{x \to x_0, x \in B} \nabla f = 0$ by Lemma 3.10. We again arrive at a contradiction with the boundary point theorem for the function $u = f$ satisfying (3-31). $\square$

References

[Atiyah et al. 1964] M. F. Atiyah, R. Bott, and A. Shapiro, "Clifford modules", *Topology* **3:** suppl. 1 (1964), 3–38. MR 29 #5250 Zbl 0146.19001

[Blažić and Gilkey 2004] N. Blažić and P. Gilkey, "Conformally Osserman manifolds and conformally complex space forms", *Int. J. Geom. Methods Mod. Phys.* **1** (2004), 97–106. MR 2005c:53034 Zbl 1076.53038

[Blažić and Gilkey 2005] N. Blažić and P. Gilkey, "Conformally Osserman manifolds and self-duality in Riemannian geometry", pp. 15–18 in *Differential geometry and its applications*, edited by J. Bureš et al., Matfyzpress, Prague, 2005. MR 2007k:53032 Zbl 1117.53028

[Blažić et al. 2005] N. Blažić, P. Gilkey, S. Nikčević, and U. Simon, "The spectral geometry of the Weyl conformal tensor", pp. 195–203 in *PDEs, submanifolds and affine differential geometry*, edited by B. Opozda et al., Banach Center Publications **69**, Polish Acad. Sci., Warsaw, 2005. MR 2006k:53044 Zbl 1091.53007

[Blažić et al. 2008] N. Blažić, P. Gilkey, S. Nikčević, and I. Stavrov, "Curvature structure of self-dual 4-manifolds", *Int. J. Geom. Methods Mod. Phys.* **5**:7 (2008), 1191–1204. MR 2010a:53148 Zbl 1163.53010

[Chi 1988] Q.-S. Chi, "A curvature characterization of certain locally rank-one symmetric spaces", *J. Differential Geom.* **28**:2 (1988), 187–202. MR 90a:53060 Zbl 0654.53053

[Derdziński and Shen 1983] A. Derdziński and C. L. Shen, "Codazzi tensor fields, curvature and Pontryagin forms", *Proc. London Math. Soc.* (3) **47** (1983), 15–26. MR 84h:53048 Zbl 0519.53015

[Fraenkel 2000] L. E. Fraenkel, *An introduction to maximum principles and symmetry in elliptic problems*, Cambridge Tracts in Math. **128**, Cambridge University Press, 2000. MR 2001c:35042 Zbl 0947.35002

[García-Río et al. 2002] E. García-Río, D. N. Kupeli, and R. Vázquez-Lorenzo, *Osserman manifolds in semi-Riemannian geometry*, Lecture Notes in Mathematics **1777**, Springer, Berlin, 2002. MR 2003e:53052 Zbl 1005.53040

[Gilkey 2001] P. B. Gilkey, *Geometric properties of natural operators defined by the Riemann curvature tensor*, World Scientific, River Edge, NJ, 2001. MR 2002k:53052 Zbl 1007.53001

[Gilkey 2007] P. B. Gilkey, *The geometry of curvature homogeneous pseudo-Riemannian manifolds*, ICP Advanced Texts in Mathematics **2**, Imperial College Press, London, 2007. MR 2008k:53151 Zbl 1128.53041

[Gilkey et al. 1995] P. Gilkey, A. Swann, and L. Vanhecke, "Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator", *Quart. J. Math. Oxford Ser.* (2) **46**:183 (1995), 299–320. MR 96h:53051 Zbl 0848.53023

[Harvey and Lawson 1982] R. Harvey and H. B. Lawson, Jr., "Calibrated geometries", *Acta Math.* **148** (1982), 47–157. MR 85i:53058 Zbl 0584.53021

[Husemoller 1975] D. Husemoller, *Fibre bundles*, 2nd ed., Graduate Texts in Mathematics **20**, Springer, New York, 1975. MR 51 #6805 Zbl 0307.55015

[Lawson and Michelsohn 1989] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series **38**, Princeton University Press, 1989. MR 91g:53001 Zbl 0688.57001

[N 2003] Y. Nikolayevsky, "Osserman manifolds and Clifford structures", *Houston J. Math.* **29**:1 (2003), 59–75. MR 2003k:53048 Zbl 1069.53042

[N 2004] Y. Nikolayevsky, "Osserman manifolds of dimension 8", *Manuscripta Math.* **115**:1 (2004), 31–53. MR 2005m:53049 Zbl 1065.53034

[N 2005] Y. Nikolayevsky, "Osserman conjecture in dimension $n \neq 8, 16$", *Math. Ann.* **331**:3 (2005), 505–522. MR 2005k:53038 Zbl 1075.53016

[N 2006] Y. Nikolayevsky, "On Osserman manifolds of dimension 16", pp. 379 – 398 in *Contemporary Geometry and Related Topics* (Belgrade, 2005), edited by N. Bokan et al., Faculty of Mathematics, University of Belgrade, 2006.

[Olszak 1989] Z. Olszak, "On the existence of generalized complex space forms", *Israel J. Math.* **65**:2 (1989), 214–218. MR 90c:53091 Zbl 0674.53061

[Osserman 1990] R. Osserman, "Curvature in the eighties", *Amer. Math. Monthly* **97**:8 (1990), 731–756. MR 91i:53001 Zbl 0722.53001

[Pfister 1995] A. Pfister, *Quadratic forms with applications to algebraic geometry and topology*, London Math. Soc. Lecture Note Ser. **217**, Cambridge University Press, 1995. MR 97c:11046 Zbl 0847.11014

Received January 31, 2009.

YURI NIKOLAYEVSKY
DEPARTMENT OF MATHEMATICS AND STATISTICS
LA TROBE UNIVERSITY
MELBOURNE, VICTORIA 3086
AUSTRALIA

y.nikolayevsky@latrobe.edu.au

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

UNFAITHFUL COMPLEX HYPERBOLIC TRIANGLE GROUPS, III: ARITHMETICITY AND COMMENSURABILITY

JULIEN PAUPERT

We prove that the so-called sporadic complex reflection triangle groups in SU(2, 1) are all nonarithmetic but one, and that they are not commensurable to Mostow or Picard lattices (with a small list of exceptions). This provides an infinite list of potential new nonarithmetic lattices in SU(2, 1).

1. Introduction

Parker and Paupert [2009] considered symmetric triangle groups Δ in SU(2, 1) generated by three complex reflections through angle $2\pi/p$ for $p \geqslant 3$; the case of order 2 was studied in [Parker 2008b]. By "symmetric", we mean that the group in question is generated by three complex reflections R_1, R_2 and R_3 with the property that there exists an isometry J of order 3 such that $R_{j+1} = JR_jJ^{-1}$, where j is taken mod 3. We study the group Γ generated by R_1 and J, which contains Δ with index 1 or 3.

This type of group was first studied by Mostow [1980] for $p = 3, 4, 5$, where an additional condition was imposed on the R_j, namely the braid relation $R_iR_jR_i = R_jR_iR_j$; these provided the first examples of nonarithmetic lattices in SU(2, 1). Further nonarithmetic lattices in SU(n, 1) for $n \leqslant 9$ were constructed in [Deligne and Mostow 1986] and [Mostow 1986] as monodromy groups of certain hypergeometric functions (the lattices from the former, in dimension 2, were known to Picard, who did not consider their arithmetic nature). These lattices are (commensurable with) groups generated by complex reflections R_j with other values of p [Mostow 1986; Sauter 1990]. Subsequently no new nonarithmetic lattices have been constructed.

In [Parker and Paupert 2009], we showed that symmetric complex reflection triangle groups $\Delta = \langle R_1, R_2, R_3\rangle$, if they are discrete and if R_1R_2 and $R_1R_2R_3$ are elliptic, come in three flavors: Mostow's lattices, subgroups thereof, and a third class, which we called *sporadic groups* (see Section 2 for a precise definition). Our main motivation is that these new groups are candidates for nonarithmetic lattices in SU(2, 1). In this paper we analyze the adjoint trace fields $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$

MSC2000: 20H10, 22E40, 51M10.
Keywords: nonarithmetic lattices, complex reflection groups, complex hyperbolic geometry.

of the sporadic groups Γ, and use this to determine which sporadic groups are arithmetic, and which ones are commensurable to Mostow or Picard lattices. The main results are Theorems 4.1 and 5.2, which say in essence that all sporadic groups are nonarithmetic, except one that was studied in [Parker and Paupert 2009], and moreover that they are not commensurable to any of the Mostow or Picard lattices, with an explicit list of possible exceptions.

The only required notions of complex hyperbolic geometry are the definitions of elliptic and regular elliptic isometries, as well as complex reflections. These are standard and can be found for instance in the book [Goldman 1999].

2. Sporadic groups

We recall the setup and main results from [Parker and Paupert 2009]. Our starting point was that groups $\Gamma = \langle R_1, J\rangle$ as defined above can be parametrized up to conjugacy by $\tau = \mathrm{Tr}(R_1 J)$; we denoted by $\Gamma(\psi, \tau)$ the group generated by a complex reflection R_1 through angle ψ and a regular elliptic isometry J of order 3 such that $\tau = \mathrm{Tr}(R_1 J)$. The generators for this group were given in the form

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{2-1}$$

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \tau & -e^{i\psi/3}\,\bar{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}. \tag{2-2}$$

These preserve the Hermitian form $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{w}^* H_\tau \mathbf{z}$ where

$$H_\tau = \begin{bmatrix} 2\sin(\psi/2) & -ie^{-i\psi/6}\tau & ie^{i\psi/6}\bar{\tau} \\ ie^{i\psi/6}\bar{\tau} & 2\sin(\psi/2) & -ie^{-i\psi/6}\tau \\ -ie^{-i\psi/6}\tau & ie^{i\psi/6}\bar{\tau} & 2\sin(\psi/2) \end{bmatrix}. \tag{2-3}$$

This always produces a subgroup Γ of $\mathrm{GL}(3, \mathbb{C})$, but the signature of H_τ depends on the values of ψ and τ. We determined the corresponding parameter space for τ for any fixed value of ψ [Parker and Paupert 2009, Sections 2.4 and 2.6]. When Γ preserves a Hermitian form of signature (2, 1), we will say that Γ is *hyperbolic*.

We found necessary conditions for these groups to be discrete, and these conditions produced, along with the groups previously studied by Mostow [1980], a list of possibly discrete such groups:

Theorem 2.1. *Let R_1 be a complex reflection of order p and J a regular elliptic isometry of order* 3 *in* $\mathrm{PU}(2, 1)$. *Suppose that $R_1 J$ and $R_1 R_2 = R_1 J R_1 J^{-1}$ are elliptic. If the group $\Gamma = \langle R_1, J\rangle$ is discrete then one of the following is true*:

- Γ *is one of Mostow's lattices.*

- Γ *is a subgroup of one of Mostow's lattices.*
- Γ *is one of the sporadic groups listed below.*

Mostow's lattices correspond to $\tau = e^{i\phi}$ for some angle ϕ, while subgroups of Mostow's lattices correspond to $\tau = e^{2i\phi} + e^{-i\phi}$ for some angle ϕ, and sporadic groups (this can be taken as a definition) are those for which τ takes one of the 18 values $\{\sigma_1, \bar{\sigma}_1, \ldots, \sigma_9, \bar{\sigma}_9\}$ where the σ_i are given in the following list:

$$\sigma_1 := e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(\pi/4), \qquad \sigma_4 := e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7},$$
$$\sigma_2 := e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(\pi/5), \qquad \sigma_5 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(2\pi/5),$$
$$\sigma_3 := e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(2\pi/5), \qquad \sigma_6 := e^{2\pi i/9} + e^{-\pi i/9}\, 2\cos(4\pi/5),$$
$$\sigma_7 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(2\pi/7),$$
$$\sigma_8 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(4\pi/7),$$
$$\sigma_9 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(6\pi/7).$$

Therefore, for each value of $p \geqslant 3$, we have a finite number of new groups to study, the $\Gamma(2\pi/p, \sigma_i)$ and $\Gamma(2\pi/p, \bar{\sigma}_i)$, which are hyperbolic. We determined which sporadic groups are hyperbolic and listed them in the table in [Parker and Paupert 2009, Section 3.3]. Notably such groups exist for all values of p, and more precisely:

Proposition 2.1. *For $p \geqslant 4$ and $\tau = \sigma_1, \sigma_2, \sigma_3, \bar{\sigma}_4, \sigma_5, \sigma_6, \sigma_7, \bar{\sigma}_8$ or σ_9, the groups $\Gamma(2\pi/p, \tau)$ are hyperbolic.*

When we study the question of arithmeticity of these groups, we will use the list of all hyperbolic sporadic groups, as well as the following normalization of the entries of our matrices:

Proposition 2.2 [Parker and Paupert 2009, Proposition 2.8]. *The maps R_1, R_2 and R_3 may be conjugated within* $\mathrm{SU}(2, 1)$ *and scaled so that their matrix entries lie in the ring $\mathbb{Z}[\tau, \bar{\tau}, e^{\pm i\psi}]$.*

Explicitly, we conjugate the previous matrices by $C = \mathrm{diag}(e^{-i\psi/3}, 1, e^{i\psi/3})$ and rescale by $e^{-i\psi/3}$. Conjugating by C and rescaling by $2\sin(\psi/2)$ also brings H_τ to a Hermitian matrix with entries in the same ring $R = \mathbb{Z}[\tau, \bar{\tau}, e^{\pm i\psi}]$. Therefore, a hyperbolic $\Gamma(\psi, \tau)$ can be realized as a subgroup of $\mathrm{SU}(H, R)$ where H is an R-defined Hermitian form of signature $(2, 1)$.

Finally, we showed that some of the hyperbolic sporadic groups are nondiscrete [Parker and Paupert 2009, Corollary 4.2, Proposition 4.5 and Corollary 6.4]:

Proposition 2.3. *For $p \geqslant 3$ and $(\tau$ or $\bar{\tau} = \sigma_3, \sigma_8$ or $\sigma_9)$, $\Gamma(2\pi/p, \tau)$ is not discrete. Also, for $p \geqslant 3$ with $p \neq 5$ and $(\tau$ or $\bar{\tau} = \sigma_6)$, $\Gamma(2\pi/p, \tau)$ is not discrete.*

3. Trace fields

The trace field $\mathbb{Q}[\operatorname{Tr}\Gamma]$ is a classical invariant for a finitely generated subgroup Γ of a linear group G. It is invariant under conjugacy, but not commensurability. (We will say that two subgroups Γ_1 and Γ_2 of G are *commensurable* if there exists $g \in G$ such that $\Gamma_1 \cap g\Gamma_2 g^{-1}$ has finite index in both Γ_1 and $g\Gamma_2 g^{-1}$). To obtain a commensurability invariant for such Γ, one can consider the trace field $\mathbb{Q}[\operatorname{Tr}\Gamma^{(n)}]$ (where $\Gamma^{(n)}$ is the subgroup of Γ generated by n-th powers for $\Gamma \subset \mathrm{GL}(n, \mathbb{C})$), as in [Maclachlan and Reid 2003] for $\mathrm{SL}(2, \mathbb{C})$ or as in [McReynolds 2006] for $\mathrm{SU}(2, 1)$. Another possibility is the adjoint trace field $\mathbb{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$, given by the adjoint representation $\operatorname{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, as in [Mostow 1980; 1986; Deligne and Mostow 1986] for $\mathrm{SU}(n, 1)$. The following result can be found for instance as [Deligne and Mostow 1986, Proposition 12.2.1]:

Proposition 3.1. $\mathbb{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$ *is a commensurability invariant.*

This is the field that we will use here, as it is more convenient for our purposes. Indeed, this invariant trace field has been computed for all known nonarithmetic lattices in $\mathrm{SU}(2, 1)$. See the lists on [Mostow 1980, page 251] and [Deligne and Mostow 1986, page 86]. Moreover it is easy to compute (or at least estimate) by the following result:

Proposition 3.2. $\operatorname{Tr}\operatorname{Ad}(\gamma) = |\operatorname{Tr}(\gamma)|^2$ *for* $\gamma \in \mathrm{SU}(2, 1)$,

This result is used several times in [Mostow 1980], where it is referred to as Lemma 4.2, but unfortunately its statement is missing from final edition.

