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GEOMETRIC STRUCTURES ASSOCIATED TO A CONTACT METRIC (κ, μ)-SPACE

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We prove that any contact metric (κ, μ)-space $(M, \varphi, \xi, \eta, g)$ admits a canonical paracontact metric structure that is compatible with the contact form η. We study this canonical paracontact structure, proving that it satisfies a nullity condition and induces on the underlying contact manifold (M, η) a sequence of compatible contact and paracontact metric structures satisfying nullity conditions. We then study the behavior of that sequence, which is related to the Boeckx invariant I_M and to the bi-Legendrian structure of $(M, \varphi, \xi, \eta, g)$. Finally we are able to define a canonical Sasakian structure on any contact metric (κ, μ)-space whose Boeckx invariant satisfies $|I_M| > 1$.

1. Introduction

A contact metric (κ, μ)-space is a contact metric manifold $(M, \varphi, \xi, \eta, g)$ such that the Reeb vector field belongs to the so-called "(κ, μ)-nullity distribution", that is, it satisfies the condition

$$R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{1-1}$$

for some real numbers κ, μ and for any $X, Y \in \Gamma(TM)$; here R denotes the curvature tensor field of the Levi-Civita connection and $2h$ the Lie derivative of the structure tensor φ in the direction of the Reeb vector field ξ. This definition was introduced by Blair, Koufogiorgos and Papantoniou [1995] as a generalization both of the Sasakian condition $R_{XY}\xi = \eta(Y)X - \eta(X)Y$ and of those contact metric manifolds satisfying $R_{XY}\xi = 0$, which were studied by Blair [1977].

Recently contact metric (κ, μ)-spaces have attracted the attention of many authors, and various papers have appeared on this topic, for example [Boeckx and Cho 2008; Cappelletti Montano et al. 2008; Koufogiorgos et al. 2008]. In fact there are many motivations for studying (κ, μ)-manifolds: the first is that, in the

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non-Sasakian case (that is for $\kappa \neq 1$), the condition (1-1) determines the curvature completely; moreover, while the values of κ and μ may change, the form of (1-1) is invariant under $\mathcal{D}$-homothetic deformations; finally, there are nontrivial examples of such manifolds, the most important being the unit tangent bundle of a Riemannian manifold of constant sectional curvature endowed with its standard contact metric structure.

Boeckx [2000] provided a complete (local) classification of non-Sasakian contact metric (κ, μ)-spaces based on the invariant

$$I_M = \frac{1-\mu/2}{\sqrt{1-\kappa}}.$$

The recent paper [Cappelletti Montano 2009b] gives a geometric interpretation of this invariant in terms of Legendre foliations.

In this paper we study mainly those (non-Sasakian) contact metric (κ, μ)-spaces such that $I_M \neq \pm 1$, showing how rich the geometry of this wide class of contact metric (κ, μ)-spaces is. In fact we prove that any such contact metric (κ, μ)-manifold is endowed with a nonflat pair of bi-Legendrian structures, a 3-web structure and a canonical family of contact and paracontact metric structures satisfying nullity conditions. Such geometric structures are related to each other and depend on the sign of the Boeckx invariant I_M.

The main part of the article is devoted to the study of the interplays between the theory of contact metric (κ, μ)-spaces and paracontact geometry. The link is given by the theory of bi-Legendrian structures. Indeed, Cappelletti Montano [2009a] proved that there is a biunivocal correspondence between the set of almost bi-Legendrian structures and the set of paracontact metric structures on the same contact manifold (M, η). This bijection maps bi-Legendrian structures onto integrable paracontact metric structures and flat bi-Legendrian structures onto para-Sasakian structures. Thus, since any contact metric (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ is canonically endowed with the bi-Legendrian structure given by the eigendistributions corresponding to the nonzero eigenvalues of the operator h, one can associate to $(M, \varphi, \xi, \eta, g)$ a paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$, which we prove is given by

$$\tilde{\varphi} := \frac{1}{2\sqrt{1-\kappa}} \mathcal{L}_\xi \varphi, \quad \tilde{g} := d\eta(\cdot, \tilde{\varphi}\,\cdot) + \eta \otimes \eta, \tag{1-2}$$

and which we call the *canonical paracontact metric structure* of the contact metric (κ, μ)-space $(M, \varphi, \xi, \eta, g)$. We study this paracontact structure and we prove that its curvature tensor field satisfies the relation

$$\tilde{R}_{XY}\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y),$$

with $\tilde{\kappa} = (1-\mu/2)^2+\kappa-2$ and $\tilde{\mu}=2$ and where $\tilde{h} := (1/2)\mathcal{L}_\xi\tilde{\varphi}$. The next step is the study of the structure defined by the Lie derivative of $\tilde{\varphi}$ in the direction of the Reeb vector field. In fact we prove that if $|I_M|<1$, the structure (φ_1,ξ,η,g_1) given by

$$\varphi_1 := \frac{1}{2\sqrt{-1-\tilde{\kappa}}}\mathcal{L}_\xi\tilde{\varphi}, \quad g_1 := -d\eta(\,\cdot\,,\varphi_1\cdot\,)+\eta\otimes\eta,$$

is a *contact* metric (κ_1,μ_1)-structure on (M,η), where $\kappa_1 = \kappa+(1-\mu/2)^2$ and $\mu_1=2$. In the case $|I_M|>1$, the structure $(\tilde{\varphi}_1,\xi,\eta,\tilde{g}_1)$, defined by

$$\tilde{\varphi}_1 := \frac{1}{2\sqrt{1+\tilde{\kappa}}}\mathcal{L}_\xi\tilde{\varphi}, \quad \tilde{g}_1 := d\eta(\,\cdot\,,\tilde{\varphi}_1\cdot\,)+\eta\otimes\eta,$$

is a *paracontact* metric $(\tilde{\kappa}_1,\tilde{\mu}_1)$-structure, with $\tilde{\kappa}_1=(1-\mu/2)^2+\kappa-2$ and $\tilde{\mu}_1=2$. Furthermore, we prove that it is just the canonical paracontact structure induced by a suitable contact metric (κ',μ')-structure on M. Then we show that this procedure can be iterated and gives rise to a sequence of contact and paracontact structures associated with the initial contact metric (κ,μ)-structure (φ,ξ,η,g). The behavior of this canonical sequence essentially depends on the Boeckx invariant I_M of the contact metric (κ,μ)-manifold (M,φ,ξ,η,g). If $|I_M|>1$, the sequence consists only of paracontact structures, whereas in the case $|I_M|<1$ we have an alternation of contact and paracontact structures; see Theorem 5.6 for all details. Moreover, all the new contact metric structures on M obtained in this way are in fact Tanaka–Webster parallel structures [Boeckx and Cho 2008], that is, the Tanaka–Webster connection parallelizes both the Tanaka–Webster torsion and the Tanaka–Webster curvature.

Thus in a contact metric (κ,μ)-space (M,φ,ξ,η,g), the k-th Lie derivative $\mathcal{L}_\xi\cdots\mathcal{L}_\xi\varphi$ of the structure tensor φ in the direction ξ, once suitably normalized, defines a new contact or paracontact structure, depending on the value of I_M. This last property shows another surprising geometric feature of the invariant I_M, linked to the paracontact geometry of the contact metric (κ,μ)-manifold M.

Finally we prove that every contact metric (κ,μ)-space such that $|I_M|>1$ admits a canonical compatible Sasakian structure, explicitly given by

$$\bar{\varphi}_- := -\frac{1}{\sqrt{(1-\mu/2)^2-(1-\kappa)}}((1-\tfrac{1}{2}\mu)\varphi+\varphi h), \quad \bar{g}_- := d\eta(\,\cdot\,,\bar{\varphi}_-\cdot\,)+\eta\otimes\eta,$$

in the case $I_M<-1$ and

$$\bar{\varphi}_+ := \frac{1}{\sqrt{(1-\mu/2)^2-(1-\kappa)}}((1-\tfrac{1}{2}\mu)\varphi+\varphi h), \quad \bar{g}_+ := -d\eta(\,\cdot\,,\bar{\varphi}_+\cdot\,)+\eta\otimes\eta,$$

in the case $I_M>1$. Such Sasakian structures are related to the paracontact structures above by the formulas $\bar{\varphi}_- = \tilde{\varphi}\circ\tilde{\varphi}_1$ and $\bar{\varphi}_+ = \tilde{\varphi}_1\circ\tilde{\varphi}$. In particular, $(\bar{\varphi}_-,\tilde{\varphi},\tilde{\varphi}_1)$ or

$(\bar{\varphi}_+, \tilde{\varphi}_1, \tilde{\varphi})$, according to $I_M < -1$ or $I_M > 1$, respectively, induce an almost anti-hypercomplex structure, and hence a 3-web, on the contact distribution of (M, η).

Therefore it appears that a further geometrical interpretation of the Boeckx invariant is the fact that any contact metric (κ, μ)-space such that $|I_M| < 1$ can admit compatible Tanaka–Webster parallel structures, whereas any contact metric (κ, μ)-space such that $|I_M| > 1$ can admit compatible Sasakian structures.

All manifolds considered here are assumed to be smooth, that is, of the class $\mathscr{C}^\infty$, and connected; we denote by $\Gamma(\cdot\,,\,)$ the set of all sections of a corresponding bundle. We use the convention that $2u \wedge v = u \otimes v - v \otimes u$.

2. Preliminaries

Contact and paracontact structures. A *contact manifold* is a $(2n+1)$-dimensional smooth manifold M that carries a 1-form η, called a *contact form*, that satisfies $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is well known that given η there exists a unique vector field ξ, called the *Reeb vector field*, such that $i_\xi \eta = 1$ and $i_\xi d\eta = 0$. In the sequel we will denote by $\mathscr{D}$ the $2n$-dimensional distribution defined by $\ker(\eta)$, called the *contact distribution*. It is easy to see that the Reeb vector field is an infinitesimal automorphism with respect to the contact distribution, and the tangent bundle of M splits as the direct sum $TM = \mathscr{D} \oplus \mathbb{R}\xi$.

Given a contact manifold (M, η) one can consider two different geometric structures associated with the contact form η, namely a contact metric structure and a paracontact metric structure.

It is well known that (M, η) admits a Riemannian metric g and a $(1, 1)$-tensor field φ such that

$$\begin{aligned} \varphi^2 &= -I + \eta \otimes \xi, \\ d\eta(X, Y) &= g(X, \varphi Y), \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y) \end{aligned} \tag{2-1}$$

for all $X, Y \in \Gamma(TM)$, from which it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and $\eta = g(\cdot\,, \xi)$. The structure (φ, ξ, η, g) is called a *contact metric structure* and the manifold M endowed with such a structure is said to be a *contact metric manifold*. In a contact metric manifold M, the $(1, 1)$-tensor field $h := (1/2)\mathscr{L}_\xi \varphi$ is symmetric and satisfies

$$h\xi = 0, \quad \eta \circ h = 0, \quad h\varphi + \varphi h = 0, \quad \nabla\xi = -\varphi - \varphi h, \quad \mathrm{tr}(h) = \mathrm{tr}(\varphi h) = 0, \tag{2-2}$$

where ∇ is the Levi-Civita connection of (M, g). The tensor field h vanishes identically if and only if the Reeb vector field is Killing, and in this case the contact metric manifold is said to be *K-contact*.

In any (almost) contact (metric) manifold, one can consider the tensor field N_φ defined by

(2-3) $$N_\varphi(X, Y) := \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + 2d\eta(X, Y)\xi.$$

The tensor field N_φ satisfies the formula, which will be very useful in the sequel,

(2-4) $$\varphi N_\varphi(X, Y) + N_\varphi(\varphi X, Y) = 2\eta(X)hY$$

for all $X, Y \in \Gamma(TM)$, from which it follows that

(2-5) $$\eta(N_\varphi(\varphi X, Y)) = 0.$$

Any contact metric manifold where N_φ vanishes identically is said to be *Sasakian*. In terms of the curvature tensor field, the Sasakian condition is expressed by the relation

(2-6) $$R_{XY}\xi = \eta(Y)X - \eta(X)Y.$$

Any Sasakian manifold is K-contact, and in dimension 3 the converse also holds; see [Blair 2002] for details. A natural generalization of the Sasakian condition (2-6) leads to the notion of "contact metric (κ, μ)-manifold" [Blair et al. 1995]. Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. If the curvature tensor field of the Levi-Civita connection satisfies

(2-7) $$R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for some $\kappa, \mu \in \mathbb{R}$, we say that $(M, \varphi, \xi, \eta, g)$ is a *contact metric* (κ, μ)*-manifold* (or that ξ belongs to the (κ, μ)-nullity distribution). This definition was introduced and deeply studied by Blair, Koufogiorgos and Papantoniou [1995], who proved the following fundamental results.

Theorem 2.1 [Blair et al. 1995]. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ)-manifold. Then necessarily $\kappa \leq 1$. Moreover, if $\kappa = 1$ then $h = 0$ and $(M, \varphi, \xi, \eta, g)$ is Sasakian; if $\kappa < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions, $\mathcal{D}(0) = \mathbb{R}\xi$, $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$, corresponding to the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

Theorem 2.2 [Blair et al. 1995]. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ)-manifold. Then the following relations hold, for any $X, Y \in \Gamma(TM)$:*

(2-8) $$(\nabla_X \varphi)Y = g(X, Y + hY)\xi - \eta(Y)(X + hX),$$

(2-9) $$(\nabla_X h)Y = ((1-\kappa)g(X, \varphi Y) + g(X, \varphi hY))\xi + \eta(Y)h(\varphi X + \varphi hX) - \mu\varphi hY,$$

(2-10) $$(\nabla_X \varphi h)Y = \big(g(X, hY) - (1-\kappa)g(X, \varphi^2 Y)\big)\xi + \eta(Y)\big(hX - (1-\kappa)\varphi^2 X\big) + \mu\eta(X)hY.$$

Given a non-Sasakian contact metric (κ, μ)-manifold M, Boeckx [2000] proved that the number $I_M := (1 - \mu/2)/\sqrt{1-\kappa}$, is an invariant of the contact metric (κ, μ)-structure, and proved that two non-Sasakian contact metric (κ, μ)-manifolds $(M_1, \varphi_1, \xi_1, \eta_1, g_1)$ and $(M_2, \varphi_2, \xi_2, \eta_2, g_2)$ are locally isometric as contact metric manifolds if and only if $I_{M_1} = I_{M_2}$. Then Boeckx used the invariant I_M for providing a full classification of contact metric (κ, μ)-spaces. The standard example of contact metric (κ, μ)-manifold is given by the tangent sphere bundle $T_1 N$ of a Riemannian manifold of constant curvature c endowed with its standard contact metric structure. In this case $\kappa = c(2-c)$, $\mu = -2c$ and $I_{T_1 N} = (1+c)/|1-c|$. Therefore as c varies over the reals, $I_{T_1 N}$ takes on every value strictly greater than -1. Moreover, one can easily find that $I_{T_1 N} < 1$ if and only if $c < 0$.

On the other hand on a contact manifold (M, η) one can consider also compatible paracontact metric structures. We recall [Kaneyuki and Williams 1985] that an *almost paracontact structure* on a $(2n+1)$-dimensional smooth manifold M is given by a $(1, 1)$-tensor field $\tilde{\varphi}$, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1$ and $\tilde{\varphi}^2 = I - \eta \otimes \xi$.

(ii) Denoting by $\mathcal{D}$ the $2n$-dimensional distribution defined by η, the tensor field $\tilde{\varphi}$ induces an almost paracomplex structure on each fiber on $\mathcal{D}$.

Recall that an almost paracomplex structure on a $2n$-dimensional smooth manifold is a tensor field J of type $(1, 1)$ such that $J \neq I$, $J^2 = I$ and the eigendistributions T^+ and T^- corresponding to the eigenvalues 1 and -1 of J, respectively, have dimension n.

As an immediate consequence of the definition, $\tilde{\varphi}\xi = 0$, $\eta \circ \tilde{\varphi} = 0$ and the field of endomorphisms $\tilde{\varphi}$ has constant rank $2n$. Any almost paracontact manifold admits a semi-Riemannian metric $\tilde{g}$ such that

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y) \tag{2-11}$$

for all $X, Y \in \Gamma(TM)$. Then $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is called an *almost paracontact metric manifold*. Any such semi-Riemannian metric is necessarily of signature $(n+1, n)$. If also $d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$ for all $X, Y \in \Gamma(TM)$, then $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a *paracontact metric manifold*. On an almost paracontact manifold one defines the tensor field

$$N_{\tilde{\varphi}}(X, Y) := \tilde{\varphi}^2[X, Y] + [\tilde{\varphi}X, \tilde{\varphi}Y] - \tilde{\varphi}[\tilde{\varphi}X, Y] - \tilde{\varphi}[X, \tilde{\varphi}Y] - 2d\eta(X, Y)\xi.$$

If $N_{\tilde{\varphi}}$ vanishes identically the almost paracontact manifold is said to be *normal*.

