A NEW PROBABILITY DISTRIBUTION WITH APPLICATIONS

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We introduce a new probability distribution, which is useful in the study of basic hypergeometric series. As applications, we give probabilistic derivations of the \( q \)-binomial theorem, the \( q \)-Gauss summation formula, a new multiple identity, and an extension of the Rogers–Ramanujan identities.

1. Introduction

The probabilistic method is a useful tool in the study of basic hypergeometric series [Chapman 2005; Evans 2002; Fulman 2001; Rawlings 1997]. In this paper, we introduce a new probability distribution and then demonstrate the applications of this distribution in \( q \)-series. We begin with recall some definitions, notations and known results in [Andrews et al. 1999; Gasper and Rahman 1990; Liu 2003]. Throughout the paper, we suppose that \( 0 < q < 1 \). The \( q \)-shifted factorials are defined as

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - a q^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k).
\]

We also adopt a compact notation for multiple \( q \)-shifted factorials:

\[
(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,
\]

where \( n \) is an integer or \( \infty \). The \( q \)-binomial coefficient is defined by

\[
\binom{n}{k} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.
\]

In 1846, Heine introduced the \( r+1 \phi_r \) basic hypergeometric series, which is defined by

\[
r+1 \phi_r \left( \begin{array}{c}
(a_1, a_2, \ldots, a_{r+1}; q)_n \; x^n \\
(b_1, b_2, \ldots, b_r; q)_n
\end{array} \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n x^n}{(q, b_1, b_2, \ldots, b_r; q)_n}.
\]

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F. H. Jackson [1910] defined the \( q \)-integral by

\[
\int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n) q^n,
\]

and

\[
\int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t.
\]

The \( q \)-integrals are important in the theory and application of basic hypergeometric series. For example, the author gives some applications of the \( q \)-integral in [Wang 2008; 2009b; 2009a; 2010b; 2010a]. The Andrews–Askey [1981] integral is

\[
\int_c^d \frac{(qt/c, qt/d, q)_{\infty}}{(at, bt; q)_{\infty}} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_{\infty}}{(ac, ad, bc, bd; q)_{\infty}},
\]

which can be derived from Ramanujan’s \( \psi_1 \) summation provided that no zero factors occur in the denominator of the integral.

The Al-Salam–Carlitz polynomials \( \phi_n^{(a)}(x | q) \) are defined by

\[
\phi_n^{(a)}(x | q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k (a; q)_k,
\]

[Srivastava and Jain 1989] and have the \( q \)-integral representation [Wang 2009b]

\[
\phi_n^{(a)}(x | q) = \frac{(ax, a; q)_{\infty}}{(1-q)(q, q/x, x; q)_{\infty}} \int_1^1 \frac{(qt/x, qt; q)_{\infty}}{(at; q)_{\infty}} t^n d_q t
\]

provided that no zero factors occur in the denominator.

We frequently use the following well-known theorems:

**Theorem** (analytic continuation theorem). If \( f \) and \( g \) are analytic at \( z_0 \) and agree at infinitely many points, which include \( z_0 \) as an accumulation point, then \( f = g \).

**Theorem** (Lebesgue’s dominated convergence theorem). Suppose that \( \{X_n, n \geq 1\} \) is a sequence of random variables such that \( X_n \to X \) pointwise almost everywhere as \( n \to \infty \), and such that \( |X_n| \leq Y \) for all \( n \), where the random variable \( Y \) is integrable. Then \( X \) is integrable, and

\[
\lim_{n \to \infty} E X_n = E X,
\]

where \( E( \cdot ) \) denotes expected value.

Tannery’s theorem is a special case of Lebesgue’s dominated convergence theorem on the sequence space \( L^1 \).
Theorem [Tannery 1904]. If \( s(n) = \sum_{k \geq 0} f_k(n) \) is a finite sum (or a convergent series) for each \( n \),
\[
\lim_{n \to \infty} f_k(n) = f_k, \quad |f_k(n)| \leq M_k, \quad \text{and} \quad \sum_{k=0}^{\infty} M_k < \infty
\]
then
\[
\lim_{n \to \infty} s(n) = \sum_{k=0}^{\infty} f_k.
\]

2. A new probability distribution

In order to use Lebesgue’s dominated convergence theorem to get \( q \)-identities, we need to find some special probability distributions. In this section, we introduce a useful probability distribution.

