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We study the orthogonal quantum groups satisfying the "easiness" assumption axiomatized in our previous paper, with the construction of some new examples and with some partial classification results. The conjectural conclusion is that the easy quantum groups consist of the previously known 14 examples, plus a hypothetical multiparameter "hyperoctahedral series", related to the complex reflection groups $H_n^s = \mathbb{Z}_s \wr S_n$. We also discuss the general structure and the computation of asymptotic laws of characters for the new quantum groups that we construct.

Introduction

One of the strengths of the theory of compact Lie groups is that these objects can be classified. It is indeed extremely useful to know that the symmetry group of a classical or a quantum mechanical system falls into an advanced classification machinery, and applications of this method abound in mathematics and physics.

The quantum groups were introduced by Drinfel'd [1987] and Jimbo [1985], in order to deal with quite complicated systems, basically coming from number theory or quantum mechanics, whose symmetry groups are not "classical". There are now available several extensions and generalizations of the Drinfel'd–Jimbo construction, all of them more or less motivated by the same philosophy. A brief account of the whole story, focusing on constructions that are of interest here, is as follows:

(1) Let $G \subset U_n$ be a compact group, and consider the algebra $A = C(G)$. The matrix coordinates $u_{ij} \in A$ satisfy the commutation relations $ab = ba$. The original idea of Drinfel'd and Jimbo, further processed by Woronowicz [1987], was that these commutation relations are in fact the $q = 1$ case of the q-commutation relations $ab = qba$, where $q > 0$ is a parameter. The algebra A itself appears then as the $q = 1$ case of an algebra A_q. While A_q is no longer commutative, we can formally write $A = C(G_q)$, where G_q is a quantum group.

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(2) Wang [1995; 1998] proposed an interesting modification of this construction. His idea was to construct a new algebra A^+, by somehow "removing" the commutation relations $ab = ba$. Once again we can formally write $A^+ = C(G^+)$, where G^+ is a so-called free quantum group. This construction, while originally coming only with a vague motivation from mathematical physics, has been studied intensively in the last 15 years. Among the partial conclusions that we have so far is the fact that the combinatorics of G^+ is definitely interesting, and should have something to do with physics. In other words, G^+, while being by definition a quite abstract object, is probably the symmetry group of something very concrete.

(3) Several variations of Wang's construction appeared in recent years, notably in connection with the construction and classification of intermediate quantum groups $G \subset G^* \subset G^+$. For instance in the case $G = O_n$, it was shown in our previous paper [BS 2009] that the commutation relations $ab = ba$ can be successfully replaced with the so-called half-commutation relations $abc = cba$, in order to obtain a new quantum group O_n^*. Some other commutation-type relations, for instance of type $(ab)^s = (ba)^s$, will be described in the present paper.

(4) As a conclusion, the general idea that tends to emerge from these considerations is that a very large class of compact quantum groups should appear in the following way: start with a compact Lie group $G \subset U_n$; build a noncommutative version of $C(G)$ by replacing the commutation relations $ab = ba$ by some weaker relations; and deform this latter algebra, by using a positive parameter $q > 0$, or more generally a whole family of such positive parameters.

This was the motivating story. In practice, now, while the construction (1) is now basically understood, thanks to about 25 years of effort, (2) is just at the very beginning of an axiomatization, (3) is still at the level of pioneering examples, and (4) is just a dream. As for the possible applications to physics, basically nothing is known so far, but the hope for such an application increases as more and more interesting formulas emerge from the study of compact quantum groups.

This paper, a continuation of [BS 2009], will advance on the classification work there, for the easy quantum groups in the orthogonal case, and will present a detailed study of the new quantum groups we find.

The objects of interest will be the compact quantum groups with $S_n \subset G \subset O_n^+$. Here O_n^+ is the free analogue of the orthogonal group, constructed by Wang [1995], and for the compact quantum groups we use Woronowicz's formalism [1987].

As in [BS 2009] we restrict attention to the "easy" case. The easiness assumption, essential to our considerations, roughly states that the tensor category of G should be spanned by certain partitions, coming from the tensor category of S_n.

This might look like a quite technical condition. The point, though, is that imposing this technical condition is the price to pay for restricting attention to the "truly easy" case.

As explained in [BS 2009], our motivating belief is that "any result that holds for S_n, O_n should have a suitable extension to all easy quantum groups". This is of course a quite vague statement, whose target is actually informed by some results at the borderline between representation theory and probability. Here, however, we would rather focus on the classification problem. The further development of our "S_n, O_n philosophy", leading perhaps to some interesting applications, will be left to future papers. See Section 8 for more comments in this direction.

So, for the purposes of the present work, the easy quantum groups can be just thought of as being a carefully chosen collection of basic objects of the theory.

There are 14 natural examples of easy quantum groups, all but one described in [BS 2009], and the remaining one to be studied in detail in this paper. In addition, there are at least two infinite series, once again to be introduced here. The list is as follows:

(1) Groups: O_n, S_n, H_n, B_n, S_n', B_n'.

(2) Free versions: O_n^+, S_n^+, H_n^+, B_n^+, $S_n'^+$, $B_n'^+$.

(3) Half liberations: O_n^*, H_n^*.

(4) Hyperoctahedral series: $H_n^{(s)}$, $H_n^{[s]}$.

This list doesn't cover all the easy quantum groups, but we will present here some partial classification results, with the conjectural conclusion that the full list should consist of (1)–(3), and of a multiparameter series unifying (4). We will also investigate the new quantum groups that we find, by using various techniques from [Banica et al. 2007a; 2007b; BS 2009; Banica and Vergnioux 2009a; 2009b].

As already mentioned, we expect the list above to be a useful, fundamental starting point for a number of representation theory and probability considerations. We also expect that the new quantum groups that we find this way will lead to some interesting applications. We have several projects here, to be discussed at the end of the paper.

The paper is organized as follows. In Sections 1 and 2 we recall our previous results from [BS 2009], and we study the quantum group H_n^* by using techniques from [BS 2009; Banica and Vergnioux 2009b]. In Sections 3 and 4, we introduce the one-parameter series, and we study their basic properties by using techniques from [Banica et al. 2007a; Banica and Vergnioux 2009a]. In Sections 5 and 6, we state and prove the classification results, by making heavy use of the capping method in [BS 2009; Banica and Vergnioux 2009b]. Sections 7 and 8 contain the computation of asymptotic laws of characters, and some concluding remarks.

Notation

As in [BS 2009], the basic object we consider will be a compact quantum group G. Concrete examples include the usual compact groups G and, to some extent, the duals of discrete groups $\widehat{\Gamma}$. In the general case, G is just a fictional object, which exists only via its associated Hopf C^*-algebra of "complex continuous functions", denoted $A = C(G)$.

For simplicity of notation, we would rather use the quantum group G instead of the Hopf algebra A. For instance $\int_G u_{i_1 j_1} \cdots u_{i_k j_k}\, du$ will denote the complex number obtained by applying the Haar functional $\varphi : A \to \mathbb{C}$ to the well-defined quantity $u_{i_1 j_1} \cdots u_{i_k j_k} \in A$.

We will use the quantum group notation depending on the setting; in cases where this can lead to confusion, we will switch back to the Hopf algebra notation.

1. Easy quantum groups

We briefly recall some notions and results from [BS 2009]. This material is here mostly for fixing the formalism and the notation.

Consider first a compact group satisfying $S_n \subset G \subset O_n$. That is, $G \subset O_n$ is a closed subgroup containing the subgroup $S_n \subset O_n$ formed by the permutation matrices.

