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SYMPLECTIC SUPERCUSPIDAL REPRESENTATIONS OF GL(2n) OVER p-ADIC FIELDS

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This is part two of the authors' work on supercuspidal representations of GL(2n) over p-adic fields. We consider the complete relations among the local theta correspondence, local Langlands transfer, and the local descent attached to a given irreducible symplectic supercuspidal representation of p-adic GL_{2n}. This is the natural extension of the work of Ginzburg, Rallis and Soudry and of Jiang and Soudry on the local descents and the local Langlands transfers. The approach undertaken in this paper is purely local. A mixed approach with both local and global methods, which works for more general classical groups, has been considered by Jiang and Soudry.

1. Introduction

Let $\mathcal{F}$ be a p-adic local field of characteristic zero. Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$. By the local Langlands conjecture for $\mathrm{GL}_{2n}(\mathcal{F})$, which is now a theorem of Harris and Taylor [2001] and of Henniart [2000], there exists an irreducible admissible $2n$-dimensional representation ϕ of the local Weil group $\mathcal{W}_{\mathcal{F}}$, that is, the local Langlands parameter

$$\phi : \mathcal{W}_{\mathcal{F}} \to \mathrm{GL}_{2n}(\mathbb{C}),$$

corresponding to τ with a set of required conditions. We say that τ is of symplectic type if the image $\phi(\mathcal{W}_{\mathcal{F}})$ is contained in the symplectic subgroup $\mathrm{Sp}_{2n}(\mathbb{C})$ of the complex dual group $\mathrm{GL}_{2n}(\mathbb{C})$ of $\mathrm{GL}_{2n}(\mathcal{F})$.

Because of their deep connection with Galois representations, symplectic supercuspidal representations (or more importantly cuspidal automorphic representations) have recently received much attention; see for instance [Ginzburg et al. 2004; Chenevier and Clozel 2009]. The symplectic irreducible unitary supercuspidal

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representations of $\mathrm{GL}_{2n}(\mathcal{F})$ were characterized in [Shahidi 1990; 1992; Jacquet and Rallis 1996; Ginzburg et al. 1999; Jiang and Soudry 2003; 2004; Jiang and Qin 2007; Jiang et al. 2008] and were discussed in detail in [Jiang et al. 2008, Section 5]. We state these results as follows; the theorem's notation and terminology will explained in Section 2.

Theorem 1.1. *Suppose τ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$. Then the following are equivalent.*

(1) *τ is of symplectic type.*

(2) *The local exterior square L-factor $L(s, \tau, \Lambda^2)$ has a pole at $s = 0$.*

(3) *The local exterior square γ-factor $\gamma(s, \tau, \Lambda^2, \psi)$ has a pole at $s = 1$.*

(4) *τ has a nonzero Shalika model.*

(5) *The unitarily induced representation $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ is reducible at $s = 1$. In this case, $\mathrm{I}^{\mathrm{SO}_{4n}}(1, \tau)$ has the unique Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$, which has a nonzero generalized Shalika model.*

(6) *τ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(\mathcal{F})$.*

(7) *τ has a nonzero linear model, that is, a $\mathrm{GL}_n(\mathcal{F}) \times \mathrm{GL}_n(\mathcal{F})$-invariant functional.*

(8) *The unitarily induced representation $\mathrm{I}^{\mathrm{Sp}_{4n}}(s, \tau)$ of $\mathrm{Sp}_{4n}(\mathcal{F})$ is reducible at $s = 1/2$, and $\mathrm{I}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ has the unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$, which has a nonzero symplectic linear model, that is, a $\mathrm{Sp}_{2n}(\mathcal{F}) \times \mathrm{Sp}_{2n}(\mathcal{F})$-invariant functional.*

(9) *τ is a local Langlands functorial ψ-transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$.*

If one of the above holds for τ, then τ is self-dual.

The local Langlands functorial ψ-transfer from an irreducible ψ-generic supercuspidal representation $\tilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to the irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ is given by the [Ginzburg et al. 1999, corollary of Section 1.5]. The local exterior square L-function and gamma factor are given by the Shahidi method.

The equivalence of the characterizations in Theorem 1.1 can be explained by Figure 1. The complex dual groups of $\mathrm{SO}_{2n+1}(\mathcal{F})$ and the double metaplectic cover $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ of $\mathrm{Sp}_{2n}(\mathcal{F})$ are the same, namely, $\mathrm{Sp}_{2n}(\mathbb{C})$. In Figure 1, the map θc is the local theta correspondence for the reductive dual pairs $(\mathrm{SO}_{4n}, \mathrm{Sp}_{4n})$ and $(\mathrm{SO}_{2n+1}, \widetilde{\mathrm{Sp}}_{2n})$. The map G-G is the local Gelfand–Graev coefficient that takes representations from SO_{4n} to SO_{2n+1}. The map F-J is the local Fourier–Jacobi coefficient that takes representations from Sp_{4n} to $\widetilde{\mathrm{Sp}}_{2n}$. The map Lq is the composition of the parabolic induction from the standard parabolic subgroups with the Levi subgroup isomorphic to GL_{2n} in SO_{4n} and Sp_{4n}, and that takes the unique

$$
\begin{tikzcd}
\mathrm{SO}_{4n} \arrow[rr, leftrightarrow, "\theta\mathrm{c}"] \arrow[dd, "\text{G-G}"'] \arrow[rr, phantom, "\text{(A)}", shift right=8ex] & & \mathrm{Sp}_{4n} \arrow[dd, "\text{F-J}"] \\
 & \mathrm{GL}_{2n} \arrow[ul, "\mathrm{Lq}"'] \arrow[ur, "\mathrm{Lq}"] \arrow[l, phantom, "\text{(B)}"] \arrow[r, phantom, "\text{(C)}"] & \\
\mathrm{SO}_{2n+1} \arrow[ur, "\mathrm{Lt}"] \arrow[rr, leftrightarrow, "\theta\mathrm{c}"'] \arrow[rr, phantom, "\text{(D)}", shift left=8ex] & & \widetilde{\mathrm{Sp}}_{2n} \arrow[ul, "\mathrm{Lt}"']
\end{tikzcd}
$$

Diagram 1

Langlands quotient from the induced representations of SO_{4n} and Sp_{4n}, respectively. It is clear that G-G $\circ$ Lq and F-J $\circ$ Lq are the local descents from GL_{2n} to SO_{2n+1} and $\widetilde{\mathrm{Sp}}_{2n}$, respectively, in the sense of Ginzburg, Rallis and Soudry. Finally the map Lt is the local Langlands functorial transfer from SO_{2n+1} to GL_{2n} and from $\widetilde{\mathrm{Sp}}_{2n}$ to GL_{2n}.

For a given irreducible unitary symplectic supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathscr{F})$, the maps in Figure 1 can be realized as in Figure 2, where notation is as follows. First, σ is an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathscr{F})$, which lifts to τ by the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}, and $\tilde{\pi}$ is an irreducible ψ-generic supercuspidal representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathscr{F})$,

$$
\begin{tikzcd}
\mathscr{L}^{\mathrm{SO}_{4n}}(1,\tau) \arrow[rr, leftrightarrow, "\theta\mathrm{c}"] \arrow[dd, "\text{G-G}"'] \arrow[rr, phantom, "\text{(A)}", shift right=8ex] & & \mathscr{L}^{\mathrm{Sp}_{4n}}(1/2,\tau) \arrow[dd, "\text{F-J}"] \\
 & \tau \arrow[ul, "\mathrm{Lq}"'] \arrow[ur, "\mathrm{Lq}"] \arrow[l, phantom, "\text{(B)}"] \arrow[r, phantom, "\text{(C)}"] & \\
\sigma \arrow[ur, "\mathrm{Lt}"] \arrow[rr, leftrightarrow, "\theta\mathrm{c}"'] \arrow[rr, phantom, "\text{(D)}", shift left=8ex] & & \tilde{\pi} \arrow[ul, "\mathrm{Lt}"']
\end{tikzcd}
$$

Diagram 2

which lifts to τ by the local Langlands functorial ψ-transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to GL_{2n}. Consider the maximal parabolic subgroup P of SO_{4n} with Levi subgroup GL_{2n}. Then the unitarily parabolic induction $\mathrm{I}^{\mathrm{SO}_{4n}}(1, \tau)$ has a unique Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$, and similarly the unitarily parabolic induction $\mathrm{I}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ has a unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$. Finally, the local Gelfand–Graev coefficient takes $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ from $\mathrm{SO}_{4n}(\mathcal{F})$ back to $\mathrm{SO}_{2n+1}(\mathcal{F})$ and the local Fourier–Jacobi coefficient takes $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ from $\mathrm{Sp}_{4n}(\mathcal{F})$ back to $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$, respectively. Detailed discussion of these maps is found in Section 2.

Theorem 1.2. *For an irreducible unitary symplectic supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, Figure 2 is commutative.*

Now we explain the relation between Theorem 1.1 and Theorem 1.2, or the commutative diagrams Figure 1 and Figure 2.

Jiang and Soudry [2003] proved that for a given irreducible unitary symplectic supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, there exists uniquely an irreducible generic supercuspidal representation σ of $\mathrm{SO}_{2n+1}(\mathcal{F})$ and an irreducible ψ-generic supercuspidal representation $\tilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$, such that the subdiagram (D) is commutative. The local Langlands functorial transfer property for τ is equivalent to the existence of a pole at $s = 0$ of the local exterior square L-factor $L(s, \tau, \Lambda^2)$, or equivalently by definition a pole at $s = 1$ of the local exterior square γ-factor $\gamma(s, \tau, \Lambda^2, \psi)$. One very interesting point is the characterization in terms of the existence of a nonzero Shalika model (or functional) or of a nonzero linear model (or functional), following the idea of relative trace formula approach to the global Langlands functorial transfers. It was proved in [Jiang et al. 2008] that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, the existence of a nonzero Shalika model for τ is equivalent to the existence of a nonzero linear model for τ, although this result had been expected for a while. Jacquet and Rallis [1996] proved that the existence of a nonzero Shalika model for τ implies the existence of a nonzero linear model for τ.

For an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, why does the existence of a nonzero linear model for τ determine the local Langlands functorial transfer from $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$ to GL_{2n}, while the existence of a nonzero Shalika model for τ determines the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}? To answer this, Ginzburg, Rallis, and Soudry [Ginzburg et al. 1999] showed that if an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ has a nonzero linear model, that is, a nonzero $\mathrm{GL}_n(\mathcal{F}) \times \mathrm{GL}_n(\mathcal{F})$-invariant functional, then the unique Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ of the unitarily parabolic induction $\mathrm{I}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ (which is reducible) has a nonzero symplectic linear model, that is, a nonzero $\mathrm{Sp}_{2n}(\mathcal{F}) \times \mathrm{Sp}_{2n}(\mathcal{F})$-invariant functional. Based on the existence of a nonzero symplectic linear model for $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$, they show that the ψ-local descent (the

Fourier–Jacobi ψ-functor in this case) yields $\tilde{\pi}$ back to $\widetilde{\mathrm{Sp}}_{2n}(\mathcal{F})$. This proves that the subdiagram (C) is commutative.

The local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ was first obtained in [Jiang and Soudry 2003] by combining the subdiagrams (C) and (D) and by using the local converse theorem. More recently, Jiang and Soudry (see [Soudry 2008]) obtained the local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ via the global theory of the automorphic descent [Ginzburg et al. 2001]. Their method works for other classical groups as well.

In [Jiang and Qin 2007; Jiang et al. 2008], we began the task of establishing the local descent $\tau \mapsto \sigma$ from $\mathrm{GL}_{2n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ by using the existence of a nonzero Shalika model for τ of $\mathrm{GL}_{2n}(\mathcal{F})$ and of a nonzero generalized Shalika model for the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$. We proved by a purely local argument in [Jiang et al. 2008, Theorem 3.1] that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, the local Gelfand–Graev coefficient (a special type of twisted Jacquet functor) of the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$, which is a representation of $\mathrm{SO}_{2r+1}(\mathcal{F})$, vanishes for all $r < n$. Here, again using a purely local argument, we show that for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, the local Gelfand–Graev coefficient of the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$ is an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$; this, Theorem 2.5, is our main result. The proof idea was suggested by the global argument as in [Ginzburg et al. 2001]. Our proof goes similarly to the case of symplectic linear models in [Ginzburg et al. 1999], but is essentially based on the existence and uniqueness of a generalized Shalika model for the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$. The technical details are of independent interest, and are found in Sections 3, 4 and 5.

One fact that needs to be shown here is that the local Gelfand–Graev coefficient on $\mathrm{SO}_{2n+1}(\mathcal{F})$ from $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ lifts to τ via the local Langlands functorial transfer. In [Jiang and Soudry 2003; Soudry 2008], a global argument is used to show that this is the case. However, one would like to prove this by a purely local argument. One way to do this is to calculate explicitly the local Rankin–Selberg integral for the tensor product L-functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_r$ by using the supercuspidal representation constructed explicitly by the local Gelfand–Graev coefficient from $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ to $\mathrm{SO}_{2n+1}(\mathcal{F})$; however we do not do this here. Hence, the subdiagram (B) is commutative by Theorem 2.5 and the result in [Jiang and Soudry 2003; Soudry 2008].

Finally, we show that the subdiagram (A) is also commutative by using results of G. Muic [2006], which show that the Langlands quotient $\mathcal{L}^{\mathrm{SO}_{4n}}(1, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ and the Langlands quotient $\mathcal{L}^{\mathrm{Sp}_{4n}}(1/2, \tau)$ of $\mathrm{Sp}_{4n}(\mathcal{F})$ correspond to each other via the local theta correspondence. By combining this with Theorem 1.1, one deduces that the generalized Shalika model on $\mathrm{SO}_{4n}(\mathcal{F})$ and the symplectic linear model of $\mathrm{Sp}_{4n}(\mathcal{F})$ are related by the local theta correspondence. It would be interesting to check directly, without using Theorem 1.1, that the local theta correspondence relates the generalized Shalika model on $\mathrm{SO}_{4n}(\mathcal{F})$ and the symplectic linear model of $\mathrm{Sp}_{4n}(\mathcal{F})$.

In future work, we will study the explicit relations between Diagrams 1 and 2 and refined structures of the corresponding local Arthur packets.

2. Main result

We introduce definitions of various models and of the local descent in the case under consideration, and then state the main result for the local descent.

2.1. ***Shalika and generalized Shalika models.*** Let $\mathcal{F}$ be a finite extension of the p-adic number field $\mathbb{Q}_p$ for some rational prime p. Take the maximal parabolic subgroup $P_{n,n} = M_{n,n}N_{n,n}$ of GL_{2n} with

$$M_{n,n} = \mathrm{GL}_n \times \mathrm{GL}_n,$$
$$N_{n,n} = \left\{ n(X) = \begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix} \in \mathrm{GL}_{2n} \right\}.$$

Let ψ be a nontrivial character of $\mathcal{F}$. Define a character $\psi_{N_{n,n}}(n(X)) = \psi(\mathrm{tr}(X))$. The stabilizer of $\psi_{N_{n,n}}$ in $M_{n,n}$ is GL_n^Δ, the diagonal embedding of GL_n into $M_{n,n}$. Denote by

$$\mathcal{S}_n = \mathrm{GL}_n^\Delta \rtimes N_{n,n}$$

the Shalika subgroup. Denote by $\psi_{\mathcal{S}_n}$ the extension of $\psi_{N_{n,n}}$ from $N_{n,n}$ to the Shalika subgroup $\mathcal{S}_n$ such that $\psi_{\mathcal{S}_n}$ is trivial on GL_n^Δ. The Shalika functionals of an irreducible admissible representation (τ, V_τ) of $\mathrm{GL}_{2n}(\mathcal{F})$ are nonzero elements of the space $\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n})$. By the Frobenius reciprocity

$$\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n}) \cong \mathrm{Hom}_{\mathrm{GL}_{2n}(\mathcal{F})}(V_\tau, \mathrm{Ind}_{\mathcal{S}_n(\mathcal{F})}^{\mathrm{GL}_{2n}(\mathcal{F})}(\psi_{\mathcal{S}_n})),$$

any nonzero Shalika functional ℓ_ψ in $\mathrm{Hom}_{\mathcal{S}_n(\mathcal{F})}(V_\tau, \psi_{\mathcal{S}_n})$ gives rise to an embedding

$$V_\tau \hookrightarrow \mathrm{Ind}_{\mathcal{S}_n(\mathcal{F})}^{\mathrm{GL}_{2n}(\mathcal{F})}(\psi_{\mathcal{S}_n}),$$

the image of which is called a local Shalika model of V_τ. Jacquet and Rallis [1996] (and also Nien [2009] by different argument) proved that the local Shalika model is unique for any irreducible admissible representation of $\mathrm{GL}_{2n}(\mathcal{F})$.

