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We propose a family of homological representations of the braid groups on surfaces. This family extends linear representations of the braid groups on a disc, such as the Burau representation and the Lawrence–Krammer–Bigelow representation.

1. Introduction

1A. ***Preliminaries and history.*** Let $\Sigma(g, p)$ be a compact, connected, orientable 2-dimensional manifold of genus g with p boundary components. Set $\Sigma = \Sigma(g, p)$. Let $\{z_1^0, \dots, z_n^0\}$ be a set of n preferred distinct points in Σ for $n \geq 0$, and let $\Sigma_n = \Sigma - \{z_1^0, \dots, z_n^0\}$. We call Σ_n the surface Σ with n punctures.

For integers $n, k \geq 0$, we consider three types of *configuration spaces* as follows: The space of k-tuples of distinct points in Σ_n is denoted by

$$P_{n,k}(\Sigma) = \{(z_1, \dots, z_k) \in \Sigma_n \times \cdots \times \Sigma_n \mid z_i \neq z_j \text{ for } i \neq j\},$$

the space of subsets of k elements in Σ_n is denoted by

$$B_{n,k}(\Sigma) = \{\{z_1, \dots, z_k\} \subset \Sigma_n\},$$

and the space $B_{n;k}(\Sigma)$ of pairs of disjoint subsets of n elements and k elements in Σ is denoted by

$$B_{n;k}(\Sigma) = \{(\{z_1, \dots, z_n\}, \{z_{n+1}, \dots, z_{n+k}\}) \mid z_i \in \Sigma,\ z_i \neq z_j \text{ for } i \neq j\}.$$

It is easy to see that $B_{n,k}(\Sigma) = P_{n,k}(\Sigma)/\mathbf{S}_k$ and $B_{n;k}(\Sigma) = P_{0,n+k}(\Sigma)/\mathbf{S}_n \times \mathbf{S}_k$, where the symmetric group $\mathbf{S}_k$ acts on $P_{n,k}(\Sigma)$ by permuting components of a k-tuple and similarly $\mathbf{S}_n \times \mathbf{S}_k \subset \mathbf{S}_{n+k}$ acts on $P_{0,n+k}(\Sigma)$.

The braid groups on a surface Σ are defined by the fundamental groups of configuration spaces. Choose a basepoint $\{z_{n+1}^0, \dots, z_{n+k}^0\}$ in $\partial\Sigma$ if $\partial\Sigma \neq \varnothing$. If Σ is

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closed, then place it anywhere in Σ_n. The *pure k-braid group on* Σ_n is defined and denoted by

$$\mathbf{P}_{n,k}(\Sigma) = \pi_1(P_{n,k}(\Sigma), (z^0_{n+1}, \dots, z^0_{n+k})).$$

Similarly, the (*full*) *k-braid group on* Σ_n is given by

$$\mathbf{B}_{n,k}(\Sigma) = \pi_1(B_{n,k}(\Sigma), \{z^0_{n+1}, \dots, z^0_{n+k}\}),$$

and the *intertwining* (n, k)*-braid group on* Σ is given by

$$\mathbf{B}_{n;k}(\Sigma) = \pi_1(B_{n;k}(\Sigma), (\{z^0_1, \dots, z^0_n\}, \{z^0_{n+1}, \dots, z^0_{n+k}\})).$$

It is sometimes easier to understand if these groups are regarded as subgroups of $\mathbf{B}_{0,n+k}(\Sigma)$. The intertwining (n, k)-braid group $\mathbf{B}_{n;k}(\Sigma)$ is the preimage of $\mathbf{S}_n \times \mathbf{S}_k$ under the canonical projection: $\mathbf{B}_{0,n+k}(\Sigma) \to \mathbf{S}_{n+k}$. In addition, $\mathbf{B}_{n,k}(\Sigma)$ is the subgroup of $(n+k)$-braids in $\mathbf{B}_{n;k}(\Sigma)$ that become trivial by forgetting the last k strands, and $\mathbf{P}_{n,k}(\Sigma)$ is the subgroup of $(n+k)$-braids in $\mathbf{B}_{n,k}(\Sigma)$ that are pure, that is, the induced permutation is trivial. If the surface Σ is the 2-disc D, we will call the braid groups *classical*. For example, $\mathbf{B}_{0,n}(D)$ denotes the classical n-braid group studied by E. Artin.

In the 60s and 70s, presentations for braid groups on various surfaces were found, on the 2-sphere and the projective plane in [Fadell and van Buskirk 1962; Van Buskirk 1966], on the torus in [Birman 1969], and on all closed surfaces in [Scott 1970]. The study of braid groups on surfaces has been revived recently. González-Meneses [2001] found new presentations of the braid groups on surfaces, and the authors of [Bellingeri 2004; Bellingeri and Godelle 2007] found positive presentations of the braid groups $\mathbf{B}_{n,k}(\Sigma)$ for all surfaces Σ, with or without boundary. Here, we are interested in braid groups on surfaces with nonempty boundary and will use Bellingeri's presentations.

Boundary components of a surface can be traded with punctures when we consider braid groups. Let $\Sigma = \Sigma(g, p)$ and $\Sigma' = \Sigma(g, p+q)$. Then there are continuous maps $i : \Sigma_q \to \Sigma'$ and $j : \Sigma' \to \Sigma_q$ that are homotopy inverses each other. The induced maps $\bar{i} : B_{n+q,k}(\Sigma) \to B_{n,k}(\Sigma')$ and $\bar{j} : B_{n,k}(\Sigma') \to B_{n+q,k}(\Sigma)$ on configuration spaces are also homotopy inverses each other and induce isomorphisms $\bar{i}_*$ and $\bar{j}_*$ on fundamental groups [Bellingeri 2004; Paris and Rolfsen 1999]. Therefore we may assume $\Sigma = \Sigma(g, 1)$ by treating all but one boundary component as a puncture whenever we deal with a surface with nonempty boundary.

We use Bellingeri's presentation [2004] for the braid group $\mathbf{B}_{n,k}(\Sigma(g, 1))$:

- The generators are $\sigma_1, \dots, \sigma_{k-1}, a_1, \dots, a_g, b_1, \dots, b_g, \zeta_1, \dots, \zeta_n$.
- The relations are

(BR$_1$) $[\sigma_i, \sigma_j]$ for $|i-j| \geq 2$;

(BR$_2$) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j| = 1$;

(CR$_1$) $[a_r, \sigma_i]$, $[b_r, \sigma_i]$, $[\zeta_t, \sigma_i]$ for $i > 1$;

(CR$_2$) $[a_r, \sigma_1 a_r \sigma_1]$, $[b_r, \sigma_1 b_r \sigma_1]$, $[\zeta_t, \sigma_1 \zeta_t \sigma_1]$;

(CR$_3$) $[a_r, \sigma_1^{-1} a_s \sigma_1]$, $[a_r, \sigma_1^{-1} b_s \sigma_1]$, $[b_r, \sigma_1^{-1} a_s \sigma_1]$, $[b_r, \sigma_1^{-1} b_s \sigma_1]$ for $r < s$, $[a_r, \sigma_1^{-1} \zeta_u \sigma_1]$, $[b_r, \sigma_1^{-1} \zeta_u \sigma_1]$, $[\zeta_t, \sigma_1^{-1} \zeta_u \sigma_1]$ for $t < u$;

(SCR) $\sigma_1 b_r \sigma_1 a_r \sigma_1 = a_r \sigma_1 b_r$.

The corresponding result for configuration spaces of pure braids by Fadell and Neuwirth can be generalized to show that the projection $B_{n;k}(\Sigma) \to B_{0,n}(\Sigma)$ is a fiber bundle with fiber $B_{n,k}(\Sigma)$. Except for the cases $\Sigma = S^2$ and $\Sigma = \mathbb{R}P^2$, Gonçalves and Guaschi [2003] completely determined when the short exact sequences of braid groups derived from Fadell–Neuwirth fibrations split. In particular, the short exact sequence derived from the fibration above, that is,

$$1 \to \mathbf{B}_{n,k}(\Sigma) \to \mathbf{B}_{n;k}(\Sigma) \to \mathbf{B}_{0,n}(\Sigma) \to 1,$$

always splits for $k \geq 1$ if Σ has nonempty boundaries.

The braid groups are closely related to the mapping class groups. Birman [1974] determined when surface braid group embeds into the corresponding mapping class group. In particular, if $\partial\Sigma$ is nonempty, $\mathbf{B}_{0,n}(\Sigma)$ embeds into the mapping class group on Σ_n and so an n-braid on Σ can be regarded as a homeomorphism of Σ that preserves the set of n punctures.

The classical braid groups have various representations that can be as simple as taking exponent sums or taking induced permutations. The braid action on the punctured disk D_n gives rise to a faithful representation into automorphism groups of free groups, and a characterization of automorphisms coming from braid actions is possible. Each representation serves its own purpose. It is common to try to construct a linear representation to have a better understanding of a given group via matrices over a certain commutative ring and their multiplications.

For the classical braid groups, linear representations are abundant. Burau in 1936 and Gassner in 1961 found linear representations of $\mathbf{B}_{0,n}(D)$ and $\mathbf{P}_{0,n}(D)$, respectively. These representations are derived from braid actions on homologies of appropriate coverings of D_n. These representations take the form of $(n-1)\times(n-1)$ matrices that can also be computed via Fox's free differential calculus on automorphisms of free groups mentioned above. The Burau representation is faithful for $n \leq 3$ but not for $n \geq 5$ [Bigelow 1999]. The faithfulness of the Gassner representation is known only for $n \leq 3$.

Lawrence [1990] discovered a family of linear representations of $\mathbf{B}_{0,n}(D)$ via a monodromy on a vector bundle over $P_{n,k}(D)$. Krammer [2000] defined a free $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module V using *forks* and relations between them, and he proved using an algebraic and combinatorial argument that the braid group acts on V faithfully for braid index 4. This representation is essentially the same as the one considered by Lawrence for $k = 2$ but uses the configuration space $B_{n,2}(D)$ instead of $P_{n,2}(D)$ and is now called the Lawrence–Krammer–Bigelow representation. Bigelow reinterpreted this representation using covering spaces and covering transformation groups instead of vector bundles and local coefficients. Then the monodromy corresponds to the braid action on homology groups of covering spaces, as it did for the Burau representation and the Gassner representation. Bigelow [2001] constructed a linear representation using homology group $H_2(\tilde{B}_{n,2}(D))$ of the covering space $\tilde{B}_{n,2}(D)$ whose covering transformation group is $\langle q\rangle \oplus \langle t\rangle$, and he proved that $\mathbb{R}\otimes V$ is isomorphic to $\mathbb{R}\otimes H_2(\tilde{B}_{n,2}(D))$. Also, Krammer [2002] and Bigelow [2001] independently proved that the Lawrence–Krammer–Bigelow representation is faithful for all $n \geq 1$, and so the classical braid groups are linear. Bardakov [2005] applied this linearity to show that the braid groups of the sphere and projective plane are linear. Bigelow and Budney [2001] proved using the Lawrence–Krammer–Bigelow representation and a suitable branched covering that the mapping class group of genus 2 surface has a faithful linear representation. However, Paoluzzi and Paris showed that there is a difference between V and $H_2(\tilde{B}_{n,2})(D)$ as a $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module for $n \geq 3$ and a found basis for a $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module $H_2(\tilde{B}_{n,2}(D))$; so the exact definition of "Lawrence–Krammer–Bigelow representation" became somewhat ambiguous.

