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**A NEW PROBABILITY DISTRIBUTION WITH APPLICATIONS**

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## A NEW PROBABILITY DISTRIBUTION WITH APPLICATIONS

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**We introduce a new probability distribution, which is useful in the study of basic hypergeometric series. As applications, we give probabilistic derivations of the  $q$ -binomial theorem, the  $q$ -Gauss summation formula, a new multiple identity, and an extension of the Rogers–Ramanujan identities.**

### 1. Introduction

The probabilistic method is a useful tool in the study of basic hypergeometric series [Chapman 2005; Evans 2002; Fulman 2001; Rawlings 1997]. In this paper, we introduce a new probability distribution and then demonstrate the applications of this distribution in  $q$ -series. We begin with recall some definitions, notations and known results in [Andrews et al. 1999; Gasper and Rahman 1990; Liu 2003]. Throughout the paper, we suppose that  $0 < q < 1$ . The  $q$ -shifted factorials are defined as

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

We also adopt a compact notation for multiple  $q$ -shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

where  $n$  is an integer or  $\infty$ . The  $q$ -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

In 1846, Heine introduced the  ${}_{r+1}\phi_r$  basic hypergeometric series, which is defined by

$${}_{r+1}\phi_r \left( \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n x^n}{(q, b_1, b_2, \dots, b_r; q)_n}.$$

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F. H. Jackson [1910] defined the  $q$ -integral by

$$(1-1) \quad \int_0^d f(t)d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n)q^n,$$

and

$$(1-2) \quad \int_c^d f(t)d_q t = \int_0^d f(t)d_q t - \int_0^c f(t)d_q t.$$

The  $q$ -integrals are important in the theory and application of basic hypergeometric series. For example, the author gives some applications of the  $q$ -integral in [Wang 2008; 2009b; 2009a; 2010b; 2010a]. The Andrews–Askey [1981] integral is

$$(1-3) \quad \int_c^d \frac{(qt/c, qt/d; q)_{\infty}}{(at, bt; q)_{\infty}} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_{\infty}}{(ac, ad, bc, bd; q)_{\infty}},$$

which can be derived from Ramanujan’s  ${}_1\psi_1$  summation provided that no zero factors occur in the denominator of the integral.

The Al-Salam–Carlitz polynomials  $\varphi_n^{(a)}(x|q)$  are defined by

$$\varphi_n^{(a)}(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k (a; q)_k,$$

[Srivastava and Jain 1989] and have the  $q$ -integral representation [Wang 2009b]

$$(1-4) \quad \varphi_n^{(a)}(x|q) = \frac{(ax, a; q)_{\infty}}{(1-q)(q, q/x, x; q)_{\infty}} \int_x^1 \frac{(qt/x, qt; q)_{\infty} t^n}{(at; q)_{\infty}} d_q t$$

provided that no zero factors occur in the denominator.

We frequently use the following well-known theorems:

**Theorem** (analytic continuation theorem). *If  $f$  and  $g$  are analytic at  $z_0$  and agree at infinitely many points, which include  $z_0$  as an accumulation point, then  $f = g$ .*

**Theorem** (Lebesgue’s dominated convergence theorem). *Suppose that  $\{X_n, n \geq 1\}$  is a sequence of random variables such that  $X_n \rightarrow X$  pointwise almost everywhere as  $n \rightarrow \infty$ , and such that  $|X_n| \leq Y$  for all  $n$ , where the random variable  $Y$  is integrable. Then  $X$  is integrable, and*

$$\lim_{n \rightarrow \infty} EX_n = EX,$$

where  $E(\cdot)$  denotes expected value.

Tannery’s theorem is a special case of Lebesgue’s dominated convergence theorem on the sequence space  $L^1$ .

**Theorem** [Tannery 1904]. *If  $s(n) = \sum_{k \geq 0} f_k(n)$  is a finite sum (or a convergent series) for each  $n$ ,*

$$\lim_{n \rightarrow \infty} f_k(n) = f_k, \quad |f_k(n)| \leq M_k, \quad \text{and} \quad \sum_{k=0}^{\infty} M_k < \infty$$

then

$$\lim_{n \rightarrow \infty} s(n) = \sum_{k=0}^{\infty} f_k.$$

### 2. A new probability distribution

In order to use Lebesgue’s dominated convergence theorem to get  $q$ -identities, we need to find some special probability distributions. In this section, we introduce a useful probability distribution.

