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ELLIPTIC PSEUDODIFFERENTIAL EQUATIONS AND SOBOLEV SPACES OVER p-ADIC FIELDS

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We study the solutions of equations of type $f(D, \alpha)u = v$, where $f(D, \alpha)$ is a p-adic pseudodifferential operator. If v is a Bruhat–Schwartz function, there exists a distribution E_α, a fundamental solution, such that $u = E_\alpha * v$ is a solution. However, it is unknown to which function space $E_\alpha * v$ belongs. We show that if $f(D, \alpha)$ is an elliptic operator, then $u = E_\alpha * v$ belongs to a certain Sobolev space, and we give conditions for the continuity and uniqueness of u. By modifying the Sobolev norm, we establish that $f(D, \alpha)$ gives an isomorphism between certain Sobolev spaces.

1. Introduction

In recent years p-adic analysis has received much attention due to its applications in mathematical physics; see [Albeverio and Karwowski 1994; Avetisov et al. 2002; Avetisov et al. 2003; Khrennikov 1994; 1997; Kochubeĭ 1993; Rammal et al. 1986; Varadarajan 1997; Vladimirov et al. 1994] and references therein. Many new mathematical matters have emerged, among them, p-adic pseudodifferential equations [Albeverio et al. 2006; Chuong and Co 2008; Khrennikov 1992; Kochubeĭ 1991; 1993; 1998; 2001; 2008, Rodríguez-Vega and Zúñiga-Galindo 2008; Vladimirov et al. 1994; Zúñiga-Galindo 2003; 2004; 2008]. Here we study the solutions of p-adic elliptic pseudodifferential equations on Sobolev spaces.

A pseudodifferential operator $f(D, \beta)$ is an operator of the form

$$(f(D,\alpha)\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}(|f(\xi)|_p^\alpha \mathcal{F}_{x\to\xi}\varphi(x)) \quad \text{for } \varphi \in S,$$

where $\mathcal{F}$ denotes the Fourier transform, α is a positive real number, S denotes the $\mathbb{C}$-vector space of Bruhat–Schwartz functions over $\mathbb{Q}_p^n$, and $f(\xi) \in \mathbb{Q}_p[\xi_1, \ldots, \xi_n]$. If $f(\xi)$ is a homogeneous polynomial of degree d satisfying $f(\xi) = 0$ if and only if $\xi = 0$, then the corresponding operator is called an elliptic pseudodifferential operator. This operator is considered to be a p-adic analogue of a linear partial elliptic differential operator with constant coefficients. A p-adic pseudodifferential

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equation is an equation of type $f(D,\alpha)u = v$. If $v \in S$, there is a distribution E_α, a fundamental solution, such that $u = E_\alpha * v$ is a solution. Zúñiga-Galindo [2003] established the existence of a fundamental solution for general pseudodifferential operators by adapting the proof given by Atiyah [1970] for the Archimedean case. However, it is unknown to which function space $E_\alpha * v$ belongs. Here, we show that if $f(D,\alpha)$ is an elliptic operator, then $u = E_\alpha * v$ belongs to a certain Sobolev space (see Theorem 19), and we give conditions for the continuity and uniqueness of u. By modifying the Sobolev norm, we can establish that $f(D,\alpha)$ gives an isomorphism between certain Sobolev spaces; see Propositions 22 and 23 and Theorem 24. Our approach is based on the explicit calculation of fundamental solutions of pseudodifferential operators on certain function spaces and the fact that elliptic pseudodifferential operators behave like the Taibleson operator when acting on certain function spaces; see Theorems 13 and 14.

2. Preliminary results

We summarize some basic facts about p-adic analysis. For a complete exposition, see [Taibleson 1975; Vladimirov et al. 1994].

Let $\mathbb{Q}_p$ be the field of the p-adic numbers, and let $\mathbb{Z}_p$ be the ring of p-adic integers. For $x \in \mathbb{Q}_p$, let $v(x) \in \mathbb{Z} \cup \{\infty\}$ denote the valuation of x normalized by the condition $v(p) = 1$. By definition $v(x) = \infty$ if and only if $x = 0$. Let $|x|_p = p^{-v(x)}$ be the normalized absolute value. Here, by definition $|x|_p = 0$ if and only if $x = 0$. We extend the p-adic absolute value to $\mathbb{Q}_p^n$ as follows:

$$\|x\|_p := \max\{|x_1|_p, \dots, |x_n|_p\} \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

We define the *exponent of local constancy* of $\varphi(x) \in S(\mathbb{Q}_p^n)$ as the smallest nonnegative integer l with the property that, for any $x \in \mathbb{Q}_p^n$,

$$\varphi(x + x') = \varphi(x) \quad \text{if } \|x'\|_p \le p^{-l}.$$

For x and y in $\mathbb{Q}_p^n$, we put $x \cdot y := \sum_{i=1}^n x_i y_i$.

Let Ψ denote an additive character of $\mathbb{Q}_p$ that is trivial on $\mathbb{Z}_p$ but not on $p^{-1}\mathbb{Z}_p$. For $\varphi \in S(\mathbb{Q}_p^n)$, we define its Fourier transform as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \Psi(x \cdot \xi)\varphi(x)\, dx,$$

where dx denotes the Haar measure of $\mathbb{Q}_p^n$ normalized so that $\mathbb{Z}_p^n$ has measure one.

We denote by χ_r for $r \in \mathbb{Z}$ the characteristic function of the polydisc $B_r(0) := (p^r\mathbb{Z}_p)^n$. For any $\varphi \in S$, we set $r_\varphi := \min\{r \in \mathbb{N} \mid \varphi|_{B_r(0)} = \varphi(0)\}$.

Definition 1. We set $\mathcal{L} := \mathcal{L}(\mathbb{Q}_p^n) = \{\varphi \in S \mid \int_{\mathbb{Q}_p^n} \varphi(x)\, dx = 0\}$, and $\mathcal{W} := \mathcal{W}(\mathbb{Q}_p^n)$ to be the $\mathbb{C}$-vector space generated by the functions χ_r for $r \in \mathbb{Z}$.

