A CONSTANT RANK THEOREM FOR LEVEL SETS OF IMMERSED HYPERSURFACES IN $\mathbb{R}^{n+1}$ WITH PRESCRIBED MEAN CURVATURE

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We prove a constant rank theorem on the second fundamental forms of level sets of immersed hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed mean curvature.

1. Introduction

Constant rank theorems have been a powerful tool in the study of convex solutions to partial differential equations. Caffarelli and Friedman [1985] first proved a constant rank theorem on solutions to a class of semilinear elliptic PDEs in two dimensions and hence proved the strict convexity of the solutions. Singer, Wong, Yau and Yau explored a similar idea [Singer et al. 1985]. Korevaar and Lewis [1987] extended Caffarelli and Friedman’s results to the $n$-dimensional case. In the last decade, the constant rank theorem has been extended to fully nonlinear elliptic PDEs [Guan and Ma 2003; Caffarelli et al. 2007; Guan et al. 2006]; these authors found important applications for it in some geometric problems. For the convexity of level sets, Korevaar proved a constant rank theorem:

**Theorem 1.1** [Korevaar 1990]. *Let $\Omega$ be a connected domain in $\mathbb{R}^n$. Let $u \in C^4(\Omega)$ solve

\[
Lu := A\left(\Delta u - \frac{u_i u_j}{|\nabla u|^2} u_{ij}\right) + B\left(\frac{u_i u_j}{|\nabla u|^2} u_{ij}\right) = f(u, |\nabla u|),
\]

where $A, B, f$ are $C^2$ functions of $u$, and $\mu := |\nabla u|$. These satisfy the structure conditions

(i) $(\sqrt{A/B})_{\mu \mu} \geq 0$, and

(ii) $(f(u, \mu)/B \mu^2)_{\mu \mu} \leq 0$.*


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Suppose that $|\nabla u| \neq 0$ and that $u$ has convex level sets $\{x \in \Omega \mid u(x) \leq c\}$. Then all the level sets of $u$ have second fundamental forms with (the same) constant rank throughout $\Omega$.

The equations in Theorem 1.1 include $p$-Laplacian equations and mean curvature equations as special cases. In these cases we respectively take
\[
A = \mu^{p-2}, \quad B = (p-1)\mu^{p-2}, \quad \text{and} \quad A = \frac{1}{\sqrt{1+\mu^2}}, \quad B = \frac{1}{(1+\mu^2)^{3/2}}.
\]

Korevaar [1990] used this theorem to prove some interesting results on the convexity of the level sets of solutions to elliptic PDEs. Recently, Xu [2008] generalized Theorem 1.1 to the case where the function $f$ in (1-1) also depends on the coordinate variable $x$, and accordingly the structure condition (ii) turns into
\[
\mu^3 \frac{f(x, u, 1/\mu)}{B(u, 1/\mu)} \text{ is convex in } (x, \mu).
\]

In this paper, we will prove an analogous result on a class of immersed hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed mean curvature.

Let $M^n$ be a smooth immersed hypersurface in $\mathbb{R}^{n+1}$, and let $X : M \rightarrow \mathbb{R}^{n+1}$ be the immersion satisfying
\[
(1-2) \quad H = -f(X, N),
\]
where $H$ and $N$ are respectively the mean curvature and unit normal vectors of $M^n$ at $X$, and $f$ is a smooth function in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let $\xi$ be a fixed unit vector in $\mathbb{R}^{n+1}$. Then the height function of $M^n$ corresponding to $\xi$ can be expressed as $u(X) = \langle X, \xi \rangle$; here $\langle \cdot, \cdot \rangle$ means the usual Euclidean inner product in $\mathbb{R}^{n+1}$. Now, the level set of $M^n$ corresponding to $\xi$ with height $c$ is defined as
\[
(1-3) \quad \Sigma_c = \{X \in M^n \mid u(X) = c\}.
\]

Suppose $u$ has no critical point on $M^n$. Then $\Sigma_c$ can be considered as a hypersurface in the hyperplane $\Pi = \{X \in \mathbb{R}^{n+1} \mid \langle X, \xi \rangle = c\}$.

With the above notations, our constant rank theorem on the level sets of an immersed hypersurface with prescribed mean curvature can be stated as follows:

**Theorem 1.2.** Let $M^n$ be an immersed hypersurface in $\mathbb{R}^{n+1}$ whose mean curvature satisfies (1-2). Assume that the height function $u$ of $M^n$ corresponding to $\xi$ has no critical point, and that the level sets are all locally convex with respect to the normal direction $-Du$, that is, their second fundamental forms are positive semidefinite. Then the second fundamental forms of all the level sets have (the same) constant rank, provided $f(X, N) = f(X) \geq 0$ and the matrix
\[
(1-4) \quad 2f - \frac{\partial^2 f}{\partial X_A \partial X_B} - 3 \frac{\partial f}{\partial X_A} \frac{\partial f}{\partial X_B}
\]
is positive semidefinite, where $1 \leq A, B \leq n + 1$. In other words, when $f$ is a positive function, the condition (1-4) simply means that $f^{-1/2}$ is a concave function in $\mathbb{R}^{n+1}$.

**Remark 1.3.** For the more general case where $H = -f(X, N)$ as in (1-2), by (3-28) and (3-29) in Section 3, we still can choose the structure conditions on $f$ to ensure the result of Theorem 1.2. For example, if $f(X, N) = \langle \xi, N \rangle^\beta$ with $\langle \xi, N \rangle > 0$ on $M^n$, then the structure condition is $\beta \geq 1$ or $\beta \leq 0$.

**Remark 1.4.** Throughout, we adapt these conventions: The hypersurface $M^n$ is orientable. We choose the unit normal vector field $N$ so that it represents the orientation of $M^n$. The unit vector field normal to the level set is obtained by projecting $N$ onto the hyperplane $\Pi = \{X \in \mathbb{R}^{n+1} \mid \langle X, \xi \rangle = c\}$.

