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THE EXISTENCE AND MONOTONICITY OF A THREE-DIMENSIONAL TRANSONIC SHOCK IN A FINITE NOZZLE WITH AXISYMMETRIC EXIT PRESSURE

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We establish the existence of a multidimensional transonic shock solution in a class of slowly varying nozzles for the three dimensional steady full Euler system with axially symmetric exit pressure in the diverging part lying in an appropriate scope. We also show that the shock position depends monotonically on the exit pressure.

1. Introduction and the main results

The transonic shock problem in a de Laval nozzle is a fundamental one in fluid dynamics and has been extensively studied by many authors under the assumption that the transonic flow is quasi-one-dimensional or the transonic shock goes through some fixed point in advance [Chen et al. 2006; Chen et al. 2007; Chen and Feldman 2003; Chen 2008; Courant and Friedrichs 1948; Embid et al. 1984; Glaz and Liu 1984; Kuz'min 2002; Liu 1982a; 1982b; Xin et al. 2009; Xin and Yin 2005; 2008a; 2008b; Yuan 2006]. Courant and Friedrichs [1948, page 386] proposed a physically more interesting transonic shock wave pattern in a de Laval nozzle as follows: Given an appropriately large end pressure $p_e(x)$, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes $p_e(x)$. This means that the position of the transonic shock should be completely free. Indeed, the assumption that the shock goes through some fixed point at the wall of the nozzle in advance may lead to overdetermined boundary conditions for the

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transonic shock problem for the full Euler system with the given exit pressure; see [Xin et al. 2009; Xin and Yin 2008a] for details. Here, we focus on the existence of a solution to this transonic shock problem for the three-dimensional full Euler system when the exit pressure $p_e(x)$ is axisymmetric and lies in an appropriate scope without other artificial constraints. In particular, we show the shock position depends monotonically on the exit pressure.

The steady and nonisentropic Euler system in three-dimensional space is

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla P = 0, \\ \operatorname{div}\big((\rho(e + \frac{1}{2}|u|^2) + P)u\big) = 0, \end{cases} \tag{1-1}$$

where $u = (u_1, u_2, u_3)$, ρ, P, e and S stand for the velocity, density, pressure, internal energy and specific entropy, respectively. The pressure function $P = P(\rho, S)$ and the internal energy function $e = e(\rho, S)$ are smooth in their arguments. It is assumed that $\partial_\rho P(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$.

For the ideal polytropic gases, the equations of state are given by

$$P = A\rho^\gamma e^{S/c_v} \quad \text{and} \quad e = \frac{P}{(\gamma - 1)\rho},$$

where A, c_v and γ are positive constants, and $1 < \gamma < 3$ (in air, $\gamma \approx 1.4$).

We now describe the class of de Laval nozzle that will be studied later on; see also [Li et al. 2010a; 2010b]. The wall Γ of the nozzle is assumed to be $C^{3,\alpha}$-regular for $X_0 - 1 \le r \equiv (x_1^2 + x_2^2 + x_3^2)^{1/2} \le X_0 + 1$, where $X_0 > 0$ is a fixed large constant, and $\alpha \in (0, 1)$ and Γ consists of two curved surfaces Π_1 and Π_2; here Π_1 includes the converging part of the nozzle, and Π_2 constructs a symmetric curved diverging part of it. See Figure 1. More precisely, Π_2 is represented by the equation $x_2^2 + x_3^2 = x_1^2 \tan^2 \theta_0$ with $x_1 > 0$ and $X_0 < r < X_0 + 1$, where θ satisfies $0 < \theta_0 < \pi/2$ and is sufficiently small. For simplicity, we assume that the $C^{3,\alpha}$-smooth supersonic incoming flow $(S_0^-, P_0^-(x), u_0^-(x))$ is spherically symmetric near $r = X_0$; here $S_0^-(x) = S_0^-$ is a constant, $P_0^-(x) = P_0^-(r)$, and $u_0^-(x) = U_0^-(r)x/r$. This assumption is easily satisfied because of the hyperbolicity of the supersonic incoming flow and the symmetry of Π_2.

Let shock Σ in the nozzle be given by $x_1 = \eta(x')$ with $x' = (x_2, x_3)$, and denote the flow field behind the shock by $(S^+(x), P^+(x), u^+(x))$. The Rankine–Hugoniot conditions on Σ imply

$$\begin{cases} [(1, -\nabla_{x'}\eta(x')) \cdot \rho u] = 0, \\ [((1, -\nabla_{x'}\eta(x')) \cdot \rho u)u] + (1, -\nabla_{x'}\eta(x'))^t [P] = 0, \\ [(1, -\nabla_{x'}\eta(x')) \cdot (\rho(e + \frac{1}{2}|u|^2) + P)u] = 0. \end{cases} \tag{1-2}$$

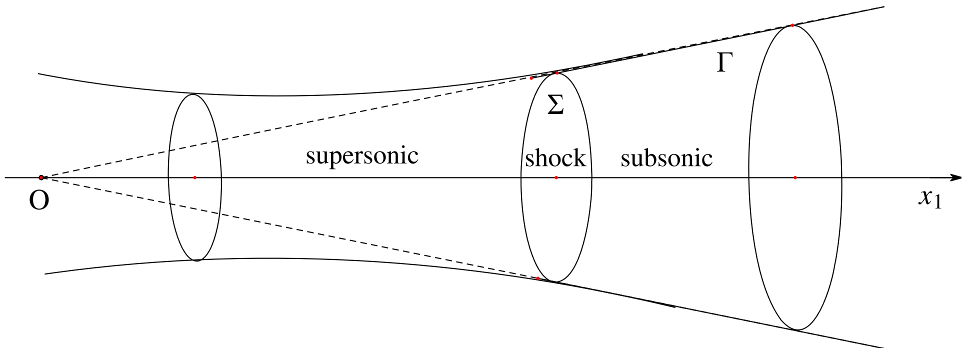

Figure 1

Here the brackets around function denotes the jump of that function across Σ.

In addition, $P^+(x)$ should satisfy the physical entropy condition

$$P^+(x) > P^-(x) \quad \text{on } x_1 = \eta(x_2, x_3); \tag{1-3}$$

see [Courant and Friedrichs 1948].

On the exit of the nozzle, the pressure is prescribed and axisymmetric:

$$P^+(x) = P_e + \varepsilon \tilde{P}(\theta) \quad \text{on } r = X_0 + 1. \tag{1-4}$$

Here P_e is a positive constant, $\varepsilon > 0$ is sufficiently small, $\theta = r^{-1} \arcsin(x_2^2 + x_3^2)^{1/2}$, and $\tilde{P}(\theta) \in C^{2,\alpha}[0, \theta_0]$ with $\tilde{P}'(0) = \tilde{P}'(\theta_0) = 0$. We require that for given exit pressure P_e, the Euler system (1-1) has a radial symmetric transonic shock lying at $r = r_0 \in (X_0, X_0 + 1)$ with supersonic incoming flow $(S_0^-, P_0^-(r), (U_0^-(r)/r)x)$ for $r \in (X_0, r_0)$. For the range of P_e and detailed information on the corresponding transonic shock solution $(S_0^\pm, P_0^\pm(r), (U_0^\pm(r)/r)x)$, see Theorem A.1.

The wall of the nozzle is assumed to be solid; thus

$$x_1 u_1^+ \tan^2\theta_0 - x_2 u_2^+ - x_3 u_3^+ = 0 \quad \text{on } \Pi_2. \tag{1-5}$$

Finally, we assume X_0 and θ_0 to be suitably large and small respectively so that

$$X_0 \theta_0 = 1 \quad \text{and} \quad \tfrac{1}{2}\eta_0 < \theta_0 < \eta_0. \tag{1-6}$$

Here $\eta_0 > 0$ is a constant.

Note that (1-6) means that the nozzle wall $\Pi_2 : x_2^2 + x_3^2 = x_1^2 \tan^2\theta_0$ is close to the cylindrical surface $x_2^2 + x_3^2 = 1$ for $X_0 \le r \le X_0 + 1$.

The main results in this paper can be stated as follows:

Theorem 1.1 (existence and monotonicity). *Under the assumptions above*, *with*

$$M_0^-(X_0) \equiv \frac{U_0^-(X_0)}{c(\rho_0^-(X_0))} > \sqrt{\frac{\gamma+3}{2}}$$

and $\varepsilon < 1/X_0^3$, *the problem* (1-1) *with the conditions* (1-2)–(1-5) *has a solution* $(S^+(x), P^+(x), u^+(x); \eta(x_2, x_3))$ *that admits the following estimates*:

(i) $\eta(x_2, x_3) \in C^{3,\alpha}(\bar{S}_e)$, with $S_e = \{(x_2, x_3) : (\eta(x_2, x_3), x_2, x_3) \in \Sigma\}$ being the projection of the shock surface Σ onto the (x_2, x_3)-plane. Moreover, there exists a constant $C_0 > 0$ (depending only on α and the supersonic incoming flow) such that

$$\|\eta(x_2, x_3) - (r_0^2 - x_2^2 - x_3^2)^{1/2}\|_{L^\infty(S_e)} \le C_0 X_0 \varepsilon,$$
$$\|\nabla_{x_2,x_3}(\eta(x_2, x_3) - (r_0^2 - x_2^2 - x_3^2)^{1/2})\|_{C^{2,\alpha}(\bar{S}_e)} \le C_0 \varepsilon.$$

(ii) Denote by

$$\Omega_+ = \{(x_1, x_2, x_3) : \eta(x_2, x_3) < x_1 < ((X_0+1)^2 - x_2^2 - x_3^2)^{1/2},\ x_2^2 + x_3^2 \le x_1^2 \tan^2 \theta_0\}$$

the subsonic region. Then $(S^+(x), P^+(x), u^+(x)) \in C^{2,\alpha}(\overline{\Omega}_+)$ satisfies

$$\|(S^+(x), P^+(x), u^+(x)) - (S_0^+, \hat{P}_0^+(r), \hat{u}_0^+(x))\|_{C^{2,\alpha}(\bar{\Omega}_+)} \le C_0 \varepsilon,$$

where $\hat{u}_0^+(x) = \hat{U}_0^+(r)x/r$, and $(S_0^+, \hat{P}_0^+(r), \hat{u}_0^+(r))$ stands for the extension of the background solution $(S_0^+, P_0^+(r), U_0^+(r)x/r)$ in Ω_+ described in more detail in Theorem A.1 and Remark A.3.

(iii) The position of the shock surface depends on the given exit pressure monotonically and continuously.

Remark 1.2. Showing that the shock position depends monotonically on the exit pressure is one of the keys to the existence result described by Theorem 1.1. When the exit pressure changes at order $O(\varepsilon)$, the shock position will change at order $X_0 O(\varepsilon)$ instead of $O(1)\varepsilon$; this will be crucial in our analysis.

Remark 1.3. The condition

$$M_0^-(X_0) \equiv \frac{U_0^-(X_0)}{c(\rho_0^-(X_0))} > \sqrt{\frac{\gamma+3}{2}}$$

on the supersonic Mach number is there to ensure that the shock position along the nozzle wall is monotonic in the subsonic pressure across the shock; this is the initial step toward showing the monotonic dependence of the shock position on the exit pressure. See (4-34), (4-36), (4-38), and (4-39) for more details.

Remark 1.4. Although in [Li et al. 2010a] we established by a completely different method (see [Li et al. 2009a] also) the existence of a three-dimensional transonic shock for a variety of conic nozzles with axisymmetric exit pressures, we did not show monotonic dependence of the shock position on the exit pressure.

There has already been much work on the steady transonic problem; see [Bers 1950; 1951; Čanić et al. 2000; Chen et al. 2006; Chen et al. 2007; Chen and Feldman 2003; Chen 2008; Courant and Friedrichs 1948; Embid et al. 1984; Glaz and Liu 1984; Kuz'min 2002; Li et al. 2009a; 2009b; 2010a; 2010b; Liu 1982a; 1982b; Morawetz 1994; Xin et al. 2009; Xin and Yin 2005; 2008b; 2008a; Yuan 2006; Zheng 2003; 2006] and the references therein. In particular, for a three-dimensional nozzle with a symmetric diverging part and a symmetric supersonic incoming flow near the diverging part of the nozzle, Xin and Yin [2008b] and Courant and Friedrichs [1948] have shown that there exist two constant pressures P_1 and P_2 with $P_1 < P_2$ such that if the exit pressure P_e is in the interval (P_1, P_2), then the transonic shock exists uniquely in the diverging part of the nozzle, and the position and the strength of the shock are completely determined by P_e and the resulting ordinary differential equations. Xin and Yin [2008b] also established global existence, stability and long time asymptotic behavior of an unsteady symmetric transonic shock under the exit pressure P_e when the initial unsteady shock lies in the symmetric diverging part of the three-dimensional nozzle; on the other hand a steady symmetric transonic shock is dynamically unstable if it lies in the symmetric converging part of the nozzle. In [Li et al. 2009b], we established for the two-dimensional steady Euler system, by a monotonicity argument on the shock position and the exit pressure, uniqueness and existence of a completely free two-dimensional transonic shock in a nozzle with variable end pressures at the exit. For the three-dimensional steady Euler system, we have shown in [Li et al. 2010b] the uniqueness of a completely free three-dimensional transonic shock solution of class $C^{3,\alpha}$ in a nozzle with general exit pressure; this regularity is higher than the $C^{2,\alpha}$ regularity of solutions in Theorem 1.1. In this paper, we will focus on the existence and monotonicity property of a completely free three-dimensional transonic shock for a certain class of the exit pressures.

Next we comment on the proofs of the main results in this paper. In almost all previous results dealing with transonic shocks in a nozzle with given exit pressure except, except for those in [Li et al. 2009b; 2010a; 2010b; Xin et al. 2009], the authors assume that the shock goes through a fixed point in advance; this plays the crucial role in the analysis, in particular, in the process of determining the shock position. However, for de Laval nozzles, this assumption is not physical since the shock position should be determined by the supersonic incoming flow, the geometry of the nozzle and the exit pressure, as pointed out by Courant and Friedrichs. Moreover, this constraint may lead in general to an over-determined problem. In [Li et al. 2009b; 2010a; 2010b], we have successfully removed this condition, and further determined the shock position and transonic flow in the nozzle. This leads to the well-posedness of the transonic shock problem in the two-dimensional case and the uniqueness of solutions to it in the three-dimensional case, as well as some new observations and techniques.

A key step in [Li et al. 2009b; 2010b] is to derive a priori gradient estimates instead of the solution itself, in order to establish that the shock position along the walls of the nozzle varies monotonically with exit pressure. This leads to the determination of a unique shock position and the desired stability estimates. However, it seems difficult to apply these methods directly to obtain the existence of the transonic shock in a three-dimensional nozzle. The main reasons are as follows: $C^{3,\alpha}$ regularity of the solution in the subsonic region plays a fundamental role in the theorems, but this higher order regularity is a source of great difficulties for nozzles with variable exit pressure. Compared with two-dimensional case, it seems much more difficult to find higher order compatibility conditions near the intersection curve of the shock surface with the wall of the nozzle, which is necessary to ensure $C^{3,\alpha}$ regularity of the solution nearby. In the two-dimensional case, higher order compatibility at the intersection points of the shock curve with the walls of the nozzle can be found directly from the Euler system together with the no-flow boundary condition of the walls of the nozzle, and Rankine–Hugoniot conditions on the shock curve. This yields naturally $C^{3,\alpha}$ regularity of the solution in [Li et al. 2009b]; similar approaches cannot be applied in the three-dimensional case; see [Xin and Yin 2008b, Lemma 6.1]. In addition, for the axially symmetric exit pressure in this paper, it is natural to introduce spherical coordinates in the space variables, which brings new technical difficulties in finding compatibility conditions on the symmetry axis and handling singularities and source terms in the transformed equations near the symmetry axis. Due to the singularity near the symmetry axis and the source terms for the Euler system in spherical coordinates, the key gradient estimate method in [Li et al. 2009b] cannot be applied here; see (2-8) and Remark 3.3.

To overcome these difficulties, our strategy is as follows: First, we will give some rather delicate computations and analysis of the three-dimensional Euler system and the related axisymmetric functions near the x_1-axis and the nozzle wall; this is to establish $C^{2,\alpha}$ regularity of the solutions; see Lemmas B.1–B.7 and Section 3. Second, to derive that the shock position is monotonic in the end pressure, we will focus directly on the first order elliptic system and how the two pressures and two shock positions (see (4-17)) differ from those in the gradient estimates of [Li et al. 2009b; 2010b]. The key step is to establish an ordinary differential-integral inequality in the difference of pressures (see (4-45)). Based on this result and the continuous dependence of the shock position on the exit pressure, we can finally complete the proof of Theorem 1.1.

The rest of the paper is organized as follows. In Section 2, we will reformulate the three-dimensional problem (1-1) with the boundary conditions (1-2)–(1-5). First we transform the nozzle wall into a cube surface, and decompose the velocity u^+ as the radial speed U_1^+ and two angular speeds U_2^+ and U_3^+. In the Euler system on $(S^+, P^+, U_1^+, U_2^+, U_3^+)$, with the exit boundary condition (1-4), it is natural to search for a solution with $U_3^+ \equiv 0$. Furthermore, we decompose the Euler system

(1-1) as a 2×2 first order elliptic system for ρ^+ and U_2^+/U_1^+, and two algebraic equations in U_1^+ and specific entropy S^+ respectively. In Section 3, we use the decomposition in Section 2 to linearize the compressible Euler system, establish an existence result under the assumption that the shock goes through some fixed point at the nozzle wall in advance, and obtain some key estimates based on the background solution. We note that this solution does not satisfy the boundary condition (1-4) unless the exit pressure is adjusted by an appropriate constant. In Section 4, we establish that the shock position is monotonic in the end pressure. In Section 5, we use the continuous dependence of the solution on the shock position to the existence result in Theorem 1.1. In Appendix A, we list some properties of the background solution. We give some useful inequalities and estimates in Appendix B. Finally, in Appendix C we give a detailed discussion of the regularity of $C^{3,\alpha}$ solutions to problem (1-1) with (1-2)–(1-5).

We will use the following conventions:

$O(\varepsilon)$ means that there exists a generic constant $C_1 > 0$ independent of X_0 and ε such that $\|O(\varepsilon)\|_{C^{1,\alpha}} \le C_1\varepsilon$.

$O(1/X_0^m)$ for $m > 0$ means that there exists a generic constant $C_2 > 0$ independent of X_0 and ε such that $\|O(1/X_0^m)\|_{C^{1,\alpha}} \le C_2/X_0^m$.

2. Reformulation of the problem

In this section, we will reformulate the nonlinear problem (1-1) with (1-2)–(1-5) to obtain a coupled first order elliptic system in the angular velocity exponent U_2^+ and the density ρ^+, and two first order equations, one in the radial velocity U_1^+ and the other in the specific entropy S^+. As in [Xin and Yin 2008b], we will need to derive relations between (ρ^+, U_1^+) and (U_2^+, U_3^+) in the shock Σ. Due to the symmetry of the nozzle wall Π_2 and the supersonic incoming flow in the diverging part, it will be more convenient to use the spherical coordinates

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta\cos\varphi, \quad x_3 = r\sin\theta\sin\varphi \tag{2-1}$$

and velocity decomposition

$$\begin{aligned} U_1^+ &= u_1^+\cos\theta + u_2^+\sin\theta\cos\varphi + u_3^+\sin\theta\sin\varphi, \\ U_2^+ &= u_1^+\sin\theta - u_2^+\cos\theta\cos\varphi - u_3^+\cos\theta\sin\varphi, \\ U_3^+ &= -u_2^+\sin\varphi + u_3^+\cos\varphi, \end{aligned} \tag{2-2}$$

where $\theta \in [0, \theta_0]$, $\varphi \in [0, 2\pi]$, and $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

In the spherical coordinates (2-1), set

$$\tilde{\nabla} := \Big(\partial_r, -\frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\varphi\Big) \quad \text{and} \quad \tilde{U} = (U_1, U_2, U_3).$$

Then (1-1) and (1-2) are transformed respectively into

$$
\begin{cases}
\tilde{\nabla}\cdot(\rho^+\tilde{U}^+)+\Big(\dfrac{2}{r},-\dfrac{1}{r}\cot\theta\Big)\rho^+\cdot(U_1^+,U_2^+)=0,\\
(\tilde{U}^+\cdot\tilde{\nabla})\tilde{U}^++\dfrac{\tilde{\nabla}P^+}{\rho^+}+\dfrac{1}{r}\begin{pmatrix}-((U_2^+)^2+(U_3^+)^2)\\ U_1^+U_2^++(U_3^+)^2\cot\theta\\ U_1^+U_2^+-U_2^+U_3^+\cot\theta\end{pmatrix}=0,\\
(\tilde{U}^+\cdot\tilde{\nabla})S^+=0,
\end{cases}
\tag{2-3}
$$

and

$$
\begin{cases}
[\rho\tilde{U}]\cdot\Big(1,\dfrac{1}{\tilde{r}}\partial_\theta\tilde{r},-\dfrac{\partial_\varphi\tilde{r}}{\tilde{r}\sin\theta}\Big)=0,\\
[\rho\tilde{U}\otimes\tilde{U}+PI]\cdot\Big(1,\dfrac{1}{\tilde{r}}\partial_\theta\tilde{r},-\dfrac{\partial_\varphi\tilde{r}}{\tilde{r}\sin\varphi}\Big)=0\\
\big[(\rho(e+\frac{1}{2}|\tilde{U}|^2)+P)\tilde{U}\big]\cdot\Big(1,\dfrac{1}{\tilde{r}}\partial_\theta\tilde{r},-\dfrac{\partial_\varphi\tilde{r}}{\tilde{r}\sin\varphi}\Big)=0,
\end{cases}
\tag{2-4}
$$

where $r=\tilde{r}(\theta,\varphi)$ is the equation of the shock surface Σ in the spherical coordinates (r,θ,φ).

