Pacific Journal of Mathematics

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Volume 245 No. 2 April 2010

PACIFIC JOURNAL OF MATHEMATICS
Vol. 245, No. 2, 2010

A CONSTANT RANK THEOREM FOR LEVEL SETS OF IMMERSED HYPERSURFACES IN $\mathbb{R}^{n+1}$ WITH PRESCRIBED MEAN CURVATURE

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We prove a constant rank theorem on the second fundamental forms of level sets of immersed hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed mean curvature.

1. Introduction

Constant rank theorems have been a powerful tool in the study of convex solutions to partial differential equations. Caffarelli and Friedman [1985] first proved a constant rank theorem on solutions to a class of semilinear elliptic PDEs in two dimensions and hence proved the strict convexity of the solutions. Singer, Wong, Yau and Yau explored a similar idea [Singer et al. 1985]. Korevaar and Lewis [1987] extended Caffarelli and Friedman's results to the n-dimensional case. In the last decade, the constant rank theorem has been extended to fully nonlinear elliptic PDEs [Guan and Ma 2003; Caffarelli et al. 2007; Guan et al. 2006]; these authors found important applications for it in some geometric problems. For the convexity of level sets, Korevaar proved a constant rank theorem:

Theorem 1.1 [Korevaar 1990]. *Let Ω be a connected domain in $\mathbb{R}^n$. Let $u \in C^4(\Omega)$ solve*

$$Lu := A\Big(\Delta u - \frac{u_i u_j}{|\nabla u|^2} u_{ij}\Big) + B\Big(\frac{u_i u_j}{|\nabla u|^2} u_{ij}\Big) = f(u, |\nabla u|), \tag{1-1}$$

where A, B, f are C^2 functions of u, and $\mu := |\nabla u|$. These satisfy the structure conditions

(i) $(\sqrt{A/B}\,)_{\mu\mu} \geq 0$, *and*

(ii) $(f(u,\mu)/B\mu^2)_{\mu\mu} \leq 0$.

MSC2000: primary 35J15; secondary 53A10.

Keywords: constant rank theorem, level sets, mean curvature.

Hu was partially supported by NSFC numbers 10871138 and 10871139. Ma and Ou were supported by NSFC numbers 10671186 and 10871187.

Suppose that $|\nabla u| \neq 0$ *and that* u *has convex level sets* $\{x \in \Omega \mid u(x) \leq c\}$. *Then all the level sets of* u *have second fundamental forms with* (*the same*) *constant rank throughout* Ω.

The equations in Theorem 1.1 include p-Laplacian equations and mean curvature equations as special cases. In these cases we respectively take

$$A = \mu^{p-2}, \quad B = (p-1)\mu^{p-2} \quad \text{and} \quad A = \frac{1}{\sqrt{1+\mu^2}}, \quad B = \frac{1}{(1+\mu^2)^{3/2}}.$$

Korevaar [1990] used this theorem to prove some interesting results on the convexity of the level sets of solutions to elliptic PDEs. Recently, Xu [2008] generalized Theorem 1.1 to the case where the function f in (1-1) also depends on the coordinate variable x, and accordingly the structure condition (ii) turns into

$$\mu^3 \frac{f(x, u, 1/\mu)}{B(u, 1/\mu)} \quad \text{is convex in } (x, \mu).$$

In this paper, we will prove an analogous result on a class of immersed hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed mean curvature.

Let M^n be a smooth immersed hypersurface in $\mathbb{R}^{n+1}$, and let $X : M \to \mathbb{R}^{n+1}$ be the immersion satisfying

$$H = -f(X, N), \tag{1-2}$$

where H and N are respectively the mean curvature and unit normal vectors of M^n at X, and f is a smooth function in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let ξ be a fixed unit vector in $\mathbb{R}^{n+1}$. Then the height function of M^n corresponding to ξ can be expressed as $u(X) = \langle X, \xi \rangle$; here $\langle \cdot, \cdot \rangle$ means the usual Euclidean inner product in $\mathbb{R}^{n+1}$. Now, the level set of M^n corresponding to ξ with height c is defined as

$$\Sigma_c = \{X \in M^n \mid u(X) = c\}. \tag{1-3}$$

Suppose u has no critical point on M^n. Then Σ_c can be considered as a hypersurface in the hyperplane $\Pi = \{X \in \mathbb{R}^{n+1} \mid \langle X, \xi \rangle = c\}$.

With the above notations, our constant rank theorem on the level sets of an immersed hypersurface with prescribed mean curvature can be stated as follows:

Theorem 1.2. *Let* M^n *be an immersed hypersurface in* $\mathbb{R}^{n+1}$ *whose mean curvature satisfies* (1-2). *Assume that the height function* u *of* M^n *corresponding to* ξ *has no critical point, and that the level sets are all locally convex with respect to the normal direction* $-Du$, *that is, their second fundamental forms are positive semidefinite. Then the second fundamental forms of all the level sets have* (*the same*) *constant rank, provided* $f(X, N) = f(X) \geq 0$ *and the matrix*

$$2f \frac{\partial^2 f}{\partial X_A \partial X_B} - 3 \frac{\partial f}{\partial X_A} \frac{\partial f}{\partial X_B} \tag{1-4}$$

is positive semidefinite, where $1 \le A, B \le n+1$. *In other words, when* f *is a positive function, the condition* (1-4) *simply means that* $f^{-1/2}$ *is a concave function in* $\mathbb{R}^{n+1}$.

Remark 1.3. For the more general case where $H = -f(X, N)$ as in (1-2), by (3-28) and (3-29) in Section 3, we still can choose the structure conditions on f to ensure the result of Theorem 1.2. For example, if $f(X, N) = \langle \xi, N\rangle^{\beta}$ with $\langle \xi, N\rangle > 0$ on M^n, then the structure condition is $\beta \ge 1$ or $\beta \le 0$.

Remark 1.4. Throughout, we adapt these conventions: The hypersurface M^n is orientable. We choose the unit normal vector field N so that it represents the orientation of M^n. The unit vector field normal to the level set Σ_c is obtained by projecting N onto the hyperplane $\Pi = \{X \in \mathbb{R}^{n+1} \mid \langle X, \xi\rangle = c\}$.

