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ON THE ASYMPTOTIC BEHAVIOR OF D-SOLUTIONS OF THE PLANE STEADY-STATE NAVIER–STOKES EQUATIONS

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We prove that all the derivatives of a D-solution $(\boldsymbol{u}, p)$ of the Navier–Stokes equations in a plane neighborhood of infinity $\complement C_{R_0}$ decay more rapidly than $|x|^{\epsilon-1/2}$ for every positive ϵ. Moreover, we show that if the flux of u through the boundary of C_{R_0} is zero, the second derivatives of p are summable over the complement of C_{R_0}.

In the theory of the steady-state Navier–Stokes equations a *D-solution* is an analytic pair $(\boldsymbol{u}, p)$[1] which satisfies the equations [Galdi 1994][2]

$$\begin{aligned} \Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p &= \boldsymbol{0}, \\ \operatorname{div}\boldsymbol{u} &= 0, \end{aligned} \tag{1}$$

in a neighborhood of infinity $\complement C_{R_0} \subset \mathbb{R}^2$ and has a finite *Dirichlet integral*:

$$\int_{\complement C_{R_0}} |\nabla\boldsymbol{u}|^2 < +\infty.$$

An open problem in viscous hydrodynamics concerns the behavior at infinity of these solutions. Thanks to the celebrated results of D. Gilbarg and H. W. Weinberger [1978] and G. P. Galdi [1994], we know that

$$\begin{aligned} |\boldsymbol{u}|^2 &= o(\log r), & \nabla\boldsymbol{u} &= o(r^{-3/4}\log^{9/8} r), \\ \nabla_{k-1} p(x) &= o(1), & \nabla_k\boldsymbol{u}(x) &= o(1), \end{aligned} \tag{2}$$

for all $k \in \mathbb{N}$.

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[1]We use a standard vector notation as in, for example, [Galdi 1994]. We set $C_R = \{x \in \mathbb{R}^2 : r = |\boldsymbol{x}| < R\}$. If f is a function defined in a neighborhood of infinity $\complement C_{R_0}$ and $\varphi(r)$ is a positive function, $f = o(\varphi)$ and $f = O(\varphi)$ mean respectively that $\lim_{r\to+\infty} f/g = 0$ and f/g is bounded in $\complement C_{R_0}$. $D^{k,q}(\complement C_{R_0}) = \{u \in L^1_{\text{loc}}(\complement C_{R_0}) : \|\nabla_k u\|_{L^q(\complement C_{R_0})} < +\infty\}$, where $k \in \mathbb{N}_0$, $q \in [1, +\infty)$ and $\nabla_k u = \nabla \ldots \nabla u$ (k times), $\nabla_0 u = u$; $\mathcal{H}^1$ denotes the Hardy space on $\mathbb{R}^2$ [Stein 1993].

[2]$\boldsymbol{u}(x)$ and $p(x)$ are the velocity field and the pressure field respectively, and $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is the vector with components $u_i\partial_i u_j$. Since our results are independent of kinematical viscosity ν, we shall put $\nu = 1$.

The aim of this paper is to improve $(2)_3$ and to establish some summability properties of the derivatives of the pressure field p. To be precise, we prove the following:

Theorem. If $(\boldsymbol{u}, p)$ is a D-solution, then

$$\nabla p = O(r^{\epsilon-1/2}) \tag{3}$$

for every positive ϵ. Moreover, if

$$\int_{\partial C_{R_0}} \boldsymbol{u}\cdot\boldsymbol{n} = 0, \tag{4}$$

then

$$p \in D^{2,1}(\complement C_{R_0}). \tag{5}$$

To prove the theorem we need some well-known results, which we state in the form of lemmas.

Lemma 1 [Galdi 1994]. Let $v(x) = \displaystyle\int_{\mathbb{R}^2} \frac{1}{|x-y|^{\lambda}|y|^{\mu}}\,\mathrm{d}a_y$, with $\lambda < 2$, $\mu < 2$. If $\lambda+\mu > 2$, then

$$v(x) = cr^{2-\lambda-\mu}$$

for a suitable constant $c = c(\lambda, \mu)$.

Lemma 2 [Stein 1993]. If $f \in \mathcal{H}^1$, then the problem

$$\Delta p = f \quad \text{in } \mathbb{R}^2, \qquad \lim_{x\to\infty} p(x) = 0,$$

admits the unique solution

$$p(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(y)\log|x-y|\,\mathrm{d}a_y \in D^{2,1}(\mathbb{R}^2)\cap D^{1,2}(\mathbb{R}^2).$$

Lemma 3 [Coifman et al. 1993]. If $\boldsymbol{u} \in D^{1,2}(\mathbb{R}^2)$ is divergence-free, then

$$\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}} \in \mathcal{H}^1.$$

Proof of (3). Taking the divergence in $(1)_1$ and taking into account $(1)_2$, we see that p satisfies the Poisson equation

$$\Delta p + \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}} = 0 \quad \text{in } \complement C_{R_0}. \tag{6}$$