The main method of this paper as follows: First, we define a probability distribution by  $q$ -shifted factorials; its expected value can be easily obtained. Then we construct a sequence of random variables with this probability distribution. Finally, we use Lebesgue’s dominated convergence theorem to obtain a  $q$ -identity.

**Lemma 2.1.** *Suppose  $x$  is a real such that  $x < 0$ ; then we have*

$$(2-1) \quad \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_{\infty} q^k}{(q, q/x, x; q)_{\infty}} \geq 0$$

and

$$(2-2) \quad \sum_{n=0}^1 \sum_{k=0}^{\infty} \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_{\infty} q^k}{(q, q/x, x; q)_{\infty}} = 1,$$

where  $n = 0, 1$  and  $k = 0, 1, 2, \dots$

*Proof.* Inequality (2-1) is obvious by the definition of the  $q$ -shifted factorials and the assumption that  $x < 0$ . We only need to prove (2-2).

Since

$$(2-3) \quad \sum_{n=0}^1 \sum_{k=0}^{\infty} \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_{\infty} q^k}{(q, q/x, x; q)_{\infty}} = \frac{1}{(1-q)(q, q/x, x; q)_{\infty}} \times \left( (1-q) \sum_{k=0}^{\infty} (q^{k+1}/x, q^{k+1}; q)_{\infty} q^k - x(1-q) \sum_{k=0}^{\infty} (q^{k+1}, xq^{k+1}; q)_{\infty} q^k \right),$$

using the definition of the  $q$ -integral gives

$$(1 - q) \sum_{k=0}^{\infty} (q^{k+1}/x, q^{k+1}; q)_{\infty} q^k = \int_0^1 (qt/x, qt; q)_{\infty} d_q t$$

and

$$x(1 - q) \sum_{k=0}^{\infty} (q^{k+1}, xq^{k+1}; q)_{\infty} q^k = \int_0^x (qt/x, qt; q)_{\infty} d_q t.$$

Consequently, we have

$$(2-4) \quad (1 - q) \sum_{k=0}^{\infty} (q^{k+1}/x, q^{k+1}; q)_{\infty} q^k \\ = \int_0^1 (qt/x, qt; q)_{\infty} d_q t - \int_0^x (qt/x, qt; q)_{\infty} d_q t = \int_x^1 (qt/x, qt; q)_{\infty} d_q t.$$

Employing the Andrews–Askey integral (1-3) gives

$$(2-5) \quad \int_x^1 (qt/x, qt; q)_{\infty} d_q t = (1 - q)(q, q/x, x; q)_{\infty}.$$

Substituting (2-4) and (2-5) into (2-3) gives (2-2). □

**Definition 2.2.** A random variable  $\zeta$  has distribution  $W(x; q)$  if

$$P(\zeta = x^n q^k) = \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_{\infty} q^k}{(q, q/x, x; q)_{\infty}},$$

where  $x < 0$ ,  $0 < q < 1$ ,  $n = 0, 1$  and  $k = 0, 1, 2, \dots$

The distribution  $W(x; q)$  has some applications in the study of basic hypergeometric series.

Before giving applications, we need the following lemmas.

**Lemma 2.3.** Let  $-1 < x < 0$  and  $|a| < 1$ . Let  $\zeta$  denote a random variable having with  $W(x; q)$ . Then we have

$$(2-6) \quad E\left(\frac{\zeta^m}{(a\zeta; q)_{\infty}}\right) = \frac{1}{(a, ax; q)_{\infty}} \varphi_m^{(a)}(x|q) \quad \text{for } m = 0, 1, 2, \dots$$

*Proof.* Using the definition of the  $q$ -integral (1-1), (1-2) and the  $q$ -integral representation of the Al-Salam–Carlitz polynomials (1-4), we have

$$\begin{aligned}
 & E\left(\frac{\zeta^m}{(a\zeta; q)_\infty}\right) \\
 &= \sum_{n=0}^1 \sum_{k=0}^\infty \frac{(-x)^n (x^{n-1}q^{k+1}, x^nq^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty} \cdot \frac{x^{nm}q^{km}}{(ax^nq^k; q)_\infty} \\
 &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \left( (1-q) \sum_{k=0}^\infty (q^{k+1}/x, q^{k+1}; q)_\infty \cdot \frac{q^{k(m+1)}}{(aq^k; q)_\infty} \right. \\
 &\quad \left. - x(1-q) \sum_{k=0}^\infty (q^{k+1}, xq^{k+1}; q)_\infty \cdot \frac{x^m q^{k(m+1)}}{(axq^k; q)_\infty} \right) \\
 &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \\
 &\quad \times \left( \int_0^1 \frac{(qt/x, qt; q)_\infty t^m}{(at; q)_\infty} d_q t - \int_0^x \frac{(qt/x, qt; q)_\infty t^m}{(at; q)_\infty} d_q t \right) \\
 &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \int_x^1 \frac{(qt/x, qt; q)_\infty t^m}{(at; q)_\infty} d_q t \\
 &= \frac{1}{(a, ax; q)_\infty} \varphi_m^{(a)}(x|q). \quad \square
 \end{aligned}$$