The main method of this paper as follows: First, we define a probability distribution by \( q \)-shifted factorials; its expected value can be easily obtained. Then we construct a sequence of random variables with this probability distribution. Finally, we use Lebesgue’s dominated convergence theorem to obtain a \( q \)-identity.

Lemma 2.1. Suppose \( x \) is a real such that \( x < 0 \); then we have
\[
(2-1) \quad \frac{(-x)^n(x^{n-1}q^{k+1}, x^n q^{k+1}; q)_{\infty} q^k}{(q, q/x, x; q)_{\infty}} \geq 0
\]
and
\[
(2-2) \quad \sum_{n=0}^{1} \sum_{k=0}^{\infty} \frac{(-x)^n(x^{n-1}q^{k+1}, x^n q^{k+1}; q)_{\infty} q^k}{(q, q/x, x; q)_{\infty}} = 1,
\]
where \( n = 0, 1 \) and \( k = 0, 1, 2, \ldots \).

Proof. Inequality (2-1) is obvious by the definition of the \( q \)-shifted factorials and the assumption that \( x < 0 \). We only need to prove (2-2).

Since
\[
(2-3) \quad \sum_{n=0}^{1} \sum_{k=0}^{\infty} \frac{(-x)^n(x^{n-1}q^{k+1}, x^n q^{k+1}; q)_{\infty} q^k}{(q, q/x, x; q)_{\infty}} = \frac{1}{(1-q)(q, q/x, x; q)_{\infty}} \times \left( (1-q) \sum_{k=0}^{\infty} (q^{k+1}/x, q^{k+1}; q)_{\infty} q^k - x (1-q) \sum_{k=0}^{\infty} (q^{k+1}, xq^{k+1}; q)_{\infty} q^k \right),
\]
using the definition of the $q$-integral gives

$$
(1 - q) \sum_{k=0}^{\infty} (q^{k+1}/x, q^{k+1}; q)_{\infty} q^k = \int_0^1 (qt/x, qt; q)_{\infty} d_q t
$$

and

$$
x (1 - q) \sum_{k=0}^{\infty} (q^{k+1}, xq^{k+1}; q)_{\infty} q^k = \int_0^x (qt/x, qt; q)_{\infty} d_q t.
$$

Consequently, we have

$$
(2-4) \quad (1 - q) \sum_{k=0}^{\infty} (q^{k+1}/x, q^{k+1}; q)_{\infty} q^k
$$

$$
= \int_0^1 (qt/x, qt; q)_{\infty} d_q t - \int_0^x (qt/x, qt; q)_{\infty} d_q t = \int_1^x (qt/x, qt; q)_{\infty} d_q t.
$$

Employing the Andrews–Askey integral (1-3) gives

$$
(2-5) \quad \int_1^x (qt/x, qt; q)_{\infty} d_q t = (1 - q)(q, q/x, x; q)_{\infty}.
$$

Substituting (2-4) and (2-5) into (2-3) gives (2-2). \hfill \Box

**Definition 2.2.** A random variable $\xi$ has distribution $W(x; q)$ if

$$
P(\xi = x^n q^k) = \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_{\infty} q^k}{(q, q/x, x; q)_{\infty}},
$$

where $x < 0$, $0 < q < 1$, $n = 0, 1$ and $k = 0, 1, 2, \ldots$

The distribution $W(x; q)$ has some applications in the study of basic hypergeometric series.

Before giving applications, we need the following lemmas.