For any $k \geq 1$, Bigelow [2004] considered the braid action on the Borel–Moore homology group $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$. He obtained a family of representations via the induced action on the image of $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$ in $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D), \partial\tilde{B}_{n,k}(D))$. For simplicity, we will consider the braid action on the free module $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$, whose basis can be easily described by forks to obtain a linear representation. We will call these representations *homology linear representations*. The Burau representation, and the Lawrence–Krammer–Bigelow representation of $\mathbf{B}_{0,n}(D)$ are homology linear representations

$$\Phi_k : \mathbf{B}_{0,n}(D) \to \mathrm{GL}\left(\binom{n+k-2}{k}, \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]\right)$$

obtained from the braid action on homologies of covers of $B_{n,k}(D)$ when $k = 1$ ($t = 1$ in this case) and $k = 2$, respectively [Bigelow 2004]. For $k \geq 3$, Zheng [2005] proved that Φ_k is faithful for all $n \geq 1$.

Overview. We construct a family of homological representations of braid groups on a surface with nonempty boundary; these extend the homological linear representations of the classical braid group. In Section 2, we first try to follow how the homology linear representations of $\mathbf{B}_{0,n}(D)$ were constructed via a covering of the configuration space $B_{n,k}(D)$. In the case of the disk, the braid action automatically commutes with covering transformations, or in other words, braids act trivially on local coefficients. However, in the case of surfaces of genus ≥ 1, this condition forces the variable q to equal 1; see Lemma 2.6. Then the braid action becomes almost trivial. For example, if $k = 1$, the action of σ_i^2 is trivial. To get around this problem, we introduce in Section 3 the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$ to replace $\mathbf{B}_{n,k}(\Sigma)$. As we mentioned earlier, this group is a semidirect product of $\mathbf{B}_{n,k}(\Sigma)$ and $\mathbf{B}_{0,n}(\Sigma)$. Although the braid action does not preserve the local coefficient given by the $\mathbf{B}_{n,k}(\Sigma)$ factor, the $\mathbf{B}_{0,n}(\Sigma)$ factor of $\mathbf{B}_{n;k}(\Sigma)$ can adjust the coefficient so that the braid action becomes compatible. We will extend the coefficient ring for homology representations to give room to control the braid action, at the expense of giving up its commutativity, so that it becomes more interesting and still preserves coefficients. Eventually we obtain in Theorem 3.2 representations of braid groups on surfaces that extend homology linear representations of the classical braid group. Also we explicitly compute the representations in the form of matrices using a geometric argument. We extend the intersection pairing between $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$ and its dual space $H_k(\tilde{B}_{n,k}(D), \partial\tilde{B}_{n,k}(D))$ and use bases for the two spaces that are described by *forks* and *noodles*; see Theorem 3.4.

In Section 4, we argue that the construction of our representations is natural and that there are no other alternatives if one wants to obtain an extension of the homological representation using covers of the configuration space $B_{n,k}(D)$. We show that the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$ is the normalizer of $\mathbf{B}_{n,k}(\Sigma)$ in $\mathbf{B}_{0,n+k}(\Sigma)$ so that the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$ is such a group that is unique and maximal up to a meaningless extension; see Theorem 4.2. The coefficient ring for our representations is the integral group ring of a quotient group of $\mathbf{B}_{n;k}(\Sigma)$. Theorem 4.3 shows that for $k \geq 3$ the quotient group is uniquely determined if one wants to extend homology linear representations of the classical braid group. For $k = 1, 2$, the quotient group is the simplest that serves our purpose. Theorem 4.4 shows that the braid action on the quotient group is virtually unique.

Our construction involving the group extension $\mathbf{B}_{n;k}(\Sigma)$ of $\mathbf{B}_{n,k}(\Sigma)$ is purely algebraic, without a good geometric interpretation. Thus, some useful geometric tools are not available. For example, the intersection pairing mentioned above is not invariant under the braid group action. This seems to make it difficult to discuss properties of our representations such as faithfulness and irreducibility. Although the corresponding representation of the classical braid group is faithful for $k = 2$ and irreducible for $k \leq 2$ [Jones 1987; Zinno 2001], the faithfulness and irreducibility of our representations are beyond the scope of this article.

2. Homology linear representations

We first review the construction of homology linear representations of the classical braid group $\mathbf{B}_{0,n}(D)$ using the configuration space $B_{n,k}(D)$; we then discuss the difficulty in extending these homology linear representations to the braid group $\mathbf{B}_{0,n}(\Sigma)$ on a surface Σ with nonempty boundary. As we noted earlier, boundary components can be traded with punctures. From now on, we assume that Σ denotes a compact, connected, oriented surface with exactly one boundary component and that n and k are positive integers.

Homology linear representations of classical braid group. Let $\phi : \mathbf{B}_{n,k}(D) \to G$ be an epimorphism onto a group G. Consider the covering $p : \tilde{B}_{n,k}(D) \to B_{n,k}(D)$ corresponding to $\operatorname{Ker}\phi$. Since the classical braid group embeds into the mapping class group of the punctured disk D_n, we may assume we have a homeomorphism $\bar{\beta} : B_{n,k}(D) \to B_{n,k}(D)$ for each $\beta \in \mathbf{B}_{0,n}$. By the lifting criteria, $\bar{\beta}$ lifts to $\tilde{\beta} : \tilde{B}_{n,k}(D) \to \tilde{B}_{n,k}(D)$ if and only if $\bar{\beta}_*(\operatorname{Ker}\phi) \subset \operatorname{Ker}\phi$. Equivalently, there is an induced automorphism $\beta_\sharp$ on G such that $\beta_\sharp\phi = \phi\bar{\beta}_*$.

Now we consider *Borel–Moore homology* [Borel and Moore 1960; Hughes and Ranicki 1996] defined by

$$H_\ell^{\mathrm{BM}}(\tilde{B}_{n,k}(D)) = \varprojlim H_\ell(\tilde{B}_{n,k}(D), p^{-1}(B_{n,k}(D) \setminus A)),$$

where the inverse limit is taken over all compact subsets A of $B_{n,k}(D)$.

The middle-dimensional homology group $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$ is a free $\mathbb{Z}[G]$-module of rank $\binom{n+k-2}{k}$ (see [Bigelow 2004]) and $\tilde{\beta}$ induces a map $\tilde{\beta}_* : H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D)) \to H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$ such that

$$\tilde{\beta}_*(yc) = \beta_\sharp(y)\tilde{\beta}_*(c) \quad \text{for } y \in G \text{ and } c \in H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D)).$$

Thus the map $\tilde{\beta}_*$ is a $\mathbb{Z}[G]$-module homomorphism if and only if $\beta_\sharp(y) = y$ for all $y \in G$ if and only if

$$(*) \qquad \phi = \phi\bar{\beta}_* \quad \text{for all } \beta \in \mathbf{B}_{0,n}.$$

Notice that the condition $(*)$ also implies $\bar{\beta}_*(\operatorname{Ker}\phi) \subset \operatorname{Ker}\phi$. Here we need to know that the induced homomorphism $\tilde{\beta}_*$ depends only on the isotopy class of the homeomorphism β. In fact, since D has a boundary, we choose the basepoint $\{z_{n+1}^0, \dots, z_{n+k}^0\}$ of $B_{n,k}(D)$ in ∂D, and then the isotopy preserves the basepoint and gives the same induced map $\tilde{\beta}_*$. Consequently, if we choose a group G and an epimorphism $\phi : \mathbf{B}_{n,k}(D) \to G$ satisfying $(*)$, we obtain a family of representations Φ_k from $\mathbf{B}_{0,n}(D)$ to $\operatorname{Aut}_{\mathbb{Z}[G]}(H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D)))$, the group of $\mathbb{Z}[G]$-module automorphisms on $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$; the Φ_k are defined by

$$\Phi_k(\beta) = \tilde{\beta}_* : H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D)) \to H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D)).$$

Because we want to get a linear representation, G should be abelian. By the presentation given in Section 1A, $\mathbf{B}_{n,k}(D)$ is generated by $\zeta_1, \dots, \zeta_n, \sigma_1, \dots, \sigma_{k-1}$.

Suppose that $\phi : \mathbf{B}_{n,k}(D) \to G$ is an epimorphism such that $(*)$ holds and G is abelian. Each generator σ_i of $\mathbf{B}_{0,n}(D)$ acts trivially on $\mathbf{B}_{n,k}(D)$ except for

$$(\bar{\sigma}_i)_*(\zeta_i) = \zeta_i \zeta_{i+1} \zeta_i^{-1} \quad \text{and} \quad (\bar{\sigma}_i)_*(\zeta_{i+1}) = \zeta_i.$$

Then the condition $(*)$ implies that $\phi(\zeta_i) = \phi((\bar{\sigma}_i)_*(\zeta_{i+1})) = \phi(\zeta_{i+1})$. Hence for $k = 1$, G is a quotient of $\langle q \rangle$, and $\phi(\zeta_i) = q$ for $i = 1, \ldots, n$. For $k \geq 2$, G is a quotient of $\langle q \rangle \oplus \langle t \rangle$, and $\phi(\zeta_i) = q$ and $\phi(\sigma_j) = t$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k-1$.

We define a group G_D and an epimorphism $\phi_D : \mathbf{B}_{n,k}(D) \to G_D$ depending only on k as follows:

$$\phi_D : \mathbf{B}_{n,k}(D) \to G_D = \begin{cases} \langle q \rangle & \text{if } k = 1, \\ \langle q \rangle \oplus \langle t \rangle & \text{if } k \geq 2. \end{cases}$$

Theorem 2.1 [Bigelow 2004; Lawrence 1990]. *Let $\phi_D : \mathbf{B}_{n,k}(D) \to G_D$ be the epimorphism defined above. Then there is a homomorphism*

$$\Phi_k : \mathbf{B}_{0,n}(D) \to \operatorname*{Aut}_{\mathbb{Z}[G_D]} (H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))).$$

In fact, Φ_1 is the Burau representation and Φ_2 is the Lawrence–Krammer–Bigelow representation.

