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**GLOBAL CLASSICAL SOLUTIONS TO HYPERBOLIC
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We consider the Cauchy problem for hyperbolic geometric flow equations introduced recently by Kong and Liu motivated by the Einstein equation and Hamilton Ricci flow, and obtain a necessary and sufficient condition for the global existence of classical solutions to this kind of flow on Riemann surfaces. The results show that the scalar curvature of the solution metric g_{ij} converges to one of flat curvature, and the hyperbolic geometric flow has the advantage that the surgery technique may be replaced by choosing a suitable initial velocity tensor.

1. Introduction

Let \mathcal{M} be an n -dimensional complete Riemann manifold with Riemann metric g_{ij} . The general evolution equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathcal{F}\left(g, \frac{\partial g}{\partial t}\right) = 0$$

for the metric g_{ij} has been recently introduced by Kong and Liu [2007] and called the generalized *hyperbolic geometric flow* (denoted by HGF). Here \mathcal{F} are some given smooth functions of the Riemann metric g and its first derivative with respect to t , and R_{ij} are the components of the Ricci curvature tensor. In this paper, we may the HGF the Kong–Liu hyperbolic geometric flow. The local existence and nonlinear stability have been proved by Dai, Kong and Liu [≥ 2010 ; 2008]. See the celebrated survey paper [Kong 2008] for progress on this new topic.

In this paper, we study the evolution of a Riemann metric g_{ij} on a Riemann surface \mathcal{M} by its Ricci curvature tensor R_{ij} under the HGF equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}.$$

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Here, we are interested in the initial metric

$$(1) \quad ds^2 = u_0(x)(dx^2 + dy^2) \quad \text{at } t = 0$$

on a surface of topological type \mathbb{R}^2 , where $u_0(x)$ is a C^2 function with bounded C^2 norm and satisfies

$$(2) \quad 0 < k \leq u_0(x) \leq M < \infty,$$

where k and M are positive constants.

On this surface, the HGF equation simplifies because all of the information about curvature is contained in the scalar curvature function R . In our notation, $R = 2K$, where K is the Gauss curvature. The Ricci curvature is given by $R_{ij} = \frac{1}{2}Rg_{ij}$, so the HGF equation simplifies to

$$(3) \quad \frac{\partial^2 g_{ij}}{\partial t^2} = -Rg_{ij}.$$

The metric for a surface can always be written (at least locally) as $g_{ij} = u(t, x, y)\delta_{ij}$, where $u(t, x, y) > 0$, and δ_{ij} is Kronecker's symbol. Therefore, we have

$$(4) \quad R = -\frac{\Delta \ln u}{u}.$$

Thus, Equation (3) reduces to $u_{tt} - \Delta \ln u = 0$.

The initial data $u_0(x)$ depends only on x and not y ; therefore, we may consider the Cauchy problem

$$(5) \quad \begin{cases} u_{tt} - (\ln u)_{xx} = 0, \\ u = u_0(x) & \text{at } t = 0, \\ u_t = u_1(x) & \text{at } t = 0, \end{cases}$$

where $u_1(x) \in C^1$ with bounded C^1 norm.

By using the transformation

$$(6) \quad \phi = \ln u,$$

Kong and Liu have proved the following theorem.

Theorem 1.1 [Kong et al. 2009]. *Suppose that $u_1(x) \geq |u'_0(x)|/\sqrt{u_0(x)}$ for all $x \in \mathbb{R}$. Then, the Cauchy problem (5) admits a unique global solution for all $t \in \mathbb{R}$.*

Moreover, if $u_1(x) \equiv u'_0(x)/\sqrt{u_0(x)}$, and there exists a point $x_0 \in \mathbb{R}$ such that $u'_0(x_0) < 0$, then the Cauchy problem (5) admits a unique classical solution only in $[0, T) \times \mathbb{R}$, where

$$T = -\frac{2}{\inf_x \{u'_0(x)u_0^{-3/2}(x)\}}.$$

In this paper, we will prove the following theorems without using (6).

Theorem 1.2. *Suppose that*

$$(7) \quad u_1(x) + \frac{u'_0(x)}{\sqrt{u_0(x)}} \geq 0 \quad \text{for all } x \in \mathbb{R},$$

$$(8) \quad u_1(x) - \frac{u'_0(x)}{\sqrt{u_0(x)}} \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

Then (5) admits a unique global solution for all $t \in \mathbb{R}$.

