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Dedicated to Professor Katsuhiro Shiohama on the occasion of his 70th birthday.

Let M be an n-dimensional complete locally conformally flat Riemannian manifold with constant scalar curvature R and $n \geq 3$. We first prove that if $R = 0$ and the $L^{n/2}$ norm of the Ricci curvature tensor of M is pinched in $[0, C_1(n))$, then M is isometric to a complete flat Riemannian manifold, which improves Pigola, Rigoli, and Setti's pinching theorem. Next, we prove that if $n \geq 6$, $R \neq 0$, and the $L^{n/2}$ norm of the trace-free Ricci curvature tensor of M is pinched in $[0, C_2(n))$, then M is isometric to a space form. Finally, we prove an L^n trace-free Ricci curvature pinching theorem for complete locally conformally flat Riemannian manifolds with constant nonzero scalar curvature. Here $C_1(n)$ and $C_2(n)$ are explicit positive constants depending only on n.

1. Introduction

The curvature pinching phenomenon plays an important role in global differential geometry. Motivated by the famous pinching theorem for minimal submanifolds in a sphere due to J. Simons [1968], C. L. Shen [1989] proved an L^p pinching theorem for embedded compact minimal hypersurfaces in $\mathbb{S}^{n+1}(1)$. Many authors have extended this result [Wang 1988; Lin and Xia 1989; Xu 1990; 1994; Bérard 1991; Shiohama and Xu 1994; Ni 2001; Xu and Gu 2007a; 2007b], but by producing extrinsic rigidity theorems for submanifolds. We are interested in intrinsic L^p pinching problems for Riemannian manifolds.

A conformally flat structure on a Riemannian manifold is a natural generalization of a conformal structure of a Riemannian surface. A Riemannian manifold (M, g) is locally conformally flat with a locally conformally flat structure on M

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if and only if there exists a coordinate chart $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ covering M such that $(\varphi_\alpha^{-1})^* g = \rho_\alpha dx^2$ for every $\alpha \in \Lambda$, where dx^2 is the Euclidean metric on $\mathbb{R}^n$ and ρ_α is a positive function on $\mathbb{R}^n$. It is well known that a Riemannian surface is always locally conformally flat. In higher dimensions, however, not every manifold admits a locally conformally flat structure, and it is difficult to give a good classification of locally conformally flat manifolds. Throughout this paper, we always assume that M is an n-dimensional complete Riemannian manifold with $n \geq 3$. According to the decomposition of the Riemannian curvature tensor, a locally conformally flat manifold has constant sectional curvature if and only if it is Einstein, that is, the trace-free Ricci tensor, defined by $\widetilde{\mathrm{Ric}} = \mathrm{Ric} - (R/n)g$, is identically equal to zero, where Ric is the Ricci curvature tensor and R is the scalar curvature. As a consequence, by the Hopf classification theorem, space forms are the only locally conformally flat Einstein manifolds.

In [1967], M. Tani showed that the universal cover of a compact oriented locally conformally flat manifold with positive Ricci curvature and constant scalar curvature is isometrically a sphere. This result has been generalized by other mathematicians to the case where M satisfies some pointwise pinching condition. Recently, S. Pigola, M. Rigoli and A. G. Setti characterized a simply connected space form with a pointwise Ricci curvature pinching condition:

Theorem A [Pigola et al. 2007]. *For $n \geq 3$, let (M, g) be a complete simply connected and locally conformally flat Riemannian n-manifold with constant scalar curvature $R > 0$. If $|\mathrm{Ric}|^2 \leq R^2/(n-1)$ on M and the strict inequality holds at some point, then M is isometric to a sphere.*

Q. M. Cheng, S. Ishikawa and K. Shiohama [1999] completely classified three-dimensional complete and locally conformally flat Riemannian manifolds whose scalar curvature and norm of the Ricci curvature tensor are positive constants. Can the pointwise pinching conditions be replaced by global pinching ones? In [2007], Pigola, Rigoli and Setti got a global pinching result that can be considered as an extension of the theorem above:

Theorem B [Pigola et al. 2007]. *For $n \geq 3$, let (M, g) be a complete simply connected and locally conformally flat Riemannian n-manifold with zero scalar curvature and $n \geq 3$. If $\|\mathrm{Ric}\|_{n/2} < C(n)$, then M is isometric to Euclidean space. Here $\|\cdot\|_k$ denotes the L^k norm and $C(n) = 2n^{-5/2}(n-1)^{1/2}(n-2)^3 \omega_n^{2/n}$, with ω_n the volume of the unit sphere $\mathbb{S}^n$,*

Suppose that M is locally conformally flat with constant scalar curvature R. In Section 3, we will first prove that if $R = 0$ and the $L^{n/2}$ norm of the Ricci curvature tensor of M is pinched in $[0, C_1(n))$ for some explicit positive constant $C_1(n)$ depending only on n, then M is isometric to a complete flat Riemannian manifold, which improves Pigola, Rigoli, and Setti's pinching theorem. Secondly, we prove that if $n \geq 6$, $R \neq 0$, and the $L^{n/2}$ norm of the trace-free Ricci curvature tensor of M is pinched in $[0, C_2(n))$ for some explicit positive constant $C_2(n)$ depending only on n, then M is isometric to a space form. Finally, we prove an L^n trace-free Ricci curvature pinching theorem for complete locally conformally flat Riemannian manifolds with nonzero constant scalar curvature.