Naive extension to braid groups on surfaces. Let Σ be a surface of genus $g \geq 1$ having one boundary component. The assumption $\partial\Sigma \neq \varnothing$ is necessary for another reason besides the two mentioned at the end of Section 1A. Suppose that $\partial\Sigma = \varnothing$ and $\beta \in \mathbf{B}_{0,n}(\Sigma)$ uniquely determines the isotopy class of a homeomorphism $\bar{\beta} : B_{n,k}(\Sigma) \to B_{n,k}(\Sigma)$. Then we must choose the basepoint $\{z_{n+1}^0, \ldots, z_{n+k}^0\}$ in the interior of Σ. We can easily find a homeomorphism $\bar{\beta} : B_{n,k}(\Sigma) \to B_{n,k}(\Sigma)$ that is isotopic to the identity via an isotopy that does not preserve the basepoint. Then β represents the identity element in $\mathbf{B}_{0,n}(\Sigma)$ but $\tilde{\beta}_* : H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma)) \to H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ may be nontrivial. Thus no representation can be obtained in this way if $\partial\Sigma = \varnothing$.

We need to define what it means to say that a representation of the braid group $\mathbf{B}_{0,n}(\Sigma)$ extends homology linear representations of the classical braid groups.

Definition 2.2. Given a ring R, let M be an R-module on which the braid group $\mathbf{B}_{0,n}(\Sigma)$ acts as R-module isomorphisms. The R-module M is an *extension* of homology linear representations of the classical braid groups $\mathbf{B}_{0,n}(D)$ if there exists a $\mathbb{Z}[G_D]$-submodule M' of M such that

(i) M' is invariant under the action by the subgroup $\mathbf{B}_{0,n}(D)$ of $\mathbf{B}_{0,n}(\Sigma)$; and

(ii) for some $k \geq 1$, R contains $\mathbb{Z}[G_D]$ as a subring and there is a $\mathbb{Z}[G_D]$-isomorphism from $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$ to M' that commutes with the $\mathbf{B}_{0,n}(D)$ action.

As in the classical braid cases, we have to look at the action of $\mathbf{B}_{0,n}(\Sigma)$ on $\mathbf{B}_{n,k}(\Sigma)$. The following lemma helps us to observe the action we want.

Lemma 2.3 [Birman 1974; Fadell and Neuwirth 1962; Gonçalves and Guaschi 2003]. *Let $\pi_n : B_{n;k}(\Sigma) \to B_{0,n}(\Sigma)$ be the projection onto the first n coordinates. Then the space $B_{n;k}(\Sigma)$ is a fiber bundle with fiber $B_{n,k}(\Sigma)$, and the induced short exact sequence*

$$1 \to \mathbf{B}_{n,k}(\Sigma) \longrightarrow \mathbf{B}_{n;k}(\Sigma) \xrightarrow{(\pi_n)_*} \mathbf{B}_{0,n}(\Sigma) \to 1$$

splits for all $k \geq 1$.

This lemma shows us how to decompose a braid $\beta \in \mathbf{B}_{n;k}(\Sigma)$ into a product $\beta = \beta_1\beta_2$ for $\beta_1 \in \mathbf{B}_{0,n}(\Sigma)$ and $\beta_2 \in \mathbf{B}_{n,k}(\Sigma)$. Let $\imath : \mathbf{B}_{0,n}(\Sigma) \to \mathbf{B}_{n;k}(\Sigma)$ be the splitting map. Then the lemma shows that $\mathbf{B}_{n;k}(\Sigma)$ can be generated by the sets

$$X_1 = \{\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}, \bar{\mu}_1, \dots, \bar{\mu}_g, \bar{\lambda}_1, \dots, \bar{\lambda}_g\},$$
$$X_2 = \{\sigma_1, \dots, \sigma_{k-1}, \zeta_1, \dots, \zeta_n, \mu_1, \dots, \mu_g, \lambda_1, \dots, \lambda_g\},$$

where the generators in X_1 are the images of generators in $\mathbf{B}_{0,n}(\Sigma)$ under the inclusion map $\imath$.

Then the action of $\mathbf{B}_{0,n}(\Sigma)$ on $\mathbf{B}_{n,k}(\Sigma)$ is equivalent to the conjugate action in $\mathbf{B}_{n;k}(\Sigma)$ if we regard these two groups as subgroups of $\mathbf{B}_{n;k}(\Sigma)$. The following easy lemma shows how $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n,k}(\Sigma)$.

Lemma 2.4. *Each generator of $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n,k}(\Sigma)$ as follows.*

(1) *For $1 \leq i \leq n-1$,*

$$(\bar{\sigma}_i)_*(\zeta_t) = \begin{cases} \zeta_i\zeta_{i+1}\zeta_i^{-1} & \text{if } t = i, \\ \zeta_i & \text{if } t = i+1. \end{cases}$$

(2) *For $1 \leq r \leq g$,*

$$(\bar{\mu}_r)_*(\zeta_1) = \mu_r\zeta_1\mu_r^{-1},$$
$$(\bar{\mu}_r)_*(\mu_s) = \begin{cases} \mu_r\zeta_1\mu_r\zeta_1^{-1}\mu_r^{-1} & \text{if } s = r, \\ [\mu_r, \zeta_1]\mu_s[\mu_r, \zeta_1]^{-1} & \text{if } r < s, \end{cases}$$
$$(\bar{\mu}_r)_*(\lambda_s) = \begin{cases} \lambda_r\mu_r\zeta_1^{-1}\mu_r^{-1} & \text{if } s = r, \\ [\mu_r, \zeta_1]\lambda_s[\mu_r, \zeta_1]^{-1} & \text{if } r < s. \end{cases}$$

(3) *For $1 \leq r \leq g$,*

$$(\bar{\lambda}_r)_*(\zeta_1) = \lambda_r\zeta_1\lambda_r^{-1},$$
$$(\bar{\lambda}_r)_*(\mu_s) = \begin{cases} \lambda_r\zeta_1\lambda_r^{-1}\mu_r\zeta_1\lambda_r\zeta_1^{-1}\lambda_r^{-1} & \text{if } s = r, \\ [\lambda_r, \zeta_1]\,\mu_s\,[\lambda_r, \zeta_1]^{-1} & \text{if } r < s, \end{cases}$$
$$(\bar{\lambda}_r)_*(\lambda_s) = \begin{cases} \lambda_r\zeta_1\lambda_r\zeta_1^{-1}\lambda_r^{-1} & \text{if } s = r, \\ [\lambda_r, \zeta_1]\,\lambda_s\,[\lambda_r, \zeta_1]^{-1} & \text{if } r < s. \end{cases}$$

(4) *All other generators act trivially.*

We can find the presentation for $\mathbf{B}_{n;k}(\Sigma)$ using this lemma as follows.

Lemma 2.5. *The braid group $\mathbf{B}_{n;k}(\Sigma)$ admits the presentation in which*

- *the generators are*

$$X_1 = \{\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}, \bar{\mu}_1, \dots, \bar{\mu}_g, \bar{\lambda}_1, \dots, \bar{\lambda}_g\},$$
$$X_2 = \{\sigma_1, \dots, \sigma_{k-1}, \zeta_1, \dots, \zeta_n, \mu_1, \dots, \mu_g, \lambda_1, \dots, \lambda_g\};$$

- *the relations are*
  (i) (BR_1) *through* (SCR) *among generators in* X_1,
  (ii) (BR_1) *through* (SCR) *among generators in* X_2, *and*
  (iii) $\bar{x}^{-1} y \bar{x} = (\bar{x}_*)(y)$ *for all* $\bar{x} \in X_1$ *and* $y \in X_2$,

where the action by $\bar{x}_*$ is given in Lemma 2.4.

Proof. By Lemma 2.3, the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$ is a semidirect product of the normal subgroup $\mathbf{B}_{n,k}(\Sigma)$ and $\mathbf{B}_{0,n}(\Sigma)$, where $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n,k}(\Sigma)$ by conjugation as shown in Lemma 2.4. Then it is easy to show that the semidirect product $\mathbf{B}_{n;k}(\Sigma)$ admits the desired presentation. □

For surfaces, the condition $(*)$ implies an undesirable consequence:

Lemma 2.6. *Let $\phi : \mathbf{B}_{n,k}(\Sigma) \to G$ be an epimorphism satisfying $\phi = \phi\bar{\beta}_*$ for any $\beta \in \mathbf{B}_{n,k}(\Sigma)$. Then $\phi(\zeta_i) = 1$ for $i = 1, \dots, n$.*

Proof. As seen earlier, the hypothesis on ϕ implies that $(\mu_r)_\sharp(y) = y$ for all $y \in G$ and $r = 1, \dots, g$. But by Lemma 2.4(2), we have

$$\begin{aligned}(\mu_r)_\sharp \phi(\lambda_r) &= \phi((\bar{\mu}_r)_*(\lambda_r)) = \phi(\lambda_r \mu_r \zeta_1^{-1} \mu_r^{-1}) \\ &= \phi(\lambda_r)\phi((\bar{\mu}_r)_*(\zeta_1^{-1})) = \phi(\lambda_r)(\mu_r)_\sharp(\phi(\zeta_1^{-1})) = \phi(\lambda_r)\phi(\zeta_1^{-1}).\end{aligned}$$

Since $(\mu_r)_\sharp(\phi(\lambda_r)) = \phi(\lambda_r)$ by hypothesis, $\phi(\zeta_1) = 1$ and so $\phi(\zeta_i) = 1$ for all $1 \le i \le n$ by Lemma 2.4(1). □

This lemma says that the condition $(*)$ forces us to set $q = 1$ in the group G_D. Thus $\mathbb{Z}[G_D]$ cannot be a subring of $\mathbb{Z}[G]$, and so a naive attempt to obtain a representation of the braid group $\mathbf{B}_{0,n}(\Sigma)$ using a covering of $B_{n,k}(\Sigma)$ corresponding to any epimorphism $\phi : \mathbf{B}_{n,k}(\Sigma) \to G$ cannot give an extension of any homology linear representation of the classical braid groups.

3. A family of proposed representations

As we have seen in the previous section, we are forced to take a rather small covering of $B_{n,k}(\Sigma)$ in order that the condition $(*)$ be satisfied, that is, the braid action commutes with covering transformations so that it preserves the coefficient. The remedy we propose in this article is to use the same configuration space $B_{n,k}(\Sigma)$ with an extended coefficient ring so that we have some room to adjust coefficients to make the braid action compatible with the coefficients. This remedy is a reasonable thing to do if we hope to construct an extension of homology linear representations of the classical braid groups. Indeed, we successfully obtain an extension that seems the most general among ones obtained from coverings of $B_{n,k}(\Sigma)$.

3A. ***Existence of extensions of homology linear representations.*** We first consider the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$. Note that $\mathbf{B}_{n;k}(\Sigma)$ is a candidate for group extension of $\mathbf{B}_{n,k}(\Sigma)$ by Lemma 2.3, and $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n;k}(\Sigma)$ by right multiplication and acts on $\mathbf{B}_{n,k}(\Sigma)$ by conjugation because $\mathbf{B}_{n;k}(\Sigma)$ is the semidirect product of $\mathbf{B}_{0,n}(\Sigma)$ and $\mathbf{B}_{n,k}(\Sigma)$.