Theorem 1.3. *If there exists a point $x_0 \in \mathbb{R}$, such that*

$$(9) \quad u_1(x_0) + \frac{u'_0(x_0)}{\sqrt{u_0(x_0)}} < 0,$$

or there exists a point $x_0 \in \mathbb{R}$ such that

$$(10) \quad u_1(x_0) - \frac{u'_0(x_0)}{\sqrt{u_0(x_0)}} < 0,$$

then (5) admits a unique classical solution only in $[0, T) \times \mathbb{R}$.

Remark 1.4. Following Theorem 1.2, it is easy to see that Cauchy problem

$$(11) \quad \begin{cases} \frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} & \text{for } i, j = 1, 2, \\ g_{ij} = u_0(x)\delta_{ij} & \text{for } t = 0, \\ \frac{\partial g_{ij}}{\partial t} = u_1(x)\delta_{ij} & \text{for } t = 0 \text{ and } i, j = 1, 2 \end{cases}$$

has a unique smooth solution for all time $t \in \mathbb{R}$, and the solution metric g_{ij} assumes the form

$$(12) \quad g_{ij} = u(x, t)\delta_{ij} \quad \text{for } i, j = 1, 2.$$

Theorems 1.2 and 1.3 will be proved in Sections 3 and 4. Using Theorem 1.2, we can further prove this:

Theorem 1.5. *Suppose that*

$$(13) \quad \inf_x \left\{ u_1(x) + \frac{u'_0(x)}{\sqrt{u_0(x)}} \right\} > 0 \quad \text{and} \quad \inf_x \left\{ u_1(x) - \frac{u'_0(x)}{\sqrt{u_0(x)}} \right\} > 0.$$

Then (11) has a unique classical solution of the form (12) for all time. Moreover the scalar curvature $R(x, t)$ corresponding to the solution metric g_{ij} satisfies

$$R(x, t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

and there exists positive constant $k_1 > 0$ independent of t and x such that

$$R(x, t) \leq k_1 \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Theorem 1.5 shows that any metric on a simply connected noncompact surface converges to one of flat curvature by choosing a suitable initial velocity tensor $\partial g_{ij}(x, 0)/\partial t$, but the general case of a metric in \mathbb{R}^2 is still open. Theorem 1.3 shows that if we do not choose suitable a initial velocity tensor $\partial g_{ij}(x, 0)/\partial t$, the solution to Cauchy problem will blow up in finite time.

2. Preliminaries

We need only discuss the classical solution on $t \geq 0$. The result for $t \leq 0$ can be obtained by changing t to $-t$.

Let

$$(14) \quad u_t = v \quad \text{and} \quad u_x = w.$$

Then, it follows from the first equation in (5) and (14) that

$$(15) \quad u_t = v, \quad w_t - v_x = 0, \quad v_t - (1/u)w_x = -w^2/u^2.$$

It is easy to see that the eigenvalues of equations (15) are

$$\lambda_1 = -\lambda, \quad \lambda_2 = 0, \quad \lambda_3 = \lambda, \quad \lambda = 1/\sqrt{u},$$

and the matrix $L(U)$ (where $U = (u, w, v)^T$) of left eigenvectors and the matrix $R(U)$ of right eigenvectors are respectively

$$L(U) = \begin{pmatrix} l_1(U) \\ l_2(U) \\ l_3(U) \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & -\lambda & 1 \end{pmatrix},$$

$$R(U) = (r_1(U), r_2(U), r_3(U)) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & 0 & -\lambda \\ 1 & 0 & 1 \end{pmatrix}.$$

Since $\nabla \lambda_i(U) r_i(U) \equiv 0$ for $i = 1, 2, 3$, (15) is a linear degenerate strict hyperbolic system.