When do the solutions of elliptic PDEs have convex level sets? Gabriel [1957] proved that the level sets of the Green function on a 3-dimensional convex domain are strictly convex. Lewis [1977] extended Gabriel's result to p-harmonic functions in higher dimensions. Caffarelli and Spruck [1982] generalized Lewis's result to a class of semilinear elliptic PDEs. For recent progress, see [Colesanti and Salani 2003] and [Cuoghi and Salani 2006]. The constant rank theorem is an important step for the concrete convexity theorem, since one can use it to prove strict convexity results, as in, for example, [Korevaar 1990]. In practice, one always runs into difficulty at the critical points of the solution (or height functions in our case). In some sense our constant rank theorem is only a local and intermediate result.

In Section 2, we will give a formula for the curvatures of the level sets of an immersed hypersurface in $\mathbb{R}^{n+1}$. We prove it by the method of moving frames. We prove our main result, Theorem 1.2, in Section 3 using a calculation similar to the one in [Xu 2008].

2. Formulas of curvature of level sets

For a C^2 function u defined in a n-dimensional domain Ω in $\mathbb{R}^n$, let $\kappa_1, \dots, \kappa_{n-1}$ be the principal curvatures of the level sets of u with respect to the normal direction $-Du$. Then the k-th curvature of the level sets, denoted by L_k, is the k-th elementary symmetric function of $\kappa_1, \dots, \kappa_{n-1}$. Clearly, L_1 and L_{n-1} are respectively the mean curvature and Gauss curvature of the level sets. If u has no critical point, that is, $|\nabla u| \ne 0$, then Trudinger [1997] (see also [Gilbarg and Trudinger 1977]) expressed L_k as

$$L_k = \frac{\partial \sigma_{k+1}(D^2 u)}{\partial u_{ij}} u_i u_j |\nabla u|^{-k-2}, \tag{2-1}$$

where we use summation convention for repeated indices, and where $\sigma_k(D^2u)$ is the k-th elementary symmetric function of the eigenvalues of the Hessian (D^2u).

There is an formula analogous to (2-1) on hypersurfaces in $\mathbb{R}^{n+1}$:

Proposition 2.1. *Let M^n be a smoothly immersed hypersurface in $\mathbb{R}^{n+1}$. Let u be its height function and Σ_c one of its level sets, with respect to a fixed unit vector ξ, as given in the last section. Then the k-th curvature of the level set Σ_c with respect to $-Du$ is*

$$L_k = \frac{\partial \sigma_{k+1}(\boldsymbol{B})}{\partial h_{ij}} u_i u_j |\nabla u|^{-(k+2)}. \tag{2-2}$$

Here $\boldsymbol{B} = (h_{ij})$ is the second fundamental form of M^n, $\sigma_k(\boldsymbol{B})$ is the k-th elementary symmetric function of the eigenvalues of $\boldsymbol{B}$, and u_i for $1 \le i \le n$ are the first order covariant derivatives of u computed in any orthonormal frame field on M^n.

Huang [1992] gave the formula (2-2) for $n = 2$. Here we give a complete proof by using moving frames. In this section, indices will run from 1 to $n-1$ when lower case and Greek; Latin indices will run from 1 to n when lower case and from 1 to $n+1$ when upper case.

For an orthonormal frame field $\{X; e_A\}$ in $\mathbb{R}^{n+1}$, we have

$$dX = \omega_A e_A \quad \text{and} \quad de_A = \omega_{A,B} e_B, \tag{2-3}$$

where $\{\omega_A\}$ is the dual frame of $\{e_A\}$, and $\{\omega_{A,B}\}$ are connection forms. Then the structure equations read as

$$d\omega_A = \omega_{A,B} \wedge \omega_B \quad \text{and} \quad d\omega_{A,B} = \omega_{A,C} \wedge \omega_{C,B}. \tag{2-4}$$

If we choose e_{n+1} to be the unit normal vector field N of M^n, then $\omega_{n+1} = 0$ on M^n, and hence by (2-4)

$$\omega_{n+1,i} \wedge \omega_i = 0. \tag{2-5}$$

Then Cartan's lemma implies $\omega_{n+1,i} = h_{ij}\omega_j$ and $h_{ij} = h_{ji}$, where $\boldsymbol{B} = (h_{ij})$ is the second fundamental form of M^n.

Proof of Proposition 2.1. First, we check that the right side of (2-2) is independent of the choice of the frame fields $\{X; e_i\}$ on M^n. Then we can just prove (2-2) in a special frame field.

Suppose $\{X; \bar{e}_i\}$ is another frame field on M^n. Then there is an orthogonal transformation T such that $(\bar{e}_1, \dots, \bar{e}_n) = (e_1, \dots, e_n)T$. Then

$$(\bar{u}_1, \dots, \bar{u}_n) = (u_1, \dots, u_n)T, \tag{2-6}$$

where $\nabla u = u_i e_i = \bar{u}_i \bar{e}_i$ is the gradient of u. Also, for the dual frame field and the connection forms we have

$$(\bar{\omega}_1, \dots, \bar{\omega}_n) = (\omega_1, \dots, \omega_n)T,$$
$$(\bar{\omega}_{1,n+1}, \dots, \bar{\omega}_{n,n+1}) = (\omega_{1,n+1}, \dots, \omega_{n,n+1})T.$$

Furthermore, for the second fundamental form we have

$$\bar{\boldsymbol{B}} = T^{-1}\boldsymbol{B}T. \tag{2-7}$$

Obviously $\sigma_k(\boldsymbol{B})$ and $|\nabla u|$ are invariant under the transformation T. Then the following equalities show that the right side of (2-2) is independent of the choice of $\{e_1, \dots, e_n\}$:

$$\begin{aligned}\frac{\partial\sigma_k(\boldsymbol{B})}{\partial h_{ij}}u_iu_j &= \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial\bar{h}_{ml}}\frac{\partial\bar{h}_{ml}}{\partial h_{ij}}u_iu_j = \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial\bar{h}_{ml}}\frac{\partial(T^{mp}h_{pq}T_{ql})}{\partial h_{ij}}u_iu_j\\ &= \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial\bar{h}_{ml}}T^{mi}T_{jl}u_iu_j = \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial\bar{h}_{ml}}T_{im}u_iT_{jl}u_j = \frac{\partial\sigma_k(\bar{\boldsymbol{B}})}{\partial\bar{h}_{ml}}\bar{u}_m\bar{u}_l.\end{aligned} \tag{2-8}$$