**Lemma 2.4.** Let  $-1 < x < 0$  and  $|a| < 1$ . Let  $\zeta$  denote a random variable having distribution  $W(x; q)$ . Then we have

$$E\left(\frac{1}{(a\zeta, b\zeta; q)_\infty}\right) = \frac{(abx, ; q)_\infty}{(a, b, ax, bx; q)_\infty}.$$

*Proof.* Using the definition of the  $q$ -integral (1-1), (1-2) and the Andrews–Askey integral (1-3), we have

$$\begin{aligned}
 & E\left(\frac{1}{(a\zeta, b\zeta; q)_\infty}\right) \\
 &= \sum_{n=0}^1 \sum_{k=0}^\infty \frac{(-x)^n (x^{n-1}q^{k+1}, x^nq^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty} \cdot \frac{1}{(ax^nq^k, bx^nq^k; q)_\infty} \\
 &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \left( (1-q) \sum_{k=0}^\infty (q^{k+1}/x, q^{k+1}; q)_\infty \cdot \frac{q^k}{(aq^k, bq^k; q)_\infty} \right. \\
 &\quad \left. - x(1-q) \sum_{k=0}^\infty (q^{k+1}, xq^{k+1}; q)_\infty \cdot \frac{q^k}{(axq^k, bx^nq^k; q)_\infty} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \left( \int_0^1 \frac{(qt/x, qt; q)_\infty}{(at, bt; q)_\infty} d_q t - \int_0^x \frac{(qt/x, qt; q)_\infty}{(at, bt; q)_\infty} d_q t \right) \\
 &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \int_x^1 \frac{(qt/x, qt; q)_\infty}{(at, bt; q)_\infty} d_q t = \frac{(abx, ; q)_\infty}{(a, b, ax, bx; q)_\infty},
 \end{aligned}$$

which completes the proof. □

**Lemma 2.5.** *Let  $|x| < 1$ . Then*

$$(2-7) \quad \lim_{n \rightarrow \infty} \varphi_n^{(a)}(x|q) = \sum_{k=0}^{\infty} \frac{(a; q)_k x^k}{(q; q)_k}.$$

*Proof.* Let  $f_k(n) = \begin{bmatrix} n \\ k \end{bmatrix} x^k (a; q)_k$  if  $k \leq n$  and  $f_k(n) = 0$  if  $k > n$ . We have

$$\varphi_n^{(a)}(x|q) = \sum_{k=0}^{\infty} f_k(n).$$

Since

$$\lim_{n \rightarrow \infty} f_k(n) = \frac{(a; q)_k x^k}{(q; q)_k}, \quad |f_k(n)| \leq \frac{|(a; q)_k x^k|}{(q; q)_\infty}, \quad \sum_{k=0}^{\infty} \frac{|(a; q)_k x^k|}{(q; q)_\infty} < \infty,$$

by Tannery’s theorem we know (2-7) holds. □

### 3. The $q$ -binomial theorem

One of the most important summation formulas for basic hypergeometric series is the  $q$ -binomial theorem, which was derived by Cauchy in 1843, Heine in 1847, and by other mathematicians. There are many proofs. By using the probability distribution  $W(x; q)$  and the Lebesgue dominated convergence theorem, we give a probabilistic derivation; see also [Andrews et al. 1999; Gasper and Rahman 1990].

**Theorem 3.1.**  $\sum_{n=0}^{\infty} \frac{(a; q)_n x^n}{(q; q)_n} = \frac{(ax; q)_\infty}{(x; q)_\infty}$  for  $|x| < 1$ .