Let H_Σ be the abstract group depending only on k that admits for $k \geq 2$ the presentation in which

- the generators are $q, t, \overline{m}_1, \dots, \overline{m}_g, \bar{\ell}_1, \dots, \bar{\ell}_g, m_1, \dots, m_g, \ell_1, \dots, \ell_g$;
- the relations are such that all generators commute except that

$$[m_r, \ell_r] = t^2 \quad \text{and} \quad [\overline{m}_r, \ell_r] = [m_r, \bar{\ell}_r] = q.$$

Define ψ_Σ to be the epimorphism from $\mathbf{B}_{n;k}(\Sigma)$ to the group H_Σ such that

$$\begin{aligned} &\psi_\Sigma(\sigma_i) = t, \qquad \psi_\Sigma(\zeta_j) = q, \qquad \psi_\Sigma(\bar{\sigma}_m) = 1, \\ &\psi_\Sigma(\mu_r) = m_r, \quad \psi_\Sigma(\lambda_r) = \ell_r, \quad \psi_\Sigma(\bar{\mu}_r) = \overline{m}_r, \quad \psi_\Sigma(\bar{\lambda}_r) = \bar{\ell}_r, \end{aligned}$$

where $1 \leq i \leq k-1$, $1 \leq j \leq n$, $1 \leq m \leq n-1$ and $1 \leq r \leq g$. If $k = 1$, then we redefine H_Σ to be the quotient of the group above by $t = 1$. Then H_D is isomorphic to G_D defined earlier for all $k \geq 1$, and is a subgroup of H_Σ for any Σ and $k \geq 1$. Even though H_Σ (or H_D) depends on whether $k = 1$ or $k \geq 2$, our notation does not show it for the sake of simplicity.

Let $\phi_\Sigma : \mathbf{B}_{n,k}(\Sigma) \to G_\Sigma$ be the restriction of ψ_Σ to $\mathbf{B}_{n,k}(\Sigma)$ onto G_Σ, the subgroup $\psi_\Sigma(\mathbf{B}_{n,k}(\Sigma))$ of H_Σ. Then G_Σ is generated by

$$\{q, t, m_1, \dots, m_g, \ell_1, \dots, \ell_g\}.$$

Since any two elements of G_Σ commute up to multiplications by central elements q and t, it is a normal subgroup of H_Σ. We can find the covering $p : \tilde{B}_{n,k}(\Sigma) \to B_{n,k}(\Sigma)$ corresponding to $\operatorname{Ker}\phi_\Sigma$. Since the braid group $\mathbf{B}_{0,n}(\Sigma)$ embeds into the mapping class group of punctured surface Σ_n, a braid $\beta \in \mathbf{B}_{0,n}(\Sigma)$ determines a homeomorphism $\bar{\beta} : B_{n,k}(\Sigma) \to B_{n,k}(\Sigma)$. Recall that the induced homomorphism $\bar{\beta}_*$ on $\mathbf{B}_{n,k}(\Sigma)$ is in fact the same as the conjugation by $\imath(\beta)$ where $\imath : \mathbf{B}_{0,n}(\Sigma) \to \mathbf{B}_{n;k}(\Sigma)$ is the splitting map in Lemma 2.3.

Lemma 3.1. *With the notation above, the homeomorphism $\bar{\beta} : B_{n,k}(\Sigma) \to B_{n,k}(\Sigma)$ lifts to a homeomorphism $\tilde{\beta} : \tilde{B}_{n,k}(\Sigma) \to \tilde{B}_{n,k}(\Sigma)$ for any $\beta \in \mathbf{B}_{0,n}(\Sigma)$, and the restriction ϕ_Σ of ψ_Σ satisfies $\beta_\sharp \phi_\Sigma = \phi_\Sigma \bar{\beta}_*$*

Proof. By the lifting criteria, $\bar{\beta}$ lifts to $\tilde{\beta}$ if and only if $\bar{\beta}_*(\operatorname{Ker} \phi_\Sigma) \subset \operatorname{Ker} \phi_\Sigma$ if and only if there is an induced automorphism $\beta_\sharp$ on G_Σ given by $\beta_\sharp \phi_\Sigma = \phi_\Sigma \bar{\beta}_*$. Thus it suffices to show that $\phi_\Sigma \bar{\beta}_*(W) = 1$ for any $W \in \operatorname{Ker} \phi_\Sigma$ and $\beta \in \mathbf{B}_{0,n}(\Sigma)$. Let W be a word in the generators $\{\mu_i, \lambda_i, \sigma_i, \zeta_i\}$ of $\mathbf{B}_{n,k}(\Sigma)$. Since the presentation for H_Σ shows that any two elements are commutative up to multiplications by central elements q and t, we have

$$\phi_\Sigma(W) = W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) = q^c t^d \prod m_i^{a_i} \ell_i^{b_i},$$

where $W(\{x_i \leftarrow y_i\})$ denotes the word obtained from W by replacing the generators x_i by the generators y_i.

Suppose $\phi_\Sigma(W) = 1$. Then $a_i = b_i = 0$ for all $1 \le i \le g$. Thus for generators $\sigma_r, \mu_r, \lambda_r$ for $\mathbf{B}_{0,n}(\Sigma)$, we have

$$\begin{aligned}
\phi_\Sigma((\bar{\sigma}_r)_*(W)) &= \phi_\Sigma(W(\zeta_r \leftarrow \zeta_r\zeta_{r+1}\zeta_r^{-1}, \zeta_{r+1} \leftarrow \zeta_r)) \\
&= W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) = 1, \\
\phi_\Sigma((\bar{\mu}_r)_*(W)) &= \phi_\Sigma(W(\lambda_r \leftarrow \lambda_r\mu_r\zeta_1^{-1}\mu_r^{-1})) \\
&= W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \lambda_r \leftarrow q^{-1}\ell_r, \sigma_i \leftarrow t, \zeta_i \leftarrow q) \\
&= q^{-b_r} W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) = 1, \\
\phi_\Sigma((\bar{\lambda}_r)_*(W)) &= \phi_\Sigma(W(\mu_r \leftarrow \lambda_r\zeta_1\lambda_r^{-1}\mu_r\zeta_1\lambda_r\zeta_1^{-1}\lambda_r^{-1})) \\
&= W(\mu_i \leftarrow m_i, \mu_r \leftarrow qm_r, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) \\
&= q^{a_r} W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) = 1.
\end{aligned}$$

Therefore $\phi_\Sigma(\bar{\beta}_*(W)) = 1$ and so $\beta_\sharp(\phi_\Sigma(\alpha)) = \phi_\Sigma(\bar{\beta}_*(\alpha))$ for all $\alpha \in \mathbf{B}_{n,k}(\Sigma)$. □

By the lemma above, we have a $\mathbb{Z}$-module automorphism $\tilde{\beta}_*$ on $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$. Note that $\tilde{\beta}_*$ is not necessarily a $\mathbb{Z}[G_\Sigma]$-module homomorphism since the condition $(*)$ may not hold, that is, the automorphism $\beta_\sharp$ of G_Σ may not be the identity.

On the other hand, $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n;k}(\Sigma)$ by right multiplication and so there is an induced action of β on H_Σ given by $\beta \cdot h = h\psi_\Sigma(\beta)$ for $h \in H_\Sigma$. It is possible to alter the induced action by multiplying with a certain function χ from $\mathbf{B}_{0,n}(\Sigma)$ to the centralizer of G_Σ in H_Σ. We will discuss this possibility in Theorem 4.4.

Using the $\mathbb{Z}$-module automorphism $\tilde{\beta}_*$ and the action on $\mathbf{B}_{n;k}(\Sigma)$ by $\mathbf{B}_{0,n}(\Sigma)$, we construct a $\mathbb{Z}[H_\Sigma]$-module automorphism $\beta \otimes \tilde{\beta}_*$ on $\mathbb{Z}[H_\Sigma] \otimes_{\mathbb{Z}[G_\Sigma]} H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ by

$$(\beta \otimes \tilde{\beta}_*)(h \otimes c) = (\beta \cdot h) \otimes \tilde{\beta}_*(c) \quad \text{for } h \in H_\Sigma \text{ and } c \in H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma)).$$

Theorem 3.2. *Let Σ be a compact, connected, oriented 2-dimensional manifold with nonempty boundary. Define the group H_Σ (depending on k) and the epimorphism $\psi_\Sigma : \mathbf{B}_{n;k}(\Sigma) \to H_\Sigma$ as above. Let ϕ_Σ be the restriction of ψ_Σ to $\mathbf{B}_{n,k}(\Sigma)$. Set $G_\Sigma = \phi_\Sigma(\mathbf{B}_{n,k}(\Sigma))$. Then there is a homomorphism*

$$\Phi_k : \mathbf{B}_{0,n}(\Sigma) \to \operatorname*{Aut}_{\mathbb{Z}[H_\Sigma]}(\mathbb{Z}[H_\Sigma] \otimes_{\mathbb{Z}[G_\Sigma]} H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))), \quad \beta \mapsto \beta \otimes \tilde{\beta}_*,$$

where the action of β on H_Σ is given by $\beta \cdot h = h\psi(\beta)$ for $h \in H_\Sigma$.

This family Φ_k of representations is an extension of a homology linear representation of the classical braid group $\mathbf{B}_{0,n}(D)$ in the sense of Definition 2.2.

Proof. Clearly Φ_k is a group homomorphism. To see the well-definedness and the $\mathbb{Z}[H_\Sigma]$-linearity of $\Phi_k(\beta)$, we claim that

$$\beta \cdot (hh') = (\beta \cdot h)\beta_\sharp(h') \quad \text{for all } h \in H_\Sigma, h' \in G_\Sigma.$$

Then, for $c \in H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$,

$$\begin{aligned}(\beta \otimes \tilde{\beta}_*)(h \otimes h'c) &= (\beta \cdot h) \otimes \tilde{\beta}_*(h'c) = (\beta \cdot h) \otimes \beta_\sharp(h')\tilde{\beta}_*(c)\\ &= (\beta \cdot h)\beta_\sharp(h') \otimes \tilde{\beta}_*(c) = (\beta \otimes \tilde{\beta}_*)(hh' \otimes c)\\ &= hh'(\beta \otimes \tilde{\beta}_*)(1 \otimes c).\end{aligned}$$

Here the last equality is clear by the definition of the action by β.