Moreover, in a paracontact metric manifold one defines a symmetric, trace-free operator $\tilde{h}$ by setting $\tilde{h} = (1/2)\mathcal{L}_\xi \tilde{\varphi}$. One can prove (see [Zamkovoy 2009]) that $\tilde{h}$ is a symmetric operator that anticommutes with $\tilde{\varphi}$ and satisfies $\tilde{h}\xi = 0$, $\eta \circ \tilde{h} = 0$ and

$\widetilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h}$, where $\widetilde{\nabla}$ denotes the Levi-Civita connection of $(M, \tilde{g})$. Furthermore, $\tilde{h}$ vanishes identically if and only if ξ is a Killing vector field and in this case $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is called a *K-paracontact manifold*. A normal paracontact metric manifold is said to be a *para-Sasakian manifold*. Also in this context, the para-Sasakian condition implies the K-paracontact condition and the converse holds in dimension 3. In terms of the covariant derivative of $\tilde{\varphi}$, the para-Sasakian condition may be expressed by

$$(\widetilde{\nabla}_X\tilde{\varphi})Y = -\tilde{g}(X, Y)\xi + \eta(Y)X. \tag{2-12}$$

On the other hand one can prove (see [Zamkovoy 2009]) that in any para-Sasakian manifold,

$$\tilde{R}_{XY}\xi = \eta(Y)X - \eta(X)Y, \tag{2-13}$$

but, unlike contact metric structures, the condition (2-13) does not necessarily imply that the manifold is para-Sasakian.

In any paracontact metric manifold Zamkovoy [2009] introduced a canonical connection that plays the same role in paracontact geometry that the generalized Tanaka–Webster connection [Tanno 1989] does in a contact metric manifold.

Theorem 2.3 [Zamkovoy 2009]. *On a paracontact metric manifold there exists a unique connection $\widetilde{\nabla}^{pc}$, called the* canonical paracontact connection, *satisfying the properties*

(i) $\widetilde{\nabla}^{pc}\eta = 0$, $\widetilde{\nabla}^{pc}\xi = 0$, $\widetilde{\nabla}^{pc}\tilde{g} = 0$;

(ii) $(\widetilde{\nabla}^{pc}_X\tilde{\varphi})Y = (\widetilde{\nabla}_X\tilde{\varphi})Y - \eta(Y)(X - \tilde{h}X) + \tilde{g}(X - \tilde{h}X, Y)\xi$;

(iii) $\tilde{T}^{pc}(\xi, \tilde{\varphi}Y) = -\tilde{\varphi}\tilde{T}^{pc}(\xi, Y)$;

(iv) $\tilde{T}^{pc}(X, Y) = 2d\eta(X, Y)\xi$ *on* $\mathscr{D} = \ker(\eta)$.

The explicit expression of this connection is

$$\widetilde{\nabla}^{pc}_X Y = \widetilde{\nabla}_X Y + \eta(X)\tilde{\varphi}Y + \eta(Y)(\tilde{\varphi}X - \tilde{\varphi}\tilde{h}X) + \tilde{g}(X - \tilde{h}X, \tilde{\varphi}Y)\xi. \tag{2-14}$$

The torsion tensor field is given by

$$\tilde{T}^{pc}(X, Y) = \eta(X)\tilde{\varphi}\tilde{h}Y - \eta(Y)\tilde{\varphi}\tilde{h}X + 2g(X, \tilde{\varphi}Y)\xi. \tag{2-15}$$

An almost paracontact structure $(\tilde{\varphi}, \xi, \eta)$ is *integrable* [Zamkovoy 2009] if the almost paracomplex structure $\tilde{\varphi}|_{\mathscr{D}}$ satisfies the condition $N_{\tilde{\varphi}}(X, Y) \in \Gamma(\mathbb{R}\xi)$ for all $X, Y \in \Gamma(\mathscr{D})$. This is equivalent to requiring that the eigendistributions $T^{\pm}$ of $\tilde{\varphi}$ satisfy $[T^{\pm}, T^{\pm}] \subset T^{\pm} \oplus \mathbb{R}\xi$. For an integrable paracontact metric manifold, the canonical paracontact connection shares many of the properties of the Tanaka–Webster connection on CR-manifolds. For instance we have the following result.

Theorem 2.4 [Zamkovoy 2009]. *A paracontact metric manifold* $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ *is integrable if and only if the canonical paracontact connection parallelizes the structure tensor* $\tilde{\varphi}$.

In particular, by Theorem 2.4 and (2-12) it follows that any para-Sasakian manifold is integrable.

Bi-Legendrian manifolds. Let (M, η) be a $(2n+1)$-dimensional contact manifold. It is well known that the contact condition $\eta \wedge (d\eta)^n \neq 0$ geometrically means that the contact distribution $\mathcal{D}$ is as far as possible from being integrable. In fact one can prove that the maximal dimension of any involutive subbundle of $\mathcal{D}$ is n. Such n-dimensional integrable distributions are called *Legendre foliations* of (M, η). More generally, a *Legendre distribution* on a contact manifold (M, η) is an n-dimensional subbundle L of the contact distribution that is not necessarily integrable but does satisfy the weaker condition that $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$.

The theory of Legendre foliations has been extensively studied in recent years from various points of view. In particular, Pang [1990] classified Legendre foliations using a bilinear symmetric form $\Pi_{\mathcal{F}}$ on the tangent bundle of the foliation $\mathcal{F}$, defined by

$$\Pi_{\mathcal{F}}(X, X') = -(\mathcal{L}_X \mathcal{L}_{X'} \eta)(\xi) = 2d\eta([\xi, X], X').$$

He called a Legendre foliation *positive* (*negative*) *definite*, *nondegenerate*, *degenerate* or *flat*, according to whether the bilinear form $\Pi_{\mathcal{F}}$ is positive (negative) definite, nondegenerate, degenerate or vanishes identically, respectively. Then for a nondegenerate Legendre foliation $\mathcal{F}$, Libermann [1991] defined a linear map $\Lambda_{\mathcal{F}} : TM \to T\mathcal{F}$, whose kernel is $T\mathcal{F} \oplus \mathbb{R}\xi$, such that

$$\Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}} Z, X) = d\eta(Z, X) \tag{2-16}$$

for any $Z \in \Gamma(TM)$, $X \in \Gamma(T\mathcal{F})$. The operator $\Lambda_{\mathcal{F}}$ is surjective and satisfies $(\Lambda_{\mathcal{F}})^2 = 0$ and $\Lambda_{\mathcal{F}}[\xi, X] = (1/2)X$ for all $X \in \Gamma(T\mathcal{F})$. Then one can extend $\Pi_{\mathcal{F}}$ to a symmetric bilinear form on TM by putting

$$\overline{\Pi}_{\mathcal{F}}(Z, Z') := \begin{cases} \Pi_{\mathcal{F}}(Z, Z') & \text{if } Z, Z' \in \Gamma(T\mathcal{F}), \\ \Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}} Z, \Lambda_{\mathcal{F}} Z') & \text{otherwise.} \end{cases}$$

If (M, η) is endowed with two transversal Legendre distributions L_1 and L_2, we say that (M, η, L_1, L_2) is an *almost bi-Legendrian manifold*. Thus, in particular, the tangent bundle of M splits up as the direct sum $TM = L_1 \oplus L_2 \oplus \mathbb{R}\xi$. When both L_1 and L_2 are integrable we refer to a *bi-Legendrian manifold*. An (almost) bi-Legendrian manifold is said to be flat, degenerate or nondegenerate if and only if both the Legendre distributions are flat, degenerate or nondegenerate, respectively. Any contact manifold (M, η) endowed with a Legendre distribution L admits a canonical almost bi-Legendrian structure. Indeed let (φ, ξ, η, g) be a compatible contact metric structure. Then the relation $d\eta(\varphi X, \varphi Y) = d\eta(X, Y)$ easily implies that $Q := \varphi L$ is a Legendre distribution on M that is g-orthogonal to L. Q is usually referred as the *conjugate Legendre distribution* of L and in general is not involutive, even if L is.

The next theorem shows the existence of a canonical connection on an almost bi-Legendrian manifold.

Theorem 2.5 [Cappelletti Montano 2005]. *Let (M, η, L_1, L_2) be an almost bi-Legendrian manifold. There exists a unique linear connection ∇^{bl}, called* bi-Legendrian connection, *satisfying*

(i) $\nabla^{bl} L_1 \subset L_1$, $\nabla^{bl} L_2 \subset L_2$,

(ii) $\nabla^{bl} \xi = 0$, $\nabla^{bl} d\eta = 0$,

(iii) $T^{bl}(X, Y) = 2d\eta(X, Y)\xi$ *for all* $X \in \Gamma(L_1)$, $Y \in \Gamma(L_2)$,
$T^{bl}(X, \xi) = [\xi, X_{L_1}]_{L_2} + [\xi, X_{L_2}]_{L_1}$ *for all* $X \in \Gamma(TM)$,

where T^{bl} denotes the torsion tensor field of ∇^{bl}, and X_{L_1} and X_{L_2} the projections of X onto the subbundles L_1 and L_2 of TM, respectively.

The behavior of the bi-Legendrian connection in the case of conjugate Legendre distributions was considered later:

Theorem 2.6 [Cappelletti Montano 2007]. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold endowed with a Legendre distribution L. Let $Q := \varphi L$ be the conjugate Legendre distribution of L and ∇^{bl} the bi-Legendrian connection associated with (L, Q). Then the following statements are equivalent*:

(i) $\nabla^{bl} g = 0$.

(ii) $\nabla^{bl} \varphi = 0$.

(iii) $\nabla^{bl}_X X' = -(\varphi[X, \varphi X'])_L$ *for all* $X, X' \in \Gamma(L)$ *and* $\nabla^{bl}_Y Y' = -(\varphi[Y, \varphi Y'])_Q$ *for all* $Y, Y' \in \Gamma(Q)$, *and the tensor field h maps the subbundle L onto L and the subbundle Q onto Q.*

(iv) *The metric g is bundlelike with respect both to the distribution $L \oplus \mathbb{R}\xi$ and to the distribution $Q \oplus \mathbb{R}\xi$.*

Furthermore, *assuming L and Q integrable*, (i)–(iv) *are equivalent to the total geodesicity* (*with respect to the Levi-Civita connection of g*) *of the Legendre foliations defined by L and Q.*

3. The foliated structure of a contact metric (κ, μ)-space

Theorem 2.1 implies that any non-Sasakian contact metric (κ, μ)-manifold is endowed with three mutually orthogonal involutive distributions $\mathcal{D}(\lambda)$, $\mathcal{D}(-\lambda)$ and $\mathcal{D}(0) = \mathbb{R}\xi$, corresponding to the eigenspaces λ, $-\lambda$ and 0 of the operator h, where

$\lambda = \sqrt{1-\kappa}$. As we pointed out in [Cappelletti Montano and Di Terlizzi 2008], $(\mathcal{D}(\lambda), \mathcal{D}(-\lambda))$ defines a bi-Legendrian structure on (M, η). We also started the study of the bi-Legendrian structure of a contact metric (κ, μ)-manifold, expressing the Pang invariant of each Legendre foliation $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ as

$$\begin{aligned} \Pi_{\mathcal{D}(\lambda)} &= (2\sqrt{1-\kappa} - \mu + 2) g|_{\mathcal{D}(\lambda)\times\mathcal{D}(\lambda)}, \\ \Pi_{\mathcal{D}(-\lambda)} &= (-2\sqrt{1-\kappa} - \mu + 2) g|_{\mathcal{D}(-\lambda)\times\mathcal{D}(-\lambda)}; \end{aligned} \tag{3-1}$$

see also [Cappelletti Montano 2009b]. It follows that only one among the following five cases may occur:

(I) Both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are positive definite.

(II) $\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is negative definite.

(III) Both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are negative definite.

(IV) $\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is flat.

(V) $\mathcal{D}(\lambda)$ is flat and $\mathcal{D}(-\lambda)$ is negative definite.

Moreover, the bi-Legendrian structure $(\mathcal{D}(\lambda), \mathcal{D}(-\lambda))$ belongs to the class (I), (II), (III), (IV), (V) if and only if $I_M > 1$, $-1 < I_M < 1$, $I_M < -1$, $I_M = 1$, $I_M = -1$, respectively.

Furthermore, the following characterization of contact metric (κ, μ)-manifolds in terms of Legendre foliations holds.

Theorem 3.1 [Cappelletti Montano and Di Terlizzi 2008]. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian contact metric manifold. Then $(M, \varphi, \xi, \eta, g)$ is a contact metric (κ, μ)-manifold if and only if it admits two mutually orthogonal Legendre distributions L and Q and a unique linear connection $\overline{\nabla}$ satisfying*

(i) $\overline{\nabla} L \subset L$, $\overline{\nabla} Q \subset Q$,

(ii) $\overline{\nabla}\eta = 0$, $\overline{\nabla} d\eta = 0$, $\overline{\nabla} g = 0$, $\overline{\nabla}\varphi = 0$, $\overline{\nabla} h = 0$,

(iii) $\overline{T}(X, Y) = 2d\eta(X, Y)\xi$ *for all* $X, Y \in \Gamma(\mathcal{D})$,
$\overline{T}(X, \xi) = [\xi, X_L]_Q + [\xi, X_Q]_L$ *for all* $X \in \Gamma(TM)$,

where $\overline{T}$ denotes the torsion tensor field of $\overline{\nabla}$ and X_L and X_Q are, respectively, the projections of X onto the subbundles L and Q of TM. Furthermore, L and Q are integrable and coincide with the eigenspaces $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ of the operator h, and $\overline{\nabla}$ coincides with the bi-Legendrian connection ∇^{bl} associated to the bi-Legendrian structure (L, Q).

In particular, from (3-1) it follows that $\nabla^{bl}\Pi_{\mathcal{D}(\lambda)} = \nabla^{bl}\Pi_{\mathcal{D}(-\lambda)} = 0$. Conversely:

Theorem 3.2 [Cappelletti Montano 2009b]. *Suppose (M, η) is a contact manifold endowed with a bi-Legendrian structure $(\mathcal{F}_1, \mathcal{F}_2)$ such that $\nabla^{bl}\Pi_{\mathcal{F}_1} = \nabla^{bl}\Pi_{\mathcal{F}_2} = 0$. Assume that one of the following conditions holds:*

(I) *$\mathcal{F}_1$ and $\mathcal{F}_2$ are positive definite and there exist two positive numbers a and b such that $\overline{\Pi}_{\mathcal{F}_1} = ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2} = ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$.*

(II) *$\mathcal{F}_1$ is positive definite and $\mathcal{F}_2$ is negative definite and there exist $a > 0$ and $b < 0$ such that $\overline{\Pi}_{\mathcal{F}_1} = ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2} = ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$.*

(III) *$\mathcal{F}_1$ and $\mathcal{F}_2$ are negative definite and there exist two negative numbers a and b such that $\overline{\Pi}_{\mathcal{F}_1} = ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2} = ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$.*

Then (M,η) admits a compatible contact metric structure (φ,ξ,η,g) such that

(i) *if $a = b$, then (M,φ,ξ,η,g) is a Sasakian manifold*;

(ii) *if $a \neq b$, then (M,φ,ξ,η,g) is a contact metric (κ,μ)-manifold whose associated bi-Legendrian structure is $(\mathcal{F}_1,\mathcal{F}_2)$, where*

$$\kappa = 1 - \tfrac{1}{16}(a-b)^2, \quad \mu = 2 - \tfrac{1}{2}(a+b). \tag{3-2}$$

4. The canonical paracontact structure of a contact metric (κ,μ)-space

[Cappelletti Montano 2009a] studied the interplay between paracontact geometry and the theory of bi-Legendrian structures, and showed the existence of a biunivocal correspondence $\Psi : \mathcal{AB} \to \mathcal{PM}$ between the set $\mathcal{AB}$ of almost bi-Legendrian structures and the set of paracontact metric structures $\mathcal{PM}$ on the same contact manifold (M,η). This bijection maps bi-Legendrian structures onto integrable paracontact structures, maps flat almost bi-Legendrian structures onto K-paracontact structures, and maps flat bi-Legendrian structures onto para-Sasakian structures. For the convenience of the reader we recall the definition of the biunivocal correspondence above. If (L_1,L_2) is an almost bi-Legendrian structure on (M,η), the corresponding paracontact metric structure $(\tilde{\varphi},\xi,\eta,\tilde{g}) = \Psi(L_1,L_2)$ is given by

$$\tilde{\varphi}|_{L_1} = I, \quad \tilde{\varphi}|_{L_2} = -I, \quad \tilde{\varphi}\xi = 0, \quad \tilde{g} := d\eta(\cdot\,,\tilde{\varphi}\,\cdot) + \eta\otimes\eta. \tag{4-1}$$

Also studied was the relationship between the bi-Legendrian and the canonical paracontact connections; in the integrable case they coincide:

Theorem 4.1 [Cappelletti Montano 2009a]. *Let (M,η,L_1,L_2) be an almost bi-Legendrian manifold, and $(\tilde{\varphi},\xi,\eta,\tilde{g}) = \Psi(L_1,L_2)$ be the paracontact metric structure induced on M by* (4-1). *Let ∇^{bl} and $\widetilde{\nabla}^{pc}$ be the corresponding bi-Legendrian and canonical paracontact connections. Then*

(a) $\nabla^{bl}\tilde{\varphi} = 0$ *and* $\nabla^{bl}\tilde{g} = 0$,

(b) *the bi-Legendrian and the canonical paracontact connections coincide if and only if the induced paracontact metric structure is integrable.*

As we stressed in Section 3, any (non-Sasakian) contact metric (κ,μ)-manifold (M,φ,ξ,η,g) carries a canonical bi-Legendrian structure $(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$, which in some sense completely characterizes the contact metric (κ,μ)-structure itself.

Definition 4.2. The paracontact metric structure $(\tilde\varphi,\xi,\eta,\tilde g):=\Psi(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$ is said to be the *canonical paracontact metric structure* of the (non-Sasakian) contact metric (κ,μ)-space (M,φ,ξ,η,g).