We note that any $\varphi \in S$ can be written uniquely as $\varphi_{\mathscr{L}} + \varphi_{\mathscr{W}}$, where

$$\varphi_{\mathscr{W}} = p^{r_\varphi n}\Big(\int_{\mathbb{Q}_p^n} \varphi(x)\,dx\Big)\chi_{r_\varphi} \in \mathscr{W} \quad \text{and} \quad \varphi_{\mathscr{L}} = \varphi - \varphi_{\mathscr{W}} \in \mathscr{L}.$$

However, S is not the direct sum of $\mathscr{L}$ and $\mathscr{W}$. The space $\mathscr{W}$ was introduced in [Zúñiga-Galindo 2004], and $\{\mathscr{F}(\varphi) \mid \varphi \in \mathscr{L}\}$ is a Lizorkin space of second class [Albeverio et al. 2006].

2.1. ***Elliptic pseudodifferential operators.*** Let $f(\xi) \in \mathbb{Q}_p[\xi_1, \dots, \xi_n]$ be a nonconstant polynomial. A pseudodifferential operator $f(D, \alpha)$ for $\alpha > 0$ with symbol $|f(\xi)|_p^\alpha$ is an operator of the form $(f(D, \alpha)\varphi) = \mathscr{F}^{-1}(|f|_p^\alpha \mathscr{F}\varphi)$ where $\varphi \in S$.

Definition 2. Let $f(\xi) \in \mathbb{Q}_p[\xi_1, \dots, \xi_n]$ be a nonconstant polynomial. We say that $f(\xi)$ is an elliptic polynomial of degree d if $f(\xi)$ is a homogeneous polynomial of degree d and if $f(\xi) = 0$ if and only if $\xi = 0$.

Lemma 3 [Zúñiga-Galindo 2008, Lemma 1]. *Let $f(\xi) \in \mathbb{Q}_p[\xi_1, \dots, \xi_n]$ be an elliptic polynomial of degree d. There exist positive constants $C_0(f)$ and $C_1(f)$ such that*

$$C_0(f)\|\xi\|_p^d \le |f(\xi)|_p \le C_1(f)\|\xi\|_p^d \quad \textit{for every } \xi \in \mathbb{Q}_p^n.$$

We note that if $f(\xi)$ is elliptic, then $cf(\xi)$ is elliptic for any $c \in \mathbb{Q}_p^\times$. For this reason, we will assume from now on that the elliptic polynomials have coefficients in $\mathbb{Z}_p$.

Lemma 4 [Zúñiga-Galindo 2008, Lemma 3]. *Let $f(\xi) \in \mathbb{Q}_p[\xi_1, \dots, \xi_n]$ be an elliptic polynomial of degree d. Let $A \subset \mathbb{Q}_p^n$ be a compact subset such that $0 \notin A$. Then there exists a positive integer $m = m(A, f)$ such that $|f(\xi)|_p \ge p^{-m}$ for any $\xi \in A$. Also, for any covering of A of the form $\bigcup_{i=1}^L B_i$ with $B_i = z_i + (p^m\mathbb{Z}_p)^n$, we have $|f(\xi)|_p = |f(z_i)|_p$ for any $\xi \in B_i$.*

Definition 5. Let $f(\xi) \in \mathbb{Z}_p[\xi_1, \dots, \xi_n]$ be an elliptic polynomial of degree d. We say that $|f|_p^\beta$ is an elliptic symbol, and that $f(D, \beta)$ is an elliptic pseudodifferential operator of order d.

2.2. ***Igusa's local zeta functions.*** Let $g(x) \in \mathbb{Q}_p[x]$ for $x = (x_1, \dots, x_n)$ be a nonconstant polynomial. Igusa's local zeta function associated to $g(x)$ is the distribution

$$\langle |g|_p^s, \varphi\rangle = \int_{\mathbb{Q}_p^n \setminus g^{-1}(0)} |g(x)|_p^s \varphi(x)\,dx,$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, where $\varphi \in S$ and dx denotes the normalized Haar measure of $\mathbb{Q}_p^n$. The local zeta functions were introduced by Weil, and their basic properties for general $g(x)$ were first studied by Igusa. A central result in the theory of local zeta functions is that $|g|_p^s$ admits a meromorphic continuation to the complex plane such that $\langle |g|_p^s, \varphi\rangle$ is rational function of p^{-s} for each $\varphi \in S$. Furthermore, there exists a finite set $\bigcup_{E\in\mathscr{E}}\{(N_E, n_E)\}$ of pairs of positive integers such that

$$\prod_{E\in\mathscr{E}}(1-p^{-n_E-N_E s})\,|g|_p^s$$

is a holomorphic distribution on S. In particular, the real parts of the poles of $|g|_p^s$ are negative rational numbers [Igusa 2000, Chapter 8]. The existence of a meromorphic continuation for the distribution $|g|_p^s$ implies that a fundamental solution exists for the pseudodifferential operator with symbol $|g|_p^\alpha$ [Zúñiga-Galindo 2003].

For a fixed $\varphi \in S$, we denote the integral $\langle |g|_p^s, \varphi\rangle$ by $Z_\varphi(s, g)$. In particular, $Z(s, g) = Z_{\chi_0}(s, g)$.

Lemma 6. *Let $f(x) \in \mathbb{Z}_p[x]$ for $x = (x_1, \dots, x_n)$ be an elliptic polynomial of degree d. Then*

$$Z(s, f) = \frac{L(p^{-s})}{1-p^{-ds-n}},$$

where $L(p^{-s})$ is a polynomial in p^{-s} with rational coefficients. Also, $s = -n/d$ is a pole of $Z(s, f)$.