Jiang and Qin [2007] introduced the *generalized Shalika model* for $\mathrm{SO}_{4n}(\mathcal{F})$. Let $\nu_1 = 1$ and inductively define

$$\nu_n = \begin{pmatrix} & 1 \\ \nu_{n-1} & \end{pmatrix} \quad \text{for } n \geq 2 \text{ and } n \in \mathbb{N}. \tag{2-1}$$

Let SO_{4n} be the even special orthogonal group attached to the nondegenerate $4n$-dimensional quadratic vector space over $\mathcal{F}$ with respect to ν_{4n}. That is,

$$\mathrm{SO}_{4n} = \{g \in \mathrm{GL}_{4n} \mid {}^t g \cdot \nu_{4n} \cdot g = \nu_{4n}\}.$$

Let $P_{2n} = M_{2n} V_{2n}$ be the Siegel parabolic subgroup of SO_{4n}, consisting of elements of the form

$$(g, X) = \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \begin{pmatrix} \mathrm{I}_n & X \\ & \mathrm{I}_n \end{pmatrix}, \tag{2-2}$$

where $g \in \mathrm{GL}_{2n}$ and $g^* = \nu_{2n} \, {}^t g^{-1} \nu_{2n}$, and X satisfies ${}^t X = -\nu_{2n} X \nu_{2n}$.

The generalized Shalika group $\mathcal{H}_{2n}$ of SO_{4n} is the subgroup of P consisting of elements (g, X) with $g \in \mathrm{Sp}_{2n}$. Here the symplectic group is given by

$$\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid {}^t g \cdot J_{2n} \cdot g = J_{2n}\}, \quad \text{where } J_{2n} = \begin{pmatrix} & \nu_n \\ -\nu_n & \end{pmatrix} \quad \text{for } n \in \mathbb{N}.$$

Define a character $\psi_{\mathcal{H}}$ of $\mathcal{H}_{2n}(\mathcal{F})$ (we write $\mathcal{H} = \mathcal{H}_{2n}$ when n is understood) by letting

$$\begin{aligned} \psi_{\mathcal{H}}((g, X)) &= \psi(\mathrm{tr}(J_{2n} X \nu_{2n})) \\ &= \psi(\mathrm{tr}((\mathrm{diag}(-\mathrm{I}_n, \mathrm{I}_n))X)). \end{aligned}$$

It is well defined. The *generalized Shalika functional* or *$\psi_{\mathcal{H}}$-functional* of an irreducible admissible representation (σ, V_σ) of $\mathrm{SO}_{4n}(\mathcal{F})$ is a nonzero functional in the space

$$\mathrm{Hom}_{\mathrm{SO}_{4n}(\mathcal{F})}(V_\sigma, \mathrm{Ind}_{\mathcal{H}_{2n}(\mathcal{F})}^{\mathrm{SO}_{4n}(\mathcal{F})}(\psi_{\mathcal{H}})) = \mathrm{Hom}_{\mathcal{H}_{2n}(\mathcal{F})}(V_\sigma, \psi_{\mathcal{H}}).$$

Nien [2010] has shown the uniqueness of the generalized Shalika model. Hence one can use a nonzero generalized Shalika functional to define a generalized Shalika model for σ. To relate the Shalika model on GL_{2n} and the generalized Shalika model on SO_{4n}, we consider the following parabolic induction.

For an irreducible, unitary, supercuspidal representation (τ, V_τ) of $\mathrm{GL}_{2n}(\mathcal{F})$, we consider the unitary representation $\mathrm{I}(s, \tau)$ of $\mathrm{SO}_{4n}(\mathcal{F})$ induced from the Siegel parabolic subgroup $P_{2n} = M_{2n} V_{2n}$, where the Levi part M_{2n} is isomorphic to GL_{2n}, via the bijection

$$a \in \mathrm{GL}_{2n} \mapsto m(a) := \mathrm{diag}(a, a^*) \in M_{2n}.$$

More precisely, a section $\phi_{\tau,s}$ in $\mathrm{I}(s, \tau)$ is a smooth function from $\mathrm{SO}_{4n}(\mathcal{F})$ to V_τ such that

$$\phi_{\tau,s}(m(a)ng) = |\det a|^{s/2+(2n-1)/2} \tau(a) \phi_{\tau,s}(g),$$

where $m(a) \in M_{2n}$ with $a \in \mathrm{GL}_{2n}(\mathcal{F})$ and $n \in V_{2n}$. In other words, one has

$$\mathrm{I}(s, \tau) = \mathrm{Ind}_{P_{2n}(\mathcal{F})}^{\mathrm{SO}_{4n}(\mathcal{F})}(|\det|^{s/2} \cdot \tau).$$

In the introduction, we used notation $\mathrm{I}^{\mathrm{SO}_{4n}}(s,\tau)$ for $\mathrm{I}(s,\tau)$ as a reminder that it is a representation of SO_{4n}. From now on, we simply use the notation $\mathrm{I}(s,\tau)$.

The relation between the Shalika model on GL_{2n} and the generalized Shalika model on SO_{4n} is given by the following theorem.

Theorem 2.2 [Jiang and Qin 2007, Theorem 3.1]. *The induced representation* $\mathrm{I}(s,\tau)$ *admits a nonzero generalized Shalika functional only when* $s=1$. *In that case*, $\mathrm{I}(1,\tau)$ *admits a nonzero generalized Shalika functional if and only if the supercuspidal datum* τ *admits a nonzero Shalika functional. The generalized Shalika functionals of* $\mathrm{I}(1,\tau)$ *are unique up to scalar*, *and if nonzero*, *they must factor through the unique Langlands quotient* $\mathscr{L}(1,\tau)$.

Again from now on we simply use $\mathscr{L}(1,\tau)$ rather than $\mathscr{L}^{\mathrm{SO}_{4n}}(1,\tau)$.

2.3. ***A family of degenerate Whittaker models.*** Degenerate Whittaker models for a reductive group G can be defined for any given nilpotent orbit in the Lie algebra $\mathfrak{g}$ of G; see [Mœglin and Waldspurger 1987]. Here, we consider a family of nilpotent orbits $\mathbb{O}_{2n,2n-k}$ of SO_{4n} corresponding to a family of partitions $[2(2n-k)+1, 1^{2k-1}]$ for $k=1,2,\dots,2n$. This family of degenerate Whittaker models on $\mathrm{SO}_{4n}(\mathscr{F})$ was considered in [Ginzburg et al. 1997] for construction of automorphic L-functions of orthogonal groups, and in [Ginzburg et al. 1999] for construction of the Ginzburg–Rallis–Soudry global descents. We take a family of unipotent subgroups N_k of SO_{4n} consisting of elements of type

$$n=n(u,b,z)=\begin{pmatrix} u & b & z \\ & \mathrm{I}_{4n-2k} & b' \\ & & u' \end{pmatrix}\in \mathrm{SO}_{4n}, \tag{2-3}$$

where $u=(u_{i,j})\in \mathrm{U}_k$, the maximal unipotent subgroup of GL_k consisting of all upper triangular unipotent matrices in GL_k, the block $b=(b_{i,j})$ is the implied size, and b' and u' are determined by b and u so that n belongs to SO_{4n}. We define a character ψ_k on N_k by

$$\psi_k(n):=\psi(u_{1,2}+\cdots+u_{k-1,k})\psi(b_{k,2n-k}+b_{k,2n-k+1}). \tag{2-4}$$

When $k=2n-1$, the subgroup N_k coincides with the unipotent radical N of the Borel subgroup of SO_{4n}, and ψ_k is the generic character of N. Let π be an irreducible admissible representation (π,V_π) of $\mathrm{SO}_{4n}(\mathscr{F})$. Then π has a nonzero ψ_k-functional if

$$\mathrm{Hom}_{\mathrm{SO}_{4n}(\mathscr{F})}(V_\pi,\mathrm{Ind}_{N_k(\mathscr{F})}^{\mathrm{SO}_{4n}(\mathscr{F})}(\psi_k))\cong \mathrm{Hom}_{N_k(\mathscr{F})}(V_\pi,\psi_k)\neq 0. \tag{2-5}$$

In this case, a nonzero element in $\mathrm{Hom}_{N_k(\mathscr{F})}(V_\pi,\psi_k)$ is called a ψ_k-functional of V_π, or more precisely, a ψ_k-degenerate Whittaker functional of V_π. For each

ψ_k-functional ℓ_{ψ_k}, we define

$$\mathcal{W}_{\psi_k,v}(g) := \ell_{\psi_k}(\pi(g)(v)) \quad \text{for } v \in V_\pi, \tag{2-6}$$

which yields a ψ_k-degenerate Whittaker model (also called an (N_k, ψ_k)-model) for V_π. In particular, when $k = 2n-1$, it produces a Whittaker model for V_π. Note that the different choices of the representatives in the $\mathcal{F}$-rational points of the unipotent orbit $\mathcal{O}_{2n,k}(\mathcal{F})$ produce different characters for $N_k(\mathcal{F})$, and hence different degenerate Whittaker models. However, the centralizers are all isomorphic, which is the $\mathcal{F}$-split $\mathrm{SO}_{4n-2k-1}(\mathcal{F})$. This is different from the case of odd orthogonal groups considered in [Jiang and Soudry 2007].

We recall the definition of Jacquet functor and module. Fix a closed subgroup $\widetilde{P} = \widetilde{N} \rtimes \widetilde{M}$ of SO_{4n} with unipotent radical $\widetilde{N}$ and a character χ on $\widetilde{N}$ normalized by $\widetilde{M}$. Then for a representation (V_π, π) of $\mathrm{SO}_{4n}(\mathcal{F})$, its Jacquet module with respect to $(\widetilde{N}, \chi)$ is defined by

$$\mathcal{J}\{\widetilde{N}, \chi\}(\pi) = V_\sigma / \operatorname{Span}\{\sigma(n)v - \chi(n)v \mid n \in \widetilde{N}, v \in V_\pi\},$$

viewed as a representation of $\widetilde{M}$. We call $\mathcal{J}\{\widetilde{N}, \chi\}$ the Jacquet functor with respect to $(\widetilde{N}, \chi)$. We write $\mathcal{J}\{\widetilde{N}\}$ for $\mathcal{J}\{\widetilde{N}, \chi\}$ when χ is trivial. For the family of ψ_k-degenerate Whittaker models, we abbreviate the corresponding family of ψ_k-twisted Jacquet modules by

$$\mathcal{J}\{\psi_k\}(V_\pi) := \mathcal{J}\{N_k, \psi_k\}(V_\pi), \tag{2-7}$$

viewed as a representation of $\mathrm{SO}_{4n-2k-1}(\mathcal{F})$.

Theorem 2.4 [Jiang et al. 2008, Theorem 3.1]. *Suppose (π, V_π) is an irreducible admissible representation of $\mathrm{SO}_{4n}(\mathcal{F})$. If π has a nonzero generalized Shalika model, then the ψ_k-twisted Jacquet modules $\mathcal{J}\{\psi_k\}(V_\pi)$ are all zero for $n \le k \le 2n$.*

For an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model, we apply the family of the ψ_k-twisted Jacquet functors to the Langlands quotient $\mathcal{L}(1, \tau)$. By Theorem 2.4, the first interesting representation we get from $\mathcal{L}(1, \tau)$ is at $k = n-1$, that is,

$$\sigma_{n-1} = \sigma_{n-1}(\tau) := \mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1, \tau)), \tag{2-8}$$

which is an admissible representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. We call σ_{n-1} the local descent of τ from GL_{2n} to SO_{2n+1}. The main result of this paper is this:

Theorem 2.5. *Suppose τ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model. Then its local descent σ_{n-1} is irreducible, generic, and a supercuspidal representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.*

We prove Theorem 2.5 in Sections 3, 4, and 5. In Section 3, we prove that the local descent σ_{n-1} as defined in (2-8) is quasisupercuspidal, which means the (nontwisted) Jacquet module $\mathcal{J}\{N\}(\sigma_{n-1})$ is trivial for the unipotent radical N of every standard proper parabolic group of SO_{2n+1}; see Theorem 3.1 for details. Hence we can write the local descent σ_{n-1} as a direct sum

$$\sigma_{n-1} = \sigma_{n-1}^1 \oplus \cdots \oplus \sigma_{n-1}^r \oplus \cdots,$$

where the σ_{n-1}^i are irreducible supercuspidal representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$. We show in Theorem 4.1(2) that the local descent σ_{n-1} has a nonzero Whittaker functional, which is unique up to a scalar. Hence among the summands σ_{n-1}^i, one and only one has a nonzero Whittaker functional, that is, it is generic. Finally, we prove in Theorem 5.1(2) that every irreducible supercuspidal summand in σ_{n-1} is generic. This implies that the local descent σ_{n-1} has only one irreducible summand, and therefore, σ_{n-1} is irreducible, generic, and supercuspidal, proving Theorem 2.5.

3. Supercuspidality of the local descent

We first prove the quasisupercuspidality of $\sigma_{n-1} = \sigma_{n-1}(\tau) = \mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1, \tau))$, as defined in (2-8) for any irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ with a nonzero Shalika model.

We relate any standard Jacquet module of σ_{n-1} to further descent σ_k of $\mathcal{L}(1, \tau)$ with $k \geq n$ in the tower of the local Gelfand–Graev models for the Langlands quotient $\mathcal{L}(1, \tau)$. Because $\mathcal{L}(1, \tau)$ has a nonzero generalized Shalika model, all standard Jacquet modules of σ_{n-1} must be zero by Theorem 2.4. The same proof can be used to show that the local descents from $\mathcal{L}(1, \tau)$ satisfy the local tower property as in [Ginzburg et al. 1999], but we omit the details here.

First we have to fix notation. Consider the embedding of elements in SO_{2k-1} into SO_{2k}, so that the embedding of unipotent elements are described explicitly.

Let $n = n(u, b, c)$ be a unipotent element of SO_{2k-1} of type

$$n = n(u, b, c) = \begin{pmatrix} u & b & c \\ & 1 & b' \\ & & u^* \end{pmatrix} \in \mathrm{SO}_{2k-1} \tag{3-1}$$

where u is in U_{k-1}, the maximal upper triangular unipotent subgroup of GL_{k-1}. Then the embedding of n under the embedding from SO_{2k-1} into SO_{2k} is given by

$$n \mapsto \iota(n) = \begin{pmatrix} u & b & -b & c \\ & 1 & 0 & -b' \\ & & 1 & b' \\ & & & u^* \end{pmatrix} \in \mathrm{SO}_{2k}\,. \tag{3-2}$$

Theorem 3.1. *Let τ be an irreducible supercuspidal representation of* $\mathrm{GL}_{2n}(\mathscr{F})$ *with $n \geq 2$, such that $L(s, \tau, \Lambda^2)$ has a pole at $s = 0$. Then $\sigma_{n-1} = \mathscr{J}\{\psi_{n-1}\}(\mathscr{L}(1, \tau))$ is a quasisupercuspidal representation of* $\mathrm{SO}_{2n+1}(\mathscr{F})$.

Proof. For simplicity, we set $\sigma := \mathscr{L}(1, \tau)$, which is an admissible representation of $\mathrm{SO}_{2n+1}(\mathscr{F})$. Denote by U_{n-1} be the maximal (upper triangular) unipotent subgroup of $\mathrm{GL}_{n-1}(\mathscr{F})$. Recall that N_{2n} is the unipotent radical of Siegel parabolic groups of SO_{4n}. For $x \in \mathscr{F}$, denote by $u_{i,j}(x)$ the unipotent matrix in SO_{4n} corresponding to $x(e_i - e_j)$, the x-multiple of root $e_i - e_j$, and let $U_{i,j} = \{u_{i,j}(x) \mid x \in \mathscr{F}\}$.

There are n unipotent radicals Q_k for $1 \leq k \leq n$ corresponding to standard maximal parabolic subgroups of SO_{2n+1}, and given by

$$Q_k = \left\{ \begin{pmatrix} \mathrm{I}_k & C & D \\ & \mathrm{I}_{2n-2k+1} & C^* \\ & & \mathrm{I}_k \end{pmatrix} \right\} \subset \mathrm{SO}_{2n+1}\,.$$

Denote by ι the embedding of elements of SO_{2n+1} into SO_{2n+2} as in (3-2).