In this section we deal with the study of the canonical paracontact metric structure of a contact metric (κ,μ)-space. The first remark is that, since $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are involutive, $(\tilde\varphi,\xi,\eta,\tilde g)$ is integrable so that, by Theorem 4.1, the connection stated in Theorem 3.1 and the canonical paracontact connection of $(\tilde\varphi,\xi,\eta,\tilde g)$ coincide.

Now we show a more explicit expression for the canonical paracontact metric structure that will be useful in the sequel.

Theorem 4.3. *Let (M,φ,ξ,η,g) be a non-Sasakian contact metric (κ,μ)-space. Then the canonical paracontact metric structure $(\tilde\varphi,\xi,\eta,\tilde g)$ of M is given by*

$$\tilde\varphi:=\frac{1}{\sqrt{1-\kappa}}h,\quad \tilde g:=\frac{1}{\sqrt{1-\kappa}}d\eta(\cdot\,,h\cdot)+\eta\otimes\eta. \tag{4-2}$$

Proof. It is well known that in any contact metric (κ,μ)-manifold one has $h^2=(\kappa-1)\varphi^2$ [Blair et al. 1995]. From this relation it follows that the tensor field $\tilde\varphi:=(1/\sqrt{1-\kappa})h$ satisfies $\tilde\varphi^2=(1/(1-\kappa))h^2=-\varphi^2=I-\eta\otimes\xi$. Moreover, $\tilde\varphi$ induces an almost paracomplex structure on the subbundle $\mathcal{D}$, given by the n-dimensional distributions $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$. Thus $\tilde\varphi$ defines an almost paracontact structure on M. Next, we define a compatible metric $\tilde g$ by setting

$$\tilde g(X,Y):=d\eta(X,\tilde\varphi Y)+\eta(X)\eta(Y) \tag{4-3}$$

for all $X,Y\in\Gamma(TM)$. In fact, by using (2-2), we have, for any $X,Y\in\Gamma(TM)$,

$$\begin{aligned}\tilde g(Y,X)&=\frac{1}{\sqrt{1-\kappa}}d\eta(Y,hX)+\eta(Y)\eta(X)=\frac{1}{\sqrt{1-\kappa}}g(Y,\varphi hX)+\eta(Y)\eta(X)\\&=\frac{1}{\sqrt{1-\kappa}}g(X,\varphi hY)+\eta(X)\eta(Y)=d\eta(X,\tilde\varphi Y)+\eta(X)\eta(Y)=\tilde g(X,Y);\end{aligned}$$

thus $\tilde g$ defines a semi-Riemannian metric. Moreover, for all $X,Y\in\Gamma(TM)$, we have

$$\begin{aligned}g(\tilde\varphi X,\tilde\varphi Y)&=d\eta(\tilde\varphi X,Y-\eta(Y)\xi)+\eta(\tilde\varphi X)\eta(\tilde\varphi Y)=d\eta(\tilde\varphi X,Y)\\&=-\tilde g(X,Y)+\eta(X)\eta(Y),\\ g(X,\tilde\varphi Y)&=d\eta(X,\tilde\varphi^2Y)+\eta(X)\eta(\tilde\varphi Y)=d\eta(X,Y-\eta(Y)\xi)=d\eta(X,Y).\end{aligned}$$

So $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ is a paracontact metric structure. Finally, the paracontact metric structure defined by (4-2) coincides with the canonical paracontact metric structure of the contact metric (κ, μ)-space $(M, \varphi, \xi, \eta, g)$ as (4-1) shows. □

The next result relates the Levi-Civita connections of (M, g) and $(M, \tilde{g})$.

Proposition 4.4. *With the hypotheses and notation of Theorem 4.3, the Levi-Civita connections ∇ and $\widetilde{\nabla}$ of g and $\tilde{g}$ are related as*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \tfrac{1}{2}\mu\big(\eta(X)\varphi Y + \eta(Y)\varphi X\big) - \frac{1}{\sqrt{1-\kappa}}\big(\eta(X)hY + \eta(Y)hX\big) + \frac{1}{2}\Big(\frac{2-\mu}{\sqrt{1-\kappa}} g(hX, Y) - 2\sqrt{1-\kappa}\, g(\varphi^2 X, Y) - 2g(X, \varphi Y) + 2X(\eta(Y)) - \eta(\nabla_X Y)\Big)\xi.$$

Proof. By using Theorem 4.3 we get for each $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_X Y, Z) &= X(\tilde{g}(Y, Z)) + Y(\tilde{g}(X, Z)) - Z(\tilde{g}(X, Y)) + \tilde{g}([X, Y], Z) + \tilde{g}([Z, X], Y) - \tilde{g}([Y, Z], X) \\ &= \frac{1}{\sqrt{1-\kappa}}\big(X(g(Y, \varphi hZ)) + Y(g(X, \varphi hZ)) - Z(g(X, \varphi hY)) \\ &\quad + g([X, Y], \varphi hZ) + g([Z, X], \varphi hY)) - g([Y, Z], \varphi hX))\big) \\ &\quad + X(\eta(Y)\eta(Z)) + Y(\eta(X)\eta(Z)) - Z(\eta(X)\eta(Y)) \\ &\quad + \eta([X, Y])\eta(Z) + \eta([Z, X])\eta(Y) - \eta([Y, Z])\eta(X). \end{aligned}$$

Hence if we apply the symmetry of $\varphi \circ h$ and the parallelism of g with respect to ∇, we obtain

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_X Y, Z) &= \frac{1}{\sqrt{1-\kappa}}\big(2g(\varphi h \nabla_X Y, Z) + g(Y, (\nabla_X \varphi h)Z) + g(X, (\nabla_Y \varphi h)Z) - g(X, (\nabla_Z \varphi h)Y)\big) \\ &\quad + 2\big(d\eta(X, Z)\eta(Y) + d\eta(Y, Z)\eta(X) - d\eta(X, Y)\eta(Z) + X(\eta(Y))\eta(Z)\big), \end{aligned}$$

so that by using (2-10), after a long but straightforward calculation

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_X Y, Z) &= g\Big(\frac{1}{\sqrt{1-\kappa}}\big(2\varphi h(\nabla_X Y) + \mu\big(\eta(X)hY + \eta(Y)hX\big)\big) - 2\big(\eta(X)\varphi Y + \eta(Y)\varphi X\big), Z\Big) \\ &\quad + 2g\Big(\Big(\frac{2-\mu}{2\sqrt{1-\kappa}} g(hX, Y) - \sqrt{1-\kappa}\, g(\varphi^2 X, Y) - g(X, \varphi Y) + X(\eta(Y))\Big)\xi, Z\Big). \end{aligned}$$

It is easy to see that $\tilde{g}(\widetilde{\nabla}_X Y, \xi) = \eta(\widetilde{\nabla}_X Y)$ and then by the previous identity and Theorem 4.3 we get

(4-4)
$$\varphi h \widetilde{\nabla}_X Y = \varphi h \nabla_X Y + \tfrac{1}{2}\mu(\eta(X)hY + \eta(Y)hX) - \sqrt{1-\kappa}(\eta(X)\varphi Y + \eta(Y)\varphi X).$$

We finally apply φh to both the sides of (4-4), use $h\varphi = -\varphi h$, $h^2 = (\kappa - 1)\varphi^2$ and straightforwardly get the claimed relation. □

We now prove that the canonical paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ satisfies a suitable nullity condition.

Lemma 4.5. *For the canonical paracontact metric structure* $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ *from Theorem 4.3, we have*

(4-5)
$$\tilde{h} = \frac{1}{2\sqrt{1-k}}((2-\mu)\varphi \circ h + 2(1-\kappa)\varphi), \quad \tilde{h}^2 = (1 - \kappa - (1 - \tfrac{1}{2}\mu)^2)\varphi^2.$$

Proof. Using the identities $\nabla\xi = -\varphi - \varphi h$, $\nabla_\xi \varphi = 0$ and $\varphi^2 h = -h$, we get

$$\begin{aligned}
2\tilde{h} &= (\mathcal{L}_\xi(\mathcal{L}_\xi \varphi))X \\
&= [\xi, (\mathcal{L}_\xi \varphi)X] - (\mathcal{L}_\xi \varphi)[\xi, X] \\
&= [\xi, [\xi, \varphi X] - 2[\xi, \varphi[\xi, X]] + \varphi[\xi, [\xi, X]] \\
&= \nabla_\xi[\xi, \varphi X] + \varphi[\xi, \varphi X] + \varphi h[\xi, \varphi X] - 2\nabla_\xi \varphi[\xi, X] \\
&\qquad -2(\varphi^2[\xi, X] + \varphi h \varphi[\xi, X]) + \varphi \nabla_\xi[\xi, X] - \varphi(-\varphi[\xi, X] - \varphi h[\xi, X]) \\
&= \nabla_\xi \nabla_\xi \varphi X - \nabla_\xi(-\varphi^2 X - \varphi h \varphi X) + \varphi \nabla_\xi \varphi X - \varphi(-\varphi^2 X - \varphi h \varphi X) + \varphi h \nabla_\xi \varphi X \\
&\qquad -\varphi h(-\varphi^2 X - \varphi h \varphi X) - 2\nabla_\xi \varphi \nabla_\xi X + 2\nabla_\xi \varphi(-\varphi X - \varphi h X) - 2\varphi^2 \nabla_\xi X \\
&\qquad +2\varphi^2(-\varphi X - \varphi h X) + 2\varphi^2 h \nabla_\xi X - 2\varphi^2 h(-\varphi X - \varphi h X) + \varphi \nabla_\xi \nabla_\xi X \\
&\qquad -\varphi \nabla_\xi(-\varphi X - \varphi h X) + \varphi^2 \nabla_\xi X - \varphi^2(-\varphi X - \varphi h X) + \varphi^2 h \nabla_\xi X \\
&\qquad -\varphi^2 h(-\varphi X - \varphi h X) \\
&= \nabla_\xi \varphi^2 X + \nabla_\xi h X + \nabla_\xi \varphi^2 X - \varphi X - h\varphi X + h\nabla_\xi X - \varphi h X + h^2 \varphi X - 2\nabla_\xi \varphi^2 X \\
&\quad -2\nabla_\xi \varphi^2 h X - 2\varphi^2 \nabla_\xi X + 2\varphi X + 2\varphi h X - 2h\nabla_\xi X - 2h\varphi X + 2h^2 \varphi X + \varphi^2 \nabla_\xi X \\
&\qquad +\varphi^2 \nabla_\xi h X + \varphi^2 \nabla_\xi X - \varphi X - \varphi h X - h \nabla_\xi X - h\varphi X + h^2 \varphi X \\
&= 2(\nabla_\xi h)X + 4h^2 \varphi X - 4h\varphi X.
\end{aligned}$$

Now since $h^2 = (\kappa - 1)\varphi^2$ and $\nabla_\xi h = \mu h \varphi$ [Blair et al. 1995], we obtain the first identity in (4-5), while the second is a straightforward consequence. □

Lemma 4.6. *Let* $(M, \varphi, \xi, \eta, g)$ *be a contact metric* (κ, μ)*-manifold and suppose* $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ *is the canonical paracontact metric structure induced on* M*, according* *to Theorem 4.3. Then the Levi-Civita connection* $\widetilde{\nabla}$ *of* $(M,\tilde{g})$ *satisfies*

$$
\begin{aligned}
&(\widetilde{\nabla}_X\tilde{\varphi})Y = -\tilde{g}(X-\tilde{h}X, Y)\xi + \eta(Y)(X-\tilde{h}X),\\
&(\widetilde{\nabla}_X\tilde{h})Y = -\eta(Y)(\tilde{\varphi}\tilde{h}X - \tilde{\varphi}\tilde{h}^2X) - 2\eta(X)\tilde{\varphi}\tilde{h}Y - \tilde{g}(X,\tilde{\varphi}\tilde{h}Y+\tilde{\varphi}\tilde{h}^2Y)\xi,
\end{aligned}
\tag{4-6}
$$

for all $X, Y \in \Gamma(TM)$.

Proof. The first identity easily follows from the integrability of $(\tilde{\varphi},\xi,\eta,\tilde{g})$, taking Theorem 2.4 into account. To prove the second, let ∇^{bl} be the bi-Legendrian connection associated to the bi-Legendrian structure $(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$. Note that ∇^{bl} coincides with the canonical paracontact connection $\widetilde{\nabla}^{pc}$, so that, by using the first formula in (4-5) and since, by Theorem 3.1, $\nabla^{bl}h = \nabla^{bl}\varphi = 0$, we have

$$
\begin{aligned}
(\widetilde{\nabla}^{pc}_X\tilde{h})Y &= (\nabla^{bl}_X\tilde{h})Y\\
&= \frac{1}{2\sqrt{1-k}}((2-\mu)(\nabla^{bl}_X\varphi h)Y + 2(1-k)(\nabla^{bl}_X\varphi)Y)\\
&= \frac{2-\mu}{2\sqrt{1-k}}((\nabla^{bl}_X\varphi)hY + \varphi(\nabla^{bl}_X h)Y) + \frac{1-k}{\sqrt{1-k}}(\nabla^{bl}_X\varphi)Y = 0.
\end{aligned}
\tag{4-7}
$$

Now, by (2-14), (4-7) and the properties of the operator $\tilde{h}$,

$$
\begin{aligned}
(\widetilde{\nabla}_X\tilde{h})Y &= \widetilde{\nabla}_X\tilde{h}Y - \tilde{h}\widetilde{\nabla}_XY\\
&= (\widetilde{\nabla}^{pc}_X\tilde{h})Y - \eta(X)\tilde{\varphi}\tilde{h}Y - \eta(\tilde{h}Y)(\tilde{\varphi}X - \tilde{\varphi}\tilde{h}X) - \tilde{g}(X,\tilde{\varphi}\tilde{h}Y)\xi + \tilde{g}(\tilde{h}X,\tilde{\varphi}\tilde{h}Y)\xi\\
&\qquad + \eta(X)\tilde{h}\tilde{\varphi}Y + \eta(Y)(\tilde{h}\tilde{\varphi}X - \tilde{h}\tilde{\varphi}\tilde{h}X) + \tilde{g}(X,\tilde{\varphi}Y)\tilde{h}\xi - \tilde{g}(\tilde{h}X,\tilde{\varphi}Y)\tilde{h}\xi\\
&= -\eta(Y)(\tilde{\varphi}\tilde{h}X - \tilde{\varphi}\tilde{h}^2X) - 2\eta(X)\tilde{\varphi}\tilde{h}Y - \tilde{g}(X,\tilde{\varphi}\tilde{h}Y + \tilde{\varphi}\tilde{h}^2Y)\xi,
\end{aligned}
$$

as claimed. □

Theorem 4.7. *Let* (M,φ,ξ,η,g) *be a contact metric* (κ,μ)*-manifold and suppose* $(\tilde{\varphi},\xi,\eta,\tilde{g})$ *is the canonical paracontact metric structure induced on* M*. Then the curvature tensor field of the Levi-Civita connection of* $(M,\tilde{g})$ *satisfies*

$$
\tilde{R}_{XY}\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y)
$$

for all $X, Y \in \Gamma(TM)$, *where*

$$
\tilde{\kappa} = \kappa - 2 + (1-\mu/2)^2 \quad \text{and} \quad \tilde{\mu} = 2. \tag{4-8}
$$

Proof. First we prove the preliminary formula

$$
\begin{aligned}
\tilde{R}_{XY}\xi = -(\widetilde{\nabla}_X\tilde{\varphi})Y + (\widetilde{\nabla}_Y\tilde{\varphi})X + (\widetilde{\nabla}_X\tilde{\varphi})\tilde{h}Y\\
+ \tilde{\varphi}((\widetilde{\nabla}_X\tilde{h})Y) - (\widetilde{\nabla}_Y\tilde{\varphi})\tilde{h}X - \tilde{\varphi}((\widetilde{\nabla}_Y\tilde{h})X).
\end{aligned}
\tag{4-9}
$$

Indeed for all $X, Y \in \Gamma(TM)$, using the identity $\widetilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h}$, we get

$$\begin{aligned}
\tilde{R}_{XY}\xi &= \widetilde{\nabla}_X\widetilde{\nabla}_Y\xi - \widetilde{\nabla}_Y\widetilde{\nabla}_X\xi - \widetilde{\nabla}_{[X,Y]}\xi \\
&= -\widetilde{\nabla}_X\tilde{\varphi}Y + \widetilde{\nabla}_X\tilde{\varphi}\tilde{h}Y + \widetilde{\nabla}_Y\tilde{\varphi}X - \widetilde{\nabla}_Y\tilde{\varphi}\tilde{h}X + \tilde{\varphi}[X,Y] - \tilde{\varphi}\tilde{h}[X,Y] \\
&= -\widetilde{\nabla}_X\tilde{\varphi}Y + \widetilde{\nabla}_X\tilde{\varphi}\tilde{h}Y + \widetilde{\nabla}_Y\tilde{\varphi}X - \widetilde{\nabla}_Y\tilde{\varphi}\tilde{h}X + \tilde{\varphi}\widetilde{\nabla}_X Y \\
&\qquad - \tilde{\varphi}\widetilde{\nabla}_Y X - \tilde{\varphi}\tilde{h}\widetilde{\nabla}_X Y + \tilde{\varphi}\tilde{h}\widetilde{\nabla}_Y X \\
&= -(\widetilde{\nabla}_X\tilde{\varphi})Y + (\widetilde{\nabla}_Y\tilde{\varphi})X + \widetilde{\nabla}_X\tilde{\varphi}\tilde{h}Y - \tilde{\varphi}\widetilde{\nabla}_X\tilde{h}Y + \tilde{\varphi}\widetilde{\nabla}_X\tilde{h}Y - \widetilde{\nabla}_Y\tilde{\varphi}\tilde{h}X + \tilde{\varphi}\widetilde{\nabla}_Y\tilde{h}X \\
&\qquad - \tilde{\varphi}\widetilde{\nabla}_Y\tilde{h}X - \tilde{\varphi}\tilde{h}\widetilde{\nabla}_X Y + \tilde{\varphi}\tilde{h}\widetilde{\nabla}_Y X \\
&= -(\widetilde{\nabla}_X\tilde{\varphi})Y + (\widetilde{\nabla}_Y\tilde{\varphi})X + (\widetilde{\nabla}_X\tilde{\varphi})\tilde{h}Y + \tilde{\varphi}((\widetilde{\nabla}_X\tilde{h})Y) - (\widetilde{\nabla}_Y\tilde{\varphi})\tilde{h}X - \tilde{\varphi}((\widetilde{\nabla}_Y\tilde{h})X).
\end{aligned}$$