Proof. If U is a unitary group (of any signature), the adjoint representation of U is isomorphic to the representation $U \otimes \overline{U}$. □

We use this to find the following bounds for $\mathbb{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma(\psi, \tau)]$:

Proposition 3.3.

$$\mathbb{Q}[\cos\psi, |\tau|^2, \operatorname{Re}\tau^3, \operatorname{Re}(e^{-i\psi}\tau^3)] \subset \mathbb{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma(\psi, \tau)] \subset \mathbb{Q}[\tau, \bar{\tau}, e^{i\psi}] \cap \mathbb{R}.$$

Proof. The second inclusion follows from Propositions 2.2 and 3.2. For the first inclusion, we use Proposition 3.2 and compute $|\operatorname{Tr}(\gamma)|^2$ for various words γ, using the table of traces from [Parker and Paupert 2009, Section 4.1]; see also formulae in [Pratoussevitch 2005]. We have

$$\begin{aligned} |\operatorname{Tr} R_1|^2 &= 5 + 4\cos\psi, \\ |\operatorname{Tr} R_1 J|^2 &= |\tau|^2 \qquad \text{(by the definition of } \tau\text{)}, \\ |\operatorname{Tr}(R_1 J)^2|^2 &= |\tau|^4 + 4|\tau|^2 - 4\operatorname{Re}\tau^3, \\ |\operatorname{Tr}(J^{-1} R_1)^2|^2 &= |\tau|^4 + 4|\tau|^2 - 4\operatorname{Re}(e^{-i\psi}\tau^3). \end{aligned}$$

□

We list the corresponding elements of $\mathbb{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma(2\pi/p,\sigma_i)]$ in the table below. Numbers in the last three columns are not the values of $|\tau|^2$, $\operatorname{Re}\tau^3$ or $\operatorname{Re}(e^{-i\psi}\tau^3)$, but rather new algebraic numbers added to $\mathbb{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma(2\pi/p,\sigma_i)]$ by these values. For example, the first four zeros in the fourth column indicate that the corresponding $\operatorname{Re}\tau^3$ is already in $\mathbb{Q}[\cos\psi, |\tau|^2]$.

	$\cos\psi$	$\lvert\tau\rvert^2$	$\operatorname{Re}\tau^3$	$\operatorname{Re}(e^{-i\psi}\tau^3)$
σ_1	$\cos 2\pi/p$	0	0	$\sqrt{2}\sin 2\pi/p$
σ_2	$\cos 2\pi/p$	$\cos\pi/5$	0	$\sin 2\pi/p$
σ_3	$\cos 2\pi/p$	$\cos 3\pi/5$	0	$\sin 2\pi/p$
σ_4	$\cos 2\pi/p$	0	0	$\sqrt{7}\sin 2\pi/p$
σ_5	$\cos 2\pi/p$	0	$\cos 2\pi/5$	$\sqrt{3}\sin 2\pi/p$
σ_6	$\cos 2\pi/p$	0	$\cos 4\pi/5$	$\sqrt{3}\sin 2\pi/p$
σ_7	$\cos 2\pi/p$	$\cos\pi/7$	0	$\sqrt{3}\sin 2\pi/p$

4. Arithmeticity

In [Parker and Paupert 2009, Propositions 6.5 and 6.6], we proved that only one of the sporadic groups with $p=3$, namely $\Gamma(2\pi/3,\bar{\sigma}_4)$, is contained in an arithmetic lattice in $\mathrm{SU}(2,1)$. (It was shown in [Parker 2008b] that all the corresponding groups with $p=2$ are arithmetic.) In this section we extend this to higher values of p, and show that in fact this group is the only such example among all higher-order sporadic groups:

Theorem 4.1. *For $p\geqslant 3$ and $\tau\in\{\sigma_1,\bar{\sigma}_1,\dots,\sigma_9,\bar{\sigma}_9\}$, the group $\Gamma(2\pi/p,\tau)$ is contained in an arithmetic lattice in* $\mathrm{SU}(2,1)$ *if and only if $p=3$ and $\tau=\bar{\sigma}_4$.*

We will use the following criterion for arithmeticity:

Proposition 4.1. *Let E be a purely imaginary quadratic extension of a totally real field F, and let H be an E-defined Hermitian form of signature* $(2,1)$ *such that a sporadic group Γ is contained in* $\mathrm{SU}(H;\mathbb{O}_E)$. *Then Γ is contained in an arithmetic lattice in* $\mathrm{SU}(2,1)$ *if and only if for all $\varphi\in\operatorname{Gal}(F)$ not inducing the identity on* $\mathbb{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$, *the form ${}^{\varphi}H$ is definite.*

This follows from [Mostow 1980, Lemma 4.1]. Hypotheses (1) and (3) of that lemma — that $\mathbb{Q}[\operatorname{Tr}\operatorname{Ad}\Gamma]$ is a totally real field, and that $\operatorname{Tr}\operatorname{Ad}\gamma$ is an algebraic integer for all $\gamma\in\Gamma$ — are verified by Propositions 2.2 and 3.2, using the special values of τ for sporadic groups.

We will prove Theorem 4.1 in several parts using this criterion. The first result follows the same lines as the corresponding one in [Parker and Paupert 2009]:

Proposition 4.2. *The sporadic group $\Gamma(2\pi/p,\tau)$ is not contained in an arithmetic lattice in* $\mathrm{SU}(2,1)$, *with the following possible exceptions*:

- $\tau = \sigma_1$ *and* $p = 4$ *or* $p \geqslant 8$;
- $\tau = \sigma_2$ *and* 3 *or* 4 *or* 5 *divides* p;
- $\tau = \bar{\sigma}_2$ *and* $p = 8, 9, 10, 12, 14, 15, 16, 18$;
- $\tau = \bar{\sigma}_4$ *and* $p = 3$ *or* $p \geqslant 7$;
- $\tau = \sigma_5$ *and* 5 *divides* p;
- $\tau = \sigma_7$ *and* 7 *divides* p.

Proof. We conjugate the generators and Hermitian form as in Proposition 2.2 so that their entries lie in the ring $\mathbb{Z}[\tau, \bar{\tau}, e^{\pm i\psi}]$, and are therefore algebraic integers in the field $\mathbb{Q}[\tau, \bar{\tau}, e^{i\psi}]$. (Recall that $\psi = 2\pi/p$ in our cases.) We then find in each case a number field E as in Proposition 4.1 containing $\mathbb{Q}[\tau, \bar{\tau}, e^{i\psi}]$, and a Galois conjugation of E that acts nontrivially on $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$ and sends the Hermitian form to another indefinite form. For the values of τ and p that are not excluded above, we can use the same argument as in [Parker and Paupert 2009], namely, that one of the Galois conjugations of E sends the parameter τ to another value for which we know that the Hermitian form is indefinite (from our description of the parameter space). This requires using a Galois conjugation fixing $e^{2i\pi/p}$. The details:

- For $\tau = \sigma_1$ or $\bar{\sigma}_1$, let $E = \mathbb{Q}[e^{i\pi/6}, e^{i\pi/4}, e^{2i\pi/p}]$. If p is not divisible by 3 or 4, σ_1 is sent to $\bar{\sigma}_1$ by the Galois conjugation that sends $e^{i\pi/6}$ to $e^{-i\pi/6}$, sends $e^{i\pi/4}$ to $e^{-i\pi/4}$, and fixes $e^{2i\pi/p}$. The corresponding Hermitian form is indefinite for $p = 3, 4, 5, 6, 7$. This works for $p = 5$ or 7, but for $p = 3$, 4 or 6 we need to find another Galois conjugation. For $p = 3$ or 6, sending $e^{i\pi/6}$ to $e^{7i\pi/6}$ (and for compatibility $e^{i\pi/4}$ to $e^{-i\pi/4}$) fixes $e^{2i\pi/3}$ (respectively $e^{2i\pi/6}$) and sends σ_1 to $e^{4i\pi/3}\bar{\sigma}_1$, which is equivalent to $\bar{\sigma}_1$. These various Galois conjugations act nontrivially on $\mathrm{Re}(e^{-i\psi}\tau^3) = 5\cos\psi + 5\sqrt{2}\sin\psi$, which is in $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$.

- For $\tau \in \{\sigma_2, \bar{\sigma}_2, \sigma_3, \bar{\sigma}_3\}$, let $E = \mathbb{Q}[e^{i\pi/6}, e^{i\pi/5}, e^{2i\pi/p}]$. If p is not divisible by 3 or 4 or 5, the Galois conjugation that sends $e^{i\pi/5}$ to $e^{3i\pi/5}$, sends $e^{i\pi/6}$ to $e^{7i\pi/6}$ and fixes $e^{2i\pi/p}$ is one that swaps σ_2 and σ_3, as well as $\bar{\sigma}_2$ and $\bar{\sigma}_3$. The Hermitian form corresponding to σ_2 and σ_3 is indefinite for all $p \geqslant 3$; for $\bar{\sigma}_2$ it is indefinite for $3 \leqslant p \leqslant 19$, and for $\bar{\sigma}_3$ it is indefinite for $3 \leqslant p \leqslant 6$. This Galois conjugation acts nontrivially on $|\tau|^2 = 2 + 2\cos(\pi/5)$ (respectively $2 + 2\cos(3\pi/5)$), which is in $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$.

  If p is not divisible by 2 or 3, the Galois conjugation sending $e^{i\pi/6}$ to $e^{-i\pi/6}$ and fixing the 2 other generators of E sends σ_2 to $\bar{\sigma}_2$. This works unless $p = 8, 9, 10, 12, 14, 15, 16, 18$.

- For $\tau = \sigma_4$ or $\bar{\sigma}_4$, let $E = \mathbb{Q}[e^{2i\pi/7}, e^{2i\pi/p}]$, which contains $i\sqrt{7} = \sigma_4 - \bar{\sigma}_4$. If p is not divisible by 7, the Galois conjugation sending $e^{2i\pi/7}$ to $e^{-2i\pi/7}$ and fixing the other generator of E sends σ_4 to $\bar{\sigma}_4$. The corresponding Hermitian form is indefinite for $p = 4, 5, 6$. This Galois conjugation acts nontrivially on $8\,\mathrm{Re}(e^{-i\psi}\tau^3) = 20\cos\psi + 4\sqrt{7}\sin\psi$, which is in $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$.

- For $\tau \in \{\sigma_5, \bar\sigma_5, \sigma_6, \bar\sigma_6\}$, let $E = \mathbb{Q}[e^{i\pi/9}, e^{2i\pi/5}, e^{2i\pi/p}]$. If p is not divisible by 5, the Galois conjugation sending $e^{2i\pi/5}$ to $e^{4i\pi/5}$ and fixing the 2 other generators of E sends σ_5 to σ_6, and $\bar\sigma_5$ to $\bar\sigma_6$. The Hermitian form corresponding to σ_5 and σ_6 is indefinite for all $p \geqslant 3$; for $\bar\sigma_5$ it is indefinite for $p = 2, 4$, and for $\bar\sigma_6$ it is indefinite for $4 \leqslant p \leqslant 29$. This Galois conjugation acts nontrivially on $\mathrm{Re}\,\tau^3 = 11/2 + 11\cos(2\pi/5)$ (respectively $11/2 + 11\cos(4\pi/5)$), which is in $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$.

  If p is not divisible by 3, the Galois conjugation sending $e^{i\pi/9}$ to $e^{-i\pi/9}$ and fixing the 2 other generators of E sends σ_6 to $\bar\sigma_6$. This works for $p = 5$ (the only case where Proposition 2.3 doesn't tell us that $\Gamma(2\pi/p, \sigma_6)$ and $\Gamma(2\pi/p, \bar\sigma_6)$ are nondiscrete).

- For $\tau \in \{\sigma_7, \bar\sigma_7, \sigma_8, \bar\sigma_8, \sigma_9, \bar\sigma_9\}$, let $E = \mathbb{Q}[e^{i\pi/9}, e^{2i\pi/7}, e^{2i\pi/p}]$. If p is not divisible by 7, the Galois conjugation sending $e^{2i\pi/7}$ to $e^{6i\pi/7}$ and fixing the 2 other generators of E sends σ_7 to σ_9 and σ_9 to σ_8, and $\bar\sigma_7$ to $\bar\sigma_9$ and $\bar\sigma_9$ to $\bar\sigma_8$. The Hermitian form corresponding to σ_7, $\bar\sigma_8$ and σ_9 is indefinite for all $p \geqslant 4$ (even 3 for σ_7, σ_9); for $\bar\sigma_7$ it is indefinite for $p = 2$, for σ_8 it is indefinite for $4 \leqslant p \leqslant 41$, and for $\bar\sigma_9$ it is indefinite for $4 \leqslant p \leqslant 8$. This Galois conjugation acts nontrivially on $|\tau|^4 + |\tau|^2 - 2\,\mathrm{Re}\,\tau^3 = 3 + 2\cos(2\pi/7)$ (respectively $3 + 2\cos(4\pi/7)$ and $3 + 2\cos(6\pi/7)$), which is in $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$.

Finally, we know from Proposition 2.3 that, for $\tau \in \{\sigma_3, \bar\sigma_3, \sigma_8, \bar\sigma_8, \sigma_9, \bar\sigma_9\}$, $\Gamma(2\pi/p, \tau)$ is nondiscrete for all p and in particular is not contained in an arithmetic lattice in SU(2, 1). □

We then examine the remaining cases, where we must now take into account the effect of our various Galois conjugations on $\psi = e^{2i\pi/p}$. In what follows, the number field E is a cyclotomic field $\mathbb{Q}[e^{2i\pi/r}]$; the Galois group of E consists of the automorphisms φ_n sending $e^{2i\pi/r}$ to $e^{2in\pi/r}$ for $(n, r) = 1$. The following criterion [Parker and Paupert 2009, Corollary 2.7] expresses the determinant κ of the Hermitian matrix H_τ in a convenient way:

Lemma 4.1. *When* $\tau = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}$ *and* $\sin(\psi/2) \geqslant 0$, *the matrix* H_τ *has signature* (2, 1) *if and only if*

$$\kappa = 8\sin(3\alpha/2 + \psi/2)\sin(3\beta/2 + \psi/2)\sin\big(-3(\alpha+\beta)/2 + \psi/2\big) < 0.$$

Proposition 4.3. $\Gamma(2\pi/p, \tau)$ *is not contained in an arithmetic lattice in* SU(2, 1) *if*

- $\tau = \sigma_1$ *and* $p = 4$ *or* $p \geqslant 8$;
- $\tau = \sigma_2$ *and* 3 *or* 4 *or* 5 *divides* p;
- $\tau = \bar\sigma_4$ *and* $p \geqslant 7$;

- $\tau = \sigma_5$ *and* 5 *divides* p; *or*
- $\tau = \sigma_7$ *and* 7 *divides* p.