**Lemma 2.3.** Let $-1 < x < 0$ and $|a| < 1$. Let $\xi$ denote a random variable having with $W(x; q)$. Then we have

$$
E\left(\frac{\xi^m}{(a\xi; q)_{\infty}}\right) = \frac{1}{(a, ax; q)_{\infty}} \varphi_m^{(a)}(x | q) \quad \text{for} \quad m = 0, 1, 2, \ldots
$$
Proof. Using the definition of the $q$-integral (1-1), (1-2) and the $q$-integral representation of the Al-Salam–Carlitz polynomials (1-4), we have
\[
E\left(\frac{\xi^m}{(a\xi, q)_\infty}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty} \cdot \frac{x^{nm} q^{km}}{(ax^n q^k; q)_\infty}
\]
\[
= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \left((1-q) \sum_{k=0}^{\infty} q^{k+1} / (q^{k+1}; q)_\infty \cdot \frac{q^{k(m+1)}}{(aq^k; q)_\infty} - x(1-q) \sum_{k=0}^{\infty} q^{k+1} / (x q^{k+1}; q)_\infty \cdot \frac{x^m q^{k(m+1)}}{(ax q^k; q)_\infty}\right)
\]
\[
= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \left(\int_0^1 \frac{(qt/x, qt; q)_\infty t^m}{(at; q)_\infty} \cdot d_q t - \int_x^1 \frac{(qt/x, qt; q)_\infty t^m}{(at; q)_\infty} \cdot d_q t\right)
\]
\[
= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \int_0^1 \frac{(qt/x, qt; q)_\infty t^m}{(at; q)_\infty} \cdot d_q t
\]
\[
= \frac{1}{(a, ax; q)_\infty} \phi_m^{(a)}(x | q).
\]

Lemma 2.4. Let $-1 < x < 0$ and $|a| < 1$. Let $\xi$ denote a random variable having distribution $W(x; q)$. Then we have
\[
E\left(\frac{1}{(a\xi, b\xi; q)_\infty}\right) = \frac{(abx, ; q)_\infty}{(a, b, ax, bx; q)_\infty}.
\]

Proof. Using the definition of the $q$-integral (1-1), (1-2) and the Andrews–Askey integral (1-3), we have
\[
E\left(\frac{1}{(a\xi, b\xi; q)_\infty}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-x)^n (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty} \cdot \frac{1}{(ax^n q^k, bx^n q^k; q)_\infty}
\]
\[
= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \left((1-q) \sum_{k=0}^{\infty} q^{k+1} / (q^{k+1}; q)_\infty \cdot \frac{q^k}{(aq^k, bq^k; q)_\infty} - x(1-q) \sum_{k=0}^{\infty} q^{k+1} / (x q^{k+1}; q)_\infty \cdot \frac{q^k}{(ax q^k, bx q^k; q)_\infty}\right)
\]
\[
\int_0^1 \frac{(qt/x, qt; q)_{\infty}}{(at, bt; q)_{\infty}} d_q t = \int_0^x \frac{(qt/x, qt; q)_{\infty}}{(at, bt; q)_{\infty}} d_q t
\]

which completes the proof. □

**Lemma 2.5.** Let \(|x| < 1\). Then

(2-7) \[
\lim_{n \to \infty} \varphi^{(a)}_n(x \mid q) = \sum_{k=0}^{\infty} \frac{(a; q)_k x^k}{(q; q)_k}.
\]

**Proof.** Let \(f_k(n) = \left[ {n \atop k} \right] x^k (a; q)_k \) if \(k \leq n\) and \(f_k(n) = 0\) if \(k > n\). We have

\[
\varphi^{(a)}_n(x \mid q) = \sum_{k=0}^{\infty} f_k(n).
\]

Since

\[
\lim_{n \to \infty} f_k(n) = \frac{(a; q)_k x^k}{(q; q)_k}, \quad |f_k(n)| \leq \frac{|(a; q)_k x^k|}{(q; q)_\infty}, \quad \sum_{k=0}^{\infty} \frac{|(a; q)_k x^k|}{(q; q)_\infty} < \infty,
\]

by Tannery’s theorem we know (2-7) holds. □

### 3. The \(q\)-binomial theorem

One of the most important summation formulas for basic hypergeometric series is the \(q\)-binomial theorem, which was derived by Cauchy in 1843, Heine in 1847, and by other mathematicians. There are many proofs. By using the probability distribution \(W(x; q)\) and the Lebesgue dominated convergence theorem, we give a probabilistic derivation; see also [Andrews et al. 1999; Gasper and Rahman 1990].

