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We prove that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) for matrices A and B with entries in $\{0, 1\}$ are continuously orbit equivalent if and only if there exists an isomorphism between the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ keeping their commutative C^*-subalgebras $C(X_A)$ and $C(X_B)$. The "if" part (and hence the "only if" part) above is equivalent to the condition that there exists a homeomorphism from X_A to X_B intertwining their topological full groups. We will also study structure of the automorphisms of $\mathcal{O}_A$ preserving the commutative C^*-algebra $C(X_A)$.

1. Introduction

The study of orbit equivalence of ergodic finite measure preserving transformations was initiated by H. Dye [1959; 1963], who proved that two such transformations are orbit equivalent. W. Krieger [1976] has proved that two ergodic nonsingular transformations are orbit equivalent if and only if the associated von Neumann crossed products are isomorphic. In topological setting, Giordano, Putnam and Skau [Giordano et al. 1995; 1999] (see also [Herman et al. 1992]) have proved that two minimal homeomorphisms on Cantor sets are strong orbit equivalent if and only if the associated C^*-crossed products are isomorphic. In a more general setting, J. Tomiyama [1996] (see [Boyle and Tomiyama 1998; Tomiyama 1998]) has proved that two topological free homeomorphisms (X, ϕ) and (Y, ψ) on compact Hausdorff spaces are continuously orbit equivalent if and only if there exists an isomorphism between the associated C^*-crossed products preserving their commutative C^*-subalgebras $C(X)$ and $C(Y)$. He also proved that it is equivalent to the condition that there exists a homeomorphism $h : X \to Y$ such that h preserves their topological full groups.

In this paper we study the relationship between the orbit structure of one-sided topological Markov shifts and the algebraic structure of the associated Cuntz–Krieger algebras. Let (X_A, σ_A) be the right one-sided topological Markov shift defined by an $N \times N$ square matrix A with entries in $\{0, 1\}$, where σ_A denotes

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the shift transformation on X_A. The one-sided topological Markov shifts are no longer homeomorphisms in general and the Cuntz–Krieger algebras cannot naturally be written as a crossed product by $\mathbb{Z}$. Hence Giordano, Putnam and Skau's and Tomiyama's method cannot be applied to study one-sided topological Markov shifts and Cuntz–Krieger algebras. However, in this paper, theorems similar to theirs will be proved in our setting by using a representation of $\mathcal{O}_A$ on a Hilbert space having its complete orthonormal basis consisting of all points of the shift space X_A.

Let $\mathfrak{D}_A$ be the C^*-subalgebra consisting of all diagonal elements of the canonical AF-algebra $\mathcal{F}_A$ inside of $\mathcal{O}_A$. It is naturally isomorphic to the commutative C^*-algebra $C(X_A)$ of all complex-valued continuous functions on X_A. Let $[\sigma_A]_c$ be the topological full group of (X_A, σ_A) whose elements consist of homeomorphisms τ on X_A such that $\tau(x)$ is contained in the orbit $\mathrm{orb}_{\sigma_A}(x)$ of x under σ_A for all $x \in X_A$ and such that its orbit cocycles are continuous. We say that (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if there exists a homeomorphism $h : X_A \to X_B$ such that $h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x))$ for $x \in X_A$ and if their orbit cocycles are continuous.

We will prove the next three theorems, where condition (I) is that of [Cuntz and Krieger 1980, page 254].

Theorem 1.1. *Let A and B be irreducible square matrices with entries in* $\{0, 1\}$ *satisfying condition* (I). *Then the following three assertions are equivalent*:

- *There exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$.*
- *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
- *There exists a homeomorphism $h : X_A \to X_B$ such that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$.*

To prove this theorem, we study the normalizer $N(\mathcal{O}_A, \mathfrak{D}_A)$ of $\mathfrak{D}_A$ in $\mathcal{O}_A$, which is defined as the group of all unitaries $u \in \mathfrak{D}_A$ such that $u\mathfrak{D}_A u^* = \mathfrak{D}_A$. We denote by $\mathcal{U}(\mathfrak{D}_A)$ the group of all unitaries in $\mathfrak{D}_A$.

Theorem 1.2. *Let A be a square matrix with entries in* $\{0, 1\}$ *satisfying condition* (I). *Then there exists a splitting short exact sequence*

$$1 \to \mathcal{U}(\mathfrak{D}_A) \xrightarrow{\mathrm{id}} N(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\tau} [\sigma_A]_c \to 1.$$

Let $\mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ be the group of automorphisms α of $\mathcal{O}_A$ such that $\alpha(\mathfrak{D}_A) = \mathfrak{D}_A$. Denote by $\mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A)$ the subgroup of $\mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ of inner automorphisms on $\mathcal{O}_A$. We set $\mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A)$ to be the quotient $\mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)/\mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A)$.

Theorem 1.3. *Let A be an irreducible square matrix with entries in* $\{0, 1\}$ *satisfying condition* (I). *Then there exist short exact sequences*

$$1 \to Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_c) \to 1,$$

$$1 \to B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} [\sigma_A]_c \to 1,$$

$$1 \to H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_c)/[\sigma_A]_c \to 1.$$

They all split. Hence $\mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A)$ *is a semidirect product*

$$\mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A) = N([\sigma_A]_c)/[\sigma_A]_c \cdot H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)).$$

where $N([\sigma_A]_c)$ denotes the normalizer of $[\sigma_A]_c$ in the group $\mathrm{Homeo}(X_A)$ *of all homeomorphisms on X_A, and $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$, $B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ and $H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ are the group of unitary one-cocycles for σ_A, the subgroup of $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ of coboundaries and the cohomology group $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))/B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ respectively.*

Similar theorems hold for the pair of the canonical AF-algebra $\mathcal{F}_A$ inside of $\mathcal{O}_A$ and its diagonal algebra $\mathfrak{D}_A$; these are studied in Section 7.

In [Matsumoto 2009], the results of this paper have been generalized.

Throughout the paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of nonnegative integers and the set of positive integers respectively.

2. Preliminaries

Let $A = [A(i, j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0, 1\}$, where $1 < N \in \mathbb{N}$. Throughout the paper, we always assume that A satisfies condition (I) in the sense of Cuntz and Krieger [1980]. We denote by X_A the shift space

$$X_A = \{(x_n)_{n\in\mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

over $\{1, \dots, N\}$ of the right one-sided topological Markov shift for A. It is a compact Hausdorff space in natural product topology. The shift transformation σ_A on X_A is defined by $\sigma_A((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$ and is a continuous surjective map on X_A. The topological dynamical system (X_A, σ_A) is called the (right one-sided) topological Markov shift for A. The condition (I) for A is equivalent to the condition that X_A is homeomorphic to a Cantor discontinuum.

A word $\mu = \mu_1 \cdots \mu_k$ for $\mu_i \in \{1, \dots, N\}$ is said to be admissible for X_A if μ appears somewhere in some element x in X_A. The length of μ is k and denoted by $|\mu|$. We denote by $B_k(X_A)$ the set of all admissible words of length $k \in \mathbb{N}$. We denote by $B_0(X_A)$ the empty word $\varnothing$. We set $B_*(X_A) = \bigcup_{k=0}^{\infty} B_k(X_A)$, the set of admissible words of X_A. For $x = (x_n)_{n\in\mathbb{N}} \in X_A$ and positive integers k, l with $k \le l$, we put the word $x_{[k,l]} = (x_k, x_{k+1}, \dots, x_l) \in B_{l-k+1}(X_A)$ and the right infinite sequence $x_{[k,\infty)} = (x_k, x_{k+1}, \dots) \in X_A$.

The Cuntz–Krieger algebra $\mathcal{O}_A$ for the matrix A has been defined by the universal C^*-algebra generated by N partial isometries $S_1, \dots, S_N$ subject to the relations

$$\sum_{j=1}^{N} S_j S_j^* = 1 \quad \text{and} \quad S_i^* S_i = \sum_{j=1}^{N} A(i, j) S_j S_j^* \quad \text{for } i = 1, \dots, N$$

[Cuntz and Krieger 1980]. If A satisfies condition (I), the algebra $\mathcal{O}_A$ is the unique C^*-algebra subject to these relations. For a word $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$, we let $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$. By the universality of the relations above, we get an action $\rho : \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_A)$, called the gauge action, from the correspondence $S_i \to e^{\sqrt{-1}t} S_i$ for $i = 1, \dots, N$ and $e^{\sqrt{-1}t} \in \mathbb{T} = \{e^{\sqrt{-1}t} \mid t \in [0, 2\pi]\}$. It is well known that the fixed point algebra of $\mathcal{O}_A$ under ρ is the AF-algebra $\mathcal{F}_A$ generated by elements $S_\mu S_\nu^*$ with $\mu, \nu \in B_*(X_A)$ and $|\mu| = |\nu|$ [Cuntz and Krieger 1980]. Let $\mathcal{F}_A^n$ be the C^*-subalgebra of $\mathcal{F}_A$ generated by elements $S_\mu S_\nu^*$, with $\mu, \nu \in B_n(X_A)$. Hence $\mathcal{F}_A^{\mathrm{alg}} = \bigcup_{n=1}^{\infty} \mathcal{F}_A^n$ is a dense $*$-subalgebra of $\mathcal{F}_A$. We denote by $E : \mathcal{O}_A \to \mathcal{F}_A$ the conditional expectation defined by $E(a) = \int_{\mathbb{T}} \rho_t(a) dt$ for $a \in \mathcal{O}_A$. Let $\mathfrak{D}_A$ be the C^*-subalgebra of $\mathcal{F}_A$ consisting of all diagonal elements of $\mathcal{F}_A$. It is generated by elements $S_\mu S_\mu^*$ for $\mu \in B_*(X_A)$ and is isomorphic to the commutative C^*-algebra $C(X_A)$ of all complex valued continuous functions on X_A through the correspondence $S_\mu S_\mu^* \in \mathfrak{D}_A \leftrightarrow \chi_\mu \in C(X_A)$, where χ_μ denotes the characteristic function on X_A for the cylinder set $U_\mu = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, x_k = \mu_k\}$ for $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$. We identify $C(X_A)$ with the subalgebra $\mathfrak{D}_A$ of $\mathcal{O}_A$. Then the following lemma is well known and basic in our further discussions.

Lemma 2.1 [Cuntz and Krieger 1980, Remark 2.18], and see [Matsumoto 2000, Proposition 3.3]. *The algebra $\mathfrak{D}_A$ is maximal abelian in $\mathcal{O}_A$.*

In [1996; 1998], Tomiyama has used the structure of pure state extensions of point evaluations of the underlying space to study the orbit structure of topological dynamical systems of homeomorphisms on compact Hausdorff spaces; see also [Tomiyama 1992a; 1992b]. However for the Cuntz–Krieger algebras, the structure of the pure state extensions of point evaluations of the underlying shift space is not clear. Instead of point evaluations, we will use a representation of the Cuntz–Krieger algebra $\mathcal{O}_A$ on a Hilbert space having the shift space X_A as a complete orthonormal basis, as follows. Let $\mathfrak{H}_A$ be the Hilbert space with complete orthonormal system e_x for $x \in X_A$. This Hilbert space is not separable. Consider the partial isometries T_i for $i = 1, \dots, N$ defined by

$$T_i e_x = \begin{cases} e_{ix} & \text{if } ix \in X_A, \\ 0 & \text{otherwise,} \end{cases}$$

where ix denotes $ix = (i, x_1, x_2, \ldots)$ for $x = (x_n)_{n\in\mathbb{N}} \in X_A$. It is easy to see that these isometries satisfy the relations

$$\sum_{j=1}^{N} T_j T_j^* = 1 \quad \text{and} \quad T_i^* T_i = \sum_{j=1}^{N} A(i, j) T_j T_j^* \quad \text{for } i = 1, \ldots, N.$$

Since A satisfies condition (I), the operator T_i is nonzero for each $i = 1, \ldots, N$, so the correspondence $S_i \to T_i$ yields a faithful representation of $\mathcal{O}_A$ on $\mathfrak{H}_A$. We regard the algebra $\mathcal{O}_A$ as the C^*-algebra generated by T_i for $i = 1, \ldots, N$ on the Hilbert space $\mathfrak{H}_A$ by this representation, and write T_i as S_i; see [Matsumoto 2000, Lemma 4.1].

3. Topological full groups of Markov shifts

For $x = (x_n)_{n\in\mathbb{N}} \in X_A$, the orbit $\mathrm{orb}_{\sigma_A}(x)$ of x under σ_A is defined by

$$\mathrm{orb}_{\sigma_A}(x) = \bigcup_{k=0}^{\infty} \bigcup_{l=0}^{\infty} \sigma_A^{-k}(\sigma_A^{l}(x)) \subset X_A.$$

Hence $y = (y_n)_{n\in\mathbb{N}} \in X_A$ belongs to $\mathrm{orb}_{\sigma_A}(x)$ if and only if there exists an admissible word $\mu_1 \cdots \mu_k \in B_k(X_A)$ such that

$$y = (\mu_1, \ldots, \mu_k, x_{l+1}, x_{l+2}, \ldots) \quad \text{for some } k, l \in \mathbb{Z}_+.$$

We denote by $\mathrm{Homeo}(X_A)$ the group of all homeomorphisms on X_A. We define the full group $[\sigma_A]$ and the topological full group $[\sigma_A]_c$ for (X_A, σ_A) as follows.