Let $H_1 = \iota(Q_k)N_{n-1}$, and denote its elements by

$$w(r, x, y, A, B) = \begin{pmatrix} r & & x & & y \\ & \begin{pmatrix} \mathrm{I}_k & A & B \\ & \mathrm{I}_{2n-2k+2} & A^* \\ & & \mathrm{I}_k \end{pmatrix} & & & x' \\ & & & & \\ & & & & r^* \end{pmatrix} \quad \text{for } r \in \mathrm{U}_{n-1}.$$

Write $r = (r_{i,j})$ and $x = (x_{i,j})$ and so on. Let ψ_{H_1} be the trivial extension of ψ_{n-1} to H_1, that is,

$$\psi_{H_1}(w(r, x, y, A, B)) = \psi(r_{1,2} + \cdots + r_{n-2,n-1})\psi(x_{n-1,n+1} + x_{n-1,n+2}).$$

To show that $\mathscr{J}\{\psi_{n-1}\}(\sigma)$ is supercuspidal, it suffices to show that

$$\mathscr{J}\{\iota(Q_k)\}(\mathscr{J}\{\psi_{n-1}\}(\sigma)) = 0 \quad \text{for all } 1 \leq k \leq n.$$

We begin by assuming to the contrary that $\mathscr{J}\{\iota(Q_k)\}(\mathscr{J}\{\psi_{n-1}\}(\sigma)) \neq 0$ for some $1 \leq k \leq n$. Then there exists a nonzero functional Φ_1 on V_σ such that

$$\Phi_1(\sigma(g)v) = \psi_{H_1}(g)\Phi_1(v) \tag{3-3}$$

holds for $g \in H_1$ and $v \in V_\sigma$.

Let H_2 be the complement of $\prod_{i=1}^{n-1} U_{i,n}$ in H_1, and define a character ψ_{H_2} on H_2 by $\psi_{H_2} = \psi_{H_1}|_{H_2}$. Then $\Phi_1(\sigma(g)v) = \psi_{H_2}(g)\Phi_1(v)$ for $g \in H_2$ and $v \in V_\sigma$. Denote by η the permutation matrix in SO_{4n} corresponding to the permutation product $(1, \ldots, n-1, n)(3n+1, \ldots, 4n)$ of two cycles. Let $H_3 = \eta H_2 \eta^{-1}$ and $\psi_{H_3}(g) = \psi_{H_2}(\eta^{-1} g \eta)$ for $g \in H_3$. Now we have a nontrivial functional Φ_3 on V_σ such that

$\Phi_3(\sigma(g)v) = \psi_{H_3}(g)\Phi_3(v)$ for $g \in H_3$ and $v \in V_\sigma$. Note that the functional Φ_3 is given by $\Phi_3(v) = \Phi_2(\eta v)$ for $v \in V_\sigma$.

Let H_4 be a subgroup of $H_3 \bigcap N_n$, consisting of elements of the form of

$$h = (h_{i,j}) = \begin{pmatrix} \mathrm{I}_n & (0_{n\times(k-1)} \mid *) & * \\ & \mathrm{I}_{2n} & \overbrace{\underbrace{\frac{*}{0}}} \\ & & \mathrm{I}_n \end{pmatrix}, \quad \text{with } h_{1,2n} = -h_{1,2n+1}.$$

Let $\psi_{H_4} = \psi_{H_3}|_{H_4}$. That is, $\psi_{H_4}(h) = \psi(h_{n,2n} + h_{n,2n+1})$.

Let $H_5 = U_{1,2n}H_4$ and let ψ_{H_5} be the character of H_5 extending ψ_{H_4} with trivial value on $U_{1,2n}$. For $u_{1,2n}(x) \in U_{1,2n}$, the adjoint action $\mathrm{ad}(u_{1,2n}(x))$ preserves H_4 and ψ_{H_4}. Therefore there exists a character χ on $U_{1,2n}$ and a functional Φ_4 on V_σ such that

$$\Phi_4(\sigma(ug)v) = \chi(u)\psi_{H_4}(g)\Phi_4(v) \tag{3-4}$$

for $u \in U_{1,2n},\ \ g \in H_4$ and $v \in V_\sigma$.

Assume that $\chi(x) = \psi(ax)$ for some $a \in \mathcal{F}$. Note that

$$\mathrm{ad}(u_{n,1}(-a))u_{1,2n}(x) = u_{1,2n}(x)u_{n,2n}(-ax).$$

Also, $\mathrm{ad}(u_{n,1}(-a))$ preserves both H_4 and ψ_{H_4}. Define $\Phi_5(v) = \Phi_4(u_{n,1}(-a)v)$. Then

$$\Phi_5(\sigma(g)v) = \psi_{H_5}(g)\Phi_5(v) \tag{3-5}$$

for $g \in H_5$ and $v \in V_\sigma$.

Let $X_0 = H_5$ and $\psi^{(0)} = \psi_{H_5}$. For $1 \le m \le n$, let $X_m = U_{m,m+1} \cdots U_{m,n+k-1}$ and write its elements by

$$X_m(\vec{x}) = \mathrm{diag}(r, \mathrm{I}_2, r^*) \quad \text{for } r = (r_{i,j}) \in \mathrm{U}_{2n-1} \text{ and } \vec{x} \in \mathcal{F}^{n+k-m-1},$$

where the m-th row of r is $(0_{m-1}, 1, \vec{x}, 0_{n-k+1})$ and $r_{i,j} = \delta_{i,j}$ for $i \ne m$. Let $\psi^{(m)}$ be the restriction of the character ψ_n of N_n to the subgroup $X_m \cdots X_1 H_5$.

For each $0 \le m \le n$, we claim in general that there exists a nontrivial functional Φ_m on V_σ such that

$$\Phi_m(uv) = \psi^{(m)}(u)\Phi_m(v) \tag{3-6}$$

for $u \in X_m \cdots X_1 H_5$ and $v \in V_\sigma$.

We proceed by induction. For $m = 0$, the claim is true by Equation (3-5). Assume that the claim is true for $0 \le j-1 \le n-2$.

Note that X_j is abelian and that $\mathrm{ad}(X_j(\vec{x}))$ preserves $X_{j-1}\cdots X_1 H_5$ and $\psi^{(j-1)}$. Hence there exists a character χ_j on X_j such that

$$\Phi_{j-1}(\sigma(ug)v)=\chi_j(u)\psi^{(j-1)}(g)\Phi_{j-1}(v) \tag{3-7}$$

holds for $u\in X_j,\ g\in X_{j-1}\cdots X_1H_5$ and $v\in V_\sigma$.

Assume that

$$\chi_j(X_j(t_1,\dots,t_{n+k-j-1}))=\psi(a_1t_1+\cdots+a_{n+k-j-1}t_{n+k-j-1})\quad \text{for } a_i\in\mathscr{F}.$$

If $a_i=0$ for all $1\le i\le n+k-j-1$, then Equation (3-7) induces a nontrivial functional on V_τ that is invariant under $\tau(u)$,

$$u\in\left\{\begin{pmatrix}\mathrm{I}_j & *\\ & \mathrm{I}_{2n-j}\end{pmatrix}\in \mathrm{GL}_{2n}\right\}.$$

This contradicts the supercuspidality of τ. Hence there exists a nonzero a_i. Let

$$m_0=\min\{1\le i\le n+k-j-1\mid a_i\neq 0\}.$$

Case 1. If $m_0=1$, define $\Phi_j(v)=\Phi_{j-1}(\tilde{\lambda}v)$, where $\tilde{\lambda}=\mathrm{diag}(\lambda,\lambda^*)\in \mathrm{SO}_{4n}$ and

$$\lambda=\begin{pmatrix}\mathrm{I}_j &&&&\\ & a_1 &&&\\ & a_2 & 1 &&\\ & \vdots & & \mathrm{I}_{n+k-j-3} &&\\ & a_{n+k-j-1} &&& 1 &\\ &&&&& \mathrm{I}_{n-k}\end{pmatrix}\in \mathrm{GL}_{2n}\,.$$

Note that

$$\begin{aligned}&\mathrm{ad}(\tilde{\lambda})X_j(t_1,\dots,t_{n+k-j-1})\\ &\qquad =X_j(-a_1^{-1}t_1-a_1^{-1}a_2t_2\cdots-a_1^{-1}a_{n+k-j-1}t_{n+k-j-1},t_2,\dots,t_{n+k-j-1}).\end{aligned}$$

Moreover, $\mathrm{ad}(\tilde{\lambda})$ preserves both $X_{j-1}\cdots X_1H_5$ and $\psi^{(j-1)}$. Hence

$$\Phi_j(\sigma(u)v)=\psi^{(j)}(u)\Phi_j(v)\quad \text{for } u\in X_j\cdots X_1H_5. \tag{3-8}$$

Case 2. If $m_0>1$, take $\theta=u_{j+1,m_0}(1)$ and $\tilde{\theta}=\mathrm{diag}(\theta,\theta^*)\in\mathrm{SO}_{4n}$, and then define $\Phi''_j(v)=\Phi_{j-1}(\tilde{\theta}v)$. Then, for $u\in X_j,\ g\in X_{j-1}\cdots X_1H_5$ and $v\in V_\sigma$,

$$\Phi''_j(\sigma(ug)v)=\chi'(u)\psi^{(j-1)}(g)\Phi''_j(v)$$

holds for some character χ' on X_j satisfying

$$\chi'(X_j(x_1,\dots,x_{n+k-j-1}))=\psi(b_1x_1+\cdots+b_{n+k-j-1}x_{n+k-j-1}),$$

with $b_1 \neq 0$. By repeating the same procedure as in the first case, we again reach the conclusion Equation (3-8).

By induction, we have shown that

$$\Phi_{n-1}(\sigma(u)v) = \psi^{(n-1)}(u)\Phi_{n-1}(v) \quad \text{for } u \in X_{n-1}\cdots X_1 H_5.$$

By similar argument, we also obtain that $\Phi'_n(\sigma(ug)v) = \chi''(u)\psi^{(n-1)}(g)\Phi'_n(v)$, where $u \in X_n$, $g \in X_{n-1}\cdots X_1 H_5$ and $v \in V_\sigma$ holds for some character χ'' on X_n satisfying $\chi''(X_n(x_1, \dots, x_{k-1})) = \psi(d_1 x_1 + \cdots + d_{k-1}x_{k-1})$.

Finally, we take $\Phi_n(v) = \Phi'_n(\mathrm{diag}(\gamma, \gamma^*)v)$ for $v \in V_\sigma$, where

$$\gamma = \begin{pmatrix} \mathrm{I}_n & & \\ & \mathrm{I}_{k-1} & \begin{pmatrix} 0, \dots, d_1 \\ \vdots \\ 0, \dots, d_{k-1} \end{pmatrix} \\ & & \mathrm{I}_{n-k+1} \end{pmatrix} \in \mathrm{GL}_{2n},$$

and obtain that $\Phi_n(\sigma(u)v) = \psi^{(n)}(u)\Phi_n(v)$ for $u \in X_n \cdots X_1 H_5$ and $v \in V_\sigma$. Since $N_n = X_n \cdots X_1 H_5$, this gives a nontrivial ψ_n-functional on V_σ, contradicting Theorem 2.4's conclusion that generalized Shalika models and (N_n, ψ_n)-models are disjoint. The initial assumption must be false, so

$$\mathscr{J}\{\imath(Q_k)\}(\mathscr{J}\{\psi_{n-1}\}(\sigma)) = 0 \quad \text{for all } 1 \le k \le n$$

and $\mathscr{J}\{\psi_{n-1}\}(\sigma)$ is quasisupercuspidal. □

4. Genericity of the local descent

By Theorem 3.1, the local descent $\sigma_{n-1} = \sigma_{n-1}(\tau) = \mathscr{J}\{\psi_{n-1}\}(\mathscr{L}(1, \tau))$ as defined in (2-8) is a quasisupercuspidal representation of $\mathrm{SO}_{2n+1}(\mathscr{F})$. We may write

$$\sigma_{n-1} = \sigma^1_{n-1} \oplus \cdots \oplus \sigma^r_{n-1} \oplus \cdots,$$

where the σ^i_{n-1} are irreducible supercuspidal representations of $\mathrm{SO}_{2n+1}(\mathscr{F})$. Note that τ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(\mathscr{F})$ with a nonzero Shalika model.

With regard to the Whittaker functional of $\sigma_{n-1} = \mathscr{J}\{\psi_{n-1}\}(\mathscr{L}(1, \tau))$, recall from (2-3) and (2-4) that

$$(4\text{-}1) \qquad N_{n-1} = \left\{ n(z, x, y) = \begin{pmatrix} z & x & y \\ & \mathrm{I}_{2n+2} & x' \\ & & z' \end{pmatrix} \;\middle|\; z \in \mathrm{U}_{n-1} \right\} \subset \mathrm{SO}_{4n}$$

and the character ψ_{n-1} of N_{n-1} is given by

$$\psi_{n-1}(n(z, x, y)) = \psi(z_{1,2} + \cdots + z_{n-2,n-1})\psi(x_{n-1,n+1} + x_{n-1,n+2}).$$

As in (2-7), the twisted Jacquet module $\sigma_{n-1} = \mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1,\tau))$ is a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. Let Z_k be the standard maximal unipotent subgroup of the split special orthogonal group SO_k consisting of upper-triangular matrices with 1 along the diagonals. That is,

$$Z_{2n+1} = \left\{ z(u,b,w) = \begin{pmatrix} u & b & w \\ & 1 & b' \\ & & u' \end{pmatrix} \in \mathrm{SO}_{2n+1} \;\middle|\; u = (u_{i,j}) \in \mathrm{U}_n \right\}. \tag{4-2}$$

We may write $b = (b_1, \dots, b_n)^t \in \mathcal{F}^n$. The Whittaker character $\psi_{Z_{2n+1}}$ of Z_{2n+1} is defined by

$$\psi_{Z_{2n+1}}(z(u,b,w)) = \psi(u_{1,2} + \cdots + u_{n-1,n} - b_n). \tag{4-3}$$

By the Frobenius reciprocity law, in order to show that σ_{n-1} has a nonzero Whittaker functional, it suffices to show that the twisted Jacquet module

$$\mathcal{J}\{Z_{2n+1}, \psi_{Z_{2n+1}}\}(\sigma_{n-1}) = \mathcal{J}\{Z_{2n+1}, \psi_{Z_{2n+1}}\}(\mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1,\tau)))$$

is nonzero.

To compose the two twisted Jacquet functors $\mathcal{J}\{Z_{2n+1}, \psi_{Z_{2n+1}}\}$ and $\mathcal{J}\{\psi_{n-1}\}$, we set $E_1 = \tilde{\imath}(Z_{2n+1})N_{n-1}$ and let ψ_{E_1} be the character of E_1 defined by

$$\psi_{E_1}(vn) = \psi_{Z_{2n+1}}(v)\psi_{n-1}(n) \quad \text{for } v \in Z_{2n+1} \text{ and } n \in N_{n-1},$$

where $\tilde{\imath} : \mathrm{SO}_{2k+1} \hookrightarrow \mathrm{SO}_{4n}$ is given by

$$g \in \mathrm{SO}_{2k+1} \mapsto \tilde{\imath}(g) = \mathrm{diag}(\mathrm{I}_{2n-k-1}, \imath(g), \mathrm{I}_{2n-k-1})$$

for any $k = 0, 1, \dots, 2n-1$, and the embedding $\imath$ is defined in (3-2). Hence

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) = \mathcal{J}\{Z_{2n+1}, \psi_{Z_{2n+1}}\} \circ \mathcal{J}\{\psi_{n-1}\}(V_\pi)$$

for any irreducible admissible representation (π, V_π) of $\mathrm{SO}_{4n}(\mathcal{F})$.

We put $k = 2n$ in the maximal unipotent subgroup of SO_{4n} defined in (2-3), so that

$$N_{2n} = \left\{ n(z,y) = \begin{pmatrix} z & y \\ & z' \end{pmatrix} \;\middle|\; z \in \mathrm{U}_{2n} \right\}. \tag{4-4}$$

Define a degenerate character $\tilde{\psi}$ of N_{2n} by

$$\tilde{\psi}(n(z,y)) = \psi(z_{1,2} + \cdots + z_{2n-1,2n}).$$

We define the twisted Jacquet module $\mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi)$ for any irreducible admissible representation (π, V_π) of $\mathrm{SO}_{4n}(\mathcal{F})$.