Proof. Let $A = \{x \in \mathbb{Z}_p^n \mid \operatorname{ord}(x_i) \geq d,\ i = 1, \dots, n\}$, and let A' be its complement in $\mathbb{Z}_p^n$, that is, $A' = \{x \in \mathbb{Z}_p^n \mid \operatorname{ord}(x_i) < d \text{ for some } i\}$. Then

$$Z(s, f) = \int_A |f(x)|_p^s\,dx + \int_{A'} |f(x)|_p^s\,dx = p^{-ds-n} Z(s, f) + \int_{A'} |f(x)|_p^s\,dx,$$

that is, $Z(s, f) = (1-p^{-ds-n})^{-1}\int_{A'}|f(x)|_p^s\,dx$. Since A' is compact, by applying Lemma 4, we can find a covering of A' by sets B_i with $i = 1, \dots, L$, where $|f|_p$ is constant on each B_i. Hence

$$\int_{A'}|f(x)|_p^s\,dx = p^{-nm}\sum_{i=1}^{L}|f(z_i)|_p^s \quad\text{and}\quad Z(s, f) = \frac{p^{-nm}\sum_{i=1}^{L}|f(z_i)|_p^s}{1-p^{-ds-n}}. \qquad\square$$

2.3. ***The Riesz kernel.*** We collect some well-known results about the Riesz kernel. See [Taibleson 1975] or [Vladimirov et al. 1994] for further details.

The p-adic *Gamma function* $\Gamma_p^{(n)}(s)$ is defined by

$$\Gamma_p^{(n)}(s) = \frac{1-p^{s-n}}{1-p^{-s}} \quad \text{for } s \in \mathbb{C} \text{ and } s \neq 0.$$

The Gamma function is meromorphic with simple zeros at $n + (2\pi i/\ln p)\mathbb{Z}$ and a unique simple pole at $s = 0$. In addition, it satisfies

$$\Gamma_p^{(n)}(s)\Gamma_p^{(n)}(n-s) = 1 \quad \text{for } s \notin \{0\} \cup \{n + (2\pi i/\ln p)\mathbb{Z}\}.$$

The *Riesz kernel* R_s is the distribution determined by the function

$$R_s(x) = \frac{\|x\|_p^{s-n}}{\Gamma_p^{(n)}(s)} \quad \text{for } \operatorname{Re}(s) > 0,\ s \notin n + (2\pi i/\ln p)\mathbb{Z} \text{ and } x \in \mathbb{Q}_p^n.$$

The Riesz kernel has, as a distribution, a meromorphic continuation to $\mathbb{C}$ given by

$$\begin{aligned}\langle R_s(x), \varphi(x)\rangle = \frac{1-p^{-n}}{1-p^{s-n}}\varphi(0) + \frac{1-p^{-s}}{1-p^{s-n}} \int_{\|x\|_p>1} \|x\|_p^{s-n}\varphi(x)\,dx \\ + \frac{1-p^{-s}}{1-p^{s-n}} \int_{\|x\|_p\le 1} \|x\|_p^{s-n}(\varphi(x)-\varphi(0))\,dx,\end{aligned}$$

with poles at $n + (2\pi i/\ln p)\mathbb{Z}$. In particular, for $\operatorname{Re}(s) > 0$,

$$\langle R_s(x), \varphi(x)\rangle = \frac{1-p^{-s}}{1-p^{s-n}} \int_{\mathbb{Q}_p^n} \varphi(x)\|x\|_p^{s-n}\,dx \quad \text{for } s \notin n + (2\pi i/\ln p)\mathbb{Z},$$

$$\langle R_{-s}(x), \varphi(x)\rangle = \frac{1-p^{s}}{1-p^{-s-n}} \int_{\mathbb{Q}_p^n} (\varphi(x)-\varphi(0))\|x\|_p^{-s-n}\,dx. \tag{1}$$

In the case $s = 0$, by passing to the limit, we obtain

$$\langle R_0(x), \varphi(x)\rangle := \lim_{s\to 0}\langle R_s(x), \varphi(x)\rangle = \varphi(0),$$

that is, $R_0(x) = \delta(x)$, the Dirac delta function. Therefore, $R_s \in S'(\mathbb{Q}_p^n)$ for $s \in \mathbb{C}\backslash\{n + (2\pi i/\ln p)\mathbb{Z}\}$.

Remark 7. For $\operatorname{Re}(s) > 0$, the distribution $\|x\|_p^s$ admits the following meromorphic continuation:

$$\begin{aligned}\langle \|x\|_p^s, \varphi(x)\rangle = \frac{1-p^{-n}}{1-p^{-s-n}}\varphi(0) + \int_{\|x\|_p>1} \|x\|_p^{s}\varphi(x)\,dx \\ + \int_{\|x\|_p\le 1} \|x\|_p^{s}(\varphi(x)-\varphi(0))\,dx \quad \text{for } \varphi \in S.\end{aligned}$$

In particular, all the poles of $\|x\|_p^s$ have real part equal to $-n$.

Lemma 8 [Taibleson 1975, Chapter III, Theorem 4.5]. *As element of $S'(\mathbb{Q}_p^n)$, $(\mathcal{F}R_s)(x)$ equals $\|x\|_p^{-s}$ for $s \notin n + (2\pi i/\ln p)\mathbb{Z}$.*

Lemma 9. *For $x = (x_1, \ldots, x_n)$, let $f(x) \in \mathbb{Q}_p[x]$ be an elliptic polynomial of degree d. Then*

$$|f|_p^s = \frac{(1-p^{ds})L(p^{-s})}{(1-p^{-n})(1-p^{-ds-n})} R_{ds+n} \quad \textit{for } s \in \mathbb{C}$$

as distributions on $\mathcal{W}$. *Here* $L(p^{-s})$ *is the numerator of* $Z(s, f)$, *which is a polynomial in* p^{-s} *with rational coefficients.*

Proof. Let $\varphi \in \mathcal{W}$, then $\varphi(x) = \sum_i c_i \chi_{r_i}(x)$, where $c_i \in \mathbb{C}$ and $r_i \in \mathbb{Z}$ (recall that $\mathcal{F}(\chi_r) = p^{-nr}\chi_{-r}$). The action of $|f|_p^s$ on $\mathcal{F}\varphi$ can be explicitly described by

$$\langle |f|_p^s, \mathcal{F}\varphi\rangle = \sum_i c_i \langle |f|_p^s, p^{-nr_i}\chi_{-r_i}\rangle.$$

However

$$\langle |f|_p^s, p^{-nr_i}\chi_{-r_i}\rangle = p^{-nr_i}\int_{\mathbb{Q}_p^n} |f(x)|_p^s \chi_{-r_i}(x)\,dx = p^{dr_i s} Z(s, f),$$

for $\mathrm{Re}(s) > 0$, so $\langle |f|_p^s, \mathcal{F}\varphi\rangle = Z(s, f)\sum_i c_i p^{dr_i s}$ for $\mathrm{Re}(s) > 0$.