Definition. Let $[\sigma_A]$ be the set of all homeomorphism $\tau \in \mathrm{Homeo}(X_A)$ such that $\tau(x) \in \mathrm{orb}_{\sigma_A}(x)$ for all $x \in X_A$. We call $[\sigma_A]$ the full group of (X_A, σ_A). Let $[\sigma_A]_c$ be the set of all τ in $[\sigma_A]$ such that there exist continuous functions $k, l : X_A \to \mathbb{Z}_+$ such that

$$\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{l(x)}(x) \quad \text{for all } x \in X_A. \tag{3-1}$$

We call $[\sigma_A]_c$ the topological full group for (X_A, σ_A). The functions k and l above are called orbit cocycles for τ, and are sometimes written as k_τ and l_τ respectively. We remark that the orbit cocyles are not necessarily uniquely determined by τ.

Examples. (i) Put $F = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Define $\tau \in \mathrm{Homeo}(X_F)$ by setting

$$\tau(x_1, x_2, \ldots) = \begin{cases} (2, 1, 1, x_4, x_5, \ldots) & \text{if } (x_1, x_2, x_3) = (1, 1, 1), \\ (1, 1, 1, x_4, x_5, \ldots) & \text{if } (x_1, x_2, x_3) = (2, 1, 1), \\ (x_1, x_2, x_3, x_4, x_5, \ldots) & \text{otherwise.} \end{cases}$$

Since $\sigma_F(\tau(x)) = \sigma_F(x)$ for all $x \in X_F$, by putting $k(x) = l(x) = 1$ for all $x \in X_F$, we see that τ belongs to $[\sigma_F]_c$.

(ii) More generally, suppose A is an $N \times N$ matrix with entries in $\{0, 1\}$. For $i \in \{1, \dots, N\}$ and $p \in \mathbb{N}$, we put

$$W_p(i) = \{(\mu_1, \dots, \mu_p) \in B_p(X_A) \mid A(\mu_p, i) = 1\}.$$

We denote by $\mathfrak{S}(W_p(i))$ the group of all permutations on the set $W_p(i)$. Put $\mathfrak{S}_p(A) = \mathfrak{S}(W_p(1)) \times \cdots \times \mathfrak{S}(W_p(N))$. Then an N-family $s = (s_1, \dots, s_N) \in \mathfrak{S}_p(A)$ of permutations defines a homeomorphism $\tau_s \in \mathrm{Homeo}(X_A)$ by setting

$$\tau_s(x_1, \dots, x_p, x_{p+1}, \dots) = (s_{x_{p+1}}(x_1, \dots, x_p), x_{p+1}, \dots) \quad \text{for } x \in X_A.$$

For all $x \in X_A$, it is easy to see that $\tau_s(x)$ belongs to $\mathrm{orb}_{\sigma_A}(x)$ and satisfies (3-1) for $k(x) = l(x) = p$. Hence τ_s yields an element of $[\sigma_A]_c$.

Let A be an arbitrary fixed $N \times N$ matrix with entries in $\{0, 1\}$ and satisfying condition (I). The following lemma is direct.

Lemma 3.1. *$[\sigma_A]$ is a subgroup of* $\mathrm{Homeo}(X_A)$ *and $[\sigma_A]_c$ is a subgroup of $[\sigma_A]$.*

Although σ_A itself does not belong to $[\sigma_A]$, the following lemma shows that σ_A locally belongs to $[\sigma_A]_c$, and the group $[\sigma_A]_c$ is not trivial in any case.

Lemma 3.2. *Assume that A is irreducible. For any $\mu \in B_2(X_A)$, there exist $\tau_\mu \in [\sigma_A]_c$ and continuous functions $k_{\tau_\mu}, l_{\tau_\mu} : X_A \to \mathbb{Z}_+$ such that*

$$\begin{cases} \sigma_A^{k_{\tau_\mu}(x)}(\tau_\mu(x)) = \sigma_A^{l_{\tau_\mu}(x)}(x) & \text{for } x \in X_A, \\ \tau_\mu(y) = \sigma_A(y) & \text{for } y \in U_\mu, \\ k_{\tau_\mu}(y) = 0, \quad l_{\tau_\mu}(y) = 1 & \text{for } y \in U_\mu. \end{cases} \tag{3-2}$$

Proof. For $\mu = (\mu_1, \mu_2) \in B_2(X_A)$, we have two cases.

Case 1: $\mu_1 = \mu_2$. Put $a = \mu_1$. Since A is irreducible, there exists $b_1 \in \{1, \dots, N\}$ such that $b_1 \neq a$ and $A(b_1, a) = 1$. Put $\{b_1, \dots, b_{N-1}\} = \{1, \dots, N\} \setminus \{a\}$. Let $\{b_{i_1}, \dots, b_{i_M}\}$ be the set of elements of $\{b_1, \dots, b_{N-1}\}$ satisfying $A(a, b_{i_1}) = \cdots = A(a, b_{i_M}) = 1$. The set $\{b_{i_1}, \dots, b_{i_M}\}$ is nonempty because A satisfies condition (I). Define a homeomorphism $\tau_\mu : X_A \to X_A$ by setting

$$\tau_\mu(x) = \begin{cases} \sigma_A(x) \in U_a & \text{if } x \in U_{aa}, \\ b_1 a b_{i_1} x_{[3,\infty)} \in U_{b_1 a b_{i_1}} & \text{if } x = a b_{i_1} x_{[3,\infty)} \in U_{a b_{i_1}}, \\ \quad\vdots & \quad\vdots \\ b_1 a b_{i_M} x_{[3,\infty)} \in U_{b_1 a b_{i_M}} & \text{if } x = a b_{i_M} x_{[3,\infty)} \in U_{a b_{i_M}}, \\ b_1 a a x_{[3,\infty)} \in U_{b_1 a a} & \text{if } x = b_1 a x_{[3,\infty)} \in U_{b_1 a}, \\ x & \text{otherwise.} \end{cases}$$

We set

$$k_{\tau_\mu}(x) = \begin{cases} 0 & \text{if } x \in U_{aa}, \\ 1 & \text{if } x \in U_{ab_{i_1}}, \\ \vdots & \vdots \\ 1 & \text{if } x \in U_{ab_{i_M}}, \\ 2 & \text{if } x \in U_{b_1 a}, \\ 0 & \text{otherwise}, \end{cases} \qquad l_{\tau_\mu}(x) = \begin{cases} 1 & \text{if } x \in U_{aa}, \\ 0 & \text{if } x \in U_{ab_{i_1}}, \\ \vdots & \vdots \\ 0 & \text{if } x \in U_{ab_{i_M}}, \\ 1 & \text{if } x \in U_{b_1 a}, \\ 0 & \text{otherwise}, \end{cases}$$

so that

$$\sigma_A^{k_{\tau_\mu}(x)}(\tau_\mu(x)) = \sigma_A^{l_{\tau_\mu}(x)}(x) \quad \text{for } x \in X_A.$$

Hence $\tau_\mu \in [\sigma_A]_c$ and $\tau_\mu(y) = \sigma_A(y)$, $k_{\tau_\mu}(y) = 0$, and $l_{\tau_\mu}(y) = 1$ for $y \in U_\mu = U_{aa}$.

Case 2: $\mu_1 \neq \mu_2$. Put $a = \mu_1$ and $b = \mu_2$. Define a homeomorphism $\tau_\mu : X_A \to X_A$ by setting

$$\tau_\mu(x) = \begin{cases} \sigma_A(x) \in U_b & \text{if } x \in U_{ab}, \\ ax \in U_{ab} & \text{if } x \in U_b, \\ x & \text{otherwise.} \end{cases}$$

We set

$$k_{\tau_\mu}(x) = \begin{cases} 0 & \text{if } x \in U_{ab}, \\ 1 & \text{if } x \in U_b, \\ 0 & \text{otherwise}, \end{cases} \qquad l_{\tau_\mu}(x) = \begin{cases} 1 & \text{if } x \in U_{ab}, \\ 0 & \text{if } x \in U_b, \\ 0 & \text{otherwise}, \end{cases}$$

so that

$$\sigma_A^{k_{\tau_\mu}(x)}(\tau_\mu(x)) = \sigma_A^{l_{\tau_\mu}(x)}(x) \quad \text{for } x \in X_A.$$

Hence $\tau_\mu \in [\sigma_A]_c$ and

$$\tau_\mu(y) = \sigma_A(y), \quad k_{\tau_\mu}(y) = 0, \quad l_{\tau_\mu}(y) = 1 \quad \text{for } y \in U_\mu = U_{ab}. \qquad \square$$

By a similar argument, this lemma holds for any word μ with any length $|\mu| \geq 2$.

Lemma 3.3. *For $x = (x_n)_{n \in \mathbb{N}} \in X_A$ and $j \in \{1, \ldots, N\}$ with $jx = (j, x_1, x_2, \ldots)$ in X_A, there exists $\tau \in [\sigma_A]_c$ such that $\tau(x) = jx$.*

Proof. If $x = j^\infty = (j, j, \ldots)$, we may choose id as τ. If $x \neq j^\infty$, there exists $k \in \mathbb{N}$ and $i \in \{1, \ldots, N\}$ with $i \neq j$ such that $x_n = j$ for $1 \leq n \leq k-1$ and $x_k = i$. Put

$$\mu = (\underbrace{j, \ldots, j}_{k-1}, i) \in B_k(X_A) \quad \text{and} \quad \nu = (\underbrace{j, \ldots, j}_{k}, i) = j\mu \in B_{k+1}(X_A),$$

so that $x \in U_\mu$. Define $\tau : X_A \to X_A$ by setting

$$\tau(y_1, y_2, y_3, \dots) = \begin{cases} (j, y_1, y_2, \dots) & \text{if } y \in U_\mu, \\ (y_2, y_3, y_4, \dots) & \text{if } y \in U_\nu, \\ (y_1, y_2, y_3, \dots) & \text{otherwise.} \end{cases}$$

Since $U_\mu \cap U_\nu = \varnothing$, we see that $\tau : X_A \to X_A$ yields an element of $[\sigma_A]_c$. □

Put $[\sigma_A]_c(x) = \{\tau(x) \in X_A \mid \tau \in [\sigma_A]_c\}$ for $x \in X_A$.

Lemma 3.4. $[\sigma_A]_c(x) = \mathrm{orb}_{\sigma_A}(x)$ *for* $x \in X_A$.

Proof. For any $\tau \in [\sigma_A]$, there exist continuous functions $k, l : X_A \to \mathbb{Z}_+$ such that $\tau(x) = (\mu_1(x), \dots, \mu_{k(x)}(x), x_{l(x)+1}, x_{l(x)+2}, \dots)$ for some $(\mu_1(x), \dots, \mu_{k(x)}(x))$ in $B_{k(x)}(X_A)$. Thus $\tau(x) \in \mathrm{orb}_{\sigma_A}(x)$ is clear, and hence $[\sigma_A]_c(x) \subset \mathrm{orb}_{\sigma_A}(x)$.

Now for the other inclusion. By the previous lemmas, for $x = (x_n)_{n\in\mathbb{N}} \in X_A$ and $j = \{1, \dots, N\}$ with $jx \in X_A$, there exist $\tau_1, \tau_2 \in [\sigma_A]_c$ such that

$$\tau_1(x) = (j, x_1, x_2, \dots) \quad \text{and} \quad \tau_2(x) = (x_2, x_3, \dots),$$

so that $[\sigma_A]_c(x) \ni (j, x_1, x_2, \dots), (x_2, x_3, \dots)$. Since $[\sigma_A]_c$ is a group, we see that

$$[\sigma_A]_c(x) \ni (\mu_1, \dots, \mu_k, x_{l+1}, x_{l+2}, \dots) \quad \text{for all } k, l \in \mathbb{Z}_+, \text{ and}$$
$$(\mu_1, \dots, \mu_k) \in B_k(X_A) \quad \text{with } (\mu_1, \dots, \mu_k, x_{l+1}, x_{l+2}, \dots) \in X_A.$$

Hence $[\sigma_A]_c(x) \supset \mathrm{orb}_{\sigma_A}(x)$. □

4. Full groups and normalizers

In this section, we will study the topological full group $[\sigma_A]_c$ and the normalizer $N(\mathcal{O}_A, \mathfrak{D}_A)$. We denote by $\mathcal{U}(\mathcal{O}_A)$ and $\mathcal{U}(\mathfrak{D}_A)$ the groups of unitaries of $\mathcal{O}_A$ and $\mathfrak{D}_A$ respectively. The normalizer $N(\mathcal{O}_A, \mathfrak{D}_A)$ of $\mathfrak{D}_A$ in $\mathcal{O}_A$ is defined by

$$N(\mathcal{O}_A, \mathfrak{D}_A) = \{v \in \mathcal{U}(\mathcal{O}_A) \mid v\mathfrak{D}_A v^* = \mathfrak{D}_A\}.$$

We will identify the algebra $C(X_A)$ with the subalgebra $\mathfrak{D}_A$ of $\mathcal{O}_A$. For $v \in \mathcal{U}(\mathcal{O}_A)$, we put $\mathrm{Ad}(v)(a) = vav^*$ for $a \in \mathcal{O}_A$.