Writing the classical Stokes formula in the shell $T = C_R \setminus C_{R_0}$ ($R \gg R_0$), we have

$$2\pi p(x) = \int_{\partial T} \partial_n p(\xi)\log|x-\xi|\,\mathrm{d}s_\xi - \int_{\partial T}\frac{p(\xi)(x-\xi)\cdot\boldsymbol{n}(\xi)}{|x-\xi|^2}\,\mathrm{d}s_\xi - \int_T (\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)\log|x-y|\,\mathrm{d}a_y,$$

where $\boldsymbol{n}$ denotes the outward unit normal to ∂T. Hence, taking the gradient shows that

$$(7)\quad 2\pi\nabla p(x) = \int_{\partial T} \frac{(x-\xi)\partial_n p(\xi)}{|x-\xi|^2}\,\mathrm{d}s_\xi - \nabla\int_{\partial T} \frac{p(\xi)(x-\xi)\cdot\boldsymbol{n}(\xi)}{|x-\xi|^2}\,\mathrm{d}s_\xi - \int_T \frac{(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)(x-y)}{|x-y|^2}\,\mathrm{d}a_y.$$

By virtue of (2), we are allowed to let $R\to+\infty$ in (7) to have

$$\begin{aligned}(8)\quad 2\pi\nabla p(x) &= \int_{\partial C_{R_0}} \frac{(x-\xi)\partial_n p(\xi)}{|x-\xi|^2}\,\mathrm{d}s_\xi - \nabla\int_{\partial C_{R_0}} \frac{p(\xi)(x-\xi)\cdot\boldsymbol{n}(\xi)}{|x-\xi|^2}\,\mathrm{d}s_\xi - \int_{\complement C_{R_0}} \frac{(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)(x-y)}{|x-y|^2}\,\mathrm{d}a_y \\ &= -\int_{\complement C_{R_0}} \frac{(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\mathrm{T}})(y)(x-y)}{|x-y|^2}\,\mathrm{d}a_y + O(r^{-1}).\end{aligned}$$

Therefore, taking into account $(2)_2$ and Lemma 1, (8) implies (3). □

Proof of (5). Let g be a regular cut-off function in $\mathbb{R}^2$, vanishing in $C_{\bar R}$ and equal to 1 outside $C_{2\bar R}$, with $\bar R \gg R_0$. By (4), the problem

$$\operatorname{div}\boldsymbol{h} + \operatorname{div}(g\boldsymbol{u}) = 0 \quad \text{in} \quad C_{2\bar R}\setminus C_{\bar R}$$

has a solution $\boldsymbol{h}\in C_0^\infty(C_{2\bar R}\setminus C_{\bar R})$ [Galdi 1994]. From (6) it follows that the function $Q = g^2 p$ is a solution of the equation

$$(9)\qquad \Delta Q + \operatorname{div}\boldsymbol{f} + \varphi = 0 \quad \text{in } \mathbb{R}^2,$$

where $\varphi\in C_0^\infty(C_{2\bar R}\setminus C_{\bar R})$ and

$$\boldsymbol{f} = (g\boldsymbol{u}+\boldsymbol{h})\cdot\nabla(g\boldsymbol{u}+\boldsymbol{h}).$$

By virtue of Lemma 2, Q is expressed by

$$\begin{aligned}(10)\qquad 2\pi Q(x) &= -\int_{\mathbb{R}^2} (\log|x-y|)\operatorname{div}\boldsymbol{f}(y)\,\mathrm{d}a_y - \int_{\mathbb{R}^2}\varphi(y)\log|x-y|\,\mathrm{d}a_y \\ &= Q_1 + Q_2.\end{aligned}$$

By Lemma 3 $\operatorname{div}\boldsymbol{f}\in\mathscr{H}^1$ so that Lemma 2 implies that $Q_1\in D^{2,1}(\mathbb{R}^2)\cap D^{1,2}(\mathbb{R}^2)$. Since Q tends to zero at infinity, we must have

$$\int_{\mathbb{R}^2}\varphi = 0,$$

otherwise $Q = O(\log r)$. It follows that $\nabla_k Q_2 = O(r^{-1-k})$, and (5) is proved. □

Remark. The higher gradients of $\boldsymbol{u}$ and p can be estimated in the same way. Precisely, it holds

$$\nabla_k p(x),\ \nabla_k \boldsymbol{u}(x) = O(r^{\epsilon-1/2}),$$

for all positive ϵ and for all $k \in \mathbb{N}$.

Remark. By the embedding theorem and $(2)_3$, (5) implies that $p \in D^{1,2}(\complement C_{R_0})$. Since by the basic calculus

$$\int_0^{2\pi} |\nabla p|(R,\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \left| \int_R^{+\infty} \partial_r \nabla p(r,\theta)\,\mathrm{d}\theta \right| \le \frac{1}{R} \int_{\complement C_R} |\nabla\nabla p|,$$

we see that if (4) holds, then

$$\int_0^{2\pi} |\nabla p|(R,\theta) = o(R^{-1}). \tag{11}$$

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