2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$, and let $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal basis of the tangent space of M. We define the Kulkarni–Nomizu product $\odot$ for symmetric 2-tensors α and β in local coordinates by

$$(\alpha \odot \beta)_{ijkl} = \alpha_{ik}\beta_{jl} + \alpha_{jl}\beta_{ik} - \alpha_{il}\beta_{jk} - \alpha_{jk}\beta_{il}.$$

The Riemannian curvature tensor can be decomposed as

$$\mathrm{Rm} = \frac{R}{2(n-1)(n-2)} g \odot g - \frac{1}{n-2} \mathrm{Ric} \odot g + W, \tag{2-1}$$

where Rm, W, Ric, and R are respectively the Riemannian curvature tensor, the Weyl curvature tensor, the Ricci curvature tensor and the scalar curvature of M. It was shown in [Eisenhart 1997] that if $n \geq 4$, then M is locally conformally flat if and only if the Weyl tensor vanishes, and if $n = 3$, then M is locally conformally flat if and only if $\nabla \mathrm{Ric}$ is totally symmetric. If M is locally conformally flat, we see from (2-1) that the Riemannian curvature tensor can be expressed in terms of the Ricci curvature tensor by

$$\begin{aligned} R_{ijkl} = {} & \frac{R}{(n-1)(n-2)} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ & - \frac{1}{n-2} (R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}), \end{aligned} \tag{2-2}$$

where R_{ijkl} and R_{ij} are components of Rm and Ric in local orthonormal frame fields. We define the trace-free Ricci curvature tensor by $\widetilde{\mathrm{Ric}} = \sum_{i,j} \tilde{R}_{ij} \omega_i \otimes \omega_j$, where $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is the frame dual to $\{e_1, e_2, \ldots, e_n\}$, and

$$\tilde{R}_{ij} = R_{ij} - (R/n)\delta_{ij}. \tag{2-3}$$

Putting $S = |\mathrm{Ric}|^2$ and $\tilde{S} = |\widetilde{\mathrm{Ric}}|^2$, we have $\tilde{S} = S - R^2/n$ from (2-3). If R is constant, then R_{ij} and $\tilde{R}_{ij}$ are Codazzi tensors, that is, $\nabla_j R_{ik} = \nabla_k R_{ij}$ and $\nabla_j \tilde{R}_{ik} = \nabla_k \tilde{R}_{ij}$ for $1 \leq i, j, k \leq n$.

Lemma 2.1. *Let (M, g) be a locally conformally flat Riemannian n-manifold with constant scalar curvature. Set $f_\tau = (\tilde{S} + n\tau^2)^{1/2}$, where $\tau \in \mathbb{R}^+$. Then*

$$|\nabla \widetilde{\mathrm{Ric}}|^2 \geq \frac{n+2}{n} |\nabla f_\tau|^2. \tag{2-4}$$

Proof. Putting $x_{ij} = \tilde{R}_{ij} + \tau\delta_{ij}$, we have $\nabla_k x_{ij} = \nabla_k \tilde{R}_{ij}$ and hence

$$\sum_{i,j,k} (\nabla_k x_{ij})^2 = \sum_{i,j,k} (\nabla_k \tilde{R}_{ij})^2. \tag{2-5}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal frame such that $\tilde{R}_{ij} = \lambda_i \delta_{ij}$ for $1 \le i, j \le n$. Since $f_\tau = (\tilde{S} + n\tau^2)^{1/2}$, we get $x_{ij} = (\lambda_i + \tau)\delta_{ij}$ and $\sum_{i,j} x_{ij}^2 = f_\tau^2$. Then

$$\begin{aligned}(2f_\tau|\nabla f_\tau|)^2 = |\nabla f_\tau^2|^2 &= 4\sum_k \Bigl(\sum_i x_{ii}\nabla_k x_{ii}\Bigr)^2 \\ &\le 4\Bigl(\sum_i x_{ii}^2\Bigr)\Bigl(\sum_{i,k} (\nabla_k x_{ii})^2\Bigr) = 4 f_\tau^2 \Bigl(\sum_{i,k} (\nabla_k x_{ii})^2\Bigr).\end{aligned} \tag{2-6}$$

On the other hand, we have

$$\sum_{i,j,k} (\nabla_k x_{ij})^2 \ge 2\sum_{i\ne k} (\nabla_k x_{ii})^2 + \sum_{i,k} (\nabla_k x_{ii})^2. \tag{2-7}$$

For each fixed k, we have

$$\begin{aligned}\sum_i (\nabla_k x_{ii})^2 &= \sum_{i\ne k} (\nabla_k x_{ii})^2 + \Bigl(\sum_i \nabla_k x_{ii} - \sum_{i\ne k} \nabla_k x_{ii}\Bigr)^2 \\ &= \sum_{i\ne k} (\nabla_k x_{ii})^2 + \Bigl(\sum_{i\ne k} \nabla_k x_{ii}\Bigr)^2 \\ &\le \sum_{i\ne k} (\nabla_k x_{ii})^2 + (n-1)\sum_{i\ne k} (\nabla_k x_{ii})^2.\end{aligned} \tag{2-8}$$