Proposition 4.1. *For* $\tau \in [\sigma_A]_c$, *there exists a unitary* $u_\tau \in N(\mathcal{O}_A, \mathfrak{D}_A)$ *such that*

$$\mathrm{Ad}(u_\tau)(f) = f \circ \tau^{-1} \quad \textit{for } f \in \mathfrak{D}_A,$$

and $\tau \in [\sigma_A]_c \to u_\tau \in N(\mathcal{O}_A, \mathfrak{D}_A)$ *is a group homomorphism.*

Proof. Let the C^*-algebra $\mathcal{O}_A$ be represented on the Hilbert space $\mathfrak{H}_A$ with complete orthonormal basis $\{e_x \mid x \in X_A\}$. Then the generating partial isometries S_i for $i = 1, \dots, N$ act on $\mathfrak{H}_A$ by $S_i e_x = e_{ix}$ if $ix \in X_A$, and otherwise $S_i e_x = 0$. Since $\tau : X_A \to X_A$ is a homeomorphism, the operator u_τ on $\mathfrak{H}_A$ defined by $u_\tau e_x = e_{\tau(x)}$

for $x \in X_A$ yields a unitary on $\mathfrak{H}_A$. We will prove that u_τ belongs to $\mathcal{O}_A$. Let $l, k : X_A \to \mathbb{Z}_+$ be continuous functions satisfying (3-1). Since both $k(X_A)$ and $l(X_A)$ are finite sets of $\mathbb{Z}_+$, there exist

$$\tilde{k} = \max\{k(x) \mid x \in X_A\} \quad \text{and} \quad \tilde{l} = \max\{l(x) \mid x \in X_A\} \quad \text{in } \mathbb{Z}_+.$$

Take $\mu(x) = (\mu_1(x), \ldots, \mu_{k(x)}(x)) \in B_{k(x)}(X_A)$ such that

$$\tau(x) = (\mu_1(x), \ldots, \mu_{k(x)}(x), x_{l(x)+1}, x_{l(x)+2}, x_{l(x)+3}, \ldots).$$

The set of words $\{(\mu_1(x), \ldots, \mu_{k(x)}(x)) \in B_{k(x)}(X_A) \mid x \in X_A\}$ is a finite subset of

$$W_{\tilde{k}}(X_A) = \bigcup_{j=0,\ldots,\tilde{k}} B_j(X_A).$$

The map $x \in X_A \to (\mu_1(x), \ldots, \mu_{k(x)}(x)) \in W_{\tilde{k}}(X_A)$ is continuous, where $W_{\tilde{k}}(X_A)$ is endowed with discrete topology. For any word $\nu = \nu_1 \cdots \nu_j \in W_{\tilde{k}}(X_A)$ with $j \le \tilde{k}$ and $0 \le n \le \tilde{l}$, the sets

$$E_\nu = \{x \in X_A \mid \mu_1(x) = \nu_1, \ldots, \mu_{k(x)}(x) = \nu_j\} \quad \text{and} \quad F_n = \{x \in X_A \mid l(x) = n\}$$

are clopen in X_A. Define the projections $Q_\nu = \chi_{E_\nu}$ and $P_n = \chi_{F_n}$ in $\mathfrak{D}_A$. Since X_A is composed of disjoint unions

$$X_A = \bigcup_{\nu \in W_{\tilde{k}}(X_A)} E_\nu = \bigcup_{n=0,\ldots,\tilde{l}} F_n,$$

we have

$$\sum_{\nu \in W_{\tilde{k}}(X_A)} Q_\nu = \sum_{n=0,\ldots,\tilde{l}} P_n = 1.$$

For $x \in X_A$ and $\nu \in W_{\tilde{k}}$ with $0 \le n \le \tilde{l}$, we have $x \in E_\nu \cap F_n$ if and only if $e_{\tau(x)} = S_\nu e_{\sigma_A^n(x)}$, so that

$$e_{\tau(x)} = \sum_{n=0,\ldots,\tilde{l}} \sum_{\nu \in W_{\tilde{k}}} \Big(S_\nu \sum_{\xi \in B_n(X_A)} S_\xi^* \Big) P_n Q_\nu e_x \quad \text{for } x \in X_A.$$

Therefore

$$u_\tau = \sum_{n=0,\ldots,\tilde{l}} \sum_{\nu \in W_{\tilde{k}}} \Big(S_\nu \sum_{\xi \in B_n(X_A)} S_\xi^* \Big) P_n Q_\nu,$$

which belongs to the algebra $\mathcal{O}_A$. The equality

$$\mathrm{Ad}(u_\tau)(f) = f \circ \tau^{-1} \quad \text{for } f \in \mathfrak{D}_A.$$

is straightforward from the definition $u_\tau e_x = e_{\tau(x)}$ for $x \in X_A$ of u_τ. $\square$

For $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$, $\mathrm{Ad}(v)$ induces an automorphism on the algebras $\mathcal{O}_A$ and $\mathfrak{D}_A$. Let τ_v denote the homeomorphism on X_A induced by $\mathrm{Ad}(v) : \mathfrak{D}_A \to \mathfrak{D}_A$ satisfying $\mathrm{Ad}(v)(f) = f \circ {\tau_v}^{-1}$ for $f \in \mathfrak{D}_A$. We will prove that τ_v gives rise to an element of $[\sigma_A]_c$. We fix $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$ for a while.

Lemma 4.2. *There exists a family v_m for $m \in \mathbb{Z}$ of partial isometries in $\mathcal{O}_A$ such that all but finitely many v_m for $m \in \mathbb{Z}$ are zero, and with these properties*:

(1) $v = \sum_{m\in\mathbb{Z}} v_m$, *where the nonzero v_m, $m \in \mathbb{Z}$ are finite.*

(2) $v_m \mathfrak{D}_A v_m^* \subset \mathfrak{D}_A$ *and* $v_m^* \mathfrak{D}_A v_m \subset \mathfrak{D}_A$ *for* $m \in \mathbb{Z}$.

(3) $v_m^* v_m$ *and* $v_m v_m^*$ *are projections in* $\mathfrak{D}_A$ *for* $m \in \mathbb{Z}$.

(4) $v_m^* v_{m'} = v_m v_{m'}^* = 0$ *for* $m \neq m'$.

(5) $v_0 \in \mathcal{F}_A$.

Proof. Put $g(t) = v^* \rho_t(v) \in \mathcal{O}_A$ for $t \in \mathbb{T}$. For $f \in \mathfrak{D}_A$, we have

$$\rho_t(v) f \rho_t(v)^* = \rho_t(v f v^*) = v f v^*,$$

so that $v^* \rho_t(v)$ commutes with each element of $\mathfrak{D}_A$. By Lemma 2.1, $g(t)$ belongs to the algebra $\mathfrak{D}_A$. We put

$$v_m = \int_0^{2\pi} \rho_t(v) e^{-\sqrt{-1}mt} dt \quad \text{and} \quad \hat{g}(m) = \int_0^{2\pi} g(t) e^{-\sqrt{-1}mt} dt \quad \text{for } m \in \mathbb{Z}.$$

Then $v_m = v\hat{g}(m)$. Since $g(t) \in \mathfrak{D}_A$, we have

$$g(t)^* = \rho_t(v^* \rho_{-t}(v)) = g(-t) \quad \text{and} \quad g(t)g(s) = v^* \rho_t(v) \rho_t(v^* \rho_s(v)) = g(t+s),$$

so that $\hat{g}(m)$ for $m \in \mathbb{Z}$ are projections in $\mathfrak{D}_A$ such that $\hat{g}(m)\hat{g}(m') = 0$ for $m \neq m'$. Regard $g(t) \in \mathfrak{D}_A$ as a function on X_A. For $x \in X_A$, we see that $|g(t)(x)|^2 = \langle g(t)e_x \mid g(t)e_x \rangle = 1$, so that by Parseval's identity

$$1 = \int_0^{2\pi} |g(t)(x)|^2 dt = \sum_{m\in\mathbb{Z}} \Big| \int_0^{2\pi} g(t)(x) e^{-\sqrt{-1}mt} dt \Big|^2 = \sum_{m\in\mathbb{Z}} \|\hat{g}(m)(x)e_x\|^2.$$

Put $E_m = \mathrm{supp}(\hat{g}(m))$ a clopen set in X_A for $m \in \mathbb{Z}$. By the equality above, we have $X_A = \bigcup_{m\in\mathbb{Z}} E_m$ and $E_m \cap E_{m'} = \varnothing$ for $m \neq m'$. By the compactness of X_A, all but finitely many E_m are empty. Then elements $v_m^* v_m = \hat{g}(m)$ and $v_m v_m^* = v\hat{g}(m)v^*$ are both projections in $\mathfrak{D}_A$. It follows that

$$v_m \mathfrak{D}_A v_m^* = v\hat{g}(m)\mathfrak{D}_A \hat{g}(m) v^* \subset \mathfrak{D}_A \quad \text{and} \quad v_m^* \mathfrak{D}_A v_m = \hat{g}(m) v^* \mathfrak{D}_A v \hat{g}(m) \subset \mathfrak{D}_A,$$

because $\hat{g}(m) \in \mathfrak{D}_A$. Therefore parts (1), (2), (3) and (4) hold. For part (5), we have

$$v_0 = v\hat{g}(0) = v \int_0^{2\pi} v^* \rho_t(v) dt = E(v) \in \mathcal{F}_A. \qquad \square$$

Lemma 4.3. *For a fixed $n \in \mathbb{N}$, there exist partial isometries $v_\mu, v_{-\mu} \in \mathcal{F}_A$ for each $\mu \in B_n(X_A)$ satisfying the following conditions*:

(1) $v_n = \sum_{\mu \in B_n(X_A)} S_\mu v_\mu$ *and* $v_{-n} = \sum_{\mu \in B_n(X_A)} v_{-\mu} S_\mu^*$.

(2) $v_\mu^* v_\mu,\ S_\mu v_\mu v_\mu^* S_\mu^*,\ S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*$ *and* $v_{-\mu} v_{-\mu}^*$ *are projections in* $\mathfrak{D}_A$ *such that*

$$v_n^* v_n = \sum v_\mu^* v_\mu, \qquad v_n v_n^* = \sum S_\mu v_\mu v_\mu^* S_\mu^*,$$
$$v_{-n}^* v_{-n} = \sum S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*, \qquad v_{-n} v_{-n}^* = \sum v_{-\mu} v_{-\mu}^*,$$

where the sums are over all $\mu \in B_n(X_n)$.

(3) $v_\mu v_\nu^* = v_{-\mu}^* v_{-\nu} = 0$ *for $\mu, \nu \in B_n(X_A)$ with $\mu \neq \nu$.*

(4) $v_\mu \mathfrak{D}_A v_\mu^*,\ v_\mu^* \mathfrak{D}_A v_\mu,\ v_{-\mu} \mathfrak{D}_A v_{-\mu}^*$ *and* $v_{-\mu}^* \mathfrak{D}_A v_{-\mu}$ *are contained in* $\mathfrak{D}_A$.

Proof. For $\mu \in B_n(X_A)$, put $v_\mu = E(S_\mu^* v)$ and $v_{-\mu} = E(v S_\mu)$. They belong to $\mathcal{F}_A$ and satisfy $S_\mu^* S_\mu v_\mu = v_\mu$ and $v_{-\mu} S_\mu^* S_\mu = v_{-\mu}$. Then we have

$$S_\mu^* v_n = \int_0^{2\pi} S_\mu^* \rho_t(v) e^{-\sqrt{-1}nt} dt = E(S_\mu^* v) = v_\mu,$$
$$v_{-n} S_\mu = \int_0^{2\pi} \rho_t(v) e^{\sqrt{-1}nt} S_\mu dt = E(v S_\mu) = v_{-\mu}.$$

Hence we have $v_n = \sum_{\mu \in B_n(X_A)} S_\mu v_\mu$ and $v_{-n} = \sum_{\mu \in B_n(X_A)} v_{-\mu} S_\mu^*$. Thus (1) holds. We then have

$$v_\mu^* v_\mu = v_n^* S_\mu S_\mu^* v_n = \hat{g}(n) v^* S_\mu S_\mu^* v \hat{g}(n),$$
$$S_\mu v_\mu v_\mu^* S_\mu^* = S_\mu S_\mu^* v_n v_n^* S_\mu S_\mu^* = S_\mu S_\mu^* v \hat{g}(n) v^* S_\mu S_\mu^*,$$
$$S_\mu v_{-\mu}^* v_{-\mu} S_\mu^* = S_\mu S_\mu^* v_{-n}^* v_{-n} S_\mu S_\mu^* = S_\mu S_\mu^* \hat{g}(-n) S_\mu S_\mu^*,$$
$$v_{-\mu} v_{-\mu}^* = v_{-n} S_\mu S_\mu^* v_{-n}^* = v \hat{g}(-n) S_\mu S_\mu^* \hat{g}(-n) v^*.$$

Since $\hat{g}(n)$ and $\hat{g}(-n)$ are projections in $\mathfrak{D}_A$, and $v \mathfrak{D}_A v^* = \mathfrak{D}_A$, the elements above are projections in $\mathfrak{D}_A$, so that (2) and (3) hold. Since

$$v_\mu = S_\mu^* v_n = S_\mu^* v \hat{g}(n) \quad \text{and} \quad v_{-\mu} = v_{-n} S_\mu = v \hat{g}(-n) S_\mu$$

the assertion (4) is immediate. □

Let $u \in \mathcal{O}_A$ be a partial isometry satisfying

$$u \mathfrak{D}_A u^* \subset \mathfrak{D}_A \quad \text{and} \quad u^* \mathfrak{D}_A u \subset \mathfrak{D}_A.$$