When do the solutions of elliptic PDEs have convex level sets? Gabriel [1957] proved that the level sets of the Green function on a 3-dimensional convex domain are strictly convex. Lewis [1977] extended Gabriel’s result to $p$-harmonic functions in higher dimensions. Caffarelli and Spruck [1982] generalized Lewis’s result to a class of semilinear elliptic PDEs. For recent progress, see [Colesanti and Salani 2003] and [Cuoghi and Salani 2006]. The constant rank theorem is an important step for the concrete convexity theorem, since one can use it to prove strict convexity results, as in, for example, [Korevaar 1990]. In practice, one always runs into difficulty at the critical points of the solution (or height functions in our case). In some sense our constant rank theorem is only a local and intermediate result.

In Section 2, we will give a formula for the curvatures of the level sets of an immersed hypersurface in $\mathbb{R}^{n+1}$. We prove it by the method of moving frames. We prove our main result, Theorem 1.2, in Section 3 using a calculation similar to the one in [Xu 2008].

## 2. Formulas of curvature of level sets

For a $C^2$ function $u$ defined in a $n$-dimensional domain $\Omega$ in $\mathbb{R}^n$, let $\kappa_1, \ldots, \kappa_{n-1}$ be the principal curvatures of the level sets of $u$ with respect to the normal direction $-Du$. Then the $k$-th curvature of the level sets, denoted by $L_k$, is the $k$-th elementary symmetric function of $\kappa_1, \ldots, \kappa_{n-1}$. Clearly, $L_1$ and $L_{n-1}$ are respectively the mean curvature and Gauss curvature of the level sets. If $u$ has no critical point, that is, $|\nabla u| \neq 0$, then Trudinger [1997] (see also [Gilbarg and Trudinger 1977]) expressed $L_k$ as

$$L_k = \frac{\partial \sigma_{k+1}(D^2u)}{\partial u_{ij}} u_i u_j |\nabla u|^{-k-2}, \quad (2-1)$$

where we use summation convention for repeated indices, and where $\sigma_k(D^2u)$ is the $k$-th elementary symmetric function of the eigenvalues of the Hessian $(D^2u)$. 

There is a formula analogous to (2-1) on hypersurfaces in $\mathbb{R}^{n+1}$.

**Proposition 2.1.** Let $M^n$ be a smoothly immersed hypersurface in $\mathbb{R}^{n+1}$. Let $u$ be its height function and $\Sigma_c$ one of its level sets, with respect to a fixed unit vector $\xi$, as given in the last section. Then the $k$-th curvature of the level set $\Sigma_c$ with respect to $-Du$ is

\begin{equation}
L_k = \frac{\partial \sigma_{k+1}(B)}{\partial h_{ij}} u_i u_j |\nabla u|^{-(k+2)}.
\end{equation}

Here $B = (h_{ij})$ is the second fundamental form of $M^n$, $\sigma_k(B)$ is the $k$-th elementary symmetric function of the eigenvalues of $B$, and $u_i$ for $1 \leq i \leq n$ are the first order covariant derivatives of $u$ computed in any orthonormal frame field on $M^n$.

Huang [1992] gave the formula (2-2) for $n = 2$. Here we give a complete proof by using moving frames. In this section, indices will run from 1 to $n-1$ when lower case and Greek; Latin indices will run from 1 to $n$ when lower case and from 1 to $n+1$ when upper case.

For an orthonormal frame field $\{X; e_A\}$ in $\mathbb{R}^{n+1}$, we have

\begin{equation}
dX = \omega_A e_A \quad \text{and} \quad de_A = \omega_{A,B} e_B,
\end{equation}

where $\{\omega_A\}$ is the dual frame of $\{e_A\}$, and $\{\omega_{A,B}\}$ are connection forms. Then the structure equations read as

\begin{equation}
d\omega_A = \omega_{A,B} \wedge \omega_B \quad \text{and} \quad d\omega_{A,B} = \omega_{A,C} \wedge \omega_{C,B}.
\end{equation}

If we choose $e_{n+1}$ to be the unit normal vector field $N$ of $M^n$, then $\omega_{n+1} = 0$ on $M^n$, and hence by (2-4)

\begin{equation}
\omega_{n+1,i} \wedge \omega_j = 0.
\end{equation}

Then Cartan’s lemma implies $\omega_{n+1,i} = h_{ij} \omega_j$ and $h_{ij} = h_{ji}$, where $B = (h_{ij})$ is the second fundamental form of $M^n$.

**Proof of Proposition 2.1.** First, we check that the right side of (2-2) is independent of the choice of the frame fields $\{X; e_i\}$ on $M^n$. Then we can just prove (2-2) in a special frame field.

Suppose $\{X; \tilde{e}_i\}$ is another frame field on $M^n$. Then there is an orthogonal transformation $T$ such that $(\tilde{e}_1, \ldots, \tilde{e}_n) = (e_1, \ldots, e_n)T$. Then

\begin{equation}
(\tilde{u}_1, \ldots, \tilde{u}_n) = (u_1, \ldots, u_n)T,
\end{equation}

where $\nabla u = u_i e_i = \tilde{u}_i \tilde{e}_i$ is the gradient of $u$. Also, for the dual frame field and the connection forms we have

$$(\tilde{\omega}_1, \ldots, \tilde{\omega}_n) = (\omega_1, \ldots, \omega_n)T,$$

$$(\tilde{\omega}_{1,n+1}, \ldots, \tilde{\omega}_{n,n+1}) = (\omega_{1,n+1}, \ldots, \omega_{n,n+1})T.$$
Furthermore, for the second fundamental form we have

\[ 2-7 \]

\[ \vec{B} = T^{-1} B T. \]