Therefore, replacing (4-6) in (4-9) and using the second formula in (4-5), we obtain

$$\begin{aligned}
\tilde{R}_{XY}\xi &= \tilde{g}(X - \tilde{h}X, Y)\xi - \eta(Y)(X - \tilde{h}X) - \tilde{g}(Y - \tilde{h}Y, X)\xi + \eta(X)(Y - \tilde{h}Y) \\
&\qquad - \tilde{g}(X - \tilde{h}X, \tilde{h}Y)\xi + \eta(\tilde{h}Y)(X - \tilde{h}X) - \eta(Y)(\tilde{\varphi}^2\tilde{h}X - \tilde{\varphi}^2\tilde{h}^2X) \\
&\qquad - 2\eta(X)\tilde{\varphi}^2\tilde{h}Y + \tilde{g}(Y - \tilde{h}Y, \tilde{h}X)\xi - \eta(\tilde{h}X)(Y - \tilde{h}Y) \\
&\qquad + \eta(X)(\tilde{\varphi}^2\tilde{h}Y - \tilde{\varphi}^2\tilde{h}^2Y) + 2\eta(Y)\tilde{\varphi}^2\tilde{h}X \\
&= \tilde{g}(X,Y)\xi - \tilde{g}(\tilde{h}X,Y)\xi - \eta(Y)X + \eta(Y)\tilde{h}X - \tilde{g}(Y,X)\xi + \tilde{g}(\tilde{h}Y,X)\xi \\
&\qquad + \eta(X)Y - \eta(X)\tilde{h}Y - \tilde{g}(X,\tilde{h}Y)\xi + \tilde{g}(\tilde{h}X,\tilde{h}Y)\xi - \eta(Y)\tilde{\varphi}^2\tilde{h}X \\
&\qquad + \eta(Y)\tilde{\varphi}^2\tilde{h}^2X - 2\eta(X)\tilde{\varphi}^2\tilde{h}Y + \tilde{g}(Y,\tilde{h}X)\xi - \tilde{g}(\tilde{h}Y,\tilde{h}X)\xi \\
&\qquad + \eta(X)\tilde{\varphi}^2\tilde{h}Y - \eta(X)\tilde{\varphi}^2\tilde{h}^2Y + 2\eta(Y)\tilde{\varphi}^2\tilde{h}X \\
&= -\eta(Y)X + \eta(Y)\tilde{h}X + \eta(X)Y - \eta(X)\tilde{h}Y - 2\eta(X)\tilde{h}Y - \eta(Y)\tilde{h}X + \eta(Y)\tilde{h}^2X \\
&\qquad + 2\eta(Y)\tilde{h}X + \eta(X)\tilde{h}Y - \eta(X)\tilde{h}^2Y \\
&= -\eta(Y)X + \eta(X)Y + \big(1 - \kappa - (1 - \mu/2)^2\big)\eta(Y)\varphi^2X \\
&\qquad - \big(1 - \kappa - (1 - \mu/2)^2\big)\eta(X)\varphi^2Y - 2\eta(X)\tilde{h}Y + 2\eta(Y)\tilde{h}X \\
&= \big(\kappa - 2 + (1 - \mu/2)^2\big)\big(\eta(Y)X - \eta(X)Y\big) + 2\big(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y\big). \qquad \square
\end{aligned}$$

Theorem 4.7 justifies the following definition. A paracontact metric manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a *paracontact metric* $(\tilde{\kappa}, \tilde{\mu})$*-manifold* if the curvature tensor field of the Levi-Civita connection satisfies

(4-10) $$\tilde{R}_{XY}\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y),$$

where $\tilde{\kappa}$, $\tilde{\mu}$ are real constants. Using (4-10) and the formula (see [Zamkovoy 2009])

(4-11) $$\tilde{R}_{\xi X}\xi + \tilde{\varphi}\tilde{R}_{\xi\tilde{\varphi}X}\xi = 2(\tilde{\varphi}^2X - \tilde{h}^2X),$$

one can easily prove that

(4-12) $$\tilde{h}^2 = (1 + \tilde{\kappa})\tilde{\varphi}^2.$$

For $\tilde{\kappa} = -1$, we get $\tilde{h}^2 = 0$ and now the analogy with contact metric (κ, μ)-manifolds breaks down because, since the metric $\tilde{g}$ is not positive definite, we cannot conclude that $\tilde{h} = 0$ and the manifold is para-Sasakian. Natural questions are whether there exist examples of paracontact metric manifolds such that $\tilde{h}^2 = 0$ but $\tilde{h} \neq 0$ and whether the $(\tilde{\kappa}, \tilde{\mu})$-nullity condition (4-10) could force the operator $\tilde{h}$ to vanish identically even if the metric $\tilde{g}$ is not positive definite. Also, though paracontact metric manifolds with $\tilde{h}^2 = 0$ have made their appearance in several contexts (see for instance [Zamkovoy 2009, Theorem 3.12]), to the knowledge of the authors not even one explicit example has been given. We now provide one.

Example 4.8. Let $\mathfrak{g}$ be the 5-dimensional Lie algebra with basis X_1, X_2, Y_1, Y_2, ξ and nonvanishing Lie brackets defined by

$$[X_1, X_2] = 2X_2, \qquad [X_1, Y_1] = 2\xi, \qquad [X_2, Y_1] = -2Y_2,$$
$$[X_2, Y_2] = 2(Y_1 + \xi), \qquad [\xi, X_1] = -2Y_1, \qquad [\xi, X_2] = -2Y_2.$$

Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. On G we define a left-invariant paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ by setting

$$\tilde{\varphi}\xi = 0, \quad \tilde{\varphi}X_i = X_i, \quad \tilde{\varphi}Y_i = -Y_i, \quad \eta(X_i) = \eta(Y_i) = 0, \quad \eta(\xi) = 1,$$

and

$$\tilde{g}(X_i, X_j) = \tilde{g}(Y_i, Y_j) = 0, \quad \tilde{g}(X_i, Y_i) = 1, \quad \tilde{g}(X_1, Y_2) = \tilde{g}(X_2, Y_1) = 0$$

for all $i, j \in \{1, 2\}$. Then a direct computation shows that $\tilde{h}^2$ vanishes identically, but $\tilde{h} \neq 0$ since, for example, $\tilde{h}X_1 = -Y_1$. Also, one can see that $(G, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-manifold, with $\tilde{\kappa} = -1$ and $\tilde{\mu} = 2$.

5. The canonical sequence of contact and paracontact metric structures associated with a contact metric (κ, μ)-space

In this section we will show that the procedure Theorem 4.3 used for defining the canonical paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ via the Lie derivative of φ can be iterated. Indeed, Lemma 4.5 suggests that the Lie derivative of $\tilde{\varphi}$ in the direction ξ could define a compatible almost contact or paracontact structure on (M, η) provided that the coefficient $1 - \kappa - (1 - \mu/2)^2$, which directly brings up the invariant I_M, is positive or negative, respectively. Furthermore, we show that this algorithm can also be applied to the new contact and paracontact structures, so that one can attach to M a canonical sequence of contact and paracontact metric structures; this sequence strictly depends on the invariant I_M and hence on the class of M according to the classification recalled in Section 3. We start by proving the following fundamental result.

Theorem 5.1. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ)-manifold and suppose $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ is the canonical paracontact metric structure of M. Then*

(i) *if $|I_M| < 1$, the paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ induces on (M, η) a canonical compatible contact metric (κ_1, μ_1)-structure $(\varphi_1, \xi, \eta, g_1)$, where*

$$\kappa_1 = \kappa + (1 - \tfrac{1}{2}\mu)^2, \quad \mu_1 = 2; \tag{5-1}$$

(ii) *if $|I_M| > 1$, the paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ induces on (M, η) a canonical compatible paracontact metric $(\tilde{\kappa}_1, \tilde{\mu}_1)$-structure $(\tilde{\varphi}_1, \xi, \eta, \tilde{g}_1)$, where*

$$\tilde{\kappa}_1 = \kappa - 2 + (1 - \tfrac{1}{2}\mu)^2, \quad \tilde{\mu}_1 = 2. \tag{5-2}$$

Proof. (i) Assume that $|I_M| < 1$. By Lemma 4.5, $\tilde{h}^2$ is proportional to φ^2 and the constant of proportionality $-(2-\mu)^2 + 4(1-\kappa)$ is positive since we are assuming that $|I_M| < 1$. Then we set

$$\begin{aligned} \varphi_1 &:= \frac{1}{\sqrt{1-\kappa-(1-\mu/2)^2}}\, \tilde{h} \\ &= \frac{1}{2\sqrt{(1-\kappa)(1-\kappa-(1-\mu/2)^2)}}((2-\mu)\varphi \circ h + 2(1-\kappa)\varphi). \end{aligned} \tag{5-3}$$

Due to (4-5) we have $\varphi_1^2 = \varphi^2 = -I + \eta \otimes \xi$; hence (φ_1, ξ, η) is an almost contact structure on M. We now look for a compatible Riemannian metric g_1 such that $d\eta = g_1(\cdot, \varphi_1 \cdot)$. Thus we set

$$g_1(X, Y) := -d\eta(X, \varphi_1 Y) + \eta(X)\eta(Y). \tag{5-4}$$

We first need to prove that g_1 is a Riemannian metric. For any $X, Y \in \Gamma(TM)$, using the symmetry of the operator $\tilde{h}$ with respect to $\tilde{g}$, we have

$$\begin{aligned} g_1(Y, X) &= -\frac{1}{\sqrt{1-\kappa-(1-\mu/2)^2}}\, d\eta(Y, \tilde{h}X) + \eta(Y)\eta(X) \\ &= -\frac{1}{\sqrt{1-\kappa-(1-\mu/2)^2}}\, \tilde{g}(Y, \tilde{\varphi}\tilde{h}X) + \eta(Y)\eta(X) \\ &= -\frac{1}{\sqrt{1-\kappa-(1-\mu/2)^2}}\, \tilde{g}(X, \tilde{\varphi}\tilde{h}Y) + \eta(X)\eta(Y) \\ &= -d\eta(X, \varphi_1 Y) + \eta(X)\eta(Y) \\ &= g_1(X, Y), \end{aligned}$$

so that g_1 is a symmetric tensor. Furthermore, directly by (5-4),

$$d\eta(X, Y) = g_1(X, \varphi_1 Y) \quad \text{and} \quad g_1(\varphi_1 X, \varphi_1 Y) = g_1(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(TM)$. Now we look for conditions ensuring the positive definiteness of g_1. Let X be a nonzero vector field on M and put

$$\alpha := \frac{1}{2\sqrt{(1-\kappa)(1-\kappa-(1-\mu/2)^2)}}.$$

Since $g(\xi,\xi) = \eta(\xi)\eta(\xi) = 1 > 0$ we can assume that $X \in \Gamma(\mathcal{D})$. Then by (5-3) and (5-4),

$$\begin{aligned} g_1(X,X) &= -\alpha(2-\mu)d\eta(X,\varphi hX) - 2\alpha(1-\kappa)d\eta(X,\varphi X) \\ &= \alpha(2-\mu)g(X,hX) + 2\alpha(1-\kappa)g(X,X) \\ &= \alpha(2-\mu)g(X_\lambda + X_{-\lambda}, h(X_\lambda + X_{-\lambda})) \\ &\qquad\qquad + 2\alpha(1-\kappa)g(X_\lambda + X_{-\lambda}, X_\lambda + X_{-\lambda}) \\ &= \alpha(2-\mu)g(X_\lambda + X_{-\lambda}, \lambda X_\lambda - \lambda X_{-\lambda}) \\ &\qquad\qquad + 2\alpha(1-\kappa)g(X_\lambda + X_{-\lambda}, X_\lambda + X_{-\lambda}) \\ &= \alpha\lambda(2\lambda-\mu+2)g(X_\lambda, X_\lambda) + \alpha\lambda(2\lambda+\mu-2)g(X_{-\lambda}, X_{-\lambda}), \end{aligned} \tag{5-5}$$

where we have decomposed the vector field $X \in \Gamma(\mathcal{D})$ into its components along $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$, and $\lambda = \sqrt{1-\kappa}$. Thus g_1 is a Riemannian metric provided that $2\lambda - \mu + 2 > 0$ and $2\lambda + \mu - 2 > 0$. In view of (3-1), the conditions above are just equivalent to the positive definiteness of the Legendre foliation $\mathcal{D}(\lambda)$ and to the negative definiteness of $\mathcal{D}(-\lambda)$, and hence to the requirement that $|I_M| < 1$. Thus, as we are assuming that $|I_M| < 1$, we conclude that g_1 is a Riemannian metric. We now prove that $(\varphi_1, \xi, \eta, g_1)$ is a contact metric (κ_1, μ_1)-structure, for some constants κ_1 and μ_1 to be found. For this purpose we first find a more explicit expression of the tensor field $h_1 := (1/2)\mathcal{L}_\xi\varphi_1$. As before, set $\alpha := 1/(2\sqrt{(1-\kappa)(1-\kappa-(1-\mu/2)^2)})$. Then by (4-2) and (4-5), we have

$$\begin{aligned} h_1 &= \tfrac{1}{2}\alpha((2-\mu)((\mathcal{L}_\xi\varphi)\circ h + \varphi\circ(\mathcal{L}_\xi h)) + 2(1-\kappa)\mathcal{L}_\xi\varphi) \\ &= \tfrac{1}{2}\alpha((2-\mu)(2h^2 + (2-\mu)\varphi^2\circ h + 2(1-\kappa)\varphi^2) + 4(1-\kappa)h) \\ &= \tfrac{1}{2}\alpha(-(2-\mu)^2 + 4(1-\kappa))h \\ &= h\sqrt{1-{I_M}^2}. \end{aligned}$$

Thus h_1 is proportional to h and hence has the eigenvalues λ_1 and $-\lambda_1$, where $\lambda_1 := \sqrt{(1-\kappa)(1-{I_M}^2)} = 1-\kappa-(1-\mu/2)^2$, and the corresponding eigendistributions coincide with the those of the operator h. Then the bi-Legendrian connection associated with $(\mathcal{D}(-\lambda_1), \mathcal{D}(\lambda_1))$ coincides with the bi-Legendrian connection ∇^{bl} associated with the bi-Legendrian structure $(\mathcal{D}(-\lambda), \mathcal{D}(\lambda))$ induced by h. We prove that ∇^{bl} preserves the tensor fields φ_1. Indeed for all $X, Y \in \Gamma(TM)$

$$(\nabla^{bl}_X\varphi_1)Y = \alpha(2-\mu)((\nabla^{bl}_X\varphi)hY + \varphi(\nabla^{bl}_X h)Y) + 2\alpha(1-\kappa)(\nabla^{bl}_X\varphi)Y = 0$$

since $\nabla^{bl}\varphi = 0$ and $\nabla^{bl}h = 0$. Moreover, as $\nabla^{bl}\varphi_1 = 0$ and $\nabla^{bl}d\eta = 0$, also $\nabla^{bl}g_1 = 0$. Therefore, since obviously also $\nabla^{bl}h_1 = 0$, ∇^{bl} satisfies all the conditions of Theorem 3.1 and we can conclude that $(\varphi_1, \xi, \eta, g_1)$ is a contact metric (κ_1, μ_1)-structure. In order to find the expression of κ_1 and μ_1, we observe that immediately $\kappa_1 = 1-\lambda_1^2 = \kappa + (1-\kappa){I_M}^2 = \kappa + (1-\mu/2)^2$. Then applying the first of (3-1) and $\Pi_{\mathscr{D}(\lambda)} = \Pi_{\mathscr{D}(\lambda_1)}$, we have, for any nonzero $X \in \Gamma(\mathscr{D}(\lambda))$,

$$(2\sqrt{1-\kappa}-\mu+2)g(X,X) = (2\sqrt{1-\kappa_1}-\mu_1+2)g_1(X,X).$$

Using (5-5) we get $2\sqrt{1-\kappa_1}-\mu_1+2 = \sqrt{-(2-\mu)^2+4(1-\kappa)}$, so that

$$\mu_1 = 2\sqrt{1-\kappa-(1-\mu/2)^2}+2-\sqrt{-(2-\mu)^2+4(1-\kappa)} = 2.$$

(ii) Assume that $|I_M| > 1$. Then we define

$$\tag{5-6} \begin{aligned}\tilde{\varphi}_1 &:= \frac{1}{\sqrt{(1-\mu/2)^2-(1-\kappa)}}\,\tilde{h}\\ &= \frac{1}{2\sqrt{(1-\kappa)((1-\mu/2)^2-(1-\kappa))}}((2-\mu)\varphi\circ h + 2(1-\kappa)\varphi).\end{aligned}$$

