GLOBAL CLASSICAL SOLUTIONS TO HYPERBOLIC GEOMETRIC FLOW ON RIEMANN SURFACES

FAGUI LIU AND YUANZHANG ZHANG

We consider the Cauchy problem for hyperbolic geometric flow equations introduced recently by Kong and Liu motivated by the Einstein equation and Hamilton Ricci flow, and obtain a necessary and sufficient condition for the global existence of classical solutions to this kind of flow on Riemann surfaces. The results show that the scalar curvature of the solution metric $g_{ij}$ converges to one of flat curvature, and the hyperbolic geometric flow has the advantage that the surgery technique may be replaced by choosing a suitable initial velocity tensor.

1. Introduction

Let $\mathcal{M}$ be an $n$-dimensional complete Riemann manifold with Riemann metric $g_{ij}$. The general evolution equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathcal{F}(g, \frac{\partial g}{\partial t}) = 0$$

for the metric $g_{ij}$ has been recently introduced by Kong and Liu [2007] and called the generalized hyperbolic geometric flow (denoted by HGF). Here $\mathcal{F}$ are some given smooth functions of the Riemann metric $g$ and its first derivative with respect to $t$, and $R_{ij}$ are the components of the Ricci curvature tensor. In this paper, we may the HGF the Kong–Liu hyperbolic geometric flow. The local existence and nonlinear stability have been proved by Dai, Kong and Liu [≥ 2010; 2008]. See the celebrated survey paper [Kong 2008] for progress on this new topic.

In this paper, we study the evolution of a Riemann metric $g_{ij}$ on a Riemann surface $\mathcal{M}$ by its Ricci curvature tensor $R_{ij}$ under the HGF equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}.$$

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Here, we are interested in the initial metric
\[ ds^2 = u_0(x)(dx^2 + dy^2) \] at \( t = 0 \)
on a surface of topological type \( \mathbb{R}^2 \), where \( u_0(x) \) is a \( C^2 \) function with bounded \( C^2 \) norm and satisfies
\[ 0 < k \leq u_0(x) \leq M < \infty, \]
where \( k \) and \( M \) are positive constants.

On this surface, the HGF equation simplifies because all of the information about curvature is contained in the scalar curvature function \( R \). In our notation, \( R = 2K \), where \( K \) is the Gauss curvature. The Ricci curvature is given by \( R_{ij} = \frac{1}{2}Rg_{ij} \), so
\[ \frac{\partial^2 g_{ij}}{\partial t^2} = -R_{ij}. \]
The metric for a surface can always be written (at least locally) as \( g_{ij} = u(t, x, y)\delta_{ij} \), where \( u(t, x, y) > 0 \), and \( \delta_{ij} \) is Kronecker’s symbol. Therefore, we have
\[ R = -\frac{\Delta \ln u}{u}. \]
Thus, Equation (3) reduces to \( u_{tt} - \Delta \ln u = 0 \).

The initial data \( u_0(x) \) depends only on \( x \) and not \( y \); therefore, we may consider the Cauchy problem
\[ \begin{cases} u_{tt} - (\ln u)_{xx} = 0, \\ u = u_0(x) \quad \text{at} \quad t = 0, \\ u_t = u_1(x) \quad \text{at} \quad t = 0, \end{cases} \]
where \( u_1(x) \in C^1 \) with bounded \( C^1 \) norm.

By using the transformation
\[ \phi = \ln u, \]
Kong and Liu have proved the following theorem.

**Theorem 1.1** [Kong et al. 2009]. Suppose that \( u_1(x) \geq |u_0'(x)|/\sqrt{u_0(x)} \) for all \( x \in \mathbb{R} \). Then, the Cauchy problem (5) admits a unique global solution for all \( t \in \mathbb{R} \).

Moreover, if \( u_1(x) \equiv u_0'(x)/\sqrt{u_0(x)} \), and there exists a point \( x_0 \in \mathbb{R} \) such that \( u_0'(x_0) < 0 \), then the Cauchy problem (5) admits a unique classical solution only in \( [0, T) \times \mathbb{R} \), where
\[ T = -\frac{2}{\inf_x \{u_0'(x)u_0^{-3/2}(x)\}}. \]

In this paper, we will prove the following theorems without using (6).
Theorem 1.2. Suppose that
\[
\begin{align*}
\partial_t u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}} &\geq 0 \quad \text{for all } x \in \mathbb{R}, \\
\partial_t u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}} &\geq 0 \quad \text{for all } x \in \mathbb{R}.
\end{align*}
\]
Then (5) admits a unique global solution for all \( t \in \mathbb{R} \).