*Proof.* Let  $\zeta$  be a random variable having distribution  $W(x; q)$ , where  $-1 < x < 0$ . We consider the sequence

$$\left\{ \frac{\zeta^n}{(a\zeta; q)_\infty} \right\}_{n=1}^{\infty} \quad \text{for } |a| < 1$$

of random variables (on a probability space). It is easy to see that  $\zeta^n$  converges to  $I_{(\zeta=1)}$ , which has Binomial distribution  $B(1, 1/(x; q)_\infty)$  and

$$\lim_{n \rightarrow \infty} \frac{\zeta^n}{(a\zeta; q)_\infty} = \frac{I_{(\zeta=1)}}{(a; q)_\infty},$$

where  $I_\Omega$  is the indicator function defined by

$$I_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Since

$$\left| \frac{\zeta^n}{(a\zeta; q)_\infty} \right| \leq \frac{1}{(|a|; q)_\infty},$$

using Lebesgue's dominated convergence theorem gives

$$(3-1) \quad \lim_{n \rightarrow \infty} E\left(\frac{\zeta^n}{(a\zeta; q)_\infty}\right) = E\left(\frac{I_{(\zeta=1)}}{(a; q)_\infty}\right).$$

Employing (1-4) and using Tannery's theorem gives

$$(3-2) \quad \begin{aligned} \lim_{m \rightarrow \infty} E\left(\frac{\zeta^m}{(a\zeta; q)_\infty}\right) &= \frac{1}{(a, ax; q)_\infty} \lim_{m \rightarrow \infty} \varphi_m^{(a)}(x|q) \\ &= \frac{1}{(a, ax; q)_\infty} \sum_{m=0}^{\infty} \frac{(a; q)_m x^m}{(q; q)_m}. \end{aligned}$$

By direct calculation,

$$(3-3) \quad E\left(\frac{I_{(\zeta=1)}}{(a; q)_\infty}\right) = \frac{1}{(a, x; q)_\infty}.$$

Substituting (3-2) and (3-3) into (3-1) gives

$$\sum_{n=0}^{\infty} \frac{(a; q)_n x^n}{(q; q)_n} = \frac{(ax; q)_\infty}{(x; q)_\infty},$$

where  $-1 < x < 0$  and  $|a| < 1$ . By analytic continuation, we may replace the assumptions  $-1 < x < 0$  by  $|a| < 1$  by  $|x| < 1$ . Thus, we get Theorem 3.1.  $\square$

#### 4. The $q$ -Gauss summation formula

In 1847, Heine derived a  $q$ -analogue of Gauss's summation formula. We show that this result can be recovered with the probability distribution  $W(x; q)$ .

**Theorem 4.1.**  ${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}$  for  $|c/(ab)| < 1$ .

*Proof.* Let  $\zeta$  and  $\eta$  denote two independent random variables having distributions  $W(x; q)$  and  $W(y; q)$ , respectively, where we set  $-1 < x, y < 0$ . We consider the following sequence of random variables (on a probability space):

$$\left\{ \frac{\eta^n}{(a\zeta\eta; q)_\infty} \right\}_{n=1}^{\infty} \quad \text{for } |a| < 1.$$

Clearly  $\eta^n$  converges to  $I_{(\eta=1)}$  having binomial distribution  $B(1, 1/((y; q)_\infty))$  and

$$\lim_{n \rightarrow \infty} \frac{\eta^n}{(a\zeta\eta; q)_\infty} = \frac{I_{(\eta=1)}}{(a\zeta; q)_\infty},$$

where  $I_\Omega$  is the indicator function.

Since

$$\left| \frac{\eta^n}{(a\zeta\eta; q)_\infty} \right| \leq \frac{1}{(|a|; q)_\infty},$$

using Lebesgue's dominated convergence theorem gives

$$(4-1) \quad \lim_{n \rightarrow \infty} E\left(\frac{\eta^n}{(a\zeta\eta; q)_\infty}\right) = E\left(\frac{I_{(\eta=1)}}{(a\zeta; q)_\infty}\right).$$

Observe that

$$\begin{aligned} E\left(\frac{\eta^n}{(a\zeta\eta; q)_\infty}\right) &= E\left(E\left(\frac{\eta^n}{(a\zeta\eta; q)_\infty} \mid \zeta\right)\right) \\ &= E\left(\frac{1}{(a\zeta, ay\zeta; q)_\infty} \varphi_n^{(a\zeta)}(x \mid q)\right) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} y^k \cdot E\left(\frac{1}{(a\zeta q^k, ay\zeta; q)_\infty}\right) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} y^k \cdot \frac{(a^2xyq^k; q)_\infty}{(aq^k, axq^k, ay, axy; q)_\infty} \\ &= \frac{(a^2xy; q)_\infty}{(a, ax, ay, axy; q)_\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \cdot \frac{(a, ax; q)_\infty y^k}{(a^2xy; q)_\infty}. \end{aligned}$$