Now we adapt the frame field above so that along the level set Σ_c, the e_α are its tangential vectors. Furthermore, we choose another frame field $\tilde{e}_A$ in $\mathbb{R}^{n+1}$ so that $\tilde{e}_{n+1} = \xi$ and $\tilde{e}_\alpha = e_\alpha$, and so that $\tilde{e}_n$ lies in the hyperplane Π and is normal to Σ_c with the same direction of the projection of $e_{n+1} = N$ on Π. With respect to this frame field, the structure equations of Σ_c are

$$d\tilde{\omega}_i = \tilde{\omega}_{i,j}\wedge\tilde{\omega}_j \quad\text{and}\quad d\tilde{\omega}_{ij} = \tilde{\omega}_{i,l}\wedge\tilde{\omega}_{l,j}. \tag{2-9}$$

On Σ_c, we have $\tilde{\omega}_n = 0$, which implies

$$\tilde{\omega}_{n,\alpha} = \tilde{h}_{\alpha\beta}\tilde{\omega}_\beta \quad\text{and}\quad \tilde{h}_{\alpha\beta} = \tilde{h}_{\beta\alpha}, \tag{2-10}$$

where $\tilde{h}_{\alpha\beta}$ is the second fundamental form of Σ_c in Π (with respect to the unit normal $\tilde{e}_n$).

Clearly e_n, e_{n+1} and $\tilde{e}_n, \tilde{e}_{n+1}$ are in the same 2-plane perpendicular to the e_α. Let ϕ be the angle between e_n and $\tilde{e}_n$. Then we have

$$\tilde{e}_n = e_n cos\phi + e_{n+1}\sin\phi \quad\text{and}\quad \tilde{e}_{n+1} = -\tilde{e}_n sin\phi + e_{n+1}\cos\phi. \tag{2-11}$$

Accordingly,

$$\tilde{\omega}_n = \omega_n\cos\phi + \omega_{n+1}\sin\phi, \quad \tilde{\omega}_{n+1} = -\omega_n\sin\phi + \omega_{n+1}\cos\phi, \quad \tilde{\omega}_\alpha = \omega_\alpha. \tag{2-12}$$

Taking the exterior derivative of (2-12), and using (2-4) and (2-12) again, we get

$$\begin{aligned} d\tilde{\omega}_n &= (d\phi + \omega_{n,n+1})\wedge\tilde{\omega}_{n+1} + ((\cos\phi)\omega_{n,\alpha} + (\sin\phi)\omega_{n+1,n})\wedge\omega_\alpha,\\ d\tilde{\omega}_{n+1} &= (-d\phi + \omega_{n+1,n})\wedge\tilde{\omega}_n + ((\cos\phi)\omega_{n+1,\alpha} - (\sin\phi)\omega_{n,\alpha})\wedge\omega_\alpha.\end{aligned} \tag{2-13}$$

Notice that $\tilde{\omega}_n = \tilde{\omega}_{n+1} = 0$ on Σ_c. Comparing (2-13) with (2-9), we have

$$\begin{aligned}\tilde{\omega}_{n,\alpha} &= (\cos\phi)\omega_{n,\alpha} + (\sin\phi)\omega_{n+1,\alpha},\\ \tilde{\omega}_{n+1,\alpha} &= (-\sin\phi)\omega_{n,\alpha} + (\cos\phi)\omega_{n+1,\alpha}.\end{aligned} \tag{2-14}$$

On the other hand, $\langle \tilde{e}_\alpha, \xi\rangle = 0$ on Σ_c, and since $d(\langle \tilde{e}_\alpha, \xi\rangle) = \langle \tilde{\omega}_{\alpha,A}\tilde{e}_A, \xi\rangle$, we have $\tilde{\omega}_{\alpha,n+1} = 0$. This together with (2-14) implies

$$\tag{2-15} \begin{aligned} \tilde{\omega}_{n,\alpha} &= \frac{\cos^2\phi}{\sin\phi}\omega_{n+1,\alpha} + (\sin\phi)\omega_{n+1,\alpha} \\ &= \frac{1}{\sin\phi}\omega_{n+1,\alpha} = \frac{1}{\sin\phi}(h_{\alpha\beta}\omega_\beta + h_{\alpha n}\omega_n). \end{aligned}$$

Combining this with (2-10) gives

$$\tag{2-16} \tilde{h}_{\alpha\beta} = \frac{1}{\sin\phi}h_{\alpha\beta} \quad \text{and} \quad h_{\alpha,n} = 0.$$

From the definition of the height function u, we can see $u_i = e_i(\langle X, \xi\rangle) = \langle e_i, \xi\rangle$; in particular, $u_n = \langle e_n, \xi\rangle$. Note that $\tilde{e}_{n+1} = \xi$, hence the second equation of (2-11) implies $u_n = -\sin\phi$ and $\langle \xi, e_{n+1}\rangle = \cos\phi$. By the decomposition

$$\xi = \sum_1^n \langle \xi, e_i\rangle e_i + \langle \xi, e_{n+1}\rangle e_{n+1}$$

we deduce that $1 = |\nabla u|^2 + \cos^2\phi$ and therefore $|\nabla u| = \pm\sin\phi$. With e_n chosen suitably we may assume $\sin\phi > 0$. Then (2-16) becomes

$$\tag{2-17} \tilde{h}_{\alpha\beta} = \frac{1}{|\nabla u|}h_{\alpha\beta} \quad \text{and} \quad h_{\alpha n} = 0.$$

From this one can easily see that

$$\tag{2-18} \begin{aligned} L_k = \sigma_k(\tilde{h}_{\alpha\beta}) &= \frac{1}{|\nabla u|^k}\sigma_k(h_{\alpha\beta}) \\ &= \frac{1}{|\nabla u|^{k+2}}\frac{\partial\sigma_{k+1}(\boldsymbol{B})}{\partial h_{nn}}u_n u_n = \frac{\partial\sigma_{k+1}(\boldsymbol{B})}{\partial h_{ij}}u_i u_j |\nabla u|^{-(k+2)}, \end{aligned}$$

where we have used $|u_n| = |\nabla u|$. □

3. Proof of Theorem 1.2

We adapt the notations in Section 2, and collect these formulas for convenience:

$$\tag{3-1} \begin{aligned} X_i &= e_i, \\ X_{ij} &= -h_{ij}e_{n+1} && \text{(Gauss formula)}, \\ e_{n+1,i} &= h_{ij}e_j && \text{(Weingarten formula)}, \\ h_{ijk} &= h_{ikj} && \text{(Codazzi equation)}, \\ R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} && \text{(Gauss equation)}, \\ h_{ijkl} &= h_{ijlk} + h_{im}R_{mjkl} + h_{jm}R_{mikl}, \end{aligned}$$

and for the smooth function u on M^n we also have the Ricci identity

$$u_{ijk} = u_{ikj} + u_m R_{mijk},$$

where R_{ijkl} is the Riemann curvature tensor, and as for the rest of this section, repeated indices are summed from 1 to n, unless otherwise stated.