To show the claim, choose $\alpha \in \mathbf{B}_{n,k}(\Sigma)$ so that $\phi_\Sigma(\alpha) = h'$. By Lemma 3.1, we have

$$\beta_\sharp(\phi_\Sigma(\alpha)) = \phi_\Sigma(\bar{\beta}_*(\alpha)) = \psi_\Sigma(\bar{\beta}_*(\alpha)) = \psi_\Sigma(\beta^{-1}\alpha\beta) = \psi_\Sigma(\beta)^{-1}\phi_\Sigma(\alpha)\psi_\Sigma(\beta).$$

Thus

$$\beta \cdot (hh') = hh'\psi_\Sigma(\beta) = (h\psi_\Sigma(\beta))(\psi_\Sigma(\beta)^{-1}h'\psi_\Sigma(\beta)) = (\beta \cdot h)\beta_\sharp(h').$$

To show that Φ_k is an extension of a homology linear representation of the classical braid groups, we regard an n punctured disk D_n as a subspace of Σ_n. Then the configuration space $B_{n,k}(D)$ is a subspace of the configuration space $B_{n,k}(\Sigma)$. For the covering $p : \tilde{B}_{n,k}(\Sigma) \to B_{n,k}(\Sigma)$ corresponding to ϕ_Σ, we denote a connected component of $p^{-1}(B_{n,k})$ by $\tilde{B}_{n,k}(D)$. Since G_D embeds into G_Σ, the restriction $p|_{\tilde{B}_{n,k}(D)} : \tilde{B}_{n,k}(D) \to B_{n,k}(D)$ is the covering over $B_{n,k}(D)$ corresponding to $\psi|_{\mathbf{B}_{n,k}(D)} : \mathbf{B}_{n,k}(D) \to G_D$. In fact, $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$ is a submodule of

$H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ as $\mathbb{Z}[G_D]$-module. One can see this more explicitly in the proof of Lemma 3.3. Each braid $\beta \in \mathbf{B}_{0,n}(D)$ gives a $\mathbb{Z}[H_D]$-module automorphism $\beta \otimes \tilde{\beta}_*$ on $\mathbb{Z}[H_D] \otimes_{\mathbb{Z}[G_D]} H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$. Since $G_D = H_D$, this automorphism is the same as $\tilde{\beta}_*$ on $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$, which is the homology linear representation of the classical braid group. □

In Section 4, we will show that if one wants to obtain a result similar to the theorem above, the extension $\mathbf{B}_{n;k}(\Sigma)$ of $\mathbf{B}_{n,k}(\Sigma)$ is determined uniquely up to redundant coefficient extension, and the quotient H_Σ is uniquely determined for $k \geq 3$ and is the simplest for $k \geq 1$ in the sense that any proper quotient of H_Σ does not contain G_D properly.

Computation of the proposed representations. We now compute explicit matrix forms of the representations described in Theorem 3.2; these turn out to be extensions of the Burau and Lawrence–Krammer–Bigelow representations of the classical braid groups. The following lemma and its proof show not only that $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ is a free $\mathbb{Z}[G_\Sigma]$-module but also how to choose a basis. The lemma is an extension of the corresponding lemma on a disk by Bigelow [2004], and we borrow the main idea of his proof.

Lemma 3.3. *The homology group $H_\ell^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ is the direct sum of*

$$\binom{2g+n+k-2}{k}$$

copies of $\mathbb{Z}[G_\Sigma]$ for $\ell = k$ and is trivial otherwise.

Proof. Let d be a metric on Σ that can be either hyperbolic or Euclidean. Suppose punctures $z_1^0, \ldots, z_n^0$ lie on a geodesic. Let γ_j be the geodesic segment joining z_j^0 and z_{j+1}^0 for $1 \leq j \leq n-1$. For $1 \leq i \leq g$, let α_i and β_i be geodesic loops based at z_1^0 that represent the meridian and the longitude of the i-th handle, so that the α_i, β_i, and γ_j are mutually disjoint. Let Γ be the union of all of these arcs, so that Γ_n is a disjoint union of open $2g+n-1$ geodesic segments. Consider

$$B_\Gamma = B_{n,k}(\Gamma) = \{\{z_1, \ldots, z_k\} \subset \Gamma_n\}.$$

Then it is not hard to see B_Γ is homeomorphic to a disjoint union of $\binom{2g+n+k-2}{k}$ open k-balls that can be parametrized by $(2g+n-1)$-tuples $(r_1, \ldots, r_{2g+n-1})$ of nonnegative integers that add up to k so that the i-th segment of Γ_n contains r_i points from $\{z_1, \ldots, z_k\}$.

Let $p : \tilde{B}_{n,k}(\Sigma) \to B_{n,k}(\Sigma)$ be the covering corresponding to the epimorphism $\phi_\Sigma : B_{n,k}(\Sigma) \to G_\Sigma$. We will be done if we can show that the map

$$H_\ell^{\mathrm{BM}}(p^{-1}(B_\Gamma)) \to H_\ell^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$$

induced by the inclusion is an isomorphism, since $H_\ell^{\mathrm{BM}}(B_\Gamma)$ is isomorphic to the direct sum of $\binom{2g+n+k-2}{k}$ copies of $H_\ell(D^k, S^{k-1})$.

Define a family of compact subsets A_ϵ of Σ_n by

$$A_\epsilon = \{\{z_1, \dots, z_k\} \in B_{n,k}(\Sigma) \mid d(z_i, z_j) \geq \epsilon \text{ for } i \neq j,\ d(z_i, z_j^0) \geq \epsilon \text{ for all } i, j\}.$$

Since any compact subset of $B_{n,k}(\Sigma)$ is contained in A_ϵ for sufficiently small $\epsilon > 0$, it suffices to show that

$$H_\ell(p^{-1}(B_\Gamma), p^{-1}(B_\Gamma - A_\epsilon)) \to H_\ell(\tilde{B}_{n,k}(\Sigma), p^{-1}(B_{n,k}(\Sigma) - A_\epsilon))$$

is an isomorphism.

Let $\Sigma_\epsilon \subset \Sigma$ be the closed ϵ-neighborhood of Γ, and let $B_\epsilon = B_{n,k}(\Sigma_\epsilon)$. Then the obvious homotopy collapsing from $B_{n,k}(\Sigma)$ to B_ϵ gives the isomorphism

$$H_\ell(p^{-1}(B_\epsilon), p^{-1}(B_\epsilon - A_\epsilon)) \to H_\ell(\tilde{B}_{n,k}(\Sigma), p^{-1}(B_{n,k}(\Sigma) - A_\epsilon)).$$

Let B be the set of $\{x_1, \dots, x_k\} \in B_\epsilon$ such that for each x_i there exists a unique nearest point in Γ_n. Then B is open and contains $A_\epsilon \cap B_\epsilon$. By excision, the inclusion induces an isomorphism

$$H_\ell(p^{-1}(B), p^{-1}(B - A_\epsilon)) \to H_\ell(p^{-1}(B_\epsilon), p^{-1}(B_\epsilon - A_\epsilon)).$$

Finally, the obvious deformation retract from B to B_Γ gives an isomorphism

$$H_\ell(p^{-1}(B), p^{-1}(B - A_\epsilon)) \to H_\ell(p^{-1}(B_\Gamma), p^{-1}(B_\Gamma - A_\epsilon)). \qquad \square$$

We remark that $B_\epsilon = B_{n,k}(\Sigma_\epsilon)$ and $B_\Gamma = B_{n,k}(\Gamma)$ do not have the same homotopy type even though Γ is a deformation retract of Σ_ϵ. This is because Γ is a 1-dimensional complex and movements of points in Γ avoiding collision are more restricted.

Let $I(n, k, g)$ be the set of $(2g+n-1)$-tuples of nonnegative integers that add up to k. The proof of Lemma 3.3 shows that a typical basis for $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ as free $\mathbb{Z}[G_\Sigma]$-module can be indexed by the set $I(n, k, g)$. The proof also shows that a basis for $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(D))$ can be chosen as a subset of a basis for $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$. Thus the homology linear representations for $\mathbf{B}_{0,n}(D)$ appears in matrix forms of proposed representations for $\mathbf{B}_{0,n}(\Sigma)$ as minors.

Krammer [2000; 2002] and Bigelow [2001; 2003; 2004] have shown that there is a natural and useful way of describing a basis geometrically. Recall the loops α_i and β_i and the arcs γ_j from the proof of Lemma 3.3. For $(r_1, \dots, r_{2g+n-1}) \in I(n, k, g)$, choose r_i disjoint duplicates of α_i or β_{i-g} or γ_{i-2g} if $1 \leq i \leq g$ or $g+1 \leq i \leq 2g$ or $2g+1 \leq i \leq 2g+n-1$, respectively. For each i, join these r_i disjoint duplicates to $\partial\Sigma$ by mutually disjoint arcs (that determine a basing). This geometric object is called a *fork*. In fact, a fork uniquely determines a k-cycle in $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ by lifting the Cartesian product of k curves together with basing

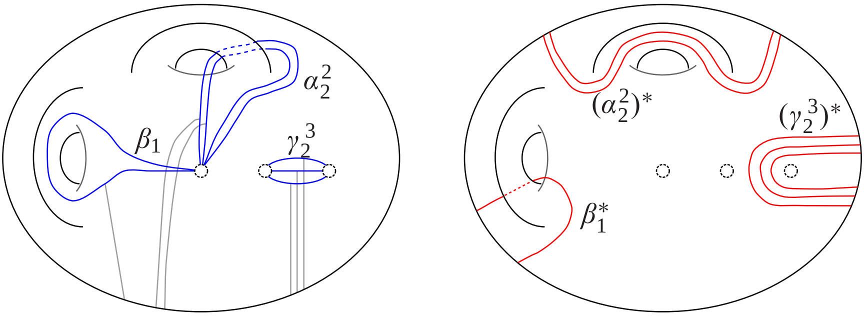

Figure 1. An example of a fork F (left) and its dual noodle N (right).

arcs in the fork. The basing is required to have a unique lift. For example, the fork corresponding to $(0, 2, 1, 0, 0, 3) \in I(3, 6, 2)$ looks like the set of curves on the left of Figure 1.

As Bigelow [2004] showed for the case of the disk, the Poincaré duality, the universal coefficient theorem, and Lemma 3.3 imply that the ordinary relative homology $H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$ is the dual space of the Borel–Moore homology $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ in the sense that there is a nonsingular sesquilinear pairing

$$\langle\,\cdot\,,\cdot\,\rangle : H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma)) \times H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma)) \to \mathbb{Z}[S],$$

where S is a skew field containing $\mathbb{Z}[G_\Sigma]$. In fact, the group G_Σ is biordered, and so it can embed into a skew field such as the Mal'cev–Neumann power series ring [Mal'cev 1948; Neumann 1949]. Explicitly, the pairing above is defined by setting, for cycles $F \in H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ and $N \in H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$ in each homology group,

$$\langle F, N\rangle = \sum_{y \in G_\Sigma} y(F, yN),$$

where $(\,\cdot\,,\cdot\,)$ counts the intersection number.