Meanwhile, (1-4) and (1-5) are correspondingly changed into

$$
P^+(r,\theta,\varphi)=P_e+\varepsilon\tilde{P}(\theta)\quad\text{on } r=X_0+1 \tag{2-5}
$$

and

$$
U_2^+=0\quad\text{on } \theta=\theta_0. \tag{2-6}
$$

For the axisymmetric exit pressure (1-4), we will search for solutions of (2-3)–(2-6) in the form

$$
(S^+,P^+,\tilde{U}^+;\tilde{r})=(S^+(r,\theta),P^+(r,\theta),U_1^+(r,\theta),U_2^+(r,\theta),0;\tilde{r}(\theta)), \tag{2-7}
$$

that is, we look for a solution and shock surface independent of the variable φ.

In this case, using the notation

$$
U\equiv(U_1,U_2),\quad U^\perp\equiv(-U_2,U_1),\quad \nabla\equiv(\partial_r,-(1/r)\partial_\theta),
$$

we can simplify (2-3) and (2-4) to

$$
\begin{cases}
\nabla\cdot(\rho^+U^+)+\dfrac{1}{r}\rho^+(2,-\cot\theta)\cdot U^+=0,\\
(U^+\cdot\nabla)U^++\dfrac{1}{\rho^+}\nabla P^++\dfrac{U_2^+}{r}(U^+)^\perp=0,\\
(U\cdot\nabla)S^+=0,
\end{cases}
\tag{2-8}
$$

and

$$\begin{cases} [\rho U]\cdot\Big(1, \dfrac{\tilde r'(\theta)}{\tilde r(\theta)}\Big) = 0, \\ [\rho U\otimes U + PI]\cdot\Big(1, \dfrac{\tilde r'(\theta)}{\tilde r(\theta)}\Big) = 0, \\ \big[(\rho(e+\tfrac12|U|^2)+P)U\big]\cdot\Big(1, \dfrac{\tilde r'(\theta)}{\tilde r(\theta)}\Big) = 0. \end{cases} \tag{2-9}$$

For convenience, we use the transformation

$$y_1 = r \quad\text{and}\quad y_2 = X_0\theta, \tag{2-10}$$

to change the fixed wall Π_2 into $y_2 = 1$.

In the following, we will drop the + superscripts for simplicity in presentation.

In this case, (2-8) and (2-9) can be rewritten respectively as

$$\begin{cases} \nabla_y\cdot(\rho U) + \dfrac{\rho}{y_1}U\cdot\Big(2, -\cot\Big(\dfrac{y_2}{X_0}\Big)\Big) = 0, \\ (U\cdot\nabla_y)U + \dfrac{1}{\rho}\nabla_y P + \dfrac{U_2}{y_1}U^{\perp} = 0, \\ (U\cdot\nabla_y)S = 0, \end{cases} \tag{2-11}$$

and

$$\begin{pmatrix} [\rho U] \\ [\rho U\otimes U + PI] \\ [(\rho(e+\frac12|U|^2)+P)U] \end{pmatrix}\cdot\begin{pmatrix} 1 \\ \dfrac{X_0\xi'(y_2)}{\xi(y_2)} \end{pmatrix} = 0, \tag{2-12}$$

where $\nabla_y \equiv (\partial_{y_1}, -(X_0/y_1)\partial_{y_2})$ and $\xi(y_2) = \tilde r(y_2/X_0)$, and (2-5) and (2-6) become respectively

$$P(y) = P_e + \varepsilon\tilde P(y_2/X_0) \quad\text{on } y_1 = X_0 + 1 \tag{2-13}$$

and

$$U_2 = 0 \quad\text{on } y_2 = 1. \tag{2-14}$$

Next, we derive boundary conditions of (P, S, U_1) on the shock surface.

It follows from (2-12) that

$$\xi'(y_2) = -\frac{\xi(y_2)}{X_0}\frac{[\rho U_1U_2]}{[\rho U_2^2 + P]}. \tag{2-15}$$

This, together with (2-12), yields on Σ that

$$
\begin{aligned}
&G_1(\rho, U, S) \equiv [\rho U_1][\rho U_2^2 + P] - \rho^2 U_1 U_2^2 = 0,\\
&G_2(\rho, U, S) \equiv [\rho U_1^2 + P][\rho U_2^2 + P] - (\rho U_1 U_2)^2 = 0,\\
&G_3(\rho, U, S) \equiv [(\rho e + \tfrac{1}{2}\rho|U|^2 + P)U_1][\rho U_2^2 + P]\\
&\qquad\qquad - \rho U_1(\rho e + \tfrac{1}{2}\rho|U|^2 + P)U_2^2 = 0.
\end{aligned}
\tag{2-16}
$$

It follows from a direct computation and the implicit function theorem that at the shock position Σ

$$
\begin{aligned}
(S - S_0^+, P - P_0^+, U_1 - \hat{U}_0^+)(r_0)\\
= (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)(U_2^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)),
\end{aligned}
\tag{2-17}
$$

where $\tilde{g}_j$ is smooth in its arguments and satisfies $\tilde{g}_j(0,0,0) = 0$ for $j = 1, 2, 3$. Moreover, by (1-6), the expected estimates in Theorem 1.1, and Remarks A.2 and A.3, it can be verified that

$$\tilde{g}_i = (O(\varepsilon) + O(1/X_0))(O(U_2) + O(\xi(y_2) - r_0)) \quad \text{for } i = 1, 2, 3.$$

This implies that on the shock surface, the influence of U_2 and $\xi(y_2) - r_0$ on $S - S_0^+$, $U_1 - \hat{U}_0^+$ and $P - \hat{P}_0^+$ can be almost neglected.

On the other hand, due to (2-1) and (2-10), the extension $(S_0^\pm, \hat{P}_0^\pm(r), \hat{U}_0^\pm(r))$ of the background solution in Appendix A will be changed into

$$(S_0^\pm, \hat{P}_0^\pm(y), \hat{U}_0^\pm(y)), \tag{2-18}$$

which satisfies for large X_0

$$\left|\frac{d^k \hat{P}_0^\pm(y_1)}{dy_1^k}\right| + \left|\frac{d^k \hat{U}_0^\pm(y_1)}{dy_1^k}\right| \le \frac{C}{X_0^k} \quad \text{for } k = 1, 2, 3, \tag{2-19}$$

where the constant $C > 0$ is independent of X_0 (see Remark A.2).

To treat the system (2-11) with (2-12)–(2-14), we introduce new coordinates

$$z_1 = \frac{y_1 - \xi(y_2)}{X_0 + 1 - \xi(y_2)} \quad \text{and} \quad z_2 = y_2, \tag{2-20}$$

which changes the free domain

$$R_+ = \{(y_1, y_2) : \xi(y_2) < y_1 < X_0 + 1,\ 0 < y_2 < 1\} \tag{2-21}$$

into a fixed square

$$E_+ = \{(z_1, z_2) : 0 < z_1 < 1,\ 0 < z_2 < 1\}. \tag{2-22}$$

There coordinates will decouple the system (2-11) with (2-12)–(2-14).

With some abuse of notation, we set

$$(S, P, U_1, U_2)(z) = (S, P, U_1, U_2)(\xi(z_2) + z_1(X_0 + 1 - \xi(z_2)), z_2), \tag{2-23}$$

$$(\hat{P}_0^+, \hat{U}_0^+)(z_1) = (\hat{P}_0^+, \hat{U}_0^+)(r_0 + z_1(X_0 + 1 - r_0)). \tag{2-24}$$

Define

$$w = U_2/U_1. \tag{2-25}$$

We now derive a first order elliptic system in w and P.

In fact,

$$\frac{1}{\rho U_1^2} \times \big((\text{the third equation in (2-11)}) - U_2 \times (\text{the first equation in (2-11)})\big),$$

together with the fourth equation in (2-11), yields

$$\partial_{y_1} w - \frac{X_0}{y_1}\left(\frac{1}{\rho U_1^2} - \frac{w^2}{\gamma P}\right)\partial_{y_2} P - \frac{w}{\gamma P}\partial_{y_1} P - \frac{w}{y_1} + \frac{w^2}{y_1}\cot\frac{y_2}{X_0} = 0.$$

While

$$\frac{y_1}{X_0 \rho U_1^2} \times \big((\text{the second equation in (2-11)}) - U_1 \times (\text{the first equation in (2-11)})\big)$$

yields

$$\partial_{y_2} w + \frac{w}{X_0}\cot\frac{y_2}{X_0} + \frac{y_1}{X_0}\left(\frac{1}{\rho U_1^2} - \frac{1}{\gamma P}\right)\partial_{y_1} P + \frac{w}{\gamma P}\partial_{y_2} P - \frac{w^2+2}{X_0} = 0.$$

In the (z_1, z_2) coordinates, we then have in E_+

$$\begin{aligned} \partial_{z_1} w - a_1 \partial_{z_2} P &= F_1(S, P, U_1, U_2; \xi), \\ \partial_{z_2} w + \frac{1}{X_0}\cot\frac{z_2}{X_0} w + a_2 \partial_{z_1} P &= F_2(S, P, U_1, U_2; \xi), \end{aligned} \tag{2-26}$$

where

$$\begin{aligned} a_1 &= \frac{X_0(X_0+1-r_0)}{r_0}\frac{1}{\hat{\rho}_0^+(0)(\hat{U}_0^+(0))^2}, \\ a_2 &= \frac{r_0}{X_0(X_0+1-r_0)}\left(\frac{1}{\hat{\rho}_0^+(0)(\hat{U}_0^+(0))^2} - \frac{1}{\gamma \hat{P}_0^+(0)}\right), \end{aligned}$$

and

$$
\begin{aligned}
&F_1(S, P, U_1, U_2; \xi) \\
&\quad = \frac{X_0}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\left(\frac{1}{\rho U_1^2}-\frac{w^2}{\gamma P}\right)\big((z_1-1)\xi'(z_2)\partial_{z_1} \\
&\qquad\qquad + (X_0+1-\xi(z_2))\partial_{z_2}\big)P + \frac{w}{\gamma P}\partial_{z_1}P - a_1\partial_{z_2}P \\
&\qquad + \frac{w(X_0+1-\xi(z_2))}{\xi(z_2)+z_1(X_0+1-\xi(z_2))} - \frac{w^2(X_0+1-\xi(z_2))}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\cot\frac{z_2}{X_0}, \\
&F_2(S, P, U_1, U_2; \xi) \\
&\quad = a_2\partial_{z_1}P - \frac{\xi(z_2)+z_1(X_0+1-\xi(z_2))}{X_0(X_0+1-\xi(z_2))}\left(\frac{1}{\rho U_1^2}-\frac{1}{\gamma P}\right)\partial_{z_1}P \\
&\qquad - \frac{w}{\gamma P}\left(\frac{(z_1-1)\xi'(z_2)}{X_0+1-\xi(z_2)}\partial_{z_1}+\partial_{z_2}\right)P + \frac{(1-z_1)\xi'(z_2)}{X_0+1-\xi(z_2)}\partial_{z_1}w + \frac{w^2+2}{X_0}.
\end{aligned}
$$

It should be noted that in (2-26),

$$\frac{w^2(X_0+1-\xi(z_2))}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\cot\frac{z_2}{X_0} \quad \text{and} \quad \frac{1}{X_0}\cot\frac{z_2}{X_0}w$$

are singular at $z_2 = 0$, and thus special care is required in our analysis.

In addition, it follows from the first equality and the fourth equality in (2-9) that

$$\left[\tfrac{1}{2}|U|^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho}\right] = 0.$$

This, together with the first and the fifth equation in (1-1) yields the Bernoulli's law

$$\tfrac{1}{2}U_1^2(1+w^2) + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \tfrac{1}{2}(U_0^-(X_0))^2 + \frac{\gamma}{\gamma-1}\frac{P_0^-(X_0)}{\rho_0^-(X_0)}. \tag{2-27}$$

In terms of the fourth equation in (2-11), the equation for the entropy becomes

$$\left(\left(1+\frac{X_0 w(1-z_1)\xi'(z_2)}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\right)\partial_{z_1} - \frac{X_0(X_0+1-\xi(z_2))w}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\partial_{z_2}\right)S = 0. \tag{2-28}$$

The related boundary conditions of (S^+, P, U_1, U_2) are

$$
\begin{aligned}
&(S, P, U_1)(0, z_2) - (S_0^+, \hat{P}_0^+, U_0^+)(0) \\
&\qquad = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)(U_2^2(0, z_2), P_0^-(\xi(z_2)) - P_0^-(r_0), U_0^-(\xi(z_2)) - U_0^-(r_0)).
\end{aligned} \tag{2-29}
$$

and

$$P(1, z_2) = P_e + \varepsilon \tilde{P}(z_2/X_0). \tag{2-30}$$

$$U_2(z_1, 1) = 0, \tag{2-31}$$

where the shock $\xi(z_2)$ is determined by

$$\xi'(z_2) = -\frac{\xi(z_2)}{X_0} \frac{(\rho U_1 U_2)(0, z_2)}{\rho(0, z_2) U_2^2(0, z_2) + P(0, z_2) - P_0^-(\xi(z_2))}. \tag{2-32}$$

Consequently, in order to show Theorem 1.1, we only need to solve the problem (2-26)–(2-28) with conditions (2-29)–(2-32).

3. The existence of a three-dimensional transonic shock for undetermined exit pressure

We will now establish the existence of a three-dimensional transonic shock in a nozzle when the transonic shock is assumed to go through some fixed point on the wall and when the end pressure $P_e + \varepsilon P_0(\theta)$ in (1-4) is adjusted by an appropriate constant. It follows from this that if one can show that the shock goes through some a point at the wall and if the corresponding adjustment constant on the end pressure is zero, then Theorem 1.1 will be proved.

Theorem 3.1. *Let the three-dimensional nozzle and the supersonic incoming flow be described as in Section 1. Assume further that*

$$\xi(1) = \tilde{r}_0, \tag{3-1}$$

where $\tilde{r}_0 \in (r_0 - \tilde{C} X_0^{3/2} \varepsilon, r_0 + \tilde{C} X_0^{3/2} \varepsilon)$ *with* $\tilde{C} > 0$ *some fixed constant. Then for* $\varepsilon < 1/X_0^3$ *and large* X_0, *there exists a constant* C_0 *such that the problem* (2-26)–(2-28) *and* (2-32) *with conditions* (2-29), (2-31) *and* (3-1) *has a* $C^{2,\alpha}(E_+)$ *transonic solution* $(S(z), P(z), U_1(z), U_2(z); \xi(z_2))$ *when* (2-30) *is replaced by*

$$P = \tilde{P}_e + \varepsilon \tilde{P}(z_2/X_0) + C_0 \quad \text{on } r = X_0 + 1. \tag{3-2}$$

Moreover,

$$\|\xi - \tilde{r}_0\|_{C^{3\alpha}[0,1]} \le C\varepsilon \tag{3-3}$$

and

$$\|(S, P, U_1) - (S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1))\|_{C^{2,\alpha}(E_+)} + \|U_2\|_{C^{2,\alpha}(E_+)} + |C_0| \le C\varepsilon. \tag{3-4}$$

Here C *is a generic nonnegative constant that is independent of* X_0 *and* ε, *and* $(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1))$ *is the background solution representing a radially symmetric transonic shock at position* $\tilde{r}_0$ *with exit pressure* $\tilde{P}_e$.

Due to singular terms in (2-26) on $\{z_2 = 0\}$, special attention must be paid to handle the possible nearby singularities of the solution. Fortunately, this difficulty can be overcome and $C^{2,\alpha}$ regularity of the subsonic flow can be established.

We define iteration spaces as follows:

$$S_\sigma = \{\xi(z_2) \in C^{3,\alpha}[0,1] : \|\xi - \tilde r_0\|_{C^{3,\alpha}[0,1]} \le \sigma,\ \xi'(0) = \xi'(1) = 0,\ \xi^{(3)}(0) = 0\} \tag{3-5}$$

and

$$\begin{aligned}\Xi_\delta = \big\{(S, P, U_1, U_2) : \|(S, P, U_1, U_2) - (S_a^+, \hat P_a^+, \hat U_a^+, 0)\|_{C^{2,\alpha}(\overline{E_+})} \le \delta,\\ \partial_{z_2}(S, P, U_1)(z_1, 0) = \partial_{z_2}(S, P, U_1)(z_1, 1) = (0, 0, 0),\\ U_2(z_1, 0) = U_2(z_1, 1) = \partial_{z_2}^2 U_2(z_1, 0) = 0\big\},\end{aligned} \tag{3-6}$$

with $\sigma > 0$ and $\delta > 0$ to be determined.

The proof of Theorem 3.1 is divided into four steps.

Step 1 (approximating shock). For $(\tilde S, P(q, \tilde S), V_1, V_2) \in \Xi_\delta$, we may by (2-32) define the approximating shock location as

$$\begin{aligned}\xi'(z_2) &= -\frac{\xi(z_2)}{X_0} \frac{(qV_1V_2)(0, z_2)}{P(q, \tilde S)(0, z_2) - P_0^-(\xi(z_2)) + (qV_2^2)(0, z_2)},\\ \xi(1) &= \tilde r_0,\end{aligned} \tag{3-7}$$

which has a unique solution $\xi(z_2) \in C^{3,\alpha}([0, 1])$. It follows from the compatibility conditions in (3-6) that $\xi(z_2)$ satisfies at $z_2 = 0, 1$ the last two conditions in (3-5), and

$$\|\xi(z_2) - \tilde r_0\|_{C^{3,\alpha}} \le C\|V_2\|_{C^{2,\alpha}} \le C\delta. \tag{3-8}$$

In addition, as in (2-29), on $z_1 = \xi(z_2)$ we may require that

$$\begin{aligned}(S, P, U_1)(0, z_2) - (S_a^+, \hat P_a^+(\tilde r_0), \hat U_a^+(\tilde r_0))\\ = (\check g_1, \check g_2, \check g_3)((V_2)^2, P_0^- - P_0^-(\tilde r_0), U_0^- - U_0^-(\tilde r_0)).\end{aligned} \tag{3-9}$$

It can be verified directly that $\partial_{z_2}(S, P, U_1)(0, 0) = \partial_{z_2}(S, P, U_1)(0, 1) = 0$.

Step 2 (approximating the specific entropy S). By (2-28), we approximate S by solving the problem

$$\begin{aligned}\left(\left(V_1 + \frac{X_0(1-z_1)\xi'(z_2)V_2}{\xi(z_2) + z_1(X_0 + 1 - \xi(z_2))}\right)\partial_{z_1} - \frac{X_0(X_0 + 1 - \xi(z_2))V_2}{\xi(z_2) + z_1(X_0 + 1 - \xi(z_2))}\partial_{z_2}\right) S = 0\\ \text{in } E_+,\\ S_a^+ + \tilde g_1((V_2)^2(0, z_2), P_0^-(\xi(z_2)) - P_0^-(\tilde r_0), U_0^-(\xi(z_2)) - U_0^-(\tilde r_0)) = S\\ \text{at } z_1 = 0.\end{aligned} \tag{3-10}$$

Due to (3-6), this problem has a unique solution $S \in C^{2,\alpha}(E_+)$. Moreover, by Remarks A.2 and A.3, we have

$$\begin{aligned}\|S - S_a^+\|_{C^{2,\alpha}} &\le C\|V_2\|^2_{C^{2,\alpha}} + \frac{C}{X_0}\|\xi - \tilde{r}_0\|_{C^{3,\alpha}} \\ &\le C\Big(\|V_2\|_{C^{2,\alpha}} + \frac{1}{X_0}\Big)\|V_2\|_{C^{2,\alpha}} \le C\Big(\delta + \frac{1}{X_0}\Big)\delta.\end{aligned} \tag{3-11}$$

Differentiating (3-10) with respect to z_2 and noting $\xi'(1) = V_2(z_1, 1) = 0$, we have

$$V_1\partial_{z_1}(\partial_{z_2}S) - \frac{X_0(X_0+1-\xi(z_2))\partial_{z_2}V}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}(\partial_{z_2}S) = 0 \text{ along } z_2 = 0 \text{ or } z_2 = 1,$$
$$\partial_{z_2}S(0,0) = \partial_{z_2}S(0,1) = 0,$$

which implies that

$$\partial_{z_2}S(z_1,0) = \partial_{z_2}S(z_1,1) = 0. \tag{3-12}$$

Thus, S belongs to Ξ_δ for small δ.

Convention 3.2. The reader may have noticed that X_0 sets the length scale for many quantities here. Since this trend will continue, we now declare that any symbol with check above it is that symbol divided by X_0. For example, $\check{z}_2 = z_2/X_2$, and $\check{1} = 1/X_0$.