Proof of Theorem 1.2. Suppose the second fundamental forms of the level sets of M^n take the minimum rank k with $k \le n-2$ at a point $P \in M^n$. We will treat the case $k > 0$ first, and then show how to modify the argument for the case $k = 0$. With the assumption that the level sets are all locally convex, we find easily that

$$\begin{aligned} L_r(P) &= 0 \quad \text{for all } r > k, \\ L_r(P) &> 0 \quad \text{for all } r \le k, \end{aligned} \tag{3-2}$$

and moreover

$$\begin{aligned} Z := \{X \in M^n \mid\ & \text{the second fundamental form} \\ & \text{of the level sets of } M^n \text{ has rank } k \text{ at } X\} \\ = \{X \in M^n \mid\ & L_{k+1}(X) = 0\}. \end{aligned} \tag{3-3}$$

Obviously Z is a closed set in M^n. If we can show that Z is also open in M^n — that is, that there is a neighborhood U_P of P in M^n such that $L_{k+1} \equiv 0$ on U_P — then $Z = M^n$, which is the result in the theorem.

Now $L_{k+1}(P) = 0 = \min_{X \in M^n} L_{k+1}(X)$, so by the strong maximum principle, we need only to show that

$$\Delta L_{k+1}(X) \le 0 \quad \text{mod } \{L_{k+1}(X), \nabla L_{k+1}(X)\} \quad \text{in } U_P, \tag{3-4}$$

where we modify the terms of L_{k+1} and its first derivatives, coefficients are locally bounded, and Δ is the Beltrami–Laplace operator on M^n.

For the rest of this section, define

$$W := (h_{ij}) \quad \text{with } i, j \le n-1, \quad L := L_{k+1}, \quad F := \sigma_{k+2}(\boldsymbol{B}),$$

and

$$F^{ij} := \frac{\partial F}{\partial h_{ij}}, \quad F^{ij,rs} := \frac{\partial^2 F}{\partial h_{ij} \partial h_{rs}}, \quad F^{ij,rs,pq} := \frac{\partial^3 F}{\partial h_{ij} \partial h_{rs} \partial h_{pq}}.$$

Hence, by (2-2),

$$|\nabla u|^{k+3} L = F^{ij} u_i u_j. \tag{3-5}$$

Taking the covariant derivative of this, we get

$$\begin{aligned} (|\nabla u|^{k+3} L)_\alpha &= |\nabla u|^{k+3} L_\alpha + (|\nabla u|^{k+3})_\alpha L, \\ (F^{ij} u_i u_j)_\alpha &= F^{ij,rs} h_{rs\alpha} u_i u_j + 2F^{ij} u_{i\alpha} u_j. \end{aligned} \tag{3-6}$$

Taking the covariant derivative again, we get

$$(|\nabla u|^{k+3}L)_{\alpha\alpha} = |\nabla u|^{k+3}L_{\alpha\alpha} + 2(|\nabla u|^{k+3})_\alpha L_\alpha + (|\nabla u|^{k+3})_{\alpha\alpha}L,$$

$$(F^{ij}u_iu_j)_{\alpha\alpha} = F^{ij,rs,pq}h_{pq\alpha}h_{rs\alpha}u_iu_j + F^{ij,rs}h_{rs\alpha\alpha}u_iu_j + 4F^{ij,rs}h_{rs\alpha}u_{i\alpha}u_j + 2F^{ij}u_{i\alpha\alpha}u_j + 2F^{ij}u_{i\alpha}u_{j\alpha}. \tag{3-7}$$

For a fixed point X_0 in U_P, choose a frame $\{e_1, \dots, e_n\}$ such that u_i through u_{n-1} vanish, $|u_n| = |\nabla u| > 0$, the form W is diagonal, and $h_{11} \ge h_{22} \ge \cdots \ge h_{n-1,n-1}$. Then by (3-2) we see that with U_P small enough

$$h_{rr}(X_0) = 0 \quad \mod \{L(X_0), \nabla L(X_0)\} \quad \text{for all } r > k,$$

$$h_{rr}(X_0) > \epsilon > 0 \quad \mod \{L(X_0), \nabla L(X_0)\} \quad \text{for all } r \le k, \tag{3-8}$$

where ϵ is a positive sufficiently small number (maybe depending on U_P).

In the following, all the calculations will be done at X_0, and the terms of $L(X_0)$ and $\nabla L(X_0)$ will be dropped, that is, all the equalities or inequalities should be understood mod$\{L(X_0), \nabla L(X_0)\}$.

Denote $G := \{h_{11}, \dots, h_{kk}\}$ and $B := \{h_{k+1,k+1}, \dots, h_{n-1,n-1}\}$. Use the same symbols for $G := \{1, \dots, k\}$ and $B := \{k+1, \dots, n-1\}$ (it won't cause confusion).

Now, by $L(P) = 0 = \min_{X\in M^n} L(X)$ we get

$$\begin{aligned} 0 &= (|\nabla u|^{k+3}L)_\alpha = (F^{ij}u_iu_j)_\alpha = F^{ij,rs}h_{rs\alpha}u_iu_j + 2F^{ij}u_{i\alpha}u_j \\ &= u_n^2F^{nn,rr}h_{rr\alpha} + 2u_nF^{in}u_{i\alpha} \\ &= u_n^2\sigma_k(G)\textstyle\sum_{r\in B} h_{rr\alpha} + 2u_nF^{nn}u_{n\alpha} + 2u_n\sum_{i=1}^{n-1} F^{in}u_{i\alpha} \\ &= u_n^2\sigma_k(G)\textstyle\sum_{r\in B} h_{rr\alpha} - 2u_n\sigma_k(G)\sum_{i\in B} h_{ni}u_{i\alpha}. \end{aligned} \tag{3-9}$$

Clearly

$$u_i = \langle X, \xi\rangle_i = \langle X_i, \xi\rangle = \langle e_i, \xi\rangle,$$

$$u_{ij} = \langle X_{ij}, \xi\rangle = -\langle h_{ij}N, \xi\rangle := h_{ij}w, \tag{3-10}$$

where $w = -\langle N, \xi\rangle = \pm\sqrt{1 - |\nabla u|^2}$.