Let $\alpha_1^*, \ldots, \alpha_g^*, \beta_1^*, \ldots, \beta_g^*, \gamma_1^*, \ldots, \gamma_g^*$ be pairwise disjoint arcs that start and end at $\partial\Sigma$, and suppose α_i^* (or β_i^*, or γ_j^*) intersects only α_i (or β_i, or γ_j) once transversely. We form a basis of $H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$ by duplicating the α_i^*, β_i^* or γ_j^*, depending on a given $(2g+n-1)$-tuple in $I(n, k, g)$. This geometric object is called a *noodle*. In fact, a noodle uniquely determines a relative k-cycle in $H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$ by lifting the Cartesian product of k arcs in the noodle. For example, the noodle corresponding to $(0, 2, 1, 0, 0, 3) \in I(3, 6, 2)$ looks like the set of arcs on the right of Figure 1.

For a k-cycle F determined by a fork and a relative k-cycle N determined by a noodle, the sesquilinear pairing $\langle F, N\rangle$ computes algebraic intersections between them with coefficients in $\mathbb{Z}[G_\Sigma]$. This pairing can easily be computed by recording intersections between the fork and the noodle on Σ. The basis determined by forks and the basis determined by noodles are dual with respect to the pairing.

In the case of a disc, Bigelow [2004] showed that this pairing is invariant under the action by $\mathbf{B}_{0,n}(D)$. However, in the case of a surface Σ of genus ≥ 1, it cannot be invariant under the action by $\mathbf{B}_{0,n}(\Sigma)$. In fact, the pairing cannot be preserved by any braid group action given by a representation Ψ into $\mathrm{Aut}_{\mathbb{Z}[G_\Sigma]}(H_k^{\mathrm{BM}}(\tilde{B}_{n,k}))$. Suppose it is preserved, that is,

$$\langle F, N\rangle = \langle \Psi(\beta)(F), \Psi(\beta)(N)\rangle.$$

for any $\beta \in \mathbf{B}_{0,n}(\Sigma)$, a k-cycle F determined by a fork, and a relative k-cycle N determined by a noodle. Then for any $y \in G_\Sigma$,

$$\begin{aligned} y\langle F, N\rangle &= \langle yF, N\rangle \\ &= \langle \Psi(\beta)(yF), \Psi(\beta)(N)\rangle \\ &= \beta_\sharp(y)\langle \Psi(\beta)(F), \Psi(\beta)(N)\rangle = \beta_\sharp(y)\langle F, N\rangle. \end{aligned}$$

The property $y = \beta_\sharp(y)$ for all $y \in G_\Sigma$ would force us to set $q = 1$ in G_Σ, and so it was abandoned.

Nonetheless, we can extend this pairing to the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{H_\Sigma}$ from

$$\mathbb{Z}[H_\Sigma]\otimes_{\mathbb{Z}[G_\Sigma]} H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma)) \times \mathbb{Z}[H_\Sigma]\otimes_{\mathbb{Z}[G_\Sigma]} H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$$

to S' defined by

$$\Big\langle \sum_i g_i F_i, \sum_j h_j N_j\Big\rangle_{H_\Sigma} = \sum_{i,j} g_i \langle F_i, N_j\rangle h_j^{-1},$$

where $g_i, h_j \in \mathbb{Z}[H_\Sigma]$ and S' is the skew field containing $\mathbb{Z}[H_\Sigma]$. Note that this extended pairing cannot be invariant under the braid group action given by Φ_k either. However it can be used to compute proposed representations Φ_k explicitly. The following theorem summarizes the above discussion.

Theorem 3.4. *Let the F_i and N_i be k-cycles and relative k-cycles in dual bases determined by forks and noodles. Then $\Phi_k(\beta)$ for each $\beta \in \mathbf{B}_{0,n}(\Sigma)$ is represented by a matrix with respect to the basis $\{F_i \mid 1 \leq i \leq \binom{2g+n+k-2}{k}\}$ whose (i, j)-th entry is given by $\psi_\Sigma(\beta)\langle\tilde{\beta}(F_i), N_j\rangle_{H_\Sigma}$, which is an element of $\mathbb{Z}[H_\Sigma]$ rather than of S'.*

As an example, we will show the matrix form of the representation Φ_1 of the 3-braid group $\mathbf{B}_{0,3}(\Sigma)$ is an extension of the Burau representation when $\Sigma = \Sigma(2, 1)$. Since $k = 1$, the basis of $H_1^{\mathrm{BM}}(\tilde{B}_{3,1}(\Sigma))$ determined by forks can be expressed by $\{\gamma_1, \gamma_2, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ and similarly the dual basis of $H_1(\tilde{B}_{3,1}(\Sigma), \partial\tilde{B}_{3,1}(\Sigma))$ determined by noodles is written by $\{\gamma_1^*, \gamma_2^*, \alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*\}$.

Figure 2 shows the action of σ_1 on the fork β_1. It also shows intersection points p_1 and p_2 with γ_1^* and p_3 with β_1^*. In the covering space, the intersection point

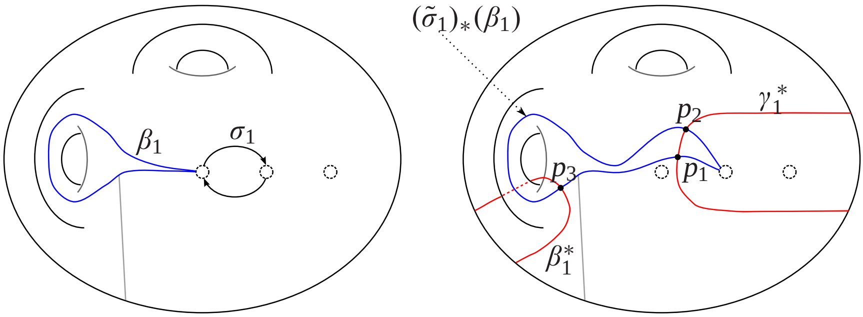

Figure 2. An example of the pairing of a fork and a noodle. At left: fork β_1 and σ_1. At right: fork and noodles.

p_1 lies on the sheet transformed by q since the fork wraps a puncture, and p_2 lies on the sheet transformed by $\ell_1 q$ since the fork contains the longitude of the first handle and wraps a puncture. We have the negative sign for p_2 since the orientation is switched. Finally, p_3 lies on the sheet containing the base point of the covering space. Therefore we have $\Phi_1(\sigma_1)(\beta_1) = \beta_1 + q(1-\ell_1)\gamma_1$. By a similar computation, we can obtain every entry of $\Phi_1(\sigma_1)$:

$$\Phi_1(\sigma_1) = \left(\begin{array}{cc|cccc} -q & 1 & q(1-m_1) & q(1-m_2) & q(1-\ell_1) & q(1-\ell_2) \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{4}{c}{I_4} \end{array}\right),$$

$$\Phi_1(\sigma_2) = \begin{pmatrix} 1 & 0 \\ q & -q \end{pmatrix} \oplus I_4,$$

$$\Phi_1(\mu_1) = \overline{m}_1 \left(\begin{array}{cc|cc|cc} \multicolumn{2}{c|}{I_2} & \multicolumn{4}{c}{\mathbf{0}} \\ \hline 1 & 0 & m_1 q & q(m_2-1) & \ell_1 - 1 & q(\ell_2-1) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c|}{\mathbf{0}} & \multicolumn{2}{c}{I_2} \end{array}\right),$$

$$\Phi_1(\mu_2) = \overline{m}_2 \left(\begin{array}{cc|cc|cc} \multicolumn{2}{c|}{I_2} & \multicolumn{4}{c}{\mathbf{0}} \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & m_1 - 1 & m_2 q & \ell_1 - 1 & \ell_2 - 1 \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} & \multicolumn{3}{|c}{I_2} \end{array}\right),$$

$$\Phi_1(\lambda_1) = \bar{\ell}_1 \left(\begin{array}{cc|cc|cc} \multicolumn{2}{c|}{I_2} & \multicolumn{4}{c}{\mathbf{0}} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & q & 0 & \multicolumn{2}{c}{\mathbf{0}} \\ & & 0 & 1 & & \\ \hline 1 & 0 & q(m_1 q - 1) & q(m_2 - 1) & \ell_1 q & q(\ell_2 - 1) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$\Phi_1(\lambda_2) = \bar{\ell}_2 \left(\begin{array}{cc|cc|cc} \multicolumn{2}{c|}{I_2} & & \mathbf{0} & & \\ \hline & & 1 & 0 & & \\ \multicolumn{2}{c|}{\mathbf{0}} & 0 & q & \multicolumn{2}{c}{\mathbf{0}} \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & m_1-1 & q(m_2q-1) & \ell_1-1 & \ell_2 q \end{array}\right).$$

Similarly, we can compute the matrix form for $k = 2$ that is the extension of Lawrence–Krammer–Bigelow representation. For $g = 1$ and $n = 3$, we have by Lemma 3.3 a 10×10 matrix for each generator. Fix a basis for $H_2^{\mathrm{BM}}(\tilde{B}_{3,2}(\Sigma))$ as shown in Figure 1. Let

$$\begin{aligned} &w_{1,1} = (0, 0, 2, 0), \quad w_{1,2} = (0, 0, 1, 1), \quad w_{2,2} = (0, 0, 0, 2), \\ &a_{0,0} = (2, 0, 0, 0), \quad a_{0,1} = (1, 0, 1, 0), \quad a_{0,2} = (1, 0, 0, 1), \\ &b_{0,0} = (0, 2, 0, 0), \quad b_{0,1} = (0, 1, 1, 0), \quad b_{0,2} = (0, 1, 0, 1), \quad z = (1, 1, 0, 0) \end{aligned}$$

in $I(3, 2, 1)$. Then the action of σ_1 on this basis is as follows:

$$\begin{aligned} \Phi_2(\sigma_1)(w_{1,1}) &= tq^2 w_{1,1}, \\ \Phi_2(\sigma_1)(w_{1,2}) &= -tqw_{1,1} - qw_{1,2}, \\ \Phi_2(\sigma_1)(w_{2,2}) &= w_{1,1} + (1+t^{-1})w_{1,2} + w_{2,2}, \\ \Phi_2(\sigma_1)(a_{0,0}) &= a_{0,0} + q(1+t^{-1})(1-m_1t)a_{0,1} + q^2(m_1^2 - (1+t)m_1 + 1)w_{1,1}, \\ \Phi_2(\sigma_1)(a_{0,1}) &= -qa_{0,1} + q^2t(m_1-1)w_{1,1}, \\ \Phi_2(\sigma_1)(a_{0,2}) &= a_{0,1} + a_{0,2} + qt(1-m_1)w_{1,1} + q(1-m_1)w_{1,2}, \\ \Phi_2(\sigma_1)(b_{0,0}) &= b_{0,0} + q(1+t^{-1})(1-\ell_1t)b_{0,1} + q^2(\ell_1^2 - (1+t)\ell_1 + 1)w_{1,1}, \\ \Phi_2(\sigma_1)(b_{0,1}) &= -qb_{0,1} + q^2t(\ell_1-1)w_{1,1}, \\ \Phi_2(\sigma_1)(b_{0,2}) &= b_{0,1} + b_{0,2} + qt(1-\ell_1)w_{1,1} + q(1-\ell_1)w_{1,2}, \\ \Phi_2(\sigma_1)(z) &= q(t^{-1} - t\ell_1)a_{0,1} + q(1-m_1)b_{0,1} + q^2(1+m_1(\ell_1-1) - t\ell_1)w_{1,1} + z. \end{aligned}$$

The correspondence between the basis $\{v_{j,k}\}$ in [Bigelow 2001] and our basis is

$$\begin{aligned} v_{1,2} &= -tq^{-4}w_{1,1}, \\ v_{1,3} &= -tq^{-4}(w_{1,1} + q(1-t^{-1})w_{1,2} + q^2 w_{2,2}), \\ v_{2,3} &= -tq^{-2}w_{2,2}. \end{aligned}$$

Then the action of Φ_2 on the basis $\{v_{j,k}\}$ together with substitution $t \mapsto -t$ is exactly that of Lawrence–Krammer–Bigelow representation in [Bigelow 2001].