On the other hand,

$$\left\langle \frac{1-p^{ds}}{1-p^{-n}} R_{ds+n}, p^{-nr_i}\chi_{-r_i}\right\rangle = \left\langle \frac{1-p^{-ds-n}}{1-p^{-n}} \|x\|_p^{ds}, p^{-nr_i}\chi_{-r_i}\right\rangle = p^{dr_i s}$$

for every $r_i \in \mathbb{Z}$ and $\mathrm{Re}(s) > 0$. Then we have

$$\langle |f|_p^s, \mathcal{F}\varphi\rangle = \frac{1-p^{ds}}{1-p^{-n}} Z(s, f)\langle R_{ds+n}, \mathcal{F}\varphi\rangle \quad \text{for } \mathrm{Re}(s) > 0.$$

Now $Z(s, f)$ and R_{ds+n} have a meromorphic continuation to the complex plane; therefore this formula extends to $\mathbb{C}$. Finally, since the Fourier transform establishes a $\mathbb{C}$-isomorphism on $\mathcal{W}$, it is possible remove the Fourier transform symbol. □

2.4. *The Taibleson operator.*

Definition 10. The Taibleson pseudodifferential operator D_T^α for $\alpha > 0$ is defined as

$$(D_T^\alpha\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}(\|\xi\|_p^\alpha \mathcal{F}_{x\to\xi}\varphi) \quad \text{for } \varphi \in S.$$

As a consequence of the Lemma 8 and (1), one gets

$$(D_T^\alpha\varphi)(x) = (k_{-\alpha} * \varphi)(x) = \frac{1-p^\alpha}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n} \|y\|_p^{-\alpha-n}(\varphi(x-y)-\varphi(x))\,dy.$$

The right side of this formula makes sense for a wider class of functions than $S(\mathbb{Q}_p)$, such as the class $\mathfrak{E}_\alpha(\mathbb{Q}_p^n)$ of locally constant functions $\varphi(x)$ satisfying

$$\int_{\|x\|_p\geq 1} \|x\|_p^{-\alpha-n}|\varphi(x)|\,dx < \infty.$$

Remark 11. As a consequence of the previous observations we may assume that the constant functions are contained in the domain of D_T^α, and that $D_T^\alpha\varphi = 0$, for any constant function.

3. Fundamental solutions for the Taibleson operator

We now consider the pseudodifferential equation

$$D_T^\alpha u = v \quad \text{with } v \in S \text{ and } \alpha > 0. \tag{2}$$

We say that $E_\alpha \in S'$ is a *fundamental solution* of (2) if $E_\alpha * v$ is a solution.

Lemma 12. *If E_α is a fundamental solution of* (2), *so is $E_\alpha + c$ for c a constant.*

Proof. Let E_α be a fundamental solution for (2). Then

$$D_T^\alpha((E_\alpha + c) * v) = D_T^\alpha((E_\alpha * v) + (c * v)) = v + D_T^\alpha(c * v) = v,$$

because u and the constant function $c * v$ are in the domain of D_T^α. □

Theorem 13. *A fundamental solution of* (2) *is*

$$E_\alpha(x) = \begin{cases} \dfrac{1 - p^{-\alpha}}{1 - p^{\alpha - n}} \|x\|_p^{\alpha - n} & \text{if } \alpha \neq n, \\ \dfrac{1 - p^n}{p^n \ln p} \ln(\|x\|_p) & \text{if } \alpha = n. \end{cases}$$

Proof. The proof is based on the ideas introduced in [Zúñiga-Galindo 2003]. The existence of a fundamental solution E_α is equivalent to that of a distribution $\mathcal{F}E_\alpha$ satisfying

$$\|x\|_p^\alpha \mathcal{F}E_\alpha = 1, \tag{3}$$

as distributions. Let $\|x\|_p^s = \sum_{m \in \mathbb{Z}} c_m (s + \alpha)^m$ be the Laurent expansion at $-\alpha$ with $c_m \in S'$ for all m. The existence of this expansion is a consequence of the completeness of S'; see for example [Igusa 2000, pages 65–66]. Since the real parts of the poles of the meromorphic continuation of $\|x\|_p^s$ are negative rational numbers (see Remark 7), $\|x\|_p^{s+\alpha} = \|x\|_p^\alpha \|x\|_p^s$ is holomorphic at $s = -\alpha$. Therefore, $\|x\|_p^\alpha c_m = 0$ for all $m < 0$ and

$$\|x\|_p^{s+\alpha} = \|x\|_p^\alpha c_0 + \sum_{m=1}^{\infty} \|x\|_p^\alpha c_m (s + \alpha)^m.$$

By using the Lebesgue dominated convergence theorem, one verifies that

$$\lim_{s \to -\alpha} \langle \|x\|_p^{s+\alpha}, \varphi \rangle = \int_{\mathbb{Q}_p^n} \varphi(x)\, dx = \langle 1, \varphi \rangle,$$

and then we can take $\mathcal{F}E_\alpha = c_0$. Furthermore, if $-\alpha$ is not a pole of $\|x\|_p^s$,

$$\mathcal{F}E_\alpha = \lim_{s \to -\alpha} \|x\|_p^s. \tag{4}$$

To calculate c_0, consider the following two cases.

Case $\alpha \neq n$. We use (4) and Lemma 8, that is

$$\langle \|x\|_p^s, (\mathcal{F}\varphi)(x)\rangle = \frac{1-p^s}{1-p^{-s-n}} \langle \|x\|_p^{-s-n}, \varphi(x)\rangle \quad \text{for } s \neq n + (2\pi i/\ln p)\mathbb{Z}.$$

Since $\alpha \neq n$ we have by (4) and Remark 7

$$\langle E_\alpha, \mathcal{F}\varphi\rangle = \lim_{s\to -\alpha} \langle \|x\|_p^s, \varphi(x)\rangle = \langle \|x\|_p^{-s}, \varphi(x)\rangle,$$

that is, $E_\alpha = \|x\|_p^{s-n} / \Gamma_p^{(n)}(\alpha)$.