Combining (2-5), (2-6), (2-7) and (2-8), we obtain

$$\sum_{i,j,k} (\nabla_k \tilde{R}_{ij})^2 \ge \frac{n+2}{n}\sum_{i,k} (\nabla_k x_{ii})^2 \ge \frac{n+2}{n}|\nabla f_\tau|^2.$$

So $|\nabla \widetilde{\mathrm{Ric}}|^2 \ge ((n+2)/n)|\nabla f_\tau|^2$. □

We see that $\mathrm{tr}(\widetilde{\mathrm{Ric}}^3) = \sum_{i,j,k} \tilde{R}_{ij}\tilde{R}_{jk}\tilde{R}_{ki}$. Following [Pigola et al. 2007], we have

$$\tfrac{1}{2}\triangle \tilde{S} = |\nabla \widetilde{\mathrm{Ric}}|^2 + \frac{n}{n-2}\,\mathrm{tr}(\widetilde{\mathrm{Ric}}^3) + \frac{R}{n-1}\tilde{S}. \tag{2-9}$$

By using the Lagrange multiplier method, we have the inequality

$$\mathrm{tr}(\widetilde{\mathrm{Ric}}^3) \ge -\frac{n-2}{\sqrt{n(n-1)}}\tilde{S}^{3/2}. \tag{2-10}$$

Putting $f_\tau = (\tilde{S} + n\tau^2)^{1/2} = (|\widetilde{\mathrm{Ric}}|^2 + n\tau^2)^{1/2}$, $f = (\tilde{S})^{1/2}$, from (2-4), (2-9) and (2-10) we have

$$\tfrac{1}{2}\triangle f^2 \geq \frac{n+2}{n}|\nabla f_\tau|^2 - \sqrt{\frac{n}{n-1}}f^3 + \frac{R}{n-1}f^2. \tag{2-11}$$

Lemma 2.2 [Hebey 1999]. *For $n \geq 3$, let (M, g) be a smooth complete locally conformally flat Riemannian n-manifold. Then for any smooth function f with compact support*,

$$\Big(\int_M |f|^{2n/(n-2)} dM\Big)^{(n-2)/n} \leq \frac{4}{n(n-2)\omega_n^{2/n}}\Big(\int_M |\nabla f|^2 dM + \frac{n-2}{4(n-1)}\int_M Rf^2 dM\Big). \tag{2-12}$$

3. L^p Ricci curvature pinching theorems

Theorem 3.1. *Let (M, g) be a complete locally conformally flat Riemannian n-manifold with constant scalar curvature R. Put*

$$C_1(n) = 2n^{-5/2}(n-1)^{1/2}(n-2)(n^2-2n+4)\omega^{2/n},$$
$$C_2(n) = \sqrt{n(n-1)}\,\omega_n^{2/n}.$$

(i) *If $n \geq 3$, $R = 0$, and $\|\mathrm{Ric}\|_{n/2} < C_1(n)$, then M is isometric to a complete flat Riemannian manifold. In particular, if M is simply connected, then M is isometric to the Euclidean space $\mathbb{R}^n$.*

(ii) *If $n \geq 6$, $R = n(n-1)c \neq 0$, and $\|\widetilde{\mathrm{Ric}}\|_{n/2} < C_2(n)$, then M is isometric to a space form. In particular, if M is simply connected, then M is isometric to either the sphere $\mathbb{S}^n(1/\sqrt{c})$ with radius $1/\sqrt{c}$ if $c > 0$, or the hyperbolic space $\mathbb{H}^n(c)$ with constant curvature c if $c < 0$.*

Proof. Since $\triangle f^2 = \triangle f_\tau^2$, from (2-11) we have

$$0 \geq -f_\tau \triangle f_\tau - \sqrt{\frac{n}{n-1}}f^3 + \frac{2}{n}|\nabla f_\tau|^2 + \frac{R}{n-1}f^2. \tag{3-1}$$

We choose a cut-off function $\phi_r \in C^\infty(M)$ such that

$$\begin{cases} \phi_r(x) = 1 & \text{if } x \in B_r(q), \\ \phi_r(x) = 0 & \text{if } x \in M \setminus B_{2r}(q), \\ \phi_r(x) \in [0, 1] \text{ and } |\nabla\phi_r| \leq 1/r & \text{if } x \in B_{2r}(q) \setminus B_r(q), \end{cases} \tag{3-2}$$

where $B_r(q)$ is the geodesic ball in M with radius r centered at $q \in M$. In particular, if M is compact, and if $r \geq d$, where d is the diameter of M, then $\phi_r \equiv 1$ on M.