4. Justification of the proposed representations

To add to the family of representations proposed in the previous section, we will now investigate the possibility that there may be other representations of the surface braid groups that extend the homology linear representations of the classical braid groups. One may try to consider alternatives in the three ways — a group extension of $\mathbf{B}_{n,k}(\Sigma)$ other than $\mathbf{B}_{n;k}(\Sigma)$, a quotient group of $\mathbf{B}_{n;k}(\Sigma)$ other than H_Σ, and an action on H_Σ by $\mathbf{B}_{0,n}(\Sigma)$ other than right multiplication via the quotient map.

Group extension of $\mathbf{B}_{n,k}(\boldsymbol{\Sigma})$**.** To make adjustment of coefficients in the most flexible way, we may try to find the largest possible group extension $\mathbf{E}_{n,k}(\Sigma)$ of $\mathbf{B}_{n,k}(\Sigma)$ such that $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{E}_{n,k}(\Sigma)$. If we regard $\mathbf{B}_{0,n}(\Sigma)$ and $\mathbf{B}_{n,k}(\Sigma)$ as subgroups of some large braid group $\mathbf{B}_{0,n+k+\ell}(\Sigma)$, then $\mathbf{B}_{0,n}(\Sigma)$ acts naturally on $\mathbf{B}_{0,n+k+\ell}(\Sigma)$ as well as on $\mathbf{B}_{n,k}(\Sigma)$ by conjugation. Thus we assume that $\mathbf{B}_{n,k}(\Sigma) \subset \mathbf{E}_{n,k}(\Sigma) \subset \mathbf{B}_{0,n+k+\ell}(\Sigma)$ for some $\ell \geq 0$.

Lemma 4.1. *Let Σ be a surface with nonempty boundary and let Σ' be a collar neighborhood of $\partial\Sigma$. Let $N(\mathbf{B}_{n,k}(\Sigma))$ denote the normalizer of $\mathbf{B}_{n,k}(\Sigma)$ in $\mathbf{B}_{0,n+k+\ell}(\Sigma)$ for some $\ell \geq 0$. Then $N(\mathbf{B}_{n,k}(\Sigma)) \cong \mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,\ell}(\Sigma')$.*

Proof. We first identify $\mathbf{B}_{n,k}(\Sigma)$ and $\mathbf{B}_{0,n}(\Sigma)$ with the corresponding subgroups of $\mathbf{B}_{0,n+k+\ell}(\Sigma)$ via the embeddings that add trivial ℓ and $k+\ell$ strands, respectively. Then we will show $N(\mathbf{B}_{n,k}(\Sigma)) = \mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,\ell}(\Sigma')$ as subgroups of $\mathbf{B}_{0,n+k+\ell}(\Sigma)$. It is clear that $\mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,\ell}(\Sigma') \subset N(\mathbf{B}_{n,k}(\Sigma))$ since $\mathbf{B}_{n,k}(\Sigma)$ is a normal subgroup of $\mathbf{B}_{n;k}(\Sigma)$ from the short exact sequence of Lemma 2.3 and since elements of $\mathbf{B}_{0,\ell}(\Sigma')$ commute with those of $\mathbf{B}_{n,k}(\Sigma)$. Conversely, let $\beta \in N(\mathbf{B}_{n,k}(\Sigma)) \subset \mathbf{B}_{0,n+k+\ell}(\Sigma)$. Any element $\alpha \in \mathbf{B}_{n,k}(\Sigma)$ and its conjugate $\beta^{-1}\alpha\beta \in \mathbf{B}_{n,k}(\Sigma)$ induce permutations that preserve the sets $\{1,\ldots,n\}$, $\{n+1,\ldots,n+k\}$ and $\{n+k+1,\ldots,n+k+\ell\}$. It is easy to see that the induced permutation of β itself must fix these three sets since α can be arbitrary in $\mathbf{B}_{n,k}(\Sigma)$. Thus $\beta \in \mathbf{B}_{n+k;l}(\Sigma)$ and the split exact sequence

$$1 \to \mathbf{B}_{n+k,l}(\Sigma) \longrightarrow \mathbf{B}_{n+k;l}(\Sigma) \xrightarrow{(\pi_{n+k})_*} \mathbf{B}_{0,n+k}(\Sigma) \to 1$$

gives a unique decomposition $\beta = \beta_1\beta_2$ for $\beta_1 \in \mathbf{B}_{0,n+k}(\Sigma)$ and $\beta_2 \in \mathbf{B}_{n+k,\ell}(\Sigma)$. In fact, $\beta_1 = (\pi_{n+k})_*(\beta) \in \mathbf{B}_{n;k}(\Sigma)$ since the epimorphism $(\pi_{n+k})_*$ forgets the last ℓ strands or replaces them by the trivial ℓ-strand braid.

For any $\alpha \in \mathbf{B}_{n,k}(\Sigma) \subset \mathbf{B}_{0,n+k}(\Sigma) \subset \mathbf{B}_{0,n+k+\ell}(\Sigma)$, we have $(\pi_{n+k})_*(\beta_2^{-1}\alpha\beta_2) = \beta_2^{-1}\alpha\beta_2$ since $\beta_2^{-1}\alpha\beta_2 \in \mathbf{B}_{0,n+k}$. On the other hand, $(\pi_{n+k})_*(\beta_2^{-1}\alpha\beta_2) = \alpha$ since $(\pi_{n+k})_*$ replaces the last ℓ strands by the trivial braid. Thus we have $\beta_2^{-1}\alpha\beta_2 = \alpha$. From the presentation of $\mathbf{B}_{0,n+k+\ell}(\Sigma)$ in Section 1A, it is easy to see that β_2 must be a local braid in order for β_2 to commute with every element of $\mathbf{B}_{n,k}(\Sigma)$. Thus we have $\beta_2 \in \mathbf{B}_{0,\ell}(\Sigma')$, where Σ' is an annulus that is a collar neighborhood of $\partial\Sigma$ in Σ. Consequently, we have shown $N(\mathbf{B}_{n,k}(\Sigma)) \subset \mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,\ell}(\Sigma')$ □

By this lemma, the extension $\mathbf{E}_{n,k}(\Sigma)$ of $\mathbf{B}_{n,k}(\Sigma)$ can be taken as a subgroup of $\mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,l}(\Sigma')$. We remark that $\mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,l}(\Sigma')$ is also a subgroup of the intertwining braid group $\mathbf{B}_{n;k+l}(\Sigma)$.

Then we follow the construction given in the discussion before Theorem 3.2 with $\mathbf{E}_{n,k}(\Sigma)$ replacing $\mathbf{B}_{n;k}(\Sigma)$.

Let $\psi : \mathbf{E}_{n,k}(\Sigma) \to H$ be an epimorphism onto a group H. If we choose an action of $\mathbf{B}_{0,n}(\Sigma)$ on the extension $\mathbf{E}_{n,k}(\Sigma)$, then the action is carried over H via ψ and it is convenient to use the convention that $(\beta_1\beta_2)\cdot h = \beta_2 \cdot (\beta_1 \cdot h)$ for $\beta_1, \beta_2 \in \mathbf{B}_{0,n}(\Sigma)$ and $h \in H$. To obtain a $\mathbb{Z}[H]$-module automorphism $\beta \otimes \tilde{\beta}_*$ on $\mathbb{Z}[H] \otimes_{\mathbb{Z}[G]} H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ that is an extension of a homology linear representation of the classical braid group, this induced action of $\mathbf{B}_{0,n}(\Sigma)$ on H needs to satisfy two conditions.

(i) Lifting criteria: $\beta_\sharp$ exists and $\beta_\sharp(\phi(\alpha)) = \phi(\bar{\beta}_*(\alpha))$ for all $\alpha \in \mathbf{B}_{n,k}(\Sigma)$, where $\phi = \psi|_{\mathbf{B}_{n,k}(\Sigma)}$.

(ii) Linearity and compatibility: $hh'(\beta\cdot 1) = \beta\cdot(hh') = (\beta\cdot h)\beta_\sharp(h')$ for all $h \in H$ and $h' \in G = \phi(\mathbf{B}_{n,k}(\Sigma))$.

As in the proof of Theorem 3.2, we then have

$$(\beta \otimes \tilde{\beta}_*)(h \otimes h'c) = (\beta \otimes \tilde{\beta}_*)(hh' \otimes c) = hh'(\beta \otimes \tilde{\beta}_*)(1 \otimes c)$$

for all $h \in H$, $h' \in G$ and $c \in H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$.

Theorem 4.2. *Suppose there are an epimorphism $\psi : \mathbf{E}_{n,k}(\Sigma) \to H$ and an action of $\mathbf{B}_{0,n}(\Sigma)$ on H satisfying the two conditions above. Let Ψ_k be the representation obtained from ψ and the action. Then*

$$\Psi_k = 1_{\mathbb{Z}[H]} \otimes_{\mathbb{Z}[H']} \Psi'_k$$

for a representation Ψ'_k obtained from an epimorphism $\psi' : \mathbf{B}_{n;k}(\Sigma) \to H' \subset H$ and an action $\mathbf{B}_{0,n}(\Sigma)$ on H', where $1_{\mathbb{Z}[H]}$ is the identity map on $\mathbb{Z}[H]$.