Theorem 1.3. If there exists a point \( x_0 \in \mathbb{R} \), such that
\[
\begin{align*}
\partial_t u_1(x_0) + \frac{u_0'(x_0)}{\sqrt{u_0(x_0)}} &< 0, \\
\partial_t u_1(x_0) - \frac{u_0'(x_0)}{\sqrt{u_0(x_0)}} &< 0,
\end{align*}
\]
then (5) admits a unique classical solution only in \([0, T) \times \mathbb{R}\).

Remark 1.4. Following Theorem 1.2, it is easy to see that Cauchy problem
\[
\begin{align*}
\partial_t^2 g_{ij} &= -2R_{ij} \quad \text{for } i, j = 1, 2, \\
g_{ij} &= u_0(x) \delta_{ij} \quad \text{for } t = 0, \\
\partial_t g_{ij} &= u_1(x) \delta_{ij} \quad \text{for } t = 0 \text{ and } i, j = 1, 2
\end{align*}
\]
has a unique smooth solution for all time \( t \in \mathbb{R} \), and the solution metric \( g_{ij} \) assumes the form
\[
\begin{align*}
g_{ij} &= u(x, t) \delta_{ij} \quad \text{for } i, j = 1, 2.
\end{align*}
\]

Theorems 1.2 and 1.3 will be proved in Sections 3 and 4. Using Theorem 1.2, we can further prove this:

Theorem 1.5. Suppose that
\[
\begin{align*}
\inf_x \left\{ u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}} \right\} > 0 \quad \text{and} \quad \inf_x \left\{ u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}} \right\} > 0.
\end{align*}
\]
Then (11) has a unique classical solution of the form (12) for all time. Moreover the scalar curvature \( R(x, t) \) corresponding to the solution metric \( g_{ij} \) satisfies
\[
R(x, t) \to 0 \quad \text{as } t \to +\infty,
\]
and there exists positive constant \( k_1 > 0 \) independent of \( t \) and \( x \) such that
\[
R(x, t) \leq k_1 \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}.
\]
Theorem 1.5 shows that any metric on a simply connected noncompact surface converges to one of flat curvature by choosing a suitable initial velocity tensor \( \partial g_{ij}(x, 0)/\partial t \), but the general case of a metric in \( \mathbb{R}^2 \) is still open. Theorem 1.3 shows that if we do not choose suitable a initial velocity tensor \( \partial g_{ij}(x, 0)/\partial t \), the solution to Cauchy problem will blow up in finite time.

2. Preliminaries

We need only discuss the classical solution on \( t \geq 0 \). The result for \( t \leq 0 \) can be obtained by changing \( t \) to \( -t \).

Let

\[
(14) \quad u_t = v \quad \text{and} \quad u_x = w.
\]

Then, it follows from the first equation in (5) and (14) that

\[
(15) \quad u_t = v, \quad w_t - v_x = 0, \quad v_t - (1/u)w_x = -w^2/u^2.
\]

It is easy to see that the eigenvalues of equations (15) are

\[ \lambda_1 = -\lambda, \quad \lambda_2 = 0, \quad \lambda_3 = \lambda, \quad \lambda = 1/\sqrt{u}, \]

and the matrix \( L(U) \) (where \( U = (u, w, v)^T \)) of left eigenvectors and the matrix \( R(U) \) of right eigenvectors are respectively

\[
L(U) = \begin{pmatrix} l_1(U) \\ l_2(U) \\ l_3(U) \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & -\lambda & 1 \end{pmatrix},
\]

\[
R(U) = (r_1(U), r_2(U), r_3(U)) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & 0 & -\lambda \\ 1 & 0 & 1 \end{pmatrix}.
\]

Since \( \nabla \lambda_i(U)r_i(U) \equiv 0 \) for \( i = 1, 2, 3 \), (15) is a linear degenerate strict hyperbolic system.