Substituting (3-10) into (3-9), using (3-8), and noting that W is diagonal, we deduce

$$\textstyle\sum_{i\in B} h_{ii\alpha} = 0 \quad \text{for all } \alpha < n,$$

$$u_n\textstyle\sum_{i\in B} h_{iin} = 2\sum_{i\in B} h_{ni}^2 w. \tag{3-11}$$

By (3-7) we have

$$(|\nabla u|^{k+3}L)_{\alpha\alpha} = |\nabla u|^{k+3}L_{\alpha\alpha} + 2(|\nabla u|^{k+3})_\alpha L_\alpha + (|\nabla u|^{k+3})_{\alpha\alpha}L = (F^{ij}u_iu_j)_{\alpha\alpha}.$$

That is,

$$|\nabla u|^{k+3}L_{\alpha\alpha} = F^{ij,rs,pq}h_{pq\alpha}h_{rs\alpha}u_iu_j + F^{ij,rs}h_{rs\alpha\alpha}u_iu_j + 4F^{ij,rs}h_{rs\alpha}u_{i\alpha}u_j + 2F^{ij}u_{i\alpha\alpha}u_j + 2F^{ij}u_{i\alpha}u_{j\alpha}$$
$$= u_n^2F^{nn,rs,pq}h_{pq\alpha}h_{rs\alpha} + u_n^2F^{nn,rs}h_{rs\alpha\alpha} + 4u_nF^{in,rs}h_{rs\alpha}u_{i\alpha} + 2u_nF^{in}u_{i\alpha\alpha} + 2F^{ij}u_{i\alpha}u_{j\alpha}, \tag{3-12}$$

which we decompose as $I + II + III + IV$, where

$$I := u_n^2F^{nn,rs,pq}h_{pq\alpha}h_{rs\alpha}, \qquad II := 4u_nF^{in,rs}h_{rs\alpha}u_{i\alpha},$$
$$III := u_n^2F^{nn,rs}h_{rs\alpha\alpha} + 2u_nF^{in}u_{i\alpha\alpha}, \quad IV := 2F^{ij}u_{i\alpha}u_{j\alpha}. \tag{3-13}$$

Next we will compute the above terms step by step. First

$$I := u_n^2F^{nn,rs,pq}h_{pq\alpha}h_{rs\alpha} = u_n^2F^{nn,rr,ss}h_{rr\alpha}h_{ss\alpha} + u_n^2F^{nn,rs,sr}h_{rs\alpha}h_{sr\alpha} =: I_1 + I_2, \tag{3-14}$$

and

$$I_1 := u_n^2F^{nn,rr,ss}h_{rr\alpha}h_{ss\alpha}$$
$$= 2u_n^2\sum_{r\in G,s\in B}F^{nn,rr,ss}h_{rr\alpha}h_{ss\alpha} + u_n^2\sum_{r,s\in B}F^{nn,rr,ss}h_{rr\alpha}h_{ss\alpha}$$
$$= 2u_n^2\sum_{r\in G,s\in B}\sigma_{k-1}(G\,|\,r)h_{rr\alpha}h_{ss\alpha} + u_n^2\sigma_{k-1}(G)\sum_{r,s\in B,r\neq s}h_{rr\alpha}h_{ss\alpha}, \tag{3-15}$$

where here and below we use the notation $\sigma_{k-1}(G\,|\,r) := \sigma_{k-1}(G\backslash\{h_{rr}\})$ and the convention $\sigma_0 = 1$. Substituting (3-11) into (3-15) yields

$$I_1 = 2u_n^2\sum_{r\in G,s\in B}\sigma_{k-1}(G\,|\,r)h_{rr\alpha}h_{ss\alpha} + u_n^2\sigma_{k-1}(G)\sum_{r\in B}h_{rr\alpha}\Big(\sum_{s\in B}h_{ss\alpha} - h_{rr\alpha}\Big)$$
$$= 4wu_n\sum_{s\in B}h_{sn}^2\sum_{r\in G}\sigma_{k-1}(G\,|\,r)h_{rrn} - u_n^2\sigma_{k-1}(G)\sum_{\alpha=1}^{n}\sum_{r\in B}h_{rr\alpha}^2 + 4w^2\sigma_{k-1}(G)\Big(\sum_{s\in B}h_{sn}^2\Big)^2.$$

For the remaining term in (3-14), we have

$$I_2 = 2u_n^2\sum_{r\in G,s\in B}F^{nn,rs,sr}h_{rs\alpha}h_{sr\alpha} + u_n^2\sum_{r,s\in B}F^{nn,rs,sr}h_{rs\alpha}h_{sr\alpha}$$
$$= -2u_n^2\sum_{\alpha=1}^{n}\sum_{r\in G,s\in B}\sigma_{k-1}(G\,|\,r)h_{rs\alpha}^2 - u_n^2\sigma_{k-1}(G)\sum_{\alpha=1}^{n}\sum_{r,s\in B,r\neq s}h_{rs\alpha}^2.$$

So for the first term in (3-13) we have

$$I = 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G|i)h_{iin}h_{jn}^2 - 2u_n^2 \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{k-1}(G|i)h_{ij\alpha}^2 + 4w^2\sigma_{k-1}(G)\Big(\sum_{j\in B} h_{jn}^2\Big)^2 - u_n^2\sigma_{k-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} h_{ij\alpha}^2. \tag{3-16}$$

To compute the second term in (3-13), first we have by using (3-10)

$$\begin{aligned} II &= 4wu_n F^{in,rs} h_{rs\alpha} h_{i\alpha} \\ &= 4wu_n F^{nn,rs} h_{rs\alpha} h_{n\alpha} + 4wu_n \sum_{i=1}^{n-1} F^{in,ni} h_{ni\alpha} h_{i\alpha} \\ &\quad + 4wu_n \sum_{i,j=1}^{n-1} F^{in,ji} h_{ji\alpha} h_{i\alpha} + 4wu_n \sum_{i=1}^{n-1} F^{in,rr} h_{rr\alpha} h_{i\alpha}. \end{aligned} \tag{3-17}$$