Define the projections $p_u = u^* u$ and $q_u = u u^* \in \mathfrak{D}_A$ and clopen sets $X_u = \mathrm{supp}(p_u)$ and $Y_u = \mathrm{supp}(q_u) \subset X_A$. Then $\mathrm{Ad}(u) : \mathfrak{D}_A p_u \to \mathfrak{D}_A q_u$ yields an isomorphism and induces a homeomorphism $h_u : X_u \to Y_u$ such that

$$\mathrm{Ad}(u)(g) = g \circ {h_u}^{-1} \in \mathfrak{D}_A q_u (= C(Y_u)) \quad \text{for } g \in \mathfrak{D}_A p_u (= C(X_u)).$$

Lemma 4.4. *Keep the notation above. For $x \in X_u$, put $y = h_u(x) \in Y_u$. Then we have*
$$\|S^*_{y_{[1,n]}} u S_{x_{[1,n]}}\| = 1 \quad \text{for all } n \in \mathbb{N}.$$

Proof. Since $S^*_{y_{[1,n]}} u S_{x_{[1,n]}}$ is a partial isometry, we see that $\|S^*_{y_{[1,n]}} u S_{x_{[1,n]}}\| = 1$ for all $n \in \mathbb{N}$ or $\|S^*_{y_{[1,n]}} u S_{x_{[1,n]}}\| = 0$ for all $n > N_0$ for some N_0. It suffices to show that $S^*_{y_{[1,n]}} u S_{x_{[1,n]}} \neq 0$ for all $n \in \mathbb{N}$. One then sees that

$$(S^*_{y_{[1,n]}} u S_{x_{[1,n]}} S^*_{x_{[1,n]}} u^* S_{y_{[1,n]}} e_{\sigma_A{}^n(y)} \mid e_{\sigma_A{}^n(y)}) = (\mathrm{Ad}(u)(\chi) e_y \mid e_y),$$

where χ denotes the characteristic function on X_A for the cylinder set $U_{x_{[1,n]}}$ of the word $x_{[1,n]}$. Since

$$\mathrm{Ad}(u)(\chi) e_y = (\chi \circ h_u^{-1})(y) e_y = \chi(x) e_y = e_y,$$

we obtain

$$(S^*_{y_{[1,n]}} u S_{x_{[1,n]}} S^*_{x_{[1,n]}} u^* S_{y_{[1,n]}} e_{\sigma_A{}^n(y)} \mid e_{\sigma_A{}^n(y)}) = (e_y \mid e_y) = 1,$$

so that $S^*_{y_{[1,n]}} u S_{x_{[1,n]}} \neq 0$. □

The following is key:

Lemma 4.5. *Keep the situation above. Assume that $u \in \mathcal{F}_A$. Then there exists $k \in \mathbb{N}$ such that for all $x = (x_n)_{n\in\mathbb{N}} \in X_u$, we have $y_n = x_n$ for all $n > k$, where $y = (y_n)_{n\in\mathbb{N}} = h_u(x)$.*

Proof. Suppose that for any $k \in \mathbb{N}$ there exist $x \in X_u$ and $N > k$ such that $y_N \neq x_N$. Now $u \in \mathcal{F}_A$, and take $u' \in \mathcal{F}_A^{k_0}$ for some k_0 such that $\|u - u'\| < \frac{1}{2}$. Take $x \in X_u$ and $N_0 > k_0$ such that $y_{N_0} \neq x_{N_0}$. Since u' belongs to $\mathcal{F}_A^{N_0-1}$, it can be written as

$$u' = \sum_{\xi,\eta \in B_{N_0-1}(X_A)} c_{\xi,\eta} S_\xi S_\eta^* \in \mathcal{F}_A^{N_0-1} \quad \text{for some } c_{\xi,\eta} \in \mathbb{C}.$$

Hence we have

$$S^*_{y_{[1,N_0-1]}} u' S_{x_{[1,N_0-1]}} = c_{y_{[1,N_0-1]},x_{[1,N_0-1]}} S^*_{y_{[1,N_0-1]}} S_{y_{[1,N_0-1]}} S^*_{x_{[1,N_0-1]}} S_{x_{[1,N_0-1]}}$$

so that

$$S^*_{y_{[1,N_0]}} u' S_{x_{[1,N_0]}} = c_{y_{[1,N_0-1]},x_{[1,N_0-1]}} S^*_{y_{N_0}} S^*_{y_{[1,N_0-1]}} S_{y_{[1,N_0-1]}} S^*_{x_{[1,N_0-1]}} S_{x_{[1,N_0-1]}} S_{x_{N_0}} = 0$$

because $y_{N_0} \neq x_{N_0}$. Hence we have $S^*_{y_{[1,n]}} u' S_{x_{[1,n]}} = 0$ for $n > N_0$. For $n > N_0$, it then follows that

$$\|S^*_{y_{[1,n]}} u S_{x_{[1,n]}}\| = \|S^*_{y_{[1,n]}} (u - u') S_{x_{[1,n]}}\| < \tfrac{1}{2}.$$

This contradicts the preceding lemma. □

Thus we have this:

Lemma 4.6. *For a partial isometry $u \in \mathcal{F}_A$ satisfying*

$$u\mathfrak{D}_A u^* \subset \mathfrak{D}_A \quad \textit{and} \quad u^*\mathfrak{D}_A u \subset \mathfrak{D}_A,$$

there exists $k_u \in \mathbb{N}$ such that the homeomorphism $h_u : \operatorname{supp}(u^*u) \to \operatorname{supp}(uu^*)$ defined by $\operatorname{Ad}(u)(g) = g \circ h_u^{-1}$ for $g \in \mathfrak{D}_A u^*u$ satisfies the condition

$$\sigma_A^{k_u}(h_u(x)) = \sigma_A^{k_u}(x) \quad \textit{for } x \in \operatorname{supp}(u^*u).$$

Proposition 4.7. *For any $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$, the homeomorphism τ_v on X_A induced by the automorphism of $\mathfrak{D}_A$ defined by the restriction of $\operatorname{Ad}(v)$ to $\mathfrak{D}_A$ gives rise to an element of the topological full group $[\sigma_A]_c$.*

Proof. For $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$, let $v_m, m \in \mathbb{Z}$ be the partial isometries in $\mathcal{O}_A$ as in Lemma 4.2. Take $K \in \mathbb{N}$ such that $v_m = 0$ for all $m \in \mathbb{Z}$ with $|m| > K$, and hence $v = \sum_{m=-K}^{K} v_m$. We have

$$\operatorname{Ad}(v)(f) = \sum_{n=1}^{K} v_n f v_n^* + v_0 f v_0^* + \sum_{n=1}^{K} v_{-n} f v_{-n}^* \quad \text{for } f \in \mathfrak{D}_A.$$

Since $v_m^* v_m$ and $v_m v_m^*$ are projections in $\mathfrak{D}_A$, we may put clopen sets

$$X_A^{(m)} = \operatorname{supp}(v_m^* v_m) \quad \text{and} \quad Y_A^{(m)} = \operatorname{supp}(v_m v_m^*) \quad \text{for } m \in \mathbb{Z} \text{ with } |m| \le K$$

in X_A such that X_A is made of disjoint unions: $X_A = \bigcup_{|m| \le K} X_A^{(m)} = \bigcup_{|m| \le K} Y_A^{(m)}$. Since $v_0 \in \mathcal{F}_A$, by Lemma 4.6, there exists $k_0 \in \mathbb{N}$ such that

$$\sigma_A^{k_0}(\tau_0(x)) = \sigma_A^{k_0}(x) \quad \text{for } x \in X_A^{(0)}, \tag{4-1}$$

where $\tau_0 : X_A^{(0)} \to Y_A^{(0)}$ is the homeomorphism satisfying $\operatorname{Ad}(v_0)(f) = f \circ \tau_0^{-1}$ for $f \in \mathfrak{D}_A v_0^* v_0$. For v_n, v_{-n} and $1 \le n \le K$, by Lemma 4.3, we have, for $f \in \mathfrak{D}_A$

$$v_n f v_n^* = \sum_{\mu \in B_n(X_A)} S_\mu v_\mu f v_\mu^* S_\mu^* \quad \text{and} \quad v_{-n} f v_{-n}^* = \sum_{\mu \in B_n(X_A)} v_{-\mu} S_\mu^* f S_\mu v_{-\mu}^*.$$

Put

$$\begin{aligned} X_A^{(n,\mu)} &= \operatorname{supp}(v_\mu^* v_\mu) & X_A^{(-n,\mu)} &= \operatorname{supp}(S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*), \\ Y_A^{(n,\mu)} &= \operatorname{supp}(S_\mu v_\mu v_\mu^* S_\mu^*), & Y_A^{(-n,\mu)} &= \operatorname{supp}(v_{-\mu} v_{-\mu}^*). \end{aligned}$$

By Lemma 4.3, $X_A^{(m)} = \bigcup_{\mu \in B_{|m|}(X_A)} X_A^{(m,\mu)}$ and $Y_A^{(m)} = \bigcup_{\mu \in B_{|m|}(X_A)} Y_A^{(m,\mu)}$ for $|m| \le K$. There exists a homeomorphism

$$\tau_{(m,\mu)} : X_A^{(m,\mu)} \to Y_A^{(m,\mu)} \quad \text{for } m \in \mathbb{Z} \text{ with } |m| \le K$$

such that

$$\begin{aligned} \mathrm{Ad}(S_\mu v_\mu)(f) &= f \circ \tau_{(n,\mu)}^{-1} \quad &&\text{for } f \in \mathfrak{D}_A v_\mu^* v_\mu, \\ \mathrm{Ad}(v_{-\mu} S_\mu^*)(g) &= g \circ \tau_{(-n,\mu)}^{-1} \quad &&\text{for } g \in \mathfrak{D}_A S_\mu v_{-\mu}^* v_{-\mu} S_\mu^* \end{aligned}$$

for $n \in \mathbb{N}$ with $1 \le n \le K$. As $v_\mu, v_{-\mu} \in \mathcal{F}_A$, there exist $k_{(n,\mu)}, k_{(-n,\mu)} \in \mathbb{N}$ such that

$$\begin{aligned} \sigma_A^{k_{(n,\mu)}}(\tau_{(n,\mu)}(x)) &= \sigma_A^{k_{(n,\mu)}+n}(x) \quad &&\text{for } x \in X_A^{(n,\mu)}, \\ \sigma_A^{k_{(n,\mu)}+n}(\tau_{(-n,\mu)}(x)) &= \sigma_A^{k_{(n,\mu)}}(x) \quad &&\text{for } x \in X_A^{(-n,\mu)}. \end{aligned}$$

Since we have

$$\tau_v(x) = \begin{cases} \tau_{(n,\mu)}(x) & \text{for } x \in X_A^{(n,\mu)}, \\ \tau_0(x) & \text{for } x \in X_A^{(0)}, \\ \tau_{(-n,\mu)}(x) & \text{for } x \in X_A^{(-n,\mu)} \end{cases}$$

and X_A is made of disjoint unions as

$$X_A = X_A^{(0)} \cup \bigcup_{1\le|m|\le K} \bigcup_{\mu\in B_{|m|}(X_A)} X_A^{(m,\mu)} = Y_A^{(0)} \cup \bigcup_{1\le|m|\le K} \bigcup_{\mu\in B_{|m|}(X_A)} Y_A^{(m,\mu)},$$

where $X_A^{(0)}, X_A^{(m,\mu)}$ and $Y_A^{(0)}, Y_A^{(m,\mu)}$ for $1 \le |m| \le K$ and $\mu \in B_{|m|}(X_A)$ are clopen sets, we conclude that $\tau_v \in [\sigma_A]_c$. ☐

There is a natural embedding id of the unitaries $\mathcal{U}(\mathfrak{D}_A)$ into $N(\mathcal{O}_A, \mathfrak{D}_A)$. For $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$, the induced homeomorphism τ_v on X_A gives rise to an element of $[\sigma_A]_c$ by the above proposition.