Step 3 (approximating P and w). By (2-26), the second equality in (3-9) and (2-30)–(2-31), the approximate pressure P and w can be obtained from the boundary value problem

$$\begin{aligned}&\partial_1 w - \bar{a}_1\partial_2 P = F_1(\tilde{S}, P(q,\tilde{S}), V_1, V_2; \xi),\\ &\partial_2 w + \check{1}\cot\check{z}_2 w + \bar{a}_2\partial_1 P = F_2(\tilde{S}, P(q,\tilde{S}), V_1, V_2; \xi),\\ &P(0,z_2) - \hat{P}_a^+(\tilde{r}_0)\\ &\qquad = \tilde{g}_2(V_2^2(0,z_2), P_0^-(\xi(z_2)) - P_0^-(\tilde{r}_0), U_0^-(\xi(z_2)) - U_0^-(\tilde{r}_0)),\\ &P(1,z_2) = \tilde{P}_e + \varepsilon\tilde{P}(\check{z}_2) + C_0,\\ &w(z_1,0) = 0, \quad w(z_1,1) = 0.\end{aligned} \tag{3-13}$$

Here $\bar{a}_1$ and $\bar{a}_2$ are defined as a_1 and a_2 in (2-26), but with $(\hat{\rho}_0^+, \hat{U}_0^+, \hat{P}_0^+; r_0)$ replaced by $(\hat{\rho}_a^+, \hat{U}_a^+, \hat{P}_a^+; \tilde{r}_0)$. Note that the boundary condition $w(z_1, 0) = 0$ comes essentially from requiring $C^{2,\alpha}$ regularity of the solution (P, w), by assuming $\tilde{P}'(0) = 0$ in (1-4). The constant C_0 will be chosen so that the solvability condition in (3-13) can be fulfilled. More concretely, it follows from the second equation in (3-13) and $w(z_1, 0) = 0$ that

$$w(z) = \frac{1}{\sin \check{z}_2} \int_0^{z_2} \sin \check{s} (F_2 - \bar{a}_2 \partial_1 P)(z_1, s) ds.$$

Since $w(z_1, 1) = 0$, we have

$$\int_0^1 \sin \check{s} \big(F_2(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) - \bar{a}_2 \partial_1 P \big)(z_1, s) ds = 0.$$

In particular,

$$\int_0^1 \sin \check{s} \big(F_2(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) - \bar{a}_2 \partial_1 P \big)(1, s) ds = 0. \tag{3-14}$$

We will take this as the solvability condition of (3-13) that determines the unknown constant C_0.

Next, since $\hat{P}_a^+(z_1)$ satisfies

$$\bar{a}_2 \partial_1 \hat{P}_a^+(z_1) - F_2(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1), 0; \tilde{r}_0) = 0 \quad \text{in } E_+ \quad \text{and} \quad \hat{P}_a^+(1) = \tilde{P}_e,$$

a direct computation yields

$$\begin{aligned}
\partial_1 w - \bar{a}_1 \partial_2 (P - \hat{P}_a^+) &= F_1(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi), \\
\partial_2 w + \check{1} \cot \check{z}_2 w + \bar{a}_2 \partial_1 (P - \hat{P}_a^+) &= F_2(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) \\
&\quad - F_2(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1), 0; \tilde{r}_0), \\
(P - \hat{P}_a^+)(0, z_2) &= \tilde{g}_2(V_2^2(0, z_2), P_0^-(\xi(z_2)) - P_a^-(\tilde{r}_0), U_0^-(\xi(z_2)) - U_a^-(\tilde{r}_0)), \\
(P - \hat{P}_a^+)(1, z_2) &= \varepsilon \tilde{P}(\check{z}_2) + C_0, \\
w(z_1, 0) = 0, &\quad w(z_1, 1) = 0.
\end{aligned} \tag{3-15}$$

Next, we derive a second order elliptic equation for $P - \hat{P}_a^+$ from (3-15).

Applying ∂_{z_1} and $-(\partial_{z_2} + \check{1} \cot(\check{z}_2))$ to the first and second equation in (3-15) respectively and adding up yields

$$\begin{aligned}
&\partial_1 \big(\bar{a}_2 \partial_1 (P - \hat{P}_a^+(z_1)) \big) + \partial_2 \big(\bar{a}_1 \partial_2 (P - \hat{P}_a^+(z_1)) \big) + \check{\bar{a}}_1 \cot \check{z}_2 \partial_2 \big(P - \hat{P}_a^+(z_1) \big) \\
&\quad = \partial_1 \big(F_2(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) - F_2(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1), 0; \tilde{r}_0) \big) \\
&\qquad - \partial_2 \big(F_1(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) \big) - \check{1} \cot \check{z}_2 F_1(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) \quad \text{in } E_+, \\
&(P - \hat{P}_a^+)(0, z_2) = \tilde{g}_2 \big(V_2^2(0, z_2), P_0^-(\xi(z_2)) - P_0^-(\tilde{r}_0), U_0^-(\xi(z_2)) - U_0^-(\tilde{r}_0) \big), \\
&(P - \hat{P}_a^+)(1, z_2) = \varepsilon \tilde{P}(\check{z}_2) + C_0, \\
&\partial_2 (P - \hat{P}_a^+(z_1)) = 0 \quad \text{on } z_2 = 0 \text{ or } z_2 = 1,
\end{aligned} \tag{3-16}$$

where the fact that $\partial_{z_2}(P-\hat{P}_a^+)(z_1,0)=\partial_{z_2}(P-\hat{P}_a^+)(z_1,1)=0$ comes from (3-15) and $F_1(\tilde{S}, P(q,\tilde{S}), V_1, V_2;\xi)(z_1,0) = F_1(\tilde{S}, P(q,\tilde{S}), V_1, V_2;\xi)(z_1,1) = 0$.

We now decompose the problem (3-16) as $P(z) = P_1(z) + P_2(z)$, with

$$\begin{aligned}
&\partial_1(\bar{a}_2\partial_1(P_1-\hat{P}_a^+(z_1))) + \partial_2(\bar{a}_1\partial_2(P_1-\hat{P}_a^+(z_1))) + \check{\bar{a}}_1\cot\check{z}_2\,\partial_2(P_1-\hat{P}_a^+(z_1))\\
&\quad = \partial_1\big(F_2(\tilde{S}, P(q,\tilde{S}), V_1, V_2;\xi) - F_2(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1), 0; \tilde{r}_0)\big)\\
&\qquad\qquad - \partial_2\big(F_1(\tilde{S}, P(q,\tilde{S}), V_1, V_2;\xi)\big) - \check{1}\cot\check{z}_2 F_1(\tilde{S}, P(q,\tilde{S}), V_1, V_2;\xi),\\
&P_1(0,z_2) - \hat{P}_a^+(0) = \tilde{g}_2(V_2^2(0,z_2), P_0^-(\xi(z_2)) - P_a^-(\tilde{r}_0), U_0^-(\xi(z_2)) - U_a^-(\tilde{r}_0)),\\
&P_1(1,z_2) - \hat{P}_a^+(1) = \varepsilon\tilde{P}(\check{z}_2),\\
&\partial_2(P_1-\hat{P}_a^+(z_1)) = 0 \quad \text{on } z_2 = 0 \text{ or } z_2 = 1,
\end{aligned}\tag{3-17}$$

and

$$\begin{aligned}
&\bar{a}_2\partial_1^2P_2 + \bar{a}_1\partial_2^2P_2 + \check{\bar{a}}_1\cot\check{z}_2\,\partial_2P_2 = 0 \quad \text{in } E_+,\\
&P_2(0,z_2) = 0,\\
&P_2(1,z_2) = C_0,\\
&\partial_2P_2 = 0 \quad \text{on } z_2 = 0 \text{ or } z_2 = 1.
\end{aligned}\tag{3-18}$$

We first treat the problem (3-17).

It follows from Lemma B.5 (for the case of $k=1$) that (3-17) has a unique $C^{2,\alpha}(E_+)$ solution $P_1(z)$ satisfying

$$\begin{aligned}
&\|P_1(z) - \hat{P}_a^+(z_1)\|_{C^{2,\alpha}}\\
&\qquad \le C\|F_2(\tilde{S}, P(q,\tilde{S}), V_1, V_2;\xi) - F_2(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1), 0; \tilde{r}_0)\|_{C^{1,\alpha}}\\
&\qquad\quad + C\|F_1(\tilde{S}, P(q,\tilde{S}), V_1, V_2;\xi)\|_{C^{1,\alpha}} + C\varepsilon\|\tilde{P}(\check{z}_2)\|_{C^{2,\alpha}}\\
&\qquad\quad + C\|\tilde{g}_2(V_2^2(0,z_2), P_0^-(\xi(z_2)) - P_0^-(\tilde{r}_0), U_0^-(\xi(z_2)) - U_0^-(\tilde{r}_0))\|_{C^{2,\alpha}}.
\end{aligned}$$

Though $(V_2^2(X_0+1-\xi(z_2)))/(\xi(z_2)+z_1(X_0+1-\xi(z_2)))\cot\check{z}_2$ may be singular in F_1, it follows from Lemma B.3 that

$$\begin{aligned}
\Big\|\frac{V_2^2(X_0+1-\xi(z_2))}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\cot\check{z}_2\Big\|_{C^{1,\alpha}} &\le C\|V_2\|_{C^{1,\alpha}}\Big\|\check{1}\cot\check{z}_2V_2\Big\|_{C^{1,\alpha}(E_+)}\\
&\le C\delta\|V_2\|_{C^{2,\alpha}}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\|P_1(z)-\hat{P}_a^+(z_1)\|_{C^{2,\alpha}}\\
&\quad\le O(\check{1})\|\tilde{S}-S_a^+\|_{C^{2,\alpha}}+O(\check{1})\|P(q,\tilde{S})-\hat{P}_a^+\|_{C^{2,\alpha}}\\
&\qquad+O(\check{1}+\delta)\|V_1-\hat{U}_a^+\|_{C^{2,\alpha}}+O(\check{1}+\delta+\varepsilon)\|V_2\|_{C^{2,\alpha}}\\
&\qquad\qquad+O(\check{1}+\delta)\|\xi-\tilde{r}_0\|_{C^{2,\alpha}}+O(\varepsilon)\\
&\quad\le C(\check{\delta}+\delta^2+\varepsilon).
\end{aligned}
\tag{3-19}
$$

Next, note that the problem (3-18) has a solution

$$
P_2(z)=C_0z_1,
\tag{3-20}
$$

which is unique by Lemma B.5.

In this case, by the second equation in (3-15), (3-14) can be changed into

$$
\begin{aligned}
\int_0^1 \sin\check{s}\big(&F_2(\tilde{S},P(q,\tilde{S}),V_1,V_2;\xi)-F_2(S_a^+,\hat{P}_a^+(z_1),\hat{U}_a^+(z_1),0;\tilde{r}_0)\\
&-\bar{a}_2(\partial_1P_1-\partial_1\hat{P}_a^+(z_1))-\bar{a}_2C_0\big)(1,s)\,ds=0.
\end{aligned}
\tag{3-21}
$$

Note that $\bar{a}_2=O(1)>0$ since $(S_a^+,\hat{P}_a^+,\hat{U}_a^+)$ is subsonic. Then we can choose a unique constant C_0 such that (3-21) holds. Moreover, it follows from (3-19) and the expression of F_2 that C_0 admits the estimate

$$
\begin{aligned}
|C_0|
&=\frac{1}{2\bar{a}_2X_0\sin^2\frac{1}{2X_0}}\bigg|\int_0^1\sin\check{s}\big(F_2(\tilde{S},P(q,\tilde{S}),V_1,V_2;\xi)\\
&\qquad-F_2(S_a^+,\hat{P}_a^+(z_1),\hat{U}_a^+(z_1),0;\tilde{r}_0)\\
&\qquad\qquad-\bar{a}_2(\partial_1P_1-\partial_1\hat{P}_a^+(z_1))\big)(1,s)\,ds\bigg|\\
&\le\|P_1(z)-\hat{P}_a^+(z_1)\|_{C^{2,\alpha}}\\
&\qquad+\frac{1}{\bar{a}_2}\big\|F_2(\tilde{S},P(q,\tilde{S}),V_1,V_2;\xi)-F_2(S_a^+,\hat{P}_a^+(z_1),\hat{U}_a^+(z_1),0;\tilde{r}_0)\big\|_{C^{1,\alpha}}\\
&\le C(\check{\delta}+\delta^2+\varepsilon).
\end{aligned}
\tag{3-22}
$$

Collecting all the estimates (3-17)–(3-22) shows that there exists a unique constant C_0 such that the second order elliptic equation (3-16) with mixed boundary conditions has a unique solution $P(z)$ satisfying

$$
\|P-\hat{P}_a^+\|_{C^{2,\alpha}}+|C_0|\le\|P_1-\hat{P}_a^+\|_{C^{2,\alpha}}+C|C_0|\le C(\check{\delta}+\delta^2+\varepsilon).
\tag{3-23}
$$

With $P(z)$ so determined, we can obtain w in E_+ by solving the problem

$$\begin{aligned} &\partial_1 w = \bar{a}_1 \partial_2 P + F_1(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi), \\ &\partial_2 w + \check{1} \cot \check{z}_2 w = F_2(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) - \bar{a}_2 \partial_1 P, \\ &w(z_1, 0) = 0. \end{aligned} \tag{3-24}$$

It follows from Lemma B.7 that (3-24) has a unique solution w due to (3-13). On the other hand, by $w(z_1, 0) = 0$, we arrive at

$$\|w\|_{C^{2,\alpha}} \le C(\|\partial_1 w\|_{C^{1,\alpha}} + \|\partial_2 w\|_{C^{1,\alpha}}). \tag{3-25}$$

We now estimate $\|\partial_1 w\|_{C^{1,\alpha}(E_+)}$ and $\|\partial_2 w\|_{C^{1,\alpha}(E_+)}$.

By the first equation in (3-15) and (3-23), we have

$$\|\partial_1 w\|_{C^{1,\alpha}} \le C\big(\|P - \hat{P}_a^+\|_{C^{2,\alpha}} + \|F_1\|_{C^{1,\alpha}}\big) \le C(\check{\delta} + \delta^2 + \varepsilon). \tag{3-26}$$

Next, it follows from the second equation in (3-15) that

$$\begin{aligned} w(z) = \frac{1}{\sin \check{z}_2} \int_0^{z_2} \sin \check{s} \big(F_2(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) \\ - F_2(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1), 0; \tilde{r}_0) \\ - \bar{a}_2(\partial_1 P - \partial_1 \hat{P}_a^+(z_1)) \big) ds. \end{aligned} \tag{3-27}$$

Furthermore, a direct but careful computation using (3-27) and (3-21) yields

$$w(z_1, 0) = \partial_{z_2}^2 w(z_1, 0) = w(1, 1) = 0. \tag{3-28}$$

Indeed, $w(z_1, 0) = w(1, 1) = 0$ comes directly from (3-21), (3-24) and (3-27), while $\partial_{z_2}^2 w(z_1, 0) = 0$ follows from the following computations:

Applying ∂_{z_2} two both sides of the second equation in (3-24) yields

$$\partial_{z_2}^2 w + \check{1} \cot \check{z}_2 \, \partial_{z_2} w - \frac{1}{X_0^2 \sin^2 \check{z}_2} w = \partial_{z_2} F_2 - \bar{a}_2 \partial_{z_1 z_2}^2 P. \tag{3-29}$$

Note that for small z_2,

$$\begin{aligned} &\partial_{z_2}^2 w + \check{1} \cot \check{z}_2 \partial_{z_2} w - \frac{1}{X_0^2 \sin^2 \check{z}_2} w \\ &\quad = \partial_{z_2}^2 w + \frac{1}{X_0^2 \sin^2 \check{z}_2} (\partial_{z_2} w X_0 \sin \check{z}_2 \cos \check{z}_2 - w) \\ &\quad = \partial_{z_2}^2 w + \frac{1}{X_0^2 \sin^2 \check{z}_2} \Big(\partial_{z_2} w X_0 (\check{z}_2 + o(\check{z}_2^2)) \big(1 - \tfrac{1}{2} \check{z}_2^2 + o(\check{z}_2^3)\big) \\ &\qquad\qquad - z_2 \int_0^1 \partial_{z_2} w(z_1, \theta z_2) d\theta \Big) \\ &\quad = \partial_{z_2}^2 w + \frac{1}{X_0^2 \sin^2 \check{z}_2} \big(\partial_{z_2} w z_2 - \partial_{z_2} w(z_1, 0) z_2 - \tfrac{1}{2} \partial_{z_2}^2 w(z_1, 0) z_2^2 + o(z_2^2) \big) \end{aligned}$$

$= \frac{3}{2}\partial_{z_2}^2 w(z_1, 0) + o(z_2),$

and it follows from $\partial_{z_2} P(z_1, 0) = 0$ and the expression of F_2 that $\partial^2_{z_1 z_2} P(z_1, 0) = 0$ and $\partial_{z_2} F_2(z_1, 0) = 0$. Consequently, (3-29) shows that $\partial_{z_2}^2 w(z_1, 0) = 0$.

In addition, because $\partial_{z_1} w(z_1, 1) = 0$, which comes from $\partial_{z_2} P(z_1, 1) = 0$ and $F_1(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi)(z_1, 1) = 0$, and $w(1, 1) = 0$, we have

$$w(z_1, 1) = 0. \tag{3-30}$$

Finally, it follows from the second equation in (3-15) and Lemma B.6 that

$$\begin{aligned}
&\|\partial_2 w\|_{C^\alpha} + \|\partial_2^2 w\|_{C^\alpha} \\
&\quad \le C\big(\|F_2(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) - F_2(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1), 0; \tilde{r}_0)\|_{C^{1,\alpha}} \\
&\qquad\qquad + \|P - \hat{P}_a^+(z_1)\|_{C^{2,\alpha}}\big) \\
&\quad \le C(\check{\delta} + \delta^2 + \varepsilon).
\end{aligned}$$

This, together with (3-26), yields

$$\begin{aligned}
&\|w\|_{C^{2,\alpha}} \\
&\quad \le C\big(\|F_2(\tilde{S}, P(q, \tilde{S}), V_1, V_2; \xi) - F_2(S_a^+, \hat{P}_a^+(z_1), \hat{U}_a^+(z_1), 0; \tilde{r}_0)\|_{C^{1,\alpha}} \\
&\qquad\qquad + \|P - \hat{P}_a^+\|_{C^{2,\alpha}} + \|F_1\|_{C^{1,\alpha}}\big) \\
&\quad \le C(\check{\delta} + \delta^2 + \varepsilon).
\end{aligned} \tag{3-31}$$

Thus, it follows from (3-16), (3-22)–(3-24), (3-28), (3-30) and (3-31) that there exists a unique constant C_0 such that the first order elliptic system (3-13) has a unique solution $(P(z), w(z))$ satisfying the estimates

$$\|P - \hat{P}_0^+\|_{C^{2,\alpha}} + \|w\|_{C^{2,\alpha}} + |C_0| \le C(\check{\delta} + \delta^2 + \varepsilon). \tag{3-32}$$

and

$$\partial_2 P(z_1, 0) = \partial_2 P(z_1, 1) = w(z_1, 0) = w(z_1, 1) = \partial_2^2 w(z_1, 0) = 0. \tag{3-33}$$

Step 4 (approximating the radial velocity U_1). Due to (2-27), the radial velocity U_1 can be uniquely determined from

$$\begin{aligned}
&U_1^2(1 + w^2) + \frac{2\gamma}{\gamma - 1}\frac{P}{\rho} - \big(\hat{U}_a^+\big)^2 - \frac{2\gamma}{\gamma - 1}\frac{\hat{P}_a^+}{\hat{\rho}_a^+} = 0, \\
&U_1(z) > 0.
\end{aligned} \tag{3-34}$$

It follows from (3-11) and (3-32) that $U_1(z)$ satisfies

$$\begin{aligned}
\|U_1 - \hat{U}_a^+\|_{C^{2,\alpha}} &\le C\big(\delta\|w\|_{C^{2,\alpha}} + \|S - S_a^+\|_{C^{2,\alpha}} + \|P - \hat{P}_a^+\|_{C^{2,\alpha}}\big) \\
&\le C(\check{\delta} + \delta^2 + \varepsilon).
\end{aligned} \tag{3-35}$$

By (3-12), (3-28) and (3-30), a direct computation yields

$$\partial_{z_2} U_1(z_1, 0) = \partial_{z_2} U_1(z_1, 1) = 0. \tag{3-36}$$

All the constants C in (3-8), (3-11), (3-32) and (3-35) depend only on the supersonic incoming flow and $\|\tilde{P}(\check{z}_2)\|_{C^{2,\alpha}}$, so we can choose $\sigma = O(1)\varepsilon > 0$ and $\delta = O(1)\varepsilon > 0$ such that $(S, P, U_1, U_2; \xi)$ obtained in Steps 1–4 belongs to the space Ξ_δ. Consequently, we can define a map T from Ξ_δ to itself by

$$T(\tilde{S}, P(q, \tilde{S}), V_1, V_2) = (S, P, U_1, U_2). \tag{3-37}$$

Proof of Theorem 3.1. It suffices to prove that the mapping T defined in (3-37) is contractible in $C^{1,\alpha}(E_+)$.