We decompose the last four terms as $II_1 + II_2 + II_3 + II_4$. By (3-11), the first can be treated as

$$\begin{aligned} II_1 &= 4wu_n F^{nn,rr} h_{rr\alpha} h_{n\alpha} = 4wu_n \sigma_k(G) \sum_{r\in B} h_{rr\alpha} h_{n\alpha} \\ &= 4wu_n \sigma_k(G) \sum_{r\in B} h_{rrn} h_{nn} = 8w^2 \sigma_k(G) h_{nn} \sum_{r\in B} h_{rn}^2. \end{aligned}$$

For the second and the third terms, straightforward calculations show that

$$II_2 = -4wu_n\sigma_k(G)\sum_{i\in B} h_{ni\alpha}h_{i\alpha} = -4wu_n\sigma_k(G)\sum_{i\in B} h_{nni}h_{in}, \tag{3-18}$$

and

$$\begin{aligned} II_3 &= 4wu_n \sum_{i,j\in B} F^{in,ji} h_{ji\alpha} h_{i\alpha} \\ &\qquad + 4wu_n \sum_{i\in G, j\in B} F^{in,ji} h_{ji\alpha} h_{i\alpha} + 4wu_n \sum_{j\in G, i\in B} F^{in,ji} h_{ji\alpha} h_{i\alpha} \\ &= 4wu_n\sigma_{k-1}(G) \sum_{i,j\in B, i\neq j} h_{jn} h_{ijn} h_{in} + 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G|i) h_{nj} h_{ji\alpha} h_{i\alpha} \\ &\qquad + 4wu_n \sum_{i\in G, j\in B} \sigma_{k-1}(G|i) h_{ni} h_{ijn} h_{jn}. \end{aligned} \tag{3-19}$$

Again by (3-11), the fourth term can be treated as

$$
\begin{aligned}
II_4 &= 4wu_n \sum_{i,r\in B} F^{in,rr} h_{rr\alpha} h_{i\alpha} + 4wu_n \sum_{i\in G,r\in B} F^{in,rr} h_{rr\alpha} h_{i\alpha} \\
&\qquad + 4wu_n \sum_{r\in G,i\in B} F^{in,rr} h_{rr\alpha} h_{i\alpha} \\
&= -4wu_n\sigma_{k-1}(G) \sum_{i,r\in B,i\neq r} h_{in}h_{rr\alpha}h_{i\alpha} - 4wu_n \sum_{i\in G,r\in B} \sigma_{k-1}(G\,|\,i)h_{ni}h_{rr\alpha}h_{i\alpha} \\
&\qquad - 4wu_n \sum_{r\in G,i\in B} \sigma_{k-1}(G\,|\,r)h_{ni}h_{rr\alpha}h_{i\alpha} \\
&= -4wu_n\sigma_{k-1}(G) \sum_{i,r\in B,i\neq r} h_{in}^2 h_{rrn} - 4wu_n \sum_{i\in G,r\in B} \sigma_{k-1}(G\,|\,i)h_{ni}h_{rri}h_{ii} \\
&\qquad - 4wu_n \sum_{i\in G,r\in B} \sigma_{k-1}(G\,|\,i)h_{ni}^2 h_{rrn} - 4wu_n \sum_{r\in G,i\in B} \sigma_{k-1}(G\,|\,r)h_{ni}^2 h_{rrn} \\
&= -4wu_n\sigma_{k-1}(G)\sum_{i\in B} h_{in}^2\Big(\sum_{r\in B} h_{rrn} - h_{iin}\Big) \\
&\qquad - 4wu_n \sum_{i\in G} \sigma_{k-1}(G\,|\,i)h_{ii}h_{ni}\sum_{r\in B} h_{rri} - 4wu_n\sum_{i\in G}\sigma_{k-1}(G\,|\,i)h_{ni}^2\sum_{r\in B} h_{rrn} \\
&\qquad - 4wu_n \sum_{r\in G,i\in B} \sigma_{k-1}(G\,|\,r)h_{ni}^2 h_{rrn} \\
&= 4wu_n\sigma_{k-1}(G)\sum_{i\in B} h_{in}^2 h_{iin} - 8w^2\sigma_{k-1}(G)\Big(\sum_{i\in B} h_{in}^2\Big)^2 \\
&\qquad - 8w^2 \sum_{i\in G,r\in B} \sigma_{k-1}(G\,|\,i)h_{in}^2 h_{rn}^2 - 4wu_n \sum_{i\in B} h_{ni}^2 \sum_{r\in G} \sigma_{k-1}(G\,|\,r)h_{rrn}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
II &= 8w^2\sigma_k(G)h_{nn}\sum_{j\in B} h_{nj}^2 - 4wu_n\sigma_k(G)\sum_{j\in B} h_{nnj}h_{nj} \\
&\qquad - 8w^2 \sum_{i\in G,\, j\in B} \sigma_{k-1}(G\,|\,i)h_{in}^2 h_{jn}^2 - 4wu_n \sum_{i\in G,\, j\in B} \sigma_{k-1}(G\,|\,i)h_{nj}^2 h_{iin} \\
&\quad + 4wu_n \sum_{i\in G,\, j\in B} \sigma_{k-1}(G\,|\,i)h_{i\alpha}h_{jn}h_{ij\alpha} + 4wu_n \sum_{i\in G,\, j\in B} \sigma_{k-1}(G\,|\,i)h_{ni}h_{nj}h_{ijn} \\
&+ 4wu_n\sigma_{k-1}(G) \sum_{i,j\in B,i\neq j} h_{ni}h_{nj}h_{ijn} + 4wu_n\sigma_{k-1}(G)\sum_{j\in B} h_{nj}^2 h_{jjn} \\
&\qquad - 8w^2\sigma_{k-1}(G)\Big(\sum_{j\in B} h_{nj}^2\Big)^2.
\end{aligned}
\tag{3-20}
$$

Now we deal with the third term in (3-13):

$$\begin{aligned} III &:= u_n^2 F^{nn,rs} h_{rs\alpha\alpha} + 2u_n F^{in} u_{i\alpha\alpha} \\ &= u_n^2 F^{nn,rr} h_{rr\alpha\alpha} + 2u_n F^{nn} u_{n\alpha\alpha} + 2u_n \sum_{i=1}^{n-1} F^{in} u_{i\alpha\alpha}. \end{aligned} \tag{3-21}$$