Proof. In each case, find a Galois conjugation φ acting nontrivially on $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$ such that two of $\varphi(e^{3i\alpha/2})$, $\varphi(e^{3i\beta/2})$, and $\varphi(e^{-3i(\alpha+\beta)/2})$ lie in the open upper half of the unit circle, and the third in the open lower half (or, in the case of $\tau = \bar{\sigma}_4$, all three in the lower half). Then this property is stable, that is, if $\varphi(\psi)$ is small enough, adding $\varphi(\psi)/2$ to each of the three angles will not change it, where we think of φ as acting on angles. The details:

- As before, for $\tau = \sigma_1$ let $E = \mathbb{Q}[e^{i\pi/6}, e^{i\pi/5}, e^{2i\pi/p}]$; we will use $\varphi \in \mathrm{Gal}(E)$ fixing σ_1 up to a cube root of unity ($\mathrm{mod}_\times\, e^{\pm 2\pi i/3}$). In the notation of Lemma 4.1, the corresponding triple $(3\alpha/2, 3\beta/2, -3(\alpha+\beta)/2)$ is $(\pi/2, \pi/8, -5\pi/8)$. We can get $\varphi_n(\sigma_1) = \sigma_1 \,\mathrm{mod}_\times\, e^{\pm 2\pi i/3}$ by sending $e^{i\pi/4}$ to $e^{\pm i\pi/4}$ and fixing $e^{i\pi/6} \,\mathrm{mod}_\times\, e^{\pm 2\pi i/3}$, or by sending $e^{i\pi/4}$ to $e^{\pm 3i\pi/4}$ and $e^{i\pi/6}$ to $e^{7i\pi/6} = -e^{i\pi/6} \,\mathrm{mod}_\times\, e^{\pm 2\pi i/3}$. This means that n is congruent to (1 or -1 mod 8) and (1 or 5 or 9 mod 12) in the first case, and to (3 or -3 mod 8) and (3 or 7 or 11 mod 12) in the second. We win if we can find such an n, coprime with p and such that $n\pi/p < \pi/2$, that is, $n \leqslant 2p+1$ (this is the largest angle by which one can rotate the 3 points on the unit circle without any of them changing sides). The first few solutions to the above congruencies are $n = (1), 3, 9, 11, 17, 19, 25, 27, 33, 35, 41$. Start with $n = 3$; this works as long as 3 doesn't divide p and $p \geqslant 7$. We check that $\varphi_5(\kappa) < 0$ (and $\varphi_5(\sqrt{2}) \neq \sqrt{2}$) for $p = 4$. Assume then that 3 divides p, and use $n = 11$; this works as long as 11 doesn't divide p and $p \geqslant 23$. This leaves $p = 9, 12, 15, 18, 21$; we check that $n = 5$ works for $p = 9, 18, 21$, that $n = 7$ works for $p = 12$, and $n = 11$ for $p = 15$. Assume then that 33 divides p, and use $n = 17$; this works as long as 17 doesn't divide p and $p \geqslant 34$. This leaves $p = 33$, where we check that $\varphi_5(\kappa) < 0$. We then go on in this fashion (skipping solutions like 27 and 33, which are divisible by 3), assuming that $3 \times 11 \times 17$ divides p and using $n = 19$ and so on. In this fashion p increases multiplicatively, whereas solutions to the above congruences increase additively; therefore such n exist by a wider and wider margin. We conclude inductively that such an n exists for p large enough (and we have checked the few exceptions for small p).
- As previously, for $\tau = \sigma_2$ or σ_3 let $E = \mathbb{Q}[e^{i\pi/6}, e^{i\pi/5}, e^{2i\pi/p}]$ and consider $\varphi \in \mathrm{Gal}(E)$ sending $e^{i\pi/5}$ to $e^{3i\pi/5}$ and $e^{i\pi/6}$ to $e^{7i\pi/6} = -e^{i\pi/6}$. Then φ swaps σ_2 and σ_3. With the notation of Lemma 4.1, the corresponding triples

$$(3\alpha/2, 3\beta/2, -3(\alpha+\beta)/2)$$

are

$$\begin{aligned} &(\pi/2, \pi/20, -11\pi/20) && \text{when } \tau = \sigma_2, \\ &(\pi/2, 7\pi/20, -17\pi/20) && \text{when } \tau = \sigma_3. \end{aligned}$$

Now when 3 or 4 or 5 divide p, φ also acts on $e^{2i\pi/p}$.

If 4 divides p, writing $p=4k$, $(e^{2i\pi/p})^k=i=(e^{i\pi/6})^3$ is sent to $-i$, so $\varphi(e^{2i\pi/p})$ must be a k-th root of $-i$; in other words, $\omega_k.e^{-i\pi/2k}$ for a k-th root of unity ω_k. In fact, if 3 or 5 don't divide p, one can send $e^{2i\pi/p}$ to any $\omega_k.e^{-i\pi/2k}$, say with $\omega_k=e^{2i\pi/k}$ (this gives a better bound on p than 1). Then $\psi/2$ is sent to $3\psi/2$ (because $-\pi/2k+2\pi/k=3\pi/2k$), and the argument works for $3\pi/p<11\pi/20$ ($p\geqslant 6$) when $\tau=\sigma_3$, and $3\pi/p<17\pi/20$ ($p\geqslant 4$) when $\tau=\sigma_2$. There remain the cases where 5 divides p, as well as $\tau=\sigma_3$ and $p=4$. In the latter case one can check that $\varphi_{13}(\kappa)<0$, with $\varphi_{13}(\cos 3\pi/5)\neq\cos 3\pi/5$.

Now suppose that 5 divides p but 3 or 4 do not, and write $p=5k$. As above, one can send $e^{2i\pi/p}$ to $e^{6i\pi/p}$, and the same argument tells us that $\varphi(\kappa)<0$ for $p\geqslant 4$ when $\tau=\sigma_2$ and $p\geqslant 6$ when $\tau=\sigma_3$. When $p=5$ and $\tau=\sigma_3$, one can again check that $\varphi_{13}(\kappa)<0$ (with $\varphi_{13}(\cos 3\pi/5)\neq\cos 3\pi/5$).

If 3 divides p, we find $\varphi\in\mathrm{Gal}(E)$ as above; specifically, we require that $\varphi(e^{i\pi/5})=e^{3i\pi/5}$ or $e^{-3i\pi/5}$ and $\varphi(e^{i\pi/6})=e^{7i\pi/6}$ up to a cube root of unity, so that φ swaps σ_2 and σ_3 (up to a cube root of unity). Such a φ is realized as a φ_n if (and only if) n is congruent to (3 or -3 mod 10) and (3 or 7 or 11 mod 12). The values of such n are 3, 7, 23, 27, 43, 47, ... Moreover, with the angle triples as above, $\varphi_n(\kappa)<0$ for $n\pi/p<17\pi/20$ (when $\tau=\sigma_2$) or $n\pi/p<\pi/2$ (when $\tau=\sigma_3$). We may use $n=7$ as long as 7 doesn't divide p, which works for $p\geqslant 9$ when $\tau=\sigma_2$, and $p\geqslant 15$ when $\tau=\sigma_3$. We then check the cases $p=6$ and $\tau=\sigma_2$, as well as $p=6,9,12$ and $\tau=\sigma_3$. It turns out that $n=7$ works for all of these (renormalizing $7\times 2\pi/6$ when $p=6$ as $2\pi/6$). Now if 7 also divides p, we use the next solution $n=23$, which works for $p\geqslant 47$ when $\tau=\sigma_2$, and $p\geqslant 28$ when $\tau=\sigma_3$, as long as 23 doesn't divide p. We check that $n=11$ works for $p=21$ for $\tau=\sigma_2,\sigma_3$ and $p=42$ for $\tau=\sigma_2$. One can then assume that 21×23 divides p, and so on. We conclude inductively as above.

- For $\tau=\bar\sigma_4$, E is as before $\mathbb{Q}[e^{2i\pi/7},e^{2i\pi/p}]$, and $(3\alpha/2,3\beta/2,-3(\alpha+\beta)/2)=(-3\pi/7,-6\pi/7,9\pi/7)$. If 7 doesn't divide p, consider $\varphi\in\mathrm{Gal}(E)$ fixing $e^{2i\pi/7}$ and sending $e^{2i\pi/p}$ to $e^{2in\pi/p}$ with $(n,p)=1$ and $1<n\leqslant 3p/7$ (this is possible as $p\geqslant 7$). Then $n\pi/p\leqslant 3\pi/7$ as required.

If 7 divides p, say $p=7k$, one can again fix $e^{2i\pi/7}$ and send $e^{2i\pi/p}$ to a k-th root of itself; when $k\geqslant 3$, letting $\varphi(e^{2i\pi/p})=e^{2i\pi(1/k+1/p)}$ works (that is, $\varphi(\kappa)<0$), because $\pi/k+\pi/p<3\pi/7$. There remain only the cases $p=7$, where one can check that $\varphi_2(\kappa)<0$ with $\varphi_2(\cos 2\pi/7)\neq\cos 2\pi/7$, and $p=14$, where one can check that $\varphi_9(\kappa)<0$ with $\varphi_9(\cos\pi/7)\neq\cos\pi/7$.

- As previously, for $\tau=\sigma_5$ or σ_6 let $E=\mathbb{Q}[e^{i\pi/9},e^{2i\pi/5},e^{2i\pi/p}]$ and consider $\varphi\in\mathrm{Gal}(E)$ sending $e^{2i\pi/5}$ to $e^{4i\pi/5}$ and fixing $e^{i\pi/9}$. Then φ swaps σ_5 and σ_6. With the notation of Lemma 4.1, the corresponding triples $(3\alpha/2,3\beta/2,-3(\alpha+\beta)/2)$ are

$(\pi/3, 13\pi/30, -23\pi/30)$ when $\varphi(\tau) = \sigma_5$, and $(\pi/3, 31\pi/30, -41\pi/30)$ when $\varphi(\tau) = \sigma_6$.

If 5 divides p, say $p = 5k$, φ must send $e^{2i\pi/p}$ to a k-th root of $e^{4i\pi/5}$, and one can choose any of these if 3 does not divide p, such as $e^{4i\pi/5k}$. When $\tau = \sigma_5$, this works for $2\pi/p \leqslant 17\pi/30$ (and $p \geqslant 4$), and when $\tau = \sigma_6$ for $2\pi/p \leqslant 11\pi/30$ (and $p \geqslant 6$). When $p = 5$ and $\tau = \sigma_6$, one can check that $\varphi_4(\kappa) < 0$ (with $\varphi_4(\sqrt{3}\sin(2\pi/5)) \neq \sqrt{3}\sin(2\pi/5)$).

Now if 3 also divides p, we must look more closely at how φ is defined above. Namely, such a φ is a φ_n if and only if n is congruent to 2 mod 5 and 1 mod 18. The smallest such n is 37. However one can relax slightly the definition of φ to allow $\varphi(e^{i\pi/9}) = \omega_3 e^{i\pi/9}$ for any cubic root of unity ω_3, as this does not affect τ. The conditions are then that n should be congruent to (2 mod 5) and (1 or 7 or 13 mod 18). We can then use $n = 7$, unless 7 divides p. In that case φ_7 would work for $7\pi/p \leqslant 17\pi/30$ (with $p \geqslant 13$) when $\tau = \sigma_6$, and for $7\pi/p \leqslant 11\pi/30$ (with $p \geqslant 20$) when $\tau = \sigma_5$. Since at this point 15 divides p, there remains only the case where $p = 15$ and $\tau = \sigma_6$, in which case one can check that $\varphi_{11}(\kappa) < 0$ with $\varphi_{11}(\cos(2\pi/15)) \neq \cos(2\pi/15)$.

Finally, if 7 also divides p (at this point p is divisible by 105), we can do the same thing. That is, we claim that there exists n congruent to (2 mod 5) and (1 or 7 or 13 mod 18), coprime with p and such that $n\pi/p \leqslant 11\pi/30$ (that is, $n \leqslant 11p/30$). For $p = 105k$, $n = 37$ satisfies these conditions for $1 \leqslant k \leqslant 36$. After that, suppose that 37 divides p and so on; we conclude inductively as above.

- As before, for $\tau = \sigma_7$ let $E = \mathbb{Q}[e^{i\pi/9}, e^{2i\pi/7}, e^{2i\pi/p}]$ and consider $\varphi_n \in \mathrm{Gal}(E)$ sending $e^{2i\pi/7}$ to $e^{6i\pi/7}$ (respectively $e^{-2i\pi/7}$) and fixing $e^{i\pi/9}$ (up to a cube root of unity). This means that n should be congruent to (3 respectively -1 mod 7) and (1 or 7 or 13 mod 18). Then $\varphi_n(\sigma_7) = \sigma_9$ (respectively σ_7), and the corresponding triple $(3\alpha/2, 3\beta/2, -3(\alpha+\beta)/2)$ is

$$(\pi/3, 47\pi/42, -61\pi/42) \quad (\text{respectively } (\pi/3, 11\pi/42, -25\pi/42)).$$

With these values, $\varphi_n(\kappa) < 0$ when $n\pi/p \leqslant 19\pi/42$ (respectively $n\pi/p \leqslant 25\pi/42$). The smallest such n is 13, which works for $p \geqslant 22$ (as long as 13 doesn't divide p). It remains to check $p = 7$, 14 or 21 (here 7 is assumed to divide p): $n = 5$ works when $p = 7$ or 21, and $n = 11$ works when $p = 14$. If 13 divides p, use the next solution $n = 31$, and so on. We conclude inductively as above. □

Lemma 4.2. *For* $\tau = \bar{\sigma}_2$ *and* $p = 8, 9, 10, 12, 14, 15, 16, 18$, $\Gamma(2\pi/p, \tau)$ *is not contained in an arithmetic lattice in* $\mathrm{SU}(2,1)$.

Proof. For each of these values we find a Galois conjugation φ_n of E such that $\varphi_n(\kappa) < 0$, where $\kappa = \det H_\tau$, and acting nontrivially on $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$. For this last condition, it suffices to check that $\varphi_n(\cos 2\pi/p) \neq \cos 2\pi/p$ (this is true for all cases below, except $n = 7$ and $p = 8$, in which case $\varphi_7(\cos\pi/5) \neq \cos\pi/5$). The condition $\varphi_n(\kappa) < 0$ can easily be checked, for instance numerically. We claim that the following φ_n satisfy these conditions when $\tau = \bar{\sigma}_2$: φ_7 works for $p = 8, 9, 10, 12$, and φ_{11} works for $p = 14, 15, 16, 18$. □

5. Commensurability

In this section we compare the adjoint trace fields of our sporadic groups with those of the previously known lattices in SU(2, 1), namely the Picard and Mostow lattices (see [Deligne and Mostow 1986; Mostow 1980; 1986; Sauter 1990; Thurston 1998; Parker 2008a] for an overview). From the lists on [Mostow 1980, p. 251; Deligne and Mostow 1986, p. 86; Thurston 1998, pp. 548–549], we see that for these lattices Γ, $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$ is always of the form $\mathbb{Q}[\cos 2\pi/d]$, where

- $d = 3, 4, 5, 6, 8, 9, 10, 12, 18$ for the arithmetic Picard lattices;
- $d = 12, 15, 20, 24$ for the nonarithmetic Picard lattices;
- $d = 1, 8, 10, 12, 15, 18$ for the arithmetic Mostow lattices;
- $d = 12, 15, 18, 20, 24, 30, 42$ for the nonarithmetic Mostow lattices.

Moreover, only two nonarithmetic noncocompact lattices are known in SU(2, 1), both with $d = 12$.

Remark 5.1. The nonarithmetic Picard and Mostow lattices in SU(2, 1) fall into at least 7 and at most 9 distinct commensurability classes.

Indeed there are 6 distinct adjoint trace fields ($d = 15$ and 30 give the same field), and for $d = 12$ there are two classes, one cocompact and the other noncocompact. Also, there are a priori 15 examples, but Mostow [1986] and Sauter [1990] find commensurabilities among some of them. See [Parker 2008a] for more details.

Now we use the values from Proposition 3.3 to distinguish commensurability classes of sporadic groups, from each other and from the Picard and Mostow lattices. We will also use the fact that arithmeticity and cocompactness are commensurability invariants. We summarize the results from this section:

Theorem 5.2. *For $p \geqslant 2$ and $\tau \in \{\sigma_1, \bar{\sigma}_1, \ldots, \sigma_9, \bar{\sigma}_9\}$, sporadic groups $\Gamma(2\pi/p, \tau)$ are not commensurable to any Picard or Mostow lattice, except possibly when*

- *$p = 2$ or 4 or 6 and τ is any sporadic value*;
- *$p = 3$ and $\tau = \sigma_7$*;
- *$p = 5$ and τ or $\bar{\tau} = \sigma_1, \sigma_2$*;
- *$p = 7$ and $\tau = \bar{\sigma}_4$*;
- *$p = 8$ and $\tau = \sigma_1$*;
- *$p = 10$ and $\tau = \sigma_1, \sigma_2, \bar{\sigma}_2$*;

- *$p = 12$ and $\tau = \sigma_1, \sigma_7$;*
- *$p = 20$ and $\tau = \sigma_1, \sigma_2$;*
- *$p = 24$ and $\tau = \sigma_1$.*

The first observation follows simply from the order of the complex reflections in the group; that is, from the fact that $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma(2\pi/p, \tau)]$ contains $\cos 2\pi/p$. The values of $p \geqslant 3$ that we rule out are the divisors of 12, 15, 18, 20, 24, 30, 42.