Let u, v be the fundamental representations of G, S_n. By functoriality we have an inclusion $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) \subset \mathrm{Hom}(v^{\otimes k}, v^{\otimes l})$ for any k and l. On the other hand, the Hom-spaces for v are well known: they are spanned by operators T_p, with p belonging to $P(k,l)$, the set of partitions between k points and l points. More precisely, if $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{C}^n$, the formula for T_p is

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1 \cdots j_l} \delta_p \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_l \end{pmatrix} e_{j_1} \otimes \cdots \otimes e_{j_l} \tag{1-1}$$

Here the δ symbol on the right is 0 or 1, depending on whether the indices "fit" or not, that is, $\delta = 1$ if all blocks of p contain equal indices, and $\delta = 0$ if not.

Thus the space $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ consists of linear combinations of operators of type T_p with $p \in P(k,l)$.

We call G easy if its tensor category is spanned by partitions.

Definition 1.1. We say a compact group $S_n \subset G \subset O_n$ is *easy* if there exist sets $D(k,l) \subset P(k,l)$ such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p \mid p \in D(k,l))$ for any k, l.

It follows from the axioms of tensor categories that the collection of sets $D(k,l)$ must be closed under certain categorical operations, notably vertical and horizontal concatenation, and upside-down turning. The corresponding algebraic structure formed by the sets $D(k,l)$, axiomatized in [BS 2009], will be called *category of crossing partitions*.

We recall that a matrix is called monomial if it has exactly one nonzero entry on each row and each column. The basic examples are the permutation matrices.

Definition 1.2. We consider the following groups:

(1) O_n, the orthogonal group;

(2) S_n, the symmetric group, formed by the permutation matrices

(3) H_n, the hyperoctahedral group, formed by monomial matrices with ± 1 entries;

(4) B_n, the bistochastic group, formed by orthogonal matrices with sum 1 on each row;

(5) $S'_n = \mathbb{Z}_2 \times S_n$, the group formed by the permutation matrices times ± 1;

(6) $B'_n = \mathbb{Z}_2 \times B_n$, the group formed by the bistochastic matrices times ± 1.

It follows from definitions that all these groups satisfy $S_n \subset G \subset O_n$. Among all these groups, only O_n and S_n are "irreducible", because we have canonical isomorphisms $H_n = \mathbb{Z}_2 \wr S_n$ and $B_n \simeq O_{n-1}$. See [BS 2009].

The partitions in $P(k,l)$ with $k+l$ even are themselves called even.

Theorem 1.3 [BS 2009]. *The only easy groups are the ones in Definition 1.2, and the corresponding categories of crossing partitions are as follows*:

(1) P_o, *all pairings*;

(2) P_s, *all partitions*;

(3) P_h, *partitions with blocks of even size*;

(4) P_b, *singletons and pairings*;

(5) $P_{s'}$, *all partitions* (*even part*);

(6) $P_{b'}$, *singletons and pairings* (*even part*).

The second assertion follows from some well-known results about the groups O_n, S_n and their versions, and the first can be proved by carefully manipulating the categorical axioms.

We now discuss the free analogue of the above results. Let O_n^+ and S_n^+ be respectively the free orthogonal and symmetric quantum groups corresponding to the Hopf algebras $A_o(n)$ and $A_s(n)$ constructed by Wang [1995; 1998]. Here and in what follows, we use Woronowicz's Hopf algebra formalism [1987] and its subsequent quantum group interpretation.

We have $S_n \subset S_n^+$, so by functoriality the Hom-spaces for S_n^+ appear as subspaces of the corresponding Hom-spaces for S_n. The Hom-spaces for S_n^+ have in fact a very simple description. They are spanned by the operators T_p with $P \in \mathrm{NC}(k,l)$, the set of noncrossing partitions between k upper points and l lower points.

Definition 1.1. has a free analogue.

Definition 1.4. A compact quantum group $S_n^+ \subset G \subset O_n^+$ is called *free* if there exist sets $D(k,l) \subset \mathrm{NC}(k,l)$ such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p \mid p \in D(k,l))$ for any k, l.

In this definition, the word "free" has a quite subtle meaning, to be fully justified later on. Forn now, let us note that the passage from Definition 1.1 to Definition 1.4 is basically done by restricting attention to the noncrossing partitions, which, according to [Speicher 1994], should indeed lead to freeness.

As in the classical case, the sets of partitions $D(k,l)$ must be stable under certain categorical operations, coming this time from the axioms in [Woronowicz 1988]. The corresponding algebraic structure, axiomatized in [BS 2009], is called the category of noncrossing partitions.

We denote by H_n^+ the hyperoctahedral quantum group constructed in [Banica et al. 2007b], and by B_n^+, $S_n'^+$ and $B_n'^+$ the free analogues of the groups B_n, S_n' and B_n' constructed in [BS 2009].

Definition 1.5. We consider the following quantum groups, all given with the defining relations between the basic coordinates $u_{ij} \in C(G)$:

(1) O_n^+, orthogonality ($u_{ij} = u_{ij}^*$ and $u^t = u^{-1}$);

(2) S_n^+, magic condition (all rows and columns of u are partitions of unity);

(3) H_n^+, cubic condition (orthogonality and $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$ for $j \neq k$);

(4) B_n^+, bistochastic condition (orthogonality and on each row the sum is 1);

(5) $S_n'^+$, cubic condition, with the same sum on rows and columns;

(6) $B_n'^+$, orthogonality, with the same sum on rows and columns;

Perhaps the very first observation is that for any of the groups G appearing in Definition 1.2 we have $C(G) = C(G^+)/I$, where $I \subset C(G^+)$ is the commutator ideal. In other words, G^+ is indeed a noncommutative version of G. We refer to [BS 2009] and to its predecessors [Banica et al. 2007b; Wang 1995; 1998] for the whole story, and for a careful treatment of all this material.

The free analogue of Theorem 1.3 is this:

Theorem 1.6 [BS 2009]. *Definition 1.5 lists the only free quantum groups. The corresponding categories of noncrossing partitions are as follows*:

(1) NC_o, *all noncrossing pairings*;

(2) NC_s, *all noncrossing partitions*;

(3) NC_h, *noncrossing partitions with blocks of even size*;

(4) NC_b, *singletons and noncrossing pairings*;

(5) $\mathrm{NC}_{s'}$, *all noncrossing partitions* (*even part*);

(6) $\mathrm{NC}_{b'}$, *singletons and noncrossing pairings* (*even part*).

The proof of this theorem follows that of Theorem 1.3. The symmetry between Theorems 1.3 and 1.6 corresponds to the liberation operation for orthogonal Lie groups, further investigated in [BS 2009].

2. Half-liberation

We consider now the general situation where we have a compact quantum group satisfying $S_n \subset G \subset O_n^+$. Once again, we can ask for the tensor category of G to be spanned by certain partitions, coming from the tensor category of S_n.

Definition 2.1. A compact quantum group $S_n \subset G \subset O_n^+$ is called easy if there exist sets $D(k,l) \subset P(k,l)$ such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p \mid p \in D(k,l))$ for any k, l.

This definition generalizes at the same time Definitions 1.1 and 1.4. Indeed, the easy quantum groups $S_n \subset G \subset O_n^+$ satisfying the extra assumption $G \subset O_n$ are the easy groups, and those satisfying the extra assumption $S_n^+ \subset G$ are the free quantum groups. This follows from definitions; see [BS 2009].

Once again, the sets of partitions $D(k,l)$ must be stable under certain categorical operations coming from the axioms in [Woronowicz 1988]. The corresponding algebraic structure, axiomatized in [BS 2009], will be called simply “category of partitions”.