Case $\alpha = n$. We compute the constant term, c_0, in the expansion

$$\langle \|x\|_p^s, \mathcal{F}\varphi\rangle = \sum_{m\in\mathbb{Z}} \langle c_m, \mathcal{F}(\varphi)\rangle (s+m)^s.$$

Now

$$\langle \|x\|_p^s, (\mathcal{F}\varphi)(x)\rangle = (1-p^s)\left\langle \frac{\|x\|_p^{-s-n}}{1-p^{-s-n}}, \varphi(x)\right\rangle = (1-p^s)\left\langle \frac{p^{v(x)(s+n)}}{1-p^{-s-n}}, \varphi(x)\right\rangle,$$

where $x = (x_1, \dots, x_n)$, $v(x) := \min_{1\le i\le n} v(x_i)$, and $\|x\|_p = p^{-v(x)}$. Therefore by expanding

$$\begin{aligned}\frac{(1-p^s)p^{v(x)(s+n)}}{1-p^{-s-n}} &= \frac{1-p^{-n}}{\ln p}(s+n)^{-1} \\ &\quad + \frac{(1-p^{-n})v(x)\ln p - \frac{\ln p}{p^n} + \frac{1}{2}(1-p^{-n})\ln p}{\ln p} + O((s+n)),\end{aligned}$$

one gets

$$\langle E_n, \varphi\rangle = \langle c_0, \varphi\rangle = \left\langle \frac{1-p^n}{p^n \ln(p)} \ln(\|x\|_p) + \frac{p^n - 3}{2p^n}, \varphi(x)\right\rangle.$$

The claim follows from the fact that a fundamental solution is determined up to the addition of a constant; see Lemma 12. □

In case $n = 1$, Theorem 13 is already known; see [Kochubeĭ 2001, Theorem 2.1], for example.

4. Fundamental solutions for elliptic operators

Theorem 14. *Let $f(D, \alpha)$ be an elliptic operator of order d. Then a fundamental solution E_α of $f(D, \alpha)u = v$ for $\alpha > 0$ and $v \in \mathcal{W}$ is given by*

$$E_\alpha(x) = \begin{cases} \dfrac{L(p^\alpha)(1-p^{-d\alpha})}{(1-p^{-n})(1-p^{d\alpha-n})}\|x\|_p^{d\alpha-n} & \text{if } \alpha \neq n/d, \\ \dfrac{L(p^{n/d})(1-p^n)}{(1-p^{-n})(p^n \ln p)} \ln(\|x\|_p) & \text{if } \alpha = n/d, \end{cases}$$

where the equality is as distributions on $\mathcal{W}$ *and* $L(p^{-s})$ *is the numerator of* $Z(s, f)$.

Proof. As mentioned previously, the existence of a fundamental solution E_α is equivalent to that of a distribution $\mathcal{F}E_\alpha$ satisfying $|f|_p^\alpha \mathcal{F}E_\alpha = 1$ in S'. By Lemma 9,

$$\langle |f|_p^\alpha, \varphi\rangle = \left\langle \frac{(1-p^{d\alpha})L(p^{-\alpha})}{(1-p^{-n})(1-p^{-d\alpha-n})} R_{d\alpha+n}, \varphi \right\rangle \quad \text{for } \varphi \in \mathcal{W} \text{ and } s \in \mathbb{C}.$$

The result follows by reasoning as in the proof of Theorem 13, and by the fact that the space $\mathcal{W}$ is invariant under the Fourier transform. □

Corollary 15. *With the hypotheses of the previous theorem, and assuming that* $\alpha \neq n/d$, *we have*

$$|\mathcal{F}(E_\alpha * \varphi)(x)| \leq C(\alpha) \|x\|_p^{-d\alpha} |\mathcal{F}(\varphi)(x)| \quad \textit{for all } x \in \mathbb{Q}_p^n \textit{ and } \varphi \in \mathcal{W}.$$

5. Solutions of elliptic pseudodifferential equations in Sobolev spaces

Given $\varphi \in S$ and l a nonnegative number, we define

$$\|\varphi\|_{H^l}^2 = \int_{\mathbb{Q}_p^n} (\max(1, \|\xi\|_p))^{2l} |\mathcal{F}(\varphi)(\xi)|^2 \, d\xi.$$

The completion of S with respect to $\|\cdot\|_{H^l}$ is the *l-Sobolev space* $H^l := H^l(\mathbb{Q}_p^n)$.

We note that H^l contains properly the space S of test functions. Indeed, consider the function

$$f(x) = \begin{cases} 0 & \text{if } \|x\|_p \leq 1, \\ \|x\|_p^{-\beta} & \text{if } \|x\|_p > 1 \end{cases}$$

with $\beta > n$. A direct calculation shows that

$$\|f\|_{H^l}^2 = \int_{\|\xi\|_p \leq 1} \left| \frac{(1-p^{-n})(1-\|\xi\|_p^{\beta-n} p^{n-\beta})}{(1-p^{n-\beta})} - p^{-\beta} \|\xi\|_p^{\beta-n} \right|^2 d\xi.$$

Thus, $\|f\|_{H^l}^2 < \infty$, but f does not have compact support.

