ON THE ASYMPTOTIC BEHAVIOR OF D-SOLUTIONS OF THE
PLANE STEADY-STATE NAVIER–STOKES EQUATIONS

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We prove that all the derivatives of a $D$-solution $(u, p)$ of the Navier–Stokes equations in a plane neighborhood of infinity $\mathbb{C}C_{R_0}$ decay more rapidly than $|x|^{\epsilon-1/2}$ for every positive $\epsilon$. Moreover, we show that if the flux of $u$ through the boundary of $C_{R_0}$ is zero, the second derivatives of $p$ are summable over the complement of $C_{R_0}$.

In the theory of the steady-state Navier–Stokes equations a $D$-solution is an analytic pair $(u, p)$ which satisfies the equations [Galdi 1994]

$$\Delta u - u \cdot \nabla u - \nabla p = 0,$$

$$\text{div } u = 0,$$

in a neighborhood of infinity $\mathbb{C}C_{R_0} \subset \mathbb{R}^2$ and has a finite Dirichlet integral:

$$\int_{\mathbb{C}C_{R_0}} |\nabla u|^2 < +\infty.$$

An open problem in viscous hydrodynamics concerns the behavior at infinity of these solutions. Thanks to the celebrated results of D. Gilbarg and H. W. Weinberger [1978] and G. P. Galdi [1994], we know that

$$|u|^2 = o(\log r), \quad \nabla u = o(r^{-3/4} \log^{9/8} r),$$

$$\nabla_{k-1} p(x) = o(1), \quad \nabla_k u(x) = o(1),$$

for all $k \in \mathbb{N}$.

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1 We use a standard vector notation as in, for example, [Galdi 1994]. We set $C_R = \{x \in \mathbb{R}^2 : r = |x| < R\}$. If $f$ is a function defined in a neighborhood of infinity $\mathbb{C}C_{R_0}$ and $\phi(r)$ is a positive function, $f = o(\phi)$ and $f = O(\phi)$ mean respectively that $\lim_{r \to +\infty} f/g = 0$ and $f/g$ is bounded in $\mathbb{C}C_{R_0}$. $D^{k,q}(\mathbb{C}C_{R_0}) = \{u \in L^1_{\text{loc}}(\mathbb{C}C_{R_0}) : \|\nabla_k u\|_{L^q(\mathbb{C}C_{R_0})} < +\infty\}$, where $k \in \mathbb{N}_0$, $q \in [1, +\infty)$ and $\nabla_k u = \nabla \ldots \nabla u (k$ times), $\nabla_0 u = u$; $\mathcal{H}^1$ denotes the Hardy space on $\mathbb{R}^2$ [Stein 1993].

2 $u(x)$ and $p(x)$ are the velocity field and the pressure field respectively, and $u \cdot \nabla u$ is the vector with components $u_i \partial_i u_j$. Since our results are independent of kinematical viscosity $\nu$, we shall put $\nu = 1$. 

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The aim of this paper is to improve (2) and to establish some summability properties of the derivatives of the pressure field \( p \). To be precise, we prove the following:

**Theorem.** If \((u, p)\) is a D-solution, then

\[
\nabla p = O(r^{\epsilon - 1/2})
\]

for every positive \( \epsilon \). Moreover, if

\[
\int_{\partial C_{R_0}} u \cdot n = 0,
\]

then

\[
p \in D^{2,1}(C_{R_0}).
\]

To prove the theorem we need some well-known results, which we state in the form of lemmas.

**Lemma 1** [Galdi 1994]. Let \( v(x) = \int_{R^2} \frac{1}{|x - y|^{\lambda} |y|^\mu} \, da_y \), with \( \lambda < 2, \mu < 2 \). If \( \lambda + \mu > 2 \), then

\[
v(x) = cr^{2-\lambda-\mu}
\]

for a suitable constant \( c = c(\lambda, \mu) \).