Set

$$(16) \quad p = v + \lambda w \quad \text{and} \quad q = v - \lambda w.$$

Lemma 2.1. *p and q satisfy*

$$(17a) \quad p_t - \lambda p_x = \frac{1}{4} \lambda^2 p(q - p),$$

$$(17b) \quad u_t = \frac{1}{2}(p + q),$$

$$(17c) \quad q_t + \lambda q_x = \frac{1}{4} \lambda^2 q(p - q).$$

Proof. Noting that $\lambda_t = -\frac{1}{2}\lambda^3v$ and $\lambda_x = -\frac{1}{2}\lambda^3w$, we calculate

$$\begin{aligned} p_t - \lambda p_x &= (v + \lambda w)_t - \lambda(v + \lambda w)_x \\ &= v_t - \lambda v_x + \lambda(w_t - \lambda w_x) + w(\lambda_t - \lambda \lambda_x) \\ &= -\frac{1}{2}\lambda^3w(v + \lambda w) = \frac{1}{4}\lambda^2(q - p)p. \end{aligned}$$

In a similar way, we can prove (17c), and (17b) is obvious. □

Let

$$r = p_x + \frac{1}{8}\lambda pq \quad \text{and} \quad s = q_x - \frac{1}{8}\lambda pq.$$

Lemma 2.2. *r and s satisfy*

$$(18) \quad r_t - \lambda r_x = \frac{1}{4}\lambda^2(2q - 3p)r + \frac{1}{32}\lambda^3p^2(4p - 5q),$$

$$(19) \quad s_t + \lambda s_x = \frac{1}{4}\lambda^2(2p - 3q)s + \frac{1}{32}\lambda^3q^2(5p - 4q).$$

Proof. Let

$$L_1 = \frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x} \quad \text{and} \quad L_2 = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}.$$

Then,

$$\begin{aligned} L_1 p_x &= \frac{1}{4}\lambda^2((2q - 3p)p_x + pq_x) + \frac{1}{8}\lambda^3p(p - q)^2, \\ L_2 q_x &= \frac{1}{4}\lambda^2((2p - 3q)q_x + qp_x) - \frac{1}{8}\lambda^3q(p - q)^2. \end{aligned}$$

Noting that

$$\begin{aligned} L_1(\lambda p) &= -\frac{1}{4}\lambda^3p(q + p), & L_2(\lambda q) &= -\frac{1}{4}\lambda^3q(p + q), \\ L_1 q &= -\frac{1}{16}\lambda^3pq^2 - \frac{1}{8}L_1(\lambda pq), & L_2 p &= \frac{1}{16}\lambda^3qp^2 + \frac{1}{8}L_2(\lambda pq), \end{aligned}$$

by a direct calculation, we easily prove (18) and (19). □

Noting that (5), (14) and (16), we write at $t = 0$

$$(20) \quad \begin{aligned} p &= p_0(x) \equiv u_1(x) + \lambda_0(x)u'_0(x), & u &= u_0(x), \\ q &= q_0(x) = u_1(x) - \lambda_0(x)u'_0(x), \end{aligned}$$

where $\lambda_0(x) = 1/\sqrt{u_0(x)}$.

Theorem 2.3 (the existence domain $D(T)$ of the classical solution to (5)). *If there exists a positive constant M_1 such that*

$$(21) \quad 0 \leq p(x, t) \leq M_1 \quad \text{and} \quad 0 \leq q(x, t) \leq M_1,$$

then on $D(T)$,

$$\begin{aligned} |u(x, t)| &\leq M(T), & |u_x(x, t)| &\leq M(T), & |u_t(x, t)| &\leq M(T), \\ |r(x, t)| &\leq M(T), & |s(x, t)| &\leq M(T), \end{aligned}$$

where $M(T)$ is a positive constant, and

$$D(T) = \{(x, t) \mid x \in \mathbb{R}, 0 \leq t \leq T, T > 0\}.$$

Then, by the local existence theorem of the classical solution to quasilinear hyperbolic systems, (5) admits a unique global classical solution on $t \geq 0$.

Remark 2.4. Theorem 2.3 shows that if a singularity occurs in a C^1 solution to Cauchy problem (17) and (20) in finite time, then the solution itself should go to infinity at the starting point of the singularity [Li and Liu 2003].