Hence, we get the left hand side of (4-1):

$$(4-2) \quad \lim_{n \rightarrow \infty} E\left(\frac{\eta^n}{(a\zeta\eta; q)_\infty}\right) = \frac{(a^2xy; q)_\infty}{(a, ax, ay, axy; q)_\infty} \sum_{k=0}^{\infty} \frac{(a, ax; q)_\infty y^k}{(q, a^2xy; q)_\infty}.$$

On the other hand, the right hand side of (4-1) equals

$$(4-3) \quad E\left(\frac{I_{(\eta=1)}}{(a\zeta; q)_\infty}\right) = p(\eta = 1)E\left(\frac{1}{(a\zeta; q)_\infty}\right) = \frac{1}{(a, ax, y; q)_\infty}.$$

Substituting (4-2) and (4-3) into (4-1) gives

$$\sum_{k=0}^{\infty} \frac{(a, ax; q)_\infty y^k}{(q, a^2xy; q)_\infty} = \frac{(ay, axy; q)_\infty}{(a^2xy, y; q)_\infty},$$

which is equivalent to the  $q$ -Gauss theorem, Theorem 4.1, by analytic continuation. □

### 5. A multiple identity

Multiple basic hypergeometric series have been investigated by various authors [Milne 1997; Wang 2009a; Zhang 2006; Zhang and Liu 2006]. We will use the distribution  $W(x; q)$  to prove the following multiple identity.

**Theorem 5.1.** *Let  $|a| < 1$ . Then for any positive integers  $m$  and  $n$ , we have*

$$(5-1) \quad \sum_{y_1+\dots+y_m \geq n} \begin{bmatrix} y_1+\dots+y_m \\ n \end{bmatrix} q^{y_2+2y_3+\dots+(m-1)y_m} a^{y_1+\dots+y_m} = \frac{a^n}{(a; q)_{n+m}} \begin{bmatrix} n+m-1 \\ n \end{bmatrix}.$$

*Proof.* Let  $\xi$  denote a random variable with distribution  $W(x; q)$ , where  $-1 < x < 0$ . For any positive integer  $m$ , we consider the sequence

$$\left\{ \frac{(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)}{(a\xi; q)_\infty} \right\}_{n=1}^\infty \quad \text{for } |a| < 1$$

of random variables (on a probability space). It is easy to see that

$$\lim_{n \rightarrow \infty} \frac{(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)}{(a\xi; q)_\infty} = \frac{1}{(a\xi; q)_\infty}.$$

Since  $|(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)/(a\xi; q)_\infty| \leq 1/(|a|; q)_\infty$ , using Lebesgue’s dominated convergence theorem gives

$$(5-2) \quad \lim_{n \rightarrow \infty} E\left(\frac{(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)}{(a\xi; q)_\infty}\right) = E\left(\frac{1}{(a\xi; q)_\infty}\right).$$

Employing (2-6), we get the right hand side of (5-2):

$$(5-3) \quad E\left(\frac{1}{(a\xi; q)_\infty}\right) = \frac{1}{(a, ax; q)_\infty}.$$

On the other hand, observing that

$$\begin{aligned} & \frac{(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)}{(a\xi; q)_\infty} \\ &= \frac{1 - (a\xi)^n}{1 - a\xi} \cdot \frac{1 - (aq\xi)^n}{1 - aq\xi} \cdots \frac{1 - (aq^{m-1}\xi)^n}{1 - aq^{m-1}\xi} \cdot \frac{1}{(aq^m\xi; q)_\infty} \\ &= \sum_{y_1=0}^{n-1} (a\xi)^{y_1} \cdot \sum_{y_2=0}^{n-1} (aq\xi)^{y_2} \cdots \sum_{y_m=0}^{n-1} (aq^{m-1}\xi)^{y_m} \cdot \frac{1}{(aq^m\xi; q)_\infty} \\ &= \sum_{0 \leq y_1, \dots, y_m \leq n-1} q^{y_2+2y_3+\dots+(m-1)y_m} a^{y_1+\dots+y_m} \cdot \frac{\xi^{y_1+\dots+y_m}}{(aq^m\xi; q)_\infty}, \end{aligned}$$

we have

$$\begin{aligned}
 E\left(\frac{[1 - (a\xi)^n][1 - (aq\xi)^n] \cdots [1 - (aq^{m-1}\xi)^n]}{(a\xi; q)_\infty}\right) &= \sum_{0 \leq y_1, \dots, y_m \leq n-1} q^{y_2+2y_3+\dots+(m-1)y_m} a^{y_1+\dots+y_m} E\left(\frac{\xi^{y_1+\dots+y_m}}{(aq^m\xi; q)_\infty}\right) \\
 &= \frac{1}{(aq^m, axq^m; q)_\infty} \\
 &\quad \times \sum_{0 \leq y_1, \dots, y_m \leq n-1} q^{y_2+2y_3+\dots+(m-1)y_m} a^{y_1+\dots+y_m} \varphi_{y_1+\dots+y_m}^{(aq^m)}(x|q).
 \end{aligned}$$