We already know that the easy quantum groups include the 6 easy groups and the 6 free quantum groups. In general, the world of easy quantum groups is quite rigid, but we can produce some more examples in the following way.

Definition 2.2. The half-liberated version of an easy group G is the quantum group G^* given by $C(G^*) = C(G^+)/I$, where I is the ideal generated by the half-commutation relations $abc = cba$, imposed on the basic matrix coordinates $u_{ij} \in C(G^+)$.

In other words, instead of removing the commutativity relations of type $ab = ba$ from the standard presentation of $C(G)$, which would produce the algebra $C(G^+)$, we replace these commutativity relations by the weaker relations $abc = cba$.

To study the half-liberated versions, we need a categorical interpretation of the relations $abc = cba$. Let us agree that the upper points of a partition $p \in P(k,l)$ are labeled $1, 2, \ldots, k$, and the lower points are labeled $1', 2', \ldots, l'$.

Lemma 2.3 [BS 2009]. *For a compact quantum group $G \subset O_n^+$, the following are equivalent*:

(1) *The basic coordinates u_{ij} satisfy $abc = cba$.*

(2) *T_p belongs to* $\mathrm{End}(u^{\otimes 3})$, *where* $p = (13')(22')(3'1)$.

Proof. By the definition (1-1) of T_p, we have $T_p(e_a \otimes e_b \otimes e_c) = e_c \otimes e_b \otimes e_a$. This gives the formulas

$$T_p u^{\otimes 3}(e_a \otimes e_b \otimes e_c) = \sum_{ijk} e_k \otimes e_j \otimes e_i \otimes u_{ia} u_{jb} u_{kc}$$

$$u^{\otimes 3} T_p(e_a \otimes e_b \otimes e_c) = \sum_{ijk} e_i \otimes e_j \otimes e_k \otimes u_{ic} u_{jb} u_{ka}$$

The identification of the right terms gives the equivalence in the statement. □

We now go back to the quantum groups G^*. Observe first that we have inclusions $G \subset G^* \subset G^+$. As pointed out in [BS 2009], the cases $G = S_n, B_n, S_n', B_n'$ are not interesting, because here we have $G = G^*$. This can be checked by a direct computation with generators and relations, or with the partition p appearing in Lemma 2.3, and will follow as well from the general classification results in Sections 5 and 6.

In the cases $G = O_n, H_n$, however, we obtain new quantum groups. Label the legs of each partition by $1, 2, 3, \ldots,$ clockwise starting from top left.

Theorem 2.4. *The half-liberated versions of O_n and H_n are easy quantum groups, and the corresponding categories of partitions are*

(1) *E_o, pairings with each string connecting an odd number to an even number*;

(2) *E_h, partitions with each block having the same number of odd and even legs.*

Proof. Our claim is that E_o and E_h are categories of partitions, corresponding respectively to the quantum groups O_n^* and H_n^*.

(1) Here E_o is nothing but the set of pairings with each string having an even number of crossings, and the result was proved in [BS 2009]. The idea is that E_o is generated in the categorical sense by the partition p appearing in Lemma 2.3.

(2) The fact that E_h is indeed a category of partitions follows from definitions. Thinking of each block as being "balanced" with respect to the odd and even labels, we see that the categorical operations preserve the balancing. For instance when checking the stability under composition, which is the crucial axiom, we see that given a connected union of blocks of the two partitions that are composed, the "balancing in the middle" is subject to canceling.

The fact that E_h corresponds to the above quantum group H_n^* can be checked in several ways. Consider for instance the diagram

$$\begin{array}{ccc} O_n^* & \subset & O_n^+ \\ \cup & & \cup \\ H_n^* & \subset & H_n^+. \end{array}$$

We know from definitions that H_n^* is obtained by putting together the relations for O_n^* and for H_n, so we have the quantum group equality $H_n^* = O_n^* \cap H_n^+$. Now by the general properties of Tannakian duality, it follows that the category of partitions of H_n^* is generated by the category of partitions for H_n^+, namely the noncrossing partitions having even blocks, and by the half-liberation partition p in Lemma 2.3.

This category is by definition included into E_h, and the reverse inclusion can be checked as well by a straightforward computation. □

The quantum group O_n^*, appearing first in [BS 2009], was further investigated in [Banica and Vergnioux 2009b]. To get some insight into the structure of H_n^*, we will use similar methods.

Definition 2.5. The projective version of a quantum group $G \subset U_n^+$ is the quantum group $PG \subset U_{n^2}^+$, having as basic coordinates the elements $v_{ij,kl} = u_{ik}u_{jl}^*$.

In other words, $C(PG) \subset C(G)$ is the algebra generated by the elements $v_{ij,kl} = u_{ik}u_{jl}^*$. In the case where G is a classical group we recover the well-known formula $PG = G/(G \cap T)$, where $T \subset U_n$ are the unitary diagonal matrices. We refer to [Banica and Vergnioux 2009b] for a full discussion and a list of concrete examples.

Consider now the compact group $K_n = \mathbb{T} \wr S_n$ consisting of monomial (that is, permutation-like) matrices, with elements on the unit circle $\mathbb{T}$ as nonzero entries.

The next result, whose first claim is from [Banica and Vergnioux 2009b], will play a key role in the study of H_n^* and the other quantum groups introduced here.

Theorem 2.6. *The projective versions of half-liberations are as*

(1) $PO_n^* = PU_n$, *and*

(2) $PH_n^* = PK_n$.

Proof. The first claim is proved using that the partitions for PO_n^* and PU_n are the same. For the second, we use a similar method. Observe first that from $H_n^* \subset O_n^*$, we get $PH_n^* \subset PO_n^* = PU_n$, so PH_n^* is indeed a classical group.

To compute this group, consider the diagram

$$\begin{array}{ccc} K_n & \subset & U_n^+ \\ \cup & & \cup \\ H_n & \subset & H_n^*. \end{array}$$

We fix $k, l \geq 0$ and consider the formal words $\alpha = (u \otimes \bar{u})^{\otimes k}$ and $\beta = (u \otimes \bar{u})^{\otimes l}$. Our claim is that the corresponding spaces $\mathrm{Hom}(\alpha, \beta)$ for our 4 quantum groups appear as span of the operators T_p, with p belonging to the following 4 sets of partitions:

$$\begin{array}{ccc} E_h(2k, 2l) & \supset & E_o(2k, 2l) \\ \cup & & \cup \\ P_h(2k, 2l) & \supset & E_h(2k, 2l). \end{array}$$

Indeed, the bottom left set is a good one, by Theorem 1.3. The bottom right set is also a good one, by Theorem 2.4. For the top right set, this follows from the equality $PO_n^* = PU_n$ and from Theorem 2.4, and for full details see [Banica and Vergnioux 2009b]. As for the top left set, this follows for instance from the various results in [Banica et al. 2007a; Banica 2008; Banica and Vergnioux 2009a] regarding K_n^+, after "adding a crossing". A direct proof can be obtained as well, by working out the categorical interpretation of the various relations defining K_n.

In summary, we have computed the relevant diagrams for the projective versions of our four algebras. So, let us look now at these projective versions:

$$\begin{array}{ccc} PK_n & \subset & PU_n^+ \\ \cup & & \cup \\ PH_n & \subset & PH_n^* . \end{array}$$

The quantum groups PH_n^* and PK_n appear as subgroups of the same quantum group, namely PU_n^+, and the discussion above tells us that these subgroups have the same diagrams. The same argument of [Banica and Vergnioux 2009b] tells us that $PH_n^* = PK_n$. □

3. The hyperoctahedral series

We now introduce a new series of quantum groups, $H_n^{(s)}$ with $s \in \{2, 3, \ldots, \infty\}$. These will intermediate between $H_n^{(2)} = H_n$ and $H_n^{(\infty)} = H_n^*$.