Theorem 4.8. *The sequence* $1 \to \mathcal{U}(\mathfrak{D}_A) \xrightarrow{\mathrm{id}} N(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\tau} [\sigma_A]_c \to 1$ *is exact and splits.*

Proof. By Proposition 4.7, the map $\tau : v \in N(\mathcal{O}_A, \mathfrak{D}_A) \to \tau_v \in [\sigma_A]_c$ defines a homomorphism. It is surjective by Proposition 4.1. Suppose that $\tau_v = \mathrm{id}$ on X_A for some $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$. This means that $\mathrm{Ad}(v) = \mathrm{id}$ on $\mathfrak{D}_A$. Hence v commutes with all of elements of $\mathfrak{D}_A$. By Lemma 2.1, v belongs to $\mathfrak{D}_A$. Therefore the sequence is exact. As in Proposition 4.1, for $\tau \in [\sigma_A]_c$, the unitary u_τ defined by setting $u_\tau e_x = e_{\tau(x)}$ for $x \in X_A$ gives rise to a section of the exact sequence. Hence the sequence splits. ☐

5. Orbit equivalence

Definition. Two topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be topologically orbit equivalent if there exists a homeomorphism $h : X_A \to X_B$ such that $h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x))$ for $x \in X_A$. In this case, $h(\sigma_A(x)) \in \mathrm{orb}_{\sigma_B}(h(x))$ for $x \in X_A$, so that $h(\sigma_A(x)) \in \bigcup_{k=0}^\infty \bigcup_{l=0}^\infty \sigma_B^{-k} \sigma_B^l(h(x))$. Hence there exist functions

$$k_1, l_1 : X_A \to \mathbb{Z}_+ \quad \text{such that } \sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)),$$

and similarly there exists functions

$$k_2, l_2 : X_B \to \mathbb{Z}_+ \quad \text{such that } \sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)).$$

We say that (X_A, σ_A) and (X_B, σ_B) are *continuously orbit equivalent* if there exists a homeomorphism $h : X_A \to X_B$ and continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ such that, for $x \in X_A$ and $y \in X_B$,

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)), \quad \sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)). \tag{5-1}$$

Example. Let $A_{[2]} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $F = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The subshift X_F is the set of all sequences $(x_n)_{n\in\mathbb{N}}$ of $1, 2$ such that the word (2, 2) is forbidden. Define a homeomorphism $h : X_F \to X_{A_{[2]}}$ by substituting the word 2 for the word (2, 1) from the leftmost in order; for example

$$\begin{aligned} &h(1, 2, 1, 1, 2, 1, 2, 1, 1, 1, 1, 2, 1, 2, 1, 2, 1, 1, 1, 1, 1, 2, 1, 1, 1, \dots) \\ &= (1, 2, 1, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 1, 2, 1, 1, \dots) \in X_{A_{[2]}}. \end{aligned}$$

For $i = 1, 2$, put

$$\begin{aligned} U_{F,i} &= \{x = (x_n)_{n\in\mathbb{N}} \in X_{A_{[2]}} \mid x_1 = i\}, \\ U_{A_{[2]},i} &= \{y = (y_n)_{n\in\mathbb{N}} \in X_{A_{[2]}} \mid y_1 = i\}. \end{aligned}$$

By setting

$$\begin{cases} k_1(x) = 0,\ l_1(x) = 1 & \text{for } x \in U_{F,1}, \\ k_1(x) = 1,\ l_1(x) = 1 & \text{for } x \in U_{F,2}, \end{cases} \qquad \begin{cases} k_2(y) = 0,\ l_2(y) = 1 & \text{for } y \in U_{A_{[2]},1}, \\ k_2(y) = 0,\ l_2(y) = 2 & \text{for } y \in U_{A_{[2]},2}, \end{cases}$$

we see that (X_F, σ_F) and $(X_{A_{[2]}}, \sigma_{A_{[2]}})$ are continuously orbit equivalent.

The following lemma is straightforward.

Lemma 5.1. *If $h : X_A \to X_B$ is a homeomorphism satisfying $\sigma_B^{k(x)}(h(\sigma_A(x))) = \sigma_B^{l(x)}(h(x))$ for $x \in X_A$ for some functions $k, l : X_A \to \mathbb{Z}_+$, then by putting*

$$k^n(x) = \sum_{i=0}^{n-1} k(\sigma_A^i(x)) \quad \text{and} \quad l^n(x) = \sum_{i=0}^{n-1} l(\sigma_A^i(x)),$$

we have

$$\sigma_B^{k^n(x)}(h(\sigma_A^n(x))) = \sigma_B^{l^n(x)}(h(x)) \quad \text{for } x \in X_A \text{ and } n \in \mathbb{N}.$$

Lemma 5.2. *If $h : X_A \to X_B$ is a homeomorphism, and $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ are continuous functions satisfying*

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{and} \quad \sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y))$$

for $x \in X_A$ and $y \in X_B$, then

$$h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x)) \quad \textit{for } x \in X_A.$$

Hence if (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, then they are topologically orbit equivalent.

Proof. By Lemma 5.1, we have

$$h(\sigma_A^n(x)) \subset \sigma_B^{-k_1^n(x)} \sigma_B^{l_1^n(x)}(h(x)) \quad \text{for } x \in X_A \text{ and } n \in \mathbb{N}$$

so that $h(\sigma_A^n(x))) \subset \mathrm{orb}_{\sigma_B}(h(x))$. For $z = (\mu_1, \dots, \mu_m, x_1, x_2, \dots) \in \sigma_A^{-m}(x)$, we have again by Lemma 5.1

$$\sigma_B^{l_1^m(z)}(h(\mu_1, \dots, \mu_m, x_1, x_2, \dots)) = \sigma_B^{k_1^m(z)}(h(\sigma_A^m(z)) = \sigma_B^{k_1^m(z)}(h(x)).$$

Hence $h(\mu_1, \dots, \mu_m, x_1, x_2, \dots) \subset \sigma_B^{-l_1^m(z)} \sigma_B^{k_1^m(z)}(h(x)) \subset \mathrm{orb}_{\sigma_B}(h(x))$. Thus we have $h(\mathrm{orb}_{\sigma_A}(x)) \subset \mathrm{orb}_{\sigma_B}(h(x))$. For the other inclusion relation, we similarly have $h^{-1}(\mathrm{orb}_{\sigma_B}(y)) \subset \mathrm{orb}_{\sigma_A}(h^{-1}(y))$ for $y \in X_B$. This implies that $\mathrm{orb}_{\sigma_B}(h(x))) \subset h(\mathrm{orb}_{\sigma_A}(x))$, so that $h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x))$. □ □

Proposition 5.3. *If $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$ for some homeomorphism $h : X_A \to X_B$, then (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*

Proof. Assume that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$. For any $y \in X_B$, put $x = h^{-1}(y)$, so that $h([\sigma_A]_c(x)) = [\sigma_B]_c(h(x))$. By Lemma 3.3, we have $[\sigma_A]_c(x) = \mathrm{orb}_{\sigma_A}(x)$ and $[\sigma_B]_c(h(x)) = \mathrm{orb}_{\sigma_B}(h(x))$, so $h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x))$.

We will next show that there exist continuous cocycle functions for h. By Lemma 3.2, For any $\mu \in B_2(X_A)$, there exist $\tau_\mu \in [\sigma_A]_c$ and $k_{\tau_\mu}, l_{\tau_\mu} : X_A \to \mathbb{Z}_+$ satisfying (3-2). Put $\tau_h = h \circ \tau_\mu \circ h^{-1} \in h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$. For $x \in U_\mu$, we have $h(\sigma_A(x)) = \tau_h(h(x))$. Since $\tau_h \in [\sigma_B]_c$, one may find $k_{\tau_h}^\mu, l_{\tau_h}^\mu : X_B \to \mathbb{Z}_+$ such that

$$\sigma_B^{k_{\tau_h}^\mu(y)}(\tau_h(y)) = \sigma_B^{l_{\tau_h}^\mu(y)}(y).$$

For $y \in h(U_\mu)$, put $x = h^{-1}(y)$ so that

$$\sigma_B^{k_{\tau_h}^\mu(h(x))}(h \circ \sigma_A(x)) = \sigma_B^{l_{\tau_h}^\mu(h(x))}(h(x)) \quad \text{for } x \in U_\mu.$$

Let $\{\mu^{(1)}, \dots, \mu^{(M)}\}$ be the set $B_2(X_A)$ of all admissible words of length 2. Define $k_1^h, l_1^h : X_A \to \mathbb{Z}_+$ by setting

$$k_1^h(x) = k_{\tau_h}^{\mu^{(i)}}(h(x)) \quad \text{and} \quad l_1^h(x) = l_{\tau_h}^{\mu^{(i)}}(h(x)) \quad \text{for } x \in U_{\mu^{(i)}}.$$

They are continuous and satisfy

$$\sigma_B^{k_1^h(x)}(h \circ \sigma_A(x)) = \sigma_B^{l_1^h(x)}(h(x)) \quad \text{for } x \in X_A.$$

Similarly there exist continuous functions $k_2^h, l_2^h : X_B \to \mathbb{Z}_+$ such that

$$\sigma_A^{k_2^h(y)}(h^{-1} \circ \sigma_B(y)) = \sigma_A^{l_2^h(y)}(h^{-1}(y)) \quad \text{for } y \in X_B.$$

Hence (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent. □

We also have the converse:

Proposition 5.4. *If (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, then there exists a homeomorphism $h : X_A \to X_B$ such that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$.*

Proof. Suppose there exists a homeomorphism $h : X_A \to X_B$, $h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x))$ for $x \in X_A$ and there exist continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfying (5-1). For $n \in \mathbb{N}$, let $k_1^n, l_1^n : X_A \to \mathbb{Z}_+$ and $k_2^n, l_2^n : X_B \to \mathbb{Z}_+$ be continuous functions as in Lemma 5.1 such that

$$\sigma_B^{k_1^n(x)}(h(\sigma_A^n(x))) = \sigma_B^{l_1^n(x)}(h(x)), \quad \sigma_A^{k_2^n(y)}(h^{-1}(\sigma_B^n(y))) = \sigma_A^{l_2^n(y)}(h^{-1}(y)) \tag{5-2}$$

for $x \in X_A$ and $y \in X_B$. For any $\tau \in [\sigma_A]_c$, there exist continuous functions $k_\tau, l_\tau : X_A \to \mathbb{Z}_+$ such that

$$\sigma_A^{k_\tau(x)}(\tau(x)) = \sigma_A^{l_\tau(x)}(x) \quad x \in X_A. \tag{5-3}$$

For $y \in X_B$, put $x = h^{-1}(y)$. We set $m = k_\tau(x) \in \mathbb{N}$. By (5-2) and (5-3), we have

$$\sigma_B^{l_1^m(\tau(x))}(h(\tau(x))) = \sigma_B^{k_1^m(\tau(x))}(h(\sigma_A^m(\tau(x)))) = \sigma_B^{k_1^m(\tau(x))}(h(\sigma_A^{l_\tau(x)}(x))).$$

We set $n = l_\tau(x) \in \mathbb{N}$. By applying $\sigma_B^{k_1^n(x)}$ to the equality above, we have by (5-2)

$$\begin{aligned}\sigma_B^{k_1^n(x)+l_1^m(\tau(x))}(h(\tau(x))) &= \sigma_B^{k_1^m(\tau(x))}\sigma_B^{k_1^n(x)}(h(\sigma_A^n(x))) \\ &= \sigma_B^{k_1^m(\tau(x))}\sigma_B^{l_1^n(x)}(h(x)) = \sigma_B^{k_1^m(\tau(x))+l_1^n(x)}(h(x))\end{aligned}$$

and hence

$$\sigma_B^{k_1^n(x)+l_1^m(\tau(x))}(h \circ \tau \circ h^{-1}(y)) = \sigma_B^{k_1^m(\tau(x))+l_1^n(x)}(y).$$

By putting

$$\begin{aligned}k_\tau^h(y) &= k_1^n(x) + l_1^m(\tau(x)) = k_1^{l_\tau(h^{-1}(y))}(h^{-1}(y)) + l_1^{k_\tau(h^{-1}(y))}(\tau(h^{-1}(y))), \\ l_\tau^h(y) &= k_1^m(\tau(x)) + l_1^n(x) = k_1^{k_\tau(h^{-1}(y))}(\tau(h^{-1}(y))) + l_1^{l_\tau(h^{-1}(y))}(h^{-1}(y)),\end{aligned}$$

we have

$$\sigma_B^{k_\tau^h(y)}(h \circ \tau \circ h^{-1}(y)) = \sigma_B^{l_\tau^h(y)}(y) \quad \text{for all } y \in X_B,$$

so that $h \circ \tau \circ h^{-1} \in [\sigma_B]_c$ and $h \circ [\sigma_A]_c \circ h^{-1} \subset [\sigma_B]_c$. Similarly $h^{-1} \circ [\sigma_B]_c \circ h \subset [\sigma_A]_c$, and we conclude that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$. □

Proposition 5.5. *If there is an isomorphism* $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ *such that* $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$, *then there is a homeomorphism* $h : X_A \to X_B$ *such that* $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$.

Proof. By Theorem 4.8, there exists a group isomorphism $\widetilde{\Psi} : [\sigma_A]_c \to [\sigma_B]_c$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{U}(\mathfrak{D}_A) & \xrightarrow{\mathrm{id}} & N(\mathcal{O}_A, \mathfrak{D}_A) & \xrightarrow{\tau} & [\sigma_A]_c & \to & 1 \\ & & \downarrow{\scriptstyle \Psi|_{\mathcal{U}(\mathfrak{D}_A)}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \widetilde{\Psi}} & & \\ 1 & \to & \mathcal{U}(\mathfrak{D}_B) & \xrightarrow{\mathrm{id}} & N(\mathcal{O}_B, \mathfrak{D}_B) & \xrightarrow{\tau} & [\sigma_B]_c & \to & 1 \end{array}$$

For any $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$, put $\mathrm{Ad}(v)(f) = vfv^*$ for $f \in \mathfrak{D}_A$. Let $\tau_v \in \mathrm{Homeo}(X_A)$ be the homeomorphism on X_A satisfying $\mathrm{Ad}(v)(f) = f \circ \tau_v^{-1}$ for $f \in \mathfrak{D}_A$. Let $h : X_A \to X_B$ be the homeomorphism satisfying $\Psi(f) = f \circ h^{-1}$ for $f \in \mathfrak{D}_A$. Since $\Psi : N(\mathcal{O}_A, \mathfrak{D}_A) \to N(\mathcal{O}_B, \mathfrak{D}_B)$ is an isomorphism and $\{\tau_v \mid v \in N(\mathcal{O}_A, \mathfrak{D}_A)\} = [\sigma_A]_c$, the identity $\Psi \circ \mathrm{Ad}(v) \circ \Psi^{-1} = \mathrm{Ad}(\Psi(v))$ implies that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$. $\square$

Proposition 5.6. *If* (X_A, σ_A) *and* (X_B, σ_B) *are continuously orbit equivalent, then there exists an isomorphism* $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ *such that* $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$.