Obviously \( \sigma_k(B) \) and \( |\nabla u| \) are invariant under the transformation \( T \). Then the following equalities show that the right side of (2-2) is independent of the choice of \( \{e_1, \ldots, e_n\} \):

\[ 2-8 \]

\[ \frac{\partial \sigma_k(B)}{\partial h_{ij}} u_i u_j = \frac{\partial \sigma_k(B)}{\partial h_{ml}} \partial h_{ml} u_i u_j = \frac{\partial \sigma_k(B)}{\partial h_{ml}} \partial (T^{mp} h_{pq} T_{ql}) \partial h_{ij} u_i u_j \]

\[ = \frac{\partial \sigma_k(B)}{\partial h_{ml}} T^{ml} T_{ji} u_i u_j = \frac{\partial \sigma_k(B)}{\partial h_{ml}} T_{im} u_i T_{jl} u_j = \frac{\partial \sigma_k(B)}{\partial h_{ml}} \bar{u}_m \bar{u}_l. \]

Now we adapt the frame field above so that along the level set \( \Sigma_c \), the \( e_\alpha \) are its tangential vectors. Furthermore, we choose another frame field \( \tilde{e}_A \) in \( \mathbb{R}^{n+1} \) so that \( \tilde{e}_{n+1} = \xi \) and \( \tilde{e}_a = e_a \), and so that \( \tilde{e}_n \) lies in the hyperplane \( \Pi \) and is normal to \( \Sigma_c \) with the same direction of the projection of \( e_{n+1} = N \) on \( \Pi \). With respect to this frame field, the structure equations of \( \Sigma_c \) are

\[ 2-9 \]

\[ d\tilde{\omega}_i = \tilde{\omega}_{i,j} \wedge \tilde{\omega}_j \quad \text{and} \quad d\tilde{\omega}_{ij} = \tilde{\omega}_{i,l} \wedge \tilde{\omega}_{l,j}. \]

On \( \Sigma_c \), we have \( \tilde{\omega}_n = 0 \), which implies

\[ 2-10 \]

\[ \tilde{\omega}_{n,a} = \tilde{h}_{a\beta} \tilde{\omega}_{\beta} \quad \text{and} \quad \tilde{h}_{a\beta} = \tilde{h}_{\beta a}, \]

where \( \tilde{h}_{a\beta} \) is the second fundamental form of \( \Sigma_c \) in \( \Pi \) (with respect to the unit normal \( \tilde{e}_n \)).

Clearly \( e_n, e_{n+1} \) and \( \tilde{e}_n, \tilde{e}_{n+1} \) are in the same 2-plane perpendicular to the \( e_a \). Let \( \phi \) be the angle between \( e_n \) and \( \tilde{e}_n \). Then we have

\[ 2-11 \]

\[ \tilde{e}_n = e_n \cos \phi + e_{n+1} \sin \phi \quad \text{and} \quad \tilde{e}_{n+1} = -e_n \sin \phi + e_{n+1} \cos \phi. \]

Accordingly,

\[ 2-12 \]

\[ \tilde{\omega}_n = \omega_n \cos \phi + \omega_{n+1} \sin \phi, \quad \tilde{\omega}_{n+1} = -\omega_n \sin \phi + \omega_{n+1} \cos \phi, \quad \tilde{\omega}_a = \omega_a. \]

Taking the exterior derivative of (2-12), and using (2-4) and (2-12) again, we get

\[ 2-13 \]

\[ d\tilde{\omega}_n = (d\phi + \omega_{n+1}) \wedge \tilde{\omega}_{n+1} + ((\cos \phi) \omega_n, + (\sin \phi) \omega_{n+1}) \wedge \omega_a, \]

\[ d\tilde{\omega}_{n+1} = (-d\phi + \omega_{n+1}) \wedge \tilde{\omega}_n + ((\cos \phi) \omega_{n+1}, - (\sin \phi) \omega_n) \wedge \omega_a. \]

Notice that \( \tilde{\omega}_n = \tilde{\omega}_{n+1} = 0 \) on \( \Sigma_c \). Comparing (2-13) with (2-9), we have

\[ 2-14 \]

\[ \tilde{\omega}_{n,a} = (\cos \phi) \omega_{n,a} + (\sin \phi) \omega_{n+1,a}, \]

\[ \tilde{\omega}_{n+1,a} = (-\sin \phi) \omega_n, + (\cos \phi) \omega_{n+1,a}. \]
On the other hand, \( \langle \tilde{e}_a, \zeta \rangle = 0 \) on \( \Sigma_c \), and since \( d(\langle \tilde{e}_a, \zeta \rangle) = \langle \tilde{\omega}_a, \tilde{\omega}_a, \zeta \rangle \), we have \( \tilde{\omega}_{a,n+1} = 0 \). This together with (2-14) implies
\[
\tilde{\omega}_{a} = \frac{\cos^2 \phi}{\sin \phi} \omega_{n+1,a} + (\sin \phi) \omega_{n+1,a} = \frac{1}{\sin \phi} \omega_{n+1,a} = \frac{1}{\sin \phi} (h_{a\beta} \omega_{\beta} + h_{a_n} \omega_n).
\]
Combining this with (2-10) gives
\[
\tilde{h}_{a\beta} = \frac{1}{\sin \phi} h_{a\beta} \quad \text{and} \quad h_{a,n} = 0.
\]
From the definition of the height function \( u \), we can see \( u_i = e_i (\langle X, \zeta \rangle) = \langle e_i, \zeta \rangle \); in particular, \( u_n = \langle e_n, \zeta \rangle \). Note that \( \tilde{e}_{n+1} = \zeta \), hence the second equation of (2-11) implies \( u_n = -\sin \phi \) and \( \langle \zeta, e_{n+1} \rangle = \cos \phi \). By the decomposition
\[
\zeta = \sum_{i=1}^n \langle \zeta, e_i \rangle e_i + \langle \zeta, e_{n+1} \rangle e_{n+1}
\]
we deduce that \( 1 = |\nabla u|^2 + \cos^2 \phi \) and therefore \( |\nabla u| = \pm \sin \phi \). With \( e_n \) chosen suitably we may assume \( \sin \phi > 0 \). Then (2-16) becomes
\[
\tilde{h}_{a\beta} = \frac{1}{|\nabla u|} h_{a\beta} \quad \text{and} \quad h_{a,n} = 0.
\]
From this one can easily see that
\[
L_k = \sigma_k(\tilde{h}_{a\beta}) = \frac{1}{|\nabla u|^2} \sigma_k(h_{a\beta})
\]
\[
= \frac{1}{|\nabla u|^{k+2}} \frac{\partial \sigma_{k+1}(B)}{\partial h_{nn}} u_n u_n = \frac{\partial \sigma_{k+1}(B)}{\partial h_{ij}} u_i u_j |\nabla u|^{-(k+2)},
\]
where we have used \( |u_n| = |\nabla u| \).