Lemma 5.1. *For $p \neq 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 21, 24, 30, 42$, the sporadic groups $\Gamma(2\pi/p, \tau)$ are not commensurable to any Picard or Mostow lattice. Moreover, they fall into infinitely many distinct commensurability classes.*

We then examine the remaining values of p, where we can rule out most cases except when $p = 3, 4$ or 6:

Lemma 5.2. *For values $p \in \{5, 7, 8, 9, 10, 12, 14, 15, 18, 20, 21, 24, 30, 42\}$, the sporadic groups $\Gamma(2\pi/p, \tau)$ are not commensurable to any Picard or Mostow lattice, except possibly when*

- *$p = 5$ and τ or $\bar{\tau} = \sigma_1, \sigma_2$;*
- *$p = 7$ and $\tau = \bar{\sigma}_4$;*
- *$p = 8$ and $\tau = \sigma_1$;*
- *$p = 10$ and $\tau = \sigma_1, \sigma_2, \bar{\sigma}_2$;*
- *$p = 12$ and $\tau = \sigma_1, \sigma_7$;*
- *$p = 20$ and $\tau = \sigma_1, \sigma_2$; or*
- *$p = 24$ and $\tau = \sigma_1$.*

Proof. We use the values found for $\mathbb{Q}[\mathrm{Tr}\,\mathrm{Ad}\,\Gamma]$ in Section 3, listed in the table at the end of that section, as well as the following criterion.

Let $p \geqslant 3$, $p \neq 6$ and $d \in \mathbb{N}$. Then $\sin 2\pi/p = \cos(p-4)\pi/2p$ is in $\mathbb{Q}[\cos 2\pi/d]$ if and only if

- p divides d (if 4 divides p);
- $2p$ divides d (if p is even but not divisible by 4); and
- $4p$ divides d (if p is odd).

This allows us to rule out the cases

- $p = 7, 9, 14, 15, 18, 21, 30, 42$ when τ or $\bar{\tau} = \sigma_1$;
- $p = 7, 8, 9, 12, 14, 15, 18, 21, 24, 30, 42$ when τ or $\bar{\tau} = \sigma_2$ or σ_3;
- $p = 5, 8, 9, 10, 12, 14, 15, 18, 20, 21, 24, 30$ when τ or $\bar{\tau} = \sigma_4$;
- $p = 5, 7, 8, 9, 10, 12, 14, 15, 18, 20, 21, 24, 30, 42$ when τ or $\bar{\tau} = \sigma_5$ or σ_6;
- $p = 5, 7, 8, 9, 10, 14, 15, 18, 20, 21, 24, 30, 42$ when τ or $\bar{\tau} = \sigma_7$. □

Lemma 5.3. $\Gamma(2\pi/3, \bar{\sigma}_4)$ *is not commensurable to any Picard or Mostow lattice.*

Proof. Recall that this is the only arithmetic sporadic group. $\mathbb{Q}[\operatorname{Tr} \operatorname{Ad} \Gamma(2\pi/3, \bar{\sigma}_4)]$ contains $\sqrt{21}$, which is not in $\mathbb{Q}[\cos 2\pi/d]$ for $d = 1, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18$. □

Lemma 5.4. *The groups* $\Gamma(2\pi/3, \sigma_1)$, $\Gamma(2\pi/3, \bar{\sigma}_1)$, $\Gamma(2\pi/3, \sigma_5)$, $\Gamma(2\pi/5, \sigma_3)$ *and* $\Gamma(2\pi/5, \bar{\sigma}_3)$ *are not commensurable to any Picard or Mostow lattice.*

Proof. In the groups $\Gamma(2\pi/3, \sigma_1)$, $\Gamma(2\pi/3, \bar{\sigma}_1)$, $\Gamma(2\pi/5, \sigma_3)$ and $\Gamma(2\pi/5, \bar{\sigma}_3)$, $R_1 R_2$ is parabolic [Parker and Paupert 2009], whereas $R_2(R_1 J)^5$ is parabolic in $\Gamma(2\pi/3, \sigma_5)$ (details to appear in a forthcoming paper). It follows from Godement's compactness criterion that such a group cannot be commensurable to a cocompact lattice. Therefore it suffices to check that these groups are not commensurable to the noncocompact Picard and Mostow lattices, which both have adjoint trace field equal to $\mathbb{Q}[\cos 2\pi/12]$. Now for $\tau = \sigma_1$ or $\bar{\sigma}_1$, $\mathbb{Q}[\operatorname{Tr} \operatorname{Ad} \Gamma(2\pi/3, \tau)]$ contains $\sqrt{2} \sin 2\pi/p = \sqrt{6}/2$, which is not in $\mathbb{Q}[\cos 2\pi/12]$, and in the three other cases $\mathbb{Q}[\operatorname{Tr} \operatorname{Ad} \Gamma]$ contains $\cos 2\pi/5$, which is not in $\mathbb{Q}[\cos 2\pi/12]$ either. □

Lemma 5.5. $\Gamma(2\pi/3, \bar{\sigma}_1)$, $\Gamma(2\pi/3, \sigma_2)$ *and* $\Gamma(2\pi/3, \bar{\sigma}_2)$ *are not discrete, and therefore not commensurable to any Picard or Mostow lattice.*

Proof. In the first of these groups $R_1(R_1 J)^4$ is elliptic of infinite order, and in the two others $R_1(R_1 J)^5$ is elliptic of infinite order (details to appear). □

Acknowledgments

The author would like to thank John Parker and Domingo Toledo for stimulating conversations concerning this work.

References

[Deligne and Mostow 1986] P. Deligne and G. D. Mostow, "Monodromy of hypergeometric functions and nonlattice integral monodromy", *Inst. Hautes Études Sci. Publ. Math.* 63 (1986), 5–89. MR 88a:22023a Zbl 0615.22008

[Goldman 1999] W. M. Goldman, *Complex hyperbolic geometry*, Oxford University Press, New York, 1999. MR 2000g:32029 Zbl 0939.32024

[Maclachlan and Reid 2003] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Math. **219**, Springer, New York, 2003. MR 2004i:57021 Zbl 1025.57001

[McReynolds 2006] D. McReynolds, "Arithmetic lattices in SU(n, 1)", preprint, 2006, Available at http://www.ma.utexas.edu/users/dmcreyn/ComplexArithmeticI.pdf.

[Mostow 1980] G. D. Mostow, "On a remarkable class of polyhedra in complex hyperbolic space", *Pacific J. Math.* **86**:1 (1980), 171–276. MR 82a:22011 Zbl 0456.22012

[Mostow 1986] G. D. Mostow, "Generalized Picard lattices arising from half-integral conditions", *Inst. Hautes Études Sci. Publ. Math.* 63 (1986), 91–106. MR 88a:22023b Zbl 0615.22009

[Parker 2008a] J. R. Parker, “Complex hyperbolic lattices”, preprint, Univeristy of Durham, 2008, Available at http://maths.dur.ac.uk/∼dma0jrp/img/Lattices.pdf.

[Parker 2008b] J. R. Parker, “Unfaithful complex hyperbolic triangle groups, I: Involutions”, *Pacific J. Math.* **238**:1 (2008), 145–169. MR 2009h:20056 Zbl 1158.20023

[Parker and Paupert 2009] J. R. Parker and J. Paupert, “Unfaithful complex hyperbolic triangle groups, II: Higher order reflections”, *Pacific J. Math.* **239**:2 (2009), 357–389. MR 2009h:20057 Zbl 1161.20046

[Pratoussevitch 2005] A. Pratoussevitch, “Traces in complex hyperbolic triangle groups”, *Geom. Dedicata* **111** (2005), 159–185. MR 2006d:32036 Zbl 1115.32015

[Sauter 1990] J. K. Sauter, Jr., “Isomorphisms among monodromy groups and applications to lattices in PU(1, 2)”, *Pacific J. Math.* **146**:2 (1990), 331–384. MR 92d:22016 Zbl 0759.22013

[Thurston 1998] W. P. Thurston, “Shapes of polyhedra and triangulations of the sphere”, pp. 511–549 in *The Epstein birthday schrift*, edited by I. Rivin et al., Geom. Topol. Monogr. **1**, Geom. Topol. Publ., Coventry, 1998. MR 2000b:57026 Zbl 0931.57010

Received February 23, 2009.

JULIEN PAUPERT
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
155 SOUTH 1400 EAST
SALT LAKE CITY, UT 84112
UNITED STATES
paupert@math.utah.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

A REMARK ON KHOVANOV HOMOLOGY AND TWO-FOLD BRANCHED COVERS

LIAM WATSON

We give examples of knots distinguished by the total rank of their Khovanov homology but sharing the same two-fold branched cover. Hence Khovanov homology does not yield an invariant of two-fold branched covers.

Mutation provides an easy method for producing distinct knots sharing a two-fold branched cover: The mutation in the branch set corresponds to a trivial surgery in the cover. Due to a result of Wehrli [2007; 2009] (see also [Bloom 2009]), this provides a range of examples of manifolds that branch cover S^3 in more than one way, but for which the distinct branch sets have identical rank in their respective Khovanov homology groups over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

From this point of view this fact is not completely surprising, as Khovanov homology is closely related to the Heegaard Floer homology of two-fold branched covers [Ozsváth and Szabó 2005]. Indeed, this is made precise in Bloom's proof of mutation invariance [2009]. More generally, there is a question posed by Ozsváth: Is Khovanov homology an invariant of the two-fold branched cover? More precisely, is the *total rank* of the reduced Khovanov homology (over $\mathbb{F}_2$) an invariant of two-fold branched covers? This short note gives a negative answer.

Theorem. *The total rank of Khovanov homology is not an invariant of two-fold branched covers.*

This theorem is proved by exhibiting manifolds that are two-fold branched covers of S^3 in two different ways, and for which the pair of branch sets is distinguished by the total rank in Khovanov homology. We work with the reduced version of Khovanov homology, denoted $\widetilde{\mathrm{Kh}}$, with $\mathbb{F}_2$ coefficients [Khovanov 2000; 2003].

Surgery on torus knots. Let $S^3_{r/s}(K)$ denote the result of (r/s)-surgery on a knot $K \hookrightarrow S^3$, and let $T_{p,q}$ denote the positive (p,q) torus knot in S^3 (with $0 < p < q$). Note that, as we will only consider torus knots, p and q are relatively prime.

Proposition 1 [Moser 1971]. *The manifold $S^3_{\pm 1/n}(T_{p,q})$ is Seifert fibered with base orbifold $S^2(p, q, pqn \mp 1)$ for $n > 0$.*

MSC2000: 57M12, 57M27.
Keywords: Khovanov homology, two-fold branched cover, Heegaard Floer homology.
Supported by a Canada Graduate Scholarship (NSERC).

Our conventions for Seifert fibered spaces follow [Boyer 2002]. Our conventions differ from those in Moser's work, resulting in the sign discrepancy between our statement and Moser's. By applying the work of Heil [1974], it is possible to give a quick proof:

Proof. Let $M = S^3 \smallsetminus \nu(T_{p,q})$, so that $M(\alpha) = S^3_{r/s}(T_{p,q})$ for a given slope $\alpha = r\mu + s\lambda$, where μ is the knot meridian and λ is the preferred longitude. As the complement of a regular fiber of a Seifert fibration of S^3, M is Seifert fibered with base orbifold $D^2(p,q)$. Let φ denote a regular fiber in ∂M; it is well known that $\varphi = pq\mu + \lambda$. Now $M(\alpha)$ is Seifert fibered with base orbifold $S^2(p,q,|\alpha\cdot\varphi|)$ whenever $\alpha \neq \varphi$, according to [Heil 1974]. In the present setting, $\alpha = \pm\mu + n\lambda$ for $n > 0$, so $M(\alpha) = S^3_{\pm 1/n}(T_{p,q})$. As a result, $M(\pm\mu + n\lambda) = S^3_{\pm 1/n}(T_{p,q})$ is Seifert fibered with base orbifold $S^2(p,q,pqn \mp 1)$ as claimed. □

Seifert involutions. For a link $L \hookrightarrow S^3$, let $\Sigma(S^3, L)$ denote the two-fold branched cover of S^3, branched over L.

Proposition 2 [Seifert 1933]. *$S^3_{\pm 1/n}(T_{2,q}) \cong \Sigma(S^3, T_{q,2qn\mp 1})$ for $n > 0$ and odd $q > 1$.*

Proof. The manifold $\Sigma(S^3, T_{q,2qn\mp 1})$ is the Brieskorn sphere $\Sigma(2,q,2qn\mp 1)$ and is Seifert fibered with base orbifold $S^2(2,q,2qn\mp 1)$ [Milnor 1975, Lemma 1.1]; see also [Seifert 1933, Zusatz zu Satz 17]. For each $n > 0$ and odd $q > 1$, there is a unique $\mathbb{Z}$-homology sphere admitting a Seifert fibered structure with base orbifold $S^2(2,q,2qn\mp 1)$; see for example [Scott 1983; Saveliev 1999, Theorem 6.7]. The result follows. □

The Montesinos trick. A knot K is called strongly invertible if there is an involution of (S^3, K) that reverses orientation on K. Thus, the complement $S^3 \smallsetminus \nu(K)$ of any strongly invertible knot admits an involution with one-dimensional fixed point set given by a pair of arcs meeting the boundary $\partial(S^3 \smallsetminus \nu(K))$ transversally in the 4 endpoints. Since the quotient of a solid torus under such an involution is a 3-ball, it follows that a fundamental domain for the action of the involution on $S^3 \smallsetminus \nu(K)$ is a 3-ball as well, since $S^3 \cong \Sigma(S^3, L)$ if and only if L is the trivial knot [Waldhausen 1969].

By keeping track of the fixed point set in the quotient, we obtain a tangle denoted by $T = (B^3, \tau')$, where τ' is a pair of arcs properly embedded in the 3-ball B^3 meeting the boundary transversally in 4 points. By construction, $S^3 \smallsetminus \nu(K)$ is realized as the two-fold branched cover of B^3, denoted $\Sigma(B^3, \tau')$, branched over the arcs τ'. In this context tangles are considered up to homeomorphism of the pair (B^3, τ') that generally need not fix the boundary sphere.

Given a strongly invertible knot, the Montesinos trick [1975] amounts to the observation that Dehn surgery in the cover may be interpreted as rational tangle

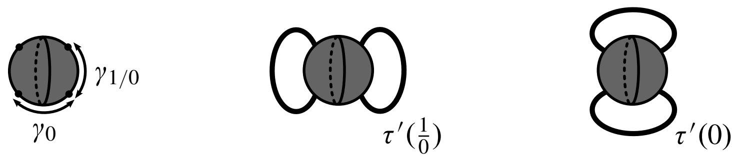

Figure 1. The arcs $\gamma_{1/0}$ and γ_0 in the boundary of a tangle (left) lifting to μ and $m\mu + \lambda$ respectively. The "denominator" (center) and "numerator" (right) closures are denoted by $\tau'(\frac{1}{0})$ and $\tau'(0)$ respectively. Note that in this context $\tau'(\frac{1}{0})$ is the trivial knot.

attachment in the base. Recall that a tangle is rational if and only if the two-fold branched cover is a solid torus. To identify the corresponding branch set to a given surgery, in Figure 1 we briefly recall the notation introduced in [Watson 2008].

By construction, it is possible to identify the trivial surgery by the unknotted branch set $\tau'(\frac{1}{0})$ (see Figure 1, and Figure 3 for a particular example). Said another way, the arc $\gamma_{1/0}$ in the boundary of this representative for the tangle, identified in Figure 1, lifts to the knot meridian μ in the cover. Thus, the link $\tau'(0)$ gives the branch set for some integer surgery; the arc γ_0 lifts to a slope $m\mu + \lambda$ in $\partial(S^3 \smallsetminus \nu(K))$ for some m, where λ is the preferred longitude.

More generally, we may represent any integer surgery by varying the number of half-twists as in Figure 2, since the half-twist lifts to a full Dehn twist about the meridian; see [Rolfsen 1976], for example. As a result it is always possible to fix a *preferred* representative, which we denote (B^3, τ), of the homeomorphism class T with the property that $S^3_0(K) \cong \Sigma(S^3, \tau(0))$. In this notation, we have that $\tau'(0) \simeq \tau(m)$, and the desired homeomorphism is determined by m half-twists. Moreover, $S^3_n(K) \cong \Sigma(S^3, \tau(n))$, where $\tau(n)$ is the link shown in Figure 2.