For any two given elements $(\tilde{S}_1, \tilde{P}_1, V_{11}, V_{21})$ and $(\tilde{S}_2, \tilde{P}_2, V_{12}, V_{22})$ in Ξ_δ, set

$$T(\tilde{S}_1, \tilde{P}_1, V_{11}, V_{21}) = (S_1, P_1, U_{11}, U_{21}),$$
$$T(\tilde{S}_2, \tilde{P}_2, V_{12}, V_{22}) = (S_2, P_2, U_{12}, U_{22}),$$

and denote the corresponding approximating shocks (obtained by solving (3-7)) by $\xi_1(z_2)$ and $\xi_2(z_2)$, respectively. Below we will use the fact that $\sigma = O(1)\varepsilon > 0$ and $\delta = O(1)\varepsilon > 0$ in (3-5) and (3-6).

Define

$$(W_1, W_2, W_3, W_4) = (S_1 - S_2, P_1 - P_2, U_{11} - U_{12}, U_{21} - U_{22}),$$
$$(\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3, \widetilde{W}_4) = (\widetilde{S}_1 - \widetilde{S}_2, \widetilde{P}_1 - \widetilde{P}_2, V_{11} - V_{12}, V_{21} - V_{22}).$$

For convenience, we set also

$$W_5 = \frac{U_{21}}{U_{11}} - \frac{U_{22}}{U_{12}}, \quad \widetilde{W}_5 = \frac{V_{21}}{V_{11}} - \frac{V_{22}}{V_{12}}, \quad W_6 = \xi_1(z_2) - \xi_2(z_2).$$

Next, we derive some useful estimates on W_i for $i = 1, 2, \cdots, 6$, so that the contractible property of T can be established.

First, it follows from (3-7) and a simple computation that

$$\begin{aligned} W_6'(z_2) &= O(\varepsilon)\widetilde{W}_1 + O(\varepsilon)\widetilde{W}_2 + O(\varepsilon)\widetilde{W}_3 \\ &\qquad + O(1)\widetilde{W}_4 + O(\check{\varepsilon})W_6 \quad \text{in } [0, 1], \\ W_6(1) &= 0. \end{aligned} \tag{3-38}$$

This yields

$$\|W_6\|_{C^{2,\alpha}[0,1]} \le C\Big(\varepsilon \sum_{i=1}^{3} \|\widetilde{W}_i\|_{C^{1,\alpha}} + \|\widetilde{W}_4\|_{C^{1,\alpha}}\Big). \tag{3-39}$$

Second, it follows from (2-28) and Lemma B.8 that

$$\|W_1\|_{C^{1,\alpha}} \le C\Big(\varepsilon \sum_{i=2}^{4} \|\widetilde{W}_i\|_{C^{1,\alpha}} + \check{1}\,\|W_6\|_{C^{2,\alpha}}\Big). \tag{3-40}$$

Next, it follows from (3-13) and (3-21) that

$$\begin{aligned}
\partial_1 W_5 - \bar{a}_1 \partial_2 W_2 &= O(\varepsilon)\widetilde{W}_1 + O(\varepsilon)\widetilde{W}_2 + O(\varepsilon)\widetilde{W}_3 \\
&\qquad + O(\check{1})\widetilde{W}_5 + O(\varepsilon)W_6 + O(\varepsilon)\partial_1\widetilde{W}_2 \\
&\qquad\qquad + O(\check{1})\partial_2\widetilde{W}_2 + O(\check{1})W_6'(z_2), \\
\partial_2 W_5 + \check{1}\cot(\check{z}_2)W_5 + \bar{a}_2\partial_1 W_2 & \\
&= O(\check{1})\widetilde{W}_1 + O(\check{1})\widetilde{W}_2 + O(\check{1})\widetilde{W}_3 \\
&\qquad + O(\varepsilon)\widetilde{W}_5 + O(\check{1})W_6 + O(\check{1})\partial_1\widetilde{W}_2 \\
&\qquad + O(\varepsilon)\partial_2\widetilde{W}_2 + O(\varepsilon)\partial_1\widetilde{W}_5 + O(\varepsilon)W_6'(z_2), \\
W_2(0,z_2) &= O(\varepsilon)\widetilde{W}_4(0,z_2) + O(\check{1})W_6(z_2), \\
W_2(1,z_2) &= \text{constant}, \\
W_5(z_1,0) &= 0, \quad W_5(z_1,1) = 0.
\end{aligned} \tag{3-41}$$

Then it follows from Lemma B.5 for the case $k=0$ and (B-31) of Lemma B.6 that

$$\|W_2\|_{C^{1,\alpha}} + \|W_5\|_{C^{1,\alpha}} + |\text{constant}| \le \check{C}\Big(\sum_{i=1}^{5} \|\widetilde{W}_i\|_{C^{1,\alpha}} + \|W_6\|_{C^{2,\alpha}}\Big). \tag{3-42}$$

Finally, it follows from the algebraic equation (2-27) that

$$W_3 = O(1)W_1 + O(1)W_2 + O(\varepsilon)W_5. \tag{3-43}$$

This yields

$$\|W_3\|_{C^{1,\alpha}} \le C(\|W_1\|_{C^{1,\alpha}} + \|W_2\|_{C^{1,\alpha}} + \varepsilon\|W_5\|_{C^{1,\alpha}}). \tag{3-44}$$

Collecting all the estimates (3-39), (3-40), (3-42) and (3-44) obtained thus far, we arrive at

$$\sum_{i=1}^{3} \|W_i\|_{C^{1,\alpha}} + \|W_5\|_{C^{1,\alpha}} \le C(\check{1}+\varepsilon) \sum_{j=1}^{5} \|\widetilde{W}_i\|_{C^{1,\alpha}}. \tag{3-45}$$

In terms of the definitions of W_4, W_5, $\widetilde{W}_4$ and $\widetilde{W}_5$, one deduces from (3-45) that

$$\sum_{i=1}^{4} \|W_i\|_{C^{1,\alpha}} \le C(\check{1}+\varepsilon) \sum_{j=1}^{4} \|\widetilde{W}_i\|_{C^{1,\alpha}}. \tag{3-46}$$

Since X_0 is large and ε is small, $C(\check{1}+\varepsilon) < 1$ holds true in (3-46). This implies that the mapping T from Ξ_δ into itself is contractible in $C^{1,\alpha}(E_+)$. Therefore, it follows from the contractible mapping theorem that there exists a unique fixed point of T in the function space Ξ_δ, which completes the proof of Theorem 3.1. □

We complete this section by pointing out some refined estimates on the solution obtained in Theorem 3.1. First, we note that by some elementary analysis for ordinary differential systems, one can verify the following fact, which has been given in [Li et al. 2009b, Proposition 5.3]:

Suppose $(S^+_{0,1}, \hat{P}^+_{0,1}(r), \hat{U}^+_{0,1}(r)$ *and* $(S^+_{0,2}, \hat{P}^+_{0,2}(r), \hat{U}^+_{0,2}(r))$, *with* $r \in [X_0, X_0+1]$ *given in Remark A.3, are two extended subsonic flows that correspond to the shock positions* $r_{0,1}$ *and* $r_{0,2}$ *with* $r_{0,i} \in (X_0, X_0+1)$, *and constant end pressures* $P_{1,e}$ *and* $P_{2,e}$ *respectively. Then there exists a uniform constant* $C > 1$ *independent of* X_0 *such that for large* X_0

$$\begin{aligned} &\|(S^+_{0,1}, \hat{P}^+_{0,2}(r), \hat{U}^+_{0,2}(r)) - (S^+_{0,2}, \hat{P}^+_{0,1}(r), \hat{U}^+_{0,1}(r))\|_{C^{4,\alpha}[X_0,X_0+1]} \\ &\qquad \le C|P_{2,e} - P_{1,e}|, \\ &(X_0/C)|P_{2,e} - P_{1,e}| \le |r_{0,2} - r_{0,1}| \le C X_0 |P_{2,e} - P_{1,e}|. \end{aligned} \tag{3-47}$$

This result combines with Theorem 3.1 to give another:

Theorem 3.1′. *Under the assumptions of Theorem 3.1, we have*

$$\|\xi - r_0\|_{L^\infty[0,1]} \le C X_0^{3/2}\varepsilon, \qquad \|\xi'\|_{C^{2,\alpha}[0,1]} \le C\varepsilon \tag{3-48}$$

and

$$\|(S, P, U_1) - (S^+_0, \hat{P}^+_0(z_1), \hat{U}^+_0(z_1))\|_{C^{2,\alpha}(E_+)} + |C_0| \le C\sqrt{X_0}\varepsilon, \tag{3-49}$$

$$\begin{aligned} \|\partial_{z_2}(S, P, U_1) - \partial_{z_2}(S^+_0, \hat{P}^+_0(z_1), \hat{U}^+_0(z_1))\|_{C^{1,\alpha}(E_+)} + \|U_2\|_{C^{2,\alpha}(E_+)} \\ \le C\varepsilon. \end{aligned} \tag{3-50}$$

Here the generic constant $C > 0$ *is independent of* X_0 *and* ε, *but may depend on* $\tilde{C}$.

Remark 3.3. In Theorems 3.1 and 3.1′ or the problem (1-1) with (1-2)–(1-5), it seems difficult to find higher order compatibility conditions at the nozzle wall so that the solutions will achieve $C^{3,\alpha}$ regularity; this is due to the source terms in (2-8). For more details, see Appendix C.

4. The monotonic dependence of the shock position on the exit pressure

The key to proving Theorem 1.1, as in [Li et al. 2009b], establishing the monotonic dependence of the shock position on the end pressure. For this end, we assume that the problem (2-26)–(2-28), (2-32) with (2-29) and (2-31), has two solutions

$$(S, P, U_1, U_2; \xi_1) \in C^{2,\alpha}(E_+) \times C^{3,\alpha}([0,1]),$$
$$(\tilde{S}, \tilde{P}, V_1, V_2; \xi_2) \in C^{2,\alpha}(E_+) \times C^{3,\alpha}([0,1])$$

when the exit pressure boundary condition (2-30) is replaced respectively by

$$P(1, z_2) = P_e + \varepsilon \tilde{P}_1(\check{z}_2), \tag{4-1}$$

$$\tilde{P}(1, z_2) = P_e + \varepsilon \tilde{P}_2(\check{z}_2). \tag{4-2}$$

Theorem 4.1. *If* $(P, \rho, U_1, U_2, S; \xi_1)$ *and* $(\tilde{P}, q, V_1, V_2, \tilde{S}; \xi_2)$ *both satisfy the estimates* (3-48)–(3-50), *and*

$$M_0^-(X_0) \equiv \frac{U_0^-(X_0)}{c(\rho_0^-(X_0))} > \sqrt{\frac{\gamma+3}{2}},$$

then

$$|\xi_2(1) - \xi_1(1)| \le C X_0 \varepsilon \|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}[0,1]}, \tag{4-3}$$

and

$$\begin{aligned} \|(S, P, U_1, U_2) - (\tilde{S}, \tilde{P}, V_1, V_2)\|_{C^{1,\alpha}(E_+)} + \|\xi_1' - \xi_2'\|_{C^{1,\alpha}[0,1]} \\ \le C\varepsilon \|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}[0,1]}. \end{aligned} \tag{4-4}$$

Furthermore, if $P_1(1, z_2) - P_2(1, z_2) = \tilde{C} = O(\sqrt{X_0}\varepsilon)$ *and* $\xi_1(1) < \xi_2(1)$, *then* $\xi_1(z_2) < \xi_2(z_2)$ *and the constant* $\tilde{C}$ *is positive. Moreover, there exists a generic constant* $C > 1$ *such that*

$$\frac{\check{1}}{C}(\xi_2(1) - \xi_1(1)) \le \tilde{C} \le \check{C}(\xi_2(1) - \xi_1(1)). \tag{4-5}$$

Proof. Without loss of generality, we assume

$$\xi_1(1) < \xi_2(1). \tag{4-6}$$

With some abuse of notation, we set

$$W_1(z) = S - \tilde{S}, \qquad W_2(z) = P - \tilde{P}, \qquad W_3(z) = U_1 - V_1,$$
$$W_4(z) = U_2 - V_2, \quad W_5(z) = \frac{U_2}{U_1} - \frac{V_2}{V_1}, \quad W_6(z_2) = \xi_1 - \xi_2.$$

The proof of Theorem 4.1 will be divided into five steps.

Step i (the estimate of W_6). It follows from (2-32) that $W_6(z_2)$ satisfies

$$
\begin{aligned}
&W_6'(z_2) = \sum_{i=1}^{3} O(\varepsilon) W_i + O(1) W_4 + O(\check{\varepsilon}) W_6, \\
&W_6(1) = \xi_1(1) - \xi_2(1)
\end{aligned}
\tag{4-7}
$$

and

$$
\begin{aligned}
W_6''(z_2) = \sum_{i=1}^{4} O(\varepsilon) W_i + O(\check{\varepsilon}) W_6 + \sum_{i=1}^{3} O(\varepsilon)\partial_2 W_i& \\
+ O(1)\partial_2 W_4 + O(\check{\varepsilon}) W_6'(z_2)&, \\
W_6'(1) = 0.&
\end{aligned}
\tag{4-8}
$$

By (4-6), we have

$$
\|W_6'(z_2)\|_{C^{1,\alpha}} \le C(\varepsilon(\xi_2(1) - \xi_1(1)) + \|\partial_2 W_4\|_{C^\alpha}) + C\varepsilon\Big(\sum_{i=1}^{4} \|W_i\|_{C^{1,\alpha}}\Big)
\tag{4-9}
$$

and

$$
\begin{aligned}
\|W_6\|_{C^{2,\alpha}} &\le C((\xi_2(1) - \xi_1(1)) + \|W_6'(z_2)\|_{C^{1,\alpha}}) \\
&\le C((\xi_2(1) - \xi_1(1)) + \|\partial_2 W_4\|_{C^\alpha}) + C\varepsilon\Big(\sum_{i=1}^{4} \|W_i\|_{C^{1,\alpha}}\Big).
\end{aligned}
\tag{4-10}
$$

Step ii (the estimate of W_1). First, we solve the first order system (2-28) in the coordinate $z = (z_1, z_2)$. Let $z_2^1(s; z)(z_2^2(s; z))$ be the characteristic going through $z = (z_1, z_2)$ and reaching $(0, \beta)((0, \tilde{\beta}))$ at $s = 0$ corresponding to the vector field $(U_1, U_2)((V_1, V_2))$, that is,

$$
\begin{aligned}
&\frac{dz_2^1(s; z)}{ds} = -\frac{X_0(X_0 + 1 - \xi_1(z_2^1))}{A_1} U_2(\xi_1(z_2^1) + s(X_0 + 1 - \xi_1(z_2^1)), z_2^1), \\
&z_2^1(z_1; z) = z_2, \quad z_2^1(0; z) = \beta,
\end{aligned}
$$

where

$$
A_1 = (\xi_1(z_2^1) + s(X_0 + 1 - \xi_1(z_2^1)))U_1 + U_2 X_0(1 - s)\xi_1'(z_2^1).
$$

Set $l(s; z) = z_2^1(s; z) - z_2^2(s; z)$, and note that $z_2^1(0; z) = \beta$ and $z_2^2(0; z) = \tilde{\beta}$. Then we have

$$
\begin{aligned}
\frac{dl}{ds} = O(\varepsilon) l + O(\varepsilon) W_3(s, z_2^1) + O(1) W_4(s, z_2^1)& \\
+ O(\varepsilon) W_6(z_2^1) + O(\varepsilon^2) W_6'(z_2^1)& \\
l(0; z) = \beta - \tilde{\beta}, \quad l(z_1; z) = 0.&
\end{aligned}
\tag{4-11}
$$

By the $C^{2,\alpha}$ regularity of solutions, we can check that the coefficients of $l(t;z)$ in (4-11) are in $C^{1,\alpha}$. Based on this, we intend to derive the $C^{1,\alpha}$ estimate of $\beta-\tilde{\beta}$. Indeed, by (4-11), we can arrive at

$$\|\beta-\tilde{\beta}\|_{L^\infty} \le C(\varepsilon\|W_3\|_{L^\infty}+\|W_4\|_{L^\infty}+\varepsilon\|W_6\|_{L^\infty}+\varepsilon^2\|W_6'(z_2)\|_{L^\infty}).$$

On the other hand,

$$z_2^1(s;z) = -\int_0^s \frac{X_0(X_0+1-\xi_1(z_2^1))}{A_1} U_2(\xi_1(z_2^1)+t(X_0+1-\xi_1(z_2^1)), z_2^1)\,dt+\beta,$$

and

$$z_2 = -\int_0^{z_1} \frac{X_0(X_0+1-\xi_1(z_{21}))}{A_1} U_2(\xi_1(z_2^1)+t(X_0+1-\xi_1(z_{21})), z_2^1)\,dt+\beta.$$

Similar relations hold for $z_2^2(s;z)$, z_2, and $\tilde{\beta}$ corresponding to (V_1, V_2). Hence, one can obtain

$$\begin{aligned}
\beta-\tilde{\beta} &= \int_0^{z_1} (O(\varepsilon)W_3(t,z_2^1)+O(1)W_4(t,z_2^1)\\
&\qquad\qquad +O(\varepsilon)W_6(z_2^1)+O(\varepsilon^2)W_6'(z_2^1)+O(\varepsilon)l(t;z))\,dt,\\
l(s;z) &= \int_{z_1}^{s} (O(\varepsilon)W_3(t,z_2^1)+O(1)W_4(t,z_2^1)\\
&\qquad\qquad +O(\varepsilon)W_6(z_2^1)+O(\varepsilon^2)W_6'(z_2^1)+O(\varepsilon)l(t;z))\,dt
\end{aligned} \tag{4-12}$$

and

$$\|\partial_{z_1}(\beta,\tilde{\beta})\|_{C^{1,\alpha}} \le C\varepsilon, \quad \|\partial_{z_2}(\beta,\tilde{\beta})\|_{C^{1,\alpha}} \le C. \tag{4-13}$$

It follows from (4-12) and (4-13) that

$$\|\beta-\tilde{\beta}\|_{C^{1,\alpha}} \le C(\varepsilon\|W_3\|_{C^{1,\alpha}}+\|W_4\|_{C^{1,\alpha}}+\varepsilon\|W_6\|_{C^{2,\alpha}}). \tag{4-14}$$

In addition, by (2-28) and the characteristics method, we have

$$\begin{aligned}
W_1(z) &= W_1(0,\beta(z_1,z_2))+O(\varepsilon)\big(\beta(z_1,z_2)-\tilde{\beta}(z_1,z_2)\big),\\
W_1(0,z_2) &= O(\varepsilon)W_4(0,z_2)+O(\check{1})W_6(z_2).
\end{aligned} \tag{4-15}$$

Combining (4-15) with (4-14) yields

$$\begin{aligned}
\|W_1\|_{C^{1,\alpha}} &\le C\big(\varepsilon\|(\varepsilon W_2,\varepsilon W_3,W_4)\|_{C^{1,\alpha}}+\check{1}\|W_6\|_{C^{2,\alpha}}+\varepsilon\|\beta-\tilde{\beta}\|_{C^{1,\alpha}}\big)\\
&\le C\big(\check{1}(\xi_2(1)-\xi_1(1))+\varepsilon\|(\varepsilon W_2,W_3,W_4)\|_{C^{1,\alpha}}\\
&\qquad\qquad +\check{1}\|W_6'(z_2)\|_{C^{1,\alpha}}\big).
\end{aligned} \tag{4-16}$$

Step iii (the estimates of W_2 and W_5). By the system (2-26) and the related boundary conditions, a direct computation yields

$$
\begin{aligned}
&\partial_1 W_5 - \tilde{a}_1 \partial_2 W_2 = O(\varepsilon)\cdot(W_1, W_2, W_3, W_6) + O(\check{1}) W_5 + O(\varepsilon)\partial_1 W_2 \\
&\qquad\qquad + O(\check{1})\partial_2 W_2 + O(\check{1}) W_6'(z_2), \\
&\partial_2 W_5 + \check{1}\cot(\check{z}_2) W_5 + \tilde{a}_2 \partial_1 W_2 \\
&\qquad = O(\check{1})\cdot(W_1, W_2, W_3, W_6, \partial_1 W_2) \\
&\qquad\qquad + O(\varepsilon)\cdot(W_5, \partial_2 W_2, \partial_1 W_5, W_6'), \\
&W_2(0, z_2) = O(\varepsilon) W_4(0, z_2) + O(\check{1}) W_6(z_2), \\
&W_2(1, z_2) = \varepsilon \tilde{P}_1(\check{z}_2) - \varepsilon \tilde{P}_2(\check{z}_2), \\
&W_5(z_1, 0) = 0, \\
&W_5(z_1, 1) = 0,
\end{aligned}
\tag{4-17}
$$

where $\tilde{a}_1$ and $\tilde{a}_2$ are positive constants that are defined like a_1 and a_2 respectively in (2-26) for the background solution, but with shock position at $r = \xi_1(1)$ rather than at $r = r_0$.

