ON THE ASYMPTOTIC BEHAVIOR OF $D$-SOLUTIONS OF THE PLANE STEADY-STATE NAVIER–STOKES EQUATIONS

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We prove that all the derivatives of a $D$-solution $(u, p)$ of the Navier–Stokes equations in a plane neighborhood of infinity $\mathbb{C}C_{R_0}$ decay more rapidly than $|x|^{\epsilon-1/2}$ for every positive $\epsilon$. Moreover, we show that if the flux of $u$ through the boundary of $C_{R_0}$ is zero, the second derivatives of $p$ are summable over the complement of $C_{R_0}$.

In the theory of the steady-state Navier–Stokes equations a $D$-solution is an analytic pair $(u, p)$ which satisfies the equations [Galdi 1994]

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\begin{align*}
\Delta u - u \cdot \nabla u - \nabla p &= 0, \\
\text{div } u &= 0,
\end{align*}

in a neighborhood of infinity $\mathbb{C}C_{R_0} \subset \mathbb{R}^2$ and has a finite Dirichlet integral:

\[ \int_{\mathbb{C}C_{R_0}} |\nabla u|^2 < +\infty. \]

An open problem in viscous hydrodynamics concerns the behavior at infinity of these solutions. Thanks to the celebrated results of D. Gilbarg and H. W. Weinberger [1978] and G. P. Galdi [1994], we know that

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\begin{align*}
|u|^2 &= o(\log r), \\
\nabla u &= o(r^{-3/4} \log^{9/8} r), \\
\nabla_{k-1} p(x) &= o(1), \\
\nabla_k u(x) &= o(1),
\end{align*}

for all $k \in \mathbb{N}$.

MSC2000: primary 76D05; secondary 35Q30, 76D03.

Keywords: steady-state Navier–Stokes equations, $D$-solutions, behavior at infinity.

1 We use a standard vector notation as in, for example, [Galdi 1994]. We set $C_R = \{ x \in \mathbb{R}^2 : r = |x| < R \}$. If $f$ is a function defined in a neighborhood of infinity $\mathbb{C}C_{R_0}$ and $\varphi(r)$ is a positive function, $f = o(\varphi)$ and $f = O(\varphi)$ mean respectively that $\lim_{r \to +\infty} f/g = 0$ and $f/g$ is bounded in $\mathbb{C}C_{R_0}$.

2 $D_k^{1, q}(\mathbb{C}C_{R_0}) = \{ u \in L^1_{\text{loc}}(\mathbb{C}C_{R_0}) : \| \nabla_k u \|_{L^q(\mathbb{C}C_{R_0})} < +\infty \}$, where $k \in \mathbb{N}_0$, $q \in [1, +\infty)$ and $\nabla_k u = \nabla \ldots \nabla u$ ($k$ times), $\nabla_0 u = u$; $H^1$ denotes the Hardy space on $\mathbb{R}^2$ [Stein 1993].

$u(x)$ and $p(x)$ are the velocity field and the pressure field respectively, and $u \cdot \nabla u$ is the vector with components $u_i \partial_i u_j$. Since our results are independent of kinematical viscosity $\nu$, we shall put $\nu = 1$. 

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The aim of this paper is to improve (2) and to establish some summability properties of the derivatives of the pressure field $p$. To be precise, we prove the following:

**Theorem.** If $(u, p)$ is a D-solution, then

$$\nabla p = O(r^{\epsilon-1/2})$$

for every positive $\epsilon$. Moreover, if

$$\int_{\partial C_{R_0}} u \cdot n = 0,$$

then

$$p \in D^{2,1}(\bar{C}_{R_0}).$$

To prove the theorem we need some well-known results, which we state in the form of lemmas.

**Lemma 1 [Galdi 1994].** Let $v(x) = \int_{\mathbb{R}^2} \frac{1}{|x-y|^\lambda |y|^\mu} \, da_y$, with $\lambda < 2$, $\mu < 2$. If $\lambda + \mu > 2$, then

$$v(x) = cr^{2-\lambda-\mu}$$

for a suitable constant $c = c(\lambda, \mu)$.

**Lemma 2 [Stein 1993].** If $f \in \mathcal{H}^1$, then the problem

$$\Delta p = f \quad \text{in } \mathbb{R}^2, \quad \lim_{x \to \infty} p(x) = 0,$$

admits the unique solution

$$p(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log |x-y| \, da_y \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2).$$