Lemma 16. *If* $l > n/2$, *then there exists an embedding of* H^l *into the space of uniformly continuous functions.*

Proof. Let $\varphi \in H^l$. Since the Fourier transform of a function in L^1 is uniformly continuous, it is sufficient to show that $\mathcal{F}(\varphi) \in L^1$. By using the Hölder inequality and the fact that

$$\int_{\mathbb{Q}_p^n} (\max(1, \|\xi\|_p))^{-2l} \, d\xi < +\infty \quad \text{for } l > n/2,$$

we have

$$\int_{\mathbb{Q}_p^n} |\mathcal{F}(\varphi)(\xi)| \, d\xi = \int_{\mathbb{Q}_p^n} \frac{(\max(1, \|\xi\|_p))^l}{(\max(1, \|\xi\|_p))^l} |\mathcal{F}(\varphi)(\xi)| \, d\xi \leq C \|\varphi\|_{H^l}. \qquad □$$

Lemma 17. *For any $\alpha > 0$ and $l \geq 0$, the mapping $f(D, \alpha) : H^{l+d\alpha} \to H^l$ is a well-defined continuous mapping between Banach spaces.*

Proof. Let $\varphi \in S$. Since $f(D, \alpha)$ is an elliptic operator, by Lemma 3, we have

$$\begin{aligned}\|f(D, \alpha)\varphi\|_{H^l}^2 &= \int_{\mathbb{Q}_p^n} (\max(1, \|\xi\|_p))^{2l} |f(\xi)|^{2\alpha} |\mathcal{F}(\varphi)(\xi)|^2 \, d\xi \\ &\leq C_1 \int_{\mathbb{Q}_p^n} (\max(1, \|\xi\|_p))^{2(l+d\alpha)} |\mathcal{F}(\varphi)(\xi)|^2 \, d\xi = C_1 \|\varphi\|_{H^{l+d\alpha}}^2. \quad \square\end{aligned}$$

The result follows from the fact that S is dense in $H^{l+d\alpha}$.

Remark 18. Let β be a positive real number, and let $I(\beta) := \int_{\|\varepsilon\|_p \leq 1} \|\varepsilon\|_p^\beta \, d\varepsilon$. Then

$$I(\beta) = \frac{1 - p^{-n}}{1 - p^{-n-n\beta}} \quad \text{for } \beta > -n.$$

Indeed,

$$I(\beta) = \int\limits_{\|\varepsilon\|_p < 1} \|\varepsilon\|_p^\beta \, d\varepsilon + \int\limits_{\|\varepsilon\|_p = 1} d\varepsilon = \int\limits_{\|\varepsilon\|_p < 1} \|\varepsilon\|_p^\beta \, d\varepsilon + 1 - p^{-n}.$$

By making the change of variables $\varepsilon_i = p x_i$ for $i = 1, \ldots, n$, we have

$$I(\beta) = p^{-n-n\beta} I(\beta) + 1 - p^{-n}.$$

Theorem 19. *Let $f(D, \alpha)$ for $0 < \alpha < n/2d$ be an elliptic pseudodifferential operator of order d. Let l be a positive real number satisfying $l > n/2$. Then $f(D, \alpha)u = v$ for $v \in S$ has a unique uniformly continuous solution $u \in H^{l+d\alpha}$.*

Proof. Let $v \in \mathcal{S}$. Then $v = v_{\mathcal{W}} + v_{\mathcal{L}}$, where $v_{\mathcal{W}} \in \mathcal{W}$ and $v_{\mathcal{L}} \in \mathcal{L}$. Thus, to prove the existence of a solution u, it is sufficient to show that the equations

$$f(D, \alpha) u_{\mathcal{W}} = v_{\mathcal{W}}, \tag{5}$$

$$f(D, \alpha) u_{\mathcal{L}} = v_{\mathcal{L}}. \tag{6}$$

have solutions.

We first consider (5). By Theorem 14, $u_{\mathcal{W}} = E_\alpha * v_{\mathcal{W}}$ is a solution of (5), and by Corollary 15, we have

$$\begin{aligned}\|u_{\mathcal{W}}\|^2_{H^{l+d\alpha}} &= \int_{\mathbb{Q}_p^n} (\max(1, \|\xi\|_p))^{2(l+d\alpha)} |\mathcal{F}(E_\alpha * v_{\mathcal{W}})(\xi)|^2 \, d\xi \\ &= C(\alpha, d, n) \int_{\mathbb{Q}_p^n} (\max(1, \|\xi\|_p))^{2(l+d\alpha)} \|\xi\|_p^{-2d\alpha} |\mathcal{F}(v_{\mathcal{W}})(\xi)|^2 \, d\xi \\ &= C(\alpha, d, n) \Big(\int_{\|\xi\|_p \le 1} \|\xi\|_p^{-2d\alpha} |\mathcal{F}(v_{\mathcal{W}})(\xi)|^2 \, d\xi \\ &\qquad\qquad + \int_{\|\xi\|_p > 1} \|\xi\|_p^{2l} |\mathcal{F}(v_{\mathcal{W}})(\xi)|^2 \, d\xi \Big).\end{aligned}$$

We now recall that $v_{\mathcal{W}}(\xi) = p^{rn} C \chi_r(\xi)$, with $r > 0$. Then, $\mathcal{F}(v_{\mathcal{W}})(\xi) = C \chi_{-r}(\xi)$ and

$$\begin{aligned}\|u_{\mathcal{W}}\|^2_{H^{l+d\alpha}} &\le C(\alpha, d, n) \Big(C^2 p^{2rn} \int_{\|\varepsilon\|_p \le 1} \|\varepsilon\|_p^{-2d\alpha} \, d\varepsilon + \|v_{\mathcal{W}}\|^2_{H^l} \Big) \\ &\le C(\alpha, d, n) \big(C_1(\alpha, d, n) + \|v_{\mathcal{W}}\|^2_{H^l} \big),\end{aligned}$$

since $-2d\alpha > -n$; see Remark 18. Therefore $u_{\mathcal{W}} \in H^{l+d\alpha}$.