Proof of Theorem 2.3. Through any point (t, x) , let

$$x = x_1(t, \beta_1), \quad x = x_2(t, \beta_2), \quad x = x_3(t, \beta_3)$$

be the λ_1, λ_2 and λ_3 characteristics, respectively, that satisfy

$$\begin{aligned} \frac{dx_1}{dt} = \lambda_1 = -\lambda, & \quad \frac{dx_2}{dt} = \lambda_2 = 0, & \quad \frac{dx_3}{dt} = \lambda_3 = \lambda, \\ x_1(0, \beta_1) = \beta_1, & \quad x_2(0, \beta_2) = \beta_2, & \quad x_3(0, \beta_3) = \beta_3. \end{aligned}$$

Along the λ_2 characteristics, integrating (17b) yields

$$(22) \quad u(x, t) = u_0(\beta_2) + \frac{1}{2} \int_0^t (p + q)(x_2(\tau, \beta_2), \tau) d\tau.$$

Then, it follows from (17b), (21) and (22) that

$$|u_t| \leq M_1(T) \quad \text{and} \quad 0 < \inf_x u_0(x) \leq u(x, t) \leq M_2(T);$$

hereafter the $M_i(T)$ for $i = 1, 2, \dots$ denote positive constants.

Along the λ_1 characteristics $x = x_1(t, \beta_1)$, we have by integrating (18) and noting (21) that

$$|r(x, t)| \leq M_2(T) + M_3(T) \int_0^t R(\tau) d\tau, \quad \text{where } R(t) = \sup_x |r(x, t)|,$$

Thus, by the Bellman lemma, we get $|r(x, t)| \leq M_4(T)$. Similarly, we can prove $|s(x, t)| \leq M_5(T)$. Noting that $(u_x)_t = \frac{1}{2}(r + s)$, it is easy to see that

$$|u_x(x, t)| \leq M_6(T). \quad \square$$

3. Global classical solution: Proof of Theorem 1.2

According to the local existence and uniqueness theorems of the classical solutions to the quasilinear hyperbolic systems [Li and Yu 1985], to prove Theorem 1.2 it suffices to establish uniform *a priori* estimates of the C^1 norms of p, q and u .

Lemma 3.1 [Hong 1995; Kong 1998]. *Consider*

$$\begin{aligned} u_t + \lambda_1(x, t)u_x &= A(x, t)(u - v), \\ v_t + \lambda_2(x, t)v_x &= B(x, t)(v - u), \end{aligned}$$

where λ_1, λ_2, A and B are continuous functions, and $\lambda_1 \leq \lambda_2$. If A and B are both nonpositive, then

$$\min(u_0(x), v_0(x)) \leq u(x, t), v(x, t) \leq \max(u_0(x), v_0(x)).$$

Lemma 3.2. *On the existence domain of the classical solution to the Cauchy problem (17) and (20), if (7) and (8) hold, then*

$$(23) \quad 0 \leq p(x, t) \leq \sup_{x \in \mathbb{R}} p_0(x),$$

$$(24) \quad 0 \leq q(x, t) \leq \sup_{x \in \mathbb{R}} q_0(x),$$

$$(25) \quad 0 < \inf_{x \in \mathbb{R}} u_0(x) \leq u(x, t) \leq \sup_x u_0(x) + Ct,$$

where $C > 0$ is a constant.

Proof. Along the λ_1 characteristics, we have

$$p(x, t) = p_0(\beta_1) \exp\left(\int_0^t \frac{1}{4} \lambda^2(q - p)(x_1(\tau, \beta_1), \tau) d\tau\right).$$

By (7) and (20), we have $p_0(x) \geq 0$ for all $x \in \mathbb{R}$. Thus, we obtain $p(x, t) \geq 0$. Similarly, we can prove $q(x, t) \geq 0$. By these two inequalities, we have

$$\frac{1}{4} \lambda^2 p \geq 0 \quad \text{and} \quad \frac{1}{4} \lambda^2 q \geq 0.$$

Thus, following Lemma 3.1, it is easy to see that

$$p(x, t) \leq \sup_x p_0(x) \quad \text{and} \quad q(x, t) \leq \sup_x q_0(x).$$

Integrating (17b), we get

$$u(x, t) = u_0(\beta_2) + \frac{1}{2} \int_0^t (p + q)(x_2(\tau, \beta_2), \tau) d\tau.$$

Thus, because $p(x, t) \geq 0$ and $q(x, t) \geq 0$, we obtain (25). □

Proof of Theorem 1.2. It follows easily now from Lemma 3.2 and Theorem 2.3. □

Remark 3.3. By (4) and (25), and under the assumptions of Theorem 1.2, we have

$$|R(x, t)| \leq M_7(T).$$

4. Blow-up of classical solutions: Proof of Theorem 1.3

We now investigate the blow-up phenomena of the hyperbolic geometric flow.