**Lemma 2** [Stein 1993]. If \( f \in \mathcal{H}^1 \), then the problem

\[
\Delta p = f \quad \text{in } \mathbb{R}^2, \quad \lim_{x \to \infty} p(x) = 0,
\]

admits the unique solution

\[
p(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log |x - y| \, da_y \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2).
\]

**Lemma 3** [Coifman et al. 1993]. If \( u \in D^{1,2}(\mathbb{R}^2) \) is divergence-free, then

\[
\nabla u \cdot \nabla u^T \in \mathcal{H}^1.
\]

**Proof of (3).** Taking the divergence in \((1)_1\) and taking into account \((1)_2\), we see that \( p \) satisfies the Poisson equation

\[
\Delta p + \nabla u \cdot \nabla u^T = 0 \quad \text{in } C_{R_0}.
\]

Writing the classical Stokes formula in the shell \( T = C_R \setminus C_{R_0} \) \((R \gg R_0)\), we have

\[
2\pi p(x) = \int_{\partial T} \partial_n p(\xi) \log |x - \xi| \, ds_\xi
\]

\[
- \int_{\partial T} p(\xi)(x - \xi) \cdot n(\xi) |x - \xi|^2 \, ds_\xi - \int_{T} (\nabla u \cdot \nabla u^T)(y) \log |x - y| \, da_y,
\]
where $n$ denotes the outward unit normal to $\partial T$. Hence, taking the gradient shows that

$$
(7) \quad 2\pi \nabla p(x) = \int_{\partial T} \frac{(x - \xi) \cdot \nabla p(\xi)}{|x - \xi|^2} \, ds_{\xi} - \int_{T} \frac{(\nabla u \cdot \nabla u^T)(y)(x - y)}{|x - y|^2} \, da_y.
$$

By virtue of (2), we are allowed to let $R \to +\infty$ in (7) to have

$$
(8) \quad 2\pi \nabla p(x) = \int_{\partial C_{R_0}} \frac{(x - \xi) \cdot \nabla p(\xi)}{|x - \xi|^2} \, ds_{\xi} - \int_{\partial C_{R_0}} \frac{(\nabla u \cdot \nabla u^T)(y)(x - y)}{|x - y|^2} \, da_y.
$$

Therefore, taking into account (2) and Lemma 1, (8) implies (3).

**Proof of (5).** Let $g$ be a regular cut-off function in $\mathbb{R}^2$, vanishing in $C_{\tilde{R}}$ and equal to 1 outside $C_{2\tilde{R}}$, with $\tilde{R} \gg R_0$. By (4), the problem

$$
\text{div} \ h + \text{div} \ (g u) = 0 \quad \text{in} \quad C_{2\tilde{R}} \setminus C_{\tilde{R}}
$$

has a solution $h \in C^\infty_0(C_{2\tilde{R}} \setminus C_{\tilde{R}})$ [Galdi 1994]. From (6) it follows that the function $Q = g^2 p$ is a solution of the equation

$$
(9) \quad \Delta Q + \text{div} \, f + \varphi = 0 \quad \text{in} \quad \mathbb{R}^2,
$$

where $\varphi \in C^\infty_0(C_{2\tilde{R}} \setminus C_{\tilde{R}})$ and

$$
\begin{align*}
\varphi & = (g u + h) \cdot \nabla (g u + h).
\end{align*}
$$

By virtue of Lemma 2, $Q$ is expressed by

$$
(10) \quad 2\pi Q(x) = -\int_{\mathbb{R}^2} (\log |x - y|) \, \text{div} \, f(y) \, da_y - \int_{\mathbb{R}^2} \varphi(y) \log |x - y| \, da_y = Q_1 + Q_2.
$$

By Lemma 3 $\text{div} \, f \in \mathcal{H}^1$ so that Lemma 2 implies that $Q_1 \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2)$. Since $Q$ tends to zero at infinity, we must have

$$
\int_{\mathbb{R}^2} \varphi = 0,
$$

otherwise $Q = O(\log r)$. It follows that $\nabla_k Q_2 = O(r^{-1-k})$, and (5) is proved.
Remark. The higher gradients of \( u \) and \( p \) can be estimated in the same way. Precisely, it holds
\[
\nabla_k p(x), \nabla_k u(x) = O(r^{\epsilon-1/2}),
\]
for all positive \( \epsilon \) and for all \( k \in \mathbb{N} \).

Remark. By the embedding theorem and (2), (5) implies that \( p \in D^{1,2}(\mathbb{C}C_{R_0}) \). Since by the basic calculus
\[
\int_0^{2\pi} |\nabla p|(R, \theta) \, d\theta = \int_0^{2\pi} \left| \int_{-\infty}^{+\infty} \partial_r \nabla p(r, \theta) \, d\theta \right| \leq \frac{1}{R} \int_{\mathbb{C}C_R} |\nabla \nabla p|,
\]
we see that if (4) holds, then
\[
(11) \quad \int_0^{2\pi} |\nabla p|(R, \theta) = o(R^{-1}).
\]

References


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