The quantum group $H_n^{(s)}$ is obtained from H_n^* by imposing the s-commutation condition $abab\cdots = baba\cdots$ (words of length s) on the basic coordinates u_{ij}.

Definition 3.1. $C(H_n^{(s)})$ is the universal C^*-algebra generated by n^2 self-adjoint variables u_{ij}, subject to the relations

(1) (orthogonality) $uu^t = u^t u = 1$, where $u = (u_{ij})$ and $u^t = (u_{ji})$;

(2) (cubic relations) $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$ for any i and any $j \neq k$;

(3) (half-commutation) $abc = cba$ for any $a, b, c \in \{u_{ij}\}$;

(4) (s-mixing relation) $abab\cdots = baba\cdots$ (length s words) for any $a, b \in \{u_{ij}\}$.

That $H_n^{(s)}$ is a quantum group follows from the elementary fact that the cubic relations are of Hopf type, that is, they allow the construction of the Hopf algebra maps Δ, ε, S. This can be checked by a routine computation.

At $s = 2$, the s-mixing is the usual commutation $ab = ba$. Since this relation is stronger than the half-commutation $abc = cba$, we are led to the algebra generated by n^2 commuting self-adjoint variables satisfying (1) and (2), which is $C(H_n)$.

In the case $s = \infty$, the s-mixing relation disappears by definition. Thus we are led to the algebra defined by the relations (1)–(3), which is $C(H_n^*)$.

Summarizing, we have $H_n^{(2)} = H_n$ and $H_n^{(\infty)} = H_n^*$, as previously claimed. In what follows we present a detailed study of $H_n^{(s)}$.

Lemma 3.2. *For a compact quantum group $G \subset H_n^*$, the following are equivalent:*

(1) *The basic coordinates u_{ij} satisfy $abab \cdots = baba \cdots$ (length s words).*

(2) *T_p belongs to* $\mathrm{End}(u^{\otimes s})$, *where* $p = (135 \cdots 2'4'6' \cdots)(246 \cdots 1'3'5' \cdots)$.

Proof. According to the definition of T_p given in (1-1), the operator associated to the partition in the statement is given by the formula

$$T_p(e_{a_1} \otimes e_{b_1} \otimes e_{a_2} \otimes e_{b_2} \otimes \cdots) = \delta(a)\delta(b) e_b \otimes e_a \otimes e_b \otimes e_a \otimes \cdots.$$

Here we use the convention $\delta(a) = 1$ if all the indices a_i are equal and $\delta(a) = 0$ if not, along with a similar convention for $\delta(b)$. The indices a and b appearing on the right are the common values of the a indices and b indices, respectively, in the case $\delta(a) = \delta(b) = 1$, and are irrelevant quantities in the remaining cases.

This gives the formulas

$$T_p u^{\otimes s}(e_{a_1} \otimes e_{b_1} \otimes e_{a_2} \otimes \cdots) = \sum_{ij} e_i \otimes e_j \otimes e_i \otimes \cdots \otimes u_{ia_1} u_{jb_1} u_{ia_2} \cdots,$$

$$u^{\otimes s} T_p(e_{a_1} \otimes e_{b_1} \otimes e_{a_2} \otimes \cdots) = \delta(a)\delta(b) \sum_{ij} e_{i_1} \otimes e_{j_1} \otimes e_{i_2} \otimes \cdots \otimes u_{i_1 b} u_{j_1 a} u_{i_2 b} \cdots.$$

Here the upper sum is over all indices i and j, and the lower sum is over all multiindices $i = (i_1, \dots, i_s)$ and $j = (j_1, \dots, j_s)$. The identification of the terms on the right, after a suitable relabeling of indices, gives the equivalence in the statement. □

We now work out the s-analogue of Theorem 2.4.

Theorem 3.3. *$H_n^{(s)}$ is an easy quantum group, and its associated category E_h^s is that of the s-*balanced *partitions, that is, partitions satisfying the conditions that*

(1) *the total number of legs is even, and*

(2) *in each block, the number of odd and even legs are equal, modulo s.*

Proof. At $s = 2$ the first condition implies the second one, so here we simply get the partitions having an even number of legs, corresponding to H_n. At $s = \infty$ we get the partitions that are balanced, in the sense of the proof of Theorem 2.4, which are known from there to correspond to the quantum group H_n^*.

We first claim that E_h^s is a category. This follows simply by adapting the $s = \infty$ argument in the proof of Theorem 2.4, just by adding "modulo s" everywhere.

It remains to prove that this category corresponds to $H_n^{(s)}$. This follows from the fact that the partition p of Lemma 3.2 generates the category of s-balanced partitions, as one can check by a routine computation. □

Consider now the complex reflection group $H_n^s = \mathbb{Z}_s \wr S_n$, consisting of monomial matrices having the s-roots of unity as nonzero entries. Observe that we have $PH_n^{(s)} = H_n^s/\mathbb{T}$.

We have the following s-analogue of Theorem 2.6.

Theorem 3.4. $PH_n^{(s)} = PH_n^s$.

Proof. This statement holds at $s = 2$, because here we have $H_n^{(2)} = H_n^2 = H_n$; it holds at $s = \infty$ due to Theorem 2.6.

In the general case, it follows by adapting the proof of Theorem 2.6. Observe first that from $H_n^{(s)} \subset H_n^*$ we get $PH_n^{(s)} \subset PH_n^* = PK_n$, so $PH_n^{(s)}$ is a classical group.

To compute this group, consider the diagram

$$\begin{array}{ccc} H_n^s & \subset & U_n^+ \\ \cup & & \cup \\ S_n & \subset & H_n^{(s)}. \end{array}$$

The corresponding sets of partitions, as in the proof of Theorem 2.6, are

$$\begin{array}{ccc} E_h^s(2k, 2l) & \supset & E_o(2k, 2l) \\ \cup & & \cup \\ P(2k, 2l) & \supset & E_h^s(2k, 2l). \end{array}$$

The bottom left set is a good one, by Theorem 1.3, as is bottom right one, by Theorem 3.3. For the top right set, this was already discussed in the proof of Theorem 2.6. For the top left set, this follows either from the results in [Banica et al. 2007a; Banica and Vergnioux 2009a] regarding the free version H_n^{s+}, after adding a crossing, or from the $s = \infty$ computation in the proof of Theorem 2.6. A direct proof can be obtained as well.

We now look at the projective versions of the above quantum groups:

$$\begin{array}{ccc} PH_n^s & \subset & PU_n^+ \\ \cup & & \cup \\ PH_n & \subset & PH_n^{(s)}. \end{array}$$

As in the proof of Theorem 2.6, we have two quantum subgroups having the same diagrams, and we conclude that $PH_n^{(s)} = PH_n^s$. □

4. The higher hyperoctahedral series

We introduce a second one-parameter series of quantum groups, $H_n^{[s]}$ with s in $\{2, 3, \ldots, \infty\}$, having as main particular case the group $H_n^{[2]} = H_n$.

Definition 4.1. $C(H_n^{[s]})$ is the universal C^*-algebra generated by n^2 self-adjoint variables u_{ij}, subject to the relations

(1) (orthogonality) $uu^t = u^t u = 1$, where $u = (u_{ij})$ and $u^t = (u_{ji})$.