3. Proof of Theorem 1.2

We adapt the notations in Section 2, and collect these formulas for convenience:
\[
X_i = e_i,
\]
\[
X_{ij} = -h_{ij} e_{n+1}
\]
(Gauss formula),
\[
e_{n+1,i} = h_{ij} e_j
\]
(Weingarten formula),
\[
h_{ijk} = h_{ikj}
\]
(Codazzi equation),
\[
R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}
\]
(Gauss equation),
\[
h_{ijkl} = h_{ijk} + h_{im} R_{mijkl} + h_{jm} R_{mikl},
\]
and for the smooth function \( u \) on \( M^n \) we also have the Ricci identity

\[
    u_{ijk} = u_{ikj} + u_m R_{mijk},
\]

where \( R_{ijkl} \) is the Riemann curvature tensor, and as for the rest of this section, repeated indices are summed from 1 to \( n \), unless otherwise stated.

**Proof of Theorem 1.2.** Suppose the second fundamental forms of the level sets of \( M^n \) take the minimum rank \( k \) with \( k \leq n - 2 \) at a point \( P \in M^n \). We will treat the case \( k > 0 \) first, and then show how to modify the argument for the case \( k = 0 \).

With the assumption that the level sets are all locally convex, we find easily that

\[
    L_r(P) = 0 \quad \text{for all } r > k,
\]

\[
    L_r(P) > 0 \quad \text{for all } r \leq k,
\]

and moreover

\[
    Z := \{ X \in M^n \mid \text{the second fundamental form of the level sets of } M^n \text{ has rank } k \text{ at } X \} = \{ X \in M^n \mid L_{k+1}(X) = 0 \}.
\]

Obviously \( Z \) is a closed set in \( M^n \). If we can show that \( Z \) is also open in \( M^n \)—that is, that there is a neighborhood \( U_P \) of \( P \) in \( M^n \) such that \( L_{k+1} \equiv 0 \) on \( U_P \)—then \( Z = M^n \), which is the result in the theorem.

Now \( L_{k+1}(P) = 0 = \min_{X \in M^n} L_{k+1}(X) \), so by the strong maximum principle, we need only to show that

\[
    \Delta L_{k+1}(X) \leq 0 \mod \{ L_{k+1}(X), \nabla L_{k+1}(X) \} \quad \text{in } U_P,
\]

where we modify the terms of \( L_{k+1} \) and its first derivatives, coefficients are locally bounded, and \( \Delta \) is the Beltrami–Laplace operator on \( M^n \).

For the rest of this section, define

\[
    W := (h_{ij}) \quad \text{with } i, j \leq n - 1, \quad L := L_{k+1}, \quad F := \sigma_{k+2}(B),
\]

and

\[
    F^{ij} := \frac{\partial F}{\partial h_{ij}}, \quad F^{ij,rs} := \frac{\partial^2 F}{\partial h_{ij} \partial h_{rs}}, \quad F^{ij,rs,pq} := \frac{\partial^3 F}{\partial h_{ij} \partial h_{rs} \partial h_{pq}}.
\]

Hence, by (2-2),

\[
    |\nabla u|^{k+3} L = F^{ij} u_i u_j.
\]

Taking the covariant derivative of this, we get

\[
    (|\nabla u|^{k+3} L)_a = |\nabla u|^{k+3} L_a + (|\nabla u|^{k+3})_a L,
\]

\[
    (F^{ij} u_i u_j)_a = F^{ij,rs} h_{rs} u_i u_j + 2 F^{ij} u_{ia} u_j.
\]
Taking the covariant derivative again, we get

\[
(|\nabla u|^{k+3} L)_{aa} = |\nabla u|^{k+3} L_{aa} + 2(|\nabla u|^{k+3})_{a} L_{a} + (|\nabla u|^{k+3})_{aa} L,
\]

(3-7) \( (F^{ij} u_{ij})_{aa} = F^{ij,rs}_{pq} h_{pq} h_{rs} u_{ai} u_{aj} + F^{ij,rs}_{k} h_{rs} u_{ia} u_{aj} + 4 F^{ij,rs}_{k} h_{rs} u_{aa} u_{ij} + 2 F^{ij}_{aa} u_{ij} + 2 F^{ij}_{ia} u_{ja}. \)

For a fixed point \( X_{0} \) in \( U_{P} \), choose a frame \( \{e_{1}, \ldots, e_{n}\} \) such that \( u_{i} \) through \( u_{n-1} \) vanish, \( |u_{n}| = |\nabla u| > 0 \), the form \( W \) is diagonal, and \( h_{11} \geq h_{22} \geq \cdots \geq h_{n-1,n-1} \). Then by (3-2) we see that with \( |\nabla u| \) small enough

\[
\begin{align*}
    h_{rr}(X_{0}) &= 0 \mod \{L(X_{0}), \nabla L(X_{0})\} \quad \text{for all } r > k, \\
    h_{rr}(X_{0}) &> \epsilon > 0 \mod \{L(X_{0}), \nabla L(X_{0})\} \quad \text{for all } r \leq k,
\end{align*}
\]

where \( \epsilon \) is a positive sufficiently small number (maybe depending on \( U_{P} \)).