Multiplying both sides of (3-1) by $\phi_r^2 f_\tau^{n/2-2}$ and integrating by parts we get

$$
\begin{aligned}
\text{(3-3)}\quad 0 \geq{}& 2\int \phi_r f_\tau^{n/2-1}\langle\nabla\phi_r, \nabla f_\tau\rangle dM + \frac{8(n-2)}{n^2}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM \\
&- \sqrt{\frac{n}{n-1}}\int \phi_r^2 f_\tau^{n/2-2} f^3 dM + \frac{R}{n-1}\int \phi_r^2 f_\tau^{n/2-2} f^2 dM \\
&+ \frac{32}{n^3}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM \\
={}& \frac{8(n^2-2n+4)}{n^3}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM + (\sigma+2)\int \phi_r f_\tau^{n/2-1}\langle\nabla\phi_r, \nabla f_\tau\rangle dM \\
&- \sigma\int \phi_r f_\tau^{n/2-1}\langle\nabla\phi_r, \nabla f_\tau\rangle dM - \sqrt{n/n-1}\int \phi_r^2 f_\tau^{n/2-2} f^3 dM \\
&+ \frac{R}{n-1}\int \phi_r^2 f_\tau^{n/2-2} f^2 dM \\
\geq{}& \frac{8(n^2-2n+4)}{n^3}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM + (\sigma+2)\int \phi_r f_\tau^{n/2-1}\langle\nabla\phi_r, \nabla f_\tau\rangle dM \\
&- \frac{8\sigma\rho}{n^2}\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM - \frac{\sigma}{2\rho}\int f_\tau^{n/2}|\nabla\phi_r|^2 dM \\
&- \sqrt{\frac{n}{n-1}}\int \phi_r^2 f_\tau^{n/2-2} f^3 dM + \frac{R}{n-1}\int \phi_r^2 f_\tau^{n/2-2} f^2 dM \\
={}& \Big(\frac{8(n^2-2n+4)}{n^3} - \frac{8\sigma\rho}{n^2}\Big)\int \phi_r^2|\nabla f_\tau^{n/4}|^2 dM \\
&+ \frac{2(\sigma+2)}{n}\int \frac{n}{2}\phi_r f_\tau^{n/2-1}\langle\nabla\phi_r, \nabla f_\tau\rangle dM \\
&- \frac{\sigma}{2\rho}\int f_\tau^{n/2}|\nabla\phi_r|^2 dM - \sqrt{\frac{n}{n-1}}\int \phi_r^2 f_\tau^{n/2-2} f^3 dM \\
&+ \frac{R}{n-1}\int \phi_r^2 f_\tau^{n/2-2} f^2 dM,
\end{aligned}
$$

for arbitrary positive constants σ and ρ, where here and below the measure dM implies integration over M.

By a direct computation, we have

$$
\text{(3-4)}\qquad |\nabla(\phi_r f_\tau^{n/4})|^2 = f_\tau^{n/2}|\nabla\phi_r|^2 + \frac{n}{2}\phi_r f_\tau^{n/2-1}\langle\nabla\phi_r, \nabla f_\tau\rangle + \phi_r^2|\nabla f_\tau^{n/4}|^2.
$$

Choose $\rho > 0$ such that

$$
\frac{8(n^2-2n+4)}{n^3} - \frac{8\sigma\rho}{n^2} = \frac{2(\sigma+2)}{n},
$$

so that $\rho=((2-\sigma)n^2-8n+16)/(4n\sigma)$. Since $\rho>0$, we have $\sigma<2(n^2-4n+8)/n^2$. By (3-3) and (3-4) we obtain

$$\begin{aligned}
0\geq{}&\frac{2(\sigma+2)}{n}\int\Big(\frac{n}{2}\phi_r f_\tau^{n/2-1}\langle\nabla\phi_r,\nabla f_\tau\rangle+\phi_r^2|\nabla f_\tau^{n/4}|^2\Big)dM\\
&-\frac{\sigma}{2\rho}\int f_\tau^{n/2}|\nabla\phi_r|^2dM-\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM\\
&\qquad\qquad+\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM\\
={}&\frac{2(\sigma+2)}{n}\int(|\nabla(\phi_r f_\tau^{n/4})|^2-f_\tau^{n/2}|\nabla\phi_r|^2)dM-\frac{\sigma}{2\rho}\int f_\tau^{n/2}|\nabla\phi_r|^2dM\\
&-\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM+\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM\\
={}&\frac{2(\sigma+2)}{n}\int|\nabla(\phi_r f_\tau^{n/4})|^2dM-\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM\\
&+\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f_\tau^{n/2}|\nabla\phi_r|^2dM.
\end{aligned}$$

This together with the Sobolev inequality in Lemma 2.2 implies

$$\begin{aligned}
0\geq{}&\frac{2(\sigma+2)}{n}\Big(\frac{n(n-2)w_n^{2/n}}{4}\|\phi_r^2 f_\tau^{n/2}\|_{n/(n-2)}-\frac{(n-2)R}{4(n-1)}\|\phi_r f_\tau^{n/4}\|_2^2\Big)\\
&-\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM+\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM\\
&\qquad\qquad-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f_\tau^{n/2}|\nabla\phi_r|^2dM\\
={}&\frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f_\tau^{n/2}\|_{n/(n-2)}-\frac{(\sigma+2)(n-2)R}{2n(n-1)}\|\phi_r^2 f_\tau^{n/2}\|_1\\
&-\sqrt{\frac{n}{n-1}}\int\phi_r^2 f_\tau^{n/2-2}f^3dM+\frac{R}{n-1}\int\phi_r^2 f_\tau^{n/2-2}f^2dM\\
&\qquad\qquad-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f_\tau^{n/2}|\nabla\phi_r|^2dM.
\end{aligned}$$

As $\tau\to 0$, this inequality becomes

$$\begin{aligned}
\text{(3-5)}\quad 0\geq{}&\frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}\\
&+\Big(\frac{R}{n-1}-\frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{n/2}\|_1\\
&-\sqrt{\frac{n}{n-1}}\int\phi_r^2 f^{n/2+1}dM-\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2dM.
\end{aligned}$$