Theorem 4.1. *Let π be an irreducible smooth representation of* SO_{4n} *that admits a nonzero generalized Shalika model.*

(1) *There exists a vector space isomorphism between the two twisted Jacquet modules*, *that is*,

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi).$$

(2) *The local descent* σ_{n-1} *has a nonzero Whittaker functional*, *which is unique up to a scalar.*

Proof. The proof of (1) needs to use the local version of the Fourier expansion for representations, in particular, the [Ginzburg et al. 1999, General Lemma]. We treat the various cases in Sections 4.2–4.12.

We show here that (2) follows from (1). Take π to be $\mathcal{L}(1,\tau)$ and consider $\mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi) = \mathcal{J}\{N_{2n}, \tilde{\psi}\}(\mathcal{L}(1,\tau))$. We may write $N_{2n} = \mathrm{U}_{2n} \ltimes V_{2n}$, where V_{2n} is the unipotent radical of the Siegel parabolic subgroup P_{2n} of SO_{4n} as defined in (2-2). Then we decompose the twisted Jacquet functor as

$$\mathcal{J}\{N_{2n}, \tilde{\psi}\} = \mathcal{J}\{\mathrm{U}_{2n}, \psi_{\mathrm{U}_{2n}}\}^{\mathrm{GL}_{2n}} \circ \mathcal{J}\{V_{2n}\}$$

where the left part of the composition is the Whittaker functor of GL_{2n} and the right is the nontwisted Jacquet functor (that is, the constant term functor along V_{2n}).

Consider first $\mathcal{J}\{V_{2n}\}(\mathcal{L}(1,\tau))$. By [Bernstein and Zelevinsky 1977, Geometric Lemma], we obtain that

$$\mathcal{J}\{V_{2n}\}(\mathcal{L}(1,\tau)) \simeq \tau \otimes |\det|^{-1/2}$$

as representations of $\mathrm{GL}_{2n}(\mathcal{F})$. By the local uniqueness of Whittaker model of τ, we see that the space

$$\mathcal{J}\{\mathrm{U}_{2n}, \psi_{\mathrm{U}_{2n}}\}^{\mathrm{GL}_{2n}} \circ \mathcal{J}\{V_{2n}\}(\mathcal{L}(1,\tau))$$

is one-dimensional. Therefore, $\mathcal{J}\{E_1, \psi_{E_1}\}(\mathcal{L}(1,\tau))$ is one-dimensional by (1); in particular, the local descent σ_{n-1} has a unique Whittaker functional. □

4.2. We start to prove (1) of Theorem 4.1 by constructing a few intermediate twisted Jacquet modules relating $\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi)$ and $\mathcal{J}\{N_{2n}, \tilde{\psi}\}(V_\pi)$. The relations are explained in terms of the local versions of Fourier expansions for representations; this is called the General Lemma in [Ginzburg et al. 1999], and also here.

In this subsection and Section 4.3, we consider the general case when (π, V_π) is any smooth representation of $\mathrm{SO}_{4n}(\mathcal{F})$.

Let

$$C_1 = \{\tilde{\imath}(v)n \mid v \in Z_{2n+1},\ n = n(z,x,y) \text{ such that } x_{n-1,1} = 0\}.$$

Let $\psi_{C_1} = \psi_{E_1}|_{C_1}$. For $i = 1, \dots, n$, let

$$X_i = \left\{ \begin{pmatrix} \mathrm{I}_{n-1} & x & 0 \\ & \mathrm{I}_{2n+2} & x' \\ & & \mathrm{I}_{n-1} \end{pmatrix} \in N_{n-1} \;\middle|\; x_{s,t} \in \delta_{s,n-1}\delta_{t,i} \cdot \mathcal{F} \right\},$$

where $\delta_{i,j}$ is defined by that $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$. For $i = 1, \dots, n-1$, set

$$Y_i = \{\mathrm{I}_{4n} + \lambda E_{n+i-1,2n+1} - \lambda E_{2n,3n+2-i} \mid \lambda \in \mathcal{F}\} \subset \mathrm{SO}_{4n},$$

where $E_{i,j} = (e_{k,l})$, $e_{k,l} = \delta_{k,i}\delta_{l,j}$, and set

$$Y_n = \left\{ \begin{pmatrix} \mathrm{I}_{2n-2} & & \\ & h & \\ & & \mathrm{I}_{2n-2} \end{pmatrix} \;\middle|\; h = \begin{pmatrix} 1 & x & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \right\} \subset \mathrm{SO}_{4n}\,.$$

Note that X_1 is the complement of C_1 in E_1, that is, $E_1 = C_1 \rtimes X_1$. Let $D_1 = C_1 \rtimes Y_1$, and letψ_{D_1} be the trivial extension of ψ_{C_1} to D_1. This forms a setting which for which the General Lemma applies. Hence we have

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{D_1, \psi_{D_1}\}(V_\pi).$$

For $i = 2, \dots, n$, define a series of subgroups C_i of $Z_{2n+2}N_{n-1}$ by

$$C_i = \left\{ vn \;\middle|\; v = \begin{pmatrix} u & t & w \\ & \iota(h) & t' \\ & & u' \end{pmatrix} \in Z_{2n+2}, \quad \begin{array}{c} u \in \mathrm{U}_{i-1},\ h \in Z_{2n+3-2i}, \\ n = n(z, x, y) \in N_{n-1}, \\ x_{n-1,1} = x_{n-1,2} = \cdots = x_{n-1,i} = 0 \end{array} \right\},$$

where Z_{2n+2} is identified with its embedding in the middle diagonal part of SO_{4n}. Let ψ^i be the character of C_i defined by

$$\psi^i(vn) = \psi_{n-1}(n)\psi(u_{1,2} + \cdots + u_{i-2,i-1} + t_{i-1,1})\psi_{Z_{2n+3-2i}}(h).$$

Then X_i and Y_i both normalize C_i and ψ^i. The trivial extensions of ψ^i to $C_i \rtimes X_i$ and $C_i \rtimes Y_i$ are still denoted by ψ^i. Let $D_i := C_i \rtimes Y_i$. Then $D_{i-1} \simeq C_i \rtimes X_i$ for $i = 2, \dots n$ and the characters ψ^{i-1} and ψ^i of D_{i-1} are equal. Again, this is the setting of the General Lemma, and we obtain

$$\mathcal{J}\{D_{i-1}, \psi^{i-1}\}(V_\pi) \simeq \mathcal{J}\{D_i, \psi^i\}(V_\pi) \quad \text{for } i = 2, \dots, n.$$

Hence we obtain a vector space isomorphism of twisted Jacquet modules:

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{D_n, \psi^n\}(V_\pi).$$

Note that

$$D_n = \left\{ \begin{pmatrix} z & y & w \\ & h & y' \\ & & z' \end{pmatrix} \;\middle|\; \begin{array}{l} h = \begin{pmatrix} 1 & \tilde{f} & -f & w \\ & 1 & 0 & f \\ & & 1 & -\tilde{f} \\ & & & 1 \end{pmatrix} \in Z_4, \\ z \in \mathrm{U}_{2n-2} \text{ with } z_{n-1,i} = 0 \text{ for } i \geq n \end{array} \right\} \subset Z_{4n}.$$

Then we also have the isomorphism $\mathcal{J}\{D_n, \psi^n\}(V_\pi) \simeq \mathcal{J}\{D_n, \psi_{D_n}\}(V_\pi)$ of vector spaces, where the character ψ_{D_n} of D_n is given by

$$\psi_{D_n}(v) = \psi(z_{1,2} + z_{2,3} + \cdots + z_{2n-3,2n-2} + y_{n-1,2} + y_{n-1,3} - f).$$

4.3. Let ν be the permutation matrix in GL_{2n} given by

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & \dots & 2n-1 & 2n \\ 2 & 4 & \dots & 2(n-1) & 1 & 3 & \dots & 2n-1 & 2n \end{pmatrix},$$

and identify it with its embedding $m(\nu)$, where $m : g \in \mathrm{GL}_{2n} \mapsto \mathrm{diag}(g, g^*) \in \mathrm{SO}_{4n}$. Let $E = \nu D_n \nu^{-1}$, and define a character ψ_E of E by

$$\psi_E(n) := \psi_{D_n}(\nu^{-1} n \nu) \quad \text{for } n \in E.$$

Let $T(n)$ be the subgroup of GL_{2n} consisting of certain elements $t = (t_{i,j})$, as follows: Let $\bar{t}_j = (t_{j+1,j}, \dots, t_{2n,j})^t$ and $t_i = (t_{i,i+1}, \dots, t_{i,2n})$ for $i, j \leq 2n - 1$.

- For $1 \leq i \leq 2n$, require $t_{i,i} = 1$.
- For $j \leq n-2$, require that the (single-element) rows of $\bar{t}_{2j-1}$ alternate between arbitrary and zero, except for the last 4, which are all zero; require that $\bar{t}_{2n-3}$ and $\bar{t}_{2n-1}$ vanish.
- For $j \leq n$, require that $\bar{t}_{2j}$ vanishes.
- For $i \leq n$, require that $t_{2i-1} = (0 \;*\; 0 \;*\; \dots \;*\; 0 \;*\; *)$.
- Require $t_{2(n-1)} = (0, *)$.

Then

$$E = \left\{ n = \begin{pmatrix} t & X \\ & t' \end{pmatrix} \;\middle|\; t \in T(n) \right\}$$

and the character ψ_E is given by

$$\psi_E(n) = \psi(t_{1,3} + t_{2,4} + \cdots + t_{2n-3,2n-1} + t_{2n-2,2n} + x_{2n-2,1} + x_{2n-1,1}). \tag{4-5}$$

Example 4.4. In the case of $n = 4$,

$$T(4) = \left\{ \begin{pmatrix} 1 & 0 & * & 0 & * & 0 & * & * \\ * & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & 0 & * & * \\ * & 0 & * & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_8 .$$

Since $\psi_E(n) = \psi_{D_n}(\nu^{-1} n \nu)$ for all $n \in E$, we have the vector space isomorphism

$$\mathcal{J}\{D_n, \psi_{D_n}\}(V_\pi) \simeq \mathcal{J}\{E, \psi_E\}(V_\pi).$$

Next, we will apply the General Lemma to fill the zeros of t_{2i-1} from right to left, using $\bar{t}_{2i-1}$.

Let

$$\begin{aligned} Y^{i,1} &= \{m(\mathrm{I}_{2n} + yE_{2i,1}) \mid y \in \mathcal{F}\} \quad \text{for } i = 1, \dots, n-2, \\ X^{1,j} &= \{m(\mathrm{I}_{2n} + xE_{1,2j}) \mid x \in \mathcal{F}\} \quad \text{for } j = 2, \dots, n-1, \\ E^{i,1} &= \{n \in E \mid n_{j,1} = 0, \forall j > 2i\} \cdot \textstyle\prod_{j=i+2}^{n} X^{1,j} \quad \text{for } i \le n-3, \\ E^{n-2,1} &= E, \\ C^{i,1} &= \{n \in E^{i,1} \mid n_{2i,1} = 0\}, \quad D^{1,i+1} = C^{i,1} X^{1,i+1}, \quad A^{1,i+1} = D^{1,i+1} Y^{i,1}. \end{aligned}$$

Define a series of characters $\psi^{i,1} = \psi_E|_{C^{i,1}}$. Extend $\psi^{i,1}$ trivially to $D^{1,i+1}$ as $\psi^{i,1}_{D^{1,i+1}}$ and to $E^{i,1}$ as $\psi^{i,1}_{E^{i,1}}$. Note that

$$D^{1,i+1} = E^{i-1,1} \quad \text{and} \quad \psi^{i,1}_{D^{1,i+1}}|_{C^{i-1,1}} = \psi^{i-1,1}.$$

By the General Lemma, we have vector space isomorphisms

$$\mathcal{J}\{E^{i,1}, \psi^{i,1}_{E^{i,1}}\}(V_\pi) \simeq \mathcal{J}\{D^{1,i+1}, \psi^{i,1}_{D^{1,i+1}}\}(V_\pi) \simeq \mathcal{J}\{E^{i-1,1}, \psi^{i-1,1}_{E^{i-1,1}}\}(V_\pi)$$

for $i = n-2, \dots, 2$. In particular, we have

$$\mathcal{J}\{E, \psi_E\}(V_\pi) \simeq \mathcal{J}\{D^{1,2}, \psi^{1,1}_{D^{1,2}}\}(V_\pi).$$

Note that the GL_{2n} part of $D^{1,2}$ looks like $\left(\begin{smallmatrix} \mathrm{I}_2 & * \\ 0 & T' \end{smallmatrix}\right)$ with $T' \in T(n-1)$. Now let

$$\begin{aligned} Y^{r,s} &= \{m(\mathrm{I}_{2n} + yE_{2r,2s-1}) \mid y \in \mathcal{F}\} \quad \text{for } 1 \le r, s \le n-2, \\ X^{r,s} &= \{m(\mathrm{I}_{2n} + xE_{2r-1,2s}) \mid x \in \mathcal{F}\} \quad \text{for } 1 \le r \le n-2 \text{ and } 1 \le s \le n-1. \end{aligned}$$

For $1 \le j \le i \le n-2$, we define

$$E^{i,j} = \tilde{E}^{i,j} \prod_{s=i+2}^{n-1} X^{j,s}, \quad \text{where } \tilde{E}^{i,j} = \left\{ \begin{pmatrix} t & X \\ & t' \end{pmatrix} \in \mathrm{SO}_{4n} \right\},$$

where $t_{\ell,2j-1} = 0$ for all $\ell > 2i$ and otherwise is of the form

$$t = \begin{pmatrix} \mathrm{I}_2 & & * & * \\ & \ddots & & \\ & & \mathrm{I}_2 & * \\ & & & Z \end{pmatrix}, \quad \text{with } Z \in T(n-j+1),$$

We further define

$$C^{i,j} = \left\{ n = \begin{pmatrix} t & X \\ & t' \end{pmatrix} \in E^{i,j} \;\middle|\; t_{2i,2j-1} = 0 \right\},$$

$$D^{j,i+1} = C^{i,j} X^{j,i+1}, \quad A^{j,i+1} = D^{j,i+1} Y^{i,j}.$$

We also define $\psi^{i,j} = \psi_E|_{C^{i,j}}$. Note that $D^{j,i+1} \simeq A^{i-1,j}$ for $i \ge j+1$ and that $D^{j,j+1} \simeq A^{n-1,j+1}$. The relations among those $\psi^{i,j}$ and their trivial extensions $\psi^{i,j}_{D^{j,i+1}}$ and $\psi^{i,j}_{A^{i,j}}$ to $D^{j,i+1}$ and $A^{i,j}$, respectively, are compatible in the sense of the General Lemma. We then have vector space isomorphisms

$$\begin{aligned} \mathcal{J}\{E, \psi\}(V_\pi) \simeq \mathcal{J}\{D^{1,2}, \psi^{1,1}_{D^{1,2}}\}(V_\pi) \simeq \cdots \simeq \mathcal{J}\{D^{j,j+1}, \psi^{j,j}_{D^{j,j+1}}\}(V_\pi) \\ \simeq \cdots \simeq \mathcal{J}\{D^{n-2,n-1}, \psi^{n-2,n-2}_{D^{n-2,n-1}}\}(V_\pi). \end{aligned}$$

Denote by B_n the standard Borel subgroup of GL_n. The subgroup $D^{n-2,n-1}$ consists of elements of the form

$$\begin{pmatrix} t & X \\ & t' \end{pmatrix} \in \mathrm{SO}_{4n}, \quad \text{with } t = \begin{pmatrix} \mathrm{I}_2 & y_1 & * & \cdots & * \\ & \mathrm{I}_2 & y_2 & \cdots & * \\ & & \ddots & & \\ & & & \mathrm{I}_2 & y_{n-1} \\ & & & & z \end{pmatrix},$$

where $y_1, \ldots, y_{n-2} \in \mathrm{Mat}_2$, $y_{n-1} \in \mathrm{B}_2$ and $z \in \mathrm{U}_2$. The character $\psi^{n-2,n-1}_{D^{n-2,n-1}}$ is given by

$$\psi^{n-2,n-2}_{D^{n-2,n-1}}(n) = \psi(\mathrm{tr}(y_1 + \cdots + y_{n-1}))\psi(x_{2n-2,1} + x_{2n-1,1}). \tag{4-6}$$

Proposition 4.5. *Let π be a smooth representation of SO_{4n}. Then there exists a vector space isomorphism between two twisted Jacquet modules given by*

$$\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{D^{n-2,n-1}, \psi^{n-2,n-1}_{D^{n-2,n-2}}\}(V_\pi).$$

So far we have only assumed π to be a smooth representation of $\mathrm{SO}_{4n}(\mathcal{F})$.