In the following, all the calculations will be done at \( X_{0} \), and the terms of \( L(X_{0}) \) and \( \nabla L(X_{0}) \) will be dropped, that is, all the equalities or inequalities should be understood \( \mod\{L(X_{0}), \nabla L(X_{0})\} \).

Denote \( G := \{h_{11}, \ldots, h_{kk}\} \) and \( B := \{h_{k+1,k+1}, \ldots, h_{n-1,n-1}\} \). Use the same symbols for \( G := \{1, \ldots, k\} \) and \( B := \{k+1, \ldots, n-1\} \) (it won’t cause confusion).

Now, by \( L(P) = 0 = \min_{X \in M^{*}} L(X) \) we get

\[
\begin{align*}
    0 &= (|\nabla u|^{k+3} L)_{a} = (F^{ij} u_{ij})_{a} = F^{ij,rs}_{k} h_{rs} u_{ai} u_{aj} + 2 F^{ij}_{aa} u_{ij} + 2 F^{ij}_{ia} u_{ja} \\
    &= u_{n}^{2} F^{nn,rr}_{k} h_{rr} + 2 u_{n} F^{in}_{ia} \\
    &= u_{n}^{2} \sigma_{k}(G) \sum_{r \in B} h_{rr} + 2 u_{n} F^{nn}_{na} + 2 u_{n} \sum_{i=1}^{n-1} F^{in}_{ia} \\
    &= u_{n}^{2} \sigma_{k}(G) \sum_{r \in B} h_{rr} - 2 u_{n} \sigma_{k}(G) \sum_{i \in B} h_{ni} u_{ia}.
\end{align*}
\]

Clearly

\[
\begin{align*}
    u_{i} &= \langle X, \xi \rangle_{i} = \langle X_{i}, \xi \rangle = (e_{i}, \xi), \\
    u_{ij} &= \langle X_{ij}, \xi \rangle = -\langle h_{ij} N, \xi \rangle := h_{ij} w,
\end{align*}
\]

where \( w = -\langle N, \xi \rangle = \pm \sqrt{1 - |\nabla u|^{2}}. \)

Substituting (3-10) into (3-9), using (3-8), and noting that \( W \) is diagonal, we deduce

\[
\begin{align*}
    \sum_{i \in B} h_{ii a} &= 0 \quad \text{for all } a < n, \\
    u_{n} \sum_{i \in B} h_{ii n} &= 2 \sum_{i \in B} h_{ni}^{2} w.
\end{align*}
\]

By (3-7) we have

\[
|\nabla u|^{k+3} L \equiv |\nabla u|^{k+3} L_{aa} + 2(|\nabla u|^{k+3})_{a} L_{a} + (|\nabla u|^{k+3})_{aa} L = (F^{ij} u_{ij})_{aa}.
\]
That is,

\[ |\nabla u|^3 L_{aa} = F^{ij,rs,pq}_{u} h_{pqa} h_{rsa} u_i u_j + F^{ij,rs}_{u} h_{rsa} u_i u_j + 4 F^{ij,rs}_{u} h_{rsa} u_i a u_j + 2 F^{ij}_{u} u_{iaa} u_j + 2 F^{ij}_{u} u_{iaa} u_j \]

\[ = u_n^2 F^{nn,rs,pq}_{u} h_{pqa} h_{rsa} + u_n^2 F^{nn,rs}_{u} h_{rsa} + 4 u_n F^{in,rs}_{u} h_{rsa} u_{iaa} + 2 u_n F^{in}_{u} u_{iaa} + 2 F^{ij}_{u} u_{iaa} u_j, \]

which we decompose as \( I + II + III + IV \), where

\[ I := u_n^2 F^{nn,rs,pq}_{u} h_{pqa} h_{rsa}, \quad II := 4 u_n F^{nn,rs}_{u} h_{rsa} u_{iaa}, \]

\[ III := u_n^2 F^{nn,rs}_{u} h_{rsa} + 2 u_n F^{in}_{u} u_{iaa}, \quad IV := 2 F^{ij}_{u} u_{iaa} u_j. \]

Next we will compute the above terms step by step. First

\[ I := u_n^2 F^{nn,rs,pq}_{u} h_{pqa} h_{rsa} = u_n^2 F^{nn,rr,ss}_{u} h_{rra} h_{ssa} + u_n^2 F^{nn,rs,ss}_{u} h_{rra} h_{ssa} =: I_1 + I_2, \]

and

\[ I_1 := u_n^2 F^{nn,rr,ss}_{u} h_{rra} h_{ssa} = 2 u_n^2 \sum_{r \in G, s \in B} F^{nn,rr,ss}_{u} h_{rra} h_{ssa} + u_n^2 \sum_{r,s \in B} F^{nn,rr,ss}_{u} h_{rra} h_{ssa} \]

\[ = 2 u_n^2 \sum_{r \in G, s \in B} \sigma_{k-1}(G|r) h_{rra} h_{ssa} + u_n^2 \sum_{r \in B, s \in G} h_{rra} h_{ssa}, \]

where here and below we use the notation \( \sigma_{k-1}(G|r) := \sigma_{k-1}(G \setminus \{h_r\}) \) and the convention \( \sigma_0 = 1 \). Substituting (3-11) into (3-15) yields

\[ I_1 = 2 u_n^2 \sum_{r \in G, s \in B} \sigma_{k-1}(G|r) h_{rra} h_{ssa} + u_n^2 \sum_{r \in B} h_{rra} \left( \sum_{s \in B} h_{ssa} - h_{rra} \right) \]

\[ = 4 w u_n \sum_{s \in B} h_{ss}^2 \sum_{r \in G} \sigma_{k-1}(G|r) h_{rra} h_{ssa} - u_n^2 \sum_{a=1}^n \sigma_{k-1}(G) \sum_{r \in B} h_{rra}^2 \]

\[ + 4 w^2 \sigma_{k-1}(G) \left( \sum_{s \in B} h_{ss}^2 \right)^2. \]