Using (4-5) and the assumption $|I_M| > 1$, one easily proves that $\tilde{\varphi}_1^2 = I - \eta\otimes\xi$, so that to conclude that $(\tilde{\varphi}_1, \xi, \eta)$ defines an almost paracontact structure we need only to prove that the eigendistributions corresponding to the eigenvalues 1 and -1 of $\tilde{\varphi}_1|_{\mathscr{D}}$ have equal dimension n. Though $\tilde{h}$ is a symmetric operator (with respect to $\tilde{g}$) it could be not necessarily diagonalizable, since $\tilde{g}$ is not positive definite. Nevertheless we now show that this is the case. Let $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ be a local orthonormal φ-basis of eigenvectors of h, that is, for $i \in \{1, \dots, n\}$,

$$X_i = -\varphi Y_i, \quad Y_i = \varphi X_i, \quad hX_i = \lambda X_i, \quad hY_i = -\lambda Y_i.$$

Then, by (4-5), for each $i \in \{1, \dots, n\}$,

$$\begin{aligned}\tilde{h}X_i &= \frac{1}{2\sqrt{1-\kappa}}((2-\mu)\varphi h X_i + 2(1-\kappa)\varphi X_i)\\ &= \frac{1}{2\sqrt{1-\kappa}}((2-\mu)\lambda Y_i + 2(1-\kappa)Y_i)\\ &= \left(1-\tfrac{1}{2}\mu+\sqrt{1-\kappa}\right)Y_i\end{aligned}$$

and, analogously, one finds $\tilde{h}Y_i = (1-\mu/2-\sqrt{1-\kappa})X_i$. Hence $\tilde{h}$ is represented with respect to the basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ by the matrix

$$\begin{pmatrix} 0_n & (1-\mu/2-\sqrt{1-\kappa})I_n & \mathbf{0}_{n1}\\ (1-\mu/2+\sqrt{1-\kappa})I_n & \mathbf{0}_n & \mathbf{0}_{n1}\\ \mathbf{0}_{1n} & \mathbf{0}_{1n} & 0 \end{pmatrix},$$

where $\mathbf{0}_n$, $\mathbf{0}_{n1}$ and $\mathbf{0}_{1n}$ denote, respectively, the $n\times n$, $n\times 1$ and $1\times n$ matrices whose entries are all 0, and I_n the identity matrix of order n. Therefore the characteristic polynomial is given by

$$\begin{aligned} p &= -\lambda\big(\lambda^2-(1-\tfrac{1}{2}\mu+\sqrt{1-\kappa})(1-\tfrac{1}{2}\mu-\sqrt{1-\kappa})\big)^n \\ &= -\lambda\big(\lambda^2-((1-\tfrac{1}{2}\mu)^2-(1-\kappa))\big)^n. \end{aligned}$$

Because of the assumption $|I_M|>1$, the number $(1-\mu/2)^2-(1-\kappa)$ is positive, so that the operator $\tilde{h}$ admits, apart from the eigenvalue 0 corresponding to the eigenvector ξ, also the eigenvalues $\tilde{\lambda}$ and $-\tilde{\lambda}$, where $\tilde{\lambda}:=\sqrt{(1-\mu/2)^2-(1-\kappa)}$. An easy computation shows that the corresponding eigendistributions are, respectively,

$$\begin{aligned} \mathcal{D}(\tilde{\lambda}) &= \operatorname{span}\left\{\sqrt{\frac{I_M-1}{I_M+1}}X_1+Y_1,\dots,\sqrt{\frac{I_M-1}{I_M+1}}X_n+Y_n\right\}, \\ \mathcal{D}(-\tilde{\lambda}) &= \operatorname{span}\left\{-\sqrt{\frac{I_M-1}{I_M+1}}X_1+Y_1,\dots,-\sqrt{\frac{I_M-1}{I_M+1}}X_n+Y_n\right\}. \end{aligned} \tag{5-7}$$

Therefore each eigendistribution $\mathcal{D}(\tilde{\lambda})$ and $\mathcal{D}(-\tilde{\lambda})$ has dimension n, and finally this implies that the eigendistributions of the operator $\tilde{\varphi}_1$ restricted to $\mathcal{D}$ are n-dimensional. Thus $(\tilde{\varphi}_1,\xi,\eta)$ is an almost paracontact structure. Next we define a compatible semi-Riemannian metric by putting, for any $X,Y\in\Gamma(TM)$,

$$\tilde{g}_1(X,Y):=d\eta(X,\tilde{\varphi}_1Y)+\eta(X)\eta(Y). \tag{5-8}$$

That $\tilde{g}_1$ is symmetric can be easily proved. Moreover, directly from (5-8) one can show that $\tilde{g}_1(\tilde{\varphi}_1X,\tilde{\varphi}_1Y)=-\tilde{g}_1(X,Y)+\eta(X)\eta(Y)$ and $d\eta(X,Y)=\tilde{g}_1(X,\tilde{\varphi}_1Y)$ for all $X,Y\in\Gamma(TM)$. Therefore $(\tilde{\varphi}_1,\xi,\eta,\tilde{g}_1)$ is a paracontact metric structure on M. Also, arguing as in the previous case, one can find that

$$\begin{aligned} \tilde{h}_1 &= \frac{1}{4\sqrt{(1-\kappa)((1-\mu/2)^2-4(1-\kappa))}}((2-\mu)\mathcal{L}_\xi(\varphi\circ h)+2(1-\kappa)\mathcal{L}_\xi\varphi) \\ &= (-\sqrt{I_M{}^2-1}\,)h. \end{aligned}$$

It remains to show that $(M,\tilde{\varphi}_1,\xi,\eta,\tilde{g}_1)$ satisfies a $(\tilde{\kappa}_1,\tilde{\mu}_1)$-nullity condition for some constants $\tilde{\kappa}_1$ and $\tilde{\mu}_1$. For this purpose we find the relationship between the Levi-Civita connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^1$ of $\tilde{g}$ and $\tilde{g}_1$, respectively. Notice that, by (5-8),

$$\begin{aligned} \tilde{g}_1(X,Y) &= \frac{1}{\sqrt{(1-\mu/2)^2-(1-\kappa)}}d\eta(X,\tilde{h}Y)+\eta(X)\eta(Y) \\ &= \beta\tilde{g}(X,\tilde{\varphi}\tilde{h}Y)+\eta(X)\eta(Y), \end{aligned} \tag{5-9}$$

where we put $\beta := 1/\sqrt{(1-\mu/2)^2-(1-\kappa)}$. Then, arguing as in Proposition 4.4, we have, for all $X, Y, Z \in \Gamma(TM)$,

$$
\begin{aligned}
&2\tilde{g}_1(\widetilde{\nabla}^1_X Y, Z)\\
&\quad = \beta\bigl(2\tilde{g}(\tilde{\varphi}\tilde{h}\widetilde{\nabla}_X Y, Z) + \tilde{g}(Y, (\widetilde{\nabla}_X\tilde{\varphi}\tilde{h})Z) + \tilde{g}(X, (\widetilde{\nabla}_Y\tilde{\varphi}\tilde{h})Z) - \tilde{g}(X, (\widetilde{\nabla}_Z\tilde{\varphi}\tilde{h})Y)\bigr)\\
&\qquad\qquad +2(d\eta(X,Z)\eta(Y) + d\eta(Y,Z)\eta(X) - d\eta(X,Y)\eta(Z) + X(\eta(Y))\eta(Z)).
\end{aligned}
$$

Using (4-6) and the identity $(\widetilde{\nabla}_X\tilde{\varphi}\tilde{h})Y = (\widetilde{\nabla}_X\tilde{\varphi})\tilde{h}Y + \tilde{\varphi}((\widetilde{\nabla}_X\tilde{h})Y)$, the previous relation becomes

$$
\begin{aligned}
\text{(5-10)}\quad &2\tilde{g}_1(\widetilde{\nabla}^1_X Y, Z)\\
&= \beta\bigl(2\tilde{g}(\tilde{\varphi}\tilde{h}\widetilde{\nabla}_X Y, Z) - \eta(Y)\tilde{g}(X, \tilde{h}Z) + \eta(Y)\tilde{g}(\tilde{h}X, \tilde{h}Z)\\
&\qquad - 2\eta(X)\tilde{g}(Y, \tilde{\varphi}^2\tilde{h}Z) - \eta(Z)\tilde{g}(Y, \tilde{\varphi}^2\tilde{h}X) + \eta(Z)\tilde{g}(Y, \tilde{\varphi}^2\tilde{h}^2X)\\
&\qquad - \eta(X)\tilde{g}(Y, \tilde{h}Z) + \eta(X)\tilde{g}(\tilde{h}Y, \tilde{h}Z) - 2\eta(Y)\tilde{g}(X, \tilde{\varphi}^2\tilde{h}Z)\\
&\qquad - \eta(Z)\tilde{g}(X, \tilde{\varphi}^2\tilde{h}Y) + \eta(Z)\tilde{g}(X, \tilde{\varphi}^2\tilde{h}^2Y) + \eta(X)\tilde{g}(Z, \tilde{h}Y)\\
&\qquad - \eta(X)\tilde{g}(\tilde{h}Z, \tilde{h}Y) + 2\eta(Z)\tilde{g}(X, \tilde{\varphi}^2\tilde{h}Y)\\
&\qquad\qquad + \eta(Y)\tilde{g}(X, \tilde{\varphi}^2\tilde{h}Z) - \eta(Y)\tilde{g}(X, \tilde{\varphi}^2\tilde{h}^2Z)\bigr)\\
&\quad + 2(d\eta(X,Z)\eta(Y) + d\eta(Y,Z)\eta(X) - d\eta(X,Y)\eta(Z) + X(\eta(Y))\eta(Z)).
\end{aligned}
$$

Notice that, by (4-8) and (4-12), $\tilde{h}^2 = (1+\tilde{\kappa})\tilde{\varphi}^2 = (\kappa-1+(1-\mu/2)^2)\tilde{\varphi}^2 = (1/\beta^2)\tilde{\varphi}^2$. Substituting this relation in (5-10) and taking the symmetry of the operator $\tilde{h}$ with respect to the semi-Riemannian metric $\tilde{g}$ into account, we get

$$
\begin{aligned}
2\tilde{g}_1(\widetilde{\nabla}^1_X Y, Z) &= \beta\Bigl(2\tilde{g}(\tilde{\varphi}\tilde{h}\widetilde{\nabla}_X Y, Z) - 2\eta(X)\tilde{g}(\tilde{h}Y, Z)\\
&\qquad + \frac{2}{\beta^2}\tilde{g}(X,Y)\eta(Z) - \frac{2}{\beta^2}\eta(X)\eta(Y)\eta(Z) - 2\eta(Y)\tilde{g}(\tilde{h}X, Z)\Bigr)\\
&\quad + 2\bigl(d\eta(X,Z)\eta(Y) + d\eta(Y,Z)\eta(X) - d\eta(X,Y)\eta(Z) + X(\eta(Y))\eta(Z)\bigr),
\end{aligned}
$$

that is, by definition of $\tilde{g}_1$,

$$
\begin{aligned}
\text{(5-11)}\quad &2(c\beta\tilde{g}(\widetilde{\nabla}^1_X Y, \tilde{\varphi}\tilde{h}Z) + \eta(\widetilde{\nabla}^1_X Y)\tilde{g}(\xi, Z))\\
&= \beta\Bigl(2\tilde{g}(\tilde{\varphi}\tilde{h}\widetilde{\nabla}_X Y, Z) - 2\eta(X)\tilde{g}(\tilde{h}Y, Z)\\
&\qquad + \frac{2}{\beta^2}\tilde{g}(X,Y)\tilde{g}(\xi, Z) - \frac{2}{\beta^2}\eta(X)\eta(Y)\tilde{g}(\xi, Z) - 2\eta(Y)\tilde{g}(\tilde{h}X, Z)\Bigr)\\
&\quad + 2\bigl(-\eta(Y)\tilde{g}(\tilde{\varphi}X, Z) - \eta(X)\tilde{g}(\tilde{\varphi}Y, Z) - \tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(\xi, Z) + X(\eta(Y))\tilde{g}(\xi, Z)\bigr).
\end{aligned}
$$

Therefore, since Z was chosen arbitrarily, we get

$$
\begin{aligned}
\text{(5-12)}\quad \beta\tilde{\varphi}\tilde{h}\widetilde{\nabla}^1_X Y + \eta(\widetilde{\nabla}^1_X Y)\xi &= \beta\tilde{\varphi}\tilde{h}\widetilde{\nabla}_X Y - \beta\eta(X)\tilde{h}Y\\
&\quad + \beta^{-1}\tilde{g}(X,Y)\xi - \beta^{-1}\eta(X)\eta(Y)\xi - \beta\eta(Y)\tilde{h}X\\
&\quad - \eta(Y)\tilde{\varphi}X - \eta(X)\tilde{\varphi}Y - \tilde{g}(X, \tilde{\varphi}Y)\xi + X(\eta(Y))\xi.
\end{aligned}
$$

Note that, since $\tilde\varphi_1=\beta\tilde h$, $\tilde h_1=-\beta^{-1}\tilde\varphi$ and $\tilde h^2=\beta^{-2}\tilde\varphi^2$,

$$
\begin{aligned}
\eta(\widetilde\nabla^1_X Y)&=\tilde g_1(\widetilde\nabla^1_X Y,\xi)\\
&=X(\tilde g_1(Y,\xi))-\tilde g_1(Y,\widetilde\nabla^1_X\xi)\\
&=X(\eta(Y))-\tilde g_1(Y,-\tilde\varphi_1X+\tilde\varphi_1\tilde h_1X)\\
&=X(\eta(Y))+d\eta(Y,X)-\tilde g_1(Y,\tilde\varphi\tilde hX)\\
&=X(\eta(Y))-\tilde g(X,\tilde\varphi Y)-\beta\tilde g(Y,\tilde\varphi\tilde h\tilde\varphi\tilde hX)\\
&=X(\eta(Y))-\tilde g(X,\tilde\varphi Y)+\beta^{-1}\tilde g(X,Y)-\beta^{-1}\eta(X)\eta(Y).
\end{aligned}\tag{5-13}
$$

Consequently, (5-12) becomes

$$
\tilde h\widetilde\nabla^1_X Y=\tilde h\widetilde\nabla_X Y-\eta(X)\tilde\varphi\tilde hY-\eta(Y)\tilde\varphi\tilde hX-\beta^{-1}\eta(Y)\tilde\varphi^2X-\beta^{-1}\eta(X)\tilde\varphi^2Y.
$$

Applying $\tilde h$ we obtain

$$
\begin{aligned}
&\widetilde\nabla^1_X Y-\eta(\widetilde\nabla^1_X Y)\xi\\
&\quad=\widetilde\nabla_X Y-\eta(\widetilde\nabla_X Y)\xi+\eta(X)\tilde\varphi Y+\eta(Y)\tilde\varphi X-\beta\eta(Y)\tilde hX-\beta\eta(X)\tilde hY.
\end{aligned}\tag{5-14}
$$

Now, a straightforward computation as in (5-13) shows that

$$
\eta(\widetilde\nabla_X Y)=X(\eta(Y))-\tilde g(X,\tilde\varphi Y)-\tilde g(X,\tilde\varphi\tilde hY).\tag{5-15}
$$

Therefore, by replacing (5-13) and (5-15) in (5-14) and recalling that we have set $\beta=1/\sqrt{(1-\mu/2)^2-(1-\kappa)}$, we finally find

$$
\begin{aligned}
\widetilde\nabla^1_X Y=\widetilde\nabla_X Y&+\eta(X)\biggl(\tilde\varphi Y-\frac{\tilde hY}{\sqrt{(1-\mu/2)^2-(1-\kappa)}}\biggr)\\
&+\eta(Y)\biggl(\tilde\varphi X-\frac{\tilde hX}{\sqrt{(1-\mu/2)^2-(1-\kappa)}}\biggr)\\
&+\bigl(\sqrt{(1-\mu/2)^2-(1-\kappa)}\bigl(\tilde g(X,Y)-\eta(X)\eta(Y)\bigr)+\tilde g(X,\tilde\varphi\tilde hY)\bigr)\xi.
\end{aligned}\tag{5-16}
$$