It is possible to determine the preferred representative directly by carefully keeping track of the image of the preferred longitude in the quotient; see for example [Bleiler 1985]. However, in practice it is straightforward to determine the appropriate homeomorphism after the fact by using the determinant of the link, given that the meridian is easy to identify in this context. Recall that $\det L =$

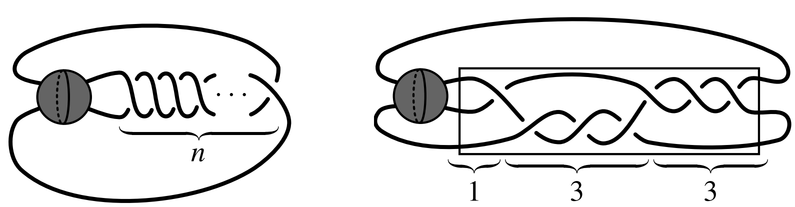

Figure 2. At left, the closure $\tau(n)$ of the preferred representative giving rise to the branch sets for integer surgeries. At right, the closure $\frac{13}{10} = [1, 3, 3]$ corresponding to $\frac{13}{10}$-surgery in the cover.

$|H_1(\Sigma(S^3, L); \mathbb{Z})|$ whenever this group is finite, and $\det L = 0$ otherwise. In particular, $\det \tau(n) = n$.

More generally, we would like to define the branch set $\tau(r/s)$ for the 3-manifold $S^3_{r/s}(K)$, continuing with the notation of [Watson 2008], so that

$$S^3_{r/s}(K) \cong \Sigma(S^3, \tau(r/s)).$$

To this end, let $[a_1, a_2, \dots, a_m]$ be a continued fraction expansion for r/s. Now $[a_1, a_2, \dots, a_m]$ encodes a rational tangle that lifts to the desired homeomorphism of the boundary; see for example [Rolfsen 1976]. A specific example is shown in Figure 2. As suggested, the desired homeomorphism is specified by an element of the 3-strand braid group $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$, where the generator σ_2 lifts to a Dehn twist about μ and the inverse σ_1^{-1} lifts to a Dehn twist about λ. For details on conventions, see [Rolfsen 1976, Chapter 10] and [Watson 2008].

Montesinos involutions. By a result of Schreier [1924], the knot $T_{p,q}$ is strongly invertible. As such, it is possible to realize the manifold $S^3_{r/s}(T_{p,q})$ as a two-fold branched cover via the Montesinos trick, as outlined above. The goal of this section is to determine the preferred representative of the tangle for which $\Sigma(B^3, \tau) \cong S^3 \smallsetminus \nu(T_{p,q})$.

In the interest of being explicit, consider the torus knot $K = T_{2,5}$, the knot 5_1 in [Rolfsen 1976]. A strong inversion on this knot is exhibited in Figure 3, together with an illustration of the isotopy of a fundamental domain to obtain a tangle with the property that $S^3 \smallsetminus \nu(K) \cong \Sigma(B^3, \tau')$.

We may fix the preferred representative (B^3, τ) for T, as in the previous section, with the properties that

(1) the denominator closure of the tangle, $\tau(\frac{1}{0})$, is unknotted and corresponds to a branch set for the trivial surgery, and

(2) the numerator closure, $\tau(0)$, gives a branch set for the zero surgery:

$$S^3_0(K) \cong \Sigma(S^3, \tau(0)).$$

This representative is shown in Figure 4, and it suffices to verify that $\det \tau(0) = 0$ (or that $\det \tau(\pm 1) = 1$) to see that this is the preferred representative as claimed. The fact that $\tau'(0)$ is a connect sum of 2-bridge links indicates that $\Sigma(S^3, \tau'(0))$ is a connect sum of lens spaces, and hence $\Sigma(S^3, \tau'(0)) \cong S^3_{10}(K)$. This results from the fact that $\varphi = 10\mu + \lambda$ for the complement of $K = T_{2,5}$ (compare the proof of Proposition 1), and explains the appearance of 10 (negative) half-twists in the preferred representative (B^3, τ) so that $\tau(10) \simeq \tau'(0)$.

See [Montesinos 1976] for a detailed discussion on Seifert fibered spaces as two-fold branched covers of S^3 in general, noting that the Montesinos links shown here encode the Seifert fiber structure in the corresponding two-fold branched cover.

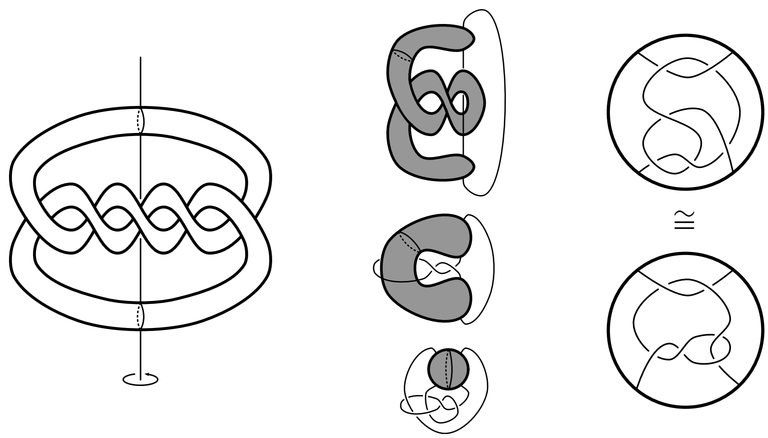

Figure 3. A strong inversion on the torus knot $T_{2,5}$ (left); isotopy of a fundamental domain (center); and two representatives of the associated quotient tangle (right). The Seifert fiber structure on the knot complement is reflected as a sum of rational tangles in the quotient, and the numerator closure in both cases is the trivial knot, identifying the image of the meridian in the quotient.

Proof of the Theorem. Continuing with $K = T_{2,5}$, by the observations above about the Seifert and Montesinos involutions, we have

$$S^3_{\pm 1/n}(K) \cong \Sigma(S^3, T_{5,10n\mp 1}) \cong \Sigma(S^3, \tau(\pm 1/n)) \quad \text{for } n > 0.$$

When $n = 1$, using the program `JavaKh` [Bar-Natan and Green 2005], we calculate

$$\operatorname{rk} \widetilde{\mathrm{Kh}}(T_{5,10\mp 1}) = 65 \mp 8 \neq 16 \mp 1 = \operatorname{rk} \widetilde{\mathrm{Kh}}(\tau(\pm 1)).$$

Similarly, when $n = 2$ we calculate

$$\operatorname{rk} \widetilde{\mathrm{Kh}}(T_{5,20\mp 1}) = 257 \mp 16 \neq 32 \mp 1 = \operatorname{rk} \widetilde{\mathrm{Kh}}(\tau(\pm \tfrac{1}{2})).$$

Each of these four pairs of examples illustrates a given manifold as a two-fold branched cover of S^3 in two different ways, with branch sets distinguished by the total rank of the reduced Khovanov homology. This proves the claim: $\operatorname{rk} \widetilde{\mathrm{Kh}}$ is not an invariant of two-fold branched covers.

Further remarks. We continue with the notation above for the preferred representative of the tangle associated to $T_{2,5}$.

Proposition 3. $\operatorname{rk} \widetilde{\mathrm{Kh}}(\tau(\pm 1/n)) \leq 16n \mp 1$ *for* $n > 0$.

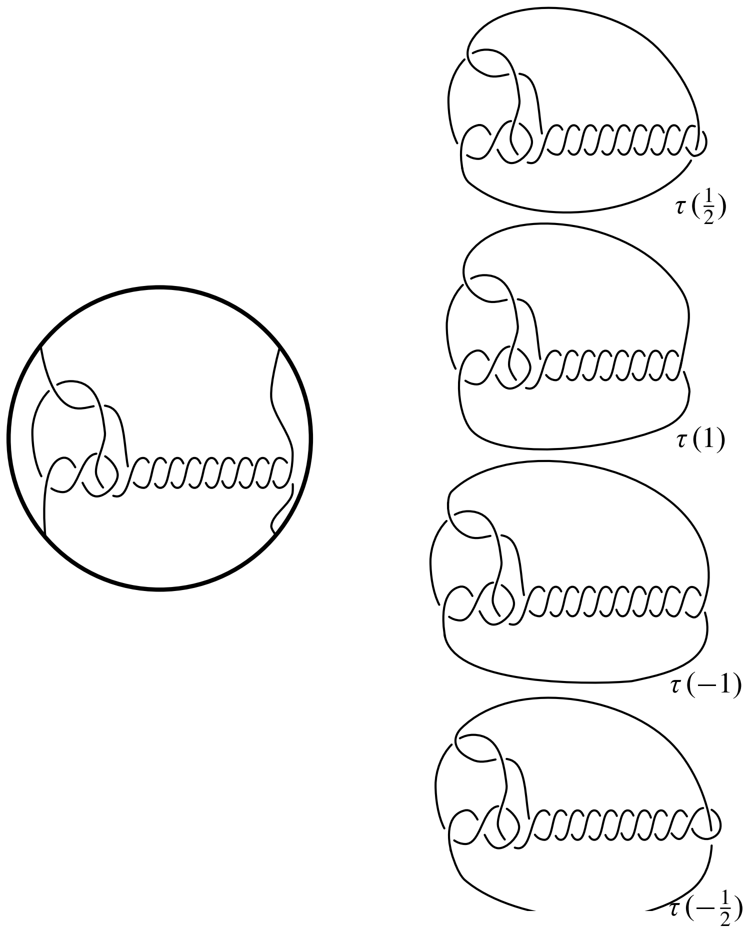

Figure 4. At left, the preferred representative of the associated quotient tangle for the torus knot $T_{2,5}$. At right, the branch sets $\tau(-\frac{1}{2})$, $\tau(-1)$, $\tau(1)$ and $\tau(\frac{1}{2})$ associated to $\{-\frac{1}{2}, -1, 1, \frac{1}{2}\}$-surgery, respectively.

Sketch of proof. We note first that $\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(\pm 1)) = 16 \mp 1$, and calculate that $\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(0)) = 16$. The result follows by induction on n: Applying the long exact sequence for Khovanov homology, we have

$$\begin{aligned}\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(1/n) &\leq \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(1/(n-1))) + \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(0)) \\ &= \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(1/(n-1))) + 16\end{aligned}$$

and

$$\begin{aligned}\operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(-1/n)) &\leq \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(-1/(n-1))) + \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(0)) \\ &= \operatorname{rk}\widetilde{\mathrm{Kh}}(\tau(-1/(n-1))) + 16.\end{aligned}$$

□

On the other hand, calculations of Khovanov homology for large torus knots are difficult to obtain. Indeed, the calculations given here were not accessible prior to the development of JavaKh. However, existing calculations suggest that $\mathrm{rk}\,\widetilde{\mathrm{Kh}}(T_{p,q})$ grows *at least* linearly in q. In particular, it seems reasonable to guess that surgery on $T_{2,5}$ provides an infinite family of examples proving the Theorem.

It would be interesting to understand the behaviour of the Khovanov homology for branch sets associated to $(1/n)$-surgery on the torus knots $T_{2,q}$ for $q \geq 5$.

References

[Bar-Natan and Green 2005] D. Bar-Natan and J. Green, "Khovanov homology: JavaKh", 2005, Available at http://katlas.math.toronto.edu/wiki/Khovanov_Homology.

[Bleiler 1985] S. A. Bleiler, "Prime tangles and composite knots", pp. 1–13 in *Knot theory and manifolds* (Vancouver, 1983), edited by D. Rolfsen, Lecture Notes in Math. **1144**, Springer, Berlin, 1985. MR 87e:57006 Zbl 0596.57003

[Bloom 2009] J. Bloom, "Odd Khovanov homology is mutation invariant", preprint, version 2, 2009. arXiv 0903.3746v2

[Boyer 2002] S. Boyer, "Dehn surgery on knots", pp. 165–218 in *Handbook of geometric topology*, edited by R. J. Daverman and R. B. Sher, North-Holland, Amsterdam, 2002. MR 2003f:57030 Zbl 1058.57004

[Heil 1974] W. Heil, "Elementary surgery on Seifert fiber spaces", *Yokohama Math. J.* **22** (1974), 135–139. MR 51 #11515 Zbl 0297.57006

[Khovanov 2000] M. Khovanov, "A categorification of the Jones polynomial", *Duke Math. J.* **101**:3 (2000), 359–426. MR 2002j:57025 Zbl 0960.57005

[Khovanov 2003] M. Khovanov, "Patterns in knot cohomology, I", *Experiment. Math.* **12**:3 (2003), 365–374. MR 2004m:57022 Zbl 1073.57007

[Milnor 1975] J. Milnor, "On the 3-dimensional Brieskorn manifolds $M(p,q,r)$", pp. 175–225 in *Knots, groups, and 3-manifolds*, edited by L. P. Neuwirth, Ann. of Math. Studies **84**, Princeton Univ. Press, 1975. MR 54 #6169 Zbl 0305.57003

[Montesinos 1975] J. M. Montesinos, "Surgery on links and double branched covers of S^3", pp. 227–259 in *Knots, groups, and* 3*-manifolds*, edited by L. P. Neuwirth, Ann. of Math. Studies **84**, Princeton Univ. Press, 1975. MR 52 #1699 Zbl 0325.55004

[Montesinos 1976] J. M. Montesinos, "Revêtements ramifiés de nœuds, espaces fibrés de Seifert et scindements de Heegaard", lecture notes, Orsay, 1976.

[Moser 1971] L. Moser, "Elementary surgery along a torus knot", *Pacific J. Math.* **38** (1971), 737–745. MR 52 #4287 Zbl 0202.54701

[Ozsváth and Szabó 2005] P. Ozsváth and Z. Szabó, "On the Heegaard Floer homology of branched double-covers", *Adv. Math.* **194**:1 (2005), 1–33. MR 2006e:57041 Zbl 1076.57013

[Rolfsen 1976] D. Rolfsen, *Knots and links*, Mathematics Lecture Series **7**, Publish or Perish, Berkeley, CA, 1976. MR 58 #24236 Zbl 0339.55004

[Saveliev 1999] N. Saveliev, *Lectures on the topology of* 3*-manifolds*, de Gruyter, Berlin, 1999. MR 2001h:57024 Zbl 0932.57001

[Schreier 1924] O. Schreier, "Über die Gruppen $A^a B^b = 1$", *Hamb. Math. Abh.* **3** (1924), 167–169. JFM 50.0070.01

[Scott 1983] P. Scott, “The geometries of 3-manifolds”, *Bull. London Math. Soc.* **15**:5 (1983), 401–487. MR 84m:57009 Zbl 0561.57001

[Seifert 1933] H. Seifert, “Topologie Dreidimensionaler Gefaserter Räume”, *Acta Math.* **60**:1 (1933), 147–238. MR 1555366 Zbl 0006.08304

[Waldhausen 1969] F. Waldhausen, “Über Involutionen der 3-Sphäre”, *Topology* **8** (1969), 81–91. MR 38 #5209 Zbl 0185.27603

[Watson 2008] L. Watson, “Surgery obstructions from Khovanov homology”, preprint, version 3, 2008. arXiv 0807.1341v3

[Wehrli 2007] S. Wehrli, “Mutation invariance of Khovanov homology over $\mathbb{Z}_2$”, lecture notes, Kyoto, 2007.

[Wehrli 2009] S. Wehrli, “Mutation invariance of Khovanov homology over $\mathbb{F}_2$”, preprint, 2009. arXiv 0904.3401

Received September 9, 2009. Revised October 20, 2009.

LIAM WATSON
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
520 PORTOLA PLAZA
LOS ANGELES, CA 90095
UNITED STATES

lwatson@math.ucla.edu
http://www.math.ucla.edu/~lwatson

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

L^p RICCI CURVATURE PINCHING THEOREMS FOR CONFORMALLY FLAT RIEMANNIAN MANIFOLDS

HONG-WEI XU AND EN-TAO ZHAO

Dedicated to Professor Katsuhiro Shiohama on the occasion of his 70th birthday.

Let M be an n-dimensional complete locally conformally flat Riemannian manifold with constant scalar curvature R and $n \geq 3$. We first prove that if $R = 0$ and the $L^{n/2}$ norm of the Ricci curvature tensor of M is pinched in $[0, C_1(n))$, then M is isometric to a complete flat Riemannian manifold, which improves Pigola, Rigoli, and Setti's pinching theorem. Next, we prove that if $n \geq 6$, $R \neq 0$, and the $L^{n/2}$ norm of the trace-free Ricci curvature tensor of M is pinched in $[0, C_2(n))$, then M is isometric to a space form. Finally, we prove an L^n trace-free Ricci curvature pinching theorem for complete locally conformally flat Riemannian manifolds with constant nonzero scalar curvature. Here $C_1(n)$ and $C_2(n)$ are explicit positive constants depending only on n.