(2) (ultracubic relations) $acb = 0$ for any $a \neq b$ on the same row or column of u.

(3) (s-mixing relation) $abab \cdots = baba \cdots$ (length s words) for any $a, b \in \{u_{ij}\}$.

That $H_n^{[s]}$ is a quantum group follows from the elementary fact that the ultracubic relations are of "Hopf type", that is, that they allow the construction of the Hopf algebra maps Δ, ε and S. This can be checked by a routine computation.

We first compare the defining relations for $H_n^{[s]}$ with those for $H_n^{(s)}$. To deal at the same time with the cubic and ultracubic relations, it is convenient to use a statement about a unifying notion, *k-cubic* relations.

Lemma 4.2. *For a compact quantum group $G \subset O_n^+$, the following are equivalent*:

(1) *The basic coordinates u_{ij} satisfy the k-cubic relations $ac_1 \cdots c_k b = 0$ for any $a \neq b$ on the same row or column of u, and for any $c_1, \ldots, c_k$.*

(2) $T_p \in \mathrm{End}(u^{\otimes k+2})$, *where* $p = (1, 1', k+2, k+2')(2, 2') \cdots (k+1, k+1')$.

Proof. According to (1-1), the operator associated to the partition in the statement is given by

$$T_p(e_a \otimes e_{c_1} \otimes \cdots \otimes e_{c_k} \otimes e_b) = \delta_{ab} e_a \otimes e_{c_1} \otimes \cdots \otimes e_{c_k} \otimes e_a.$$

This gives the formulas

$$T_p u^{\otimes k+2}(e_a \otimes e_{c_1} \otimes \cdots \otimes e_{c_k} \otimes e_b) = \sum_{ij} e_i \otimes e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes e_i \otimes u_{ia} u_{j_1 c_1} \cdots u_{j_k c_k} u_{ib},$$

$$u^{\otimes k+2} T_p(e_a \otimes e_{c_1} \otimes \cdots \otimes e_{c_k} \otimes e_b = \delta_{ab} \sum_{ijl} e_i \otimes e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes e_l \otimes u_{ia} u_{j_1 c_1} \cdots u_{j_k c_k} u_{la},$$

Here the sums are over all indices i and l, and over all multiindices $j = (j_1, \ldots, j_k)$. The identification of the terms on the right gives the desired equivalence. □

We can now establish the precise relationship between $H_n^{[s]}$ and $H_n^{(s)}$ and also show that no further series can appear in this way.

Proposition 4.3. *For $k \geq 1$, the k-cubic relations are all equivalent to the ultracubic relations*, *and they imply the cubic relations.*

Proof. This follows from two observations.

First, the k-cubic relations imply the $2k$-cubic relations. Indeed, one can connect two copies of the partition p in Lemma 4.2, by gluing them with two semicircles in the middle, and the resulting partition is the one implementing the $2k$-cubic relations.

Second, the k-cubic relations imply the $(k-1)$-cubic relations. By capping the partition p in Lemma 4.2 with a semicircle at bottom right, we get a partition $p' \in P(k+2,k)$, and by rotating the upper right leg of this partition we get the partition $p'' \in P(k+1,k+1)$ implementing the $(k-1)$-cubic relations. □

Proposition 4.3 shows that replacing in Definition 4.1 the ultracubic condition by any of the k-cubic conditions with $k \geq 2$ won't change the resulting quantum group. The other consequences of Proposition 4.3 are summarized as follows.

Proposition 4.4. *The quantum groups $H_n^{[s]}$ have the properties that*

(1) $H_n^{(s)} \subset H_n^{[s]} \subset H_n^+$;

(2) $H_n^{[2]} = H_n^{(s)} = H_n$ *at* $s = 2$;

(3) $H_n^{(s)} \neq H_n^{[s]}$ *at* $s \geq 3$.

Proof. All the assertions basically follow from Lemma 4.2:

(1) For the first inclusion, we need to show half-commutation plus cubic implies ultracubic; this can be done by placing the half-commutation partition next to the cubic partition, then using 2 semicircle cappings in the middle. The second inclusion follows from Proposition 4.3, because the ultracubic relations (1-cubic relations) imply the cubic relations (0-cubic relations).

(2) At $s = 2$ the s-commutation is the usual commutation $ab = ba$. Thus we are led here to the algebra generated by n^2 commuting self-adjoint variables satisfying the cubic condition, which is $C(H_n)$.

(3) Finally, $H_n^{(s)} \neq H_n^{[s]}$ will be a consequence of Theorem 4.5 below, because at $s \geq 3$ the half-commutation partition $p = (14)(25)(36)$ is s-balanced but not locally s-balanced. □

Theorem 4.5. *$H_n^{[s]}$ is an easy quantum group, and its associated category is that of the* locally s-balanced *partitions, that is, partitions having the property that each of their subpartitions (that is, partitions obtained by removing certain blocks) are s-balanced.*

Proof. At $s = 2$ the locally s-balancing condition is automatic for a partition having blocks of even size, so we get indeed the category corresponding to H_n.

In the general case, we first claim that the locally s-balanced partitions form a category. This follows simply by adapting the proof of Theorem 3.3, just by adding "locally" everywhere.

It remains to prove that this category corresponds to $H_n^{[s]}$. This follows from Lemma 3.2 and from the fact that the partition generating the category of locally balanced partitions, namely, $p = (1346)(25)$, is nothing but the one implementing the ultracubic relations, as one can check by a routine computation. □

5. Classification: General strategy

In this section and the next we advance the classification work started in [BS 2009]. We will prove that the easy quantum groups constructed so far are the only ones, modulo a hypothetical multiparameter "hyperoctahedral series", unifying the series constructed in the previous sections, and still waiting to be constructed.

Let G be an easy quantum group with category of partitions denoted P_g. It follows from definitions that $P_g \cap \mathrm{NC}$ is a category of noncrossing partitions; by the results in Section 1, this latter category must come from a free quantum group K^+. Since $\mathrm{NC}_k = P_g \cap \mathrm{NC}$ is included into P_g, we have $G \subset K^+$.

Definition 5.1. Associated to an easy quantum group G is the easy group K given by the equality of categories $P_g \cap \mathrm{NC} = \mathrm{NC}_k$.

According to the easy group classification in Theorem 1.3, there are six cases to be studied; five of these will be studied in Section 6, and the remaining case, $K = H_n$, will be left open.

The reason these cases are separated comes from the question, Do we have $K \subset G$? In the reminder of this section we will try to answer this question.

We begin with the technical lemma, valid in the general case. Let $\Lambda_g, \Lambda_k \subset \mathbb{N}$ be the set of the possible sizes of blocks of elements of P_g, NC_k.

Lemma 5.2. *Let G and K be as above.*

(1) $\Lambda_k \subset \Lambda_g \subset \Lambda_k \cup (\Lambda_k - 1)$.

(2) $1 \in \Lambda_g$ *implies* $1 \in \Lambda_k$.

(3) *If* NC_k *is even, so is* P_g.

Proof. We will heavily use the various abstract notions and results in [BS 2009].

(1) The first inclusion follows from $\mathrm{NC}_k \subset P_g$. The second is equivalent to the statement, "If b is a block of a partition $p \in P_g$, then there exists a certain block b' of a certain partition $p' \in P_g \cap \mathrm{NC}$, having size $\#b$ or $\#b - 1$." This then follows by using the capping method in [BS 2009]. We can cap p with semicircles, so that b remains unchanged, and we end up with a partition p' consisting of b and some extra points, at most one point between any two legs of b, which may or may not be connected. Since the semicircle capping is a categorical operation, this partition p' remains in P_g.