The explicit expression (5-16) of the Levi-Civita connection of $\tilde g_1$ in terms that of $\tilde g$ allows us to prove that $(M,\tilde\varphi_1,\xi,\eta,\tilde g_1)$ is a paracontact metric $(\tilde\kappa_1,\tilde\mu_1)$-manifold for some $\tilde\kappa_1,\tilde\mu_1\in\mathbb{R}$. Indeed, from (5-16), after some long but straightforward computations, we obtain

$$
\begin{aligned}
&(\widetilde\nabla^1_X\tilde\varphi_1)Y\\
&\quad=\biggl(-\frac{1}{\sqrt{(1-\mu/2)^2-(1-\kappa)}}\tilde g(X,\tilde\varphi\tilde hY)-\eta(X)\eta(Y)+\tilde g(X,\tilde hY)\biggr)\xi\\
&\qquad\qquad+\eta(Y)\bigl(X+\sqrt{(1-\mu/2)^2-(1-\kappa)}\tilde\varphi X\bigr)\\
&\quad=-\tilde g_1(X-\tilde h_1X,Y)\xi+\eta(Y)(X-\tilde h_1X),
\end{aligned}\tag{5-17}
$$

and

$$
\begin{aligned}
\text{(5-18)}\quad & (\widetilde{\nabla}^1_X \tilde{h}_1)Y \\
&= \sqrt{(1-\mu/2)^2-(1-\kappa)}\,\eta(Y)\tilde{h}X - 2\eta(X)\tilde{\varphi}\tilde{h}Y - \eta(Y)\tilde{\varphi}\tilde{h}X \\
&\qquad + \sqrt{(1-\mu/2)^2-(1-\kappa)}\big(\tilde{g}(X,Y)-\eta(X)\eta(Y) \\
&\qquad\qquad - \sqrt{(1-\mu/2)^2-(1-\kappa)}\,\tilde{g}(X,\tilde{\varphi}Y)\big)\xi \\
&= -\eta(Y)(\tilde{\varphi}_1\tilde{h}_1X - \tilde{\varphi}_1\tilde{h}_1^2X) - 2\eta(X)\tilde{\varphi}_1\tilde{h}_1Y - \tilde{g}_1(X,\tilde{\varphi}_1\tilde{h}_1Y + \tilde{\varphi}_1\tilde{h}_1^2Y)\xi.
\end{aligned}
$$

Then by (4-9), (5-17), (5-18) and $\tilde{h}_1^2 = ((1-\mu/2)^2-(1-\kappa))\tilde{\varphi}^2$, we get

$$
\begin{aligned}
\tilde{R}^1_{XY}\xi &= -(\widetilde{\nabla}^1_X\tilde{\varphi}_1)Y + (\widetilde{\nabla}^1_Y\tilde{\varphi}_1)X + (\widetilde{\nabla}^1_X\tilde{\varphi}_1)\tilde{h}Y \\
&\qquad + \tilde{\varphi}_1((\widetilde{\nabla}^1_X\tilde{h})Y) - (\widetilde{\nabla}^1_Y\tilde{\varphi}_1)\tilde{h}_1X - \tilde{\varphi}_1((\widetilde{\nabla}^1_Y\tilde{h}_1)X) \\
&= -\eta(Y)X + \eta(X)Y + \eta(Y)\tilde{h}_1^2X - \eta(X)\tilde{h}_1^2Y - 2\eta(X)\tilde{h}_1Y + 2\eta(Y)\tilde{h}_1X \\
&= -\eta(Y)X + \eta(X)Y + \big((1-\tfrac{1}{2}\mu)^2-(1-\kappa)\big)\big(\eta(Y)\tilde{\varphi}^2X - \eta(X)\tilde{\varphi}^2Y\big) \\
&\qquad - 2\eta(X)\tilde{h}_1Y + 2\eta(Y)\tilde{h}_1X \\
&= \big(\kappa-2+(1-\tfrac{1}{2}\mu)^2\big)\big(\eta(Y)X-\eta(X)Y\big) + 2\big(\eta(Y)\tilde{h}_1X - \eta(X)\tilde{h}_1Y\big).
\end{aligned}
$$

Thus $(\tilde{\varphi}_1, \xi, \eta, \tilde{g}_1)$ is paracontact metric $(\tilde{\kappa}_1, \tilde{\mu}_1)$-structure with

$$\tilde{\kappa}_1 = \kappa - 2 + (1-\tfrac{1}{2}\mu)^2 \quad\text{and}\quad \tilde{\mu}_1 = 2. \qquad \square$$

A *Tanaka–Webster parallel space*, introduced by Boeckx and Cho [2008], is a contact metric manifold whose generalized Tanaka–Webster torsion $\hat{T}$ and curvature $\hat{R}$ satisfy $\hat{\nabla}\hat{T} = 0$ and $\hat{\nabla}\hat{R} = 0$, that is, $\hat{\nabla}$ is invariant by parallelism (in the sense of [Kobayashi and Nomizu 1963]). Boeckx and Cho [2008, Theorem 12] proved that a contact metric manifold M is a Tanaka–Webster parallel space if and only if M is a Sasakian locally φ-symmetric space or a non-Sasakian $(\kappa, 2)$-space. Thus, we deduce that the contact metric (κ_1, μ_1)-structure $(\varphi_1, \xi, \eta, g_1)$ in Theorem 5.1(i) is in fact a Tanaka–Webster parallel structure.

Corollary 5.2. *Every non-Sasakian contact metric (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ such that $|I_M| < 1$ admits a compatible Tanaka–Webster parallel structure.*

Remark 5.3. In proving Theorem 5.1 we have proved that, even if the metric $\tilde{g}$ is not positive definite, in the case $|I_M| > 1$, the operator $\tilde{h}$ is diagonalizable and has an eigenvalue 0 of multiplicity 1 and eigenvalues $\tilde{\lambda}$ and $-\tilde{\lambda}$, where $\tilde{\lambda} = \sqrt{(1-\mu/2)^2-(1-\kappa)}$, both of multiplicity n. The eigendistributions $\mathfrak{D}(\tilde{\lambda})$ and $\mathfrak{D}(-\tilde{\lambda})$ are expressed in terms of a local φ-basis of eigenvectors of h by the relations (5-7). We now show that $\mathfrak{D}(\tilde{\lambda})$ and $\mathfrak{D}(-\tilde{\lambda})$ are in fact Legendre foliations. Indeed, for any $X, X' \in \Gamma(\mathcal{D}(\tilde{\lambda}))$ we have

$$\tilde{g}(X, \tilde{\varphi}X') = \frac{1}{\tilde{\lambda}}\tilde{g}(X, \tilde{\varphi}\tilde{h}X') = -\frac{1}{\tilde{\lambda}}\tilde{g}(X, \tilde{h}\tilde{\varphi}X') = -\frac{1}{\tilde{\lambda}}\tilde{g}(\tilde{h}X, \tilde{\varphi}X') = -\tilde{g}(X, \tilde{\varphi}X'),$$

so that $\tilde{g}(X, \tilde{\varphi}X') = 0$ and consequently $d\eta(X, X') = 0$. Analogously, $d\eta(Y, Y') = 0$ for any $Y, Y' \in \Gamma(\mathcal{D}(-\tilde{\lambda}))$. This proves that $\mathcal{D}(\tilde{\lambda})$ and $\mathcal{D}(-\tilde{\lambda})$ are Legendre distributions. Now, observe that the almost bi-Legendrian structure given by $\mathcal{D}(\tilde{\lambda})$ and $\mathcal{D}(-\tilde{\lambda})$, by definition of $\tilde{\varphi}_1$, coincides with the almost bi-Legendrian structure induced by the paracontact metric structure $(\tilde{\varphi}_1, \xi, \eta, \tilde{g}_1)$ in Theorem 5.1, which is integrable because of (5-17) and Theorem 2.4. Thus

$$[X, X'] \in \Gamma(\mathcal{D}(\tilde{\lambda}) \oplus \mathbb{R}\xi) \qquad \text{for all } X, X' \in \Gamma(\mathcal{D}(\tilde{\lambda})),$$
$$[Y, Y'] \in \Gamma(\mathcal{D}(-\tilde{\lambda}) \oplus \mathbb{R}\xi) \quad \text{for all } Y, Y' \in \Gamma(\mathcal{D}(-\tilde{\lambda})).$$

On the other hand, since $\mathcal{D}(\tilde{\lambda})$ and $\mathcal{D}(-\tilde{\lambda})$ are Legendre distributions, we have that $\eta([X, X']) = X(\eta(X')) - X'(\eta(X)) - 2d\eta(X, X') = 0$ and $\eta([Y, Y']) = 0$, so that $[X, X'] \in \Gamma(\mathcal{D})$ and $[Y, Y'] \in \Gamma(\mathcal{D})$ for all $X, X' \in \Gamma(\mathcal{D}(\tilde{\lambda}))$ and $Y, Y' \in \Gamma(\mathcal{D}(-\tilde{\lambda}))$. Hence $\mathcal{D}(\tilde{\lambda})$ and $\mathcal{D}(-\tilde{\lambda})$ are involutive.

Thus any contact metric (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ with $|I_M| > 1$ admits a supplementary bi-Legendrian structure given by the eigendistributions of the operator $\tilde{h}$ of the canonical paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ induced by (φ, ξ, η, g). The surprising fact is that such a structure $(\mathcal{D}(\tilde{\lambda}), \mathcal{D}(-\tilde{\lambda}))$ comes from a (new) contact metric (κ', μ')-structure:

Theorem 5.4. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ)-manifold such that $|I_M| > 1$, and let $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ be the canonical paracontact metric structure induced on M. Then the operator $\tilde{h} := (1/2)\mathcal{L}_\xi\tilde{\varphi}$ is diagonalizable and has eigenvalues 0 of multiplicity 1 and $\pm\tilde{\lambda}$ of multiplicity n, where $\tilde{\lambda} := \sqrt{(1-\mu/2)^2 - (1-\kappa)}$. Furthermore, denoting by $\mathcal{D}(\tilde{\lambda})$ and $\mathcal{D}(-\tilde{\lambda})$ the eigendistributions corresponding to $\tilde{\lambda}$ and $-\tilde{\lambda}$, respectively, there exists a family of compatible contact metric $(\kappa'_{a,b}, \mu'_{a,b})$-structures $(\varphi'_{a,b}, \xi, \eta, g'_{a,b})$ whose associated bi-Legendrian structure coincides with $(\mathcal{D}(\tilde{\lambda}), \mathcal{D}(-\tilde{\lambda}))$, where*

$$\kappa'_{a,b} = 1 - \tfrac{1}{16}(a-b)^2, \quad \mu'_{a,b} = 2 - \tfrac{1}{2}(a+b), \tag{5-19}$$

and a and b are any two positive real numbers such that

$$ab = \tfrac{1}{4}((1 - \tfrac{1}{2}\mu)^2 - (1-\kappa)). \tag{5-20}$$

Moreover, the Boeckx invariant of $(M, \varphi'_{a,b}, \xi, \eta, g'_{a,b})$ has absolute value strictly greater than 1, so that $(\varphi'_{a,b}, \xi, \eta, g'_{a,b})$ belongs to the same class as (φ, ξ, η, g), according to the classification in Section 3.

Proof. The first part of the theorem has been already proven in Theorem 5.1 and Remark 5.3. The remaining part of the proof consists in showing that the bi-Legendrian structure $(\mathscr{D}(-\tilde{\lambda}), \mathscr{D}(\tilde{\lambda}))$ satisfies the hypotheses of Theorem 3.2. First we find the expression of the invariants $\Pi_{\mathscr{D}(\tilde{\lambda})}$ and $\Pi_{\mathscr{D}(-\tilde{\lambda})}$. For any $X, X' \in \Gamma(\mathscr{D}(\tilde{\lambda}))$ we have

$$\begin{aligned}\Pi_{\mathscr{D}(\tilde{\lambda})}(X, X') = 2d\eta([\xi, X], X') &= 2\tilde{g}_1([\xi, X], \tilde{\varphi}_1 X') \\ &= 2\tilde{g}_1([\xi, X], X') = 2\tilde{g}_1(\tilde{h}_1 X, X'),\end{aligned}$$

and, analogously, for any $Y, Y' \in \Gamma(\mathscr{D}(-\tilde{\lambda}))$,

$$\begin{aligned}\Pi_{\mathscr{D}(-\tilde{\lambda})}(Y, Y') = 2d\eta([\xi, Y], Y') &= 2\tilde{g}_1([\xi, Y], \tilde{\varphi}_1 Y') \\ &= -2\tilde{g}_1([\xi, Y], Y') = 2\tilde{g}_1(\tilde{h}_1 Y, Y'),\end{aligned}$$

where we used the easy relations $\tilde{h}_1 X = [\xi, X]_{\mathscr{D}(-\tilde{\lambda})}$ and $\tilde{h}_1 Y = -[\xi, Y]_{\mathscr{D}(\tilde{\lambda})}$ for any $X \in \Gamma(\mathscr{D}(\tilde{\lambda}))$ and $Y \in \Gamma(\mathscr{D}(-\tilde{\lambda}))$. We prove that $\nabla'^{bl}\Pi_{\mathscr{D}(\tilde{\lambda})} = \nabla'^{bl}\Pi_{\mathscr{D}(-\tilde{\lambda})} = 0$, where ∇'^{bl} denotes the bi-Legendrian connection associated to the bi-Legendrian structure $(\mathscr{D}(-\tilde{\lambda}), \mathscr{D}(\tilde{\lambda}))$. Indeed, notice that, by Theorem 4.1 and the integrability of $(\tilde{\varphi}_1, \xi, \eta, \tilde{g}_1)$, ∇'^{bl} coincides with the canonical paracontact connection $\widetilde{\nabla}^{1pc}$ of $(M, \tilde{\varphi}_1, \xi, \eta, \tilde{g}_1)$. In particular, by (2-14) and (5-18), for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned}(\nabla'^{bl}_X \tilde{h}_1)Y &= (\widetilde{\nabla}^{1pc}_X \tilde{h}_1)Y \\ &= (\widetilde{\nabla}^1_X \tilde{h}_1)Y + \eta(X)\tilde{\varphi}_1\tilde{h}_1 Y + \tilde{g}_1(X - \tilde{h}_1 X, \tilde{\varphi}_1\tilde{h}_1 Y)\xi - \eta(Y)\tilde{h}_1\tilde{\varphi}_1 Y \\ &\qquad + \eta(Y)(\tilde{\varphi}_1\tilde{h}_1 X - \tilde{\varphi}_1\tilde{h}_1^2 X) \\ &= 0,\end{aligned}$$

where $\widetilde{\nabla}^1$ denotes the Levi-Civita connection of $(M, \tilde{g}_1)$. Consequently, for any $X, X' \in \Gamma(\mathscr{D}(\tilde{\lambda}))$ and $Z \in \Gamma(TM)$,

$$\begin{aligned}(\nabla'^{bl}_Z \Pi_{\mathscr{D}(\tilde{\lambda})})(X, X') &= 2Z(\tilde{g}_1(\tilde{h}_1 X, X')) - 2\tilde{g}_1(\tilde{h}_1 \nabla'^{bl}_Z X, X') - 2\tilde{g}_1(\tilde{h}_1 X, \nabla'^{bl}_Z X') \\ &= 2\big(Z(\tilde{g}_1(\tilde{h}_1 X, X')) - \tilde{g}_1(\nabla'^{bl}_Z \tilde{h}_1 X, X') - \tilde{g}_1(\tilde{h}_1 X, \nabla'^{bl}_Z X')\big) \\ &= 2(\nabla'^{bl}_Z \tilde{g}_1)(\tilde{h}_1 X, X') \\ &= 2(\widetilde{\nabla}^{1pc}_Z \tilde{g}_1)(\tilde{h}_1 X, X') = 0.\end{aligned}$$

In a similar way one can prove that $\nabla'^{bl}\Pi_{\mathscr{D}(-\tilde{\lambda})} = 0$. Next, we check whether $\mathscr{D}(\tilde{\lambda})$ and $\mathscr{D}(-\tilde{\lambda})$ are positive definite or negative definite Legendre foliations, according to the assumptions of Theorem 3.2. We consider the local g-orthonormal bases for $\mathscr{D}(\tilde{\lambda})$ and $\mathscr{D}(-\tilde{\lambda})$ in (5-7). As in the proof of Theorem 5.1, to simplify the notation we put $\beta := 1/\sqrt{(1 - \mu/2) - (1 - \kappa)}$. Notice that, for any $i, j \in \{1, \dots, n\}$, by

(5-9), (4-5) and (4-3),

$$\begin{aligned}\tilde{g}_1(X_i,X_j)&=\beta\tilde{g}(X_i,\tilde{\varphi}\tilde{h}X_j)\\&=-\beta\tilde{g}(X_i,\tilde{h}X_j)\\&=-\frac{\beta}{2\sqrt{1-\kappa}}((2-\mu)\tilde{g}(X_i,\varphi hX_j)+2(1-\kappa)\tilde{g}(X_i,\varphi X_j))\\&=-\frac{\beta}{2(1-\kappa)}(\lambda(2-\mu)+2(1-\kappa))g(X_i,\varphi hY_j)\\&=\beta(I_M+1)\lambda g(X_i,\varphi Y_j)=-\beta(I_M+1)\lambda\delta_{ij}.\end{aligned}$$

Similar computations yield $\tilde{g}_1(X_i,Y_j)=0$ and $\tilde{g}_1(Y_i,Y_j)=\beta(I_M-1)\lambda\delta_{ij}$. Hence

$$\begin{aligned}&\Pi_{\mathcal{D}(\tilde{\lambda})}\Bigl(\sqrt{\frac{I_M-1}{I_M+1}}X_i+Y_i,\sqrt{\frac{I_M-1}{I_M+1}}X_j+Y_j\Bigr)\\&\quad=2\frac{I_M-1}{I_M+1}\tilde{g}_1(\tilde{h}_1X_i,X_j)\\&\qquad\qquad+2\sqrt{\frac{I_M-1}{I_M+1}}\bigl(\tilde{g}_1(\tilde{h}_1X_i,Y_j)+\tilde{g}_1(\tilde{h}_1Y_i,X_j)\bigr)+2\tilde{g}_1(\tilde{h}_1Y_i,Y_j)\\&\quad=-\frac{2(I_M-1)}{\beta(I_M+1)}\tilde{g}_1(\tilde{\varphi}X_i,X_j)\\&\qquad\qquad-\frac{2}{\beta}\sqrt{\frac{I_M-1}{I_M+1}}\bigl(\tilde{g}_1(\tilde{\varphi}X_i,Y_j)+\tilde{g}_1(\tilde{\varphi}Y_i,X_j)\bigr)-\frac{2}{\beta}\tilde{g}_1(\tilde{\varphi}Y_i,Y_j)\\&\quad=-\frac{2(I_M-1)}{\beta(I_M+1)}\tilde{g}_1(X_i,X_j)-\frac{2}{\beta}\sqrt{\frac{I_M-1}{I_M+1}}\bigl(\tilde{g}_1(X_i,Y_j)-\tilde{g}_1(Y_i,X_j)\bigr)+\frac{2}{\beta}\tilde{g}_1(Y_i,Y_j)\\&\quad=4\lambda(I_M-1)\delta_{ij}.\end{aligned}$$