Proof. Although the proof is essentially same as that of Proposition 4.1, we give a complete proof for completeness. Let $h : X_A \to X_B$ be a homeomorphism giving rise to continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B). Take continuous functions $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ satisfying (5-1). Represent $\mathcal{O}_A$ on $\mathfrak{H}_A$ and $\mathcal{O}_B$ on $\mathfrak{H}_B$ as usual. We will prove that there exists a unitary $u_h : \mathfrak{H}_A \to \mathfrak{H}_B$ such that

$$\mathrm{Ad}(u_h)(\mathcal{O}_A) = \mathcal{O}_B \quad \text{and} \quad \mathrm{Ad}(u_h)(f) = f \circ h^{-1} \quad \text{for } f \in \mathfrak{D}_A.$$

We respectively denote by e^A_x for $x \in X_A$ and e^B_y for $y \in X_B$ the complete orthonormal systems on $\mathfrak{H}_A$ and $\mathfrak{H}_B$ coming from the shift spaces. Define the unitary $u_h : \mathfrak{H}_A \to \mathfrak{H}_B$ by setting $u_h e^A_x = e^B_{h(x)}$ for $x \in X_A$. We will first prove that $\mathrm{Ad}(u_h)(\mathcal{O}_A) = \mathcal{O}_B$. Denote by S^A_i and S^B_i the canonical generating partial isometries for S_i in $\mathcal{O}_A$ and in $\mathcal{O}_B$ respectively. For $y \in X_B$, we have

$$u_h S^A_i u^*_h e^B_y = \begin{cases} e^B_{h(ih^{-1}(y))} & \text{if } ih^{-1}(y) \in X_A, \\ 0 & \text{otherwise.} \end{cases}$$

Set $X^{(i)}_B = \{y \in X_B \mid ih^{-1}(y) \in X_A\}$. Put $z = ih^{-1}(y) \in X_A$. By the equality $h(\sigma_A(z)) = y$ with (5-1), we have $h(z) \in \sigma_B^{-l_1(z)}(\sigma_B^{k_1(z)}(y))$. Thus

$$h(z) = (\mu_1(z), \dots, \mu_{l_1(z)}(z), y_{k_1(z)+1}, y_{k_2(z)+1}, \dots)$$

for some $\mu_1(z) \cdots \mu_{l_1(z)}(z) \in B_{l_1(z)}(X_B)$. Since both the maps $k_1, l_1 : X_A \to \mathbb{Z}_+$ and the map $y \to z = ih^{-1}(y)$ are continuous, there exist finite numbers

$$\tilde{k}_1 = \max\{k_1(z) \mid y \in X^{(i)}_B\} \quad \text{and} \quad \tilde{l}_1 = \max\{l_1(z) \mid y \in X^{(i)}_B\}.$$

The set $\{(\mu_1(z), \dots, \mu_{l_1(z)}(z)) \in B_{l_1(z)}(X_B) \mid y \in X_B^{(i)}\}$ of words is a finite subset of $W_{\tilde{l}_1}(X_B) = \bigcup_{j=0,\dots,\tilde{l}_1} B_j(X_B)$. The map

$$y \in X_B^{(i)} \to (\mu_1(z), \dots, \mu_{l_1(z)}(z)) \in W_{\tilde{l}_1}(X_B)$$

is continuous, where $W_{\tilde{l}_1}(X_B)$ is endowed with discrete topology. For a word $\nu = \nu_1 \cdots \nu_j \in W_{\tilde{l}_1}(X_B)$ and $0 \le n \le \tilde{k}_1$, the sets

$$E_\nu^{(i)} = \{y \in X_B^{(i)} \mid \mu_1(z) = \nu_1, \dots, \mu_{l_1(z)}(z) = \nu_j\} \quad \text{and} \quad F_n^{(i)} = \{y \in X_B^{(i)} \mid k_1(z) = n\}$$

are clopen in $X_B^{(i)}$, where $z = ih^{-1}(y)$. We define projections in $\mathfrak{D}_B$:

$$Q_\nu^{(i)} = \chi_{E_\nu^{(i)}}, \quad P_n^{(i)} = \chi_{F_n^{(i)}}, \quad P^{(i)} = \chi_{X_B^{(i)}}.$$

Since we have disjoint unions

$$X_B^{(i)} = \bigcup_{\nu \in W_{\tilde{l}_1}(X_B)} E_\nu^{(i)} = \bigcup_{n=0,\dots,\tilde{k}_1} F_n^{(i)},$$

we have

$$P^{(i)} = \sum_{\nu \in W_{\tilde{l}_1}(X_B)} Q_\nu^{(i)} = \sum_{n=0,\dots,\tilde{k}_1} P_n^{(i)}.$$

For $y \in X_B^{(i)}$ and $\nu \in W_{\tilde{l}_1}(X_B)$ with $0 \le n \le \tilde{k}_1$, we have $y \in E_\nu^{(i)} \cap F_n^{(i)}$ if and only if $h(ih^{-1}(y)) = \nu\sigma_B^n(y)$, and the latter condition is equivalent to the condition that

$$e^B_{h(ih^{-1}(y))} = S_\nu^B e^B_{\sigma_B^n(y)}.$$

Since $y \in E_\nu^{(i)} \cap F_n^{(i)}$ if and only if $P_n^{(i)} Q_\nu^{(i)} e_y^B = e_y^B$, and $e^B_{\sigma_B^n(y)} = \sum_{\xi \in B_n(X_B)} S_\xi^{B*} e_y^B$, we have

$$e^B_{h(ih^{-1}(y))} = \sum_{n=0,\dots,\tilde{k}_1} \sum_{\nu \in W_{\tilde{l}_1}(X_B)} \Big(S_\nu^B \sum_{\xi \in B_n(X_B)} S_\xi^{B*} \Big) P_n^{(i)} Q_\nu^{(i)} e_y^B \quad \text{for } y \in X_B^{(i)}.$$

Hence

$$u_h S_i^A u_h^* e_y^B = \sum_{n=0,\dots,\tilde{k}_1} \sum_{\nu \in W_{\tilde{l}_1}(X_B)} \Big(S_\nu^B \sum_{\xi \in B_n(X_B)} S_\xi^{B*} \Big) P_n^{(i)} Q_\nu^{(i)} e_y^B \quad \text{for } y \in X_B^{(i)}.$$

Therefore we have

$$u_h S_i^A u_h^* = \sum_{n=0,\dots,\tilde{k}_1} \sum_{\nu \in W_{\tilde{l}_1}(X_B)} \Big(S_\nu^B \sum_{\xi \in B_n(X_B)} S_\xi^{B*} \Big) P_n^{(i)} Q_\nu^{(i)} P^{(i)}.$$

Since $P_n^{(i)}$, $Q_\nu^{(i)}$ and $P^{(i)}$ are projections in $\mathfrak{D}_B$, we have $\mathrm{Ad}(u_h)(S_i^A) \in \mathcal{O}_B$, so that $\mathrm{Ad}(u_h)(\mathcal{O}_A) \subset \mathcal{O}_B$. Since $u_h^* = u_{h^{-1}}$, we symmetrically have $\mathrm{Ad}(u_h^*)(\mathcal{O}_B) \subset \mathcal{O}_A$, so that $\mathrm{Ad}(u_h)(\mathcal{O}_A) = \mathcal{O}_B$.

It is direct to see that $\mathrm{Ad}(u_h)(f) = f \circ h^{-1}$ for $f \in \mathfrak{D}_A$ from the definition $u_h e_x^A = e_{h(x)}^B$ for $x \in X_A$, so we have $\mathrm{Ad}(u_h)(\mathfrak{D}_A) = \mathfrak{D}_B$. □

Therefore we have also proved Theorem 1.1.

6. Normalizers of the full groups and automorphisms of $\mathcal{O}_A$

In this section, we will study the normalizer subgroup

$$N([\sigma_A]_c) = \{\varphi \in \mathrm{Homeo}(X_A) \mid \varphi \circ \tau \circ \varphi^{-1} \in [\sigma_A]_c \text{ for all } \tau \in [\sigma_A]_c\}$$

of $[\sigma_A]_c$ in $\mathrm{Homeo}(X_A)$, which is related to the automorphism group $\mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$. We set

$$N[\sigma_A] = \{h \in \mathrm{Homeo}(X_A) \mid h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_A}(h(x)) \text{ for } x \in X_A\},$$

$$\begin{aligned} N_c[\sigma_A] = \{h \in \mathrm{Homeo}(X_A) \mid {} & \text{there exist continuous functions} \\ & k_1, l_1, k_2, l_2 : X_A \to \mathbb{Z}_+ \text{ such that, for } x \in X_A, \\ & \sigma_A^{k_1(x)}(h(\sigma_A(x))) = \sigma_A^{l_1(x)}(h(x)), \\ & \sigma_A^{k_2(x)}(h^{-1}(\sigma_A(x))) = \sigma_A^{l_2(x)}(h^{-1}(x))\} \end{aligned}$$

Lemma 6.1. *$N_c[\sigma_A]$ is a subgroup of $N[\sigma_A]$.*

Proof. By Lemma 5.2 for $X_A = X_B$, we see that $N_c[\sigma_A]$ is a subset of $N[\sigma_A]$. It remains to show that for $\varphi, \psi \in N_c[\sigma_A]$, the composition $\psi \circ \varphi$ belongs to $N_c[\sigma_A]$. For $n \in \mathbb{N}$, take continuous functions $k_{1,\varphi}^n, l_{1,\varphi}^n, k_{1,\psi}^n, l_{1,\psi}^n : X_A \to \mathbb{Z}_+$ such that

$$\sigma_A^{k_{1,\varphi}^n(x)}(\varphi(\sigma_A^n(x))) = \sigma_A^{l_{1,\varphi}^n(x)}(\varphi(x)), \tag{6-1}$$

$$\sigma_A^{k_{1,\psi}^n(x)}(\psi(\sigma_A^n(x))) = \sigma_A^{l_{1,\psi}^n(x)}(\psi(x)). \tag{6-2}$$

As in Lemma 5.1, we write $k_{1,\varphi}^n, l_{1,\varphi}^n, k_{1,\psi}^n, l_{1,\psi}^n$ as $k_\varphi^n, l_\varphi^n, k_\psi^n, l_\psi^n$ respectively. By applying (6-2) for $\varphi(\sigma_A(x))$ as x, we have

$$\sigma_A^{k_\psi^n(\varphi(\sigma_A(x)))}(\psi(\sigma_A^n(\varphi(\sigma_A(x))))) = \sigma_A^{l_\psi^n(\varphi(\sigma_A(x)))}(\psi(\varphi(\sigma_A(x)))).$$

Put $n = k_\varphi^1(x)$ and $m = l_\varphi^1(x)$. By (6-1) for $n = 1$, we have $\sigma_A^n(\varphi(\sigma_A(x))) = \sigma_A^m(\varphi(x))$ so that

$$\sigma_A^{k_\psi^n(\varphi(\sigma_A(x)))}(\psi(\sigma_A^m(\varphi(x)))) = \sigma_A^{l_\psi^n(\varphi(\sigma_A(x)))}(\psi(\varphi(\sigma_A(x)))),$$

and hence

$$\sigma_A^{k_\psi^n(\varphi(\sigma_A(x)))}\sigma_A^{k_\psi^m(\varphi(x))}(\psi(\sigma_A^m(\varphi(x)))) = \sigma_A^{k_\psi^m(\varphi(x))+l_\psi^n(\varphi(\sigma_A(x)))}(\psi(\varphi(\sigma_A(x)))).$$

By (6-2) we have

$$\sigma_A^{k_\psi^n(\varphi(\sigma_A(x)))}\sigma_A^{l_\psi^m(\varphi(x))}(\psi(\varphi(x))) = \sigma_A^{k_\psi^m(\varphi(x))+l_\psi^n(\varphi(\sigma_A(x)))}(\psi(\varphi(\sigma_A(x)))).$$

We put

$$k_{\psi\varphi}(x) = k_\psi^m(\varphi(x)) + l_\psi^n(\varphi(\sigma_A(x))) \quad \text{and} \quad l_{\psi\varphi}(x) = l_\psi^m(\varphi(x)) + k_\psi^n(\varphi(\sigma_A(x))),$$

where $n = k_\varphi^1(x)$ and $m = l_\varphi^1(x)$. The functions $k_{\psi\varphi}, l_{\psi\varphi} : X_A \to \mathbb{Z}_+$ are continuous and satisfy

$$\sigma_A^{k_{\psi\varphi}(x)}(\psi\varphi(\sigma_A(x))) = \sigma_A^{l_{\psi\varphi}(x)}(\psi\varphi(x)).$$

Similarly, we may find continuous functions $k_{\varphi^{-1}\psi^{-1}}, l_{\varphi^{-1}\psi^{-1}} : X_A \to \mathbb{Z}_+$ satisfying

$$\sigma_A^{k_{\varphi^{-1}\psi^{-1}}(x)}(\varphi^{-1}\psi^{-1}(\sigma_A(x))) = \sigma_A^{l_{\varphi^{-1}\psi^{-1}}(x)}(\varphi^{-1}\psi^{-1}(x)),$$

so that $\psi \circ \varphi \in N_c[\sigma_A]$. $\square$

Lemma 6.2. $N_c[\sigma_A] = N([\sigma_A]_c)$.