4.6. The next step is to eliminate the character place $x_{2n-2,1}$ in (4-6). We need two auxiliary results, Propositions 4.7 and 4.11. We assume that V_π is an irreducible admissible representation of $\mathrm{SO}_{4n}(\mathcal{F})$ with a nonzero generalized Shalika model.

We define

$$D = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \middle| T = \begin{pmatrix} t_1 & z_1 & \cdots & \cdots & * \\ & t_2 & z_2 & \cdots & * \\ & & \cdots & & \cdots \\ & & & t_{n-1} & z_{n-1} \\ & & & & t_n \end{pmatrix}, t_i \in \mathrm{U}_2,\ z_i \in \mathrm{B}_2 \right\}, \tag{4-7}$$

and a character $\psi_D(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{2n-2,1} + x_{2n-1,1})$ of D.

Proposition 4.7. *Let π be an irreducible smooth representation of SO_{4n} admitting a nonzero generalized Shalika model. Then there exists a vector space isomorphism*

$$\mathcal{J}\{D^{n-2,n-1}, \psi^{n-1,n-1}_{D^{n-2,n-1}}\}(V_\pi) \simeq \mathcal{J}\{D, \psi_D\}(V_\pi).$$

Proof. After applying the General Lemma $n-2$ times, we have the vector space isomorphisms

$$\mathcal{J}\{D^{n-2,n-1}, \psi^{n-1,n-1}_{D^{n-2,n-1}}\}(V_\pi) \simeq \mathcal{J}\{H_1, \psi_{H_1}\}(V_\pi),$$

where

$$H_1 = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \middle| T = \begin{pmatrix} \mathrm{I}_2 & z_1 & \cdots & \cdots & * \\ & t_2 & z_2 & \cdots & * \\ & & \cdots & \cdots & \\ & & & t_{n-1} & z_{n-1} \\ & & & & t_n \end{pmatrix}, t_i \in \mathrm{U}_2,\ z_i \in \mathrm{B}_2 \right\},$$

$$\psi_{H_1}(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{2n-2,1} + x_{2n-1,1}) \quad \text{for } n \in H_1.$$

Note that the group

$$m\left(\left\{\left(\begin{array}{cc:c} 1 & * & 0 \\ & 1 & 0 \\ \hdashline & & \mathrm{I}_{2n-2} \end{array}\right)\right\}\right) \subset m(\mathrm{GL}_{2n}) \subset \mathrm{SO}_{4n}$$

normalizes H_1 and ψ_{H_1}.

For $\lambda \in \mathcal{F}^*$, define a character $\psi'_{D,\lambda}$ of D by

$$\psi'_{D,\lambda}(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{n-2,1} + x_{n-1,1})\psi(\lambda t),$$

where $t_1 = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ as in H_1. By the conclusion of the next lemma, Lemma 4.8, the only twisted Jacquet module that remains is the one corresponding to $\lambda = 0$. In this case we have $\psi'_{D,0} = \psi_D$, and therefore $\mathcal{J}\{E_1, \psi_{E_1}\}(V_\pi) \simeq \mathcal{J}\{D, \psi_D\}(V_\pi)$. $\square$

Lemma 4.8. *Assume that π is an irreducible representation of* SO_{4n} *admitting a nonzero generalized Shalika model. Then*

$$\mathcal{J}\{D, \psi'_{D,\lambda}\}(V_\pi) = 0 \quad \textit{for all } \lambda \in \mathcal{F}^*.$$

Proof. First we consider the case of $\lambda = 1$. Let $\psi'_D := \psi'_{D,1}$. Then for $n = \begin{pmatrix} T & X \\ & T' \end{pmatrix} \in D$ we have

$$\psi'_D(n) = \psi(T_{1,2} + T_{1,3} + \textstyle\sum_{i=2}^{n-2} T_{i,i+2})\psi(x_{2n-2,1} + x_{2n-1,1}).$$

Let

$$z_1 = \mathrm{diag}(Z, \mathrm{I}_{2n-3}) \in \mathrm{GL}_{2n}, \qquad \text{with } Z = \begin{pmatrix} 1 & & \\ & 1 & 0 \\ & 1 & 1 \end{pmatrix}.$$

Then z_1 normalizes D. Let $\psi_{D,1}$ be the character of D defined by

$$\psi_{D,1}(n) = \psi'_D(z_1 n z_1^{-1}) = \psi(T_{1,2} + T_{2,4} + \textstyle\sum_{i=2}^{n-2} T_{i,i+2})\psi(x_{2n-2,1} + x_{2n-1,1}). \tag{4-8}$$

Clearly there exists a vector space isomorphism

$$\mathcal{J}\{D, \psi'_D\}(V_\pi) \simeq \mathcal{J}\{D, \psi_{D,1}\}(V_\pi). \tag{4-9}$$

For $i = 2, \ldots, n-1$, let $z_i = \mathrm{I}_{2n} + E_{2i+1,2i} \in \mathrm{GL}_{2n}$, and let $\psi_{D,i}$ be the character of D defined by $\psi_{D,i}(n) := \psi_{D,i-1}(z_i n z_i^{-1})$. Then we have

$$\psi_{D,i}(n) = \begin{cases} \psi(T_{1,2} + T_{2i,2i+3} + \sum_{j=2}^{n-2} T_{j,j+2})\psi(x_{2n-2,1} + x_{2n-1,1}) & \\ & \text{if } 2 \le i \le n-2, \\ \psi(T_{1,2} + \sum_{j=2}^{n-2} T_{j,j+2})\psi(x_{2n-1,1} + 2x_{2n-2,1}) & \text{if } i = n-1. \end{cases}$$

It is clear that

$$\mathcal{J}\{D, \psi_{D,i}\}(V_\pi) \simeq \mathcal{J}\{D, \psi_{D,i+1}\}(V_\pi) \quad \text{for } i = 2, \ldots, n-2. \tag{4-10}$$

From (4-9) and (4-10), we have the vector space isomorphism

$$\mathcal{J}\{D, \psi'_D\}(V_\pi) \simeq \mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi).$$

Now we assume to the contrary that

$$\mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi) \neq 0. \tag{4-11}$$

Then by the Frobenius reciprocity law, there exists a nonzero functional ℓ on V_π such that

$$\ell(\pi(n)v) = \psi_{D,n-1}(n)\ell(v) \quad \text{for } n \in D \text{ and } v \in V_\pi. \tag{4-12}$$

Such a functional ℓ on V_π factors through $\mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi)$. Hence the nonvanishing of $\mathcal{J}\{D, \psi_{D,n-1}\}(V_\pi)$ is equivalent to the nonvanishing of such ℓ.

Let μ be the permutation matrix in GL_{2n} given by

$$\begin{aligned} \mu(1) &= 1, & \mu(2i-2) &= i & &\text{for } i = 2, \dots, n, \\ \mu(2n) &= 2n, & \mu(2i-1) &= n+i-1 & &\text{for } i = 2, \dots, n, \end{aligned}$$

which can be identified with its embedding $m(\mu)$ in SO_{4n}. Denote by Ni_k the set of nilpotent elements in GL_k. Then

$$F := \mu D \mu^{-1} = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \,\middle|\, T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \begin{array}{l} \alpha, \delta, \gamma \in \mathrm{B}_n \cap \mathrm{Ni}_n,\ \beta \in \mathrm{B}_n, \\ \gamma_{i,i+1} = 0 \ \text{ for } i = 1, \dots, n-1 \end{array} \right\}.$$

Example 4.9. When $n = 4$, the T in F are of the form

$$\left(\begin{array}{cccc:cccc} 1 & * & * & * & * & * & * & * \\ 0 & 1 & * & * & 0 & * & * & * \\ 0 & 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ \hdashline 0 & 0 & * & * & 1 & * & * & * \\ 0 & 0 & 0 & * & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Let ψ_F be the character of F defined by

$$\psi_F(n) = \psi_{D,n-1}(\mu^{-1} n \mu) = \psi\bigl(\textstyle\sum_{i=1, i\neq n}^{2n-2} T_{i,i+1} + T_{n,2n} + 2X_{n,1} + X_{2n-1,1}\bigr).$$

Define a linear functional on V_π by $\ell_F(v) = \ell(\pi(\mu^{-1})v)$ for $v \in V_\pi$. Then ℓ_F is a nonzero functional on V_π satisfying $\ell_F(\pi(n)v) = \psi_F(n)\ell_F(v)$ for $n \in F$. Since ℓ_F factors through $\mathcal{J}\{F, \psi_F\}(V_\pi)$, the latter must be nonzero.

Again, by the General Lemma, we get $\mathcal{J}\{F, \psi_F\}(V_\pi) \simeq \mathcal{J}\{F', \psi_{F'}\}(V_\pi)$, where

$$F' = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \,\middle|\, T \in \mathrm{U}_{2n},\ T_{n,n+i} = 0 \ \text{ for } i = 1, \dots, n-1 \right\}$$

and the character $\psi_{F'}$ is given by

$$\psi_{F'}(n) = \psi\bigl(\textstyle\sum_{i=1, i\neq n}^{2n-2} T_{i,i+1} + T_{n,2n} + 2X_{n,1} + X_{2n-1,1}\bigr). \tag{4-13}$$

Example 4.10. The T in F' are of the form

$$\left(\begin{array}{cccc:cccc} 1 & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ \hdashline 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

(Compare this form with the one in Example 4.9 to see how the General Lemma works.)

Since $\mathcal{J}\{F, \psi_F\}(V_\pi) \simeq \mathcal{J}\{F', \psi_{F'}\}(V_\pi) \neq 0$, there is a nonzero linear functional $\ell_{F'}$ on V_π such that $\ell_{F'}(\pi(n)\,v) = \psi_{F'}(n)\ell_{F'}(v)$ for $n \in F'$.

Next, we consider the intersection $F'_n := F' \cap N_n$. Then

$$F'_n = \left\{ \begin{pmatrix} \alpha & \beta & x & y \\ & \mathrm{I}_n & 0 & x' \\ & & \mathrm{I}_n & \beta' \\ & & & \alpha' \end{pmatrix} \;\middle|\; \begin{array}{l} \alpha \in \mathrm{U}_n,\ \beta \in \mathrm{B}_n, \\ \beta_{n,i} = 0 \ \text{ for } i = 1, \dots, n-1 \end{array} \right\}. \tag{4-14}$$

and $\ell_{F'}$ is a nonzero linear functional on V_π such that

$$\ell_{F'}(\pi(n)\,v) = \psi_{F'}(n)\ell_{F'}(v) \quad \text{for } n \in F'_n.$$

Note that F'_n differs from N_n by the requirements on their elements at the β entries of (4-14). Now we will apply the local version of Fourier expansion to "fill the zeros of β".

Define a series of subgroups $F'_n \subset F'_{n-1} \subset \cdots \subset F'_1 = N_n$ as follows. Let

$$F'_i = \left\{ \begin{pmatrix} \alpha & \beta & x & y \\ & \mathrm{I}_n & 0 & x' \\ & & \mathrm{I}_n & \beta' \\ & & & \alpha' \end{pmatrix} \in N_n \;\middle|\; \alpha \in \mathrm{U}_n,\ \beta_{n,j} = 0 \ \text{ for } j = 1, \dots, i-1 \right\}. \tag{4-15}$$

Let $\psi_{F'_i}$ be the character of F'_i defined by the same formula of (4-13), that is, by

$$\psi_{F'_i}(n) = \psi(\alpha_{1,2} + \cdots + \alpha_{n-1,n} + \beta_{n,2n} + 2x_{n,1}).$$

Now we use induction in reversed order. The case of $i = n$ is shown in (4-14). Assume for some $2 \le i \le n$ that we have a nonzero linear functional ℓ_i on V_π

satisfying the quasiinvariance property

$$\ell_i(\pi(n)\,v) = \psi_{F_i'}(n)\ell_i(v) \quad \text{for } n \in F_i'. \tag{4-16}$$

We show that the functional ℓ_{i-1} is an extension of ℓ_i such that (4-16) holds with i replaced by $i-1$.

Note that the root group of $e_n - e_{i-1}$ normalizes the character $\psi_{F_i'}$. There are two possibilities:

(i) The ℓ_i having the $(F_i', \psi_{F_i'})$-quasiinvariance property can be trivially extended to ℓ_{i-1} with the $(F_{i-1}', \psi_{F_{i-1}'})$-quasiinvariance property, and we are done.

(ii) The ℓ_i can be nontrivially extended to a nonzero linear functional ℓ_{i-1}' with the $(F_{i-1}', \psi_{F_{i-1}'})$-quasiinvariance property, such that

$$\ell_{i-1}'(\pi(n)\,v) = \tilde{\psi}_{F_{i-1}'}(n)\ell_{i-1}'(v) \quad \text{for } n \in F_{i-1}'.$$

Then

$$\tilde{\psi}_{F_i'}(n) = \psi(\alpha_{1,2} + \cdots + \alpha_{n-1,n} + \beta_{n,2n} + 2x_{n,1})\psi(c\,\beta_{n,i}) \quad \text{for some } c \in \mathscr{F}^*.$$

Let $z = \mathrm{I}_{2n} + \alpha\; E_{n+i,2n} \in \mathrm{GL}_{2n}$. Then we can choose a certain $\alpha \in \mathscr{F}^*$ such that z normalizes F_i' and changes $\tilde{\psi}_{F_{i-1}'}$ back to the character $\psi_{F_{i-1}'}$. Hence we get (4-16) for ℓ_{i-1}.

By induction, we get a nonzero linear functional ℓ_1 on V_π that factors through $\mathscr{J}\{N_n, \psi_n\}(V_\pi)$.

By assumption, V_π has a nonzero generalized Shalika model. It follows from Theorem 2.4 that such a V_π has no nonzero twisted Jacquet module $\mathscr{J}\{N_n, \psi_n\}(V_\pi)$. Hence ℓ_1 must be zero.

Therefore, the assumption (4-11) must be wrong and $\mathscr{J}\{D, \psi_{D,n-1}\}(V_\pi)$ must be zero. The proves the case when $\lambda = 1$.

If $\lambda \neq 1$, conjugation by $m(a)$ with $a = \mathrm{diag}(\lambda^{-1}, 1, \lambda^{-1}, 1, \ldots, \lambda^{-1}, 1) \in \mathrm{GL}_{2n}$ will give a vector space isomorphism $\mathscr{J}\{D, \psi_{D,\lambda}'\}(V_\pi) \simeq \mathscr{J}\{D, \psi_{D,\lambda}\}(V_\pi)$, where $\psi_{D,\lambda}$ is almost the same with the character of D defined in (4-8) except that the coefficient of $x_{2n-1,1}$ is λ^{-1}. In the proof of the case when $\lambda = 1$, we see that the coefficients of $x_{2n-1,1}$ and $x_{2n-2,1}$ play no role and a similar argument applies. □

Proposition 4.11. *Let π be a smooth representation of* SO_{4n}. *Then*

$$\mathscr{J}\{D, \psi_D\}(V_\pi) \simeq \mathscr{J}\{D, \tilde{\psi}_D\}(V_\pi),$$

where $\tilde{\psi}$ is the character of D defined (*in the notation of* (4-7)) *by*

$$\tilde{\psi}_D(n) = \psi(\mathrm{tr}(z_1 + \cdots + z_{n-1}) + x_{2n-1,1}).$$

Proof. The proof is almost the same as that of Lemma 4.8. We give only a sketch.

First, let $\bar{\mathrm{B}}_n$ denote the opposite standard Borel subgroup of GL_n. By the General Lemma, we have the vector space isomorphism

$$\mathcal{J}\{D, \tilde{\psi}_D\}(V_\pi) \simeq \mathcal{J}\{\tilde{D}, \tilde{\psi}_{\tilde{D}}\}(V_\pi),$$

where

$$\tag{4-17} \tilde{D} = \left\{ \begin{pmatrix} T & X \\ & T' \end{pmatrix} \;\middle|\; T = \begin{pmatrix} 1 & * & * & \cdots & & \cdots & * \\ & t_1 & z_1 & * & \cdots & & * \\ & & t_2 & z_2 & * & \cdots & * \\ & & & \ddots & & \vdots & \\ & & & & t_{n-2} & z_{n-2} & * \\ & & & & & \mathrm{I}_2 & * \\ & & & & & & 1 \end{pmatrix},\ t_i \in \mathrm{U}_2,\ z_i \in \bar{\mathrm{B}}_2 \right\}$$

and $\tilde{\psi}_{\tilde{D}}(n) = \psi(\sum_{i_1}^{2n-2} T_{i,i+2})\psi(x_{2n-2,1} + x_{2n-1,1})$ is the character of $\tilde{D}$.