Arguing in the same way for $\mathcal{D}(-\tilde{\lambda})$ one can prove that

$$\Pi_{\mathcal{D}(-\tilde{\lambda})}\Bigl(-\sqrt{\frac{I_M-1}{I_M+1}}X_i+Y_i,-\sqrt{\frac{I_M-1}{I_M+1}}X_j+Y_j\Bigr)=4\lambda(I_M-1)\delta_{ij}.$$

Thus, because of the assumption $|I_M|>1$, we conclude that both $\Pi_{\mathcal{D}(\tilde{\lambda})}$ and $\Pi_{\mathcal{D}(-\tilde{\lambda})}$ are positive definite. Finally, in order to check the last hypothesis of Theorem 3.2, we find the explicit expression of the Libermann operators $\Lambda_{\mathcal{D}(\tilde{\lambda})}:TM\to\mathcal{D}(\tilde{\lambda})$ and $\Lambda_{\mathcal{D}(-\tilde{\lambda})}:TM\to\mathcal{D}(-\tilde{\lambda})$. Let us consider $X\in\Gamma(\mathcal{D}(\tilde{\lambda}))$ and $Y\in\Gamma(\mathcal{D}(-\tilde{\lambda}))$. Then, by applying (2-16),

$$2\tilde{g}_1(\tilde{h}_1\Lambda_{\mathcal{D}(\tilde{\lambda})}Y,X)=\Pi_{\mathcal{D}(\tilde{\lambda})}(\Lambda_{\mathcal{D}(\tilde{\lambda})}Y,X)=d\eta(Y,X)=\tilde{g}_1(Y,\tilde{\varphi}_1X)=\tilde{g}_1(Y,X),$$

from which it follows that $2\tilde{h}_1\Lambda_{\mathcal{D}(\tilde{\lambda})}Y = Y$. Applying $\tilde{h}_1$ and since $\tilde{h}_1 = -(1/\beta)\tilde{\varphi}$, we get $\Lambda_{\mathcal{D}(\tilde{\lambda})}Y = (1/2)\beta^2\tilde{h}_1 Y$. Thus

$$\Lambda_{\mathcal{D}(\tilde{\lambda})} = \begin{cases} 0 & \text{on } \mathcal{D}(\tilde{\lambda}) \oplus \mathbb{R}\xi, \\ \dfrac{1}{2\sqrt{(1-\mu/2)^2-(1-\kappa)}}\,\tilde{h}_1 & \text{on } \mathcal{D}(-\tilde{\lambda}). \end{cases} \tag{5-21}$$

In the same way one can prove that

$$\Lambda_{\mathcal{D}(-\tilde{\lambda})} = \begin{cases} -\dfrac{1}{2\sqrt{(1-\mu/2)^2-(1-\kappa)}}\,\tilde{h}_1 & \text{on } \mathcal{D}(\tilde{\lambda}), \\ 0 & \text{on } \mathcal{D}(-\tilde{\lambda}) \oplus \mathbb{R}\xi. \end{cases} \tag{5-22}$$

Hence, for any $Y, Y' \in \Gamma(\mathcal{D}(-\tilde{\lambda}))$,

$$\begin{aligned} \overline{\Pi}_{\mathcal{D}(\tilde{\lambda})}(Y, Y') &= \Pi_{\mathcal{D}(\tilde{\lambda})}\big(\Lambda_{\mathcal{D}(\tilde{\lambda})}Y, \Lambda_{\mathcal{D}(\tilde{\lambda})}Y'\big) \\ &= \tfrac{1}{4}\beta^4\Pi_{\mathcal{D}(\tilde{\lambda})}(\tilde{h}_1 Y, \tilde{h}_1 Y') = \tfrac{1}{2}\beta^2\tilde{g}_1(Y, \tilde{h}_1 Y') \end{aligned}$$

and for any $X, X' \in \Gamma(\mathcal{D}(\tilde{\lambda}))$

$$\begin{aligned} \overline{\Pi}_{\mathcal{D}(-\tilde{\lambda})}(X, X') &= \Pi_{\mathcal{D}(-\tilde{\lambda})}\big(\Lambda_{\mathcal{D}(-\tilde{\lambda})}X, \Lambda_{\mathcal{D}(-\tilde{\lambda})}X'\big) \\ &= \tfrac{1}{4}\beta^4\Pi_{\mathcal{D}(-\tilde{\lambda})}(\tilde{h}_1 X, \tilde{h}_1 X') = \tfrac{1}{2}\beta^2\tilde{g}_1(X, \tilde{h}_1 X'). \end{aligned}$$

In contrast, $\Pi_{\mathcal{D}(-\tilde{\lambda})}(Y, Y') = 2\tilde{g}_1(\tilde{h}_1 Y, Y')$ and $\Pi_{\mathcal{D}(\tilde{\lambda})}(X, X') = 2\tilde{g}_1(\tilde{h}_1 X, X')$, so that $\overline{\Pi}_{\mathcal{D}(\tilde{\lambda})} = (4/\beta^2)\overline{\Pi}_{\mathcal{D}(-\tilde{\lambda})}$ on $\mathcal{D}(\tilde{\lambda})$ and $\overline{\Pi}_{\mathcal{D}(-\tilde{\lambda})} = (4/\beta^2)\overline{\Pi}_{\mathcal{D}(\tilde{\lambda})}$ on $\mathcal{D}(-\tilde{\lambda})$. Since the constant $4/\beta^2$ is positive, the bi-Legendrian structure $(\mathcal{D}(\tilde{\lambda}), \mathcal{D}(-\tilde{\lambda}))$ satisfies all the assumptions of Theorem 3.2 and so, for any two positive constants a and b such that $ab = 4/\beta^2$, there is a contact metric $(\kappa'_{a,b}, \mu'_{a,b})$-structure $(\varphi'_{a,b}, \xi, \eta, g'_{a,b})$ whose associated bi-Legendrian structure coincides with $(\mathcal{D}(\tilde{\lambda}), \mathcal{D}(-\tilde{\lambda}))$, where $\kappa'_{a,b}$ and $\mu'_{a,b}$ are given by (5-19). Finally, the Boeckx invariant of the new contact metric $(\kappa'_{a,b}, \mu'_{a,b})$-structure $(\varphi'_{a,b}, \xi, \eta, g'_{a,b})$ is given by

$$(1 - \mu'_{a,b}/2)/\sqrt{1-\kappa'_{a,b}} = (a+b)/|a-b|.$$

Hence, as $a > 0$ and $b > 0$, we have $|I'_M| > 1$ and we conclude that $(\varphi'_{a,b}, \xi, \eta, g'_{a,b})$ is in the same class as (φ, ξ, η, g). □

Remark 5.5. As expected, all the various contact metric $(\kappa'_{a,b}, \mu'_{a,b})$-structures in the Theorem 5.4 induce, by Theorem 4.3, the same paracontact metric $(\tilde{\kappa}_1, \tilde{\mu}_1)$-structure $(\tilde{\varphi}_1, \xi, \eta, \tilde{g}_1)$. In other words, $\tilde{\kappa}_1$ and $\tilde{\mu}_1$ do not depend on the arbitrarily chosen constants a and b satisfying (5-20). Indeed, by applying Theorem 4.7, we get

$$\begin{aligned} \tilde{\kappa}_1 = \kappa'_{a,b} - 2 + (1 - \tfrac{1}{2}\mu'_{a,b})^2 &= -1 + \tfrac{1}{4}\big(\tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2\big) \\ &= -1 + \tfrac{1}{4}ab = \kappa - 2 + (1 - \tfrac{1}{2}\mu)^2 \end{aligned}$$

and $\tilde{\mu}_1 = 2$.

Now we are able to iterate the procedure of Theorem 4.3 and Theorem 5.1 and hence to define on a contact metric (κ, μ)-manifold M a canonical sequence of contact/paracontact metric structures as stated in the following theorem.

Theorem 5.6. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ)-manifold.*

(i) *If $|I_M| < 1$, then M admits a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of tensor fields and a sequence $(G_n)_{n\in\mathbb{N}}$ of $(0, 2)$-tensors defined by*

$$\varphi_0 = \varphi, \quad \varphi_1 = \frac{1}{2\sqrt{1-\kappa}} \mathcal{L}_\xi \varphi_0, \tag{5-23}$$

$$\begin{aligned} \varphi_{2n} &= \frac{1}{2\sqrt{1-\kappa-(1-\mu/2)^2}} \mathcal{L}_\xi \varphi_{2n-1}, \\ \varphi_{2n+1} &= \frac{1}{2\sqrt{1-\kappa-(1-\mu/2)^2}} \mathcal{L}_\xi \varphi_{2n}, \end{aligned} \tag{5-24}$$

$$G_{2n} = -d\eta(\cdot\,, \varphi_{2n}) + \eta\otimes\eta, \quad G_{2n+1} = d\eta(\cdot\,, \varphi_{2n+1}) + \eta\otimes\eta, \tag{5-25}$$

such that, for each $n \in \mathbb{N}$, $(\varphi_{2n}, \xi, \eta, G_{2n})$ is a contact metric (κ_{2n}, μ_{2n})-structure and $(\varphi_{2n+1}, \xi, \eta, G_{2n+1})$ is a paracontact metric $(\kappa_{2n+1}, \mu_{2n+1})$-structure, where

$$\kappa_0 = \kappa, \qquad \kappa_{2n} = \kappa + (1-\mu/2)^2, \qquad \mu_{2n} = 2, \tag{5-26}$$

$$\kappa_{2n+1} = \kappa - 2 + (1-\mu/2)^2, \qquad \mu_{2n+1} = 2. \tag{5-27}$$

Moreover, for each $n \in \mathbb{N}$, $(\varphi_{2n}, \xi, \eta, G_{2n})$ is a Tanaka–Webster parallel structure on M, and $(\varphi_{2n+1}, \xi, \eta, G_{2n+1})$ is the canonical paracontact metric structure induced by $(\varphi_{2n}, \xi, \eta, G_{2n})$ according to Theorem 4.3.

(ii) *If $|I_M| > 1$, then M admits a sequence $(\varphi_n, \xi, \eta, G_n)_{n\geq 1}$ of paracontact metric structures defined by*

$$\varphi_1 = \frac{1}{2\sqrt{1-\kappa}} \mathcal{L}_\xi \varphi, \quad \varphi_n = \frac{1}{2\sqrt{(1-\mu/2)^2-(1-\kappa)}} \mathcal{L}_\xi \varphi_{n-1},$$
$$G_n = d\eta(\cdot\,, \varphi_n) + \eta\otimes\eta,$$

such that, for each $n \geq 1$, $(\varphi_n, \xi, \eta, G_n)$ is a paracontact metric (κ_n, μ_n)-structure with

$$\kappa_n = \kappa - 2 + (1-\mu/2)^2, \quad \mu_n = 2.$$

Moreover, $(\varphi_1, \xi, \eta, G_1)$ is the canonical paracontact structure induced by (φ, ξ, η, g) and, for each $n \geq 2$, $(\varphi_n, \xi, \eta, G_n)$ is the canonical paracontact structure induced by a contact metric (κ'_n, μ'_n)-structure $(\varphi'_n, \xi, \eta, g'_n)$ on M with

$$\kappa'_n = 1 - \tfrac{1}{16}(a_n - b_n)^2, \quad \mu'_n = 2 - \tfrac{1}{2}(a_n + b_n), \tag{5-28}$$

and a_n and b_n being two constants such that

$$a_n b_n = \tfrac{1}{4}((1-\tfrac{1}{2}\mu)^2-(1-\kappa)). \tag{5-29}$$

Proof. We argue by induction on n.

(i) We distinguish the even and the odd case. The result is trivially true for $n=0$ since (M,φ,ξ,η,g) is supposed to be a contact metric (κ,μ)-manifold and for $n=1$ because of Theorem 4.7. Suppose that the assertion holds for $(\varphi_{2n},\xi,\eta,G_{2n})$ with $n\geq 2$. We have to prove that the structure $(\varphi_{2n+1},\xi,\eta,G_{2n+1})$ defined by (5-24) is a paracontact metric $(\kappa_{2n+1},\mu_{2n+1})$-structure, where κ_{2n+1} and μ_{2n+1} are given by (5-27). Notice that

$$\varphi_{2n+1}=\frac{1}{2\sqrt{1-\kappa-(1-\mu/2)^2}}\mathcal{L}_\xi\varphi_{2n}=\frac{1}{2\sqrt{1-\kappa_{2n}}}\mathcal{L}_\xi\varphi_{2n},$$

so that, according to Theorem 4.3, $(\varphi_{2n+1},\xi,\eta,G_{2n+1})$ coincides with the canonical paracontact metric structure induced on M by the contact metric (κ_{2n},μ_{2n})-structure $(\varphi_{2n},\xi,\eta,G_{2n})$. Then, by Theorem 4.7, $(\varphi_{2n+1},\xi,\eta,G_{2n+1})$ is a paracontact metric $(\tilde\kappa,\tilde\mu)$-structure, where

$$\begin{aligned}\tilde\kappa=\kappa_{2n}-2+(1-\tfrac{1}{2}\mu_{2n})^2&=\kappa+(1-\tfrac{1}{2}\mu)^2-2+(1-\tfrac{2}{2})^2\\ &=\kappa-2+(1-\tfrac{1}{2}\mu)^2=\kappa_{2n+1}\end{aligned}$$

and $\tilde\mu=2=\mu_{2n+1}$. Now we study the odd case. Assume that the assertion holds for $(\varphi_{2n+1},\xi,\eta,G_{2n+1})$. We have to prove that $(\varphi_{2n+2},\xi,\eta,G_{2n+2})$ is a contact metric $(\kappa_{2n+2},\mu_{2n+2})$-structure, where κ_{2n+2} and μ_{2n+2} are given by (5-26). By the induction hypothesis, $(\varphi_{2n+1},\xi,\eta,G_{2n+1})$ is the canonical paracontact metric structure induced by the contact metric (κ_{2n},μ_{2n})-structure $(\varphi_{2n},\xi,\eta,G_{2n})$. Since the Boeckx invariant of $(M,\varphi_{2n},\xi,\eta,G_{2n})$ is 0, we can apply Theorem 5.1 to the contact metric (κ_{2n},μ_{2n})-manifold $(M,\varphi_{2n},\xi,\eta,G_{2n})$ and conclude that the paracontact metric structure $(\varphi_{2n+1},\xi,\eta,G_{2n+1})$ induces on M a contact metric structure $(\bar\varphi_1,\xi,\eta,\bar g_1)$ given by (5-3) and (5-4). Notice that

$$\begin{aligned}\bar\varphi_1&=\frac{1/2}{\sqrt{1-\kappa_{2n}-(1-\frac{1}{2}\mu_{2n})^2}}\mathcal{L}_\xi\varphi_{2n+1}=\frac{1/2}{\sqrt{1-\kappa-(1-\frac{1}{2}\mu)^2-(1-\frac{2}{2})^2}}\mathcal{L}_\xi\varphi_{2n+1}\\ &=\frac{1/2}{\sqrt{1-\kappa-(1-\mu/2)^2}}\mathcal{L}_\xi\varphi_{2n+1}=\varphi_{2n+2}.\end{aligned}$$

Therefore $(\varphi_{2n+2},\xi,\eta,G_{2n+2})$ is a contact metric $(\bar\kappa_1,\bar\mu_1)$-structure, where, by Theorem 5.1,

$$\bar\kappa_1=\kappa_{2n}+(1-\mu_{2n}/2)^2=\kappa_{2n}=\kappa+(1-\mu/2)^2=\kappa_{2n+2}$$

and $\bar{\mu}_1 = 2 = \mu_{2n+2}$. Finally, since $\mu_{2n} = 2$ for each $n \in \mathbb{N}$, we conclude by applying [Boeckx and Cho 2008, Theorem 12] that $(M, \varphi_{2n}, \xi, \eta, G_{2n})$ is a Tanaka–Webster parallel space.