We decompose the last three terms as $III_1+III_2+III_3$. Using the exchange formula in (3-1), we can calculate

$$\begin{aligned} III_1 &= u_n^2 \sigma_k(G) \sum_{r\in B} h_{rr\alpha\alpha} \\ &= u_n^2 \sigma_k(G) \sum_{r\in B} (h_{r\alpha\alpha r} + h_{rm} R_{m\alpha r\alpha} + h_{\alpha m} R_{mrr\alpha}) \\ &= u_n^2 \sigma_k(G) \sum_{r\in B} h_{\alpha\alpha rr} \\ &\quad + u_n^2 \sigma_k(G) \sum_{r\in B} (h_{rm}(h_{mr}h_{\alpha\alpha} - h_{m\alpha}h_{\alpha r}) + h_{\alpha m}(h_{mr}h_{r\alpha} - h_{m\alpha}h_{rr})) \\ &= u_n^2 \sigma_k(G) \sum_{r\in B} H_{rr} + u_n^2 \sigma_k(G) \sum_{r\in B} (H h_{rm} h_{mr} - h_{rr} h_{m\alpha} h_{\alpha m}) \\ &= u_n^2 \sigma_k(G) \sum_{j\in B} H_{jj} + u_n^2 H \sigma_k(G) \sum_{j\in B} h_{jn}^2, \end{aligned}$$

and $III_2 = 2u_n \sigma_{k+1}(W) u_{n\alpha\alpha} = 0$. For the third term, we have

$$\begin{aligned} III_3 &= -2u_n \sigma_k(G) \sum_{i\in B} h_{in} u_{i\alpha\alpha} \\ &= -2u_n \sigma_k(G) \sum_{i\in B} h_{in} (u_{\alpha\alpha i} + u_m R_{\alpha m\alpha i}) \\ &= -2u_n \sigma_k(G) \sum_{i\in B} h_{in} (Hw)_i - 2u_n \sigma_k(G) \sum_{i\in B} h_{in} u_m (h_{\alpha\alpha} h_{mi} - h_{\alpha i} h_{m\alpha}) \\ &= -2u_n \sigma_k(G) \sum_{i\in B} h_{in} (H_i w - H h_{ij} u_j) \\ &\qquad - 2u_n^2 \sigma_k(G) \sum_{i\in B} h_{in}^2 H + 2u_n^2 \sigma_k(G) \sum_{i\in B} h_{in}^2 h_{nn} \\ &= -2w u_n \sigma_k(G) \sum_{j\in B} h_{in} H_j + 2u_n^2 \sigma_k(G) h_{nn} \sum_{j\in B} h_{jn}^2. \end{aligned}$$

We have used in the calculations above that

$$w_i = -\langle N, \xi \rangle_i = -\langle N_i, \xi \rangle = -\langle h_{ij} e_j, \xi \rangle = -h_{ij} u_j.$$

Substituting our results for III_1, III_2, and III_3 into (3-21) yields

$$III = u_n^2\sigma_k(G)\sum_{j\in B} H_{jj} + u_n^2\sigma_k(G)H\sum_{j\in B} h_{jn}^2 - 2wu_n\sigma_k(G)\sum_{j\in B} h_{jn}H_j + 2u_n^2\sigma_k(G)h_{nn}\sum_{j\in B} h_{jn}^2. \tag{3-22}$$

We decompose the final term in (3-13) as $IV_1 + IV_2 + IV_3 + IV_4$ by

$$\begin{aligned} IV &:= 2F^{ij}u_{i\alpha}u_{j\alpha} \\ &= 2F^{nn}u_{n\alpha}u_{n\alpha} + 4\sum_{i=1}^{n-1} F^{in}u_{i\alpha}u_{n\alpha} + 2\sum_{i=1}^{n-1} F^{ii}u_{i\alpha}u_{i\alpha} + 2\sum_{\substack{i,j=1\\ i\neq j}}^{n-1} F^{ij}u_{i\alpha}u_{j\alpha} \end{aligned}$$

It follows that $IV_1 = 2F^{nn}u_{n\alpha}u_{n\alpha} = 2\sigma_{k+1}(W)u_{n\alpha}u_{n\alpha} = 0$, and

$$\begin{aligned} IV_2 &= -4\sum_{i=1}^{n-1}\sigma_k(W\,|\,i)h_{in}u_{i\alpha}u_{n\alpha} = -4\sigma_k(G)\sum_{i\in B} h_{in}u_{i\alpha}u_{n\alpha} \\ &= -4w^2\sigma_k(G)\sum_{i\in B} h_{in}^2 h_{nn}. \end{aligned} \tag{3-23}$$

For the last two terms, we have

$$\begin{aligned} IV_3 &= 2\sum_{i\in G} F^{ii}u_{i\alpha}u_{i\alpha} + 2\sum_{i\in B} F^{ii}u_{i\alpha}u_{i\alpha} \\ &= -2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{jn}^2 u_{i\alpha}u_{i\alpha} + 2\sigma_k(G)\sum_{i\in B} h_{nn}u_{i\alpha}u_{i\alpha} \\ &\qquad - 2\sum_{i,j\in B, i\neq j}\sigma_k(G)h_{jn}^2 u_{i\alpha}u_{i\alpha} - 2\sum_{j\in G, i\in B}\sigma_{k-1}(G\,|\,j)h_{jn}^2 u_{i\alpha}u_{i\alpha} \\ &= -2w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{jn}^2 h_{ii}^2 - 2w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{jn}^2 h_{in}^2 \\ &\qquad + 2w^2\sigma_k(G)\sum_{i\in B} h_{nn}h_{in}^2 - 2w^2\sigma_{k-1}(G)\sum_{i,j\in B, i\neq j} h_{in}^2 h_{jn}^2 \\ &\qquad - 2w^2\sum_{j\in G, i\in B}\sigma_{k-1}(G\,|\,j)h_{jn}^2 h_{in}^2 \\ &= -2w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{ii}^2 h_{jn}^2 - 4w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{in}^2 h_{jn}^2 \\ &\qquad + 2w^2\sigma_k(G)\sum_{i\in B} h_{nn}h_{in}^2 - 2w^2\sigma_{k-1}(G)\sum_{i,j\in B, i\neq j} h_{in}^2 h_{jn}^2, \end{aligned}$$

and

$$\begin{aligned} IV_4 &= 2\sum_{i,j\in G, i\neq j} F^{ij}u_{i\alpha}u_{j\alpha} + 4\sum_{i\in G, j\in B} F^{ij}u_{i\alpha}u_{j\alpha} + 2\sum_{i,j\in B, i\neq j} F^{ij}u_{i\alpha}u_{j\alpha} \\ &= 0 + 4\sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i)h_{in}h_{jn}u_{i\alpha}u_{j\alpha} + 2\sigma_{k-1}(G)\sum_{i,j\in B, i\neq j} h_{in}h_{jn}u_{i\alpha}u_{j\alpha} \\ &= 4w^2\sum_{i\in G, j\in B} \sigma_{k-1}(G\,|\,i)h_{in}^2h_{jn}^2 + 2w^2\sigma_{k-1}(G)\sum_{i,j\in B, i\neq j} h_{in}^2h_{jn}^2. \end{aligned}$$