For the remaining term in (3-14), we have

\[ I_2 = 2 u_n^2 \sum_{r \in G, s \in B} F^{nn,rs,ss}_{u} h_{rsa} h_{rsa} + u_n^2 \sum_{r,s \in B} F^{nn,rs,ss}_{u} h_{rsa} h_{rsa} \]

\[ = -2 u_n^2 \sum_{a=1}^n \sum_{r \in G, s \in B} \sigma_{k-1}(G|r) h_{rra}^2 - u_n^2 \sigma_{k-1}(G) \sum_{a=1}^n \sum_{r,s \in B, r \neq s} h_{rsa}^2. \]
So for the first term in (3-13) we have

\[
I = 4w_u \sum_{i \in G, j \in B} \sigma_{k-1}(G \mid i)h_{ii}h_{jj}^2 - 2u_n^2 \sum_{a=1}^{n} \sum_{i \in G, j \in B} \sigma_{k-1}(G \mid i)h_{ij}^2
\]

\[
+ 4w^2 \sigma_{k-1}(G) \left( \sum_{j \in B} h_{jn}^2 \right)^2 - u_n^2 \sigma_{k-1}(G) \sum_{a=1}^{n} \sum_{i, j \in B} h_{ij}^2.
\]

To compute the second term in (3-13), first we have by using (3-10)

\[
II = 4w_u F^{in,rs} h_{rs} h_{ia}
\]

\[
= 4w_u F^{nn,rs} h_{rs} h_{na} + 4w_u \sum_{i=1}^{n-1} F^{in,ni} h_{nia} h_{ia}
\]

\[
+ 4w_u \sum_{i, j=1}^{n-1} F^{in,ji} h_{ija} h_{ia} + 4w_u \sum_{i=1}^{n-1} F^{in,rr} h_{rra} h_{ia}.
\]

We decompose the last four terms as \(II_1 + II_2 + II_3 + II_4\). By (3-11), the first can be treated as

\[
II_1 = 4w_u F^{nn,rr} h_{rra} h_{na} = 4w_u \sigma_k(G) \sum_{r \in B} h_{rra} h_{na}
\]

\[
= 4w_u \sigma_k(G) \sum_{r \in B} h_{rrn} h_{nn} = 8w^2 \sigma_k(G) h_{nn} \sum_{r \in B} h_{rrn}^2.
\]

For the second and the third terms, straightforward calculations show that

\[
II_2 = -4w_u \sigma_k(G) \sum_{i \in B} h_{nia} h_{ia} = -4w_u \sigma_k(G) \sum_{i \in B} h_{nni} h_{in},
\]

and

\[
II_3 = 4w_u \sum_{i, j \in B} F^{in,ji} h_{ija} h_{ia}
\]

\[
+ 4w_u \sum_{i \in G, j \in B} F^{in,ji} h_{ija} h_{ia} + 4w_u \sum_{j \in G, j \in B} F^{in,ji} h_{ija} h_{ia}
\]

\[
= 4w_u \sigma_{k-1}(G) \sum_{i, j \in B, i \neq j} h_{jn} h_{ij} h_{in} + 4w_u \sum_{i \in G, j \in B} \sigma_{k-1}(G \mid i)h_{ij} h_{ija} h_{ia}
\]

\[
+ 4w_u \sum_{i \in G, j \in B} \sigma_{k-1}(G \mid i)h_{ij} h_{ijn} h_{jn}.
\]
Again by (3-11), the fourth term can be treated as

$$II_4 = 4wu_n \sum_{i,r \in B} F_{in,rr} h_{iia} + 4wu_n \sum_{i \in G, r \in B} F_{in,rr} h_{iia} + 4wu_n \sum_{r \in G, i \in B} F_{in,rr} h_{iia}$$

$$= -4wu_n \sigma_{k-1}(G) \sum_{i,r \in B, i \neq r} h_{iia} h_{rra} - 4wu_n \sum_{i \in G, r \in B} \sigma_{k-1}(G | i) h_{ni} h_{rra}$$

$$= -4wu_n \sigma_{k-1}(G) \sum_{i,r \in B, i \neq r} h_{iia} h_{rra} - 4wu_n \sum_{i \in G, r \in B} \sigma_{k-1}(G | i) h_{ni} h_{rra}$$

$$= -4wu_n \sigma_{k-1}(G) \sum_{i \in G, r \in B} \sigma_{k-1}(G | i) h_{ni} h_{rra}$$

$$= -4wu_n \sigma_{k-1}(G) \sum_{i \in G} h_{iia} \left( \sum_{r \in B} h_{rra} - h_{iia} \right)$$

$$= 4wu_n \sigma_{k-1}(G) \sum_{i \in B} h_{iia}^2 - 8wu_n \sigma_{k-1}(G) \left( \sum_{i \in B} h_{iia}^2 \right)^2$$

$$\text{It follows that}$$

$$II = 8w^2 \sigma_k(G) h_{in} \sum_{j \in B} h_{nj}^2 - 4wu_n \sigma_k(G) \sum_{j \in B} h_{nj} h_{nj}$$

$$- 8w^2 \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{iia}^2 h_{jja} - 4wu_n \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{ni} h_{nij} h_{iia}$$

$$+ 4wu_n \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{iia} h_{jja} + 4wu_n \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{ni} h_{nij} h_{iia}$$

$$+ 4wu_n \sigma_{k-1}(G) \sum_{i,j \in B, i \neq j} h_{nia} h_{nij} + 4wu_n \sigma_{k-1}(G) \sum_{j \in B} h_{nj}^2 h_{jja}$$

$$\quad - 8w^2 \sigma_{k-1}(G) \left( \sum_{j \in B} h_{nj}^2 \right)^2.$$
Now we deal with the third term in (3-13):

\[
\text{III} := u_n^2 F^{nn, rs} h_{rsaa} + 2u_n F^{in} u_{iaa}
\]