1. Introduction

The curvature pinching phenomenon plays an important role in global differential geometry. Motivated by the famous pinching theorem for minimal submanifolds in a sphere due to J. Simons [1968], C. L. Shen [1989] proved an L^p pinching theorem for embedded compact minimal hypersurfaces in $\mathbb{S}^{n+1}(1)$. Many authors have extended this result [Wang 1988; Lin and Xia 1989; Xu 1990; 1994; Bérard 1991; Shiohama and Xu 1994; Ni 2001; Xu and Gu 2007a; 2007b], but by producing extrinsic rigidity theorems for submanifolds. We are interested in intrinsic L^p pinching problems for Riemannian manifolds.

A conformally flat structure on a Riemannian manifold is a natural generalization of a conformal structure of a Riemannian surface. A Riemannian manifold (M, g) is locally conformally flat with a locally conformally flat structure on M

MSC2000: primary 53C20; secondary 53C25.

Keywords: conformally flat manifold, rigidity, Ricci curvature tensor, L^p pinching problem, space form.

Supported by the NSFC, grant 10771187; the Trans-Century Training Programme Foundation for Talents by the Ministry of Education of China; the Natural Science Foundation of Zhejiang Province, grant 101037; and the China Postdoctoral Science Foundation, grant 20090461379.

if and only if there exists a coordinate chart $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ covering M such that $(\varphi_\alpha^{-1})^* g = \rho_\alpha dx^2$ for every $\alpha \in \Lambda$, where dx^2 is the Euclidean metric on $\mathbb{R}^n$ and ρ_α is a positive function on $\mathbb{R}^n$. It is well known that a Riemannian surface is always locally conformally flat. In higher dimensions, however, not every manifold admits a locally conformally flat structure, and it is difficult to give a good classification of locally conformally flat manifolds. Throughout this paper, we always assume that M is an n-dimensional complete Riemannian manifold with $n \geq 3$. According to the decomposition of the Riemannian curvature tensor, a locally conformally flat manifold has constant sectional curvature if and only if it is Einstein, that is, the trace-free Ricci tensor, defined by $\widetilde{\mathrm{Ric}} = \mathrm{Ric} - (R/n)g$, is identically equal to zero, where Ric is the Ricci curvature tensor and R is the scalar curvature. As a consequence, by the Hopf classification theorem, space forms are the only locally conformally flat Einstein manifolds.

In [1967], M. Tani showed that the universal cover of a compact oriented locally conformally flat manifold with positive Ricci curvature and constant scalar curvature is isometrically a sphere. This result has been generalized by other mathematicians to the case where M satisfies some pointwise pinching condition. Recently, S. Pigola, M. Rigoli and A. G. Setti characterized a simply connected space form with a pointwise Ricci curvature pinching condition:

Theorem A [Pigola et al. 2007]. *For $n \geq 3$, let (M, g) be a complete simply connected and locally conformally flat Riemannian n-manifold with constant scalar curvature $R > 0$. If $|\mathrm{Ric}|^2 \leq R^2/(n-1)$ on M and the strict inequality holds at some point, then M is isometric to a sphere.*

Q. M. Cheng, S. Ishikawa and K. Shiohama [1999] completely classified three-dimensional complete and locally conformally flat Riemannian manifolds whose scalar curvature and norm of the Ricci curvature tensor are positive constants. Can the pointwise pinching conditions be replaced by global pinching ones? In [2007], Pigola, Rigoli and Setti got a global pinching result that can be considered as an extension of the theorem above:

Theorem B [Pigola et al. 2007]. *For $n \geq 3$, let (M, g) be a complete simply connected and locally conformally flat Riemannian n-manifold with zero scalar curvature and $n \geq 3$. If $\|\mathrm{Ric}\|_{n/2} < C(n)$, then M is isometric to Euclidean space. Here $\|\cdot\|_k$ denotes the L^k norm and $C(n) = 2n^{-5/2}(n-1)^{1/2}(n-2)^3 \omega_n^{2/n}$, with ω_n the volume of the unit sphere $\mathbb{S}^n$,*

Suppose that M is locally conformally flat with constant scalar curvature R. In Section 3, we will first prove that if $R = 0$ and the $L^{n/2}$ norm of the Ricci curvature tensor of M is pinched in $[0, C_1(n))$ for some explicit positive constant $C_1(n)$ depending only on n, then M is isometric to a complete flat Riemannian manifold, which improves Pigola, Rigoli, and Setti's pinching theorem. Secondly, we prove that if $n \geq 6$, $R \neq 0$, and the $L^{n/2}$ norm of the trace-free Ricci curvature tensor of M is pinched in $[0, C_2(n))$ for some explicit positive constant $C_2(n)$ depending only on n, then M is isometric to a space form. Finally, we prove an L^n trace-free Ricci curvature pinching theorem for complete locally conformally flat Riemannian manifolds with nonzero constant scalar curvature.

2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$, and let $\{e_1, e_2, \dots, e_n\}$ be a local orthonormal basis of the tangent space of M. We define the Kulkarni–Nomizu product $\odot$ for symmetric 2-tensors α and β in local coordinates by

$$(\alpha \odot \beta)_{ijkl} = \alpha_{ik}\beta_{jl} + \alpha_{jl}\beta_{ik} - \alpha_{il}\beta_{jk} - \alpha_{jk}\beta_{il}.$$

The Riemannian curvature tensor can be decomposed as

$$\mathrm{Rm} = \frac{R}{2(n-1)(n-2)} g \odot g - \frac{1}{n-2} \mathrm{Ric} \odot g + W, \tag{2-1}$$

where Rm, W, Ric, and R are respectively the Riemannian curvature tensor, the Weyl curvature tensor, the Ricci curvature tensor and the scalar curvature of M. It was shown in [Eisenhart 1997] that if $n \geq 4$, then M is locally conformally flat if and only if the Weyl tensor vanishes, and if $n = 3$, then M is locally conformally flat if and only if $\nabla \mathrm{Ric}$ is totally symmetric. If M is locally conformally flat, we see from (2-1) that the Riemannian curvature tensor can be expressed in terms of the Ricci curvature tensor by

$$\begin{aligned} R_{ijkl} = \frac{R}{(n-1)(n-2)} &(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ &- \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}), \end{aligned} \tag{2-2}$$

where R_{ijkl} and R_{ij} are components of Rm and Ric in local orthonormal frame fields. We define the trace-free Ricci curvature tensor by $\widetilde{\mathrm{Ric}} = \sum_{i,j} \tilde{R}_{ij}\omega_i \otimes \omega_j$, where $\{\omega_1, \omega_2, \dots, \omega_n\}$ is the frame dual to $\{e_1, e_2, \dots, e_n\}$, and

$$\tilde{R}_{ij} = R_{ij} - (R/n)\delta_{ij}. \tag{2-3}$$

Putting $S = |\mathrm{Ric}|^2$ and $\tilde{S} = |\widetilde{\mathrm{Ric}}|^2$, we have $\tilde{S} = S - R^2/n$ from (2-3). If R is constant, then R_{ij} and $\tilde{R}_{ij}$ are Codazzi tensors, that is, $\nabla_j R_{ik} = \nabla_k R_{ij}$ and $\nabla_j \tilde{R}_{ik} = \nabla_k \tilde{R}_{ij}$ for $1 \leq i, j, k \leq n$.

Lemma 2.1. *Let (M, g) be a locally conformally flat Riemannian n-manifold with constant scalar curvature. Set $f_\tau = (\tilde{S} + n\tau^2)^{1/2}$, where $\tau \in \mathbb{R}^+$. Then*

$$|\nabla \widetilde{\mathrm{Ric}}|^2 \geq \frac{n+2}{n} |\nabla f_\tau|^2. \tag{2-4}$$

Proof. Putting $x_{ij} = \tilde{R}_{ij} + \tau\delta_{ij}$, we have $\nabla_k x_{ij} = \nabla_k \tilde{R}_{ij}$ and hence

$$\sum_{i,j,k} (\nabla_k x_{ij})^2 = \sum_{i,j,k} (\nabla_k \tilde{R}_{ij})^2. \tag{2-5}$$

Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal frame such that $\tilde{R}_{ij} = \lambda_i \delta_{ij}$ for $1 \le i, j \le n$. Since $f_\tau = (\tilde{S} + n\tau^2)^{1/2}$, we get $x_{ij} = (\lambda_i + \tau)\delta_{ij}$ and $\sum_{i,j} x_{ij}^2 = f_\tau^2$. Then

$$\begin{aligned}(2f_\tau|\nabla f_\tau|)^2 = |\nabla f_\tau^2|^2 &= 4\sum_k \Big(\sum_i x_{ii}\nabla_k x_{ii}\Big)^2 \\ &\le 4\Big(\sum_i x_{ii}^2\Big)\Big(\sum_{i,k}(\nabla_k x_{ii})^2\Big) = 4f_\tau^2\Big(\sum_{i,k}(\nabla_k x_{ii})^2\Big).\end{aligned} \tag{2-6}$$

On the other hand, we have

$$\sum_{i,j,k} (\nabla_k x_{ij})^2 \ge 2\sum_{i\ne k} (\nabla_k x_{ii})^2 + \sum_{i,k} (\nabla_k x_{ii})^2. \tag{2-7}$$

For each fixed k, we have

$$\begin{aligned}\sum_i (\nabla_k x_{ii})^2 &= \sum_{i\ne k} (\nabla_k x_{ii})^2 + \Big(\sum_i \nabla_k x_{ii} - \sum_{i\ne k} \nabla_k x_{ii}\Big)^2 \\ &= \sum_{i\ne k} (\nabla_k x_{ii})^2 + \Big(\sum_{i\ne k} \nabla_k x_{ii}\Big)^2 \\ &\le \sum_{i\ne k} (\nabla_k x_{ii})^2 + (n-1)\sum_{i\ne k} (\nabla_k x_{ii})^2.\end{aligned} \tag{2-8}$$

Combining (2-5), (2-6), (2-7) and (2-8), we obtain

$$\sum_{i,j,k} (\nabla_k \tilde{R}_{ij})^2 \ge \frac{n+2}{n} \sum_{i,k} (\nabla_k x_{ii})^2 \ge \frac{n+2}{n} |\nabla f_\tau|^2.$$

So $|\nabla\widetilde{\mathrm{Ric}}|^2 \ge ((n+2)/n)|\nabla f_\tau|^2$. □

We see that $\mathrm{tr}(\widetilde{\mathrm{Ric}}^3) = \sum_{i,j,k} \tilde{R}_{ij}\tilde{R}_{jk}\tilde{R}_{ki}$. Following [Pigola et al. 2007], we have

$$\tfrac{1}{2}\triangle\tilde{S} = |\nabla\widetilde{\mathrm{Ric}}|^2 + \frac{n}{n-2}\,\mathrm{tr}(\widetilde{\mathrm{Ric}}^3) + \frac{R}{n-1}\tilde{S}. \tag{2-9}$$

By using the Lagrange multiplier method, we have the inequality

$$\mathrm{tr}(\widetilde{\mathrm{Ric}}^3) \ge -\frac{n-2}{\sqrt{n(n-1)}}\tilde{S}^{3/2}. \tag{2-10}$$

Putting $f_\tau = (\tilde{S} + n\tau^2)^{1/2} = (|\widetilde{\mathrm{Ric}}|^2 + n\tau^2)^{1/2}$, $f = (\tilde{S})^{1/2}$, from (2-4), (2-9) and (2-10) we have

$$\tfrac{1}{2}\triangle f^2 \geq \frac{n+2}{n}|\nabla f_\tau|^2 - \sqrt{\frac{n}{n-1}}f^3 + \frac{R}{n-1}f^2. \tag{2-11}$$

Lemma 2.2 [Hebey 1999]. *For $n \geq 3$, let (M, g) be a smooth complete locally conformally flat Riemannian n-manifold. Then for any smooth function f with compact support*,

$$\Big(\int_M |f|^{2n/(n-2)} dM\Big)^{(n-2)/n} \leq \frac{4}{n(n-2)w_n^{2/n}}\Big(\int_M |\nabla f|^2 dM + \frac{n-2}{4(n-1)}\int_M Rf^2 dM\Big). \tag{2-12}$$

3. L^p Ricci curvature pinching theorems

Theorem 3.1. *Let (M, g) be a complete locally conformally flat Riemannian n-manifold with constant scalar curvature R. Put*

$$C_1(n) = 2n^{-5/2}(n-1)^{1/2}(n-2)(n^2-2n+4)w^{2/n},$$
$$C_2(n) = \sqrt{n(n-1)}\,w_n^{2/n}.$$

(i) *If $n \geq 3$, $R = 0$, and $\|\mathrm{Ric}\|_{n/2} < C_1(n)$, then M is isometric to a complete flat Riemannian manifold. In particular, if M is simply connected, then M is isometric to the Euclidean space $\mathbb{R}^n$.*

(ii) *If $n \geq 6$, $R = n(n-1)c \neq 0$, and $\|\widetilde{\mathrm{Ric}}\|_{n/2} < C_2(n)$, then M is isometric to a space form. In particular, if M is simply connected, then M is isometric to either the sphere $\mathbb{S}^n(1/\sqrt{c})$ with radius $1/\sqrt{c}$ if $c > 0$, or the hyperbolic space $\mathbb{H}^n(c)$ with constant curvature c if $c < 0$.*

Proof. Since $\triangle f^2 = \triangle f_\tau^2$, from (2-11) we have

$$0 \geq -f_\tau \triangle f_\tau - \sqrt{\frac{n}{n-1}}f^3 + \frac{2}{n}|\nabla f_\tau|^2 + \frac{R}{n-1}f^2. \tag{3-1}$$

We choose a cut-off function $\phi_r \in C^\infty(M)$ such that

$$\begin{cases} \phi_r(x) = 1 & \text{if } x \in B_r(q), \\ \phi_r(x) = 0 & \text{if } x \in M \setminus B_{2r}(q), \\ \phi_r(x) \in [0, 1] \text{ and } |\nabla\phi_r| \leq 1/r & \text{if } x \in B_{2r}(q) \setminus B_r(q), \end{cases} \tag{3-2}$$

where $B_r(q)$ is the geodesic ball in M with radius r centered at $q \in M$. In particular, if M is compact, and if $r \geq d$, where d is the diameter of M, then $\phi_r \equiv 1$ on M.

Multiplying both sides of (3-1) by $\phi_r^2 f_\tau^{n/2-2}$ and integrating by parts we get

$$
\begin{aligned}
\text{(3-3)}\quad 0 \ge 2\int \phi_r f_\tau^{n/2-1}\langle\nabla\phi_r,\nabla f_\tau\rangle dM + \frac{8(n-2)}{n^2}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM \\
-\sqrt{\frac{n}{n-1}}\int \phi_r^2 f_\tau^{n/2-2} f^3 dM + \frac{R}{n-1}\int \phi_r^2 f_\tau^{n/2-2} f^2 dM \\
+\frac{32}{n^3}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM \\
= \frac{8(n^2-2n+4)}{n^3}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM + (\sigma+2)\int \phi_r f_\tau^{n/2-1}\langle\nabla\phi_r,\nabla f_\tau\rangle dM \\
-\sigma\int \phi_r f_\tau^{n/2-1}\langle\nabla\phi_r,\nabla f_\tau\rangle dM - \sqrt{n/n-1}\int \phi_r^2 f_\tau^{n/2-2} f^3 dM \\
+\frac{R}{n-1}\int \phi_r^2 f_\tau^{n/2-2} f^2 dM \\
\ge \frac{8(n^2-2n+4)}{n^3}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM + (\sigma+2)\int \phi_r f_\tau^{n/2-1}\langle\nabla\phi_r,\nabla f_\tau\rangle dM \\
-\frac{8\sigma\rho}{n^2}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM - \frac{\sigma}{2\rho}\int f_\tau^{n/2}|\nabla\phi_r|^2 dM \\
-\sqrt{\frac{n}{n-1}}\int \phi_r^2 f_\tau^{n/2-2} f^3 dM + \frac{R}{n-1}\int \phi_r^2 f_\tau^{n/2-2} f^2 dM \\
= \Big(\frac{8(n^2-2n+4)}{n^3} - \frac{8\sigma\rho}{n^2}\Big)\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM \\
+\frac{2(\sigma+2)}{n}\int \frac{n}{2}\phi_r f_\tau^{n/2-1}\langle\nabla\phi_r,\nabla f_\tau\rangle dM \\
-\frac{\sigma}{2\rho}\int f_\tau^{n/2}|\nabla\phi_r|^2 dM - \sqrt{\frac{n}{n-1}}\int \phi_r^2 f_\tau^{n/2-2} f^3 dM \\
+\frac{R}{n-1}\int \phi_r^2 f_\tau^{n/2-2} f^2 dM,
\end{aligned}
$$

for arbitrary positive constants σ and ρ, where here and below the measure dM implies integration over M.