Second, let

$$z = \begin{pmatrix} \mathrm{I}_{2n-3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which normalizes $\tilde{D}$ and changes $\tilde{\psi}_{\tilde{D}}$ to $\tilde{\psi}'_{\tilde{D}}$, defined in the notation of (4-17) by

$$\tilde{\psi}'_{\tilde{D}}(n) = \psi(\textstyle\sum_{i_1}^{2n-2} T_{i,i+2})\psi(x_{2n-1,1}).$$

Finally, use the General Lemma to transform the $\bar{\mathrm{B}}_2$ of the first part into B_2. □

4.12. We are ready to prove Theorem 4.1(1). The proof is similar to that of [Ginzburg et al. 1999, Theorem 4.2.1], employing the local version of the Fourier expansion of representations. Let ν be the permutation matrix in GL_{4n} such that $\nu_{i,2i-1} = 1$ and $\nu_{2n+i,2i} = 1$ for $i = 1, \ldots, 2n$, and $\nu_{i,j} = 0$ otherwise. Let $B = \nu D \nu^{-1}$, and define a character ψ_B of B by $\psi_B(e) = \tilde{\psi}_D(\nu^{-1} e \nu)$ for $e \in B$. Then we have the vector space isomorphism $\mathcal{J}\{D, \tilde{\psi}_D\}(V_\pi) \simeq \mathcal{J}\{B, \psi_B\}(V_\pi)$. Note that

$$\tag{4-18} B = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \begin{array}{l} \alpha, \delta \in \mathrm{U}_{2n},\ \beta \in \mathrm{B}_{2n}, \\ \gamma \in \mathrm{B}_{2n} \cap \mathrm{Ni}_{2n} \text{ and } \gamma_{i,i+1} = 0 \text{ for } i = 1, \ldots, 2n \end{array} \right\},$$

and the character ψ_B is $\psi_B(e) = \psi(\alpha_{1,2} + \cdots + \alpha_{n,n+1} - \alpha_{n+1,n+2} - \cdots - \alpha_{2n-1,2n})$.

Example 4.13. For $n = 4$, elements in B are of the form

$$\left(\begin{array}{cccccccc:cccccccc}
1 & \boxed{\ast} & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
 & 1 & \boxed{\ast} & \ast & \ast & \ast & \ast & \ast & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
 & & 1 & \boxed{\ast} & \ast & \ast & \ast & \ast & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast \\
 & & & 1 & \boxed{\ast} & \ast & \ast & \ast & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast \\
 & & & & 1 & \boxed{\ast} & \ast & \ast & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast \\
 & & & & & 1 & \boxed{\ast} & \ast & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast \\
 & & & & & & 1 & \boxed{\ast} & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast \\
 & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast \\
\hdashline
0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & 1 & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & 0 & 1 & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & 0 & 0 & 1 & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & 0 & 0 & 0 & 1 & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & 0 & 0 & 0 & 0 & 1 & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & 0 & 0 & 0 & 0 & 0 & 1 & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right),$$

where the boxes indicate the nontrivial character positions of ψ_B.

Our goal is to "fatten" β in (4-18), using the entries of γ, by successive applications of the General Lemma, until we transform $\mathcal{J}\{B, \psi_B\}$ into $\mathcal{J}\{V_{2n}, \tilde{\psi}\}$. Let

$$\mathscr{X} = \left\{ x \in \mathrm{Mat}_{2n}(\mathscr{F}) \;\middle|\; \begin{pmatrix} \mathrm{I}_{2n} & x \\ & \mathrm{I}_{2n} \end{pmatrix} \in \mathrm{SO}_{4n} \right\}.$$

For $x \in \mathscr{X}$, write

$$\epsilon(x) = \begin{pmatrix} \mathrm{I}_{2n} & x \\ & \mathrm{I}_{2n} \end{pmatrix} \quad \text{and} \quad \bar{\epsilon}(x) = \begin{pmatrix} \mathrm{I}_{2n} & 0 \\ x & \mathrm{I}_{2n} \end{pmatrix}.$$

For a subspace $S \subset \mathscr{X}$, define

$$\epsilon(S) = \{\epsilon(x) \mid x \in S\} \quad \text{and} \quad \bar{\epsilon}(S) = \{\bar{\epsilon}(x) \mid x \in S\}.$$

Put

$$\mathscr{X}_0 = \{x \in \mathscr{X} \mid x \in \mathrm{B}_{2n}\},$$
$$\mathscr{Y}_0 = \{x \in \mathscr{X} \mid x \in \mathrm{B}_{2n} \cap \mathrm{Ni}_{2n} \text{ and } x_{i,i+1} = 0 \text{ for } i = 1, \ldots, n-1\}.$$

For $1 \le i < j-1$, define

$$\mathcal{Y}_{i,j} = \{x \in \mathcal{X}_0 \mid x_{r,l} = 0 \text{ for } r, l < j-1 \text{ and } x_{r,j} = 0 \text{ for } r \ge i\},$$
$$\mathcal{Y}^{i,j} = \mathrm{I} + \mathcal{F}(E_{i,j} - E_{2n+1-j,2n+1-i}).$$

Then elements in B can be written in the form

$$v = \epsilon(x)m(z)\bar{\epsilon}(y), \tag{4-19}$$

with $x \in \mathcal{X}_0$, $y \in \mathcal{Y}_0$ and $z \in \mathrm{U}_{2n}$. Let $\mathcal{Y}_{1,3} = \{x \in \mathcal{X}_0 \mid x_{1,3} = 0\}$. Let $C^{1,3}$ be the subgroup of the form (4-19) such that $y \in \mathcal{Y}_{1,3}$. Then $C^{1,3} = \epsilon(\mathcal{X}_0)m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{13})$. Let $Y^{1,3} = \bar{\epsilon}(\mathcal{Y}^{1,3})$. Denote by $X^{2,1} = \epsilon(\mathcal{X}^{2,1})$, where $\mathcal{X}^{2,1} = \mathcal{F}(e_{2,1} - e_{2n,2n-1})$. Let $\psi_B^{1,3} = \psi_B|_{C^{1,3}}$, $B^{1,3} = B$, and $D^{1,3} = C^{1,3}X^{2,1}$. Put $\mathcal{X}_{2,1} = \mathcal{X}_0 \oplus \mathcal{X}^{2,1}$. Then $D^{1,3} = \epsilon(\mathcal{X}_{2,1})m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{1,3})$. By the General Lemma, we conclude that

$$\mathcal{J}\{B^{1,3}, \psi^{1,3}_{B^{1,3}}\}(V_\pi) \simeq \mathcal{J}\{D^{1,3}, \psi^{1,3}_{D^{1,3}}\}(V_\pi),$$

where $\psi^{1,3}_{D^{1,3}}$ is the character of $D^{1,3}$, which is trivial on $\epsilon(\mathcal{X}_{2,1}) \cdot \overline{\mathcal{Y}}_{1,3}$.

Define $\mathcal{X}^{r,s} = \mathrm{I} + \mathcal{F}(E_{r,s} - E_{2n+1-s,2n+1-r})$ for $1 \le s < r \le 2n$. Let

$$\mathcal{X}_{r,s} = \mathcal{X}_0 \oplus \Big(\bigoplus_{q<l\le r-1} \mathcal{X}^{l,q}\Big) \oplus \Big(\bigoplus_{q=s}^{r-1} \mathcal{X}^{r,q}\Big) \quad \text{for } 1 \le s < r \le n.$$

For $1 \le i < j-1$ and $j \le n+1$, let $C^{i,j} = \epsilon(\mathcal{X}_{j-1,i+1})m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{i,j})$ if $i+1 \le j-1$. For $1 \le i < j \le n+1$, we define $Y^{i,j} = \bar{\epsilon}(\mathcal{Y}^{i,j})$ and $X^{j,i} = \epsilon(\mathcal{X}^{j,i})$, and also define

$$B^{i,j} = C^{i,j}Y^{i,j}, \quad D^{i,j} = C^{i,j}X^{j-1,i}, \quad A^{i,j} = D^{i,j}Y^{i,j}.$$

Let $\psi^{i,j}$ be the character of $C^{i,j}$, which is trivial on $\epsilon(\mathcal{X}_{j-1,i+1}) \cdot \bar{\epsilon}(\mathcal{Y}_{i,j})$. Then by the General Lemma, we have the vector space isomorphism

$$\mathcal{J}\{B^{i,j}, \psi^{i,j}_{B^{i,j}}\}(V_\pi) \simeq \mathcal{J}\{D^{i,j}, \psi^{i,j}_{D^{i,j}}\}(V_\pi)$$

for all $1 \le i < j-1$, $j \le n+1$.

Note that for $2 \le i < j-1$, $j \le n+1$, we have

$$D^{i,j} = B^{i-1,j} \quad \text{and} \quad \psi^{i,j}_{D^{i,j}} = \psi^{i-1,j}_{B^{i-1,j}},$$

and for $j = 3, \dots, n+1$, we have

$$D^{1,j} = B^{j-1,j+1} \quad \text{and} \quad \psi^{1,j}_{D^{1,j}} = \psi^{j-1,j+1}_{B^{j-1,j+1}}.$$

We conclude by the General Lemma again that

$$\mathcal{J}\{B^{1,3}, \psi^{1,3}_{B^{1,3}}\}(V_\pi) \simeq \mathcal{J}\{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi) \tag{4-20}$$

as vector spaces. Note that $D^{1,n+1} = \epsilon(\mathcal{X}_{n,1})m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{1,n+1})$.

So far in this proof, we have not used any particular property of V_π. We are now going to use the property that V_π has a nonzero generalized Shalika model.

For $n+1 \le r \le 2n-1$ and $1 \le s \le 2n-r$, define

$$\mathscr{X}_{r,s} = \mathscr{X}_{n,1} \oplus \Big(\bigoplus_{\substack{n+1 \le l \le r-1 \\ 1 \le q \le 2n-l}} \mathscr{X}_{l,q} \Big) \oplus \Big(\bigoplus_{q=s}^{2n-r} \mathscr{X}^{r,q} \Big).$$

Then $X^{n+1,n-1}$ normalizes $D^{1,n+1}$ and $\psi^{1,n+1}_{D^{1,n+1}}$. Considering its action on the right side of (4-20), we claim that for any nontrivial character ξ of $X^{n+1,n-1}$,

$$\mathscr{J}\{X^{n+1,n-1}, \xi\}(\mathscr{J}\{D^{1,n+1}, \psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi)) = 0,$$

and hence we must have the trivial character for this action. We assume to the contrary that, by the Frobenius reciprocity law, there exists ℓ a nonzero linear functional on V_π such that

$$\ell(\pi(xn)v) = \psi^{1,n+1}_{1,n+1}(n)\xi(x)\ell(v) \quad \text{for all } x \in X^{n+1,n-1},\ n \in D^{1,n+1} \text{ and } v \in V_\pi.$$

We may assume that there is a $\lambda \in \mathscr{F}^*$ such that $\xi(x(t)) = \psi(\lambda\, t)$, where $x(t) = \mathrm{I}_{4n} + t(E_{n+1,3n-1} - E_{n+2,3n})$. Then ℓ is a nonzero linear functional on V_π such that

$$\ell(\pi(n)v) = \psi^{1,n+1}_{D^{1,n+1}}(n)\ell(v) \quad \text{for } n \in X^{n+1,n-1}D^{1,n+1} \cap N_{n+1}.$$

Note that $X^{n+1,n-1}D^{1,n+1} \cap N_{n+1}$ consists of elements of the form

$$\begin{pmatrix} z & y & w \\ & \mathrm{I}_{2n-2} & y' \\ & & z' \end{pmatrix} \in \mathrm{SO}_{4n}, \tag{4-21}$$

with $z \in \mathrm{U}_{n+1}$ and $y \in \mathrm{Mat}_{n+1,2n-2}$ such that $y_{n+1,n+i} = 0$ for $i = 1, \dots, n-1$.

Now the situation is similar to that of (4-14). The same argument shows that ℓ can be extended trivially to N_{n+1} so that

$$\ell(\pi(n)v) = \psi^{1,n+1}_{N_{n+1}}(n)\ell(v) \quad \text{for } n \in N_{n+1},$$

where $\psi^{1,n+1}_{N_{n+1}}$ is the trivial extension of restriction of $\psi^{1,n+1}_{D^{1,n+1}}$ to $D^{1,n+1} \cap N_{n+1}$.

Note that for an element $n \in N_{n+1}$ of the form (4-21),

$$\psi^{1,n+1}_{D^{1,n+1}}(n) = \psi(z_{1,2} + \cdots + z_{n,n+1})\psi(y_{n+1,1} + y_{n+1,2n-2}).$$

Let ν' be the permutation matrix in GL_{2n} defined by

$$\nu'(i) = \begin{cases} i & \text{if } i = 1, \dots, n+1, \\ 2n & \text{if } i = n+2, \\ i-1 & \text{if } i = n+3, \dots, 2n, \end{cases}$$

which is identified with its embedding $m(\nu')$ in SO_{4n}. Then ν' normalizes N_{n+1} and transforms $\psi^{1,n+1}_{N_{n+1}}$ into ψ_{n+1}. Hence we obtain a nonzero linear functional that factors through $\mathcal{J}\{\psi_{n+1}\}(V_\pi)$. In particular, we have $\mathcal{J}\{\psi_{n+1}\}(V_\pi)\neq 0$.

On the other hand, V_π has a nonzero generalized Shalika model by assumption. Following Theorem 2.4, $\mathcal{J}\{\psi_{n+1}\}(V_\pi)$ must be zero. We get a contradiction. Hence $X^{n+1,n-1}$ must act trivially on $\mathcal{J}\{D^{1,n+1},\psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi)$.

Next we continue this process. Define

$$B^{n-2,n+2}=D^{1,n+1}X^{n+1,n-1},$$

and extend $\psi^{1,n+1}_{D^{1,n+1}}$ to a character $\psi^{n-2,n+2}_{B^{n-2,n+2}}$ on $B^{n-2,n+2}$ by making it trivial on $X^{n+1,n-1}$. Thus we have

$$\mathcal{J}\{B^{n-2,n+2},\psi^{n-2,n+2}_{B^{n-2,n+2}}\}(V_\pi)\simeq\mathcal{J}\{D^{1,n+1},\psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi).$$

Now we can repeat the argument as before, by replacing the $n-2$ coordinates of $\bigoplus_{i=1}^{n-2}\mathcal{Y}_{i,n+2}$ with $\bigoplus_{i=1}^{n-2}\mathcal{X}_{n+1,i}$. For $1\le i\le n-2$ and $j\ge n+2$, define $C^{i,j}=\epsilon(\mathcal{X}_{j-1,i+1})m(\mathrm{U}_{2n})\bar{\epsilon}(\mathcal{Y}_{i,j})$ and

$$B^{i,j}=C^{i,j}Y^{i,j},\quad D^{i,j}=C^{i,j}X^{j-1,i},\quad A^{i,j}=D^{i,j}Y^{i,j}.$$

Let $\psi^{i,n+2}$ be the character of $C^{i,n+2}$, which is trivial on $\ell(C_{n+1,i+1})\overline{\ell}(Y_{i,n+2})$. By the General Lemma, we conclude that

$$\mathcal{J}\{D^{1,n+1},\psi^{1,n+1}_{D^{1,n+1}}\}(V_\pi)\simeq\mathcal{J}\{D^{1,n+2},\psi^{1,n+2}_{D^{1,n+2}}\}(V_\pi)\tag{4-22}$$

as vector spaces. Then, by using the property that V_π has a nonzero generalized Shalika model, we show that $X^{n+2,n-2}$ acts trivially on the right side of (4-22). As before, we get

$$\mathcal{J}\{D^{1,n+2},\psi^{1,n+2}_{D^{1,n+2}}\}(V_\pi)\simeq\cdots\simeq\mathcal{J}\{D^{1,2n-1},\psi^{1,2n-1}_{D^{1,2n-1}}\}(V_\pi)$$

as vector spaces. Note that $D^{1,2n-1}=N_{2n}$ and $\psi^{1,2n-1}_{1,2n-1}=\tilde{\psi}$. We conclude that

$$\mathcal{J}\{D^{1,2n-1},\psi^{1,2n-1}_{D^{1,2n-1}}\}(V_\pi)=\mathcal{J}\{N_{2n},\tilde{\psi}\}(V_\pi).$$

This concludes the proof of part (1) of Theorem 4.1.