(3-21)

\[
= u_n^2 F^{nn, rr} h_{rraa} + 2u_n F^{nn} u_{nnaa} + 2u_n \sum_{i=1}^{n-1} F^{in} u_{iaa}.
\]

We decompose the last three terms as \(\text{III}_1 + \text{III}_2 + \text{III}_3\). Using the exchange formula in (3-1), we can calculate

\[
\text{III}_1 = u_n^2 \sigma_k(G) \sum_{r \in B} h_{rraa}
\]

\[
= u_n^2 \sigma_k(G) \sum_{r \in B} (h_{raar} + h_{rm} R_{mara} + h_{am} R_{mera})
\]

\[
= u_n^2 \sigma_k(G) \sum_{r \in B} h_{aarr}
\]

\[
+ u_n^2 \sigma_k(G) \sum_{r \in B} (h_{rm} (h_{ma} h_{ar} + h_{am} (h_{mr} h_{ra} - h_{ma} h_{rr}))
\]

\[
= u_n^2 \sigma_k(G) \sum_{r \in B} H_{rr} + u_n^2 \sigma_k(G) \sum_{r \in B} (H h_{rm} h_{mr} - h_{rr} h_{ma} h_{am})
\]

\[
= u_n^2 \sigma_k(G) \sum_{j \in B} H_{jj} + u_n^2 \sigma_k(G) \sum_{j \in B} h_{jn}^2,
\]

and \(\text{III}_2 = 2u_n \sigma_{k+1}(W) u_{nnaa} = 0\). For the third term, we have

\[
\text{III}_3 = -2u_n \sigma_k(G) \sum_{i \in B} h_{in} u_{iaa}
\]

\[
= -2u_n \sigma_k(G) \sum_{i \in B} h_{in} (u_{aa} + u_m R_{maa})
\]

\[
= -2u_n \sigma_k(G) \sum_{i \in B} h_{in} (H w)_i - 2u_n \sigma_k(G) \sum_{i \in B} h_{in} u_m (h_{aa} h_{mi} - h_{ai} h_{ma})
\]

\[
= -2u_n \sigma_k(G) \sum_{i \in B} h_{in} (H w - H h_{ij} u_{ij})
\]

\[
= -2u_n \sigma_k(G) \sum_{i \in B} h_{in}^2 H + 2u_n^2 \sigma_k(G) \sum_{i \in B} h_{in}^2 h_{nn}
\]

\[
= -2w u_n \sigma_k(G) \sum_{j \in B} h_{jn} H_j + u_n^2 \sigma_k(G) h_{nn} \sum_{j \in B} h_{jn}^2.
\]

We have used in the calculations above that

\[
\omega_i = -\langle N, \zeta \rangle_i = -\langle N_i, \zeta \rangle = -\langle h_{ij} e_j, \zeta \rangle = -h_{ij} u_{ij}.
\]
Substituting our results for \(III_1, III_2, \) and \(III_3\) into (3-21) yields

\[
(3-22) \quad IIIV = u^2_n \sigma_k(G) \sum_{j \in B} H_{jj} + u^2_n \sigma_k(G) H \sum_{j \in B} h_{jn}^2 - 2wu_n \sigma_k(G) \sum_{j \in B} h_{jn}H_j + 2u^2_n \sigma_k(G) h_{nn} \sum_{j \in B} h_{jn}^2.
\]

We decompose the final term in (3-13) as \(IV_1 + IV_2 + IV_3 + IV_4\) by

\[
IV := 2F^{ij} u_{ia} u_{ja}
\]

\[
= 2F^{mn} u_{na} u_{na} + 4 \sum_{i=1}^{n-1} F^{in} u_{ia} u_{na} + 2 \sum_{i=1}^{n-1} F^{ii} u_{ia} u_{ia} + 2 \sum_{i,j=1}^{n-1} F^{ij} u_{ia} u_{ja}
\]

It follows that \(IV_1 = 2F^{mn} u_{na} u_{na} = 2\sigma_{k+1}(W) u_{na} u_{na} = 0\), and

\[
(3-23) \quad IV_2 = -4 \sum_{i=1}^{n-1} \sigma_k(W | i) h_{in} u_{ia} u_{na} = -4\sigma_k(G) \sum_{i \in B} h_{in} u_{ia} u_{na}
\]

\[
= -4u^2 \sigma_k(G) \sum_{i \in B} h_{in}^2 h_{nn}.
\]

For the last two terms, we have

\[
IV_3 = 2 \sum_{i \in G} F^{ii} u_{ia} u_{ia} + 2 \sum_{i \in B} F^{ii} u_{ia} u_{ia}
\]

\[
= -2 \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{jn}^2 u_{ia} u_{ia} - 2\sigma_k(G) \sum_{i \in B} h_{nn} u_{ia} u_{ia}
\]

\[
- 2 \sum_{i,j \in B, i \neq j} \sigma_k(G) h_{jn}^2 u_{ia} u_{ia} - 2 \sum_{i,j \in B, j \neq i} \sigma_{k-1}(G | j) h_{jn}^2 u_{ia} u_{ia}
\]

\[
= -2u^2 \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{jn}^2 h_{ii} - 2u^2 \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{jn}^2 h_{jn}^2
\]

\[
+ 2u^2 \sigma_k(G) \sum_{i \in B} h_{in}^2 h_{jn}^2 - 2u^2 \sigma_{k-1}(G) \sum_{i,j \in B, i \neq j} h_{in}^2 h_{jn}^2
\]

\[
- 2u^2 \sum_{j \in G, i \in B} \sigma_{k-1}(G | j) h_{jn}^2 h_{in}^2
\]