**Theorem 3.1.** \[
\sum_{n=0}^{\infty} \frac{(a; q)_n x^n}{(q; q)_n} = \frac{(ax; q)_\infty}{(x; q)_\infty} \quad \text{for } |x| < 1.
\]

**Proof.** Let \(\xi\) be a random variable having distribution \(W(x; q)\), where \(-1 < x < 0\). We consider the sequence

\[
\left\{ \frac{\xi^n}{(a \xi; q)_\infty} \right\}_{n=1}^{\infty} \quad \text{for } |a| < 1
\]

of random variables (on a probability space). It is easy to see that \(\xi^n\) converges to \(I_{(\xi=1)}\), which has Binomial distribution \(B(1, 1/(x; q)_\infty)\) and

\[
\lim_{n \to \infty} \frac{\xi^n}{(a \xi; q)_\infty} = \frac{I_{(\xi=1)}}{(a; q)_\infty},
\]
where \( I_\Omega \) is the indicator function defined by

\[
I_\Omega(x) = \begin{cases} 
1 & \text{if } x \in \Omega, \\
0 & \text{if } x \notin \Omega.
\end{cases}
\]

Since

\[
\frac{\xi^n}{(a_\xi; q)_\infty} \leq \frac{1}{(|a|; q)_\infty},
\]

using Lebesgue’s dominated convergence theorem gives

\[
(3-1) \quad \lim_{n \to \infty} E\left( \frac{\xi^n}{(a_\xi; q)_\infty} \right) = E\left( \frac{I_{\xi=1}}{(a; q)_\infty} \right).
\]

Employing (1-4) and using Tannery’s theorem gives

\[
(3-2) \quad \lim_{m \to \infty} E\left( \frac{\xi^m}{(a_\xi; q)_\infty} \right) = \frac{1}{(a, ax; q)_\infty} \lim_{m \to \infty} \phi_m^{(a)}(x | q)
= \frac{1}{(a, ax; q)_\infty} \sum_{m=0}^{\infty} \frac{(a; q)_m x^m}{(q; q)_m}.
\]

By direct calculation,

\[
(3-3) \quad E\left( \frac{I_{\xi=1}}{(a; q)_\infty} \right) = \frac{1}{(a, x; q)_\infty}.
\]

Substituting (3-2) and (3-3) into (3-1) gives

\[
\sum_{n=0}^{\infty} \frac{(a; q)_n x^n}{(q; q)_n} = \frac{(ax; q)_\infty}{(x; q)_\infty},
\]

where \(-1 < x < 0\) and \(|a| < 1\). By analytic continuation, we may replace the assumptions \(-1 < x < 0\) by \(|a| < 1\) by \(|x| < 1\). Thus, we get Theorem 3.1. \(\square\)

4. The \(q\)-Gauss summation formula

In 1847, Heine derived a \(q\)-analogue of Gauss’s summation formula. We show that this result can be recovered with the probability distribution \(W(x; q)\).

**Theorem 4.1.** \(2\phi_1\left( a,c \frac{c/ab}{ab} ; q, \frac{c}{ab} \right) = \frac{\left( c/a, c/b; q \right)_\infty \left( c, c/ab; q \right)_\infty}{(c, c/ab; q)_\infty} \) for \(|c/(ab)| < 1\).

**Proof.** Let \(\xi\) and \(\eta\) denote two independent random variables having distributions \(W(x; q)\) and \(W(y; q)\), respectively, where we set \(-1 < x, y < 0\). We consider the following sequence of random variables (on a probability space):

\[
\left\{ \frac{\eta^n}{(a_\xi\eta; q)_\infty} \right\}_{n=1}^{\infty} \text{ for } |a| < 1.
\]
Clearly $\eta^n$ converges to $I_{(\eta=1)}$ having binomial distribution $B(1, 1/((y; q)_{\infty}))$ and

$$
\lim_{n \to \infty} \frac{\eta^n}{(a\xi \eta; q)_{\infty}} = \frac{I_{(\eta=1)}}{(a\xi; q)_{\infty}},
$$

where $I_{\Omega}$ is the indicator function.

Since

$$
\left| \frac{\eta^n}{(a\xi \eta; q)_{\infty}} \right| \leq \frac{1}{(|a|; q)_{\infty}},
$$

using Lebesgue’s dominated convergence theorem gives

$$(4-1) \quad \lim_{n \to \infty} E\left( \frac{\eta^n}{(a\xi \eta; q)_{\infty}} \right) = E\left( \frac{I_{(\eta=1)}}{(a\xi; q)_{\infty}} \right).$$

Observe that

$$
E\left( \frac{\eta^n}{(a\xi \eta; q)_{\infty}} \right) = E\left( \frac{\eta^n}{(a\xi \eta; q)_{\infty}} | \xi \right) \phi^{(a\xi)}(x | q)
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} y^k \cdot \frac{1}{(a\xi q^k, ay \xi; q)_{\infty}}
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} y^k \cdot \frac{(a^2xyq^k; q)_{\infty}}{(aq^k, axq^k, ay, axy; q)_{\infty}}
$$

$$
= \frac{(a^2xy; q)_{\infty}}{(a, ax, ay, axy; q)_{\infty}} \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(a, ax; q)_{\infty}y^k}{(a^2xy; q)_{\infty}}.
$$