Proof. For $\varphi \in N_c[\sigma_A]$ and $\tau \in [\sigma_A]_c$, we will first prove that $\varphi \circ \tau \circ \varphi^{-1} \in [\sigma_A]_c$. For $n \in \mathbb{N}$, take continuous functions $k_1^n, l_1^n, k_2^n, l_2^n : X_A \to \mathbb{Z}_+$ satisfying

$$\sigma_A^{k_1^n(x)}(\varphi(\sigma_A^n(x)) = \sigma_A^{l_1^n(x)}(\varphi(x)), \tag{6-3}$$

$$\sigma_A^{k_2^n(x)}(\varphi^{-1}(\sigma_A^n(x)) = \sigma_A^{l_2^n(x)}(\varphi^{-1}(x)) \tag{6-4}$$

for all $x \in X_A$. For $\tau \in [\sigma_A]_c$, let $k_\tau : X_A \to \mathbb{Z}_+$ be a continuous function satisfying (3-1). By (6-3) we have

$$\sigma_A^{k_1^n(\tau\varphi^{-1}(x))}(\varphi(\sigma_A^n(\tau\varphi^{-1}(x)))) = \sigma_A^{l_1^n(\tau\varphi^{-1}(x))}(\varphi(\tau\varphi^{-1}(x))).$$

Put $y = \varphi^{-1}(x)$, $n = k_\tau(y)$ and $m = l_\tau(y)$. By (3-1), we have $\sigma_A^n(\tau(y)) = \sigma_A^m(y)$ so that

$$\sigma_A^{l_1^n(\tau(y))}(\varphi(\tau(y))) = \sigma_A^{k_1^n(\tau(y))}(\varphi(\sigma_A^m(y))).$$

By applying $\sigma_A^{k_1^m(y)}$ to the equality above, we have by (6-3)

$$\sigma_A^{k_1^m(y)+l_1^n(\tau(y))}(\varphi(\tau(y)) = \sigma_A^{k_1^n(\tau(y))}\sigma_A^{k_1^m(y)}(\varphi(\sigma_A^m(y))) = \sigma_A^{k_1^n(\tau(y))}\sigma_A^{l_1^m(y)}(\varphi(y)).$$

Put

$$k_{\varphi\tau\varphi^{-1}}(x) = k_1^m(y) + l_1^n(\tau(y)) \quad \text{and} \quad l_{\varphi\tau\varphi^{-1}}(x) = k_1^n(\tau(y)) + l_1^m(y),$$

where $y = \varphi^{-1}(x)$, $n = k_\tau(y)$, $m = l_\tau(y)$. The functions $k_{\varphi\tau\varphi^{-1}}, l_{\varphi\tau\varphi^{-1}} : X_A \to \mathbb{Z}_+$ are continuous and satisfy

$$\sigma_A^{k_{\varphi\tau\varphi^{-1}}(x)}(\varphi(\tau(\varphi^{-1}(x))) = \sigma_A^{l_{\varphi\tau\varphi^{-1}}(x)}(x).$$

Hence $\varphi \circ \tau \circ \varphi^{-1} \in [\sigma_A]_c$, so that $\varphi \in N([\sigma_A]_c)$.

We will next prove the other inclusion $N_c[\sigma_A] \supset N([\sigma_A]_c)$. For $\varphi \in N([\sigma_A]_c)$ we have $\varphi \circ [\sigma_A]_c \circ \varphi^{-1}(y) = [\sigma_A]_c(y)$ for all $y \in X_A$. Put $x = \varphi^{-1}(y)$. By Lemma 3.4, we see that

$$\varphi(\mathrm{orb}_{\sigma_A}(x)) = [\sigma_A]_c(\varphi(x)) = \mathrm{orb}_{\sigma_A}(\varphi(x)).$$

Let $\{\mu^{(1)}, \dots, \mu^{(M)}\}$ be the set $B_2(X_A)$. For each word $\mu^{(i)}$, Lemma 3.2 shows that there exist $\tau_i \in [\sigma_A]_c$ and continuous functions $k^{(i)}, l^{(i)} : X_A \to \mathbb{Z}_+$ such that

$$\tau_i(y) = \sigma_A(y) \quad \text{for } y \in U_{\mu^{(i)}} \quad \text{and} \quad \sigma_A^{k^{(i)}(z)}(\tau_i(z)) = \sigma_A^{l^{(i)}(z)}(z) \quad \text{for } z \in X_A.$$

Put $\hat{\tau} = \varphi \circ \tau_i \circ \varphi^{-1}$, so that

$$\varphi \circ \sigma_A(y) = \hat{\tau}(\varphi(y)) \quad \text{for } y \in U_{\mu^{(i)}}.$$

Since $\hat{\tau} \in [\sigma_A]_c$, we may find continuous functions $k_{\hat{\tau}}, l_{\hat{\tau}} : X_A \to \mathbb{Z}_+$ such that

$$\sigma_A^{k_{\hat{\tau}}(z)}(\hat{\tau}(z)) = \sigma_A^{l_{\hat{\tau}}(z)}(z) \quad \text{for } z \in X_A.$$

Hence we have

$$\sigma_A^{k_{\hat{\tau}}(y)}(\varphi \circ \sigma_A(y)) = \sigma_A^{l_{\hat{\tau}}(y)}(\varphi(y)) \quad \text{for } y \in U_{\mu^{(i)}}.$$

Define $k_1^{\varphi}, l_1^{\varphi} : X_A \to \mathbb{Z}_+$ by setting

$$k_1^{\varphi}(y) = k_{\hat{\tau}}(y) \quad \text{and} \quad l_1^{\varphi}(y) = l_{\hat{\tau}}(y) \quad \text{for } y \in U_{\mu^{(i)}}.$$

Since $U_{\mu^{(i)}}$ is clopen and X_A is a disjoint union $\bigcup_{i=1}^{M} U_{\mu^{(i)}}$, the functions $k_1^{\varphi}, l_1^{\varphi}$ are both continuous and satisfy

$$\sigma_A^{k_1^{\varphi}(y)}(\varphi \circ \sigma_A(y)) = \sigma_A^{l_1^{\varphi}(y)}(\varphi(y)) \quad \text{for } y \in X_A.$$

Similarly we may find continuous functions $k_2^{\varphi}, l_2^{\varphi} : X_A \to \mathbb{Z}_+$ that satisfy

$$\sigma_A^{k_2^{\varphi}(y)}(\varphi^{-1} \circ \sigma_A(x)) = \sigma_A^{l_2^{\varphi}(y)}(\varphi^{-1}(x)) \quad \text{for } x \in X_A,$$

so that $\varphi \in N_c[\sigma_A]$. Therefore $N_c[\sigma_A] \supset N([\sigma_A]_c)$ and hence $N_c[\sigma_A] = N([\sigma_A]_c)$. $\square$

Proposition 6.3. *For a homeomorphism $h \in N_c([\sigma_A])$ there is an automorphism $\alpha_h \in \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ such that $\alpha_h(f) = f \circ h^{-1}$ for $f \in \mathfrak{D}_A$. The correspondence $h \in N_c([\sigma_A]) \to \alpha_h \in \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ is a homomorphism.*

Proof. Since a homomorphism $h \in N_c([\sigma_A])$ gives rise to a continuous orbit equivalence on (X_A, σ_A), the claim follows from Proposition 5.6 and its proof. $\square$

Conversely, for any automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$, we denote by ϕ_α the homeomorphism on X_A induced by the restriction of α to $\mathfrak{D}_A$ such that $\alpha(f) = f \circ \phi_\alpha^{-1}$ for $f \in \mathfrak{D}_A$.

Proposition 6.4. *ϕ_α belongs to $N([\sigma_A]_c)$.*

Proof. For $\tau \in [\sigma_A]_c$, define $u_\tau \in N(\mathcal{O}_A, \mathfrak{D}_A)$ to be the unitary constructed in Proposition 4.1 such that $\mathrm{Ad}(u_\tau)(f) = f \circ \tau^{-1}$ for $f \in \mathfrak{D}_A$. Since $\mathrm{Ad}(\alpha(u_\tau)) = \alpha \circ \mathrm{Ad}(u_\tau) \circ \alpha^{-1}$ on $\mathcal{O}_A$, the condition $\alpha(\mathfrak{D}_A) = \mathfrak{D}_A$ implies $\alpha(u_\tau) \in N(\mathcal{O}_A, \mathfrak{D}_A)$. We see that

$$\mathrm{Ad}(\alpha(u_\tau))(f) = \alpha \circ \mathrm{Ad}(u_\tau) \circ \alpha^{-1}(f) = f \circ (\phi_\alpha \circ \tau^{-1} \circ \phi_\alpha^{-1}).$$

Since the homeomorphism $\tau_{\alpha(u_\tau)}$ defined by $\alpha(u_\tau) \in N(\mathcal{O}_A, \mathfrak{D}_A)$ belongs to $[\sigma_A]_c$ and satisfies $\mathrm{Ad}(\alpha(u_\tau))(f) = f \circ \tau_{\alpha(u_\tau)}^{-1}$, we conclude that

$$\tau_{\alpha(u_\tau)}^{-1} = (\phi_\alpha \circ \tau^{-1} \circ \phi_\alpha^{-1})^{-1} = \phi_\alpha \circ \tau \circ \phi_\alpha^{-1},$$

which belongs to $[\sigma_A]_c$. □

We denote by $\varphi_A : \mathfrak{D}_A \to \mathfrak{D}_A$ the homomorphism defined by

$$\varphi_A(a) = \sum_{i=1}^{N} S_i a S_i^* \quad \text{for } a \in \mathfrak{D}_A.$$

In identifying $\mathfrak{D}_A$ with $C(X_A)$ as usual, we see $\varphi_A(f) = f \circ \sigma_A$ for $f \in C(X_A)$. A unitary one-cocycle for φ_A is a $\mathcal{U}(\mathfrak{D}_A)$-valued function $U : \mathbb{Z}_+ \to \mathcal{U}(\mathfrak{D}_A)$ satisfying

$$U(k+l) = U(k)\varphi_A^k(U(l)) \quad \text{for } k, l \in \mathbb{Z}_+ \quad \text{(see [Matsumoto 2000]).}$$

Let $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ be the set of all unitary one-cocycles for φ_A; it is an abelian group in natural way. As in [Matsumoto 2000] (see also [Cuntz 1980; Katayama and Takehana 1998]), for $U \in Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$, put

$$\lambda(U)(S_\mu) = U(k)S_\mu \quad \text{for } \mu \in B_k(X_A).$$

Then $\lambda(U)$ gives rise to an automorphism of $\mathcal{O}_A$ such that $\lambda(U)|_{\mathfrak{D}_A} = \mathrm{id}$. We note that the correspondence $U \in Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \to U(1) \in \mathcal{U}(\mathfrak{D}_A)$ yields an isomorphism of abelian groups, and hence we may identify $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ with $\mathcal{U}(\mathfrak{D}_A)$. By [Matsumoto 2000, Lemma 4.8], $\lambda : Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \to \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ is an injective homomorphism of groups.

Let $V : \mathbb{Z}_+ \to \mathcal{U}(\mathfrak{D}_A)$ be a $\mathcal{U}(\mathfrak{D}_A)$-valued function on $\mathbb{Z}_+$ satisfying

$$V(k) = v\varphi_A^k(v^*) \quad \text{for } k \in \mathbb{Z}_+$$

for some unitary $v \in \mathcal{U}(\mathfrak{D}_A)$. Then V is called a coboundary for φ_A. Since

$$V(k)\varphi_A^k(V(l)) = v\varphi_A^k(v^*)\varphi_A^k(v\varphi_A^l(v^*)) = V(k+l),$$

a coboundary V for φ_A is a unitary one-cocycle for φ_A. Let $B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ be the set of all coboundaries for φ_A. It is easy to see that $B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ is a subgroup of $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$. We remark that if $U \in Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ satisfies $U(1) = v\varphi_A(v^*)$ for some $v \in \mathcal{U}(\mathfrak{D}_A)$, then $U(k) = v\varphi_A^k(v^*)$ for $k \in \mathbb{N}$, and hence $U \in B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$.

Define $H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ by the quotient group $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))/B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$, called the cohomology group for φ_A.

Theorem 6.5. *There exist short exact sequences*

(1) $1 \to Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_c) \to 1$,

(2) $1 \to B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} [\sigma_A]_c \to 1$,

(3) $1 \to H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_c)/[\sigma_A]_c \to 1$.

They all split. Hence $\mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A)$ *is a semidirect product*

$$\mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A) = N([\sigma_A]_c)/[\sigma_A]_c \cdot H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)).$$

Proof. (1) Since $N([\sigma_A]_c) = N_c[\sigma_A]$ by Lemma 6.2, Propositions 6.3 and 6.4 imply that the homomorphism $\phi : \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A) \to N([\sigma_A]_c)$ is defined and is surjective. By [Matsumoto 2000, Lemma 4.8], the map $\lambda : Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \to \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ is injective. Let $\alpha \in \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ be such that $\phi_\alpha = \mathrm{id}$ and hence $\alpha|_{\mathfrak{D}_A} = \mathrm{id}$. By [Matsumoto 2000, Corollary 4.7], $\alpha|_{\mathfrak{D}_A} = \mathrm{id}$ if and only if $\alpha = \lambda(U)$ for some $U \in Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$. Hence we have $\mathrm{Ker}(\phi) = Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$. By Proposition 6.3, for $\varphi \in N_c[\sigma_A]$, there exists an automorphism $\alpha_\varphi \in \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$, which is of the form $\alpha_\varphi = \mathrm{Ad}(u_\varphi)$, where $u_\varphi : \mathfrak{H}_A \to \mathfrak{H}_A$ is a unitary as defined in the proof of Proposition 5.6. It is clear to see that $\phi_{\alpha_\varphi} = \varphi$. Hence the sequence splits.