5. Irreducibility of the local descent

To finish the proof of Theorem 2.5, it remains to show that σ_{n-1} is irreducible. In Sections 3 and 4, we proved that, as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, the local descent $\sigma_{n-1}=\mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1,\tau))$, as defined in (2-8), is quasisupercuspidal and has a unique nonzero Whittaker functional. Hence it is enough to show that any irreducible summand of σ_{n-1} is generic, that is, has a nonzero Whittaker functional. This is proved in Theorem 5.1(2). Theorem 5.1, whose proof is standard, may be viewed as a generalization of the geometric lemma of Bernstein and Zelevinsky [1977] for the twisted Jacquet functor $\mathcal{J}\{\psi_{n-1}\}$ applied to $\mathcal{L}(1,\tau)$. For a similar discussion for the metaplectic and symplectic groups, see [Ginzburg et al. 1999]

For a given irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$, recall that $\mathrm{I}(s,\tau)$ is the induced representation of $\mathrm{SO}_{4n}(\mathcal{F})$ from the supercuspidal datum (P_{2n},τ) as defined in Section 2.1. The unique Langlands quotient of $\mathrm{I}(s,\tau)$ at $s=1$ is $\mathcal{L}(1,\tau)$.

Theorem 5.1. *Suppose (V_σ,σ) is an irreducible supercuspidal representation of* $\mathrm{SO}_{2n+1}(\mathcal{F})$.

(1) *If* $\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\tau)),V_\sigma)$ *is nonzero for any $s\in\mathbb{C}$, then σ is generic.*

(2) *If* $\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{n-1}\}(\mathcal{L}(1,\tau)),V_\sigma)$ *is nonzero, then σ is generic.*

Clearly part (2) follows from part (1) by the exactness of the twisted Jacquet functors. Part (1) is proved in Section 5.7.

We start by investigating the structure of $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\tau))$ to determine the genericity of σ. We realize the irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(\mathcal{F})$ by its Whittaker model $\mathcal{W}(\tau,\psi)$, and realize the induced representation $\mathrm{I}(s,\tau)$ as $\mathrm{I}(s,\mathcal{W}(\tau,\psi))$. Then we consider $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$.

5.2. ***The twisted Jacquet module $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$.*** We consider first the orbital structure of the closed subgroup $\mathrm{SO}_{2n+2}\cdot N_{n-1}$ acting on the generalized flag variety $P_{2n}\backslash\mathrm{SO}_{4n}$ over the p-adic field $\mathcal{F}$, and then consider the semisimplification of $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\mathcal{W}(\tau,\psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

For $j=1,\dots,2n$, let

$$P_j=\left\{\begin{pmatrix} h & * & * \\ & g & * \\ & & h^* \end{pmatrix}\ \middle|\ h\in\mathrm{GL}_j,\ g\in\mathrm{SO}_{4n-2j}\right\}$$

be the standard maximal parabolic subgroup of SO_{4n}. Then the generalized Bruhat decomposition $P_{2n}\setminus\mathrm{SO}_{4n}/P_{n-1}$ has a complete set of representatives given by $\{\gamma_i\mid i\in 2\mathbb{N},\ n\le i\le 2n\}$, where for $i\in 2\mathbb{N}$ with $n\le i\le 2n$,

$$\gamma_i=\begin{pmatrix} & & \nu_{2n-i} \\ & \mathrm{I}_{2i} & \\ \nu_{2n-i} & & \end{pmatrix}$$

and ν_j is as defined in (2-1). For $k=0,1,\dots,n-1$, let M_k be the standard maximal parabolic subgroup of GL_{n-1} corresponding to the partition $(k,n-k-1)$

of $n-1$ such that the Levi part is $\mathrm{GL}_k \times \mathrm{GL}_{n-k-1}$ and the unipotent radical is

$$L_k = \left\{ \begin{pmatrix} \mathrm{I}_k & \\ A & \mathrm{I}_{n-1-k} \end{pmatrix} \in \mathrm{GL}_{n-1} \;\middle|\; A \in \mathrm{Mat}_{n-1-k,k} \right\}.$$

Lemma 5.3. *The orbits of the closed subgroup* $\mathrm{SO}_{2n+2} \cdot N_{n-1}$ *acting on the generalized flag variety* $P_{2n} \setminus \mathrm{SO}_{4n}$ *are represented by elements of the form* $\gamma_i w$, *where* $n \le i \le 2n$ *is even and the* w *are elements of* $W(\mathrm{GL}_{n-1})$ *given by*

$$\begin{cases} w \in [W(\mathrm{GL}_{2n-i}) \times W(\mathrm{GL}_{i-n-1})]\backslash W(\mathrm{GL}_{n-1}) & \text{if } i \neq n, \\ w = \mathrm{id} & \text{if } i = n. \end{cases}$$

Here $W(\mathrm{GL}_m)$ *denotes the Weyl group of* GL_m.

Proof. Clearly, we have $\mathrm{SO}_{2n+2}\, N_{n-1} \subset P_{n-1}$. Hence we can choose $\gamma_i w$ to be the representative of any double cosets in $P_{2n} \backslash \mathrm{SO}_{4n} / [\mathrm{SO}_{2n+2}\, N_{n-1}]$, for some

$$w \in [\gamma_i^{-1} P_{2n} \gamma_i \cap P_{n-1}] \backslash P_{n-1} / [\mathrm{SO}_{2n+2}\, N_{n-1}].$$

Since $M_{2n-i} \subset \gamma_i^{-1} P_{2n} \gamma_i \cap P_{n-1}$, we may choose a set of representatives for $[\gamma_i^{-1} P_{2n} \gamma_i \cap P_{n-1}] \backslash P_{n-1} / [\mathrm{SO}_{2n+2}\, N_{n-1}]$ from $M_{2n-i} \backslash \mathrm{GL}_{n-1} / N_{n-1}$. Then a complete set of representatives for $M_{2n-i} \backslash \mathrm{GL}_{n-1} / N_{n-1}$ can be chosen from

$$[W(\mathrm{GL}_{2n-i}) \times W(\mathrm{GL}_{i-n-1})] \backslash W(\mathrm{GL}_{n-1}). \qquad \square$$

Let $\alpha_1, \ldots, \alpha_{n-2}$ denote the simple roots of GL_{n-1} with respect to N_{n-1}. Let

$$\{x_{\alpha_j}(t) = \mathrm{I}_{n-1} + t E_{j,j+1} \mid t \in \mathcal{F}\}$$

denote the one parameter unipotent subgroup of N_{n-1} corresponding to the root α. We will take $w = \mathrm{id}$ to be the representative of the coset $W(\mathrm{GL}_k) \times W(\mathrm{GL}_{n-1-k})$ in $W(\mathrm{GL}_{n-1})$.

Lemma 5.4 [Ginzburg et al. 1999, Lemma 4.3]. *If a Weyl group element* w *belongs to* $[W(\mathrm{GL}_k) \times W(\mathrm{GL}_{n-1-k})] \backslash W(\mathrm{GL}_{n-1})$ *and is the identity, then there exists a simple root* α_j *such that* $w x_{\alpha_j}(t) w^{-1} \in L_k$ *for all* $t \in \mathcal{F}$.

Next we consider the semisimplification of the module $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s, \mathcal{W}(\tau, \psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$. It is a standard process to decompose the representation by spaces of functions on $\mathrm{SO}_{4n}(\mathcal{F})$ according the orbital decomposition obtained in Lemma 5.3.

It is clear that among the orbits

$$\mathcal{O}_{i,w} = [P_{2n}]\gamma_i w [\mathrm{SO}_{2n+2}\, N_{n-1}] \quad \text{for } i \in 2\mathbb{N} \text{ with } n \le i \le 2n,$$

the orbit $\mathcal{O}_{2[(n+1)/2],\mathrm{id}}$ is the unique open orbit. Let E be a union of orbits $\mathcal{O}_{i,\omega}$. We denote by $\mathcal{S}(E, \tau_s)$ the space of smooth functions ϕ on E that are compactly supported modulo P_{2n}, have values in the Whittaker model $\mathcal{W}(\tau, \psi)$ and are such that

$$\phi\left(\begin{pmatrix} a & * \\ & a^* \end{pmatrix} g, r\right) = |\det a|^{s/2+n-1/2}\phi(g, ra) \quad \text{for } g \in \mathrm{SO}_{4n} \text{ and } a, r \in \mathrm{GL}_{2n}.$$

We may arrange the orbits in a sequence

$$P_{2n}\,\mathrm{SO}_{2n+2}\,N_{n-1} = \Omega_1,\ \ldots,\ \Omega_l = \mathbb{O}_{2[(n+1)/2],\mathrm{id}}$$

such that $F_i = \bigcup_{j=1}^{i} \Omega_j$ is closed in SO_{4n}. It is clear that Ω_i is open in F_i and F_{i-1} is closed in F_i. We obtain the exact sequence

$$0 \to \mathcal{S}(\Omega_{i+1}, \tau_s) \xrightarrow{e} \mathcal{S}(F_{i+1}, \tau_s) \xrightarrow{r} \mathcal{S}(F_i, \tau_s) \to 0, \tag{5-1}$$

where the map e is the natural embedding and r is the restriction to F_i. Apply the twisted Jacquet functor $\mathcal{J}\{\psi_{n-1}\}$ to the exact sequence (5-1). Since the Jacquet functors are exact, we obtain another exact sequence

$$0 \to \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\Omega_{i+1}, \tau_s)) \to \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(F_{i+1}, \tau_s)) \to \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(F_i, \tau_s)) \to 0.$$

We obtain the semisimplification of $\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s, \mathcal{W}(\tau, \psi)))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$ as $\bigoplus_{i=1}^{l} \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\Omega_i, \tau_s))$.

Next, we study the space $\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\Omega_i, \tau_s))$ for $i = 1, 2, \ldots, l$. We assume for the rest of this section unless stated otherwise that all inductions are unnormalized.

As $\mathrm{SO}_{2n+2}\,N_{n-1}$ module, we have

$$\mathcal{S}(\mathbb{O}_{i,w}, \tau_s) \simeq \text{c-Ind}_{P_{2n}^{\gamma_{i,w}}}^{\mathrm{SO}_{2n+2}\,N_{n-1}} (\delta_{P_{2n}}^{1/2}\tau_s)^{\gamma_{i,w}},$$

where c-Ind denotes the compact induction and

$$R_{i,w} = P_{2n}^{\gamma_{i,w}} := (\gamma_i w)^{-1} P_{2n} \gamma_i w \cap \mathrm{SO}_{2n+1}\,N_{n-1}.$$

Lemma 5.5. *With notation above, the following vanishing properties hold.*

(1) *For* $w \neq \mathrm{id}$,

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{R_{i,w}}^{\mathrm{SO}_{2n+2}\,N_{n-1}} (\delta_{P_{2n}}^{1/2}\tau_s)^{\gamma_{i,w}}) = 0 \quad \textit{for } i \geq 2[(n+1)/2].$$

(2) *For* $w = \mathrm{id}$,

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{R_{i,\mathrm{id}}}^{\mathrm{SO}_{2n+2}\,N_{n-1}} (\delta_{P_{2n}}^{1/2}\tau_s)^{\gamma_{i,\mathrm{id}}}) = 0 \quad \textit{for } i > 2[(n+1)/2].$$

Proof. When $w \neq \mathrm{id}$, by Lemma 5.4, there is a simple root subgroup $x(t)$ inside N_{n-1} such that $\gamma_i w x(t)(\gamma_i w)^{-1}$ lies in the unipotent radical of P_{2n}. This shows that

$$x(t) \in R_{i,w} \cap N_{n-1} \quad \text{and} \quad (\delta_{P_{2n}}^{1/2}\tau_s)^{\gamma_{i,w}}(x(t)) = \mathrm{id},$$

while $\psi_{n-1}(x(t)) = \psi(t)$.

When $w = \mathrm{id}$ and $i > 2[(n+1)/2]$, the root subgroup $x_\alpha(t)$ of SO_{4n} is for $\alpha = e_{n-1} + e_{2n}$ invariant under the conjugation by $\gamma_i'^{-1}$. Hence

$$x(t) \in R_{i,w} \cap N_{n-1} \quad \text{and} \quad (\delta_{P_{2n}}^{1/2} \tau_s)^{\gamma_{i,w}}(x(t)) = \mathrm{id},$$

while $\psi_{n-1}(x(t)) = \psi(t)$. □

Therefore, we are left with $\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\Omega_l, \tau_s))$ for the Zariski open orbit $\Omega_l = \mathbb{O}_{2[(n+1)/2],\mathrm{id}}$. To summarize:

Proposition 5.6. *We have*

$$\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s, \mathcal{W}(\tau, \psi))) \simeq \mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\mathbb{O}_{2[(n+1)/2],\mathrm{id}}, \tau_s))$$

for all $s \in \mathbb{C}$ as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

5.7. ***Proof of Theorem 5.1(1).*** Keep the previous notation. By Proposition 5.6,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}&(\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s, \tau)), V_\sigma) \\ &\simeq \mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\mathbb{O}_{2[(n+1)/2],\mathrm{id}}, \tau_s)), V_\sigma),\end{aligned}$$

reducing the proof to understanding the structure of $\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\mathbb{O}_{2[(n+1)/2],\mathrm{id}}, \tau_s))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

It is more convenient to choose ν_{4n} as representative of the orbit $\mathbb{O} = \mathbb{O}_{2[(n+1)/2],\mathrm{id}}$ than the original $\gamma_{2[(n+1)/2],\mathrm{id}}$. Then

$$\mathcal{S}(\mathbb{O}, \tau_s) \simeq \text{c-Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2} N_{n-1}} (\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}},$$

where $P_{2n}^{\nu_{4n}} = \nu_{4n}^{-1} P_{2n} \nu_{4n} \cap \mathrm{SO}_{2n+2} N_{n-1}$. Let $\mathfrak{Q}_{n+1}$ be the maximal standard parabolic subgroup of SO_{2n+2} whose Levi component is isomorphic to GL_{n+1}, and let $\mathfrak{Q}_{n+1}^-$ be the opposite parabolic subgroup. Then we have

$$\begin{aligned} P_{2n}^{\nu_{4n}} &= \left\{ m\left(\begin{pmatrix} z & c \\ & \mathrm{I}_{n+1} \end{pmatrix}\right) \in \mathrm{SO}_{4n} \;\middle|\; z \in \mathrm{U}_{n-1} \right\} \cdot \overline{\mathfrak{Q}}_{n+1} \\ &:= m(\mathrm{U}_{2n,n-1}) \cdot \mathfrak{Q}_{n+1}^-, \end{aligned}$$

where $\mathrm{U}_{2n,j}$ is the subgroup of the unipotent radical U_{2n} of the standard Borel subgroup of GL_{2n} consisting of elements of type

$$\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{2n-j} \end{pmatrix} \in \mathrm{U}_{2n} \quad \text{with } z \in \mathrm{U}_j.$$

For

$$\phi \in \text{c-Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2} N_{n-1}} (\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}} \quad \text{and} \quad q = \begin{pmatrix} a & 0 \\ * & a^* \end{pmatrix} \in \mathfrak{Q}_{n+1}^-,$$

we have

$$(5\text{-}2)\quad (\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}}\,(\mathrm{diag}(\mathrm{I}_{n-1}, q, \mathrm{I}_{n-1}))\,(\phi)(g,r) = |\det a|^{-(s/2+n-1/2)}\phi(g, r(\mathrm{diag}\,\mathrm{I}_{n-1}, a)),$$

and for $\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{n+1} \end{pmatrix} \in \mathrm{U}_{2n,n-1}$, we have

$$(5\text{-}3)\qquad (\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}}\Big(m\Big(\begin{pmatrix} z & c \\ & \mathrm{I}_{n+1} \end{pmatrix}\Big)\Big)(\phi)(g,r) = \phi\Big(g, r\begin{pmatrix} z & c \\ 0 & \mathrm{I}_{n+1} \end{pmatrix}\Big).$$

To understand $\mathcal{J}\{\psi_{n-1}\}(\mathcal{S}(\mathbb{O}, \tau_s))$ as a representation of $\mathrm{SO}_{2n+1}(\mathcal{F})$, we consider the double coset decomposition $P_{2n}^{\nu_{4n}} \backslash \mathrm{SO}_{2n+2}\cdot N_{n-1}/ \mathrm{SO}_{2n+1}\cdot N_{n-1}$, which reduces the proof to the computation of the double cosets

$$\mathfrak{Q}_{n+1}^{-} \backslash \mathrm{SO}_{2n+2} / \mathrm{SO}_{2n+1}\,.$$

Next proposition shows that it has only one orbit.