Hence, we get the left hand side of (4-1):

$$(4-2) \quad \lim_{n \to \infty} E\left( \frac{\eta^n}{(a\xi \eta; q)_{\infty}} \right) = \frac{(a^2xy; q)_{\infty}}{(a, ax, ay, axy; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a, ax; q)_{\infty}y^k}{(q, a^2xy; q)_{\infty}}.
$$

On the other hand, the right hand side of (4-1) equals

$$(4-3) \quad E\left( \frac{I_{(\eta=1)}}{(a\xi; q)_{\infty}} \right) = p(\eta = 1)E\left( \frac{1}{(a\xi; q)_{\infty}} \right) = \frac{1}{(a, ax, y; q)_{\infty}}.
$$

Substituting (4-2) and (4-3) into (4-1) gives

$$
\sum_{k=0}^{\infty} \frac{(a, ax; q)_{\infty}y^k}{(q, a^2xy; q)_{\infty}} = \frac{(ay, axy; q)_{\infty}}{(a^2xy, y; q)_{\infty}},
$$

which is equivalent to the $q$-Gauss theorem, Theorem 4.1, by analytic continuation.
5. A multiple identity

Multiple basic hypergeometric series have been investigated by various authors [Milne 1997; Wang 2009a; Zhang 2006; Zhang and Liu 2006]. We will use the distribution \( W(x; q) \) to prove the following multiple identity.

**Theorem 5.1.** Let \(|a| < 1\). Then for any positive integers \( m \) and \( n \), we have

\[
(5-1) \quad \sum_{y_1 + \cdots + y_m \geq n} \left[ \frac{y_1 + \cdots + y_m}{n} \right] q^{y_2 + 2y_3 + \cdots + (m-1)y_m} a^{y_1 + \cdots + y_m} = \frac{a^n}{(a; q)_{n+m}} \left[ \frac{n+m-1}{n} \right].
\]

**Proof.** Let \( \xi \) denote a random variable with distribution \( W(x; q) \), where \(-1 < x < 0\). For any positive integer \( m \), we consider the sequence

\[
\left\{ \frac{(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)}{(a\xi; q)_\infty} \right\}_{n=1}^{\infty}
\]

for \(|a| < 1\)

of random variables (on a probability space). It is easy to see that

\[
\lim_{n \to \infty} \frac{(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)}{(a\xi; q)_\infty} = \frac{1}{(a\xi; q)_\infty}.
\]

Since \(|(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)/(a\xi; q)_\infty| \leq 1/(|a|; q)_\infty\), using Lebesgue’s dominated convergence theorem gives

\[
(5-2) \quad \lim_{n \to \infty} E\left( \frac{(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)}{(a\xi; q)_\infty} \right) = E\left( \frac{1}{(a\xi; q)_\infty} \right).
\]

Employing (2-6), we get the right hand side of (5-2):

\[
(5-3) \quad E\left( \frac{1}{(a\xi; q)_\infty} \right) = \frac{1}{(a, ax; q)_\infty}.
\]

On the other hand, observing that

\[
(1 - (a\xi)^n)(1 - (aq\xi)^n) \cdots (1 - (aq^{m-1}\xi)^n)
\]

\[
= \frac{(a\xi; q)_\infty}{1 - (a\xi)^n} \cdot \frac{1 - (aq\xi)^n}{1 - aq\xi} \cdots \frac{1 - (aq^{m-1}\xi)^n}{1 - aq^{m-1}\xi} \cdot \frac{1}{(aq^m\xi; q)_\infty}
\]

\[
= \sum_{y_1=0}^{\infty} (a\xi)^{y_1} \cdot \sum_{y_2=0}^{\infty} (aq\xi)^{y_2} \cdots \sum_{y_m=0}^{\infty} (aq^{m-1}\xi)^{y_m} \cdot \frac{1}{(aq^m\xi; q)_\infty}
\]