As in (3-16)–(3-18) and (3-21), we decompose $W_2 = W_{21} + W_{22}$ so that

$$
\begin{aligned}
&\tilde{a}_2 \partial_1^2 W_{21} + \tilde{a}_1 \partial_2^2 W_{21} + (\tilde{a}_1/X_0)\cot(\check{z}_2)\partial_2 W_{21} \\
&\qquad = \partial_1\big(O(\check{1})\cdot(W_1, W_2, W_3, \partial_1 W_2) \\
&\qquad\qquad + O(\varepsilon)\cdot(W_5, \partial_2 W_2, \partial_1 W_5, W_6') + a_3(z) W_6\big) \\
&\qquad\quad - \partial_2\big(O(\varepsilon)\cdot(W_1, W_2, W_3, W_6, \partial_1 W_2) + O(\check{1})\cdot(W_5, \partial_2 W_2, W_6')\big) \\
&\qquad\quad - X_0^{-1}\cot(\check{z}_2) \\
&\qquad\quad\ \times \big(O(\varepsilon)\cdot(W_1, W_2, W_3, W_6, \partial_1 W_2) + O(\check{1})\cdot(W_5, \partial_2 W_2, W_6')\big), \\
&W_{21}(0, z_2) = O(\varepsilon) W_4(0, z_2) + O(\check{1}) W_6(z_2), \\
&W_{21}(1, z_2) = 0, \\
&\partial_2 W_{21}(z_1, 0) = 0, \\
&\partial_2 W_{21}(z_1, 1) = 0
\end{aligned}
\tag{4-18}
$$

and

$$
\begin{aligned}
&\tilde{a}_2 \partial_1^2 W_{22} + \tilde{a}_1 \partial_2^2 W_{22} + (\tilde{a}_1/X_0)\cot(\check{z}_2)\partial_2 W_{22} = 0, \\
&W_{22}(0, z_2) = 0, \\
&W_{22}(1, z_2) = \varepsilon \tilde{P}_1(\check{z}_2) - \varepsilon \tilde{P}_2(\check{z}_2), \\
&\partial_2 W_{22}(z_1, 0) = 0, \\
&\partial_2 W_{22}(z_1, 1) = 0
\end{aligned}
\tag{4-19}
$$

and

$$\int_0^1 \sin\check{s}\big(O(\check{1})\cdot(W_1, W_2, W_3, \partial_1 W_2) + O(\varepsilon)\cdot(W_5, \partial_2 W_2, \partial_1 W_5, W_6') + a_3(z)W_6 - \tilde{a}_2\partial_1 W_{21} - \tilde{a}_2\partial_1 W_{22}\big)(1,s)\,ds = 0, \tag{4-20}$$

where $a_3(z_2) = O(\check{1})$. In particular, due to the estimates (3-48)–(3-50), we have

$$\begin{aligned} a_3(z) = &-\Big(\frac{1}{\rho U_1^2} - \frac{1}{\gamma P}\Big) \\ &\times \partial_1 P\bigg(\frac{1-z_1}{X_0(X_0+1-\xi_1(z_2))} + \frac{\xi_2(z_2)+z_1(X_0+1-\xi_2(z_2))}{X_0(X_0+1-\xi_1(z_2))(X_0+1-\xi_2(z_2))}\bigg) \\ &+ O(\varepsilon) < 0. \end{aligned} \tag{4-21}$$

Similar to the estimates in (3-42), by (B-20) in Lemma B.5 for the case $k = 0$, we have

$$\|W_{21}\|_{C^{1,\alpha}(E_1)} \le \check{C}\sum_{i=1}^{6}\|W_i\|_{C^{1,\alpha}(E_1)}, \tag{4-22}$$

$$\|W_{22}\|_{C^{1,\alpha}(E_1)} \le C\varepsilon\|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}[0,1]}. \tag{4-23}$$

In particular, for the case of $P(1,z_2) - \tilde{P}(1,z_2) = \tilde{C}$, we can determine $W_{22} = \tilde{C}z_1$ as in Section 3. Thus it follows from (4-20) and $\tilde{a}_2(z) = O(1) > 0$ that

$$\begin{aligned} \tilde{C} \le C\big(&\check{1}(\xi_2(1)-\xi_1(1)) + \check{1}\|W_1\|_{C^{1,\alpha}} + \|W_{21}\|_{C^{1,\alpha}} + \check{1}\|W_3\|_{C^{1,\alpha}} \\ &+ \varepsilon\|W_5\|_{C^{1,\alpha}} + \check{1}\|W_6'(z_2)\|_{C^{1,\alpha}}\big). \end{aligned} \tag{4-24}$$

Similar to the estimates for (3-21), (3-26) and (3-31), together with (4-9) and (4-22)–(4-23), we get

$$\begin{aligned} \|W_{21}\|_{C^{1,\alpha}} \le \check{C}\big((\xi_2(1)-\xi_1(1)) + \|(W_1, W_3, W_5, W_6')\|_{C^{1,\alpha}}\big) \\ + \check{C}\varepsilon\|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}}, \end{aligned} \tag{4-25}$$

$$\begin{aligned} \|W_5\|_{C^{1,\alpha}} \le \check{C}\big((\xi_2(1)-\xi_1(1)) + \|(W_1, W_{21}, W_3, W_6')\|_{C^{1,\alpha}}\big) \\ + \check{C}\varepsilon\|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}}. \end{aligned} \tag{4-26}$$

Thus, combining (4-25) and (4-26) with (4-23) yields

$$\begin{aligned} \|W_2\|_{C^{1,\alpha}} + \|W_5\|_{C^{1,\alpha}} \le \check{C}\big((\xi_2(1)-\xi_1(1)) + \|(W_1, W_3, W_6')\|_{C^{1,\alpha}}\big) \\ + C\varepsilon\|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}}. \end{aligned} \tag{4-27}$$

Step iv (the estimate of W_3). It follows from (2-27) that

$$W_3 = O(\varepsilon)W_5 + O(1)W_1 + O(1)W_2. \tag{4-28}$$

This yields

$$\|W_3\|_{C^{1,\alpha}} \le C\big(\|W_1\|_{C^{1,\alpha}} + \|W_2\|_{C^{1,\alpha}} + \varepsilon\|W_5\|_{C^{1,\alpha}}\big). \tag{4-29}$$

Step v (the estimate of $W_2(0, z_2)$). Note that the supersonic background solution (ρ_0^-, P_0^-, U_0^-) satisfies the system (2-11), that is,

$$\begin{aligned} \frac{d(\rho_0^- U_0^-)}{dy_1} &= -\frac{2\rho_0^- U_0^-}{y_1},\\ \frac{d(\rho_0^-(U_0^-)^2 + P_0^-)}{dy_1} &= -\frac{2\rho_0^-(U_0^-)^2}{y_1}. \end{aligned} \tag{4-30}$$

Set

$$\begin{aligned} &m_0(y_1) = (\rho_0^- U_0^-)^2,\\ &m_1(y_1) = \rho_0^-(U_0^-)^2 + P_0^-, \quad m_2 = \frac{\gamma}{\gamma-1}\frac{P_0^-}{\rho_0^-} + \tfrac{1}{2}(U_0^-)^2. \end{aligned}$$

It follows from Bernoulli's law, (2-27), that m_2 is a constant.

In addition, by (2-16) and (2-27), we have on $z_1 = 0$

$$\begin{aligned} \rho U_1 &= \sqrt{m_0} + \frac{\rho^2 U_1 U_2^2}{[\rho U_2^2 + P]},\\ \rho U_1^2 + P &= m_1 + \frac{(\rho U_1 U_2)^2}{[\rho U_2^2 + P]}, \qquad m_2 = \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \tfrac{1}{2}(U_1^2 + U_2^2). \end{aligned} \tag{4-31}$$

This implies

$$\begin{aligned} \rho &= \frac{\big(\sqrt{m_0} + \rho^2 U_1 U_2^2/[\rho U_2^2 + P]\big)^2}{m_1 - P + (\rho U_1 U_2)^2/[\rho U_2^2 + P]},\\ U_1 &= \frac{m_1 - P}{\sqrt{m_0}}, \qquad m_2 = \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \tfrac{1}{2}(U_1^2 + U_2^2). \end{aligned} \tag{4-32}$$

Substituting the first two expressions in (4-32) into the third equality in (4-32) yields on $z_1 = 0$

$$\begin{aligned} &\tfrac{1}{2}(m_1 - P)^2 + \frac{\gamma}{\gamma-1} P(m_1 - P) - m_2 m_0\\ &= m_2\sqrt{m_0}\frac{\rho^2 U_1 U_2^2}{[\rho U_2^2 + P]} - \tfrac{1}{2} m_0 U_2^2 - \tfrac{1}{2}\sqrt{m_0}\frac{\rho^2 U_1 U_2^2}{[\rho U_2^2 + P]}\bigg(\frac{(m_1 - P)^2}{m_0} + U_2^2\bigg). \end{aligned} \tag{4-33}$$

Since $(S, P, U_1, U_2; \xi_1)$ and $(\tilde{S}, \tilde{P}, V_1, V_2; \xi_2)$ both satisfy (4-33), it follows from a direct computation and the estimates (3-48)–(3-50) for $(S, P, U_1, U_2; \xi_1)$

and $(\tilde{S}, \tilde{P}, V_1, V_2; \xi_2)$ that

$$a_4(z_2)W_2 = a_5(z_2)W_6(z_2) + O(\varepsilon^2)W_1 + O(\varepsilon^2)W_2 + O(\varepsilon^2)W_3 + O(\varepsilon)W_4 + O(\varepsilon^2 X_0^{-1})W_6, \tag{4-34}$$

where

$$\begin{aligned} a_4(z_2) &= \frac{\gamma}{\gamma-1}m_1(\xi_1) - \tfrac{1}{2}(m_1(\xi_1)+m_1(\xi_2)-P-\tilde{P}) - \frac{\gamma}{\gamma-1}(P+\tilde{P}) \\ &= \frac{\gamma}{\gamma-1}m_1(r_0) - (m_1(r_0) - \hat{P}_0^+(r_0)) - \frac{2\gamma}{\gamma-1}\hat{P}_0^+(r_0) + O(\sqrt{X_0}\varepsilon) \\ &= \frac{1}{\gamma-1}\hat{\rho}_0^+(r_0)(\hat{U}_0^+(r_0))^2 - \frac{\gamma}{\gamma-1}\hat{P}_0^+(r_0) + O(\sqrt{X_0}\varepsilon) \\ &= \frac{1}{\gamma-1}\hat{\rho}_0^+(r_0)\big((\hat{U}_0^+(r_0))^2 - c^2(\hat{\rho}_0^+(r_0))\big) + O(\sqrt{X_0}\varepsilon) < 0 \end{aligned} \tag{4-35}$$

and

$$\begin{aligned} a_5(z_2) &= m_2\int_0^1 m_0'(\xi_2+s(\xi_1-\xi_2))ds \\ &\qquad - \tfrac{1}{2}(m_1(\xi_1)+m_1(\xi_2)-P-\tilde{P})\int_0^1 m_1'(\xi_2+s(\xi_1-\xi_2))ds \\ &\qquad - \frac{\gamma}{\gamma-1}\tilde{P}\int_0^1 m_1'(\xi_2+s(\xi_1-\xi_2))ds \\ &= m_2m_0'(r_0) - (m_1(r_0)-\hat{P}_0^+(r_0))m_1'(r_0) - \frac{\gamma}{\gamma-1}\hat{P}_0^+(r_0)m_1'(r_0) \\ &\qquad + O(\sqrt{X_0}\varepsilon) \\ &= -2\frac{(\rho_0^-(U_0^-)^2)(r_0)}{(\gamma-1)r_0}((\gamma+1)P_0^-(r_0) - \hat{P}_0^+(r_0)) + O(\sqrt{X_0}\varepsilon). \end{aligned} \tag{4-36}$$

Next, we analyze the sign of $a_5(z_2)$ for small ε and especially the sign of $(\gamma+1)P_0^-(r_0) - \hat{P}_0^+(r_0)$.

In fact, by (4-32), $\hat{P}_0^+(r_0)$ is a solution of the algebraic equation

$$F(s) = \tfrac{1}{2}(m_1(r_0)-s)^2 + \frac{\gamma}{\gamma-1}s(m_1(r_0)-s) - m_2m_0(r_0) = 0. \tag{4-37}$$

Since

$$\begin{aligned} F(P_0^-(r_0)) &= 0, \qquad F''(s) = -\frac{\gamma+1}{\gamma-1} < 0, \\ F'(P_0^-(r_0)) &= \frac{1}{\gamma-1}\big((\rho_0^-(U_0^-)^2)(r_0) - \gamma P_0^-(r_0)\big) \\ &= \frac{\rho_0^-(r_0)}{\gamma-1}((U_0^-(r_0))^2 - c^2(\rho_0^-(r_0))) > 0, \end{aligned}$$

which follows from direct computations, $F(s)$ is a concave function and $P_0^-(r_0)$ is a left zero point of $F(s)$.

Using the assumption $M_0^-(X_0) > \sqrt{(\gamma+3)/2}$ on the Mach number for the supersonic incoming flow, we have

$$F((\gamma+1)P_0^-(r_0)) = \frac{(\rho_0^-(r_0))^2 c^2(\rho_0^-(r_0))}{2(\gamma-1)}\big(2(U_0^-(r_0))^2 - (\gamma+3)c^2(\rho_0^-(r_0))\big) > 0.$$

This shows that

$$\hat{P}_0^+(r_0) > (\gamma+1)P_0^-(r_0). \tag{4-38}$$

Combining (4-38) with (4-36), we have

$$a_5(z_2) = O(\check{1}) \quad \text{and} \quad a_5(z_2) > 0. \tag{4-39}$$

On the other hand, by the estimates (4-9), (4-10), (4-16), (4-27) and (4-29), we have

$$\sum_{i=1}^{4} \|W_i\|_{C^{1,\alpha}} + \|W_6'(z_2)\|_{C^{1,\alpha}} \\ \le \check{C}|\xi_1(1)-\xi_2(1)| + C\varepsilon\|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}}. \tag{4-40}$$

This, together with (4-34)–(4-36), yields

$$W_2(0,z_2) \ge \check{b}_1(\xi_2(1)-\xi_1(1)) - b_2\varepsilon\|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}[0,1]}, \tag{4-41}$$

where b_i for $i=1,2$ is a generic positive constant of order $O(1)$.

Based on Steps i–v, we can prove Theorem 4.1.

Using (4-21) and substituting (4-40) into (4-20) (noting that (4-20) holds for all $z_1 \in [0,1]$), we have, for all $z_1 \in [0,1]$,

$$\int_0^1 \sin\check{s}\big(\check{b}_3(\xi_2(1)-\xi_1(1)) \\ - b_4\varepsilon\|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}} - \partial_1 W_{21}\big)(z_1,s)\,ds \le 0, \tag{4-42}$$

where b_i for $i=3,4$ is a generic positive constant. In particular,

$$b_3 \ge C(-X_0 a_3(z) + \check{C}) = O(1) > 0$$

because $a_3(z) = O(\check{1}) < 0$ in (4-21).

If we assume

$$\varepsilon\|\tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}} < \min\left\{\frac{\check{b}_1}{2b_2}, \frac{\check{b}_3}{2b_4}\right\}(\xi_2(1)-\xi_1(1)), \tag{4-43}$$

is false (that is, that this statement is true with "≥" instead of "<"), then (4-3) has been shown. If we assume (4-43) is true, then this means $W_2(0, z_2) > 0$. Due to $W_2(0, z_2) = W_{21}(0, z_2) + W_{22}(0, z_2)$ and $W_{22}(0, z_2) = 0$ in (4-18), we then get

$$W_{21}(0, z_2) > 0. \tag{4-44}$$

On the other hand, it follows from (4-42) and (4-43) that for $z_1 \in [0, 1]$

$$\partial_1\Big(\int_0^1 s W_{21}(z_1, s) \sin \check{s} ds\Big) > 0. \tag{4-45}$$

Combining (4-44) with (4-45) yields

$$\int_0^1 W_{21}(1, s) \sin \check{s} ds > 0.$$

However, this contradicts that $W_{21}(1, z_2) = 0$ in (4-18). Thus (4-43) does not hold, that is, we have shown that there exists a constant $C > 0$ such that

$$|\xi_2(1) - \xi_1(1)| \le C X_0 \varepsilon \| \tilde{P}_1(\check{z}_2) - \tilde{P}_2(\check{z}_2)\|_{C^{1,\alpha}}.$$

Combining this with (4-40), we complete the proof of (4-3) and (4-4).

Finally, by (4-24) and (4-25) and an argument analogous to the one for (4-3) and (4-4), we can also show (4-5). We omit the details. □

Remark 4.2. From (4-3) of Theorem 4.1, we have established that the position of the shock depends continuously on the exit pressure. If the condition (2-30) is replaced by $P(1, z_2) = P_e + \varepsilon \tilde{P}(\check{z}_2) + C$, then (4-5) establishes that the corresponding position of the shock depends monotonically on the exit pressure. Thus, the constant C_0 in Theorem 3.1′ can be considered as a function of the variable $y_1 \in (X_0, X_0 + 1)$, which is denoted by $C_0(y_1)$. Furthermore, it follows from (4-5) that the function $C_0(y_1)$ is Lipschitz continuous and decreasing.

5. Proof of Theorem 1.1.

First, we prove that the system (2-26)–(2-28), (2-32) with (2-29)–(2-31) has a solution.

Denote by $\bar{P}_1 = P_e - \sqrt{X_0}\varepsilon$ and $\bar{P}_2 = P_e + \sqrt{X_0}\varepsilon$ the exit pressures of the symmetric transonic shock solutions with corresponding shock positions at $y_1 = r_1$ and $y_1 = r_2$, respectively. Then it follows from (4-5) in Theorem 4.1 that $r_1 > r_2$ holds true.

For each fixed point $(y_1^*, 1)$ with $y_1^* \in [r_2, r_1]$, it follows from Theorem 3.1′ and Remark 4.2 that there exists a constant $C_0(y_1^*)$ such that problem (2-26)–(2-28), (2-32) with (2-29), (2-31) and the exit pressure $P = P_e + \varepsilon P_0(\theta) + C_0(y_1^*)$ has a unique solution $(S, P, U_1, U_2; \xi(z_2))$ that admits the estimates in Theorem 3.1′.

If $y_1^* = r_2$, it follows from (3-4) and (3-47) that

$$|C_0(r_2) - \sqrt{X_0}\varepsilon| \le C\varepsilon. \tag{5-1}$$

This implies that $C_0(r_2) > 0$. Analogously, we have $C_0(r_1) < 0$. Therefore, in terms of Theorem 4.1 and Remark 4.2, there exists a unique point $y_1^0 \in (r_2, r_1)$ such that $C_0(y_1^0) = 0$, that is, the system (2-26)–(2-28), (2-32) with (2-29)–(2-31) has a unique angular symmetric solution $(S, P, \rho, U_1, U_2; \xi)$. Also, by Theorem 3.1, this solution also satisfies the estimates

$$\|\xi - r_0\|_{L^\infty[0,1]} \le C X_0 \varepsilon, \qquad \|\xi'\|_{C^{2,\alpha}[0,1]} \le C\varepsilon \tag{5-2}$$

and

$$\|(S, P, U_1) - (S_0^+, \hat{P}_0^+(z_1), \hat{U}_0^+(z_1))\|_{C^{2,\alpha}(E_+)} \le C\varepsilon. \tag{5-3}$$

According to the constructions of the spaces of S_σ and Ξ_δ in Section 3, we can derive that

$$\begin{aligned} \partial_{z_2} S(z_1, 0) = \partial_{z_2} P(z_1, 0) = \partial_{z_2} U_1(z_1, 0) = 0, \\ \partial_{z_2} S(z_1, 1) = \partial_{z_2} P(z_1, 1) = \partial_{z_2} U_1(z_1, 1) = 0, \\ U_2(z_1, 0) = \partial_{z_2}^2 U_2(z_1, 0) = U_2(z_1, 1) = 0, \\ \xi'(0) = \xi^{(3)}(0) = \xi'(1) = 0. \end{aligned} \tag{5-4}$$

Next, we verify that the axisymmetric solution $(S, P, U_1, U_2; \xi)$ satisfies all the estimates in Theorem 1.1 in the (x_1, x_2, x_3) coordinate system.

The transformation (2-20) keeps the equivalence of $C^{2,\alpha}$ norms between the coordinates (y_1, y_2) and (z_1, z_2). Denoting the solution by $((S, P, U_1, U_2)(y); \xi(y_2))$ in the coordinates (y_1, y_2), we have

$$|\xi(y_2) - r_0| \le C X_0 \varepsilon, \quad \|\xi'(y_2)\|_{C^{2,\alpha}[0,1]} \le C\varepsilon \tag{5-5}$$

and

$$\|(S, P, U_1, U_2) - (S_0^+, \hat{P}_0^+(y_1), \hat{U}_0^+(y_1), 0)\|_{C^{2,\alpha}(R_+)} \le C\varepsilon. \tag{5-6}$$

In addition, it follows from (5-4) and a direct computation that

$$\begin{aligned} \partial_{y_2} S(y_1, 0) = \partial_{y_2} P(y_1, 0) = \partial_{y_2} U_1(y_1, 0) = 0, \\ \partial_{y_2} S(y_1, 1) = \partial_{y_2} P(y_1, 1) = \partial_{y_2} U_1(y_1, 1) = 0, \\ U_2(y_1, 0) = \partial_{y_2}^2 U_2(y_1, 0) = U_2(y_1, 1) = 0, \\ \xi'(0) = \xi^{(3)}(0) = \xi'(1) = 0. \end{aligned} \tag{5-7}$$

Therefore, by the inverse transformations of (2-1) and (2-2), the solution to the problem (1-1) with (1-2)–(1-5) has the form

$$(S,P)(x_1,x_2,x_3)=(S,P)\left((x_1^2+x_2^2+x_3^2)^{1/2},\ X_0\arcsin\left(\frac{(x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^{1/2}}\right)\right),$$

and

$$\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}(x_1,x_2,x_3)=\frac{1}{(x_1^2+x_2^2+x_3^2)^{1/2}}\begin{bmatrix}x_1 & (x_2^2+x_3^2)^{1/2}\\ x_2 & -x_1x_2/(x_2^2+x_3^2)^{1/2}\\ x_3 & -x_1x_3/(x_2^2+x_3^2)^{1/2}\end{bmatrix}$$
$$\cdot\begin{pmatrix}U_1\\U_2\end{pmatrix}\left((x_1^2+x_2^2+x_3^2)^{1/2},\ X_0\arcsin\big(\frac{(x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^{1/2}}\big)\right),$$

and the shock position $x_1=\eta(x_2,x_3)$ is given by the implicit function

$$G(x_1,x_2,x_3)\equiv(x_1^2+x_2^2+x_3^2)^{1/2}-\xi\left(X_0\arcsin\left(\frac{(x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^{1/2}}\right)\right)=0, \tag{5-8}$$

where we have for small ε

$$\partial_{x_1}G=\frac{x_1}{(x_1^2+x_2^2+x_3^2)^{1/2}}+\xi'\left(X_0\arcsin\big(\frac{(x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^{1/2}}\big)\right)\frac{X_0(x_2^2+x_3^2)^{1/2}}{x_1^2+x_2^2+x_3^2}>0$$

because $|\xi'|\le C\varepsilon$.