(i) When $R = 0$, (3-5) implies

$$\begin{aligned}
0 &\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2+1}dM \\
&\qquad\qquad - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM \\
&\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \sqrt{\frac{n}{n-1}}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}\|f\|_{n/2} \\
&\qquad\qquad - \frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}dM \\
&\geq \Big(\frac{(\sigma+2)(n-2)w_n^{2/n}}{2} - \sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\|\phi_r^2 f^{n/2}\|_{n/(n-2)} \\
&\qquad\qquad - \frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}dM.
\end{aligned}$$

Put $\sigma = 2(n^2-4n+8)/n^2-\varepsilon$, where ε is a positive constant. It follows from the assumption $\int f^{n/2}dM < \infty$ that

$$\lim_{r\to+\infty}\frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}dM = 0.$$

Combining the last two results, we get

$$0 \geq \Big(\frac{(4(n^2-2n+4)-n^2\varepsilon)(n-2)w_n^{2/n}}{2n^2} - \sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}$$

for any $\varepsilon > 0$. As $\varepsilon \to 0$, we have

$$0 \geq \Big(\frac{4(n^2-2n+4)(n-2)w_n^{2/n}}{2n^2} - \sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)},$$

which implies

$$0 \geq (C_1(n) - \|f\|_{n/2})\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}.$$

Hence $\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} = 0$, that is, $f \equiv 0$. This means that M is an Einstein manifold and is therefore a flat Riemannian manifold. In particular, if M is simply connected, then M is isometric to the Euclidean space $\mathbb{R}^n$.

(ii) When $R \neq 0$, set

$$\frac{1}{n-1} = \frac{(\sigma+2)(n-2)}{2n(n-1)},$$

so that $\sigma = 4/(n-2)$. Since $n \geq 6$, we have $\sigma < 2(n^2-4n+8)/n^2$. Then (3-5) becomes

$$\begin{aligned}
0 &\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2+1} dM \\
&\qquad\qquad - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM \\
&\geq \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} - \sqrt{\frac{n}{n-1}}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}\|f\|_{n/2} \\
&\qquad\qquad - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM \\
&= \Big(\frac{(\sigma+2)(n-2)w_n^{2/n}}{2} - \sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\|\phi_r^2 f^{n/2}\|_{n/(n-2)} \\
&\qquad\qquad - \Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM.
\end{aligned}$$

Since $|\nabla\phi_r| \leq 1/r$ for any $r > 0$, this can be rewritten as

$$\begin{aligned}
0 &\geq \Big(n w_n^{2/n} - \sqrt{\frac{n}{n-1}}\|f\|_{n/2}\Big)\|\phi_r^2 f^{n/2}\|_{n/(n-2)} \\
&\qquad\qquad - \frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{n/2} dM.
\end{aligned} \tag{3-6}$$

Since ρ and σ are constants depending only on n, so is $2(\sigma+2)/n+\sigma/(2\rho)$. From the assumption that f has finite $L^{n/2}$ norm, we get

$$\lim_{r\to+\infty}\frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n}+\frac{\sigma}{2\rho}\Big)\int_M f^{n/2} dM = 0. \tag{3-7}$$

We see from (3-6) and (3-7)

$$0 \geq (C_2(n) - \|f\|_{n/2}) \lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)},$$

which implies $\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} = 0$, that is, $f = 0$. Hence M is an Einstein manifold and a space form. In particular, if M is simply connected, it is isometric to the sphere $\mathbb{S}^n(1/\sqrt{c})$ if $c > 0$ or the hyperbolic space $\mathbb{H}^n(c)$ if $c < 0$. □

Remark 3.2. When $R = 0$, the pinching constant $C_1(n)$ is better than Pigola, Rigoli and Setti's constant.

Corollary 3.3. *For $n \geq 6$, suppose (M, g) is a complete locally conformally flat Riemannian n-manifold with constant scalar curvature, and let $C_2(n)$ be as in Theorem 3.1. If $\|\mathrm{Ric}\|_{n/2} < C_2(n)$, then M is isometric to a complete flat Riemannian manifold. In particular, if M is simply connected, then M is isometric to Euclidean $\mathbb{R}^n$.*

Lemma 3.4. *For $n \geq 3$, let (M, g) be a complete locally conformally flat Riemannian n-manifold with constant scalar curvature. If $\int_M (S - R^2/n)^{n/2} < +\infty$, then for any $\varepsilon > 0$, there is a compact set Ω_ε such that $\tilde{S} < \varepsilon$ in $M \setminus \Omega_\varepsilon$.*

Proof. By (2-9) and (2-10), we have in the sense of distribution the inequality

$$-\triangle f \leq \sqrt{\frac{n}{n-1}} f^2 - \frac{R}{n-1} f \leq \frac{\sqrt{n}\varepsilon}{2\sqrt{n-1}} f^3 + \Big(\frac{\sqrt{n}}{2\varepsilon\sqrt{n-1}} - \frac{R}{n-1}\Big) f.$$