\[
= \sum_{0 \leq y_1, \ldots, y_m \leq n-1} q^{y_2 + 2y_3 + \cdots + (m-1)y_m} a^{y_1 + \cdots + y_m} \cdot \frac{\xi^{y_1 + \cdots + y_m}}{(aq^m\xi; q)_\infty},
\]
we have

\[
E \left( \frac{[1 - (a \xi)^n][1 - (aq \xi)^n] \cdots [1 - (aq^{m-1} \xi)^n]}{(a \xi; q)_{\infty}} \right)
\]

\[
= \sum_{0 \leq y_1, \ldots, y_m \leq n-1} q^{y_2 + 2y_3 + \cdots + (m-1)y_m} a^{y_1 + \cdots + y_m} E \left( \frac{\xi^{y_1 + \cdots + y_m}}{(aq^{m} \xi; q)_{\infty}} \right)
\]

\[
= \frac{1}{(aq^m, ax q^m; q)_{\infty}} \times \sum_{0 \leq y_1, \ldots, y_m \leq n-1} q^{y_2 + 2y_3 + \cdots + (m-1)y_m} a^{y_1 + \cdots + y_m} \varphi_{y_1 + \cdots + y_m}^{(aq^m)}(x | q).
\]

Hence, we get the left hand side of (5-2):

\[
(5-4) \quad \lim_{n \to \infty} E \left( \frac{[1 - (a \xi)^n][1 - (aq \xi)^n] \cdots [1 - (aq^{m-1} \xi)^n]}{(a \xi; q)_{\infty}} \right)
\]

\[
= \frac{1}{(aq^m, ax q^m; q)_{\infty}} \times \sum_{y_1, \ldots, y_m \geq 0} q^{y_2 + 2y_3 + \cdots + (m-1)y_m} a^{y_1 + \cdots + y_m} \varphi_{y_1 + \cdots + y_m}^{(aq^m)}(x | q).
\]

Substituting (5-3) and (5-4) into (5-2) gives

\[
(5-5) \quad \sum_{y_1, \ldots, y_m \geq 0} q^{y_2 + 2y_3 + \cdots + (m-1)y_m} a^{y_1 + \cdots + y_m} \varphi_{y_1 + \cdots + y_m}^{(aq^m)}(x | q) = \frac{1}{(a, ax; q)_m}.
\]

Using Theorem 3.1 with \( a = q^m \) and \( x = ax \) gives

\[
(5-6) \quad \sum_{k=0}^{\infty} \binom{m+k-1}{k} a^k x^k = \frac{1}{(ax; q)_m}.
\]

Substituting (5-6) into (5-5) and comparing the coefficients of \( x^n \) gives (5-1). \( \square \)

6. An extension of the Rogers–Ramanujan identities

The well-known Rogers–Ramanujan identities are

\[
(6-1) \quad \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q; q)_m} = \frac{1}{(q, q^4; q^5)_{\infty}},
\]

\[
(6-2) \quad \sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q; q)_m} = \frac{1}{(q^2, q^3; q^5)_{\infty}}.
\]

There are many proofs of this beautiful pair of identities. Baxter’s [1982] is based on the statistical mechanics, and the proof of Lepowsky and Milne [1978]
uses the character formula on an infinite dimensional Lie algebra. We use our probability distribution to derive an extension of the Rogers–Ramanujan identities.

**Theorem 6.1.** We have

\[
\sum_{m=n}^{\infty} \frac{q^{m^2}}{(q; q)_m} = \frac{1}{(q; q)_{\infty}} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2} + 2k}}{(q; q)_k} \cdot \frac{a^{2k}}{(aq^k; q)_{\infty}}
\]

\[
- \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2} + 4k}}{(q; q)_k} \cdot \frac{\xi^{2k+1}}{(\xi q^k; q)_{\infty}}
\]

for \(|a| \leq 1\).