We now consider Equation (6). Since $\mathcal{F}(u_{\mathcal{L}}) = \mathcal{F}(v_{\mathcal{L}}) |f|_p^{-\alpha}$ and f is elliptic,

$$|\mathcal{F}(u_{\mathcal{L}})(\xi)| \le C \|\xi\|^{-d\alpha} |\mathcal{F}(v_{\mathcal{L}})(\xi)| \quad \text{(see Lemma 3).}$$

Then

$$\|u_{\mathcal{L}}\|^2_{H^{l+d\alpha}} \le \int_{\|\xi\|_p \le 1} \|\xi\|_p^{-2d\alpha} |\mathcal{F}(v_{\mathcal{L}})(\xi)|^2 \, d\xi + \int_{\|\xi\|_p > 1} \|\xi\|_p^{2l} |\mathcal{F}(v_{\mathcal{L}})(\xi)|^2 \, d\xi.$$

The second integral is bounded by $\|v_{\mathcal{L}}\|^2_{H^l}$. For the first integral, we observe that if $0 < \alpha < n/2d$, then

$$\int_{\|\xi\|_p \le 1} \|\xi\|_p^{-2d\alpha} |\varphi(\xi)|^2 \, d\xi \le C \|\varphi\|_{L^2} \quad \text{for any } \varphi \in S.$$

Therefore,

$$\|u_{\mathcal{L}}\|^2_{H^{l+d\alpha}} \le C \|\mathcal{F}(v_{\mathcal{L}})\|_{L^2} + \|v_{\mathcal{L}}\|^2_{H^l}.$$

In this way, we see there exists a $u \in H^{l+d\alpha}$ that is uniformly continuous by Lemma 16 and is such that $f(D, \alpha)u = v$ for any $v \in S$. Finally, we show that u is unique. Indeed, if $f(D, \alpha)u' = v$, then

$$f(D, \alpha)(u - u') = 0, \quad \text{that is,} \quad |f|_p^\alpha \mathcal{F}(u - u') = 0,$$

and thus $\mathcal{F}(u - u')(\xi) = 0$ if $\xi \ne 0$, since f is elliptic. Then $\Psi(x \cdot \xi)(u - u')(\xi) = 0$ almost everywhere, and a fortiori $(u - u')(\xi) = 0$ almost everywhere, and by the continuity of $u - u'$, we have $u(\xi) = u'(\xi)$ for any $\xi \in \mathbb{Q}_p^n$. □

6. Solutions of elliptic pseudodifferential equations in singular Sobolev spaces

In this section, we modify the Sobolev norm to obtain spaces of functions on which $f(D,\alpha)$ gives a surjective mapping.

Definition 20. Given $\varphi \in S$ and l a nonnegative number, we set

$$\|\varphi\|^2_{\mathcal{H}^l} := \int_{\mathbb{Q}_p^n} \|\xi\|_p^{2l} |\mathcal{F}(\varphi)(\xi)|^2 \, d\xi.$$

We call the completion of S with respect to $\|\cdot\|_{\mathcal{H}^l}$ the *l-singular Sobolev space* $\mathcal{H}^l := \mathcal{H}^l(\mathbb{Q}_p^n)$. Note that $H^l \subseteq \mathcal{H}^l$ for $l \geq 0$ since $\|\varphi\|_{\mathcal{H}^l} \leq \|\varphi\|_{H^l}$.

Lemma 21. *For any $\alpha > 0$ and $l \geq 0$, the mapping $f(D,\alpha) : \mathcal{H}^{l+d\alpha} \to \mathcal{H}^l$ is a well-defined continuous mapping between Banach spaces.*

Proof. The proof is similar to that of Lemma 17. □

We denote by $\mathcal{L}^l$ and $\mathcal{W}^l$ the respective completions of $\mathcal{L}$ and $\mathcal{W}$ with respect to $\|\cdot\|_{\mathcal{H}^l}$. We set $\mathcal{H}_0^l := \mathcal{L}^l + \mathcal{W}^l \subseteq \mathcal{H}^l$.

Proposition 22. *Let $f(D,\alpha)$ for $\alpha > 0$ be an elliptic pseudodifferential operator of order d, and let l be a nonnegative real number. Then $f(D,\alpha) : \mathcal{H}^{l+d\alpha} \to \mathcal{W}^l$ is a surjective mapping between Banach spaces.*

Proof. By Lemma 21, the mapping is well defined. Let $v \in \mathcal{W}^l$, and let $\{v_n\}$ be a Cauchy sequence in $\mathcal{W}$ converging to v. By Theorem 14, there exists a sequence $\{u_n\}$ in $H^{l+d\alpha}$ such that $f(D,\alpha)u_n = v_n$. We see that $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}^{l+d\alpha}$ because

$$\begin{aligned} \|u_n - u_m\|^2_{\mathcal{H}^{l+d\alpha}} &\leq C \int_{\mathbb{Q}_p^n} \|\xi\|_p^{2(l+d\alpha)} \|\xi\|_p^{-2d\alpha} |\mathcal{F}(v_n - v_m)(\xi)|^2 \, d\xi \\ &\leq C \|v_n - v_m\|^2_{\mathcal{H}^l}. \end{aligned} \tag{7}$$

Thus, there exists $u \in \mathcal{H}^{l+d\alpha}$ such that $u_n \to u$, and by the continuity of $f(D,\alpha)$, we have $f(D,\alpha)u = v$. □

Proposition 23. *Let $f(D,\alpha)$ for $\alpha > 0$ be an elliptic pseudodifferential operator of order d, and let l be a nonnegative real number. Then $f(D,\alpha) : \mathcal{H}^{l+d\alpha} \to \mathcal{L}^l$ is a surjective mapping between Banach spaces.*

Proof. By Lemma 21, the mapping is well defined. Let $v \in \mathcal{L}^l$, and let $\{v_n\}$ be a Cauchy sequence in $\mathcal{L}$ converging to v. By the same reasoning given in proof Theorem 19 for establishing the existence of a solution to Equation (6), we obtain a sequence $\{u_n\}$ in $H^{l+d\alpha}$ such that $f(D,\alpha)u_n = v_n$. To show that $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}^{l+d\alpha}$, we use $|\mathcal{F}(u_n)(\xi)| \leq C\|\xi\|^{-d\alpha}|\mathcal{F}(v_n)(\xi)|$. Then we recover (7), and the proof finishes as before. □

From the previous two lemmas we obtain the following result.