Set

\[
(16) \quad p = v + \lambda w \quad \text{and} \quad q = v - \lambda w.
\]

Lemma 2.1. \( p \) and \( q \) satisfy

\[
(17a) \quad p_t - \lambda p_x = \frac{1}{4}\lambda^2 p(q - p),
\]

\[
(17b) \quad u_t = \frac{1}{2}(p + q),
\]

\[
(17c) \quad q_t + \lambda q_x = \frac{1}{4}\lambda^2 q(p - q).
\]
Proof. Noting that \( \lambda_t = -\frac{1}{2}\lambda^3 v \) and \( \lambda_x = -\frac{1}{2}\lambda^3 w \), we calculate

\[
p_t - \lambda p_x = (v + \lambda w)_t - \lambda (v + \lambda w)_x
\]

\[
= v_t - \lambda v_x + \lambda (w_t - \lambda w_x) + w(\lambda_t - \lambda \lambda_x)
\]

\[
= -\frac{1}{2}\lambda^3 w(v + \lambda w) = \frac{1}{4}\lambda^2 (q - p)p.
\]

In a similar way, we can prove (17c), and (17b) is obvious. □

Let

\[ r = p_x + \frac{1}{8}\lambda pq \quad \text{and} \quad s = q_x - \frac{1}{8}\lambda pq. \]

**Lemma 2.2.** \( r \) and \( s \) satisfy

\[ r_t - \lambda r_x = \frac{1}{4}\lambda^2 (2q - 3p)r + \frac{1}{32}\lambda^3 p^2(4p - 5q), \]

\[ s_t + \lambda s_x = \frac{1}{4}\lambda^2 (2p - 3q)s + \frac{1}{32}\lambda^3 q^2(5p - 4q). \]

**Proof.** Let

\[ L_1 = \frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x} \quad \text{and} \quad L_2 = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}. \]

Then,

\[ L_1 p_x = \frac{1}{4}\lambda^2 ((2q - 3p) p_x + pq_x) + \frac{1}{8}\lambda^3 p(p - q)^2, \]

\[ L_2 q_x = \frac{1}{4}\lambda^2 ((2p - 3q) q_x + q p_x) - \frac{1}{8}\lambda^3 q(p - q)^2. \]

Noting that

\[
L_1(\lambda p) = -\frac{1}{4}\lambda^3 p(q + p), \quad L_2(\lambda q) = -\frac{1}{2}\lambda^3 q(p + q),
\]

\[
L_1 q = -\frac{1}{16}\lambda^3 pq^2 - \frac{1}{8}L_1(\lambda pq), \quad L_2 p = \frac{1}{16}\lambda^3 qp^2 + \frac{1}{8}L_2(\lambda pq),
\]

by a direct calculation, we easily prove (18) and (19). □

Noting that (5), (14) and (16), we write at \( t = 0 \)

\[ p = p_0(x) \equiv u_1(x) + \lambda_0(x) u_0'(x), \quad u = u_0(x), \]

\[ q = q_0(x) = u_1(x) - \lambda_0(x) u_0'(x), \]

where \( \lambda_0(x) = 1/\sqrt{u_0(x)} \).

**Theorem 2.3** (the existence domain \( D(T) \) of the classical solution to (5)). If there exists a positive constant \( M_1 \) such that

\[ 0 \leq p(x, t) \leq M_1 \quad \text{and} \quad 0 \leq q(x, t) \leq M_1, \]

then on \( D(T) \),

\[ |u(x, t)| \leq M(T), \quad |u_x(x, t)| \leq M(T), \quad |u_t(x, t)| \leq M(T), \]

\[ |r(x, t)| \leq M(T), \quad |s(x, t)| \leq M(T), \]
where $M(T)$ is a positive constant, and

$$D(T) = \{(x, t) \mid x \in \mathbb{R}, \ 0 \leq t \leq T, \ T > 0\}.$$ 

Then, by the local existence theorem of the classical solution to quasilinear hyperbolic systems, (5) admits a unique global classical solution on $t \geq 0$.

**Remark 2.4.** Theorem 2.3 shows that if a singularity occurs in a $C^1$ solution to Cauchy problem (17) and (20) in finite time, then the solution itself should go to infinity at the starting point of the singularity [Li and Liu 2003].