Now by further capping p' with semicircles, so as to get rid of the extra points, the size of b can only increase, and we end up with a one-block partition having size at least that of b. This one-block partition is obviously noncrossing, and by capping it again with semicircles we can reduce the number of legs up to $\#b$ or $\#b - 1$, and we are done.

(2) The condition $1 \in \Lambda_g$ means that there exists $p \in P_g$ having a singleton. By capping p with semicircles outside this singleton, we can obtain a singleton or a double singleton. Since both these partitions are noncrossing and have a singleton, we are done.

(3) Assume that P_g is not even, and consider a partition $p \in P_g$ having an odd number of legs. By capping p with enough semicircles we ensure ending up with a singleton, and since this singleton is by definition in $P_g \cap \mathrm{NC}$, we are done. $\square$

We are now in position of splitting the classification.

Proposition 5.3. *Let G, K be as above.*

(1) *If $K \neq H_n$, then $K \subset G \subset K^+$.*

(2) *If $K = H_n$, then $S_n' \subset G \subset H_n^+$.*

Proof. Recall that the inclusion $G \subset K^+$ follows from definitions. For the other inclusion, we have 6 cases, depending on the exact value of the easy group K:

(1a) $K = O_n$. Here $\Lambda_k = \{2\}$, so by Lemma 5.2(1) we get $\{2\} \subset \Lambda_g \subset \{1, 2\}$. Moreover, from Lemma 5.2(2), we get $\Lambda_g = \{2\}$. Thus $P_g \subset P_o$, which gives $O_n \subset G$.

(1b) $K = S_n$. Here there is nothing to prove, since $S_n \subset G$ by definition.

(1c) $K = B_n$. Here $\Lambda_k = \{1, 2\}$, so by Lemma 5.2(1) we get $\Lambda_g = \{1, 2\}$. Thus we have $P_g \subset P_b$, which gives $B_n \subset G$.

(1d) $K = S_n'$. Here $P_g \subset P_s$ by definition, and by using Lemma 5.2(3) we deduce that $P_g \subset P_{s'}$, which gives $S_n' \subset G$.

(1e) $K = B_n'$. Here $\Lambda = \{1, 2\}$, so by Lemma 5.2(1) we get $\Lambda_g = \{1, 2\}$. This gives $P_g \subset P_b$, and by Lemma 5.2(3), we get $P_g \subset P_{b'}$, which gives $B_n' \subset G$.

(2) $K = H_n$. Here $P_g \subset P_s$ by definition, and by using Lemma 5.2(3) we deduce that $P_g \subset P_{s'}$, which gives $S_n' \subset G$. $\square$

With a little more care, one can prove that the easy group K in statement (1) is nothing but the classical version of G, obtained as dual object to the commutative Hopf algebra $C(G)/I$, where $I \subset C(G)$ is the commutator ideal.

Statement (2) cannot be improved. The reason is that for the quantum group $H_n^{(s)}$ with s odd, we have $K = H_n$, and $K \not\subset G$.

6. The nonhyperoctahedral case

We classify the easy quantum groups, under the nonhyperoctahedral assumption $K \neq H_n$. Here K is as usual the easy group from Definition 5.1.

We know from Proposition 5.3 that our easy quantum group G appears as an intermediate quantum group, $K \subset G \subset K^+$. To classify these intermediate quantum groups, we use the method in [Banica and Vergnioux 2009b], where the problem was solved in the case $G = O_n$. For uniformity, we will also include this case.

Definition 6.1. Let $p \in P(k,l)$ be a partition, with the points counted modulo $k+l$ counterclockwise starting from bottom left.

(1) We call *semicircle capping* of p any partition obtained from p by connecting with a semicircle a pair of consecutive neighbors.

(2) We call *singleton capping* of p any partition obtained from p by capping one of its legs with a singleton.

(3) We call *doubleton capping* of p any partition obtained from p by capping two of its legs with singletons.

The semicircle, singleton and doubleton cappings are elementary operations on partitions that lower the total number of legs by 2, 1 and 2 respectively. There are $k+l$ possibilities for placing the semicircle or the singleton, and $(k+l)(k+l-1)/2$ possibilities for placing the double singleton. In the case of semicircle cappings at left or at right, the semicircle in question is in fact a vertical bar, but we will still call it semicircle.

The various cappings of p will be generically denoted p'.

Consider now the $5+5+1 = 11$ categories of partitions P_x, NC_x, E_x, with $x = o, s, b, s', b'$ described in Sections 1 and 2.

Lemma 6.2. *Let p be a partition having j legs.*

(1) *If $p \in P_o - E_o$ and $j > 4$, there exists a semicircle capping $p' \in P_o - E_o$.*

(2) *If $p \in E_o - \mathrm{NC}_o$ and $j > 6$, there exists a semicircle capping $p' \in E_o - \mathrm{NC}_o$.*

(3) *If $p \in P_s - \mathrm{NC}_s$ and $j > 4$, there exists a singleton capping $p' \in P_s - \mathrm{NC}_s$.*

(4) *If $p \in P_b - \mathrm{NC}_b$ and $j > 4$, there exists a singleton capping $p' \in P_b - \mathrm{NC}_b$.*

(5) *If $p \in P_{s'} - \mathrm{NC}_{s'}$ and $j > 4$, there exists a doubleton capping $p' \in P_{s'} - \mathrm{NC}_{s'}$.*

(6) *If $p \in P_{b'} - \mathrm{NC}_{b'}$ and $j > 4$, there exists a doubleton capping $p' \in P_{b'} - \mathrm{NC}_{b'}$.*

Proof. We write $p \in P(k,l)$, so that the number of legs is $j = k+l$. In the cases where our partition is a pairing, we use as well the number of strings, $s = j/2$.

Let us agree that all partitions are drawn to have a minimal number of crossings.

We use the same idea for all the proofs, namely to isolate a block of p having a crossing, or an odd number of crossings, then to cap p as in the statement, so this block remains crossing, or with an odd number of crossings. Here we use the observation that the balancing condition that defines the categories E_o and E_h can be interpreted as saying that each block has an even number of crossings when the picture of the partition is drawn so that this number of crossings is minimal.

(1) The assumption $p \notin E_o$ means that p has strings having an odd number of crossings. We fix such a string, and we try to cap p so that this string remains odd in the resulting partition p'. An examination of all possible pictures shows that this is possible, provided that our partition has $s > 2$ strings.

(2) The assumption $p \notin \mathrm{NC}_o$ means that p has crossing strings. We fix such a pair of strings, and we try to cap p so these strings remain crossing in p'. Once again, looking at all possible pictures shows that this is possible, provided that our partition has $s > 3$ strings.

(3) Since p is crossing, we can choose two of its blocks that are intersecting. If there are some other blocks left, we can cap one of their legs with a singleton, and we are done. If not, this means that our two blocks have a total of $j' \geq j > 4$ legs, so at least one of them has $j'' > 2$ legs. One of these j'' legs can always be capped with a singleton, so the capped partition remains crossing, and we are done.

(4) Here we can simply cap with a singleton, as in (3).

(5)–(6) Here we can cap with a doubleton, by proceeding twice as in (3). □

For a collection of subsets $X(k,l) \subset P(k,l)$ we denote by $\langle X \rangle \subset P$ the category of partitions generated by X. In other words, the elements of $\langle X \rangle$ come from those of X via the categorical operations for the categories of partitions, which are the vertical and horizontal concatenation and the upside-down turning. See [BS 2009].