Thanks to (5-7) and Lemmas B.3 and B.4, we know that

$$(S^+(x),\ P^+(x),\ u_1^+(x),\ u_2^+(x),\ u_3^+(x))$$

belongs to $C^{2,\alpha}(\overline{\Omega}_+)$ and satisfies the estimates in Theorem 1.1.

Finally, we show that $\eta(x_2,x_3)\in C^{3,\alpha}(\bar{S}_e)$ and satisfies Theorem 1.1(i).

Since the shock surface $x_1=\eta(x_2,x_3)$ is determined by (5-8),

$$\|\eta-(r_0^2-x_2^2-x_3^2)^{1/2}\|_{L^\infty(S_e)}\le C\|\xi-r_0\|_{L^\infty[0,1]}\le CX_0\varepsilon. \tag{5-9}$$

In addition, $\eta(x_2,x_3)$ satisfies the Rankine–Hugoniot conditions (1-2), so we have

$$\begin{aligned}\partial_{x_2}\eta&=\frac{[\rho u_1u_2][P+\rho u_3^2]-[\rho u_1u_3][\rho u_2u_3]}{[P+\rho u_2^2][P+\rho u_3^2]-[\rho u_2u_3]^2},\\ \partial_{x_3}\eta&=\frac{[\rho u_1u_3][P+\rho u_2^2]-[\rho u_1u_2][\rho u_2u_3]}{[P+\rho u_2^2][P+\rho u_3^2]-[\rho u_2u_3]^2}.\end{aligned}\tag{5-10}$$

Similarly, $\eta_0(x_2,x_3)=(r_0^2-x_2^2-x_3^2)^{1/2}$ also satisfies (5-10) when the solution $(\rho^\pm,P^\pm,u^\pm)$ is replaced by the corresponding background solution in (5-10).

Therefore, by Remark A.2, (5-9) and the interpolation theorem in Hölder space, we have

$$\begin{aligned}
&\|\nabla_{x_2,x_3}\big(\eta(x_2,x_3)-(r_0^2-x_2^2-x_3^2)^{1/2}\big)\|_{C^{2,\alpha}(\bar{S}_e)}\\
&\quad\le C(\varepsilon+\|\nabla_x(S_0^+,\hat{P}_0^+,\hat{u}_{1,0}^+,\hat{u}_{2,0}^+,\hat{u}_{3,0}^+)\|_{C^{2,\alpha}}\|\eta-(r_0^2-x_2^2-x_3^2)^{1/2}\|_{C^{2,\alpha}(\bar{S}_e)})\\
&\quad\le C\varepsilon.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Appendix A.

In this appendix, we will describe the transonic solution of the problem (1-1) with (1-2)–(1-5), when the exit pressure is a suitable constant P_e under the assumptions given in Section 1 on the nozzle walls and the supersonic incoming flow. Such a solution, called a background solution, can be obtained by solving the related ordinary differential equations. Related analysis has been given in [Courant and Friedrichs 1948, Section 147] and the details can be seen in [Xin and Yin 2008b]. For the reader's convenience and because it's needed for the computations in this paper, we will give a detailed statement.

Theorem A.1. *If the three-dimensional nozzle wall* Γ *and the supersonic incoming flow are as defined in Section 1, then there exist two constant pressures* $\tilde{P}_1$ *and* $\tilde{P}_2$ *with* $\tilde{P}_1 < \tilde{P}_2$ *such that if the exit pressure* $\tilde{P}_e \in (\tilde{P}_1, \tilde{P}_2)$, *then the system* (1-2) *has a symmetric transonic shock solution*

$$(P,u_1,u_2,u_3,S)=\begin{cases}(P_0^-(r),u_{1,0}^-(x),u_{2,0}^-(x),u_{3,0}^-(x),S_0^-) & \text{for } r<r_0,\\ (P_0^+(r),u_{1,0}^+(x),u_{2,0}^+(x),u_{3,0}^+(x),S_0^+) & \text{for } r>r_0,\end{cases}$$

where $u_{i,0}^+(x)=U_0^+(r)x_i/r$ *for* $i=1,2,3$, $X_0<r_0<X_0+1$, S_0^+ *is a constant, and* $(P_0^+(r),U_0^+(r))$ *is* C^3*-smooth.*

See Theorem 1.1 in [Xin and Yin 2008b] for the proof.

Next, we cite two useful remarks, which were stated in [Xin and Yin 2008b].

Remark A.2. By the assumption (1-6), we have for $r_0\le r\le X_0+1$

$$\left|\frac{d^kU_0^+(r)}{dr^k}\right|+\left|\frac{d^kP_0^+(r)}{dr^k}\right|\le\frac{C_k}{X_0^k}\quad\text{for } k=1,2,3.$$

Remark A.3. One can obtain an extension $(\hat{\rho}_0^+(r),\hat{U}_0^+(r))$ of $(\rho_0^+(r),U_0^+(r))$ for $r\in(X_0,X_0+1)$ by solving the Euler system.

Appendix B.

We now give some elementary facts and computations often used in Section 5. Compared with the similar results in [Li et al. 2010a, Appendix B], the estimates here are more delicate since we require them to be independent of X_0. Here and in what follows, X_0 is defined as in Section 1 and C stands for a generic positive constant that is independent of X_0.

For the convenience, we set

$$\begin{aligned}
E_1 &= \{(z_1, z_2) \in R^2 : 0 < z_1 < 1,\ 0 < z_2 < 1\},\\
E_2 &= \{(x_1, x_2, x_3) \in R^3 : 0 < x_1 < 1,\ x_2^2 + x_3^2 < 1\},\\
E_3 &= \{(z_1, z_2, z_3) \in R^3 : 0 < z_1 < 1,\ 0 < z_2 < 1,\ 0 \le z_3 < 2\pi\},\\
E_4 &= \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 \le 1\}.
\end{aligned}$$

Lemma B.1. *Let*

$$\phi(x_1, x_2) = \left(\frac{1}{X_0}\cot\left(\frac{(x_1^2 + x_2^2)^{1/2}}{X_0}\right) - \frac{1}{(x_1^2+x_2^2)^{1/2}}\right)\frac{x_2}{(x_1^2 + x_2^2)^{1/2}}.$$

Then we have

$$\|\phi\|_{C^{0,1}(E_4)} \le C. \tag{B-1}$$

Proof. Note that $\phi(x_1, x_2)$ can be rewritten as

$$\begin{aligned}
\phi(x_1, x_2) &= \frac{\int_0^1 (\cos(\check{\rho}) - \cos(s\check{\rho}))ds}{X_0 \sin\check{\rho}}\frac{x_2}{\rho}\\
&= \frac{2x_2}{\rho}\frac{\int_0^1 (\sin(\frac{1}{2}(1+s)\check{\rho})\sin(\frac{1}{2}(s-1)\check{\rho}))ds}{X_0\sin(\check{\rho})},
\end{aligned}$$

where $\rho = (x_1^2 + x_2^2)^{1/2}$

It is easy to see that

$$\|\phi\|_{L^\infty(E_4)} \le C. \tag{B-2}$$

For any two distinct points (x_{11}, x_{21}) and (x_{12}, x_{22}) in E_4, it follows from a direct computation that

$$\phi(x_{11}, x_{21}) - \phi(x_{12}, x_{22}) = I_1 + I_2 + I_3, \tag{B-3}$$

where, with $a = (x_{11}^2 + x_{21}^2)^{1/2}$ and $b = (x_{12}^2 + x_{22}^2)^{1/2}$,

$$\begin{aligned}
I_1 &= \left(\check{1}\cot(\check{a}) - \frac{1}{a}\right)\frac{x_{21} - x_{22}}{a},\\
I_2 &= -\left(\check{1}\cot(\check{a}) - \frac{1}{a}\right)\frac{x_{22}\big((x_{11} - x_{12})(x_{11} + x_{12}) + (x_{21} - x_{22})(x_{21} + x_{22})\big)}{ab\big(a+b\big)},\\
I_3 &= \frac{x_{12}}{b}\left(\check{1}\cot(\check{a}) - \frac{1}{a} - \check{1}\cot(\check{b}) + \frac{1}{b}\right).
\end{aligned}$$

Now we only estimate I_3 since the treatments on I_1 and I_2 are analogous or even simpler.

Assume that $a \geq b$ without loss of generality. Then a direct computation yields

$$|I_3| \leq \left| \frac{ab\sin(\check{b}-a) - X_0\sin(\check{a})\sin(\check{b})(b-a)}{X_0\sin(\check{a})\sin(\check{b})ab} \right|.$$

Since

$$\begin{aligned}|ab\sin(\check{b}-a) &- X_0\sin(\check{a})\sin(\check{b})(b-a)| \\ &= \left| \check{b}-aab\int_0^1\int_0^1 (\cos(s\check{b}-a) - \cos(s\check{a})\cos(t\check{b}))\,ds\,dt \right| \\ &\leq \frac{|b-a|}{X_0} ab\big(\sin(\check{a})\sin(\check{b}) + 2\sin(\check{b})\sin(\check{2}b)\big),\end{aligned}$$

we have $|I_3| \leq C|a-b|$ and hence

$$|\phi(x_{11}, x_{21}) - \phi(x_{12}, x_{22})| \leq C|a-b|. \tag{B-4}$$

Combining (B-4) with (B-2) yields Lemma B.1. □

Remark B.2. By the computation of $|I_3|$, we show that

$$X_0^{-1}\cot((x_2^2+x_3^2)^{1/2}X_0^{-1}) - (x_2^2+x_3^2)^{-1/2}$$

is in $C^{0,1}(E_4)$ and is no greater than C.

Lemma B.3. (i) *For $\phi(z_1, z_2) \in C^\alpha(E_1)$ with $0 < \alpha < 1$, there exists a constant $C > 1$ such that*

$$\frac{1}{C}\|\phi(x_1, (x_2^2+x_3^2)^{1/2})\|_{C^\alpha(E_2)} \leq \|\phi\|_{C^\alpha(E_1)} \leq C\|\phi(x_1, (x_2^2+x_3^2)^{1/2})\|_{C^\alpha(E_2)}. \tag{B-5}$$

If $\phi(z_1, z_2) \in C^{k,\alpha}(E_1)$ for some $k \in \{1, 2\}$ and $\partial_{z_2}\phi(z_1, 0) = 0$, then there exists a constant $C > 1$ such that

$$\begin{aligned}\frac{1}{C}\|\phi(x_1, (x_2^2+x_3^2)^{1/2})\|_{C^{k,\alpha}(E_2)} &\leq \|\phi\|_{C^{k,\alpha}(E_1)} \\ &\leq C\|\phi(x_1, (x_2^2+x_3^2)^{1/2})\|_{C^{k,\alpha}(E_2)}.\end{aligned} \tag{B-6}$$

(ii) *If $\phi(z_1, z_2) \in C^{k,\alpha}(E_1)$ with some $k \in \{1, 2\}$ and $\phi(z_1, 0) = 0$, then there exists a constant $C_2 > 1$ such that*

$$\|\check{1}\cot(\check{z}_2)\phi\|_{C^{k-1,\alpha}(E_1)} \leq C\|\phi\|_{C^{k,\alpha}(E_1)}. \tag{B-7}$$

Proof. Since (B-5) and (B-6) can be verified directly, we omit the proof. Next we show (B-7).

Using $\phi(z_1, 0) = 0$, we have

$$\check{1}\cot(\check{z}_2)\phi(z_1, z_2) = \frac{\check{z}_2\cos(\check{z}_2)}{\sin\check{z}_2}\int_0^1 \partial_{z_2}\phi(z_1, sz_2)ds$$
$$= \cos(\check{z}_2)\Big(1 + \frac{\check{z}_2 - \sin\check{z}_2}{\sin\check{z}_2}\Big)\int_0^1 \partial_{z_2}\phi(z_1, sz_2)ds,$$

this yields for $k = 1$ or 2

$$\text{(B-8)}\quad \|\check{1}\cot(\check{z}_2)\phi(z_1, z_2)\|_{C^{k-1,\alpha}(E_1)} \le C\Big(1 + \Big\|\frac{\check{z}_2 - \sin\check{z}_2}{\sin\check{z}_2}\Big\|_{C^{k-1,\alpha}[0,1]}\Big)\|\partial_{z_2}\phi\|_{C^{k-1,\alpha}(E_1)}.$$

Since

$$\frac{\check{z}_2 - \sin\check{z}_2}{\sin\check{z}_2} = \frac{\check{z}_2}{\sin\check{z}_2}\int_0^1 (1 - \cos(s\check{z}_2))ds = \frac{2\check{z}_2}{\sin\check{z}_2}\int_0^1 \big(\sin(\tfrac{1}{2}s\check{z}_2)\big)^2 ds,$$

$$\frac{d}{dz_2}\Big(\frac{\check{z}_2 - \sin(\check{z}_2)}{\sin\check{z}_2}\Big) = \frac{2\sin\check{z}_2 - 2\check{z}_2\cos\check{z}_2}{X_0\sin^2\check{z}_2}\int_0^1 (\sin(\tfrac{1}{2}s\check{z}_2))^2 ds + \frac{\check{z}_2}{X_0\sin\check{z}_2}\int_0^1 \sin(s\check{z}_2)sds,$$

$$\frac{d^2}{dz_2^2}\Big(\frac{\check{z}_2 - \sin\check{z}_2}{\sin\check{z}_2}\Big) = \frac{2\check{z}_2 + 2\check{z}_2\cos^2\check{z}_2 - \sin(2\check{z}_2)}{X_0^2\sin^3\check{z}_2}\int_0^1 (\sin(\tfrac{1}{2}s\check{z}_2))^2 ds$$
$$+ \frac{2\sin\check{z}_2 - 2\check{z}_2\cos(\check{z}_2)}{X_0\sin^2\check{z}_2}\int_0^1 \sin(s\check{z}_2)sds + \frac{\check{z}_2}{X_0^2\sin\check{z}_2}\int_0^1 \cos(s\check{z}_2)s^2 ds,$$

and because

$$\int_0^1 (\sin(\tfrac{1}{2}s\check{z}_2))^2 ds \le \tfrac{1}{4}\check{z}_2^2, \quad \text{and} \quad \int_0^1 \sin(\tfrac{1}{2}s\check{z}_2)ds \le \tfrac{1}{2}\check{z}_2,$$

we have

$$\Big\|\frac{z_2 - X_0\sin\check{z}_2}{X_0\sin\check{z}_2}\Big\|_{C^{1,1}[0,1]} \le C.$$

Combining this with (B-8) yields (B-7) for $k = 1$ or $k = 2$. □

Lemma B.4. (i) *For* $\phi(z_1, z_2) \in C^{k,\alpha}(E_1)$ *with some* $k = \{0, 1\}$ *and* $\phi(z_1, 0) = 0$,

$$\text{(B-9)} \qquad \sum_{i=2}^{3} \left\| \frac{x_i}{(x_2^2 + x_3^2)^{1/2}} \phi(x_1, (x_2^2 + x_3^2)^{1/2}) \right\|_{C^{k,\alpha}(E_2)} \le C \|\phi(z_1, z_2)\|_{C^{k,\alpha}(E_1)}.$$

(ii) *For* $\phi \in C^{2,\alpha}(E_1)$ *and* $\phi(z_1, 0) = \partial_{z_2}^2 \phi(z_1, 0) = 0$,

$$\text{(B-10)} \qquad \sum_{i=2}^{3} \left\| \frac{x_i}{(x_2^2 + x_3^2)^{1/2}} \phi(x_1, (x_2^2 + x_3^2)^{1/2}) \right\|_{C^{2,\alpha}(E_2)} \le C \|\phi(z_1, z_2)\|_{C^{2,\alpha}(E_1)}.$$

Proof. Put $\rho = (x_2^2 + x_3^2)^{1/2}$. Set

$$V_i(x_1, x_2, x_3) = (x_i/\rho)\phi(x_1, (x_2^2 + x_3^2)^{1/2}) \quad \text{for } i = 2, 3.$$

Then

$$\text{(B-11)} \qquad \|V_i\|_{L^\infty(E_2)} \le \|\phi(r)\|_{L^\infty(E_1)} \quad \text{for } i = 2, 3.$$

Since V_2 and V_3 have the analogous forms, it suffices to treat V_2.

(i) First we show (B-9).

For any two distinct points (x_{11}, x_{21}, x_{31}) and (x_{12}, x_{22}, x_{32}) in E_2, we may assume without loss of generality that $|x_{21}| \ge |x_{22}|$. Put $a = (x_{21}^2 + x_{31}^2)^{1/2}$ and $b = (x_{22}^2 + x_{32}^2)^{1/2}$. Then

$$\text{(B-12)} \qquad \begin{aligned} V_2(x_{11}, x_{21}, x_{31}) - V_2(x_{12}, x_{22}, x_{32}) &= \frac{x_{21}}{a}\phi(x_{11}, a) - \frac{x_{22}}{b}\phi(x_{12}, b) \\ &= J_1 + J_2 + J_3. \end{aligned}$$

where

$$\begin{aligned} J_1 &= \frac{x_{21} - x_{22}}{a}\phi(x_{11}, a), \\ J_2 &= -\frac{x_{22}\big((x_{21} - x_{22})(x_{21} + x_{22}) + (x_{31} - x_{32})(x_{31} + x_{32})\big)}{ab(a + b)}\phi(x_{11}, a), \\ J_3 &= \frac{x_{22}}{b}\big(\phi(x_{11}, a) - \phi(x_{12}, b)\big). \end{aligned}$$

By $\phi(z_1, 0) = 0$ and the assumption $|x_{21}| \ge |x_{22}|$, a direct computation yields

$$\text{(B-13)} \qquad \begin{aligned} |J_1| &\le [\phi]_\alpha \frac{(|x_{21}| + |x_{22}|)^{1-\alpha}}{a^{1-\alpha}} |x_{21} - x_{22}|^\alpha \le 2^{1-\alpha}[\phi]_\alpha |x_{21} - x_{22}|^\alpha, \\ |J_2| &\le 2[\phi]_\alpha \frac{|x_{22}|\big(|x_{21} - x_{22}| + |x_{31} - x_{32}|\big)}{a^{1-\alpha} b} \\ &\le 2^{2-\alpha}[\phi]_\alpha \big(|x_{21} - x_{22}|^\alpha + |x_{31} - x_{32}|^\alpha\big), \\ |J_3| &\le [\phi]_\alpha \big((x_{11} - x_{12})^2 + (x_{21} - x_{22})^2 + (x_{31} - x_{32})^2\big)^{\alpha/2}. \end{aligned}$$

Here $[\phi]_\alpha$ denotes the Hölder seminorm with exponent α.

Combining (B-13) with (B-12) and (B-11) yields

$$\|V_2\|_{C^\alpha(E_2)} \le C\|\phi\|_{C^\alpha(E_1)}. \tag{B-14}$$

If $\phi \in C^{1,\alpha}(E_2)$ and $\phi(z_1, 0) = 0$, we have

$$\begin{aligned}
\partial_{x_1} V_2 &= (x_2/\rho)\partial_{z_1}\phi(x_1,\rho),\\
\partial_{x_2} V_2 &= \frac{x_3^2}{\rho^3}\phi(x_1,\rho) + \frac{x_2^2}{\rho^2}\partial_{z_2}\phi(x_1,\rho),\\
\partial_{x_3} V_2 &= -\frac{x_2x_3}{\rho^3}\phi(x_1,\rho) + \frac{x_2x_3}{\rho^2}\partial_{z_2}\phi(x_1,\rho),
\end{aligned}$$

Next, we only analyze $\partial_{x_2} V_2$ since the treatment of $\partial_{x_1} V_2$ and $\partial_{x_3} V_2$ is similar. Rewrite $\partial_{x_2} V_2$ as $\partial_{x_2} V_2 = J_5 + J_6$, where

$$J_5 = \frac{x_3^2}{\rho^2}\int_0^1 \big(\partial_{z_2}\phi(x_1,\theta\rho) - \partial_{z_2}\phi(x_1,\rho)\big)d\theta \quad \text{and} \quad J_6 = \partial_{z_2}\phi(x_1,\rho).$$

For convenience, we set

$$\overline{V}(x_1,\rho) = \int_0^1 (\partial_{z_2}\phi(x_1,\theta\rho) - \partial_{z_2}\phi(x_1,\rho))d\theta.$$

Then $\overline{V}(x_1, 0) = 0$. Applying the same argument as for (B-14) yields

$$\|J_5\|_{C^\alpha(E_2)} \le C\|\phi\|_{C^{1,\alpha}(E_1)}. \tag{B-15}$$

In addition, by (B-5), we have

$$\|J_6\|_{C^\alpha(E_2)} \le C\|\phi\|_{C^{1,\alpha}(E_1)}. \tag{B-16}$$

Thus, combining (B-15) and (B-16) with (B-14) yields (B-9).