**Proof of Theorem 2.3.** Through any point $(t, x)$, let

$$x = x_1(t, \beta_1), \quad x = x_2(t, \beta_2), \quad x = x_3(t, \beta_3)$$

be the $\lambda_1$, $\lambda_2$ and $\lambda_3$ characteristics, respectively, that satisfy

$$\frac{dx_1}{dt} = \lambda_1 = -\lambda, \quad \frac{dx_2}{dt} = \lambda_2 = 0, \quad \frac{dx_3}{dt} = \lambda_3 = \lambda,$$

$$x_1(0, \beta_1) = \beta_1, \quad x_2(0, \beta_2) = \beta_2, \quad x_3(0, \beta_3) = \beta_3.$$

Along the $\lambda_2$ characteristics, integrating (17b) yields

$$u(x, t) = u_0(\beta_2) + \frac{1}{2} \int_0^t (p + q)(x_2(\tau, \beta_2), \tau)d\tau. \tag{22}$$

Then, it follows from (17b), (21) and (22) that

$$|u_t| \leq M_1(T) \quad \text{and} \quad 0 < \inf_x u_0(x) \leq u(x, t) \leq M_2(T);$$

hereafter the $M_i(T)$ for $i = 1, 2, \ldots$ denote positive constants.

Along the $\lambda_1$ characteristics $x = x_1(t, \beta_1)$, we have by integrating (18) and noting (21) that

$$|r(x, t)| \leq M_2(T) + M_3(T) \int_0^t R(\tau)d\tau, \quad \text{where} \ R(t) = \sup_x |r(x, t)|,$$

Thus, by the Bellman lemma, we get $|r(x, t)| \leq M_4(T)$. Similarly, we can prove $|s(x, t)| \leq M_5(T)$. Noting that $(u_x)_t = \frac{1}{2}(r + s)$, it is easy to see that

$$|u_x(x, t)| \leq M_6(T). \square$$

### 3. Global classical solution: Proof of Theorem 1.2

According to the local existence and uniqueness theorems of the classical solutions to the quasilinear hyperbolic systems [Li and Yu 1985], to prove Theorem 1.2 it suffices to establish uniform a priori estimates of the $C^1$ norms of $p$, $q$ and $u$. 
Lemma 3.1 [Hong 1995; Kong 1998]. Consider

$$u_t + \lambda_1(x, t)u_x = A(x, t)(u - v),$$
$$v_t + \lambda_2(x, t)v_x = B(x, t)(v - u),$$

where $\lambda_1, \lambda_2, A$ and $B$ are continuous functions, and $\lambda_1 \leq \lambda_2$. If $A$ and $B$ are both nonpositive, then

$$\min(u_0(x), v_0(x)) \leq u(x, t), v(x, t) \leq \max(u_0(x), v_0(x)).$$

Lemma 3.2. On the existence domain of the classical solution to the Cauchy problem (17) and (20), if (7) and (8) hold, then

(23) $0 \leq p(x, t) \leq \sup_{x \in \mathbb{R}} p_0(x),$
(24) $0 \leq q(x, t) \leq \sup_{x \in \mathbb{R}} q_0(x),$
(25) $0 < \inf_{x \in \mathbb{R}} u_0(x) \leq u(x, t) \leq \sup_x u_0(x) + Ct,$

where $C > 0$ is a constant.

Proof. Along the $\lambda_1$ characteristics, we have

$$p(x, t) = p_0(\beta_1) \exp\left(\int_0^t \frac{1}{4} \beta^2_1 (q - p)(x_1(\tau, \beta_1), \tau) d\tau\right).$$

By (7) and (20), we have $p_0(x) \geq 0$ for all $x \in \mathbb{R}$. Thus, we obtain $p(x, t) \geq 0$. Similarly, we can prove $q(x, t) \geq 0$. By these two inequalities, we have

$$\frac{1}{4} \beta^2_1 p \geq 0 \quad \text{and} \quad \frac{1}{4} \beta^2_1 q \geq 0.$$

Thus, following Lemma 3.1, it is easy to see that

$$p(x, t) \leq \sup_x p_0(x) \quad \text{and} \quad q(x, t) \leq \sup_x q_0(x).$$

Integrating (17b), we get

$$u(x, t) = u_0(\beta_2) + \frac{1}{2} \int_0^t (p + q)(x_2(\tau, \beta_2), \tau) d\tau.$$

Thus, because $p(x, t) \geq 0$ and $q(x, t) \geq 0$, we obtain (25). □

Proof of Theorem 1.2. It follows easily now from Lemma 3.2 and Theorem 2.3. □

Remark 3.3. By (4) and (25), and under the assumptions of Theorem 1.2, we have

$$|R(x, t)| \leq M_7(T).$$
4. Blow-up of classical solutions: Proof of Theorem 1.3

We now investigate the blow-up phenomena of the hyperbolic geometric flow.