Proposition 5.8. *Over any field k of characteristic zero, the generalized flag variety $\mathfrak{Q}_{n+1}^{-}(k) \backslash \mathrm{SO}_{2n+2}(k)$ has only one orbit under the action of $\mathrm{SO}_{2n+1}(k)$.*

Proof. Let $X = k^{2n+2}$ be a k-vector space, written with its elements as column vector, with a quadratic form q defined by $\frac{1}{2}\nu_{2n+2}$. Then $\mathrm{SO}(X) \simeq \mathrm{SO}_{2n+2}$ Let $e_1, \dots, e_{2n+2}$ be the standard basis of X, $v_0 = e_{n+1} + e_{n+2}$. Let $Y = (k\cdot v_0)^{\perp}$. Then $\dim Y = 2n+1$ and $\mathrm{SO}(Y) = \mathrm{SO}_{2n+1}$. Note that Y has a basis

$$(5\text{-}4)\qquad e_{n+1} - e_{n+2}, e_1, e_2 \dots, e_n, e_{n+3}, \dots, e_{2n+1}, e_{2n+2}.$$

Then a basis of X can be chosen to be

$$(5\text{-}5)\qquad e_{n+1} + e_{n+2}, e_{n+1} - e_{n+2}, e_1, e_2, \dots, e_n, e_{n+3}, \dots, e_{2n+1}, e_{2n+2}.$$

Let $g \in \mathrm{SO}(X)$ such that $g(v_0) = v_0$. Then $g(Y) = Y$. Assume that the matrix of $g|_Y$ in the basis (5-4) is A_g. Then g in the basis (5-5) is $\mathrm{diag}(1, A_g)$. As $\det g = 1$, we must have $\det(A_g) = 1$; hence $g \in \mathrm{SO}(Y)$, so the stabilizer of v_0 is $\mathrm{SO}(Y)$.

Note that $q(v_0) = 1$. Let $Z = \{v \in X \mid q(v) = 1\}$. Then SO_{2n+2} acts transitively on Z. To show the proposition, we only need to show that $\mathfrak{Q}_{n+1}^{-}$ acts on Z transitively. In fact, if $\mathfrak{Q}_{n+1}^{-}$ acts transitively on Z, then, letting $h \in \mathrm{SO}(X)$, there exists $t \in \mathfrak{Q}_{n+1}^{-}$ such that $h\cdot v_0 = t\cdot v_0$. Hence $(t^{-1}h)\cdot v_0 = v_0$, and then $t^{-1}h \in \mathrm{SO}_{2n+1}$ and $h \in \mathfrak{Q}_{n+1}^{-}\,\mathrm{SO}_{2n+1}$. This means that $\mathrm{SO}_{2n+2} = \mathfrak{Q}_{n+1}^{-}\,\mathrm{SO}_{2n+1}$.

Now we show that $\mathfrak{Q}_{n+1}^{-}$ acts transitively on Z. We only need to show that any element of Z can be moved to v_0 under the action of some element in $\mathfrak{Q}_{n+1}^{-}$. Let $v = (v_1, v_2) \in X$ with $v_1, v_2 \in k^{n+1}$. Take $g \in \mathfrak{Q}_{n+1}^{-}$ to be

$$g = \begin{pmatrix} a & 0 \\ b & a^{*} \end{pmatrix} \quad \text{with } a \in \mathrm{GL}_{n+1}\,.$$

Then the action of g on v is given by $g \cdot v = (av_1, bv_1 + a^* v_2)^t$.

Assume now $q(v) = 1$. Then $v_1 \neq 0$, otherwise $q(v) = 0$. Then there is $a \in \mathrm{GL}_{n+1}$ such that $av_1 = (0, \dots, 0, 1)^t$. For this a, there exists $b \in \mathrm{Mat}_{n+1}(k)$ such that $bv_1 = (1, 0, \dots, 0)^t - a^* v_2$, since $v_1 \neq 0$. Now

$$g = \begin{pmatrix} a & 0 \\ b & a^* \end{pmatrix} \in \mathcal{Q}_{n+1}^- \quad \text{and} \quad g \cdot v = v_0. \qquad \square$$

It follows that $P_{2n}^{\nu_{4n}} \backslash \mathrm{SO}_{2n+2} N_{n-1} = [P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}] \backslash \mathrm{SO}_{2n+1} N_{n-1}$. By restriction to $\mathrm{SO}_{2n+1} N_{n-1}$, we have

$$\text{c-Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}}) \simeq \text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F}) \ltimes N_{n-1}(\mathcal{F})$. Hence

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}}}^{\mathrm{SO}_{2n+2} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})) \\ \simeq \mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}}))$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$.

Define $\psi_{\mathrm{U}_{2n,n-1}}(u(z, c)) := \psi_{n-1}|_{\mathrm{U}_{2n,n-1}}(u(z, c))$.

Proposition 5.9. *With notation above*,

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})) \\ \simeq \text{c-Ind}_{\mathcal{P}_n^-}^{\mathrm{SO}_{2n+1}} (\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau') |\det|^{-s/2-1/2})$$

as representations of $\mathrm{SO}_{2n+1}(\mathcal{F})$, *where* $\mathcal{P}_n^- := \mathcal{Q}_{n+1}^- \cap \mathrm{SO}_{2n+1}$, *the representation* τ' *is obtained by restriction to* $\mathcal{P}_n^-(\mathcal{F})$ *of the representation of* $\mathcal{Q}_{n+1}^-(\mathcal{F})$ *given by* (5-2) *and* (5-3), *and* $\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')$ *denotes the twisted Jacquet module of* τ' *along* $(\mathrm{U}_{2n,n-1}, \psi_{\mathrm{U}_{2n,n-1}})$.

Proof. Let f be a section in

$$\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}}).$$

Consider the restriction of f to $\mathrm{SO}_{2n+1}(\mathcal{F})$. It is clear that this restriction map factors through

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1} N_{n-1}}^{\mathrm{SO}_{2n+1} N_{n-1}} ((\delta_{P_{2n}}^{1/2} \tau_s)^{\nu_{4n}})),$$

so we still denote the restriction by $f \mapsto f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$. By (5-2) and (5-3), the restriction $f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$ belongs to the space

$$\text{c-Ind}_{\mathcal{P}_n^-}^{\mathrm{SO}_{2n+1}} (\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau') |\det|^{-s/2-1/2}).$$

By using the orbital decomposition in Proposition 5.8 and (5-3), it is not hard to check that $f \mapsto f|_{\mathrm{SO}_{2n+1}(\mathcal{F})}$ is in fact injective. The argument is the same as in the proof of [Ginzburg et al. 1999, formula (6.5)] and similar to that of [Kudla 1986, Lemma 5.3]. We omit the details.

The surjectivity can be verified as follows. Assume that we have a smooth $\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(V_{\tau'})$-valued function g on SO_{2n+1}, compactly supported modulo $\mathcal{P}_n^-$, satisfying

$$g(qy) = \mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')|\det|^{-s/2-1/2}(q)g(y) \quad \text{for } q \in \mathcal{P}_n^- \text{ and } y \in \mathrm{SO}_{2n+1}.$$

Since g is locally constant, we may pull back g to a smooth $V_{\tau'}$-valued function g' on SO_{2n+1}, compactly supported modulo $\mathcal{P}_n^-$, satisfying

$$g'(qy) = \tau'(q)|\det|^{-(s/2+n-1/2)}g'(y) \quad \text{for } q \in \mathcal{P}_n^- \text{ and } y \in \mathrm{SO}_{2n+1}.$$

The unipotent subgroup N_{n-1} can be written as $N_{n-1} = m(\mathrm{U}_{2n,n-1}) \ltimes N''$, where N'' is the intersection of N_{n-1} with the unipotent radical V_{2n} of P_{2n}. Then

$$\mathrm{SO}_{2n+1}\, N_{n-1} = \mathrm{SO}_{2n+1}\, \mathrm{U}''_{2n,n-1},$$

which is in fact a homeomorphism. Indeed, let $z'y'x' = zyx$ with $x, x' \in \mathrm{SO}_{2n+1}$, $z, z' \in B_{n-1}$ and $y, y' \in N''$. Then $y = (z^{-1}z')y'(x'x^{-1}) \in N''$. Hence $x = x'$, $z = z'$ and $y = y'$.

Then we can pull back g to a section f in

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1}\, N_{n-1}}^{\mathrm{SO}_{2n+1}\, N_{n-1}}((\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}})),$$

which is defined as follows. Choose a compactly supported smooth function ϕ on N'' that has a nonzero projection under the twisted Jacquet functor with respect to $(N'', \psi_{n-1}|_{N''})$, and define $f'(uyx, r) := \phi(y)g'(x, ru)$, for all $x \in \mathrm{SO}_{2n+1}$, $u \in \mathrm{U}_{2n,n-1}$, $y \in N''$, and $r \in \mathrm{GL}_{2n}$. It is clear that f' is a nonzero section in

$$\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1}\, N_{n-1}}^{\mathrm{SO}_{2n+1}\, N_{n-1}}((\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}}).$$

By checking the action of N_{n-1} on f', it is clear that f' factors through

$$\mathcal{J}\{\psi_{n-1}\}(\text{c-Ind}_{P_{2n}^{\nu_{4n}} \cap \mathrm{SO}_{2n+1}\, N_{n-1}}^{\mathrm{SO}_{2n+1}\, N_{n-1}}((\delta_{P_{2n}}^{1/2}\tau_s)^{\nu_{4n}})),$$

whose image f has the restriction to $\mathrm{SO}_{2n+1}(\mathcal{F})$ equal to g. □

The elements of $\mathcal{P}_n^-$ have the form

$$\begin{pmatrix} b & & & \\ & x & 1 & 0 & \\ & -x & 0 & 1 & \\ & y' & -x' & x' & b^* \end{pmatrix} \in \mathrm{SO}_{2n+2}(\mathcal{F}),$$

which is identified (following the embedding we assumed as before) with

$$\begin{pmatrix} b & & \\ x & 1 & \\ y & x' & b^* \end{pmatrix} \in \mathrm{SO}_{2n+1}(\mathcal{F}).$$

Following the discussions above, we deduce that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}&(\mathcal{J}\{\psi_{n-1}\}(\mathrm{I}(s,\tau)), V_\sigma)\\ &\simeq \mathrm{Hom}_{\mathrm{SO}_{2n+1}(\mathcal{F})}(\text{c-Ind}_{\mathcal{P}_n^-(\mathcal{F})}^{\mathrm{SO}_{2n+1}(\mathcal{F})}(\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')|\det|^{-s/2-1/2}), V_\sigma).\end{aligned}$$

By the Frobenius reciprocity law, the last space is isomorphic to

$$\mathrm{Hom}_{\mathcal{P}_n^-(\mathcal{F})}(\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')|\det|^{-s/2-1/2}, V_\sigma).$$

By the assumption of Theorem 5.1, the last space is nonzero. Since the argument below only uses the genericity of τ' and the supercuspidality of σ and does not depend on the value s, we may consider, for simplicity, only the nonzero space $\mathrm{Hom}_{\mathcal{P}_n^-(\mathcal{F})}(\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau'), V_\sigma)$. Any nonzero element ξ in it is a $\mathcal{P}_n^-(\mathcal{F})$-equivariant, linear map from $\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')$ to V_σ. In particular, for any $v \in V_{\tau'}$, we have

$$\sigma\left(\begin{pmatrix} a & & \\ x & 1 & \\ y & x' & a^* \end{pmatrix}\right)(\xi(v)) = \xi(\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')\left(\begin{pmatrix} \mathrm{I}_{n-1} & & \\ & a & \\ & x & 1 \end{pmatrix}\right)(v)). \tag{5-6}$$

Take $a = \mathrm{I}_n$ and consider the action of the unipotent radical of $\mathcal{P}_n^-(\mathcal{F})$, which is denoted by $\mathcal{V}_n^-(\mathcal{F})$ and consists of elements of the form

$$v^-(x,y) := \begin{pmatrix} \mathrm{I}_n & & \\ x & 1 & \\ y & x' & \mathrm{I}_n \end{pmatrix}.$$

Then (5-6) implies that the center (the elements of type $v^-(0,y)$) of $\mathcal{V}_n^-(\mathcal{F})$ acts on V_σ trivially. Since V_σ is supercuspidal, there is a nonzero vector $v \in \mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(V_{\tilde{\tau}})$ such that the unipotent radical of $\mathcal{V}_n^-(\mathcal{F})$ acts on $\xi(v)$ by a nontrivial character. Since the $\mathrm{GL}_n(\mathcal{F})$ acts on the x-part (more precisely, the quotient of $\mathcal{V}_n^-(\mathcal{F})$ modulo the center) with two orbits, we may assume that

$$\sigma(v^-(x,y))(\xi(v)) = \psi_{\mathcal{V}_n^-}(v^-(x,y))\xi(v) = \psi(x_n)\xi(v) \quad \text{for } x = (x_1, \cdots, x_n)$$

where $\psi_{\mathcal{V}_n^-}$ is a nonzero character of $\mathcal{V}_n^-(\mathcal{F})$. In other words, the map ξ descends to a map from $\mathcal{J}\{\psi_{\mathrm{U}_{2n,n-1}}\}(\tau')$ to $\mathcal{J}\{\psi_{\mathcal{V}_n^-}\}(V_\sigma)$.

By (5-6), we have

$$\xi(\mathcal{J}\{\psi_{n-1}\}(\tau')\left(\begin{pmatrix}\mathrm{I}_{n-1} & & \\ & a & \\ & x & 1\end{pmatrix}\right)(v)) = \psi(x_n)\xi(v).$$

Now consider the subgroup $B_{2n,n}$ of GL_{2n} consisting of elements of the form

$$b(z, c, e, y, d) := \begin{pmatrix} z & c & e \\ 0 & 1 & y \\ 0 & 0 & d \end{pmatrix} \quad \text{with } d \in \mathrm{GL}_n(\mathscr{F}) \text{ and } z \in \mathrm{U}_{n-1}.$$

Let μ be the Weyl element of GL_{2n} that corresponds to the elementary matrix $\mathrm{diag}(\mathrm{I}_{n-1}, \nu_{n+1})$. Then it is easy to see that

$$\xi(\mathcal{J}\{\psi_{n-1}\}(\tau')(b(z, c, e, y, \mathrm{I}_n))(\mu v)) = \psi_{U_{n-1}}(z)\psi(c_{n-1})\psi(y_1)\xi(\mu v).$$

This means that the map ξ factors through the n-th derivative $\tilde{\tau}^{(n)}$ in the sense of [Bernstein and Zelevinsky 1976]. Therefore, we can view ξ as a map from the n-th derivative $\tilde{\tau}^{(n)}$ to $\mathcal{J}\{\psi_{\mathcal{V}_n^-}\}(V_\sigma)$, which has the equivalence property, for $a \in \mathrm{GL}_{n-1}$, that

$$\mathcal{J}\{\psi_{\mathcal{V}_n^-}\}(\sigma)\left(\begin{pmatrix} a & 0 \\ x & 1\end{pmatrix}\right)\xi(v) = \xi\left((\tau')^{(n)}\left(\begin{pmatrix}\mathrm{I}_n & & \\ & 1 & x^* \\ & 0 & \nu_{n-1}a\nu_{n-1}\end{pmatrix}\right)\right)(\mu v),$$

where $x^* = (x_{n-1}, x_{n-2}, \dots, x_1)$ if $x = (x_1, \dots, x_{n-1})$.

Now we come back to the situation of (5-6) with $a \in \mathrm{GL}_{n-1}$. We repeat the same process with the supercuspidality of σ and the genericity of τ. Eventually, we arrive at the $2n$-th derivative of τ', which is the twisted Jacquet module of Whittaker type. The equivalence property in this last case shows that V_σ has a nonzero Whittaker functional. Hence it is generic. This finishes the proof of Theorem 5.1(1).

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