Then letting \(a = \zeta\) gives

\[
\sum_{m=0}^{\infty} \frac{q^{m^2} \zeta^m}{(q; q)_m} = \frac{1}{(\zeta q; q)_{\infty}} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2} + 2k}}{(q; q)_k} \cdot \frac{\zeta^{2k}}{(\zeta q^k; q)_{\infty}}
\]

\[- \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2} + 4k}}{(q; q)_k} \cdot \frac{\zeta^{2k+1}}{(\zeta q^k; q)_{\infty}}.\]

where \(\zeta\) is a random variable with distribution \(W(x; q)\) and \(-1 < x < 0\). Applying the expectation operator \(E\) to both sides of the above, we get

\[
(6-3) \quad E\left(\sum_{m=0}^{\infty} \frac{q^{m^2} \zeta^m}{(q; q)_m}\right) = E\left(\frac{1}{(\zeta q; q)_{\infty}}\right) + \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2} + 2k}}{(q; q)_k} \cdot \frac{\zeta^{2k}}{(\zeta q^k; q)_{\infty}}
\]

\[
- \sum_{k=1}^{\infty} \frac{(-1)^k q^{\binom{k}{2} + 4k}}{(q; q)_k} \cdot \frac{\zeta^{2k+1}}{(\zeta q^k; q)_{\infty}}.
\]
Since $|q^{m^2} z^m / (q; q)_m| \leq q^{m^2} / (q; q)_m$ and the series $\sum_{m=0}^{\infty} q^{m^2} / (q; q)_m$ converges absolutely, using Lebesgue’s dominated convergence theorem and (2-6) gives the left hand side of (6-3):

\begin{equation}
E\left(\sum_{m=0}^{\infty} \frac{q^{m^2} z_m}{(q; q)_m}\right) = \sum_{m=0}^{\infty} \frac{q^{m^2} E\{z_m\}}{(q; q)_m} = \sum_{m=0}^{\infty} \frac{q^{m^2} h_m(x | q)}{(q; q)_m}.
\end{equation}

On the other hand, using (2-6) gives

\begin{align}
E\left(\frac{1}{(\zeta q; q)_\infty}\right) &= \frac{1}{(q, qx; q)_\infty}, \\
E\left(\frac{\zeta^{2k}}{(\zeta q^k; q)_\infty}\right) &= \frac{1}{(q^k, q^kx; q)_\infty} \phi_{2k}^{(q^k)}(x | q), \\
E\left(\frac{\zeta^{2k+1}}{(\zeta q^{k+1}; q)_\infty}\right) &= \frac{1}{(q^k, q^kx; q)_\infty} \phi_{2k+1}^{(q^k)}(x | q).
\end{align}

It is easy to see that

\begin{equation}
\left|\frac{(-1)^k q^5(\xi)_k + 4k}{(q; q)_k} \cdot \frac{\zeta^{2k+1}}{(\zeta q^k; q)_\infty}\right| \leq \left|\frac{(-1)^k q^5(\xi)_k + 2k}{(q; q)_k} \cdot \frac{\zeta^{2k}}{(\zeta q^k; q)_\infty}\right| \leq \frac{q^5(\xi)_k + 2k}{(q; q)_k(q; q)_\infty},
\end{equation}

and the series $\sum_{k=0}^{\infty} q^5(\xi)_k + 2k / ((q; q)_k(q; q)_\infty)$ is converges absolutely. Using Lebesgue’s dominated convergence theorem and (6-5), (6-6) and (6-7) gives the right hand side of (6-3):

\begin{align}
E\left(\frac{1}{(\zeta q; q)_\infty}\right) + E\left(\sum_{k=1}^{\infty} \frac{(-1)^k q^5(\xi)_k + 2k}{(q; q)_k} \cdot \frac{\zeta^{2k}}{(\zeta q^k; q)_\infty}\right) - E\left(\sum_{k=1}^{\infty} \frac{(-1)^k q^5(\xi)_k + 4k}{(q; q)_k} \cdot \frac{\zeta^{2k+1}}{(\zeta q^k; q)_\infty}\right) \\
= E\left(\frac{1}{(\zeta q; q)_\infty}\right) + \sum_{k=1}^{\infty} \frac{(-1)^k q^5(\xi)_k + 2k}{(q; q)_k} E\left(\frac{\zeta^{2k}}{(\zeta q^k; q)_\infty}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k q^5(\xi)_k + 4k}{(q; q)_k} E\left(\frac{\zeta^{2k+1}}{(\zeta q^k; q)_\infty}\right) \\
= \frac{1}{(q, qx; q)_\infty} + \sum_{k=1}^{\infty} \frac{(-1)^k q^5(\xi)_k + 2k}{(q; q)_k} \frac{1}{(q^k, q^kx; q)_\infty} \phi_{2k}^{(q^k)}(x | q) - \sum_{k=1}^{\infty} \frac{(-1)^k q^5(\xi)_k + 4k}{(q; q)_k} \frac{1}{(q^k, q^kx; q)_\infty} \phi_{2k+1}^{(q^k)}(x | q)
\end{align}
A NEW PROBABILITY DISTRIBUTION WITH APPLICATIONS 253