Hence, we get the left hand side of (5-2):

$$\begin{aligned}
 (5-4) \quad \lim_{n \rightarrow \infty} E\left(\frac{[1 - (a\xi)^n][1 - (aq\xi)^n] \cdots [1 - (aq^{m-1}\xi)^n]}{(a\xi; q)_\infty}\right) &= \frac{1}{(aq^m, axq^m; q)_\infty} \\
 &\quad \times \sum_{y_1, \dots, y_m \geq 0} q^{y_2+2y_3+\dots+(m-1)y_m} a^{y_1+\dots+y_m} \varphi_{y_1+\dots+y_m}^{(aq^m)}(x|q).
 \end{aligned}$$

Substituting (5-3) and (5-4) into (5-2) gives

$$(5-5) \quad \sum_{y_1, \dots, y_m \geq 0} q^{y_2+2y_3+\dots+(m-1)y_m} a^{y_1+\dots+y_m} \varphi_{y_1+\dots+y_m}^{(aq^m)}(x|q) = \frac{1}{(a, ax; q)_m}.$$

Using Theorem 3.1 with  $a = q^m$  and  $x = ax$  gives

$$(5-6) \quad \sum_{k=0}^{\infty} \begin{bmatrix} m+k-1 \\ k \end{bmatrix} a^k x^k = \frac{1}{(ax; q)_m}.$$

Substituting (5-6) into (5-5) and comparing the coefficients of  $x^n$  gives (5-1).  $\square$

### 6. An extension of the Rogers–Ramanujan identities

The well-known Rogers–Ramanujan identities are

$$(6-1) \quad \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q; q)_m} = \frac{1}{(q, q^4; q^5)_\infty},$$

$$(6-2) \quad \sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q; q)_m} = \frac{1}{(q^2, q^3; q^5)_\infty}.$$

There are many proofs of this beautiful pair of identities. Baxter’s [1982] is based on the statistical mechanics, and the proof of Lepowsky and Milne [1978]

uses the character formula on an infinite dimensional Lie algebra. We use our probability distribution to derive an extension of the Rogers–Ramanujan identities.

**Theorem 6.1.** *We have*

$$\sum_{m=n}^{\infty} \frac{q^{m^2}}{(q; q)_{m-n}} = \frac{1}{(q; q)_{\infty}} \times \left( q^n + \sum_{k=1}^{\infty} \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix} \frac{(-1)^k (q^k; q)_l (q; q)_{2k}}{(1-q^k)(q; q)_{2k+1-l}} q^{5\binom{k}{2}+k(n+2-l)} (1-q^{2k}+q^{4k+1}) - \sum_{k=1}^{\infty} \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix} \frac{(-1)^k (q^k; q)_l (q; q)_{2k}}{(1-q^k)(q; q)_{2k+1-l}} q^{5\binom{k}{2}+k(n+4-l)+1-l} \right).$$

*Proof.* By Watson’s  $q$ -Whipple transformation formula [Watson 1929],

$${}_8\phi_7 \left( \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, q^{n+1}a \end{matrix}; q, \frac{q^{2+n}a^2}{bcde} \right) = \frac{(qa, qa/bc; q)_n}{(qa/b, qa/c)_n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, b, c, qa/de \\ qa/d, qa/e, q^{-n}bc/a \end{matrix}; q, q \right).$$

Letting  $b, c, d, e, n \rightarrow \infty$  in this equation gives

$$\sum_{m=0}^{\infty} \frac{q^{m^2} a^m}{(q; q)_m} = \frac{1}{(aq; q)_{\infty}} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+2k}}{(q; q)_k} \cdot \frac{a^{2k}}{(aq^k; q)_{\infty}} - \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+4k}}{(q; q)_k} \cdot \frac{a^{2k+1}}{(aq^k; q)_{\infty}} \quad \text{for } |a| \leq 1.$$