(ii) We now show (B-10).

For $\phi(z) \in C^{2,\alpha}(E_1)$ with $\phi(z_1, 0) = \partial_{z_2}^2\phi(z_1, 0) = 0$, we have

$$\begin{aligned}
\partial_{x_1}^2 V_2 &= \frac{x_2}{\rho}\partial_{z_1}^2\phi(x_1,\rho),\\
\partial_{x_1x_2}^2 V_2 &= \frac{x_3^2}{\rho^3}\partial_{z_1}\phi(x_1,\rho) + \frac{x_2^2}{\rho^2}\partial_{z_1z_2}^2\phi(x_1,\rho)\\
&= \frac{x_3^2}{\rho^2}\int_0^1 (\partial_{z_1z_2}^2\phi(x_1,\theta\rho) - \partial_{z_1z_2}^2\phi(x_1,\rho))d\theta + \partial_{z_1z_2}^2\phi(x_1,\rho),\\
\partial_{x_1x_3}^2 V_2 &= -\frac{x_2x_3}{\rho^3}\partial_{z_1}\phi(x_1,\rho) + \frac{x_2x_3}{\rho^2}\partial_{z_1z_2}^2\phi(x_1,\rho).
\end{aligned}$$

It follows from $\phi(z_1, 0) = 0$, $\partial^2_{z_1}\phi(z_1, 0) = 0$ and (B-9) that

$$\|\partial^2_{x_1} V_2\|_{C^\alpha(E_1)} \le C\|\phi\|_{C^{2,\alpha}(E_1)}. \tag{B-17}$$

In a similar proof as for (B-15) and (B-16), we have

$$\sum_{i=2}^{3} \|\partial^2_{x_1 x_i} V_2\|_{C^\alpha(E_1)} \le C\|\phi\|_{C^{2,\alpha}(E_1)}, \tag{B-18}$$

The quantities $\partial^2_{x_i x_j} V_2$ for $i, j = 2, 3$ can also be estimated in the same way. Therefore, due to (B-17), (B-18) and (B-9), we have proved (B-10). □

Lemma B.5. *Let $k = 0$ or $k = 1$. If $f_i(z_1, z_2) \in C^{k,\alpha}(E_1)$ and $g_i(z_2) \in C^{k+1,\alpha}[0, 1]$ with $g_i'(0) = g_i'(1) = 0$ for $i = 1, 2$ and $\partial_{z_2} f_1(z_1, 0) = f_2(z_1, 0) = 0$, then the problem*

$$\begin{aligned}
\partial^2_{z_1} U + \partial^2_{z_2} U + \check{1}\cot(\check{z}_2)\partial_{z_2} U &= \partial_{z_1} f_1(z_1, z_2) + \partial_{z_2} f_2(z_1, z_2)\\
&\quad + \check{1}\cot(\check{z}_2) f_2(z_1, z_2) \quad \text{in } E_1,\\
U(0, z_2) &= g_1(z_2),\\
U(1, z_2) &= g_2(z_2),\\
\partial_{z_2} U(z_1, 0) &= 0,\\
\partial_{z_2} U(z_1, 1) &= 0
\end{aligned} \tag{B-19}$$

has a unique solution $U(z) \in C^{k+1,\alpha}(E_1)$ that admits the estimate

$$\|U(z)\|_{C^{k+1,\alpha}(E_1)} \le C \sum_{i=1}^{2} \big(\|f_i(z)\|_{C^{k,\alpha}(E_1)} + \|g_i\|_{C^{k+1,\alpha}[0,1]}\big). \tag{B-20}$$

Proof. Again let $\rho = (x_2^2 + x_3^2)^{1/2}$. First, we consider the elliptic problem

$$\begin{aligned}
(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}) U_1 + b_1(x_1, x_2, x_3)\partial_{x_2} U_1&\\
+ b_2(x_1, x_2, x_3)\partial_{x_3} U_1 &= \sum_{i=1}^{2} F_i(x_1, x_2, x_3) \quad \text{in } E_2,\\
U_1(0, x_2, x_3) &= g_1(\rho),\\
U_1(1, x_2, x_3) &= g_2(\rho),\\
\rho^{-1}(x_2\partial_{x_2} + x_3\partial_{x_3}) U_1(x_1, x_2, x_3) &= 0 \quad \text{on } \rho = 1.
\end{aligned} \tag{B-21}$$

where

$$
\text{(B-22)} \qquad \begin{aligned}
b_i(x_1, x_2, x_3) &= (\check{1}\cot(\check{\rho}) - \rho^{-1})x_{i+1}/\rho \quad \text{for } i = 1, 2,\\
F_1(x_1, x_2, x_3) &= \partial_{x_1} f_1(x_1, \rho),\\
F_2(x_1, x_2, x_3) &= \rho^{-1}(x_2\partial_{x_2} + x_3\partial_{x_3}) f_2(x_1, \rho)\\
&\qquad + \check{1}\cot(\rho/X_0) f_2(x_1, \rho).
\end{aligned}
$$

We turn to the existence and uniqueness of the solution to the problem (B-21). According to the theory on second order elliptic equations with cornered boundaries and mixed type boundary conditions (see [Azzam 1980; 1981; Gilbarg and Hörmander 1980; Gilbarg and Trudinger 1983; Lieberman 1986; Vekua 1952]), we need to analyze the regularity of $b_i(x_1, x_2, x_3)$ and $F_i(x_1, x_2, x_3)$ for $i = 1, 2$.

First, it follows from Lemma B.1 that $b_i(x_1, x_2, x_3)$ satisfies

$$
\text{(B-23)} \qquad \|b_i(x_1, x_2, x_3)\|_{C^\alpha(E_2)} \le C.
$$

In addition, $F_2(x_1, x_2, x_3)$ can be rewritten as

$$
\text{(B-24)} \qquad F_2(x_1, x_2, x_3) = \sum_{i=2}^{3} \partial_{x_i}\Big(\frac{x_i}{\rho} f_2(x_1, \rho)\Big) + (\check{1}\cot(\hat{\rho}) - \rho^{-1}) f_2(x_1, \rho).
$$

Since $f_2(z_1, 0) = 0$, it follows from Lemma B.4 that

$$
\text{(B-25)} \qquad \sum_{i=2}^{3} \Big\|\frac{x_i}{\rho} f_2(x_1, \rho)\Big\|_{C^{k,\alpha}(E_2)} \le C\|f_2\|_{C^{k,\alpha}(E_1)} \qquad \text{for } k = 0, 1.
$$

On the other hand, by Remark B.2, we have

$$
\text{(B-26)} \qquad \|(\check{1}\cot(\hat{\rho}) - \rho^{-1}) f_2(x_1, \rho)\|_{C^\alpha(E_2)} \le C\|f_2\|_{C^\alpha(E_1)}.
$$

Because $g_i'(0) = g_i'(1) = 0$ for $i = 1, 2$ and $\partial_{z_2} f_1(z_1, 0) = 0$, the compatible conditions at the corners for the problem (B-21) are satisfied. Moreover, by using (B-5) and (B-6) in Lemma B.3, we have

$$
\text{(B-27)} \qquad \begin{aligned}
&\|g_i\|_{C^{j,\alpha}(E_4)} \le C\|g_i\|_{C^{j,\alpha}([0,1])} \quad \text{for } i = 1, 2 \text{ and } j = 1, 2,\\
&\|f_1\|_{C^{l,\alpha}(E_2)} \le C\|f_1\|_{C^{l,\alpha}(E_1)} \quad \text{for } l = 0, 1.
\end{aligned}
$$

Then by the results in [Lieberman 1986], the problem (B-21), which has the divergence form of a seconder order elliptic equation and the regularities of (B-23)–(B-26), has a unique solution $U_1(x_1, x_2, x_3)$ such that

$$
\text{(B-28)} \qquad \|U_1(x)\|_{C^{1,\alpha}(E_2)} \le C\sum_{i=1}^{2}\big(\|f_i(z)\|_{C^\alpha(E_1)} + \|g_i\|_{C^{1,\alpha}[0,1]}\big).
$$

Furthermore, for $f_i(z) \in C^{1,\alpha}(E_1)$ and $g_i \in C^{2,\alpha}[0,1]$, due to the compatibility conditions at the corners, it follows from [Xin et al. 2009, Lemma A] that $U_1(x_1, x_2, x_3)$ is in $C^{2,\alpha}(E_2)$ and satisfies the estimate

$$\|U_1(x)\|_{C^{2,\alpha}(E_2)} \le C \sum_{i=1}^{2} \big(\|f_i(z)\|_{C^{1,\alpha}(E_1)} + \|g_i\|_{C^{2,\alpha}[0,1]}\big). \tag{B-29}$$

Next, we prove that the solution $U_1(x_1, x_2, x_3)$ in (B-21) is cylindrically symmetric. We use the transformation

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2 \cos\gamma_0 + x_3 \sin\gamma_0, \quad \bar{x}_3 = -x_2 \sin\gamma_0 + x_3 \cos\gamma_0,$$

with $\gamma_0 \in [0, 2\pi]$ being any fixed constant.

It is easy to verify that $U_1(\bar{x})$ also solves the problem (B-21). Thus, by the arbitrariness of γ_0, $U_1(x)$ is cylindrically symmetric with respect to (x_2, x_3), that is, $U_1(x)$ has the form $U_1(x) = U_1(x_1, \rho)$.

In addition, using the coordinate transformation

$$x_1 = z_1, \quad x_2 = z_2 \cos z_3, \quad x_3 = z_2 \sin z_3, \tag{B-30}$$

$U_1(x)$ can be expressed as $U_1 = U_1(z_1, z_2)$. Finally, it follows from (B-28)–(B-29) and Lemma B.3 that

$$\begin{aligned} \|U_1(z_1, z_2)\|_{C^{k,\alpha}(E_1)} &\le \|U(x_1, \rho)\|_{C^{k,\alpha}(E_2)} \\ &\le C \sum_{i=1}^{2} \big(\|f_i(z)\|_{C^{k-1,\alpha}(E_1)} + \|g_i\|_{C^{k,\alpha}[0,1]}\big) \qquad \text{for } k=1 \text{ or } k=2. \end{aligned} \tag{B-31}$$

□

Lemma B.6. *If $F(z) \in C^{\alpha}(E_1)$, then the function*

$$U(z) = \frac{1}{\sin \check{z}_2} \int_0^{z_2} \sin \check{s} F(z_1, s) ds$$

satisfies

$$\|\partial_{z_2} U(z)\|_{C^{\alpha}(E_1)} \le C \|F(z)\|_{C^{\alpha}(E_1)}. \tag{B-32}$$

Further, if $F(z) \in C^{1,\alpha}(E_1)$ and $\partial_{z_2} F(z_1, 0) = 0$, then $U(z)$ satisfies

$$\|\partial_{z_2}^2 U(z)\|_{C^{\alpha}(E_1)} \le C \|F(z)\|_{C^{1,\alpha}(E_1)}. \tag{B-33}$$

Proof. First, $U(z)$ can be rewritten as

$$U(z) = \frac{1}{\sin \check{z}_2} \int_0^{z_2} \sin \check{s} (F(z_1, s) - F(z_1, 0)) ds + X_0 \tan(\tfrac{1}{2}\check{z}_2) F(z_1, 0). \tag{B-34}$$

A direct computation yields

$$\|\partial_{z_2}^k \big(X_0 \tan(\tfrac{1}{2}\check{z}_2) F(z_1, 0)\big)\|_{C^{\alpha}(E_1)} \le C \|F\|_{C^{\alpha}(E_1)} \quad \text{for } k=1 \text{ or } k=2. \tag{B-35}$$

Based on (B-34)–(B-35), in order to show Lemma B.6, it suffices to consider the case of $F(z_1,0)=0$ in (B-32) and $F(z_1,0)=\partial_{z_2}F(z_1,0)=0$ in (B-33).

First, we prove (B-32) with $F(z_1,0)=0$.

It follows from a direct computation that

$$\begin{aligned}\partial_{z_2}U(z_1,z_2)&=F(z_1,z_2)-\frac{\cos\check{z}_2}{X_0\sin^2\check{z}_2}\int_0^{z_2}\sin(\check{s})F(z_1,s)ds\\&=\frac{1}{X_0\sin^2\check{z}_2}\int_0^{z_2}\big(\sin\check{z}_2\cos\check{s}\,F(z_1,z_2)-\sin\check{s}\cos\check{z}_2\,F(z_1,s)\big)ds.\end{aligned}$$

This easily implies

$$\|\partial_{z_2}U(z)\|_{L^\infty(E_1)}\le C\|F(z)\|_{L^\infty(E_1)}. \tag{B-36}$$

We now estimate $[\partial_{z_2}U(z)]_\alpha$ in E_1.

For any two different points (z_{11},z_{21}) and (z_{12},z_{22}) in E_1, we may assume without loss of generality that $z_{21}\ge z_{22}$. Then

$$\partial_{z_2}U(z_{11},z_{21})-\partial_{z_2}U(z_{12},z_{22})=K_1+K_2+K_3, \tag{B-37}$$

where

$$\begin{aligned}K_1&=\frac{1}{X_0\sin^2\check{z}_{21}}\int_0^{z_{21}}\big(\cos(\check{s})(\sin\check{z}_{21}F(z_{11},z_{21})-\sin\check{z}_{22}F(z_{12},z_{22}))\\&\qquad\qquad-\sin\check{s}\big(\cos(\check{z}_{21})F(z_{11},s)\\&\qquad\qquad\qquad-\cos(\check{z}_{22})F(z_{12},s)\big)\big)ds,\\K_2&=\frac{1}{X_0\sin\check{z}_{21}}\int_{z_{22}}^{z_{21}}\big(\sin\check{z}_{22}\cos(\check{s})F(z_{12},z_{22})-\sin\check{s}\cos(\check{z}_{22})F(z_{12},s)\big)ds,\\K_3&=\frac{(\sin\check{z}_{22}-\sin\check{z}_{21})\big(\sin\check{z}_{22}+\sin\check{z}_{21}\big)}{X_0(\sin\check{z}_{21}\sin\check{z}_{22})^2}\\&\qquad\times\int_0^{z_{22}}\big(\sin\check{z}_{22}\cos(\check{s})F(z_{12},z_{22})-\sin\check{s}\cos(\check{z}_{22})F(z_{12},s)\big)ds.\end{aligned}$$

It follows from $F(z_{11},0)=0$ and a direct computation that

$$\begin{aligned}|K_1|\le{}&\frac{1}{X_0\sin^2\check{z}_{21}}\big(|\sin\check{z}_{21}-\sin\check{z}_{22}|z_{21}^{1+\alpha}[F]_\alpha\\&\qquad+\sin(\check{z}_{22})((z_{11}-z_{12})^2+(z_{21}-z_{22})^2)^{\alpha/2}z_{21}[F]_\alpha\\&\quad+\sin(\check{z}_{21})|\cos\check{z}_{21}-\cos\check{z}_{22}|z_{21}^{1+\alpha}[F]_\alpha+\sin(\check{z}_{21})z_{21}|z_{11}-z_{12}|^\alpha[F]_\alpha\big)\\&\le C[F]_\alpha((z_{11}-z_{12})^2+(z_{21}-z_{22})^2)^{\alpha/2},\end{aligned}$$

$$|K_2| \le \frac{1}{\sin^2 \check{z}_{21}}(\sin(\check{z}_{22})|\sin \check{z}_{21} - \sin \check{z}_{22}|z_{22}^{\alpha}[F]_\alpha + |\cos(\check{z}_{21}) - \cos(\check{z}_{22})|z_{21}^{\alpha}[F]_\alpha)$$
$$\le C[F]_\alpha((z_{11}-z_{12})^2 + (z_{21}-z_{22})^2)^{\alpha/2},$$
$$|K_3| \le \frac{|\sin \check{z}_{22} - \sin \check{z}_{21}|(\sin \check{z}_{22} + \sin \check{z}_{21})}{X_0(\sin \check{z}_{21} \sin \check{z}_{22})^2} \sin(\check{z}_{22}) z_{22}^{1+\alpha}[F]_\alpha$$
$$\le C[F]_\alpha\big((z_{11}-z_{12})^2 + (z_{21}-z_{22})^2\big)^{\alpha/2}.$$

This implies

$$[\partial_{z_2} U(z)]_\alpha \le C[F]_\alpha. \tag{B-38}$$

Combining (B-38) with (B-35) and (B-36) yields (B-32).

Second, we prove (B-33) in the case of $F(z_1, 0) = \partial_{z_2} F(z_1, 0) = 0$. Note that

$$\partial_{z_2}^2 U(z) = \partial_{z_2} F(z_1, z_2) - \check{1} \cot(\check{z}_2) F(z_1, z_2) + \frac{1+\cos^2 \check{z}_2}{X_0^2 \sin^3 \check{z}_2} \int_0^{z_2} \sin(\check{s}) F(z_1, s) ds.$$

By (B-7) of Lemma B.3, we have

$$\|\partial_{z_2} F(z_1, z_2) - \check{1} \cot(\check{z}_2) F(z_1, z_2)\|_{C^\alpha(E_1)} \le C\|F\|_{C^{1,\alpha}(E_1)}. \tag{B-39}$$

In addition, a direct computation yields

$$\frac{1+\cos^2 \check{z}_{21}}{X_0^2 \sin^3 \check{z}_{21}} \int_0^{z_{21}} \sin(\check{s}) F(z_{11}, s) ds - \frac{1+\cos^2 \check{z}_{22}}{X_0^2 \sin^3 \check{z}_{22}} \int_0^{z_{22}} \sin(\check{s}) F(z_{12}, s) ds$$
$$= K_4 + K_5 + K_6 + K_7,$$

where

$$K_4 = \frac{1+\cos^2(\check{z}_{21})}{X_0^2 \sin^3 \check{z}_{21}} \int_0^{z_{21}} \sin(\check{s}) s \Big(\int_0^1 (\partial_{z_2} F(z_{11}, \theta s) - \partial_{z_2} F(z_{12}, \theta s)) d\theta\Big) ds,$$
$$K_5 = \frac{1+\cos^2 \check{z}_{21}}{X_0^2 \sin^3 \check{z}_{21}} \int_{z_{22}}^{z_{21}} \sin(\check{s}) s \Big(\int_0^1 \partial_{z_2} F(z_{12}, \theta s) d\theta\Big) ds,$$
$$K_6 = \frac{(\cos \check{z}_{21} - \cos \check{z}_{22})(\cos \check{z}_{21} + \cos \check{z}_{22})}{X_0^2 \sin^3 \check{z}_{21}} \times \int_0^{z_{22}} \sin(\check{s}) s \Big(\int_0^1 \partial_{z_2} F(z_{12}, \theta s) d\theta\Big) ds,$$
$$K_7 = \frac{(1+\cos^2 \check{z}_{22})(\sin \check{z}_{22} - \sin \check{z}_{21})(\sin^2 \check{z}_{22} + \sin \check{z}_{22} \sin \check{z}_{21} + \sin^2 \check{z}_{21})}{X_0^2 (\sin \check{z}_{21} \sin \check{z}_{22})^3} \times \int_0^{z_{22}} \sin(\check{s}) s \Big(\int_0^1 \partial_{z_2} F(z_{12}, \theta s) d\theta\Big) ds.$$

Hence, by using $F(z_1,0)=\partial_{z_2}F(z_1,0)=0$, we have

$$
\begin{aligned}
|K_4| &\le \frac{\sin(\check z_{21})z_{21}^2}{X_0^2\sin^3\check z_{21}}[\partial_{z_2}F]_\alpha|z_{11}-z_{12}|^\alpha \le C[\partial_{z_2}F]_\alpha|z_{11}-z_{12}|^\alpha,\\
|K_5| &\le \frac{\sin(\check z_{21})z_{21}|z_{21}-z_{22}|}{X_0^2\sin^3\check z_{21}}[\partial_{z_2}F]_\alpha z_{12}^\alpha \le C[\partial_{z_2}F]_\alpha|z_{21}-z_{22}|^\alpha,\\
|K_6| &\le \frac{2\sin(\frac12(\check z_{21}-\check z_{22}))}{X_0^2\sin^3\check z_{21}}[\partial_{z_2}F]_\alpha z_{22}^{2+\alpha}\sin(\check z_{22}) \le C[\partial_{z_2}F]_\alpha|z_{21}-z_{22}|^\alpha,\\
|K_7| &\le \frac{3\sin^2\check z_{21}|\sin\check z_{22}-\sin\check z_{21}|}{X_0^2(\sin\check z_{21}\sin\check z_{22})^3}\sin(\check z_{22})[\partial_{z_2}F]_\alpha z_{22}^{2+\alpha} \le C[\partial_{z_2}F]_\alpha|z_{21}-z_{22}|^\alpha.
\end{aligned}
$$

This leads to

$$
[\partial_{z_2}^2U(z)]_\alpha \le C[\partial_{z_2}F]_\alpha. \tag{B-40}
$$

Combining (B-40) with (B-39) and (B-32), we complete the proof of (B-33). Therefore, the proof of Lemma B.6 is completed. □

Lemma B.7. *The problem*

$$
\begin{aligned}
\partial_1 w &= a_1\partial_2 P + F_1(\tilde S, P(q,\tilde S), V_1, V_2;\xi) \quad \text{in } E_1,\\
\partial_2 w + \check 1\cot(\check z_2)w &= F_2(\tilde S, P(q,\tilde S), V_1, V_2;\xi) - a_2\partial_1 P \quad \text{in } E_1,\\
w(z_1,0) &= 0
\end{aligned} \tag{B-41}
$$

is well posed if

$$
\begin{aligned}
&\big(\partial_{z_2}+\check 1\cot(\check z_2)\big)(a_1\partial_2 P + F_1(\tilde S, P(q,\tilde S), V_1, V_2;\xi))\\
&\qquad - \partial_{z_1}(F_2(\tilde S, P(q,\tilde S), V_1, V_2;\xi) - a_2\partial_1 P) = 0,\\
&(a_1\partial_2 P + F_1(\tilde S, P(q,\tilde S), V_1, V_2;\xi))(z_1,0) = 0.
\end{aligned}
$$

Proof. Define w_i for $i=1,2$ as

(B-42)

$$
\begin{aligned}
\partial_1 w_1 &= a_1\partial_2 P + F_1(\tilde S, P(q,\tilde S), V_1, V_2;\xi) \quad \text{in } E_1,\\
w_1(1,z_2) &= \frac{1}{\sin(\check z_2)}\int_0^{z_2}\sin(\check s)(F_2(\tilde S, P(q,\tilde S), V_1, V_2;\xi) - a_2\partial_1 P)(1,s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_2 w_2 + \check 1\cot(\check z_2)w_2 &= F_2(\tilde S, P(q,\tilde S), V_1, V_2;\xi) - a_2\partial_1 P \quad \text{in } E_1,\\
w_2(z_1,0) &= 0.
\end{aligned} \tag{B-43}
$$

Obviously, w_1 and w_2 can be determined uniquely.