**Lemma 3 [Coifman et al. 1993].** If $u \in D^{1,2}(\mathbb{R}^2)$ is divergence-free, then

$$\nabla u \cdot \nabla u^T \in \mathcal{H}^1.$$

**Proof of (3).** Taking the divergence in $(1)_1$ and taking into account $(1)_2$, we see that $p$ satisfies the Poisson equation

$$\Delta p + \nabla u \cdot \nabla u^T = 0 \quad \text{in } \bar{C}_{R_0}.$$

Writing the classical Stokes formula in the shell $T = C_R \setminus C_{R_0}$ ($R \gg R_0$), we have

$$2\pi p(x) = \int_{\partial T} \partial_n p(\xi) \log |x-\xi| \, ds_\xi$$

$$- \int_{\partial T} \frac{p(\xi)(x-\xi) \cdot n(\xi)}{|x-\xi|^2} \, ds_\xi - \int_T (\nabla u \cdot \nabla u^T)(y) \log |x-y| \, da_y,$$
where \( \mathbf{n} \) denotes the outward unit normal to \( \partial T \). Hence, taking the gradient shows that

\[
2\pi \nabla p(x) = \int_{\partial T} \frac{(x - \xi) \mathbf{n} p(\xi)}{|x - \xi|^2} \, ds_\xi - \nabla \int_{\partial C_{R_0}} \frac{p(\xi)(x - \xi) \cdot \mathbf{n}(\xi)}{|x - \xi|^2} \, ds_\xi
\]

\[
- \int_{\partial C_{R_0}} \frac{\nabla \mathbf{u} \cdot \nabla \mathbf{u}^T(y)(x - y)}{|x - y|^2} \, da_y.
\]

By virtue of (2), we are allowed to let \( R \to +\infty \) in (7) to have

\[
2\pi \nabla p(x) = \int_{\partial C_{R_0}} \frac{(x - \xi) \mathbf{n} p(\xi)}{|x - \xi|^2} \, ds_\xi - \nabla \int_{\partial C_{R_0}} \frac{p(\xi)(x - \xi) \cdot \mathbf{n}(\xi)}{|x - \xi|^2} \, ds_\xi
\]

\[
- \int_{\partial C_{R_0}} \frac{\nabla \mathbf{u} \cdot \nabla \mathbf{u}^T(y)(x - y)}{|x - y|^2} \, da_y - \int_{\partial T} \frac{(\nabla \mathbf{u} \cdot \nabla \mathbf{u}^T)(y)(x - y)}{|x - y|^2} \, da_y + O(r^{-1}).
\]

Therefore, taking into account (2) and Lemma 1, (8) implies (3). \( \square \)

**Proof of (5).** Let \( g \) be a regular cut-off function in \( \mathbb{R}^2 \), vanishing in \( C_{\bar{R}} \) and equal to 1 outside \( C_{2\bar{R}} \), with \( \bar{R} \gg R_0 \). By (4), the problem

\[
\text{div} \mathbf{h} + \text{div}(g \mathbf{u}) = 0 \quad \text{in} \quad C_{2\bar{R}} \setminus C_{\bar{R}}
\]

has a solution \( \mathbf{h} \in C_0^\infty(C_{2\bar{R}} \setminus C_{\bar{R}}) \) [Galdi 1994]. From (6) it follows that the function \( Q = g^2 p \) is a solution of the equation

\[
\Delta Q + \text{div} \mathbf{f} + \varphi = 0 \quad \text{in} \quad \mathbb{R}^2,
\]

where \( \varphi \in C_0^\infty(C_{2\bar{R}} \setminus C_{\bar{R}}) \) and

\[
\mathbf{f} = (g \mathbf{u} + \mathbf{h}) \cdot \nabla(g \mathbf{u} + \mathbf{h}).
\]

By virtue of Lemma 2, \( Q \) is expressed by

\[
2\pi Q(x) = - \int_{\mathbb{R}^2} \log |x - y| \, \text{div} \mathbf{f}(y) \, da_y - \int_{\mathbb{R}^2} \varphi(y) \log |x - y| \, da_y
\]

\[
= Q_1 + Q_2.
\]

By Lemma 3 \( \text{div} \mathbf{f} \in \mathcal{H}^1 \) so that Lemma 2 implies that \( Q_1 \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2) \). Since \( Q \) tends to zero at infinity, we must have

\[
\int_{\mathbb{R}^2} \varphi = 0,
\]

otherwise \( Q = O(\log r) \). It follows that \( \nabla_k Q_2 = O(r^{-1-k}) \), and (5) is proved. \( \square \)
Remark. The higher gradients of $u$ and $p$ can be estimated in the same way. Precisely, it holds
\[ \nabla_k p(x), \nabla_k u(x) = O(r^{\epsilon-1/2}), \]
for all positive $\epsilon$ and for all $k \in \mathbb{N}$.

Remark. By the embedding theorem and (2)_{3}, (5) implies that $p \in D^{1,2}(\mathbb{C}C_{R_0})$. Since by the basic calculus
\[ \int_0^{2\pi} |\nabla p|(R, \theta) \, d\theta = \int_0^{2\pi} \left| \int_{\mathbb{C}C_R} \partial_r \nabla p(r, \theta) \, d\theta \right| \leq \frac{1}{R} \int_{\mathbb{C}C_R} |\nabla \nabla p|, \]
we see that if (4) holds, then
\[ \int_0^{2\pi} |\nabla p|(R, \theta) = o(R^{-1}). \]

References


Received September 22, 2009.

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