By a direct computation, we have

$$
\text{(3-4)}\qquad |\nabla(\phi_r f_\tau^{n/4})|^2 = f_\tau^{n/2}|\nabla\phi_r|^2 + \frac{n}{2}\phi_r f_\tau^{n/2-1}\langle\nabla\phi_r,\nabla f_\tau\rangle + \phi_r^2|\nabla f_\tau^{n/4}|^2.
$$

Choose $\rho > 0$ such that

$$
\frac{8(n^2-2n+4)}{n^3} - \frac{8\sigma\rho}{n^2} = \frac{2(\sigma+2)}{n},
$$

so that $\rho=((2-\sigma)n^2-8n+16)/(4n\sigma)$. Since $\rho>0$, we have $\sigma<2(n^2-4n+8)/n^2$. By (3-3) and (3-4) we obtain

$$\begin{aligned}
0 &\geq \frac{2(\sigma+2)}{n}\int\Big(\frac{n}{2}\phi_r f_\tau^{n/2-1}\langle\nabla\phi_r,\nabla f_\tau\rangle+\phi_r^2|\nabla f_\tau^{n/4}|^2\Big)dM \\
&\qquad -\frac{\sigma}{2\rho}\int f_\tau^{n/2}|\nabla\phi_r|^2dM-\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM \\
&\qquad\qquad +\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM \\
&= \frac{2(\sigma+2)}{n}\int(|\nabla(\phi_r f_\tau^{n/4})|^2-f_\tau^{n/2}|\nabla\phi_r|^2)dM-\frac{\sigma}{2\rho}\int f_\tau^{n/2}|\nabla\phi_r|^2dM \\
&\qquad -\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM+\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM \\
&= \frac{2(\sigma+2)}{n}\int|\nabla(\phi_r f_\tau^{n/4})|^2dM-\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM \\
&\qquad +\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f_\tau^{n/2}|\nabla\phi_r|^2dM.
\end{aligned}$$

This together with the Sobolev inequality in Lemma 2.2 implies

$$\begin{aligned}
0 &\geq \frac{2(\sigma+2)}{n}\Big(\frac{n(n-2)w_n^{2/n}}{4}\|\phi_r^2 f_\tau^{n/2}\|_{n/(n-2)}-\frac{(n-2)R}{4(n-1)}\|\phi_r f_\tau^{n/4}\|_2^2\Big) \\
&\qquad -\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM+\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM \\
&\qquad\qquad -\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f_\tau^{n/2}|\nabla\phi_r|^2dM \\
&= \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f_\tau^{n/2}\|_{n/(n-2)}-\frac{(\sigma+2)(n-2)R}{2n(n-1)}\|\phi_r^2 f_\tau^{n/2}\|_1 \\
&\qquad -\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM+\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM \\
&\qquad\qquad -\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f_\tau^{n/2}|\nabla\phi_r|^2dM.
\end{aligned}$$

As $\tau\to 0$, this inequality becomes

$$\begin{aligned}
0 &\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} \\
&\qquad +\Big(\frac{R}{n-1}-\frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{n/2}\|_1 \\
&\qquad -\sqrt{\frac{n}{n-1}}\int\phi_r^2 f^{n/2+1}dM-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2dM.
\end{aligned}\tag{3-5}$$

(i) When $R=0$, (3-5) implies

$$
\begin{aligned}
0 \geq{}& \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2+1} dM \\
&\qquad - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM \\
\geq{}& \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \sqrt{\frac{n}{n-1}}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}\|f\|_{n/2} \\
&\qquad - \frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2} dM \\
\geq{}& \Big(\frac{(\sigma+2)(n-2)w_n^{2/n}}{2} - \sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\|\phi_r^2 f^{n/2}\|_{n/(n-2)} \\
&\qquad - \frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2} dM.
\end{aligned}
$$

Put $\sigma = 2(n^2-4n+8)/n^2-\varepsilon$, where ε is a positive constant. It follows from the assumption $\int f^{n/2}dM<\infty$ that

$$\lim_{r\to+\infty}\frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}dM = 0.$$

Combining the last two results, we get

$$0\geq\Big(\frac{(4(n^2-2n+4)-n^2\varepsilon)(n-2)w_n^{2/n}}{2n^2}-\sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}$$

for any $\varepsilon>0$. As $\varepsilon\to 0$, we have

$$0\geq\Big(\frac{4(n^2-2n+4)(n-2)w_n^{2/n}}{2n^2}-\sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)},$$

which implies

$$0\geq (C_1(n)-\|f\|_{n/2})\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}.$$

Hence $\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}=0$, that is, $f\equiv 0$. This means that M is an Einstein manifold and is therefore a flat Riemannian manifold. In particular, if M is simply connected, then M is isometric to the Euclidean space $\mathbb{R}^n$.

(ii) When $R\neq 0$, set

$$\frac{1}{n-1}=\frac{(\sigma+2)(n-2)}{2n(n-1)},$$

so that $\sigma = 4/(n-2)$. Since $n \geq 6$, we have $\sigma < 2(n^2-4n+8)/n^2$. Then (3-5) becomes

$$\begin{aligned}
0 &\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2+1} dM \\
&\qquad\qquad - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM \\
&\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \sqrt{\frac{n}{n-1}}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}\|f\|_{n/2} \\
&\qquad\qquad - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM \\
&= \Big(\frac{(\sigma+2)(n-2)w_n^{2/n}}{2} - \sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\|\phi_r^2 f^{n/2}\|_{n/(n-2)} \\
&\qquad\qquad - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM.
\end{aligned}$$

Since $|\nabla\phi_r| \leq 1/r$ for any $r > 0$, this can be rewritten as

$$0 \geq \Big(nw_n^{2/n} - \sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{n/2} dM. \tag{3-6}$$

Since ρ and σ are constants depending only on n, so is $2(\sigma+2)/n+\sigma/(2\rho)$. From the assumption that f has finite $L^{n/2}$ norm, we get

$$\lim_{r\to+\infty}\frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{n/2} dM = 0. \tag{3-7}$$

We see from (3-6) and (3-7)

$$0 \geq (C_2(n) - \|f\|_{n/2})\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)},$$

which implies $\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} = 0$, that is, $f = 0$. Hence M is an Einstein manifold and a space form. In particular, if M is simply connected, it is isometric to the sphere $\mathbb{S}^n(1/\sqrt{c})$ if $c > 0$ or the hyperbolic space $\mathbb{H}^n(c)$ if $c < 0$. □

Remark 3.2. When $R=0$, the pinching constant $C_1(n)$ is better than Pigola, Rigoli and Setti's constant.

Corollary 3.3. *For $n \geq 6$, suppose (M, g) is a complete locally conformally flat Riemannian n-manifold with constant scalar curvature, and let $C_2(n)$ be as in Theorem 3.1. If $\|\mathrm{Ric}\|_{n/2} < C_2(n)$, then M is isometric to a complete flat Riemannian manifold. In particular, if M is simply connected, then M is isometric to Euclidean $\mathbb{R}^n$.*

Lemma 3.4. *For $n \geq 3$, let (M, g) be a complete locally conformally flat Riemannian n-manifold with constant scalar curvature. If $\int_M (S - R^2/n)^{n/2} < +\infty$, then for any $\varepsilon > 0$, there is a compact set Ω_ε such that $\tilde{S} < \varepsilon$ in $M \setminus \Omega_\varepsilon$.*

Proof. By (2-9) and (2-10), we have in the sense of distribution the inequality

$$-\triangle f \leq \sqrt{\frac{n}{n-1}} f^2 - \frac{R}{n-1} f \leq \frac{\sqrt{n}\varepsilon}{2\sqrt{n-1}} f^3 + \left(\frac{\sqrt{n}}{2\varepsilon\sqrt{n-1}} - \frac{R}{n-1}\right) f.$$

Putting $\varepsilon = \sqrt{n-1}/(2\sqrt{n})$, we have $-\triangle f \leq af^3 + bf$, where $a = 1/4$ and $b = (n-R)/(n-1)$. On the other hand, we have the inequality

$$\left(\int |f|^{2n/(n-2)} dM\right)^{(n-2)/n} \leq \frac{4}{n(n-2)w_n^{2/n}} \left(\int |\nabla f|^2 dM + \frac{n-2}{4(n-1)} \int Rf^2 dM\right).$$

By the proof of [Bérard et al. 1998, Theorem 4.1], we conclude that, for any $\varepsilon > 0$, there is a compact set Ω_ε such that $\tilde{S} < \varepsilon$ in $M \setminus \Omega_\varepsilon$. □

Lemma 3.5. *For $n \geq 3$, let (M, g) be a complete locally conformally flat Riemannian n-manifold with positive constant scalar curvature. If $\|\widetilde{\mathrm{Ric}}\|_n < +\infty$, then M must be compact.*

Proof. Take a local orthonormal frame $\{e_i\}$ such that $R_{ij} = \lambda_i \delta_{ij}$. From (2-2) we have

$$R_{ijij} = \frac{\tilde{\lambda}_i + \tilde{\lambda}_j}{n-2} + \frac{R}{n(n-1)},$$

where $\tilde{\lambda}_i = \lambda_i - R/n$ for $i = 1, 2, \ldots, n$ are eigenvalues of $\widetilde{\mathrm{Ric}}$. Note that R is positive. We see from Lemma 3.4 that there is a positive constant δ such that $K_M > \delta$ in $M \setminus \Omega$ for some compact set Ω.

Since M is complete, it suffices to show that M is bounded. Otherwise, there is a point $p_1 \in M$ such that $d(p_1, \Omega) = \inf_{q \in \Omega} d(p_1, q) > \pi/\sqrt{\delta}$. Since Ω is compact, there is a point $p_2 \in M$ such that $d(p_1, p_2) = d(p_1, \Omega)$. Let $\gamma : [0, s_1] \to M$ be a minimizing geodesic parameterized by arclength such that $\gamma(0) = p_1$ and $\gamma(s_1) = p_2$, where $s_1 = d(p_1, p_2)$. Then $\gamma(t) \in M \setminus \Omega$ for $t < s_1$. Pick $p_3 \in \gamma$ so that $\pi/\sqrt{\delta} < d(p_1, p_3) = s_2 < s_1$. Then $\gamma : [0, s_2] \to M$ is also a minimizing geodesic with $\gamma(s_2) = p_3$. Let $E(s)$ for $s \in [0, s_2]$ be a parallel field along $\gamma : [0, s_2] \to M$ such that $E(0) \perp \gamma'(0)$ and $|E(0)| = 1$. According to [Wu et al. 1989], there exists a piecewise smooth function $\psi : [0, \sqrt{\delta} s_2] \to \mathbb{R}$ satisfying

$$\int_0^{\sqrt{\delta} s_2} (\psi')^2 dt < \int_0^{\sqrt{\delta} s_2} \psi^2 dt,$$

where $\sqrt{\delta}s_2 > \pi$. Setting $X(t) = \psi(\sqrt{\delta}t)E(t)$, we have

$$\begin{aligned} I(X,X) &= \int_0^{s_2} (\langle X'(t), X'(t)\rangle - \langle R(\gamma'(t), X(t))X(t), \gamma'(t)\rangle)dt \\ &= \int_0^{s_2} \big(\delta\psi'(\sqrt{\delta}t)^2 - K(\gamma'(t), E(t))\psi^2(\sqrt{\delta}t)\big)dt, \end{aligned}$$

where $K(\gamma'(t), E(t))$ is the sectional curvature of the tangent plane spanned by $\gamma'(t)$ and $E(t)$. Since $K_M > \delta$ in $M \setminus \Omega$, we have

$$\begin{aligned} I(X,X) &\le \int_0^{s_2} \delta((\psi'(\sqrt{\delta}t))^2 - \psi^2(\sqrt{\delta}t))dt \\ &= \sqrt{\delta}\int_0^{\sqrt{\delta}s_2} ((\psi')^2 - \psi^2)dt < 0. \end{aligned}$$

On the other hand, since $\gamma : [0, s_2] \to M$ is a minimizing geodesic, we have $I(X, X) \ge 0$, which is a contradiction. Hence M is bounded and compact. □

Corollary 3.6 (of Lemma 3.5). *For $n \ge 3$, let (M, g) be a complete noncompact locally conformally flat Riemannian n-manifold with nonnegative constant scalar curvature. If $\|\widetilde{\mathrm{Ric}}\|_n < +\infty$, then M must be scalar flat.*

Theorem 3.7. *Let (M, g) be a complete locally conformally flat Riemannian n-manifold with constant scalar curvature R. Put*

$$\begin{aligned} C_3(n) &= 2n^{-5/2}(n-1)^{1/2}(n-2)^{1/2}(n^2-n+4)^{1/2}(3n^2-4n+4)^{1/2}w_n^{1/n}, \\ C_4(n) &= 2\sqrt{2}n^{-5/2}(n-1)^{1/2}(n-2)^{1/2}(n^2-2n+4)^{1/2} \\ &\qquad \cdot (n^3-8n^2+16n-16)^{1/2}w_n^{1/n}. \end{aligned}$$

(i) *If $n \ge 3$, $R = n(n-1)$, and $\|\widetilde{\mathrm{Ric}}\|_n < C_3(n)$, then M is isometric to a spherical space form. In particular, if M is simply connected, then M is isometric to $\mathbb{S}^n$.*

(ii) *If $n \ge 6$, $R = -n(n-1)$, $\|\widetilde{\mathrm{Ric}}\|_n < C_4(n)$ and $\|\widetilde{\mathrm{Ric}}\|_{n/2} < +\infty$, then M is isometric to a hyperbolic space form. In particular, if M is simply connected, then M is isometric to $\mathbb{H}^n$.*

Proof. (i) When $R = n(n-1)$, we see from Lemma 3.5 that M is compact. Since $\triangle f^2 = \triangle f_\tau^2$, we have from (2-11)

$$\tfrac{1}{2}\triangle f_\tau^2 \ge \frac{n+2}{n}|\nabla f_\tau|^2 - \sqrt{\frac{n}{n-1}}f^3 + \frac{R}{n-1}f^2. \tag{3-8}$$

Multiplying both sides of (3-8) by f_τ^{n-2} and integrating by parts we get

$$
\begin{aligned}
0 &\geq \frac{1}{2}\int \langle \nabla f_\tau^{n-2}, \nabla f_\tau^2\rangle dM + \frac{4(n+2)}{n^3}\int |\nabla f_\tau^{n/2}|^2 dM \\
&\qquad\qquad - \sqrt{\frac{n}{n-1}}\int f_\tau^{n-2} f^3 dM + \frac{R}{n-1}\int f_\tau^{n-2} f^2 dM \\
&= \frac{4(n-2)}{n^2}\int |\nabla f_\tau^{n/2}|^2 dM + \frac{4(n+2)}{n^3}\int |\nabla f_\tau^{n/2}|^2 dM \\
&\qquad - \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\int f_\tau^{n-2} f^4 dM + \frac{R}{n-1}\int f_\tau^{n-2} f^2 dM \\
&\qquad\qquad - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\int f_\tau^{n-2} f^2 dM \\
&\geq \frac{4(n^2-n+2)}{n^3}\int |\nabla f_\tau^{n/2}|^2 dM - \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\|f_\tau^{n-2} f^2\|_{n/(n-2)} \\
&\qquad + \frac{R}{n-1}\int f_\tau^{n-2} f^2 dM - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\int f_\tau^{n-2} f^2 dM,
\end{aligned}
\tag{3-9}
$$

for any $\varepsilon > 0$. By applying (2-12) to $f_\tau^{n/2}$, we get

$$
\int_M |\nabla f_\tau^{n/2}|^2 dM \geq \frac{n(n-2)w_n^{2/n}}{4}\|f_\tau^n\|_{n/(n-2)} - \frac{(n-2)R}{4(n-1)}\|f_\tau^n\|_1. \tag{3-10}
$$

Substituting (3-10) into (3-9) and letting $\tau \to 0$, we have

$$
\begin{aligned}
0 \geq \left(\frac{(n-2)(n^2-n+4)w_n^{2/n}}{n^2} - \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\right)\|f^n\|_{n/(n-2)} \\
+ \left(\frac{(3n^2-4n+4)R}{n^3(n-1)} - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\right)\|f^n\|_1.
\end{aligned}
\tag{3-11}
$$

Set $\varepsilon = 2n^{-5/2}(n-2)^{1/2}(3n^2-4n+4)$. Since $R = n(n-1)$, from (3-11) we get

$$
0 \geq (C_3(n)^2 - \|f^2\|_{n/2})\|f^n\|_{n/(n-2)},
$$

which implies $\|f^n\|_{n/(n-2)} = 0$, that is, $f \equiv 0$. Hence M is an Einstein manifold, which implies that M is isometric to a spherical space form. In particular, if M is simply connected, then M is isometric to $\mathbb{S}^n$.