Lemma 6.3. *Let p be a partition.*

(1) *If $p \in P_o - E_o$, then $\langle p, \mathrm{NC}_o \rangle = P_o$.*

(2) *If $p \in E_o - \mathrm{NC}_o$, then $\langle p, \mathrm{NC}_o \rangle = E_o$.*

(3) *If $p \in P_s - \mathrm{NC}_s$, then $\langle p, \mathrm{NC}_s \rangle = P_s$.*

(4) *If $p \in P_b - \mathrm{NC}_b$, then $\langle p, \mathrm{NC}_b \rangle = P_b$.*

(5) *If $p \in P_{s'} - \mathrm{NC}_{s'}$, then $\langle p, \mathrm{NC}_{s'} \rangle = P_{s'}$.*

(6) *If $p \in P_{b'} - \mathrm{NC}_{b'}$, then $\langle p, \mathrm{NC}_{b'} \rangle = P_{b'}$.*

Proof. We use Lemma 6.2 and the observation that the "capping partition" appearing there is always in the good category.

That is, we use that the semicircle is in $\mathrm{NC}_o, \mathrm{NC}_{s'}$, the singleton is in $\mathrm{NC}_s, \mathrm{NC}_b$, and the doubleton is in $\mathrm{NC}_{b'}$. This observation tells us that, in each of the cases under consideration, the category to be computed can only decrease when replacing p by one of its cappings p'. For the singleton and doubleton cappings this is clear from definitions; for the semicircle capping this is also clear from definitions, except in the case where the capping semicircle is actually a bar added at left or at right, where we can use a categorical rotation operation as in [BS 2009].

(1)–(2) These claims can be proved by recurrence on the number $s = (k+l)/2$ of strings. Indeed, by using Lemma 6.2(1)–(2), for $s > 3$ we have a descent procedure $s \to s-1$, and this leads to the situation $s \in \{1, 2, 3\}$, where the statement is clear.

(3) We can proceed by recurrence on the number of legs of p. If the number of legs is $j = 4$, then p is a basic crossing, and we have $\langle p \rangle = P_s$. If the number of legs is $j > 4$ we can apply Lemma 6.2(3), and the result follows from $\langle p \rangle \supset \langle p' \rangle = P_s$.

(4)–(6) This is similar to the proof of (1)–(2), by using Lemma 6.2(4)–(6). □

Lemma 6.4. *Let p be a partition.*

1. *If $p \in P_o$, then $\langle p, \mathrm{NC}_o \rangle \in \{P_o, E_o, \mathrm{NC}_o\}$.*
2. *If $p \in P_s$, then $\langle p, \mathrm{NC}_s \rangle \in \{P_s, \mathrm{NC}_s\}$.*
3. *If $p \in P_b$, then $\langle p, \mathrm{NC}_b \rangle \in \{P_b, \mathrm{NC}_b\}$.*
4. *If $p \in P_{s'}$, then $\langle p, \mathrm{NC}_{s'} \rangle \in \{P_{s'}, \mathrm{NC}_{s'}\}$.*
5. *If $p \in P_{b'}$, then $\langle p, \mathrm{NC}_{b'} \rangle \in \{P_{b'}, \mathrm{NC}_{b'}\}$.*

Proof. This follows by rearranging the results in Lemma 6.3. □

We may now state our main result. We call *nonhyperoctahedral* any easy quantum group G such that $K \neq H_n$.

Theorem 6.5. *There are exactly* 11 *nonhyperoctahedral easy quantum groups*:

1. *O_n, O_n^* and O_n^+, the orthogonal quantum groups*;
2. *S_n and S_n^+, the symmetric quantum groups*;
3. *B_n and B_n^+, the bistochastic quantum groups*;
4. *S_n' and $S_n'^+$, the modified symmetric quantum groups*;
5. *B_n' and $B_n'^+$, the modified bistochastic quantum groups.*

Proof. By Proposition 5.3, we have to classify the easy quantum groups satisfying $K \subset G \subset K^+$. More precisely, we have to prove that for $K = S_n, B_n, S_n', B_n'$ there is no such partial liberation, and that for $K = O_n$ there is only one partial liberation, namely the quantum group K^* mentioned above. This follows from Lemma 6.4, via the Tannakian results in [BS 2009]. □

The classification in the hyperoctahedral case seems to be a difficult problem, which we have to leave open.

7. Laws of characters

We discuss the computation of the asymptotic law of the fundamental character $\chi = \mathrm{Tr}(u)$, and of its truncated versions $\chi_t = \sum_{i=1}^{[tn]} u_{ii}$ with $t \in (0, 1]$.

These computations, which might seem quite technical, are in fact of great relevance in the general context of representation theory. Given a compact group

$G \subset U_n$, or more generally a compact quantum group $G \subset U_n^+$, the main problem in representation theory is to classify the irreducible representations of G. By the Peter–Weyl theory, these irreducible representations appear in the tensor powers $u^{\otimes k}$ of the fundamental representation, and they can be in fact identified with the minimal projections of the algebra $\mathrm{End}(u^{\otimes k})$.

The exact computation of $\mathrm{End}(u^{\otimes k})$ is generally quite difficult. However, an easier problem whose answer is generally extremely useful is the computation of the dimension of this algebra. Since this dimension can be simply obtained by integrating χ^{2k}, we are led to the fundamental problem of computing the law of χ.

In the quantum group context, the difference between the law of χ and the corresponding classical result can be quite puzzling. The problem appears for instance with S_n and S_n^+, where the law of χ is respectively Poisson with $n \to \infty$, and free Poisson with $n \geq 4$. The lack of symmetry was conceptually understood in [Banica and Collins 2007], where it was shown that the correct invariant to look at is the law of the truncated character χ_t, with $t \in (0, 1]$.

Definition 7.1. Associated to an easy quantum group $G \subset U_n^+$ is the truncated character

$$\chi_t = \sum_{i=1}^{[tn]} u_{ii},$$

where $u = (u_{ij})$ is the matrix of standard coordinates, and $t \in (0, 1]$.

Recall some basic results from [BS 2009]. Let G be an easy quantum group, and denote by $D_k \subset P(0, k)$ the corresponding sets of diagrams, having no upper points. We define the Gram matrix to be $G_{kn}(p, q) = n^{b(p \vee q)}$, where $b(\cdot)$ is the number of blocks. The Weingarten matrix is by definition its inverse, $W_{kn} = G_{kn}^{-1}$. In order for this inverse to exist, n must be big enough, and the assumption $n \geq k$ is sufficient. In the general case the notion of quasiinverse must be used; see [Collins and Matsumoto 2009] for a detailed discussion.

Theorem 7.2. *The Haar integration over G is given by*

$$\int_G u_{i_1 j_1} \cdots u_{i_k j_k} \, du = \sum_{p,q \in D_k} \delta_p(i) \delta_q(j) W_{kn}(p, q),$$

where the δ symbols are 0 *or* 1, *depending on whether the indices fit or not.*

Proof. This is proved in [BS 2009], using the idea that the integrals on the left, taken altogether, form the orthogonal projection on $\mathrm{Fix}(u^{\otimes k}) = \mathrm{span}(D_k)$. □

The Weingarten formula is particularly effective in the classical and free cases, where complete computations were performed in [BS 2009].