(ii) The result is true for $n = 1$ due to Theorem 4.7 and for $n = 2$ due to Theorem 5.1 and Theorem 5.4. Now assuming that the assertion holds for $(\varphi_n, \xi, \eta, G_n)$ with $n \geq 3$, we prove that it holds also for $(\varphi_{n+1}, \xi, \eta, G_{n+1})$. By the induction hypothesis, $(\varphi_n, \xi, \eta, G_n)$ is the canonical paracontact metric structure induced by a contact metric (κ'_n, μ'_n)-manifold, κ'_n and μ'_n being given by (5-28), whose Boeckx invariant, given by $(a + b)/|a - b|$, has absolute value strictly greater than 1. Hence we can apply Theorem 5.1 and conclude that $(\varphi_n, \xi, \eta, G_n)$ induces on M a paracontact metric $(\tilde{\kappa}'_1, \tilde{\mu}'_1)$-structure $(\tilde{\varphi}'_1, \xi, \eta, \tilde{g}'_1)$, where $\tilde{\varphi}'_1, \tilde{g}'_1$ are given by (5-6) and (5-8) and $\tilde{\kappa}'_1, \tilde{\mu}'_1$ are given by (5-2). Note that

$$\begin{aligned}\tilde{\varphi}'_1 &= \frac{1}{2\sqrt{(1-\mu'_n/2)^2-(1-\kappa'_n)}}\mathcal{L}_\xi\varphi_n\\ &= \frac{1}{\sqrt{(a_n+b_n)^2/4-(a_n-b_n)^2/4}}\mathcal{L}_\xi\varphi_n = \frac{1}{\sqrt{a_nb_n}}\mathcal{L}_\xi\varphi_n\\ &= \frac{1}{2\sqrt{(1-\mu/2)^2-(1-\kappa)}}\mathcal{L}_\xi\varphi_n = \varphi_{n+1}.\end{aligned}$$

Finally, in view of Remark 5.5, we get $\tilde{\kappa}_1 = \kappa - 2 + (1 - \mu/2)^2 = \kappa_{n+1}$ and $\tilde{\mu}_1 = 2 = \mu_{n+1}$. □

6. Canonical Sasakian structures on contact metric (κ, μ)-spaces

As pointed out in Remark 5.3, in proving Theorem 5.1 we have proven that any (non-Sasakian) contact metric (κ, μ)-space such that $|I_M| > 1$ admits a supplementary bi-Legendrian structure $(\mathcal{D}(\tilde{\lambda}), \mathcal{D}(-\tilde{\lambda}))$ given by the eigendistributions of the operator $\tilde{h} := (1/(4\sqrt{1-\kappa}))\mathcal{L}_\xi\mathcal{L}_\xi\varphi$ corresponding to the eigenvalues $\pm\tilde{\lambda}$, where $\tilde{\lambda} := \sqrt{(1-\mu/2)^2 - (1-\kappa)}$. We now prove that in fact any three of the distributions $\mathcal{D}(\lambda), \mathcal{D}(-\lambda), \mathcal{D}(\tilde{\lambda}), \mathcal{D}(-\tilde{\lambda})$ define a 3-web on the contact distribution of (M, η). Recall that a triple of distributions $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ on a smooth manifold M is called an *almost* 3-*web structure* if $TM = \mathcal{D}_i \oplus \mathcal{D}_j$ is satisfied for any two different $i, j \in \{1, 2, 3\}$. If $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are involutive, then $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ is said to be simply a 3-*web* [Nagy 1988]. Now, obviously one has that $\mathcal{D} = \mathcal{D}(\lambda) \oplus \mathcal{D}(-\lambda)$ and $\mathcal{D} = \mathcal{D}(\tilde{\lambda}) \oplus \mathcal{D}(-\tilde{\lambda})$, so that it suffices to prove that $\mathcal{D} = \mathcal{D}(\pm\lambda) \oplus \mathcal{D}(\pm\tilde{\lambda})$ for all choices of $\pm$. Let $\{X_1, \dots, X_n, Y_1 := \varphi X_1, \dots, Y_n := \varphi X_n, \xi\}$ be a (local) orthonormal φ-basis of eigenvectors of h. Then

$$\mathcal{D}(\lambda) = \operatorname{span}\{X_1, \dots, X_n\} \quad \text{and} \quad \mathcal{D}(-\lambda) = \operatorname{span}\{Y_1, \dots, Y_n\}$$

and $\mathcal{D}(\tilde{\lambda})$ and $\mathcal{D}(-\tilde{\lambda})$ are given by (5-7). Using these local expressions, if follows from some elementary linear algebra that, putting $\gamma := \sqrt{(I_M - 1)/(I_M + 1)}$,

$$\{X_1, \dots, X_n, \gamma X_1 + Y_1, \dots, \gamma X_n + Y_n\},$$
$$\{X_1, \dots, X_n, -\gamma X_1 + Y_1, \dots, -\gamma X_n + Y_n\},$$
$$\{Y_1, \dots, Y_n, \gamma X_1 + Y_1, \dots, \gamma X_n + Y_n\},$$
$$\{Y_1, \dots, Y_n, -\gamma X_1 + Y_1, \dots, -\gamma X_n + Y_n\},$$

are all local bases of the contact distribution $\mathcal{D}$. The assertion follows.

As shown in [Marchiafava and Nagy 2003], one can associate to any almost 3-web a canonical almost antihypercomplex structure, that is, a triple (I_1, I_2, I_3) consisting of an almost complex structure I_1 and two anticommuting almost product structures I_2 and I_3 satisfying $I_2 I_3 = I_1$ (hence $I_2 I_1 = -I_1 I_2 = I_3$, $I_1 I_3 = -I_3 I_1 = I_2$). Conversely, any almost antihypercomplex structure determines four almost 3-webs given by the eigendistributions of I_2 and I_3 corresponding to the eigenvalues ± 1. Consequently, any contact metric (κ, μ)-manifold such that $|I_M| > 1$ admits a canonical antihypercomplex structure on the contact distribution via the 3-webs above. Such antihypercomplex structure is in fact given by $(\bar{\varphi}_-|_{\mathcal{D}}, \tilde{\varphi}|_{\mathcal{D}}, \tilde{\varphi}_1|_{\mathcal{D}})$ in the case $I_M < -1$ and by $(\bar{\varphi}_+|_{\mathcal{D}}, \tilde{\varphi}_1|_{\mathcal{D}}, \tilde{\varphi}|_{\mathcal{D}})$ in the case $I_M > 1$, where $\tilde{\varphi}$, $\tilde{\varphi}_1$ are given, respectively, by (4-2), (5-6), and

$$\bar{\varphi}_{\pm} := \pm \frac{1}{\sqrt{(1-\frac{1}{2}\mu)^2 - (1-\kappa)}} ((1 - \tfrac{1}{2}\mu)\varphi + \varphi h).$$

Indeed using (4-2), (5-6) and the relations $h^2 = (\kappa - 1)\varphi^2$, $\varphi h = -h\varphi$, one can easily check by a straightforward computation that $\tilde{\varphi}$ and $\tilde{\varphi}_1$ induce two anticommuting almost product structures on $\mathcal{D}$ and that $\tilde{\varphi}\tilde{\varphi}_1 = \bar{\varphi}_-$ and $\tilde{\varphi}_1\tilde{\varphi} = \bar{\varphi}_+$. We prove that $\bar{\varphi}_-$ and $\bar{\varphi}_+$ are almost contact structures compatible with η. Indeed

$$\begin{aligned}
{\bar{\varphi}_-}^2 &= \frac{1}{(1-\frac{1}{2}\mu)^2 - (1-\kappa)} ((1 - \tfrac{1}{2}\mu)^2\varphi^2 + \varphi h \varphi h + (1 - \tfrac{1}{2}\mu)\varphi^2 h + (1 - \tfrac{1}{2}\mu)\varphi h \varphi) \\
&= \frac{1}{(1-\frac{1}{2}\mu)^2 - (1-\kappa)} ((1 - \tfrac{1}{2}\mu)^2\varphi^2 - \varphi^2 h^2) \\
&= \frac{1}{(1-\frac{1}{2}\mu)^2 - (1-\kappa)} ((1 - \tfrac{1}{2}\mu)^2\varphi^2 - (1-\kappa)\varphi^2) \\
&= \varphi^2 = -I + \eta \otimes \xi.
\end{aligned}$$

Analogously one can prove that ${\bar{\varphi}_+}^2 = -I + \eta \otimes \xi$. Moreover, for each almost contact structure $(\bar{\varphi}_{\pm}, \xi, \eta)$ one can define an associated metric $\bar{g}_{\pm}$ by

$$\bar{g}_{\pm}(X, Y) = -d\eta(X, \bar{\varphi}_{\pm} Y) + \eta(X)\eta(Y). \tag{6-1}$$

We prove that $\bar{g}_\pm$ is a Riemannian metric compatible with the almost contact structure $(\bar{\varphi}_\pm, \xi, \eta)$ (respecting the choice of $\pm$). By (6-1) it straightforwardly follows that $\bar{g}_-$ is nondegenerate, symmetric and satisfies

$$\bar{g}_-(\bar{\varphi}_- X, \bar{\varphi}_- Y) = \bar{g}_-(X, Y) - \eta(X)\eta(Y).$$

We prove that it positive definite. By (6-1) we have that $\bar{g}_-(\xi, \xi) = 1$, so that it suffices to prove that $\bar{g}_-(X, X) > 0$ for any $X \in \Gamma(\mathcal{D})$ with $X \neq 0$. We decompose X into its components X_λ and $X_{-\lambda}$ according to the decomposition $\mathcal{D} = \mathcal{D}(\lambda) \oplus \mathcal{D}(-\lambda)$. To simplify the notation, as in Section 5, we put $\beta := 1/\sqrt{(1-\mu/2)^2 - (1-\kappa)}$. Then we have

$$\begin{aligned}
\bar{g}_-(X, X) &= \beta((1-\tfrac{1}{2}\mu)d\eta(X, \varphi X) + d\eta(X, \varphi h X)) \\
&= -\beta((1-\tfrac{1}{2}\mu)g(X, X) + g(X, hX)) \\
&= -\beta((1-\tfrac{1}{2}\mu)(g(X_\lambda, X_\lambda) + g(X_{-\lambda}, X_{-\lambda})) \\
&\qquad\qquad\qquad + \lambda g(X_\lambda, X_\lambda) - \lambda g(X_{-\lambda}, X_{-\lambda})) \\
&= -\beta((1-\tfrac{1}{2}\mu + \sqrt{1-\kappa})g(X_\lambda, X_\lambda) + (1-\tfrac{1}{2}\mu - \sqrt{1-\kappa})g(X_{-\lambda}, X_{-\lambda})).
\end{aligned}$$

Since we are assuming $I_M < -1$, we have

$$1 - \mu/2 + \sqrt{1-\kappa} < 0 \quad \text{and} \quad 1 - \mu/2 - \sqrt{1-\kappa} < 0,$$

so that $\bar{g}_-(X, X) > 0$. Analogous arguments work for $\bar{g}_+$, using the assumption $I_M > 1$. Finally, directly from (6-1) it follows that $d\eta(\cdot, \cdot) = \bar{g}_\pm(\cdot, \bar{\varphi}_\pm)$, and we conclude that $(\bar{\varphi}_-, \xi, \eta, \bar{g}_-)$ and $(\bar{\varphi}_+, \xi, \eta, \bar{g}_+)$ are contact metric structures. We prove that they are in fact Sasakian structures. We argue on $(\bar{\varphi}_-, \xi, \eta, \bar{g}_-)$, since the same arguments work also for $(\bar{\varphi}_+, \xi, \eta, \bar{g}_+)$. We first prove that the contact metric structure is K-contact, that is, the tensor field $\bar{h}_- := (1/2)\mathcal{L}_\xi \bar{\varphi}_-$ vanishes identically. Indeed, by using (4-5), we have

$$\begin{aligned}
2\bar{h}_- &= -\beta((1 - \tfrac{1}{2}\mu)\mathcal{L}_\xi \varphi + \mathcal{L}_\xi(\varphi h)) \\
&= -\beta((1 - \tfrac{1}{2}\mu)\mathcal{L}_\xi \varphi + (\mathcal{L}_\xi \varphi) \circ h + \varphi \circ (\mathcal{L}_\xi h)) \\
&= -\beta((2-\mu)h + 2h^2 + (2-\mu)\varphi^2 h + 2(1-\kappa)\varphi^2) = 0.
\end{aligned}$$

Now we observe that $\bar{\varphi}_- \mathcal{D}(\lambda) = \mathcal{D}(-\lambda)$ and $\bar{\varphi}_- \mathcal{D}(-\lambda) = \mathcal{D}(\lambda)$. Thus the Legendre foliations $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are conjugate with respect to $\bar{\varphi}_-$, and thus they are mutually orthogonal with respect to $\bar{g}_-$. Then we can apply Theorem 2.6. Note that $\nabla^{bl}\bar{\varphi}_- = -\beta\big((1-\mu/2)\nabla^{bl}\varphi + \nabla^{bl}(\varphi h)\big) = 0$, since $\nabla^{bl}\varphi = \nabla^{bl} h = 0$. Hence, by Theorem 2.6, we have $\nabla^{bl}_X X' = -(\bar{\varphi}_-[X, \bar{\varphi}_- X'])_{\mathcal{D}(\lambda)}$ for all $X, X' \in \Gamma(\mathcal{D}(\lambda))$.

Hence

$$\begin{aligned}(N_{\bar\varphi_-}(X,X'))_{\mathcal{D}(\lambda)} &= -[X,X'] - (\bar\varphi_-[\bar\varphi_- X,X'])_{\mathcal{D}(\lambda)} - (\bar\varphi_-[X,\bar\varphi_- X'])_{\mathcal{D}(\lambda)}\\ &= -[X,X'] - \nabla^{bl}_{X'}X + \nabla^{bl}_{X}X'\\ &= T^{bl}(X,X')\\ &= 2d\eta(X,X')\xi = 0.\end{aligned}$$

Analogously, $(N_{\bar\varphi_-}(Y,Y'))_{\mathcal{D}(-\lambda)} = 0$ for all $Y, Y' \in \Gamma(\mathcal{D}(-\lambda))$. For all $X, X' \in \Gamma(\mathcal{D}(\lambda))$, we also have

$$\begin{aligned}N_{\bar\varphi_-}(\bar\varphi_- X,\bar\varphi_- X') &= -[\bar\varphi_- X,\bar\varphi_- X'] + [\bar\varphi_-{}^2X,\bar\varphi_-{}^2X']\\ &\qquad -\bar\varphi_-[\bar\varphi_-{}^2X,\bar\varphi_- X'] - \bar\varphi_-[\bar\varphi_- X,\bar\varphi_-{}^2X']\\ &= -[\bar\varphi_- X,\bar\varphi_- X'] + [X,X'] + \bar\varphi_-[X,\bar\varphi_- X'] + \bar\varphi_-[\bar\varphi_- X,X']\\ &= -N_{\bar\varphi_-}(X,X');\end{aligned}$$

hence $(N_{\bar\varphi_-}(X,X'))_{\mathcal{D}(-\lambda)} = -(N_{\bar\varphi_-}(\bar\varphi_- X,\bar\varphi_- X'))_{\mathcal{D}(-\lambda)} = 0$. Next, $N_{\bar\varphi_-}(X,X')$ has zero component also in the direction of ξ by (2-5), so $N_{\bar\varphi_-}(X,X') = 0$. In the same way one can show that $N_{\bar\varphi_-}(Y,Y') = 0$ for all $Y, Y' \in \Gamma(\mathcal{D}(-\lambda))$. Moreover, (2-4) implies that $N_{\bar\varphi_-}(X,Y) = 0$ for all $X \in \Gamma(\mathcal{D}(\lambda))$ and $Y \in \Gamma(\mathcal{D}(-\lambda))$. Finally, directly by (2-3) we have $\eta(N_{\bar\varphi_-}(Z,\xi)) = 0$ for all $Z \in \Gamma(\mathcal{D})$, and from (2-4) it follows that $\bar\varphi_-(N_{\bar\varphi_-}(Z,\xi)) = 0$. Hence $N_{\bar\varphi_-}(Z,\xi) \in \ker(\eta) \cap \ker(\bar\varphi_-) = \{0\}$. Thus the tensor field $N_{\bar\varphi_-}$ vanishes identically and so $(\bar\varphi_-, \xi, \eta, \bar g_-)$ is a Sasakian structure.

Theorem 6.1. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian contact metric (κ, μ)-space with $|I_M| > 1$. Then (M, η) admits a compatible Sasakian structure $(\bar\varphi_-, \xi, \eta, \bar g_-)$ or $(\bar\varphi_+, \xi, \eta, \bar g_+)$, depending on whether $I_M < -1$ or $I_M > 1$, where*

$$\bar\varphi_\pm := \pm\frac{1}{\sqrt{(1-\mu/2)^2-(1-\kappa)}}\left((1-\tfrac{1}{2}\mu)\varphi + \varphi h\right), \quad \bar g_\pm := -d\eta(\,\cdot\,, \bar\varphi_\pm\cdot\,) + \eta\otimes\eta.$$

Furthermore, the triple $(\bar\varphi_-, \tilde\varphi, \tilde\varphi_1)$ in the case $I_M < -1$ or $(\bar\varphi_+, \tilde\varphi_1, \tilde\varphi)$ in the case $I_M > 1$ induces an almost antihypercomplex structure on the contact distribution of (M, η), where $\tilde\varphi$ and $\tilde\varphi_1$ are given, respectively, by (4-2) *and* (5-6).

Remark 6.2. Theorem 6.1 should be compared with [Cappelletti Montano 2009b, Corollary 3.7], where a similar result was found by completely different methods. There, however, the explicit expression of the Sasakian structure was not given.

Remark 6.3. In view of Corollary 5.2 and Theorem 6.1, it appears that a possible geometric interpretation of the Boeckx invariant I_M is related to the existence on the manifold of compatible Tanaka–Webster parallel structures or Sasakian structures, depending on whether $|I_M| < 1$ or $|I_M| > 1$, respectively. In contrast, there is not much one can say about those contact metric (κ, μ)-spaces such that $I_M = \pm 1$, which seem to have a completely different geometric behavior and so deserve to be studied in a subsequent paper.

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