\[
= -2u^2 \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{jn}^2 h_{jn}^2 - 4u^2 \sum_{i \in G, j \in B} \sigma_{k-1}(G | i) h_{jn}^2 h_{jn}^2
\]

\[
+ 2u^2 \sigma_k(G) \sum_{i \in B} h_{in}^2 h_{jn}^2 - 2u^2 \sigma_{k-1}(G) \sum_{i,j \in B, i \neq j} h_{in}^2 h_{jn}^2.
\]
and

\[ IV_4 = 2 \sum_{i,j \in G, i \neq j} F^{ij} u_{ia} u_{ja} + 4 \sum_{i \in G, j \in B} F^{ij} u_{ia} u_{ja} + 2 \sum_{i \in G, j \neq i} F^{ij} u_{ia} u_{ja} \]

\[ = 0 + 4 \sum_{i \in G, j \in B} \sigma_{k-1}(G \mid i) h_{in} h_{jn} u_{ia} u_{ja} + 2 \sigma_{k-1}(G) \sum_{i, j \in B, i \neq j} h_{in} h_{jn} u_{ia} u_{ja} \]

\[ = 4w^2 \sum_{i \in G, j \in B} \sigma_{k-1}(G \mid i) h_{in}^2 h_{jn}^2 + 2w^2 \sigma_{k-1}(G) \sum_{i, j \in B, i \neq j} h_{in}^2 h_{jn}^2. \]

Our final result for IV is then

\[ (3-24) \quad IV = -2w^2 \sigma_k(G) h_{nn} \sum_{j \in B} h_{jn}^2 - 2w^2 \sum_{i \in G, j \in B} \sigma_{k-1}(G \mid i) h_{in}^2 h_{jn}^2. \]

Combining (3-16), (3-20), (3-22) and (3-24) with (3-12) we have

\[ (3-25) \quad |\nabla u|^{k+3} L_{aa} := I + III + IV := A + B + C, \]

where

\[ C := \sigma_{k-1}(G) \left( 4w u_n \sum_{i, j \in B} h_{ni} h_{nj} h_{ijn} + 4w u_n \sum_{j \in B} h_{nj}^2 h_{jnn} \right. \]

\[ - 4w^2 \left( \sum_{j \in B} h_{nj}^2 \right)^2 - u_n^2 \sum_{a=1}^n \sum_{i, j \in B} h_{ija}^2 \right) \]

\[ = -\sigma_{k-1}(G) \sum_{a=1}^n \sum_{i, j \in B} (u_n h_{ija} - 2w h_{nj} h_{ija})^2, \]

and

\[ A := \sigma_k(G) \left( u_n^2 \sum_{j \in B} H_{jj} - 2w u_n \sum_{j \in B} h_{jn} H_j - 4w u_n \sum_{j \in B} h_{nnj} h_{nj} \right. \]

\[ + u_n^2 H \sum_{j \in B} h_{jn}^2 + 2u_n^2 h_{nn} \sum_{j \in B} h_{jn}^2 + 6w^2 h_{nn} \sum_{j \in B} h_{nj}^2 \right) \]

\[ = \sigma_k(G) \left( u_n^2 \sum_{j \in B} H_{jj} - 6w u_n \sum_{j \in B} h_{jn} H_j + (3u_n^2 + 6w^2) H \sum_{j \in B} h_{jn}^2 \right) \]

\[ + \sigma_k(G) \left( -(6w^2 + 2u_n^2) \sum_{i \in G, j \in B} h_{ii} h_{jnn}^2 + 4w u_n \sum_{i \in G, j \in B} h_{ii} h_{nj} \right). \]
The summand $B$ is grouped in terms of $\sigma_{k-1}(G|i)$. We decompose the last two terms as $A_1 + A_2$. It follows that

$$B + A_2 = \sum_{a=1}^{n} \sum_{i \in G, j \in B} \sigma_{k-1}(G|i)(-8w^2h^2_{ia}h^2_{ja} + 8w u_n h_{ia}h_{ja}H_{ja} - 2u_n^2 h_{ija}^2 - 2u_n^2 h_{ija}^2)$$

$$= -2 \sum_{a=1}^{n} \sum_{i \in G, j \in B} \sigma_{k-1}(G|i)(u_n h_{ija} - 2w h_{ija})^2$$

Combining (3-25) with (3-26), we finally get

$$\nabla u^{k+3} L_{aa} = \sigma_k(G)(u_n^2 \sum_{j \in B} H_{jj} - 6w u_n \sum_{j \in B} h_{jn} H_{j})$$

$$+ (3u_n^2 + 6w^2) H \sum_{j \in B} h_{jn}^2$$

$$- 2 \sum_{a=1}^{n} \sum_{i \in G, j \in B} \sigma_{k-1}(G|i)(u_n h_{ija})$$

$$- 2w h_{ija} h_{ja})^2 - 2u_n^2 \sigma_k(G) \sigma_1(G) \sum_{j \in B} h_{jn}^2$$

$$- \sigma_{k-1}(G) \sum_{a=1}^{n} \sum_{i, j \in B} (u_n h_{ija} - 2w h_{ja} h_{ia})^2.$$
and \( f \geq 0 \). Since \( 0 < u_n^2 \leq 1 \), we reduce the structure conditions on \( f \) to

\[
(3-31) \quad f \geq 0 \quad \text{and} \quad 3f_j^2 \leq 2ff_{jj} \quad \text{for all} \quad j \in B.
\]

So the structure conditions is \( f \geq 0 \) and the matrix

\[
2f \frac{\partial^2 f}{\partial X_A \partial X_B} - 3 \frac{\partial f}{\partial X_A} \frac{\partial f}{\partial X_B}
\]

is positive semidefinite, where \( 1 \leq A, B \leq n+1 \). Clearly (3-27) implies (3-4) under these conditions, which proves the case in which \( k > 0 \).

In case \( k = 0 \), only \( A_1 \) appears in (3-25), so this obviously finishes the proof of Theorem 1.2.

\[ \square \]

References


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