From (B-43), w_2 has the expression

(B-44)
$$w_2(z_1,z_2)=\frac{1}{\sin\check{z}_2}\int_0^{z_2}\sin(\check{s})(F_2(\tilde{S},P(q,\tilde{S}),V_1,V_2;\xi)-a_2\partial_1 P)(z_1,s)ds.$$

Also it follows from (B-42) and the second equality in B.7 that $w_1(z_1,0)=0$. By (B-43) and (B-44), we arrive at

$$w_2(z_1,0)=0 \quad\text{and}\quad w_1(1,z_2)=w_2(1,z_2).$$

Next, we show $w_1=w_2$ in E_1.

Note that $(\partial_{z_2}+\check{1}\cot(\check{z}_2))$ times the first equation in (B-42) minus ∂_{z_1} applied to the first equation in (B-43) yields

(B-45)
$$\begin{gathered}(\partial_{z_2}+\check{1}\cot(\check{z}_2))\partial_{z_1}(w_1-w_2)=0 \quad\text{in } E_1,\\ w_1(z_1,0)=w_2(z_1,0)=0,\quad w_1(1,z_2)=w_2(1,z_2).\end{gathered}$$

One concludes easily that $w_1=w_2$ holds true in E_1, completing the proof. □

Lemma B.8. *Let* $(\tilde{S}_1,\tilde{P}_1,V_{11},V_{21})$ *and* $(\tilde{S}_2,\tilde{P}_2,V_{12},V_{22})$ *be in* Ξ_δ *such that*

$$\begin{aligned}T(\tilde{S}_1,\tilde{P}_1,V_{11},V_{21})&=(S_1,P_1,U_{11},U_{21}),\\ T(\tilde{S}_2,\tilde{P}_2,V_{12},V_{22})&=(S_2,P_2,U_{12},U_{22}),\end{aligned}$$

where the mapping T *is defined in* (3-37). *Denote by* $\xi_1(z_2)$ *and* $\xi_2(z_2)$ *the corresponding approximate shocks by solving* (3-7). *Define* W_i *for* $i=1,2,3,4$ *as in Section 3 with respect to* (S_1,P_1,U_{11},U_{21}) *and* (S_2,P_2,U_{12},U_{22}), *and define* $\widetilde{W}_i (i=1,2,3,4)$ *with respect to* $(\tilde{S}_1,\tilde{P}_1,V_{11},V_{21})$ *and* $(\tilde{S}_2,\tilde{P}_2,V_{12},V_{22})$.

Set

$$W_5=\frac{U_{21}}{U_{11}}-\frac{U_{22}}{U_{12}},\quad \widetilde{W}_5=\frac{V_{21}}{V_{11}}-\frac{V_{22}}{V_{12}},\quad W_6=\xi_1(z_2)-\xi_2(z_2).$$

Then under the assumptions of Theorem 3.1, we have

(B-46)
$$\|W_1\|_{C^{1,\alpha}(E_+)}\le C\Big(\delta\sum_{i=2}^{4}\|\widetilde{W}_i\|_{C^{1,\alpha}(E_+)}+\check{1}\|W_6\|_{C^{1,\alpha}(E_+)}\Big).$$

Proof. In the coordinate $z=(z_1,z_2)$, the characteristics $z_2^1(s;z)$ and $z_2^2(s;z)$, which go through the point (z_1,z_2) and correspond to the vector fields (V_{11},V_{21}) and (V_{12},V_{22}) in the right hand side of (2-28) respectively, can be defined as

$$\begin{aligned}&\frac{dz_2^i(s;z)}{ds}=-\frac{X_0(X_0+1-\xi_1(z_2^i))}{A_i}V_{2i}\big(\xi_1(z_2^i)+s(X_0+1-\xi_1(z_2^i)),z_2^i\big),\\ &z_2^i(z_1;z)=z_2,\quad z_2^1(0;z)=\beta,\quad z_2^2(0,z)=\tilde{\beta}\quad\text{for } i=1,2,\end{aligned}$$

where

$$A_i = (\xi_i(z_2^i) + s(X_0 + 1 - \xi_i(z_2^i)))V_{1i} + V_{2i}X_0(1-s)\xi_i'(z_2^i) \quad \text{for } i = 1, 2.$$

Set $l(s; z) = z_2^1(s; z) - z_2^2(s; z)$. Noting that $z_2^1(0; z) = \beta$ and $z_2^2(0; z) = \tilde{\beta}$, we have

$$\text{(B-47)} \qquad \begin{aligned} \frac{dl}{ds} &= O(\delta)l + O(\delta)\widetilde{W}_3(s, z_2^1) + O(1)\widetilde{W}_4(s, z_2^1) \\ &\qquad + O(\delta)W_6(z_2^1) + O(\delta^2)W_6'(z_2^1), \\ l(0; z) &= \beta - \tilde{\beta}, \qquad l(z_1; z) = 0, \end{aligned}$$

where we point out that the coefficients of $l(t; z)$ are in $C^{1,\alpha}$, which will be used to derive the $C^{1,\alpha}$ estimate of $\beta - \tilde{\beta}$.

By (B-47), we can arrive at

$$\|\beta - \tilde{\beta}\|_{L^\infty} \le \|l\|_{L^\infty} \le C(\delta\|\widetilde{W}_3\|_{L^\infty} + \|\widetilde{W}_4\|_{L^\infty} + \delta\|W_6\|_{L^\infty} + \delta^2\|W_6'(z_2)\|_{L^\infty}).$$

Note that

$$z_2^1(s; z) = -\int_0^s \frac{X_0(X_0 + 1 - \xi_1(z_2^1))}{A_1} V_{21}\big(\xi_1(z_2^1) + t(X_0 + 1 - \xi_1(z_2^1)), z_2^1\big)dt + \beta,$$

which implies, in particular, that

$$z_2 = -\int_0^{z_1} \frac{X_0(X_0 + 1 - \xi_1(z_{21}))}{A_1} V_{21}\big(\xi_1(z_2^1) + t(X_0 + 1 - \xi_1(z_{21})), z_2^1\big))dt + \beta.$$

Similar expressions hold for $z_2^2(s; z)$ and z_2 with β replaced by $\tilde{\beta}$. Thus, we may obtain

$$\text{(B-48)} \qquad \begin{aligned} \beta - \tilde{\beta} &= \int_0^{z_1} \big(O(\delta)\widetilde{W}_3(t, z_2^1) + O(1)\widetilde{W}_4(t, z_2^1) + O(\delta)W_6(z_2^1) \\ &\qquad + O(\delta^2)W_6'(z_2^1) + O(\delta)l(t; z)\big)dt, \\ l(s; z) &= \int_{z_1}^s \big(O(\delta)\widetilde{W}_3(t, z_2^1) + O(1)\widetilde{W}_4(t, z_2^1) + O(\delta)W_6(z_2^1) \\ &\qquad + O(\delta^2)W_6'(z_2^1) + O(\delta)l(t; z)\big)dt \end{aligned}$$

and

$$\text{(B-49)} \qquad \|\partial_{z_1}(\beta, \tilde{\beta})\|_{C^{1,\alpha}} \le C\varepsilon, \quad \|\partial_{z_2}(\beta, \tilde{\beta})\|_{C^{1,\alpha}} \le C\varepsilon.$$

In addition, it follows from (B-47) and (B-48) that

$$\text{(B-50)} \qquad \|\beta - \tilde{\beta}\|_{C^{1,\alpha}} \le C(\delta\|\widetilde{W}_3\|_{C^{1,\alpha}} + \|\widetilde{W}_4\|_{C^{1,\alpha}} + \delta\|W_6\|_{C^{2,\alpha}}).$$

This, together with (2-28) and the characteristics method, yields

$$W_1(z_1,z_2)=W_1(0,\beta(z_1,z_2))+O(\delta)\big(\beta(z_1,z_2)-\tilde\beta(z_1,z_2)\big),$$

$$W_1(0,z_2)=O(\delta^2)\widetilde W_2(0,z_2)+O(\delta^2)\widetilde W_3(0,z_2)+O(\delta)\widetilde W_4(0,z_2)+O(\check{1})W_6(z_2), \tag{B-51}$$

and

$$\|\beta(z)-\widetilde{\beta(z)}\|_{C^{1,\alpha}(E_+)}\le C\big(\delta\|\widetilde W_3\|_{C^{1,\alpha}(E_+)}+\|\widetilde W_4\|_{C^{1,\alpha}(E_+)}+\delta\|W_6\|_{C^{2,\alpha}[0,1]}\big). \tag{B-52}$$

Combining (B-52) with (B-51) yields (B-46), proving Lemma B.8. □

Remark B.9. If we choose

$$(\tilde S_2,\tilde P_2,V_{12},V_{22})=(S_2,P_2,U_{12},U_{22})=(S_0^+,\hat P_0^+,\hat U_0^+,0),$$

where $(S_0^+,\hat P_0^+,\hat U_0^+,0)$ is the background solution given in Appendix A with the exit pressure P_e, then, by the $C^{3,\alpha}$ regularity of $(S_0^+,\hat P_0^+,\hat U_0^+,0)$, we can conclude that

$$\|W_1\|_{C^{2,\alpha}(E_+)}\le C\Big(\delta\sum_{i=2}^{4}\|\widetilde W_i\|_{C^{2,\alpha}(E_+)}+\check{1}\|W_6\|_{C^{3,\alpha}(E_+)}\Big). \tag{B-53}$$

In fact, in this case, the corresponding coefficients of $l(s;z)$ in (B-47) and (B-48) are in $C^{2,\alpha}$. As in (B-50), we can derive that

$$\|\beta(z)-\widetilde{\beta(z)}\|_{C^{2,\alpha}(E_+)}\le C(\delta\|\widetilde W_3\|_{C^{2,\alpha}(E_+)}+\|\widetilde W_4\|_{C^{2,\alpha}(E_+)}+\delta\|W_6\|_{C^{3,\alpha}[0,1]}). \tag{B-54}$$

Subsequently, (B-53) can be shown by using (B-51) and (B-54).

Appendix C.

Here, for the problem (1-1) with (1-2)–(1-5), we give a detailed discussion of the higher order compatibility conditions on the nozzle wall and address the crucial difficulty in obtaining $C^{3,\alpha}$ regularities of solutions — that is, the appearance of the source terms in (2-8).

Due to the right hand conditions (1-2), the following expressions hold:

$$\begin{aligned}
G_1(\rho,U,S)&\equiv[\rho U_1][\rho U_2^2+P]-\rho^2U_1U_2^2=0,\\
G_2(\rho,U,S)&\equiv[\rho U_1^2+P][\rho U_2^2+P]-(\rho U_1U_2)^2=0,\\
G_3(\rho,U,S)&\equiv[(\rho e+\tfrac12\rho|U|^2+P)U_1][\rho U_2^2+P]\\
&\qquad-\rho U_1(\rho e+\tfrac12\rho|U|^2+P)U_2^2=0.
\end{aligned} \tag{C-1}$$

Since $U_2 = \partial_{z_2} P = \partial_{z_2} S = \partial_{z_2}\rho = \partial_{z_2} U_1 = 0$, at the point $M_0 := (z_1, z_2) = (0, 1)$, taking the tangential derivatives of second order and third order respectively along the shock surface yields at M_0

$$
\begin{aligned}
(\rho\partial_{z_2}^2 U_1 + U_1\partial_{z_2}^2\rho)[P] - 2\rho^2 U_1(\partial_{z_2}U_2)^2 = 0,\\
(2\rho U_1\partial_{z_2}^2 U_1 + U_1^2\partial_{z_2}^2\rho + \partial_{z_2}^2 P)][P] - 2\rho^2 U_1^2(\partial_{z_2}U_2)^2 = 0,\\
\Big(\Big(\frac{\gamma}{\gamma-1}P + \tfrac{1}{2}\rho U_1^2\Big)\partial_{z_2}^2 U_1 \\
+ U_1\Big(\frac{\gamma}{\gamma-1}\partial_{z_2}^2 P + \tfrac{1}{2}U_1^2\partial_{z_2}^2\rho + \rho U_1\partial_{z_2}^2 U_1 + \rho(\partial_{z_2}U_2)^2\Big)\Big)[P]\\
- 2\rho U_1\Big(\frac{\gamma}{\gamma-1}P + \tfrac{1}{2}\rho U_1^2\Big)(\partial_{z_2}U_2)^2 = 0.
\end{aligned}
\tag{C-2}
$$

and

$$
\begin{aligned}
(\rho\partial_{z_2}^3 U_1 + U_1\partial_{z_2}^3\rho)[P] - 6\rho^2 U_1\partial_{z_2}U_2\partial_{z_2}^2 U_2 = 0,\\
(2\rho U_1\partial_{z_2}^3 U_1 + U_1^2\partial_{z_2}^3\rho + \partial_{z_2}^3 P)][P] - 6\rho^2 U_1^2\partial_{z_2}U_2\partial_{z_2}^2 U_2 = 0,\\
\Big(\Big(\frac{\gamma}{\gamma-1}P + \tfrac{1}{2}\rho U_1^2\Big)\partial_{z_2}^3 U_1 \\
+ U_1\Big(\frac{\gamma}{\gamma-1}\partial_{z_2}^3 P + \tfrac{1}{2}U_1^3\partial_{z_2}^3\rho + \rho U_1\partial_{z_2}^3 U_1 + 3\rho\partial_{z_2}U_2\partial_{z_2}^2 U_2\Big)\Big)[P]\\
-6\rho U_1\Big(\frac{\gamma}{\gamma-1}P + \tfrac{1}{2}\rho U_1^2\Big)\partial_{z_2}U_2\partial_{z_2}^2 U_2 = 0.
\end{aligned}
\tag{C-3}
$$

From the first two equations in (C-2) and (C-3), we have at M_0

$$
\partial_{z_2}^2 P + \rho U_1\partial_{z_2}^2 U_1 = 0, \quad \partial_{z_2}^3 P + \rho U_1\partial_{z_2}^3 U_1 = 0.
\tag{C-4}
$$

It follows from the first and the third equations in (C-2) and (C-3) that at M_0

$$
\begin{aligned}
\Big(\frac{\gamma}{\gamma-1}P\partial_{z_2}^2 U_1 + U_1\Big(\frac{\gamma}{\gamma-1}\partial_{z_2}^2 P + \rho U_1\partial_{z_2}^2 U_1 + \rho(\partial_{z_2}U_2)^2\Big)\Big)[P]\\
-\frac{2\gamma}{\gamma-1}\rho U_1 P(\partial_{z_2}U_2)^2 = 0,\\
\Big(\frac{\gamma}{\gamma-1}P\partial_{z_2}^3 U_1 + U_1\Big(\frac{\gamma}{\gamma-1}\partial_{z_2}^3 P + \rho U_1\partial_{z_2}^3 U_1 + 3\rho\partial_{z_2}U_2\partial_{z_2}^2 U_2\Big)\Big)[P]\\
-\frac{6\gamma}{\gamma-1}\rho U_1 P\partial_{z_2}U_2\partial_{z_2}^2 U_2 = 0.
\end{aligned}
\tag{C-5}
$$

Since $\partial_{z_2}^2 U_2 + \check{1}\cot(\check{1})\partial_{z_2}U_2 = 0$ at M_0 due to (3-5) and (3-6) and by the expression of F_2 in (2-26), this together with (C-4) and (C-5) yields

$$
Q\big(\partial_{z_2}^3 P + \check{1}\cot(\check{1})\partial_{z_2}^2 P\big) = \Big(\frac{4\gamma}{\gamma-1}\rho U_1 P - 2\rho U_1[P]\Big)\partial_{z_2}U_2\partial_{z_2}^2 U_2 \quad \text{at } M_0
\tag{C-6}
$$

where

$$Q = \frac{\rho U_1^2 - \gamma P}{(\gamma-1)\rho U_1} < 0. \tag{C-7}$$

On the other hand, it follows from the first equation in (2-26), the expressions of F_1 and F_2, and (3-6) and (3-7) that $\partial_{z_2} P = \partial_{z_2} F_2 = F_1 = 0$ at M_0 and

$$a_1(\partial_{z_2}^3 P + \check{1}\cot(\check{1})\partial_{z_2}^2 P) = -\partial_{z_2}^2 F_1 - \check{1}\partial_{z_2} F_1 \quad \text{at } M_0. \tag{C-8}$$

Also, since

$$\xi^{(3)}(1) + \check{1}\cot(\check{1})\xi^{(2)}(1) = 0 \quad \text{and} \quad \partial_{z_2}^2 U_2 + \check{1}\cot(\check{1})\partial_{z_2} U_2 = 0$$

at M_0, Equation (C-8) yields

$$\frac{X_0(X_0+1-\xi)}{\xi\rho U_1^2}(\partial_{z_2}^3 P + \check{1}\cot(\check{1})\partial_{z_2}^2 P) = \frac{2(\partial_{z_2} U_2)^2(X_0+1-\xi)}{\xi U_1^2}\cot(\check{1})$$

at M_0, so that

$$\partial_{z_2}^3 P + \check{1}\cot(\check{1})\partial_{z_2}^2 P = \check{2}\rho\cot(\check{1})(\partial_{z_2} U_2)^2 \tag{C-9}$$

at M_0. Thus, it follows from (C-9) and (C-6) that

$$Q + \frac{2\gamma}{\gamma-1}U_1 P - U_1[P] = 0 \quad \text{or} \quad \partial_{z_2} U_2 = 0 \quad \text{at } M_0. \tag{C-10}$$

Meanwhile, in the general case,

$$\begin{aligned} Q + \frac{2\gamma}{\gamma-1}U_1 P - U_1[P] &= \frac{1}{(\gamma-1)\rho U_1}(\rho U_1^2 - \gamma P) + \frac{2\gamma}{\gamma-1}U_1 P - U_1[P] \\ &= \frac{1}{\gamma-1}U_1 + U_1\hat{P}_0^- + \frac{\gamma+1}{\gamma-1}U_1 P - \frac{\gamma}{(\gamma-1)\rho U_1}P \\ &= \frac{1}{\gamma-1}U_1 + U_1\hat{P}_0^- + \frac{P}{(\gamma-1)\rho U_1}((\gamma+1)\rho U_1^2 - \gamma) \neq 0. \end{aligned}$$

Thus, combining this with (C-9) yields $\partial_{z_2} U_2 = 0$ at M_0 if the solution is in $C^{3,\alpha}$.

On the other hand, it follows from (2-26) that

$$\frac{\partial_{z_2} U_2}{U_1} + \frac{\xi}{X_0(X_0+1-\xi)}\partial_{z_1}(P - \tilde{P}_0^+) = 0,$$

where $\tilde{P}_0^+$ denotes the background pressure when the shock position lies at $r = \xi(1)$. However, it seems to be rather difficult to show $\partial_{z_1}(P - \tilde{P}_0^+) = 0$ at the point M_0 in general (although $\partial_{z_2}(P - \tilde{P}_0^+) = 0$ there by (3-6)).

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