Putting $\varepsilon = \sqrt{n-1}/(2\sqrt{n})$, we have $-\triangle f \leq af^3 + bf$, where $a = 1/4$ and $b = (n-R)/(n-1)$. On the other hand, we have the inequality

$$\Big(\int |f|^{2n/(n-2)} dM\Big)^{(n-2)/n} \leq \frac{4}{n(n-2)\omega_n^{2/n}} \Big(\int |\nabla f|^2 dM + \frac{n-2}{4(n-1)} \int Rf^2 dM\Big).$$

By the proof of [Bérard et al. 1998, Theorem 4.1], we conclude that, for any $\varepsilon > 0$, there is a compact set Ω_ε such that $\tilde{S} < \varepsilon$ in $M \setminus \Omega_\varepsilon$. □

Lemma 3.5. *For $n \geq 3$, let (M, g) be a complete locally conformally flat Riemannian n-manifold with positive constant scalar curvature. If $\|\widetilde{\mathrm{Ric}}\|_n < +\infty$, then M must be compact.*

Proof. Take a local orthonormal frame $\{e_i\}$ such that $R_{ij} = \lambda_i \delta_{ij}$. From (2-2) we have

$$R_{ijij} = \frac{\tilde{\lambda}_i + \tilde{\lambda}_j}{n-2} + \frac{R}{n(n-1)},$$

where $\tilde{\lambda}_i = \lambda_i - R/n$ for $i = 1, 2, \ldots, n$ are eigenvalues of $\widetilde{\mathrm{Ric}}$. Note that R is positive. We see from Lemma 3.4 that there is a positive constant δ such that $K_M > \delta$ in $M \setminus \Omega$ for some compact set Ω.

Since M is complete, it suffices to show that M is bounded. Otherwise, there is a point $p_1 \in M$ such that $d(p_1, \Omega) = \inf_{q \in \Omega} d(p_1, q) > \pi/\sqrt{\delta}$. Since Ω is compact, there is a point $p_2 \in M$ such that $d(p_1, p_2) = d(p_1, \Omega)$. Let $\gamma : [0, s_1] \to M$ be a minimizing geodesic parameterized by arclength such that $\gamma(0) = p_1$ and $\gamma(s_1) = p_2$, where $s_1 = d(p_1, p_2)$. Then $\gamma(t) \in M \setminus \Omega$ for $t < s_1$. Pick $p_3 \in \gamma$ so that $\pi/\sqrt{\delta} < d(p_1, p_3) = s_2 < s_1$. Then $\gamma : [0, s_2] \to M$ is also a minimizing geodesic with $\gamma(s_2) = p_3$. Let $E(s)$ for $s \in [0, s_2]$ be a parallel field along $\gamma : [0, s_2] \to M$ such that $E(0) \perp \gamma'(0)$ and $|E(0)| = 1$. According to [Wu et al. 1989], there exists a piecewise smooth function $\psi : [0, \sqrt{\delta} s_2] \to \mathbb{R}$ satisfying

$$\int_0^{\sqrt{\delta} s_2} (\psi')^2 dt < \int_0^{\sqrt{\delta} s_2} \psi^2 dt,$$

where $\sqrt{\delta}s_2 > \pi$. Setting $X(t) = \psi(\sqrt{\delta}t)E(t)$, we have

$$\begin{aligned} I(X, X) &= \int_0^{s_2} (\langle X'(t), X'(t)\rangle - \langle R(\gamma'(t), X(t))X(t), \gamma'(t)\rangle)dt \\ &= \int_0^{s_2} \big(\delta\psi'(\sqrt{\delta}t)^2 - K(\gamma'(t), E(t))\psi^2(\sqrt{\delta}t)\big)dt, \end{aligned}$$

where $K(\gamma'(t), E(t))$ is the sectional curvature of the tangent plane spanned by $\gamma'(t)$ and $E(t)$. Since $K_M > \delta$ in $M \setminus \Omega$, we have

$$\begin{aligned} I(X, X) &\le \int_0^{s_2} \delta((\psi'(\sqrt{\delta}t))^2 - \psi^2(\sqrt{\delta}t))dt \\ &= \sqrt{\delta}\int_0^{\sqrt{\delta}s_2} ((\psi')^2 - \psi^2)dt < 0. \end{aligned}$$

On the other hand, since $\gamma : [0, s_2] \to M$ is a minimizing geodesic, we have $I(X, X) \ge 0$, which is a contradiction. Hence M is bounded and compact. □

Corollary 3.6 (of Lemma 3.5). *For $n \ge 3$, let (M, g) be a complete noncompact locally conformally flat Riemannian n-manifold with nonnegative constant scalar curvature. If $\|\widetilde{\mathrm{Ric}}\|_n < +\infty$, then M must be scalar flat.*

Theorem 3.7. *Let (M, g) be a complete locally conformally flat Riemannian n-manifold with constant scalar curvature R. Put*

$$\begin{aligned} C_3(n) &= 2n^{-5/2}(n-1)^{1/2}(n-2)^{1/2}(n^2-n+4)^{1/2}(3n^2-4n+4)^{1/2}\omega_n^{1/n}, \\ C_4(n) &= 2\sqrt{2}n^{-5/2}(n-1)^{1/2}(n-2)^{1/2}(n^2-2n+4)^{1/2} \\ &\qquad \cdot (n^3-8n^2+16n-16)^{1/2}\omega_n^{1/n}. \end{aligned}$$