Let

$$(26) \quad m = \sqrt{\lambda}p \quad \text{and} \quad n = \sqrt{\lambda}q.$$

Noting that

$$\frac{1}{4}\lambda^2q = \frac{1}{4}((\ln u)_t - \lambda(\ln u)_x) \quad \text{and} \quad \frac{1}{4}\lambda^2p = \frac{1}{4}((\ln u)_t + \lambda(\ln u)_x),$$

then, following (14), (16), and (17), we can prove the following lemma.

Lemma 4.1. *m and n satisfy*

$$(27) \quad m_t - \lambda m_x = -\frac{1}{4}\lambda^{3/2}m^2,$$

$$(28) \quad n_t + \lambda n_x = -\frac{1}{4}\lambda^{3/2}n^2.$$

At $t = 0$, set

$$(29) \quad \begin{aligned} m &= m_0(x) = \frac{1}{\sqrt[4]{u_0(x)}} \left(u_1(x) + \frac{u'_0(x)}{\sqrt{u_0(x)}} \right), \\ n &= n_0(x) = \frac{1}{\sqrt[4]{u_0(x)}} \left(u_1(x) - \frac{u'_0(x)}{\sqrt{u_0(x)}} \right). \end{aligned}$$

Proof of Theorem 1.3. Without loss of generality, we suppose that (9) holds; if (10) holds, we proceed similarly.

It follows from (27) and (28) that $m_t - \lambda m_x \leq 0$ and $n_t + \lambda n_x \leq 0$. Thus, it is easy to see that

$$(30) \quad m(x, t) + n(x, t) \leq M_0 \quad \text{and} \quad M_0 \equiv \sup m_0(x) + \sup n_0(x).$$

Noting that $u_0(x) \geq k > 0$, it follows from (9) and (29) that $m_0(x_0) < 0$.

Integrating (27), along λ_1 characteristics, we get

$$(31) \quad m(x_0, t) = m_0(x_0)/F(t, x_0),$$

where

$$(32) \quad F(t, x_0) = 1 + m_0(x_0)/4 \int_0^t \lambda^{3/2}(x_1(x_0, \tau), \tau) d\tau \quad \text{and} \quad \lambda^{3/2} = u^{-3/4}.$$

By (17b) and (26), it is easy to see that $(u^{3/4})_t = \frac{3}{8}(m + n)$. Therefore, we have

$$(33) \quad u^{3/4}(x, t) = u_0^{3/4}(x_0) + \frac{3}{8} \int_0^t (m + n)(x_2(x_0, \tau), \tau) d\tau.$$

By (2), (30) and (33), we have

$$(34) \quad u^{3/4}(x, 0) \geq k^{3/4} \quad \text{and} \quad u^{3/4}(x, t) \leq M^{3/4} + \frac{3}{8}M_0t.$$

Case (i). If $M_0 < 0$, then there exists $\tau_0 = 8M^{3/4}/(3(-M_0)) > 0$, such that

$$u(x, t) \leq 0 \quad \text{and} \quad t \geq \tau_0.$$

This implies the system in (5) is meaningless for $t \geq \tau_0$, that is, it admits a unique local classical solution.

Case (ii). If $M_0 = 0$, then, by (32) and (34), it is easy to find that

$$F(x_0, t) \leq 1 + \frac{1}{4}m_0(x_0)M^{-3/4}t.$$

Since $F(x_0, 0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_0 = 4M^{3/4}/(-m_0(x_0)) > 0$, such that

$$(35) \quad F(x_0, t) \rightarrow 0^+ \quad \text{as} \quad t \rightarrow t_0^-.$$

So that there exists finite time $T = T(x_0) > 0$, such that

$$(36) \quad m(x_0, t) \rightarrow -\infty \quad \text{as} \quad t \rightarrow T^-.$$

Case (iii). If $M_0 > 0$, then, it follows from (32) and (34) that

$$F(x_0, t) \leq 1 + \frac{2m_0(x_0)}{3M_0} \ln\left(1 + \frac{3M_0}{8M^{3/4}}t\right).$$

Thus, noting that $F(x_0, 0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_* > 0$ such that $F(x_0, t) \rightarrow 0^+$ as $t \rightarrow t_*^-$, and then (36) follows. \square

5. Asymptotic behavior: Proof of Theorem 1.5

We next will give the asymptotic behavior of the scalar curvature $R(x, t)$.