Theorem 7.3. *The asymptotic law of $\chi_t = \sum_{i=1}^{[tn]} u_{ii}$ with $t \in (0, 1]$ is as follows*:

(1) *For O_n, S_n, H_n and B_n, we get the Gaussian, Poisson, Bessel and shifted Gaussian laws, which form convolution semigroups.*

(2) *For O_n^+, S_n^+, H_n^+ and B_n^+ we get the semicircular, free Poisson, free Bessel and shifted semicircular laws, which form free convolution semigroups.*

(3) *For S_n', B_n', $S_n'^+$ and $B_n'^+$ we get symmetrized versions of the laws for S_n, B_n, S_n^+ and B_n^+, which do not form classical or free convolution semigroups.*

Proof. This is proved in [BS 2009] by using the Weingarten formula and cumulants. Note that the semigroups in (1) and (2) are in Bercovici–Pata [1999] bijection. □

We should mention that the measures in (3), while not forming semigroups due to the canonical copy of $\mathbb{Z}_2$, which produces a "correlation", are very close to forming some kind of semigroup. We come back to this question in our next papers [Banica et al. 2009a; 2009b].

In the remaining cases, the Weingarten formula is less effective, because counting partitions and their blocks is a delicate task. In the case of half-liberations and of the hyperoctahedral series we will use instead the projective versions computed in the previous sections, which reduce the problem to a classical computation.

Definition 7.4. We use the following complex probability measures:

(1) The complex Gaussian law of parameter $t > 0$ is the law of $x + iy$, where x and y are independent Gaussian variables of parameter t.

(2) The s-Bessel law of parameter $t > 0$ is the law of $\sum_{r=1}^{s} e^{2\pi i r/s} x_i$, where $x_1, \dots, x_s$ are independent Poisson variables of parameter t/s.

The complex Gaussian laws are well known to form a convolution semigroup. The same holds for the s-Bessel laws, and we refer to [Banica et al. 2007a] for a complete discussion. The "Bessel" terminology comes from the fact that at $s = 2$, the density of the corresponding discrete measure on $\mathbb{R}$ is given by a Bessel function of the first kind.

Definition 7.5. Given a complex probability measure μ, we call *squeezed version* of it the law of $\sqrt{zz^*}$, where z follows the law μ.

This law doesn't depend of course on the choice of z.

For example, the squeezed version of the complex Gaussian law of parameter 1 is the Rayleigh law. This is because with $z = x + iy$, we have $zz^* = x^2 + y^2$.

Another interesting example, of key relevance in free probability, is the fact that the squeezed version of Voiculescu's circular law is Wigner's semicircle law. See for example [Nica and Speicher 2006].

Theorem 7.6. *The asymptotic law of $\chi_t = \sum_{i=1}^{[tn]} u_{ii}$ with $t \in (0, 1]$ is as follows:*

(1) *For O_n^*, we get the squeezed complex Gaussian semigroup.*

(2) *For $H_n^{(s)}$, we get the squeezed s-Bessel semigroup.*

Proof. The Weingarten formula shows that the odd moments of the variables in the statement are all 0, so all computations actually take place over the projective versions. With this remark in hand, the results simply follow from the well-known fact that χ_t is asymptotically complex Gaussian for U_n and s-Bessel for H_n^s. See [Banica et al. 2007a]. □

The squeezed s-Bessel laws seem to have a quite interesting combinatorics, but this is beyond the purposes of this paper. We would like however to present one such combinatorial statement, in the simplest case, $s = \infty$ and $t = 1$.

Proposition 7.7. *The asymptotic even moments of the character $\chi \in C(H_n^*)$ satisfy*

$$c_k = \sum_{s=0}^{k-1} \binom{k}{s} \binom{k-1}{s} c_s$$

and are equal to the number of games of simple patience with n cards.

Proof. This follows from Theorem 7.6, but we will present below a direct proof, which we found at an early stage of this work. According to the general theory, the numbers in the statement are given by $c_k = \#E_h(2k)$, that is, they count the partitions of $\{1, \ldots, 2k\}$ having the property that each block has the same number of odd and even legs.

It is convenient to do the following manipulation: We keep the sequence of odd legs fixed, and we pull downwards the sequence of even legs. In this way, $E_h(2k)$ becomes the set of partitions between an upper and a lower sequence of k points, such that each block is balanced in the sense that it has the same number of upper and lower legs.

Now observe that these partitions can be obtained as follows: pick a number $r \in \{1, \ldots, n\}$; connect the first point on the upper line to some $r - 1$ other points on the upper line; choose r points on the lower line, and connect them to the already connected upper r points; and finally connect the remaining $k - r$ upper points to the remaining $k - r$ lower points by means of a balanced partition.

With $s = k - r$ this gives the formula in the statement. For the patience game interpretation, see Aldous and Diaconis [1999] and Sloane's comments [2008] about the sequence A023998, which is the sequence of moments of χ. □

For the higher hyperoctahedral quantum group $H_n^{[s]}$, our standard methods simply do not work. We don't know if this quantum group produces the squeezed version of some known semigroup.

8. Concluding remarks

We have seen in this paper that the easy quantum groups consist in principle of 6 groups, their free versions, 2 half-liberations, and one infinite series still waiting to be constructed. The construction of this hypothetical multiparameter "hyper-octahedral series", and the continuation and completion of our classification work, are of course the main two questions that we would like to address here.

The situation here, which is unexpectedly complex, brings to mind the algebraic difficulty and subtlety of the usual complex reflection groups [Broué et al. 1998].

At the level of applications, as explained in the introduction, we intend to use the easy quantum group list we know of as input for a number of representation theory and probability considerations; again, we believe that "any result that holds for S_n and O_n should have a suitable extension to all easy quantum groups".

In the noneasy case, there are of course of large number of results, classical or even free, having something to do with diagrams and with the easy quantum group technology in general, and that might fall one day into an extension of our formalism.

Here is a list of topics waiting to be developed:

(1) De Finetti theorems. These are available for S_n, O_n from the book [Kallenberg 2005], for S_n^+ from [Köstler and Speicher 2009] and then [Curran 2009a], and for O_n^+ from [Curran 2009b]. We develop a global approach to the problem by using easy quantum groups in our forthcoming paper [Banica et al. 2009a].

(2) Eigenvalue computations. The key results of Diaconis and Shahshahani [1994] about S_n, O_n can also be obtained by using Weingarten functions and cumulants; this is extended to all easy quantum groups in the preprint [Banica et al. 2009b].

(3) Invariant theory. The groups S_n, O_n and their versions S_n^+, O_n^+, O_n^* have served as a guiding example for the study of many invariants; see [Collins and Śniady 2006; Banica and Collins 2007; Novak 2007; Banica and Vergnioux 2009a; 2009b; Collins and Matsumoto 2009]. Some of these results are expected to extend to all easy quantum groups.

(4) Geometric aspects. The groups S_n, O_n and their free versions S_n^+, O_n^+ were also involved in many other "classical versus free" considerations. Let us mention here the Poisson boundary results in [Vaes and Vergnioux 2007], and the quantum isometry groups in [Bhowmick and Goswami 2009]. Once again, the easy quantum groups can lead to some new results here.

(5) Generalizations. One interesting question would be to understand the twisting and deformation of the easy quantum groups, say with the goal of extending our formalism to the $S^2 \neq \mathrm{id}$ case, via monoidal equivalence [Bichon et al. 2006]. Another question is whether the half-liberation operation can be applied to locally

compact real algebraic groups $G \subset M_n(\mathbb{R})$, so as to fit into the general axioms in [Kustermans and Vaes 2000].

In addition to these questions, one basic problem is to classify the intermediate quantum groups $K \subset G \subset K^+$, where K is a fixed easy group. This looks like a quite difficult question; but a possible way forward comes from a conjecture in [Banica et al. 2007c], stating that there is no intermediate quantum group $S_n \subset G \subset S_n^+$. This is actually a quite subtle question, whose study leads straight into the core of the "noneasy" problems.

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