(i) *If $n \ge 3$, $R = n(n-1)$, and $\|\widetilde{\mathrm{Ric}}\|_n < C_3(n)$, then M is isometric to a spherical space form. In particular, if M is simply connected, then M is isometric to $\mathbb{S}^n$.*

(ii) *If $n \ge 6$, $R = -n(n-1)$, $\|\widetilde{\mathrm{Ric}}\|_n < C_4(n)$ and $\|\widetilde{\mathrm{Ric}}\|_{n/2} < +\infty$, then M is isometric to a hyperbolic space form. In particular, if M is simply connected, then M is isometric to $\mathbb{H}^n$.*

Proof. (i) When $R = n(n-1)$, we see from Lemma 3.5 that M is compact. Since $\triangle f^2 = \triangle f_\tau^2$, we have from (2-11)

$$\tfrac{1}{2}\triangle f_\tau^2 \ge \frac{n+2}{n}|\nabla f_\tau|^2 - \sqrt{\frac{n}{n-1}}f^3 + \frac{R}{n-1}f^2. \tag{3-8}$$

Multiplying both sides of (3-8) by f_τ^{n-2} and integrating by parts we get

$$
\begin{aligned}
0 &\geq \frac{1}{2}\int \langle \nabla f_\tau^{n-2}, \nabla f_\tau^2\rangle dM + \frac{4(n+2)}{n^3}\int |\nabla f_\tau^{n/2}|^2 dM \\
&\qquad\qquad - \sqrt{\frac{n}{n-1}}\int f_\tau^{n-2} f^3 dM + \frac{R}{n-1}\int f_\tau^{n-2} f^2 dM \\
&= \frac{4(n-2)}{n^2}\int |\nabla f_\tau^{n/2}|^2 dM + \frac{4(n+2)}{n^3}\int |\nabla f_\tau^{n/2}|^2 dM \\
&\qquad - \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\int f_\tau^{n-2} f^4 dM + \frac{R}{n-1}\int f_\tau^{n-2} f^2 dM \\
&\qquad\qquad - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\int f_\tau^{n-2} f^2 dM \\
&\geq \frac{4(n^2-n+2)}{n^3}\int |\nabla f_\tau^{n/2}|^2 dM - \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\|f_\tau^{n-2} f^2\|_{n/(n-2)} \\
&\qquad + \frac{R}{n-1}\int f_\tau^{n-2} f^2 dM - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\int f_\tau^{n-2} f^2 dM,
\end{aligned}
\tag{3-9}
$$

for any $\varepsilon > 0$. By applying (2-12) to $f_\tau^{n/2}$, we get

$$
\int_M |\nabla f_\tau^{n/2}|^2 dM \geq \frac{n(n-2)w_n^{2/n}}{4}\|f_\tau^n\|_{n/(n-2)} - \frac{(n-2)R}{4(n-1)}\|f_\tau^n\|_1. \tag{3-10}
$$

Substituting (3-10) into (3-9) and letting $\tau \to 0$, we have

$$
\begin{aligned}
0 \geq \Bigg(\frac{(n-2)(n^2-n+4)w_n^{2/n}}{n^2} &- \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\Bigg)\|f^n\|_{n/(n-2)} \\
&+ \Bigg(\frac{(3n^2-4n+4)R}{n^3(n-1)} - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\Bigg)\|f^n\|_1.
\end{aligned}
\tag{3-11}
$$

Set $\varepsilon = 2n^{-5/2}(n-2)^{1/2}(3n^2-4n+4)$. Since $R = n(n-1)$, from (3-11) we get

$$
0 \geq (C_3(n)^2 - \|f^2\|_{n/2})\|f^n\|_{n/(n-2)},
$$

which implies $\|f^n\|_{n/(n-2)} = 0$, that is, $f \equiv 0$. Hence M is an Einstein manifold, which implies that M is isometric to a spherical space form. In particular, if M is simply connected, then M is isometric to $\mathbb{S}^n$.

(ii) When $R = -n(n-1)$, we choose a cut-off function $\phi_r \in C^\infty(M)$ satisfying the conditions of (3-2). Following the proof of Theorem 3.1, we have

$$
\begin{aligned}
0 \geq{}& \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} + \Big(\frac{R}{n-1} - \frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{n/2}\|_1 \\
&\qquad - \sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2+1} dM - \Big(\frac{2(\sigma+2)}{n} + \frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r|^2 dM \\
\geq{}& \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} + \Big(\frac{R}{n-1} - \frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{n/2}\|_1 \\
&\qquad - \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2} dM - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2+2} dM \\
&\qquad - \Big(\frac{2(\sigma+2)}{n} + \frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r^2|^2 dM \\
\geq{}& \frac{(\sigma+2)(n-2)w_n^{2/n}}{2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} + \Big(\frac{R}{n-1} - \frac{(\sigma+2)(n-2)R}{2n(n-1)}\Big)\|\phi_r^2 f^{n/2}\|_1 \\
&\qquad - \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\int \phi_r^2 f^{n/2} dM - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\|\phi_r^2 f^{n/2}\|_{n/(n-2)} \\
&\qquad - \Big(\frac{2(\sigma+2)}{n} + \frac{\sigma}{2\rho}\Big)\int f^{n/2}|\nabla\phi_r^2|^2 dM \\
\geq{}& \Big(\frac{(\sigma+2)(n-2)w_n^{2/n}}{2} - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\Big)\|\phi_r^2 f^{n/2}\|_{n/(n-2)} \\
&\qquad + \Big(\frac{(4-(n-2)\sigma)R}{2n(n-1)} - \frac{1}{2\varepsilon}\sqrt{\frac{n}{n-1}}\Big)\|\phi_r^2 f^{n/2}\|_1 \\
&\qquad - \frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n} + \frac{\sigma}{2\rho}\Big)\int f^{n/2} dM.
\end{aligned}
$$