Proof of Theorem 1.5. If (13) holds, then, by Lemmas 3.1 and 3.2, we have

$$C_1 \leq p(x, t) \leq C_2 \quad \text{and} \quad C_1 \leq q(x, t) \leq C_2,$$

where here and below C_i for $i = 1, 2, \dots$ denote positive constants independent of t and x . Thus, by (2) and (22), one can get

$$(37) \quad C_3(1+t) \leq u(x, t) \leq C_4(1+t),$$

and then

$$C_3^{3/4}(1+t)^{3/4} \leq u^{3/4}(x, t) \leq C_4^{3/4}(1+t)^{3/4} \leq C_4^{3/4}(1+t).$$

It follows from (32) and (37) that

$$C_5(1 + \ln(1+t)) \leq F(x, t) \leq C_6(1 + (1+t)^{1/4}).$$

Therefore, by (13) and (31), we obtain

$$(38) \quad 0 \leq \frac{C_7}{1 + (1+t)^{1/4}} \leq m(x, t) \leq \frac{C_8}{1 + \ln(1+t)}.$$

Similarly, we have

$$(39) \quad 0 \leq \frac{C_7}{1 + (1+t)^{1/4}} \leq n(x, t) \leq \frac{C_8}{1 + \ln(1+t)}.$$

Thus, $m(x, t) \rightarrow 0$ and $n(x, t) \rightarrow 0$ as $t \rightarrow +\infty$.

Noting (37), (38) and (39), we have

$$p = \frac{1}{\sqrt{\lambda}}m = u^{1/4}m, \quad q = \frac{1}{\sqrt{\lambda}}n = u^{1/4}n, \quad u_x = \frac{p-q}{2\lambda} = \frac{1}{2}u^{3/4}(m-n),$$

and

$$(40) \quad -\frac{C_8}{1 + \ln(1+t)} \leq m(x, t) - n(x, t) \leq \frac{C_8}{1 + \ln(1+t)}.$$

Thus, it is easy to see that

$$(41) \quad |u_x| \leq C_9 \frac{(1+t)^{3/4}}{1 + \ln(1+t)}.$$

It is easy to derive that

$$(42) \quad u_{xx} = \frac{p_x - q_x}{2\lambda} + \frac{1}{2}\lambda^2 u^2 = \frac{1}{2}u^{1/2}(p_x - q_x) + \frac{1}{2u}u_x^2.$$

Let $\bar{p} = p/u$ and $\bar{q} = q/u$. Then, following [Kong et al. 2009], we have

$$(43) \quad L_1 \bar{p}_x = -A_1 \bar{p}_x - B_1 \bar{q}_x,$$

$$(44) \quad L_2 \bar{q}_x = -A_2 \bar{p}_x - B_2 \bar{q}_x,$$

where

$$A_1 = \frac{1}{4}(2\bar{q} + 3\bar{q}), \quad B_1 = \frac{3}{4}\bar{p}, \quad A_2 = \frac{3}{4}\bar{q}, \quad B_2 = \frac{1}{4}(2\bar{p} + 3\bar{q}).$$

Thus, by [Kong et al. 2009], one can get

$$(45) \quad |\bar{p}_x(x, t)|, |\bar{q}_x(x, t)| \leq C_{10}.$$

Noting that $p_x - q_x = u_x(p - q)/u + u(\bar{p}_x - \bar{q}_x)$, and using (4) and (42), we have

$$(46) \quad R = \frac{1}{u^3}(u_x^2 - uu_{xx}) = \frac{u_x^2}{2u^3} - \frac{p_x - q_x}{2u^{3/2}} = \frac{u_x^2}{2u^3} - \frac{\bar{p}_x - \bar{q}_x}{2u^{1/2}} - \frac{u_x(m-n)}{2u^3}.$$

Therefore, it follows from (34), (40), (41), (45) and (46) that

$$|R(x, t)| \leq \frac{C_{11}}{(1 + \ln(1+t))^2(1+t)^{3/2}} + \frac{C_{12}}{(1+t)^{1/2}} + \frac{C_{13}(1+t)^{3/4}}{(1+t)^3(1 + \ln(1+t))^2}.$$

Therefore, $R(x, t) \rightarrow 0$ as $t \rightarrow +\infty$. □

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