Our final result for IV is then

$$IV = -2w^2\sigma_k(G)h_{nn}\sum_{j\in B} h_{jn}^2 - 2w^2\sum_{i\in G, j\in B}\sigma_{k-1}(G\,|\,i)h_{ii}^2h_{jn}^2. \tag{3-24}$$

Combining (3-16), (3-20), (3-22) and (3-24) with (3-12) we have

$$|\nabla u|^{k+3}L_{\alpha\alpha} := I + II + III + IV := A + B + C, \tag{3-25}$$

where

$$\begin{aligned} C &:= \sigma_{k-1}(G)\Big(4wu_n\sum_{\substack{i,j\in B\\ i\neq j}} h_{ni}h_{nj}h_{ijn} + 4wu_n\sum_{j\in B} h_{nj}^2h_{jjn} \\ &\qquad\qquad - 4w^2\Big(\sum_{j\in B} h_{nj}^2\Big)^2 - u_n^2\sum_{\alpha=1}^{n}\sum_{i,j\in B} h_{ij\alpha}^2\Big) \\ &= -\sigma_{k-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B}(u_nh_{ij\alpha} - 2wh_{nj}h_{i\alpha})^2, \end{aligned}$$

and

$$\begin{aligned} A &:= \sigma_k(G)\Big(u_n^2\sum_{j\in B} H_{jj} - 2wu_n\sum_{j\in B} h_{jn}H_j - 4wu_n\sum_{j\in B} h_{nnj}h_{nj} \\ &\qquad + u_n^2H\sum_{j\in B} h_{jn}^2 + 2u_n^2h_{nn}\sum_{j\in B} h_{jn}^2 + 6w^2h_{nn}\sum_{j\in B} h_{nj}^2\Big) \\ &= \sigma_k(G)\Big(u_n^2\sum_{j\in B} H_{jj} - 6wu_n\sum_{j\in B} h_{jn}H_j + (3u_n^2 + 6w^2)H\sum_{j\in B} h_{jn}^2\Big) \\ &\quad + \sigma_k(G)\Big(-(6w^2 + 2u_n^2)\sum_{i\in G, j\in B} h_{ii}h_{jn}^2 + 4wu_n\sum_{i\in G, j\in B} h_{iij}h_{nj}\Big). \end{aligned}$$

The summand B is grouped in terms of $\sigma_{k-1}(G|i)$. We decompose the last two terms as $A_1 + A_2$. It follows that

$$B + A_2 = \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{k-1}(G|i)(-8w^2 h_{i\alpha}^2 h_{jn}^2 + 8wu_n h_{i\alpha} h_{jn} h_{ij\alpha} - 2u_n^2 h_{ij\alpha}^2 - 2u_n^2 h_{ii}^2 h_{jn}^2)$$
$$= -2\sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{k-1}(G|i)(u_n h_{ij\alpha} - 2w h_{i\alpha} h_{jn})^2 - 2u_n^2 \sigma_k(G)\sigma_1(G) \sum_{j\in B} h_{jn}^2. \tag{3-26}$$

Combining (3-25) with (3-26), we finally get

$$|\nabla u|^{k+3} L_{\alpha\alpha} = \sigma_k(G)\Big(u_n^2 \sum_{j\in B} H_{jj} - 6wu_n \sum_{j\in B} h_{jn} H_j + (3u_n^2 + 6w^2) H \sum_{j\in B} h_{jn}^2\Big) - 2\sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{k-1}(G|i)(u_n h_{ij\alpha} - 2w h_{i\alpha} h_{jn})^2 - 2u_n^2 \sigma_k(G)\sigma_1(G) \sum_{j\in B} h_{jn}^2 - \sigma_{k-1}(G) \sum_{\alpha=1}^{n} \sum_{i,j\in B} (u_n h_{ij\alpha} - 2w h_{nj} h_{i\alpha})^2. \tag{3-27}$$

Then, for $H = -f(X, N)$, the structure conditions on f is

$$-u_n^2 f_{jj} + 6wu_n h_{nj} f_j - (6 - 3u_n^2) f h_{nj}^2 \le 0 \quad \text{for each } j \in B, \tag{3-28}$$

where we have used $w^2 + u_n^2 = 1$. Now we can use the following formulas to get the structure condition on f. Following Guan, Lin, and Ma [Guan et al. 2006], we have for each $i \in \{1, 2, \dots, n\}$

$$f_i = \sum_{A=1}^{n+1} f_{X_A} e_i^A + f_{e_{n+1}} (e_{n+1})_i,$$
$$f_{ii} = \sum_{A,C=1}^{n+1} f_{X_A X_C} e_i^A e_i^C + \sum_{A=1}^{n+1} f_{X_A} X_{ii}^A + 2\sum_{A=1}^{n+1} f_{X_A e_{n+1}} e_i^A (e_{n+1})_i + f_{e_{n+1}, e_{n+1}} (e_{n+1})_i (e_{n+1})_i + f_{e_{n+1}} (e_{n+1})_{ii}. \tag{3-29}$$

For example, if $f(X, N) = f(X)$, then f satisfies

$$3(1 - u_n^2) f_j^2 \le (2 - u_n^2) f f_{jj} \tag{3-30}$$

and $f \geq 0$. Since $0 < u_n^2 \leq 1$, we reduce the structure conditions on f to

$$f \geq 0 \quad \text{and} \quad 3f_j^2 \leq 2ff_{jj} \quad \text{for all } j \in B. \tag{3-31}$$

So the structure conditions is $f \geq 0$ and the matrix

$$2f\frac{\partial^2 f}{\partial X_A \partial X_B} - 3\frac{\partial f}{\partial X_A}\frac{\partial f}{\partial X_B}$$

is positive semidefinite, where $1 \leq A, B \leq n+1$. Clearly (3-27) implies (3-4) under these conditions, which proves the case in which $k > 0$.

In case $k = 0$, only A_1 appears in (3-25), so this obviously finishes the proof of Theorem 1.2. □

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Received August 17, 2008. Revised April 14, 2009.

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