(2) Theorem 4.8 implies the homomorphism $\phi : \mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A) \to [\sigma_A]_c$ is defined and surjective. For $\alpha \in \mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A)$, take $v \in \mathcal{U}(\mathcal{O}_A)$ such that $\alpha = \mathrm{Ad}(v)$. Hence v belongs to $N(\mathcal{O}_A, \mathfrak{D}_A)$. Suppose that $\phi_{\mathrm{Ad}(v)} = \mathrm{id}$ in $[\sigma_A]_c$. By (1), there exists a cocycle $U \in Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ such that $\mathrm{Ad}(v) = \lambda(U)$. By [Matsumoto 2000, Lemma 5.14], we see that $v \in \mathcal{U}(\mathfrak{D}_A)$ and $U(1) = v\varphi_A(v^*)$. Hence U belongs to $B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$. Since the sequence (1) splits, the section in (1) yields a section in (2). Hence (2) splits.

(3) The exact sequence follows from (1) and (2), and splits. □

7. Orbit equivalence and AF-algebras

In this section, we will show that the discussions in the previous sections can be applied to the pair $(\mathcal{F}_A, \mathfrak{D}_A)$ of the AF-algebra $\mathcal{F}_A$ and its diagonal algebra $\mathfrak{D}_A$, instead of the pair $(\mathcal{O}_A, \mathfrak{D}_A)$ that we have studied. For $x = (x_n)_{n\in\mathbb{N}} \in X_A$, the *uniform orbit* $\mathrm{orb}_{\sigma_A}[x]$ of x under σ_A is defined by

$$\mathrm{orb}_{\sigma_A}[x] = \bigcup_{k=0}^{\infty} \sigma_A^{-k}(\sigma_A^k(x)) \subset X_A.$$

Hence $y = (y_n)_{n\in\mathbb{N}} \in X_A$ belongs to $\mathrm{orb}_{\sigma_A}[x]$ if and only if there exist $k \in \mathbb{Z}_+$ and an admissible word $\mu_1 \cdots \mu_k \in B_k(X_A)$ such that

$$y = (\mu_1, \ldots, \mu_k, y_{k+1}, y_{k+2}, \ldots).$$

Let $[\![\sigma_A]\!]$ be the set of all $\tau \in \mathrm{Homeo}(X_A)$ such that $\tau(x) \in \mathrm{orb}_{\sigma_A}[x]$ for all $x \in X_A$. Let $[\sigma_A]_{\mathrm{AF}}$ be the set of all τ in $[\![\sigma_A]\!]$ such that there exists a continuous function $k : X_A \to \mathbb{Z}_+$ such that

$$\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{k(x)}(x) \quad \text{for all } x \in X_A. \tag{7-1}$$

We call $[\sigma_A]_{\mathrm{AF}}$ the AF-full group for (X_A, σ_A). Since X_A is compact, a homeomorphism $\tau \in \mathrm{Homeo}(X_A)$ belongs to $[\sigma_A]_{\mathrm{AF}}$ if and only if there exists a constant $k \in \mathbb{Z}_+$ such that $\sigma_A^k(\tau(x)) = \sigma_A^k(x)$ for all $x \in X_A$. We set, for $x \in X_A$, $[\sigma_A]_{\mathrm{AF}}(x) = \{\tau(x) \mid \tau \in [\sigma_A]_{\mathrm{AF}}\}$. It is immediate that $[\sigma_A]_{\mathrm{AF}}(x) = \mathrm{orb}_{\sigma_A}[x]$. Let $N(\mathcal{F}_A, \mathfrak{D}_A)$ be the normalizer of $\mathfrak{D}_A$ in $\mathcal{F}_A$, which is defined as the group of all unitaries $u \in \mathcal{F}_A$ such that $u\mathfrak{D}_A u^* = \mathfrak{D}_A$. The algebra $\mathfrak{D}_A$ is also maximal abelian in $\mathcal{F}_A$. By an argument similar to the proof of Proposition 4.1, we have this:

Lemma 7.1. *For any $\tau \in [\sigma_A]_{\mathrm{AF}}$, there exists a unitary $u_\tau \in N(\mathcal{F}_A, \mathfrak{D}_A)$ such that*

$$\mathrm{Ad}(u_\tau)(f) = f \circ \tau^{-1} \quad \textit{for } f \in \mathfrak{D}_A,$$

and the correspondence $\tau \in [\sigma_A]_{\mathrm{AF}} \to u_\tau \in N(\mathcal{F}_A, \mathfrak{D}_A)$ is a group homomorphism.

By Lemma 4.5 we have the following:

Lemma 7.2. *For $u \in N(\mathcal{F}_A, \mathfrak{D}_A)$, let $h_u \in \mathrm{Homeo}(X_A)$ be the homeomorphism on X_A induced by the restriction of $\mathrm{Ad}(u)$ to $\mathfrak{D}_A$ such that $\mathrm{Ad}(u)(f) = f \circ h_u^{-1}$ for $f \in \mathfrak{D}_A$. Then there exists a number $k \in \mathbb{N}$ such that $\sigma_A^k(h_u(x)) = \sigma_A^k(x)$ for $x \in X_A$. Namely $h_u \in [\sigma_A]_{\mathrm{AF}}$.*

Therefore by a proof similar to that of Theorem 4.8, we have this:

Proposition 7.3. *There exists a short exact sequence*

$$1 \to \mathcal{U}(\mathfrak{D}_A) \xrightarrow{\mathrm{id}} N(\mathcal{F}_A, \mathfrak{D}_A) \xrightarrow{\tau} [\sigma_A]_{\mathrm{AF}} \to 1$$

that splits.

We say that (X_A, σ_A) and (X_B, σ_B) are *uniformly orbit equivalent* if there exists a homeomorphism $h : X_A \to X_B$ such that $h(\mathrm{orb}_{\sigma_A}[x]) = \mathrm{orb}_{\sigma_B}[h(x)]$ for $x \in X_A$ and for $\tau_1 \in [\sigma_A]_{\mathrm{AF}}$ and $\tau_2 \in [\sigma_B]_{\mathrm{AF}}$ there exist constants $k_1, k_2 \in \mathbb{Z}_+$ such that

$$\sigma_B^{k_1}(h(\tau_1(x)) = \sigma_B^{k_1}(h(x)) \quad \text{and} \quad \sigma_A^{k_2}(h^{-1}(\tau_2(y)) = \sigma_A^{k_2}(h^{-1}(y))$$

for $x \in X_A$ and $y \in X_B$. The next theorem then follows from an argument similar to those in the proofs of Propositions 5.3, 5.4, 5.5 and 5.6 and Theorem 1.1.

Theorem 7.4. *The following three assertions are equivalent*:

- *There exists an isomorphism* $\Psi : \mathcal{F}_A \to \mathcal{F}_B$ *such that* $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$.
- (X_A, σ_A) *and* (X_B, σ_B) *are uniformly orbit equivalent.*
- *There is a homeomorphism* $h : X_A \to X_B$ *such that* $h \circ [\sigma_A]_{\mathrm{AF}} \circ h^{-1} = [\sigma_B]_{\mathrm{AF}}$.

Let $\mathrm{Aut}(\mathcal{F}_A, \mathfrak{D}_A)$ be the group of all $\alpha \in \mathrm{Aut}(\mathcal{F}_A)$ such that $\alpha(\mathfrak{D}_A) = \mathfrak{D}_A$. Denote by $\mathrm{Inn}(\mathcal{F}_A, \mathfrak{D}_A)$ the subgroup of $\mathrm{Aut}(\mathcal{F}_A, \mathfrak{D}_A)$ of inner automorphisms on $\mathcal{F}_A$. We set $\mathrm{Out}(\mathcal{F}_A, \mathfrak{D}_A)$ to be the quotient group $\mathrm{Aut}(\mathcal{F}_A, \mathfrak{D}_A)/\mathrm{Inn}(\mathcal{F}_A, \mathfrak{D}_A)$. We may argue as in Section 6, to obtain this:

Theorem 7.5. *There exist short exact sequences*

- $1 \to Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Aut}(\mathcal{F}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_{\mathrm{AF}}) \to 1,$
- $1 \to B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Inn}(\mathcal{F}_A, \mathfrak{D}_A) \xrightarrow{\phi} [\sigma_A]_{\mathrm{AF}} \to 1,$
- $1 \to H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Out}(\mathcal{F}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_{\mathrm{AF}})/[\sigma_A]_{\mathrm{AF}} \to 1.$

They all split. Hence $\mathrm{Out}(\mathcal{F}_A, \mathfrak{D}_A)$ *is a semidirect product*

$$\mathrm{Out}(\mathcal{F}_A, \mathfrak{D}_A) = N([\sigma_A]_{\mathrm{AF}})/[\sigma_A]_{\mathrm{AF}} \cdot H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)),$$

where $N([\sigma_A]_{\mathrm{AF}})$ *is the normalizer subgroup of* $[\sigma_A]_{\mathrm{AF}}$ *in* $[\![\sigma_A]\!]$.

Concluding remarks. After the December 2007 submission of this paper, related results have appeared in [Matui 2009; Matsumoto 2009; 2007; 2010]. The last paper shows that if the sizes of the matrices A, B are less than or equal to three, then the topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if and only if the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic.

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References

[Boyle and Tomiyama 1998] M. Boyle and J. Tomiyama, "Bounded topological orbit equivalence and C^*-algebras", *J. Math. Soc. Japan* **50**:2 (1998), 317–329. MR 99d:46088 Zbl 0940.37004

[Cuntz 1980] J. Cuntz, "Automorphisms of certain simple C^*-algebras", pp. 187–196 in *Quantum fields — algebras, processes* (Bielefeld, 1978), edited by L. Streit, Springer, Vienna, 1980. MR 82d:46089 Zbl 0475.46046

[Cuntz and Krieger 1980] J. Cuntz and W. Krieger, "A class of C^*-algebras and topological Markov chains", *Invent. Math.* **56**:3 (1980), 251–268. MR 82f:46073a Zbl 0434.46045

[Dye 1959] H. A. Dye, "On groups of measure preserving transformations, I", *Amer. J. Math.* **81** (1959), 119–159. MR 24 #A1366 Zbl 0087.11501

[Dye 1963] H. A. Dye, "On groups of measure preserving transformations, II", *Amer. J. Math.* **85** (1963), 551–576. MR 28 #1275 Zbl 0191.42803

[Giordano et al. 1995] T. Giordano, I. F. Putnam, and C. F. Skau, "Topological orbit equivalence and C^*-crossed products", *J. Reine Angew. Math.* **469** (1995), 51–111. MR 97g:46085 Zbl 0834.46053

[Giordano et al. 1999] T. Giordano, I. F. Putnam, and C. F. Skau, "Full groups of Cantor minimal systems", *Israel J. Math.* **111** (1999), 285–320. MR 2000g:46096 Zbl 0942.46040

[Herman et al. 1992] R. H. Herman, I. F. Putnam, and C. F. Skau, "Ordered Bratteli diagrams, dimension groups and topological dynamics", *Internat. J. Math.* **3**:6 (1992), 827–864. MR 94f:46096 Zbl 0786.46053

[Katayama and Takehana 1998] Y. Katayama and H. Takehana, "On automorphisms of generalized Cuntz algebras", *Internat. J. Math.* **9**:4 (1998), 493–512. MR 99f:46083 Zbl 0918.46056

[Krieger 1976] W. Krieger, "On ergodic flows and the isomorphism of factors", *Math. Ann.* **223**:1 (1976), 19–70. MR 54 #3430 Zbl 0332.46045

[Matsumoto 2000] K. Matsumoto, "On automorphisms of C^*-algebras associated with subshifts", *J. Operator Theory* **44**:1 (2000), 91–112. MR 2001g:46147 Zbl 0994.46021

[Matsumoto 2007] K. Matsumoto, "Orbit equivalence of one-sided subshifts and the associated C^*-algebras", preprint, 2007. To appear in *Yokohama Math. J.* arXiv 0709.1185

[Matsumoto 2009] K. Matsumoto, "Orbit equivalence in C^*-algebras defined by actions of symbolic dynamical systems", *Contemporary Math.* **503** (2009), 121–140.

[Matsumoto 2010] K. Matsumoto, "Some remarks on orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras", preprint, 2010.

[Matui 2009] H. Matui, "Homology and topological full groups of étale groupoids on totally disconnected spaces", preprint, 2009. arXiv 0909.1624

[Tomiyama 1992a] J. Tomiyama, *The interplay between topological dynamics and theory of C^*-algebras*, Lecture Notes Series **2**, Res. Inst. Math. Seoul Nat. Univ., Seoul, 1992. MR 93h:46097 Zbl 0892.46070

[Tomiyama 1992b] J. Tomiyama, *The interplay between topological dynamics and theory of C^*-algebras*, Lecture Notes Series **2**, Res. Inst. Math. Seoul Nat. Univ., Seoul, 1992. MR 93h:46097 Zbl 0892.46070

[Tomiyama 1996] J. Tomiyama, "Topological full groups and structure of normalizers in transformation group C^*-algebras", *Pacific J. Math.* **173**:2 (1996), 571–583. MR 97i:46124 Zbl 0861.54032

[Tomiyama 1998] J. Tomiyama, "Representations of topological dynamical systems and C^*-algebras", pp. 351–364 in *Operator algebras and operator theory* (Shanghai, 1997), edited by L. Ge et al., Contemp. Math. **228**, Amer. Math. Soc., 1998. MR 99m:46158 Zbl 0932.46051

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