Then letting  $a = \zeta$  gives

$$\sum_{m=0}^{\infty} \frac{q^{m^2} \zeta^m}{(q; q)_m} = \frac{1}{(\zeta q; q)_{\infty}} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+2k}}{(q; q)_k} \cdot \frac{\zeta^{2k}}{(\zeta q^k; q)_{\infty}} - \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+4k}}{(q; q)_k} \cdot \frac{\zeta^{2k+1}}{(\zeta q^k; q)_{\infty}}.$$

where  $\zeta$  is a random variable with distribution  $W(x; q)$  and  $-1 < x < 0$ . Applying the expectation operator  $E$  to both sides of the above, we get

$$(6-3) \quad E \left( \sum_{m=0}^{\infty} \frac{q^{m^2} \zeta^m}{(q; q)_m} \right) = E \left( \frac{1}{(\zeta q; q)_{\infty}} \right) + E \left( \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+2k}}{(q; q)_k} \cdot \frac{\zeta^{2k}}{(\zeta q^k; q)_{\infty}} \right) - E \left( \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+4k}}{(q; q)_k} \cdot \frac{\zeta^{2k+1}}{(\zeta q^k; q)_{\infty}} \right).$$

Since  $|q^{m^2} \zeta^m / (q; q)_m| \leq q^{m^2} / (q; q)_m$  and the series  $\sum_{m=0}^{\infty} q^{m^2} / (q; q)_m$  converges absolutely, using Lebesgue's dominated convergence theorem and (2-6) gives the left hand side of (6-3):

$$(6-4) \quad E\left(\sum_{m=0}^{\infty} \frac{q^{m^2} \zeta^m}{(q; q)_m}\right) = \sum_{m=0}^{\infty} \frac{q^{m^2} E\{\zeta^m\}}{(q; q)_m} = \sum_{m=0}^{\infty} \frac{q^{m^2} h_m(x|q)}{(q; q)_m}.$$

On the other hand, using (2-6) gives

$$(6-5) \quad E\left(\frac{1}{(\zeta q; q)_{\infty}}\right) = \frac{1}{(q, qx; q)_{\infty}},$$

$$(6-6) \quad E\left(\frac{\zeta^{2k}}{(\zeta q^k; q)_{\infty}}\right) = \frac{1}{(q^k, q^k x; q)_{\infty}} \varphi_{2k}^{(q^k)}(x|q),$$

$$(6-7) \quad E\left(\frac{\zeta^{2k+1}}{(\zeta q^k; q)_{\infty}}\right) = \frac{1}{(q^k, q^k x; q)_{\infty}} \varphi_{2k+1}^{(q^k)}(x|q).$$

It is easy to see that

$$\left| \frac{(-1)^k q^{5\binom{k}{2}+4k}}{(q; q)_k} \cdot \frac{\zeta^{2k+1}}{(\zeta q^k; q)_{\infty}} \right| \leq \left| \frac{(-1)^k q^{5\binom{k}{2}+2k}}{(q; q)_k} \cdot \frac{\zeta^{2k}}{(\zeta q^k; q)_{\infty}} \right| \leq \frac{q^{5\binom{k}{2}+2k}}{(q; q)_k (q; q)_{\infty}},$$

and the series  $\sum_{k=0}^{\infty} q^{5\binom{k}{2}+2k} / ((q; q)_k (q; q)_{\infty})$  is converges absolutely. Using Lebesgue's dominated convergence theorem and (6-5), (6-6) and (6-7) gives the right hand side of (6-3):

$$\begin{aligned} & E\left(\frac{1}{(\zeta q; q)_{\infty}}\right) + E\left(\sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+2k}}{(q; q)_k} \cdot \frac{\zeta^{2k}}{(\zeta q^k; q)_{\infty}}\right) \\ & \quad - E\left(\sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+4k}}{(q; q)_k} \cdot \frac{\zeta^{2k+1}}{(\zeta q^k; q)_{\infty}}\right) \\ & = E\left(\frac{1}{(\zeta q; q)_{\infty}}\right) + \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+2k}}{(q; q)_k} E\left(\frac{\zeta^{2k}}{(\zeta q^k; q)_{\infty}}\right) \\ & \quad - \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+4k}}{(q; q)_k} E\left(\frac{\zeta^{2k+1}}{(\zeta q^k; q)_{\infty}}\right) \\ & = \frac{1}{(q, qx; q)_{\infty}} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+2k}}{(q; q)_k} \frac{1}{(q^k, q^k x; q)_{\infty}} \varphi_{2k}^{(q^k)}(x|q) \\ & \quad - \sum_{k=1}^{\infty} \frac{(-1)^k q^{5\binom{k}{2}+4k}}{(q; q)_k} \frac{1}{(q^k, q^k x; q)_{\infty}} \varphi_{2k+1}^{(q^k)}(x|q) \end{aligned}$$

$$= \frac{1}{(q, qx; q)_\infty} + \frac{1}{(q, x; q)_\infty} \sum_{k=1}^\infty \frac{(-1)^k (x; q)_k}{1 - q^k} q^{5\binom{k}{2} + 2k} (\varphi_{2k}^{(q^k)}(x|q) - q^{2k} \varphi_{2k+1}^{(q^k)}(x|q)).$$