Let

\[ m = \sqrt{\lambda} p \quad \text{and} \quad n = \sqrt{\lambda} q. \]

Noting that

\[ \frac{1}{4} \lambda^2 q = \frac{1}{4}((\ln u)_t - \lambda (\ln u)_x) \quad \text{and} \quad \frac{1}{4} \lambda^2 p = \frac{1}{4}((\ln u)_t + \lambda (\ln u)_x), \]

then, following (14), (16), and (17), we can prove the following lemma.

**Lemma 4.1.** \( m \) and \( n \) satisfy

\[ m_t - \lambda m_x = -\frac{1}{4} \lambda^{3/2} m^2, \]

\[ n_t + \lambda n_x = -\frac{1}{4} \lambda^{3/2} n^2. \]

At \( t = 0 \), set

\[ m = m_0(x) = \frac{1}{\sqrt{u_0(x)}} \left( u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}} \right), \]

\[ n = n_0(x) = \frac{1}{\sqrt{u_0(x)}} \left( u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}} \right). \]

**Proof of Theorem 1.3.** Without loss of generality, we suppose that (9) holds; if (10) holds, we proceed similarly.

It follows from (27) and (28) that \( m_t - \lambda m_x \leq 0 \) and \( n_t + \lambda n_x \leq 0 \). Thus, it is easy to see that

\[ m(x, t) + n(x, t) \leq M_0 \quad \text{and} \quad M_0 \equiv \sup m_0(x) + \sup n_0(x). \]

Noting that \( u_0(x) \geq k > 0 \), it follows from (9) and (29) that \( m_0(x_0) < 0 \).

Integrating (27), along \( \lambda_1 \) characteristics, we get

\[ m(x_0, t) = m_0(x_0) / F(t, x_0), \]

where

\[ F(t, x_0) = 1 + m_0(x_0)/4 \int_0^t \lambda^{3/2}(x_1(x_0, \tau), \tau) d\tau \quad \text{and} \quad \lambda^{3/2} = u^{-3/4}. \]

By (17b) and (26), it is easy to see that \( (u^{3/4})_t = \frac{3}{8}(m + n) \). Therefore, we have

\[ u^{3/4}(x, t) = u_0^{3/4}(x_0) + \frac{3}{8} \int_0^t (m + n)(x_2(x_0, \tau), \tau) d\tau. \]

By (2), (30) and (33), we have

\[ u^{3/4}(x, 0) \geq k^{3/4} \quad \text{and} \quad u^{3/4}(x, t) \leq M^{3/4} + \frac{3}{8} M_0 t. \]
Case (i). If $M_0 < 0$, then there exists $\tau_0 = 8M^{3/4}/(3(-M_0)) > 0$, such that
\[ u(x, t) \leq 0 \quad \text{and} \quad t \geq \tau_0. \]
This implies the system in (5) is meaningless for $t \geq \tau_0$, that is, it admits a unique local classical solution.

Case (ii). If $M_0 = 0$, then, by (32) and (34), it is easy to find that
\[ F(x_0, t) \leq 1 + \frac{1}{4}m_0(x_0)M^{-3/4}t. \]
Since $F(x_0, 0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_0 = 4M^{3/4}/(-m_0(x_0)) > 0$, such that
\[ F(x_0, t) \to 0^+ \quad \text{as} \quad t \to t_0^-. \]
So that there exists finite time $T = T(x_0) > 0$, such that
\[ m(x_0, t) \to -\infty \quad \text{as} \quad t \to T^-. \]

Case (iii). If $M_0 > 0$, then, it follows from (32) and (34) that
\[ F(x_0, t) \leq 1 + \frac{2m_0(x_0)}{3M_0} \ln \left(1 + \frac{3M_0}{8M^{3/4}t}\right). \]
Thus, noting that $F(x_0, 0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_* > 0$ such that
$F(x_0, t) \to 0^+$ as $t \to t_*^-$, and then (36) follows. \qed

5. Asymptotic behavior: Proof of Theorem 1.5

We next will give the asymptotic behavior of the scalar curvature $R(x, t)$.