(ii) When $R=-n(n-1)$, we choose a cut-off function $\phi_r \in C^\infty(M)$ satisfying the conditions of (3-2). Following the proof of Theorem 3.1, we have

$$\begin{aligned}
0 &\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}+\Big(\frac{R}{n-1}-\frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{n/2}\|_1\\
&\qquad -\sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2+1}dM-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM\\
&\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}+\Big(\frac{R}{n-1}-\frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{n/2}\|_1\\
&\qquad -\frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2}dM-\frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2+2}dM\\
&\qquad -\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r^2|^2 dM\\
&\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}+\Big(\frac{R}{n-1}-\frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{n/2}\|_1\\
&\qquad -\frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2}dM-\frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}\\
&\qquad -\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r^2|^2 dM\\
&\geq \Big(\frac{(\sigma+2)(n-2)w_n^{2/n}}{2}-\frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\Big)\|\phi_r^2 f^{n/2}\|_{n/(n-2)}\\
&\qquad +\Big(\frac{(4-(n-2)\sigma)R}{2n(n-1)}-\frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\Big)\|\phi_r^2 f^{n/2}\|_1\\
&\qquad -\frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}dM.
\end{aligned}$$

Put

$$\sigma=\frac{2(n^2-4n+8)}{n^2}-\eta \quad\text{and}\quad \varepsilon=\frac{1}{2}\sqrt{\frac{n}{n-1}}\times\frac{2n(n-1)}{(4-(n-2)\sigma)R},$$

where η is a positive constant. We see that if $n\geq 6$, then $\varepsilon>0$ for sufficiently small η. When $n\geq 6$ and η is sufficiently small, the second term of the right side of the last calculation vanishes. Since f has finite $L^{n/2}$ norm, we have

$$\lim_{r\to+\infty}\frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{n/2}dM=0.$$

By combining this with the previous calculation, we obtain

$$0\geq\Big(\frac{(\sigma+2)(n-2)w_n^{2/n}}{2}-\frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}.$$

Noting that $R = -n(n-1)$ and letting $\eta \to 0$, this becomes

$$0 \geq (C_4(n))^2 - \|f^2\|_{n/2}) \lim_{r\to+\infty} \|\phi_r^2 f^{n/2}\|_{n/(n-2)},$$

which implies that $\lim_{r\to+\infty} \|\phi_r^2 f^{n/2}\|_{n/(n-2)} = 0$, that is, $f = 0$. Hence M is an Einstein manifold and is isometric to a hyperbolic space form. In particular, if M is simply connected, then M is isometric to $\mathbb{H}^n$. □

Corollary 3.8 (of Theorem 3.7). *For $n \geq 3$, let (M, g) be a complete simply connected and locally conformally flat Riemannian n-manifold with constant scalar curvature $n(n-1)$. Then there exists an explicit constant $C_3(n)$ depending only on n such that if $\| |\mathrm{Ric}|^2 - |\mathrm{Ric}_{\mathbb{S}^n}|^2 \|_{n/2} < C_3(n)$, where $\mathrm{Ric}_{\mathbb{S}^n}$ is the Ricci curvature tensor of $\mathbb{S}^n$, then M is isometric to $\mathbb{S}^n$.*

4. Questions

Theorems 3.1 and 3.7 can be considered as isolation phenomena for the Ricci curvature norm of conformally flat manifolds with constant scalar curvature. With our results in mind, we review the related $L^{n/2}$ pinching theorem obtained by Shiohama and Xu [1997]. For a compact Riemannian manifold (M, g), they defined a new curvature tensor and its $L^{n/2}$ norm by

$$\widetilde{\mathrm{Rm}} = \sum_{i,j,k,l} \tilde{R}_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l \quad \text{and} \quad \tilde{R}(M) = \int_M |\widetilde{\mathrm{Rm}}|^{n/2} dM,$$

where $\tilde{R}_{ijkl} = R_{ijkl} - R(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})/(n(n-1))$.

Theorem C [Shiohama and Xu 1997]. *For $n \geq 3$, let M be a closed Riemannian n-manifold that can be isometrically immersed in Euclidean $\mathbb{R}^{n+1}$. If $\tilde{R}(M) < C_5(n)$, where $C_5(n)$ is an explicit positive constant depending only on n, then M is homeomorphic to the sphere.*

Motivated by the result above and the striking differentiable pinching theorem due to Brendle and Schoen [2009], we propose the following question.

Question 4.1. For $n \geq 3$, let M be a compact Riemannian n-manifold. Denote by d and V the diameter and volume of M. Does there exist a positive constant ε_1 depending on n, d and V such that if $\tilde{R}(M) < \varepsilon_1$, then M is diffeomorphic to a compact space form?

When M is locally conformally flat, we see from (2-2) that Riemannian curvature tensor can be expressed in terms of the Ricci curvature tensor. By a direct computation we have $|\widetilde{\mathrm{Rm}}|^2 = (4/(n-2))|\widetilde{\mathrm{Ric}}|^2$. Another question then arises out of our L^p pinching theorems for conformally flat manifolds:

Question 4.2. For $n \geq 3$, let (M, g) be a complete locally conformally flat Riemannian n-manifold. Does there exists a positive constant ε_2 depending only on n such that if $\|\widetilde{\mathrm{Ric}}\|_{n/2} < \varepsilon_2$, then M is diffeomorphic to a complete space form? In particular, if M is simply connected, is M diffeomorphic to either $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$?

References

[Bérard 1991] P. Bérard, "Remarques sur l'équation de J. Simons", pp. 47–57 in *Differential geometry*, edited by B. Lawson and K. Tenenblat, Pitman Monogr. Surveys Pure Appl. Math. **52**, Longman Sci. Tech., Harlow, 1991. MR 93g:53082 Zbl 0731.53054

[Bérard et al. 1998] P. Bérard, M. do Carmo, and W. Santos, "Complete hypersurfaces with constant mean curvature and finite total curvature", *Ann. Global Anal. Geom.* **16**:3 (1998), 273–290. MR 2000d:53093 Zbl 0921.53027

[Brendle and Schoen 2009] S. Brendle and R. Schoen, "Manifolds with $1/4$-pinched curvature are space forms", *J. Amer. Math. Soc.* **22**:1 (2009), 287–307. MR 2010a:53045

[Cheng et al. 1999] Q.-M. Cheng, S. Ishikawa, and K. Shiohama, "Conformally flat 3-manifolds with constant scalar curvature", *J. Math. Soc. Japan* **51**:1 (1999), 209–226. MR 2000b:53038 Zbl 0949.53023

[Eisenhart 1997] L. P. Eisenhart, *Riemannian geometry*, Princeton University Press, Princeton, NJ, 1997. MR 98h:53001 Zbl 0174.53303

[Hebey 1999] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, Courant Lecture Notes in Mathematics **5**, New York University Courant Institute of Mathematical Sciences, 1999. MR 2000e:58011 Zbl 0981.58006

[Lin and Xia 1989] J. M. Lin and C. Y. Xia, "Global pinching theorems for even-dimensional minimal submanifolds in the unit spheres", *Math. Z.* **201**:3 (1989), 381–389. MR 90i:53048 Zbl 0651.53044

[Ni 2001] L. Ni, "Gap theorems for minimal submanifolds in $\mathbb{R}^{n+1}$", *Comm. Anal. Geom.* **9**:3 (2001), 641–656. MR 2002m:53097 Zbl 1020.53041

[Pigola et al. 2007] S. Pigola, M. Rigoli, and A. G. Setti, "Some characterizations of space-forms", *Trans. Amer. Math. Soc.* **359**:4 (2007), 1817–1828. MR 2008a:53036 Zbl 1123.53020

[Shen 1989] C. L. Shen, "A global pinching theorem of minimal hypersurfaces in the sphere", *Proc. Amer. Math. Soc.* **105**:1 (1989), 192–198. MR 90c:53162 Zbl 0679.53049

[Shiohama and Xu 1994] K. Shiohama and H. W. Xu, "Rigidity and sphere theorems for submanifolds", *Kyushu J. Math.* **48**:2 (1994), 291–306. MR 95f:53101 Zbl 0826.53045

[Shiohama and Xu 1997] K. Shiohama and H. W. Xu, "Lower bound for $L^{n/2}$ curvature norm and its application", *J. Geom. Anal.* **7**:3 (1997), 377–386. MR 2000d:53057 Zbl 0960.53022

[Simons 1968] J. Simons, "Minimal varieties in riemannian manifolds", *Ann. of Math.* (2) **88** (1968), 62–105. MR 38 #1617 Zbl 0181.49702

[Tani 1967] M. Tani, "On a conformally flat Riemannian space with positive Ricci curvature", *Tôhoku Math. J.* (2) **19** (1967), 227–231. MR 36 #3279 Zbl 0166.17405

[Wang 1988] H. Wang, "Some global pinching theorems for submanifolds of a sphere", *Acta Math. Sinica* **31**:4 (1988), 503–509. MR 90e:53060 Zbl 0679.53050

[Wu et al. 1989] H. Wu, C. L. Shen, and Y. L. Yu, *An Introduction to Riemannian Geometry*, Beijing University Press, 1989.

[Xu 1990] H. W. Xu, *Pinching theorems, global pinching theorems and eigenvalues for Riemannian submanifolds*, thesis, Fudan University, 1990.

[Xu 1994] H. W. Xu, "$L_{n/2}$-pinching theorems for submanifolds with parallel mean curvature in a sphere", *J. Math. Soc. Japan* **46**:3 (1994), 503–515. MR 95d:53070 Zbl 0824.53048

[Xu and Gu 2007a] H. W. Xu and J. R. Gu, "A general gap theorem for submanifolds with parallel mean curvature in $\mathbb{R}^{n+p}$", *Comm. Anal. Geom.* **15**:1 (2007), 175–193. MR 2008g:53070 Zbl 1122.53033

[Xu and Gu 2007b] H. W. Xu and J. R. Gu, "L^2-isolation phenomenon for complete surfaces arising from Yang–Mills theory", *Lett. Math. Phys.* **80**:2 (2007), 115–126. MR 2008g:53076 Zbl 1129.53037

Received December 18, 2008.

HONG-WEI XU
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU 310027
CHINA
xuhw@cms.zju.edu.cn

EN-TAO ZHAO
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU 310027
CHINA
zhaoet@cms.zju.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

ACKNOWLEDGEMENT

The editors gratefully acknowledge the valuable advice of the referees who helped them select and better the papers appearing in 2009 in the *Pacific Journal of Mathematics* (reports dated November 26, 2008 through December 31, 2009).

Marco Abate, Christopher Allday, Jianbei An, Michael Anshelevich, Benjamin Audoux, John Baez, Dubravka Ban, Thierry Barbot, Refik İnanç Baykur, Julia Bergner, Bruce C. Berndt, Marie-Françoise Bidaut-Véron, David Borthwick, Winfried Bruns, Doug Bullock, Igor Burban, Esther Cabezas-Rivas, Jae Choon Cha, Wai Kiu Chan, Bing-Long Chen, Jingyi Chen, Wenxiong Chen, Xiuxiong Chen, Qing-Ming Cheng, Jong Taek Cho, Bruno Colbois, Shaun Cooper, François Dahmani, Benoît Daniel, Guy David, Matthew B. Day, Ryan Derby-Talbot, Ryszard Deszcz, Franki Dillen, Qing Ding, Chongying Dong, Olivier Druet, Jérôme Dubois, Jean-Paul Dufour, Ricardo G. Durán, Andrew G Earnest, Benjamin Enriquez, Nicholas Ercolani, Sam Evens, Kevin Ford, Max Forester, Philip Foth, Jixiang Fu, Hiroshi Fujita, Jason Fulman, David Futer, Wee Teck Gan, Yun Gao, Peter B. Gilkey, Yair Glasner, Hiroshi Goda, Simon Goodwin, Maria Gorelik, Rod Gover, Dimitar Grantcharov, Jacob Greenstein, Daniel Guan, Walter Gubler, Nigel Higson, David Hoffman, Cynthia Hog-Angeloni, Min-Chun Hong, Mark Hovey, Elton Hsu, Yulij Ilyashenko, Jens Carsten Jantzen, Jerzy Jezierski, Shulim Kaliman, Yasuyuki Kawahigashi, Dexing Kong, Steven Krantz, Daniel Krashen, Ernst Kuwert, Joshua Lansky, Y. K. Lau, H. Blaine Lawson, Jr., Yong Hah Lee, Graham Leuschke, Tian-Jun Li, Yuxiang Li, Yi Lin, Kefeng Liu, Martin Lorenz, Jiang-Hua Lu, Peng Lu, Zhiqin Lu, Tao Luo, Alexander Lytchak, Li Ma, Zhengyu Mao, Michael McCooey, John McCuan, Michael McLendon, Vikram B. Mehta, Anders Melin, Vicente Miquel, Frank Morgan, Adriano Moura, Alexey Muranov, Camil Muscalu, Erhard Neher, André Neves, Patrick Ng, Yuri Nikolayevsky, Paul Norbury, Cezar Oniciuc, John R. Parker, Vern I. Paulsen, Pedro L. Q. Pergher, Thomas Petersen, Victor Petrogradsky, Hendryk Pfeiffer, Yehuda Pinchover, Martin Pinsonnault, Wolfgang Pitsch, Saminathan Ponnusamy, Gopal Prasad, Jie Qing, Qiu Ruifeng, Hans-Bert Rademacher, Vicentiu Radulescu, K. M. Rangaswamy, Don Redmond, Marc A. Rieffel, Igor Rivin, Steve Rosenberg, Daniel Ruberman, David Ruiz, Claude Sabbah, Makoto Sakaki, Paolo Salani, Emil Saucon, Gordan Savin, Travis Schedler, Ralf Schmidt, George Seelinger, Prasad Senesi, Sergey Shadrin, Mei-Chi Shaw, Chunli Shen, Zhongmin Shen, David Siegel, Miles Simon, Eric Sommers, Maria Specovius-Neugebauer, Melanie Stein, Emil J. Straube, Neil Strickland, András Szűcs, Ryo Takahashi, Peter Tingley, Filippo Tolli, Maria Transirico, Cristina Trombetti, Bernardo Uribe, M. van der Put, Werner Varnhorn, Alexander Vasiliev, Monica Vazirani, David Vogan, Thomas Vogel, Dietmar Vogt, José Felipe Voloch, Theodore Voronov, Paweł Walczak, Jean-Loup Waldspurger, Cormac Walsh, Genevieve S. Walsh, Changping Wang, Jiaping Wang, McKenzie Wang, Liam Watson, Hans Weinberger, David Weisbart, Joseph Wolf, Peter N. Wong, Yuanlong Xin, Rugang Ye, Imed Zaguia, Lei Zhang, Qi S. Zhang, Yuan Zhang, Xiaohua Zhu, Xiping Zhu, Yongchang Zhu.

CONTENTS

Volume 245, no. 1 and no. 2

Guidelines for Authors

Authors may submit manuscripts at pjm.math.berkeley.edu/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 245 No. 2 April 2010

0030-8730(201004)245:2;1-E