Substituting this and (6-4) into (6-3) gives

$$\sum_{m=0}^\infty \frac{q^{m^2} h_m(x|q)}{(q; q)_m} = \frac{1}{(q, qx; q)_\infty} + \frac{1}{(q, x; q)_\infty} \sum_{k=1}^\infty \frac{(-1)^k (x; q)_k}{1 - q^k} q^{5\binom{k}{2} + 2k} (\varphi_{2k}^{(q^k)}(x|q) - q^{2k} \varphi_{2k+1}^{(q^k)}(x|q)),$$

where  $-1 < x < 0$ . By analytic continuation, we may replace the assumption  $-1 < x < 0$  by  $|x| < 1$ .

Substituting the expansion

$$\frac{1}{(z; q)_\infty} = \sum_{l=0}^\infty \frac{z^l}{(q; q)_l}$$

into the last, we have

$$\sum_{m=0}^\infty \frac{q^{m^2} h_m(x|q)}{(q; q)_m} = \frac{1}{(q; q)_\infty} \sum_{l=0}^\infty \frac{q^l x^l}{(q; q)_l} + \frac{1}{(q; q)_\infty} \sum_{k=1}^\infty \sum_{l=0}^\infty \left( \frac{q^{kl} x^l}{(q; q)_l} \cdot \frac{(-1)^k q^{5\binom{k}{2} + 2k}}{1 - q^k} (\varphi_{2k}^{(q^k)}(x|q) - q^{2k} \varphi_{2k+1}^{(q^k)}(x|q)) \right).$$

Comparing the coefficients of  $x^n$  in this identity gives

$$\sum_{m=n}^\infty \frac{q^{m^2} [m]}{(q; q)_m} = \frac{q^n}{(q; q)_\infty (q; q)_n} + \frac{1}{(q; q)_\infty} \sum_{k=1}^\infty \sum_{l=0}^n \frac{(-1)^k (q^k; q)_l}{(1 - q^k)(q; q)_{n-l}} q^{5\binom{k}{2} + k(n+2-l)} \left( \begin{bmatrix} 2k \\ l \end{bmatrix} + q^{2k} \begin{bmatrix} 2k+1 \\ l \end{bmatrix} \right),$$

which can be written as Theorem 6.1. □

The Rogers–Ramanujan identities are special cases of Theorem 6.1. Letting  $n = 0$  and then applying the Jacobi triple product identity [Andrews et al. 1999]

$$\sum_{n=-\infty}^\infty (-1)^n q^{\binom{n}{2}} x^n = (q, x, q/x; q)_\infty$$

leads to the Rogers–Ramanujan identity (6-1). In fact, when  $n = 0$ , we have

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q; q)_m} &= \frac{1}{(q; q)_{\infty}} \left( 1 + \sum_{k=1}^{\infty} (-1)^k (1 + q^k) q^{5\binom{k}{2} + 2k} \right) \\ &= \frac{1}{(q; q)_{\infty}} \sum_{k=-\infty}^{\infty} (-1)^k q^{5\binom{k}{2} + 2k} \\ &= \frac{(q^5, q^2, q^3; q^5)_{\infty}}{(q; q)_{\infty}} = \frac{1}{(q, q^4; q^5)_{\infty}}. \end{aligned}$$

Similarly, the case  $n = 1$  of Theorem 6.1 results in another identity due to Rogers and Ramanujan:

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q; q)_m} &= \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q; q)_m} - \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q; q)_{m-1}} \\ &= \frac{1}{(q; q)_{\infty}} \left( 1 + \sum_{k=1}^{\infty} (1 - q^{2k+1}) q^{5\binom{k}{2} + 4k} \right) \\ &= \frac{1}{(q; q)_{\infty}} \sum_{k=-\infty}^{\infty} (-1)^k q^{5\binom{k}{2} + 4k} \\ &= \frac{(q, q^4, q^5; q^5)_{\infty}}{(q; q)_{\infty}} = \frac{1}{(q^2, q^3; q^5)_{\infty}}. \end{aligned}$$

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