Theorem 24. *Let $f(D,\alpha)$ be an elliptic pseudodifferential operator of order d. Let l be a positive real number. Then the equation $f(D,\alpha)u = v$ for $v \in \mathcal{H}_0^l$ has a unique solution $u \in \mathcal{H}^{l+d\alpha}$.*

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References

[Albeverio and Karwowski 1994] S. Albeverio and W. Karwowski, "A random walk on p-adics — the generator and its spectrum", *Stochastic Process. Appl.* **53**:1 (1994), 1–22. MR 96g:60088 Zbl 0810.60065

[Albeverio et al. 2006] S. Albeverio, A. Y. Khrennikov, and V. M. Shelkovich, "Harmonic analysis in the p-adic Lizorkin spaces: Fractional operators, pseudo-differential equations, p-adic wavelets, Tauberian theorems", *J. Fourier Anal. Appl.* **12**:4 (2006), 393–425. MR 2007g:47076 Zbl 1110.46049

[Atiyah 1970] M. F. Atiyah, "Resolution of singularities and division of distributions", *Comm. Pure Appl. Math.* **23** (1970), 145–150. MR 41 #815 Zbl 0188.19405

[Avetisov et al. 2002] V. A. Avetisov, A. H. Bikulov, S. V. Kozyrev, and V. A. Osipov, "p-adic models of ultrametric diffusion constrained by hierarchical energy landscapes", *J. Phys. A* **35**:2 (2002), 177–189. MR 2003j:82070 Zbl 1038.82077

[Avetisov et al. 2003] V. A. Avetisov, A. K. Bikulov, and V. A. Osipov, "p-adic description of characteristic relaxation in complex systems", *J. Phys. A* **36**:15 (2003), 4239–4246. MR 2004f:82055 Zbl 1049.82051

[Chuong and Co 2008] N. M. Chuong and N. V. Co, "The Cauchy problem for a class of pseudodifferential equations over p-adic field", *J. Math. Anal. Appl.* **340**:1 (2008), 629–645. MR 2009h:35475 Zbl 1153.35095

[Igusa 2000] J.-i. Igusa, *An introduction to the theory of local zeta functions*, AMS/IP Studies in Advanced Mathematics **14**, American Mathematical Society, Providence, RI, 2000. MR 2001j:11112 Zbl 0959.11047

[Khrennikov 1992] A. Y. Khrennikov, "Fundamental solutions over the field of p-adic numbers", *Algebra i Analiz* **4**:3 (1992), 248–266. In Russian; translated in *St. Petersburg Math. J.* **4**:3 (1993), 613–628. MR 93k:46060 Zbl 0828.35003

[Khrennikov 1994] A. Khrennikov, *p-adic valued distributions in mathematical physics*, Mathematics and its Applications **309**, Kluwer, Dordrecht, 1994. MR 96c:46080 Zbl 0833.46061

[Khrennikov 1997] A. Khrennikov, *Non-Archimedean analysis: Quantum paradoxes, dynamical systems and biological models*, Mathematics and its Applications **427**, Kluwer Academic Publishers, Dordrecht, 1997. MR 2001h:81004 Zbl 0920.11087

[Kochubeĭ 1991] A. N. Kochubeĭ, "Parabolic equations over the field of p-adic numbers", *Izv. Akad. Nauk SSSR Ser. Mat.* **55**:6 (1991), 1312–1330. In Russian; translated in *Math. USSR Izvestiya* **39** (1992), 1263–1280. MR 93e:35050

[Kochubeĭ 1993] A. N. Kochubeĭ, "A Schrödinger-type equation over the field of p-adic numbers", *J. Math. Phys.* **34**:8 (1993), 3420–3428. MR 94f:47059 Zbl 0780.35088

[Kochubeĭ 1998] A. N. Kochubeĭ, "Fundamental solutions of pseudodifferential equations associated with p-adic quadratic forms", *Izv. Ross. Akad. Nauk Ser. Mat.* **62**:6 (1998), 103–124. In Russian; translated in *Izvestiya Math.*, **62**:6 (1998), 1169–1188. MR 2000f:11158 Zbl 0934.35218

[Kochubeĭ 2001] A. N. Kochubeĭ, *Pseudo-differential equations and stochastics over non-Archimedean fields*, Pure and Applied Mathematics **244**, Marcel Dekker, New York, 2001. MR 2003b:35220 Zbl 0984.11063

[Kochubeĭ 2008] A. N. Kochubeĭ, "A non-Archimedean wave equation", *Pacific J. Math.* **235**:2 (2008), 245–261. MR 2009e:35305 Zbl 05366328

[Rammal et al. 1986] R. Rammal, G. Toulouse, and M. A. Virasoro, "Ultrametricity for physicists", *Rev. Modern Phys.* **58**:3 (1986), 765–788. MR 87k:82105

[Rodríguez-Vega and Zúñiga-Galindo 2008] Rodríguez-Vega, J. J., Zúñiga-Galindo, W. A., "Taibleson operators, p-adic parabolic equations and ultrametric diffusion", *Pacific J. Math.* **237**:2 (2008), 327–347. MR 2009i:35364 Zbl 05366373

[Taibleson 1975] M. H. Taibleson, *Fourier analysis on local fields*, Princeton University Press, 1975. MR 58 #6943 Zbl 0319.42011

[Varadarajan 1997] V. S. Varadarajan, "Path integrals for a class of p-adic Schrödinger equations", *Lett. Math. Phys.* **39**:2 (1997), 97–106. MR 98m:81083 Zbl 0868.47047

[Vladimirov et al. 1994] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-adic analysis and mathematical physics*, Series on Soviet and East European Mathematics **1**, World Scientific, River Edge, NJ, 1994. MR 95k:11155 Zbl 0812.46076

[Zúñiga-Galindo 2003] W. A. Zúñiga-Galindo, "Fundamental solutions of pseudo-differential operators over p-adic fields", *Rend. Sem. Mat. Univ. Padova* **109** (2003), 241–245. MR 2004e:11137 Zbl 1072.35215

[Zúñiga-Galindo 2004] W. A. Zúñiga-Galindo, "Pseudo-differential equations connected with p-adic forms and local zeta functions", *Bull. Austral. Math. Soc.* **70**:1 (2004), 73–86. MR 2005g:11237 Zbl 1074.46051

[Zúñiga-Galindo 2008] W. A. Zúñiga-Galindo, "Parabolic equations and Markov processes over p-adic fields", *Potential Anal.* **28**:2 (2008), 185–200. MR 2008k:35522 Zbl 1134.35005

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