Put

$$\sigma = \frac{2(n^2-4n+8)}{n^2} - \eta \quad \text{and} \quad \varepsilon = \frac{1}{2}\sqrt{\frac{n}{n-1}} \times \frac{2n(n-1)}{(4-(n-2)\sigma)R},$$

where η is a positive constant. We see that if $n \geq 6$, then $\varepsilon > 0$ for sufficiently small η. When $n \geq 6$ and η is sufficiently small, the second term of the right side of the last calculation vanishes. Since f has finite $L^{n/2}$ norm, we have

$$\lim_{r\to+\infty} \frac{1}{r^2}\Big(\frac{2(\sigma+2)}{n} + \frac{\sigma}{2\rho}\Big)\int_M f^{n/2} dM = 0.$$

By combining this with the previous calculation, we obtain

$$0 \geq \Big(\frac{(\sigma+2)(n-2)w_n^{2/n}}{2} - \frac{\varepsilon}{2}\sqrt{\frac{n}{n-1}}\|f^2\|_{n/2}\Big)\lim_{r\to+\infty}\|\phi_r^2 f^{n/2}\|_{n/(n-2)}.$$

Noting that $R = -n(n-1)$ and letting $\eta \to 0$, this becomes

$$0 \geq (C_4(n)^2 - \|f^2\|_{n/2}) \lim_{r\to+\infty} \|\phi_r^2 f^{n/2}\|_{n/(n-2)},$$

which implies that $\lim_{r\to+\infty} \|\phi_r^2 f^{n/2}\|_{n/(n-2)} = 0$, that is, $f = 0$. Hence M is an Einstein manifold and is isometric to a hyperbolic space form. In particular, if M is simply connected, then M is isometric to $\mathbb{H}^n$. □

Corollary 3.8 (of Theorem 3.7). *For $n \geq 3$, let (M, g) be a complete simply connected and locally conformally flat Riemannian n-manifold with constant scalar curvature $n(n-1)$. Then there exists an explicit constant $C_3(n)$ depending only on n such that if $\| |\mathrm{Ric}|^2 - |\mathrm{Ric}_{\mathbb{S}^n}|^2 \|_{n/2} < C_3(n)$, where $\mathrm{Ric}_{\mathbb{S}^n}$ is the Ricci curvature tensor of $\mathbb{S}^n$, then M is isometric to $\mathbb{S}^n$.*

4. Questions

Theorems 3.1 and 3.7 can be considered as isolation phenomena for the Ricci curvature norm of conformally flat manifolds with constant scalar curvature. With our results in mind, we review the related $L^{n/2}$ pinching theorem obtained by Shiohama and Xu [1997]. For a compact Riemannian manifold (M, g), they defined a new curvature tensor and its $L^{n/2}$ norm by

$$\widetilde{\mathrm{Rm}} = \sum_{i,j,k,l} \tilde{R}_{ijkl} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l \quad \text{and} \quad \tilde{R}(M) = \int_M |\widetilde{\mathrm{Rm}}|^{n/2} dM,$$

where $\tilde{R}_{ijkl} = R_{ijkl} - R(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})/(n(n-1))$.

Theorem C [Shiohama and Xu 1997]. *For $n \geq 3$, let M be a closed Riemannian n-manifold that can be isometrically immersed in Euclidean $\mathbb{R}^{n+1}$. If $\tilde{R}(M) < C_5(n)$, where $C_5(n)$ is an explicit positive constant depending only on n, then M is homeomorphic to the sphere.*

Motivated by the result above and the striking differentiable pinching theorem due to Brendle and Schoen [2009], we propose the following question.

Question 4.1. For $n \geq 3$, let M be a compact Riemannian n-manifold. Denote by d and V the diameter and volume of M. Does there exist a positive constant ε_1 depending on n, d and V such that if $\tilde{R}(M) < \varepsilon_1$, then M is diffeomorphic to a compact space form?

When M is locally conformally flat, we see from (2-2) that Riemannian curvature tensor can be expressed in terms of the Ricci curvature tensor. By a direct computation we have $|\widetilde{\mathrm{Rm}}|^2 = (4/(n-2))|\widetilde{\mathrm{Ric}}|^2$. Another question then arises out of our L^p pinching theorems for conformally flat manifolds:

Question 4.2. For $n \geq 3$, let (M, g) be a complete locally conformally flat Riemannian n-manifold. Does there exists a positive constant ε_2 depending only on n such that if $\|\widetilde{\mathrm{Ric}}\|_{n/2} < \varepsilon_2$, then M is diffeomorphic to a complete space form? In particular, if M is simply connected, is M diffeomorphic to either $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$?

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