\[
= \frac{1}{(q, qx; q)_\infty} + \frac{1}{(q, x; q)_\infty} \sum_{k=1}^{\infty} \frac{(-1)^k (x; q)_k}{1 - q^k} q^{5\binom{k}{2} + 2k} (\varphi_{2k}^{(q^k)} (x | q) - q^{-2k} \varphi_{2k+1}^{(q^k)} (x | q)).
\]

Substituting this and (6-4) into (6-3) gives

\[
\sum_{m=0}^{\infty} \frac{q^{m^2} h_m(x | q)}{(q; q)_m} = \frac{1}{(q, qx; q)_\infty} + \frac{1}{(q, x; q)_\infty} \sum_{k=1}^{\infty} \frac{(-1)^k (x; q)_k}{1 - q^k} q^{5\binom{k}{2} + 2k} (\varphi_{2k}^{(q^k)} (x | q) - q^{-2k} \varphi_{2k+1}^{(q^k)} (x | q)),
\]

where \(-1 < x < 0\). By analytic continuation, we may replace the assumption \(-1 < x < 0\) by \(|x| < 1\).

Substituting the expansion

\[
\frac{1}{(z; q)_\infty} = \sum_{l=0}^{\infty} \frac{z^l}{(q; q)_l}
\]

into the last, we have

\[
\sum_{m=0}^{\infty} \frac{q^{m^2} h_m(x | q)}{(q; q)_m} = \frac{1}{(q; q)_\infty} \sum_{l=0}^{\infty} \frac{q^l x^l}{(q; q)_l} + \frac{1}{(q; q)_\infty} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \left( \frac{q^{kl} x^l}{(q; q)_l} \cdot \frac{(-1)^k q^{5\binom{k}{2} + 2k}}{1 - q^k} (\varphi_{2k}^{(q^k)} (x | q) - q^{-2k} \varphi_{2k+1}^{(q^k)} (x | q)) \right).
\]

Comparing the coefficients of \(x^n\) in this identity gives

\[
\sum_{m=n}^{\infty} \frac{q^{m^2} [m]_n}{(q; q)_m} = \frac{q^n}{(q; q)_\infty (q; q)_n} + \frac{1}{(q; q)_\infty} \sum_{k=1}^{n} \sum_{l=0}^{n-k} \frac{(-1)^k (q^k; q)_l}{(1 - q^k) (q; q)_n-l} q^{5\binom{k}{2} + k(n+2-l)} \left( \binom{2k}{l} + q^{2k} \binom{2k+1}{l} \right),
\]

which can be written as Theorem 6.1.

The Rogers–Ramanujan identities are special cases of Theorem 6.1. Letting \(n = 0\) and then applying the Jacobi triple product identity [Andrews et al. 1999]

\[
\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (q, x, q/x; q)_\infty
\]
leads to the Rogers–Ramanujan identity (6-1). In fact, when \( n = 0 \), we have
\[
\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q; q)_m} = \frac{1}{(q; q)_\infty} \left( 1 + \sum_{k=1}^{\infty} (-1)^k (1 + q^k) q^{5\binom{k}{2} + 2k} \right)
\]
\[
= \frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{5\binom{k}{2} + 2k}
\]
\[
= \frac{(q^5, q^2, q^3; q^5)_\infty}{(q; q)_\infty} = \frac{1}{(q, q^4; q^5)_\infty}.
\]
Similarly, the case \( n = 1 \) of Theorem 6.1 results in another identity due to Rogers and Ramanujan:
\[
\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q; q)_m} = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q; q)_m} - \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q; q)_{m-1}}
\]
\[
= \frac{1}{(q; q)_\infty} \left( 1 + \sum_{k=1}^{\infty} (1 - q^{2k+1}) q^{5\binom{k}{2} + 4k} \right)
\]
\[
= \frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{5\binom{k}{2} + 4k}
\]
\[
= \frac{(q, q^4, q^5; q^5)_\infty}{(q; q)_\infty} = \frac{1}{(q^2, q^3; q^5)_\infty}.
\]

References


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