Proof of Theorem 1.5. If (13) holds, then, by Lemmas 3.1 and 3.2, we have
\[ C_1 \leq p(x, t) \leq C_2 \quad \text{and} \quad C_1 \leq q(x, t) \leq C_2, \]
where here and below $C_i$ for $i = 1, 2, \ldots$ denote positive constants independent of $t$ and $x$. Thus, by (2) and (22), one can get
\[ C_3(1 + t) \leq u(x, t) \leq C_4(1 + t), \]
and then
\[ C_3^{3/4}(1 + t)^{3/4} \leq u^{3/4}(x, t) \leq C_4^{3/4}(1 + t)^{3/4} \leq C_4^{3/4}(1 + t). \]
It follows from (32) and (37) that
\[ C_5(1 + \ln(1 + t)) \leq F(x, t) \leq C_6(1 + (1 + t)^{1/4}). \]
Therefore, by (13) and (31), we obtain
\[
0 \leq \frac{C_7}{1 + (1 + t)^{1/4}} \leq m(x, t) \leq \frac{C_8}{1 + \ln(1 + t)}.
\]
Similarly, we have
\[
0 \leq \frac{C_7}{1 + (1 + t)^{1/4}} \leq n(x, t) \leq \frac{C_8}{1 + \ln(1 + t)}.
\]
Thus, \(m(x, t) \to 0\) and \(n(x, t) \to 0\) as \(t \to +\infty\).

Noting (37), (38) and (39), we have
\[
p = \frac{1}{\sqrt{\lambda}} m = u^{1/4} m, \quad q = \frac{1}{\sqrt{\lambda}} n = u^{1/4} n, \quad u_x = \frac{p - q}{2\lambda} = \frac{1}{2} u^{3/4} (m - n),
\]
and
\[
- \frac{C_8}{1 + \ln(1 + t)} \leq m(x, t) - n(x, t) \leq \frac{C_8}{1 + \ln(1 + t)}.
\]
Thus, it is easy to see that
\[
|u_x| \leq C_9 \frac{(1 + t)^{3/4}}{1 + \ln(1 + t)}.
\]
It is easy to derive that
\[
u_{xx} = \frac{p_x - q_x}{2\lambda} + \frac{1}{2} \lambda^2 u_x^2 = \frac{1}{2} u^{1/2} (p_x - q_x) + \frac{1}{2 u^2} u_x^2.
\]
Let \(\bar{p} = p/u\) and \(\bar{q} = q/u\). Then, following [Kong et al. 2009], we have
\[
L_1 \bar{p}_x = -A_1 \bar{p}_x - B_1 \bar{q}_x,
\]
\[
L_2 \bar{q}_x = -A_2 \bar{q}_x - B_2 \bar{q}_x,
\]
where
\[
A_1 = \frac{1}{4} (2\bar{q} + 3\bar{q}), \quad B_1 = \frac{3}{4} \bar{p}, \quad A_2 = \frac{3}{4} \bar{q}, \quad B_2 = \frac{1}{4} (2\bar{p} + 3\bar{q}).
\]
Thus, by [Kong et al. 2009], one can get
\[
|\bar{p}_x(x, t)|, |\bar{q}_x(x, t)| \leq C_{10}.
\]
Noting that \(p_x - q_x = u_x (p - q)/u + u(\bar{p}_x - \bar{q}_x)\), and using (4) and (42), we have
\[
R = \frac{1}{u^2} (u_x^2 - uu_{xx}) = \frac{u_x^2}{2 u^3} - \frac{p_x - q_x}{2 u^{3/2}} = \frac{u_x^2}{2 u^3} - \frac{\bar{p}_x - \bar{q}_x}{2 u^{1/2}} - \frac{u_x (m - n)}{2 u^3}.
\]
Therefore, it follows from (34), (40), (41), (45) and (46) that
\[
|R(x, t)| \leq \frac{C_{11}}{(1 + \ln(1 + t))^2 (1 + t)^{3/2}} + \frac{C_{12}}{(1 + t)^{1/2}} + \frac{C_{13} (1 + t)^{3/4}}{(1 + t)^3 (1 + \ln(1 + t))^2}.
\]
Therefore, \( R(x, t) \to 0 \) as \( t \to +\infty \). □

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References


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FA GU I LIU
NORTH CHINA INSTITUTE OF WATER CONSERVANCY AND HYDROELECTRIC POWER
ZHENGZHOU 450011
CHINA
liufagui@ncwu.edu.cn

YUANZHANG ZHANG
NORTH CHINA INSTITUTE OF WATER CONSERVANCY AND HYDROELECTRIC POWER
ZHENGZHOU 450011
CHINA
zhangyuanzhang@ncwu.edu.cn
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EDITORS

V. S. Varadarajan (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Darren Long
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
long@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Jonathan Rogawski
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
jongr@math.ucla.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

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