Proof. Let $H' = \{\beta\cdot 1 \in H \mid \beta \in \mathbf{B}_{0,n}(\Sigma)\}\phi(\mathbf{B}_{n,k}(\Sigma))$ and $\psi' : \mathbf{B}_{n;k}(\Sigma) \to H'$ be a surjection defined by $\psi'(\beta) = \beta\cdot 1$ for $\beta \in \mathbf{B}_{0,n}(\Sigma)$ and $\psi' = \phi$ on $\mathbf{B}_{n,k}(\Sigma)$. Then since

$$\psi'(\beta_1\beta_2) = (\beta_1\beta_2)\cdot 1 = \beta_2\cdot(\beta_1\cdot 1) = (\beta_1\cdot 1)(\beta_2\cdot 1) = \psi'(\beta_1)\psi'(\beta_2)$$

for all $\beta_1, \beta_2 \in \mathbf{B}_{0,n}(\Sigma)$, the surjection ψ' becomes a homomorphism that preserves the semidirect product structure. Also we have

$$\phi' = \psi'|_{\mathbf{B}_{n,k}(\Sigma)} = \psi|_{\mathbf{B}_{n,k}(\Sigma)} = \phi$$

and so ϕ and ϕ' induce the same homology group $H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$, and two $\mathbb{Z}$-module automorphisms obtained from β coincide.

Consider two representations Ψ_k and Ψ'_k corresponding to ψ and ψ', respectively. Then $\Psi_k(\beta)$ gives a $\mathbb{Z}[H]$-homomorphism on $\mathbb{Z}[H]\otimes_{\mathbb{Z}[G]} H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ and $\Psi'_k(\beta)$ gives a $\mathbb{Z}[H']$-homomorphism on $\mathbb{Z}[H']\otimes_{\mathbb{Z}[G]} H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$. Since $\mathbb{Z}[H]=\mathbb{Z}[H]\otimes_{\mathbb{Z}[H']}\mathbb{Z}[H']$, the representation $\Psi_k(\beta)$ is a $\mathbb{Z}[H]$-homomorphism on $\mathbb{Z}[H]\otimes_{\mathbb{Z}[H']}\mathbb{Z}[H']\otimes_{\mathbb{Z}[G]} H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$ defined by

$$\Psi_k(\beta)(hh'\otimes c)=hh'(\beta\cdot 1)\otimes\tilde{\beta}_*(c)=h\otimes h'(\beta\cdot 1)\otimes\tilde{\beta}_*(c)$$

for all $h\in\mathbb{Z}[H]$, $h'\in\mathbb{Z}[H']$ and $c\in H_k^{\mathrm{BM}}(\tilde{B}_{n,k}(\Sigma))$. As claimed, this is equal to $1_{\mathbb{Z}[H]}\otimes_{\mathbb{Z}[H']}\Psi'_k(\beta)$. □

This theorem implies that we may assume that $\mathbf{B}_{n;k}(\Sigma)\subset\mathbf{E}_{n,k}(\Sigma)$ without loss of generality. Then by Lemma 4.1, $\mathbf{E}_{n,k}(\Sigma)=\mathbf{B}_{n;k}(\Sigma)\times\mathbf{B}$ for some subgroup $\mathbf{B}$ of $\mathbf{B}_{0,\ell}(\Sigma')$ and the theorem says that any family of representations obtained by using $\mathbf{E}_{n,k}(\Sigma)$ is merely a trivial extension of the family of representations proposed in Section 3.

Quotient of $\mathbf{B}_{n;k}(\Sigma)$**.** According to the scheme described in Theorem 3.2, it is important to find a good epimorphism $\psi:\mathbf{B}_{n;k}(\Sigma)\to H$ onto some group H.

Since Σ is not a sphere, the inclusion $B_{n,k}(D)\hookrightarrow B_{n,k}(\Sigma)$ induces a monomorphism $\mathbf{B}_{n,k}(D)\hookrightarrow\mathbf{B}_{n,k}(\Sigma)$; see [Birman 1974]. Similarly, $B_{n;k}(D)\hookrightarrow B_{n;k}(\Sigma)$ induces a monomorphism $\mathbf{B}_{n;k}(D)\hookrightarrow\mathbf{B}_{n;k}(\Sigma)$ (to be regarded as an inclusion).

We first determine an epimorphism $\psi_D:\mathbf{B}_{n;k}(D)\to H_D$ to extend the map $\phi_D:\mathbf{B}_{n,k}(D)\to G_D$ for the classical braid groups. Since we want to obtain homology linear representations for the classical braid groups, we should use that $H_D=G_D$, and all of the extra generators $\bar{\sigma}_1,\dots,\bar{\sigma}_{n-1}$ of $\mathbf{B}_{n;k}(D)$ should be sent to the identity by ψ_D, as we have seen earlier in Section 3A. Then $\psi_D|_{\mathbf{B}_{n,k}(D)}=\phi_D$. For some extension H of G_D, let $\psi:\mathbf{B}_{n;k}(\Sigma)\to H$ be an epimorphism. To obtain an extension of homology linear representations of the classical braid groups via ψ, we require the condition

$$\psi|_{\mathbf{B}_{n;k}(D)}=\psi_D \tag{†}$$

This condition is nothing but a reinterpretation of Definition 2.2 and is necessary to make the diagram

$$\begin{tikzcd}
\mathbf{B}_{n,k}(D) \arrow[r, hook] \arrow[d, "\phi_D"] & \mathbf{B}_{n;k}(D) \arrow[r, hook] \arrow[d, "\psi_D"] & \mathbf{B}_{n;k}(\Sigma) \arrow[d, "\psi"] \\
G_D \arrow[r, equal] & H_D \arrow[r, hook] & H
\end{tikzcd}$$

commutative, so that $\psi|_{\mathbf{B}_{n,k}(D)} = \phi_D$ and we can then apply the construction of Theorem 3.2. We first show that the condition (†) imposes restrictions on the choice of H.

Theorem 4.3. *Let $\psi_\Sigma : \mathbf{B}_{n;k}(\Sigma) \to H_\Sigma$ be the epimorphism defined in Section 3A.*

(1) *Let $h : H_\Sigma \to H$ be an epimorphism such that $h \circ \psi_\Sigma$ satisfies (†). Then h is an isomorphism.*

(2) *Let $\psi : \mathbf{B}_{n;k}(\Sigma) \to H$, an arbitrary epimorphism onto a group H, satisfy (†). Then ψ_Σ factors through ψ, and H is isomorphic to H_Σ for $k \geq 3$.*

Proof. (1) It suffices to show that $h(W) = 1$ implies $W = 1$ for any word W in generators of H_Σ. Assume $k \geq 2$. Using the relations of H_Σ, a given word W can be put into the form

$$W = q^c t^d \prod_{i=1}^{g} W_i, \quad \text{where } W_i = m_i^{a_i} \ell_i^{b_i} \overline{m}_i^{\bar{a}_i} \bar{\ell}_i^{\bar{b}_i}.$$

First consider $[W, \bar{\ell}_r]$. Note that W_r commutes with the other W_i as well as q and t. Since $\bar{\ell}_r$ commutes with all generators except m_r and only W_r contains m_r, we have

$$\begin{aligned}
[W, \bar{\ell}_r] &= \Big(q^c t^d \prod_i W_i\Big) \bar{\ell}_r \Big(q^c t^d \prod_i W_i\Big)^{-1} \bar{\ell}_r^{-1} \\
&= W_r \Big(q^c t^d \prod_{i \neq r} W_i\Big) \bar{\ell}_r \Big(q^c t^d \prod_{i \neq r} W_i\Big)^{-1} W_r^{-1} \bar{\ell}_r^{-1} \\
&= W_r \bar{\ell}_r W_r^{-1} \bar{\ell}_r^{-1} \\
&= \big(m_r^{a_r} \ell_r^{b_r} \overline{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r}\big) \bar{\ell}_r \big(m_r^{a_r} \ell_r^{b_r} \overline{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r}\big)^{-1} \bar{\ell}_r^{-1} \\
&= m_r^{a_r} \big(\ell_r^{b_r} \overline{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r}\big) \bar{\ell}_r \big(\ell_r^{b_r} \overline{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r}\big)^{-1} m_r^{-a_r} \bar{\ell}_r^{-1} \\
&= m_r^{a_r} \bar{\ell}_r m_r^{-a_r} \bar{\ell}_r^{-1} = q^{a_r}.
\end{aligned}$$

The last equality follows from the relation $[m_r, \bar{\ell}_r] = q$. By applying h and using $h(W) = 1$, we have

$$h(q^{a_r}) = h([W, \bar{\ell}_r]) = h(W) h(\bar{\ell}_r) h(W)^{-1} h(\bar{\ell}_r)^{-1} = 1.$$

By (†), h is the identity on G_D that is the subgroup generated by q and t, and q and t are of infinite order. Thus $h(q^{a_r}) = q^{a_r} = 1$ implies $a_r = 0$. Similarly, $b_r = \bar{a}_r = \bar{b}_r = 0$ by considering $[W, \overline{m}_r]$, $[W, \ell_r]$, and $[W, m_r]$. Therefore $W_r = 1$. Since r is arbitrary other than $1 \leq r \leq g$, we now have $W = q^c t^d$. Then $1 = h(W) = q^c t^d$ implies $c = d = 0$. Consequently, $W = 1$.

For the case $k = 1$, the proof is similar but simpler since $t = 1$ in H_Σ.

(2) Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{B}_{n;k}(\Sigma) \arrow[r, "\psi"] \arrow[d, "\psi_\Sigma"] & H \cong \mathbf{B}_{n;k}(\Sigma)/\operatorname{Ker}\psi \arrow[d, "q"] \\
H_\Sigma \cong \mathbf{B}_{n;k}(\Sigma)/\operatorname{Ker}\psi_\Sigma \arrow[r, "h"] & \mathbf{B}_{n;k}(\Sigma)/(\operatorname{Ker}\psi\cdot\operatorname{Ker}\psi_\Sigma),
\end{tikzcd}$$

which consists of obvious quotient homomorphisms. Note that the condition (†) is equivalent to $\mathbf{B}_{n;k}(D)/(\operatorname{Ker}\psi\cap\mathbf{B}_{n;k}(D))\cong G_D$. Thus $\operatorname{Ker}\psi\cap\mathbf{B}_{n;k}(D)=\operatorname{Ker}\psi_\Sigma\cap\mathbf{B}_{n;k}(D)$ since ψ_Σ also satisfies (†). Then

$$\begin{aligned}(\operatorname{Ker}\psi\cdot\operatorname{Ker}\psi_\Sigma)\cap\mathbf{B}_{n;k}(D)&=(\operatorname{Ker}\psi\cap\mathbf{B}_{n;k}(D))\cdot(\operatorname{Ker}\psi_\Sigma\cap\mathbf{B}_{n;k}(D))\\&=\operatorname{Ker}\psi_\Sigma\cap\mathbf{B}_{n;k}(D).\end{aligned}$$

Thus $h\circ\psi_\Sigma$ satisfies (†) and h is an isomorphism by part (1). Therefore ψ_Σ factors through ψ via $h^{-1}\circ q$ for $k\ge 1$.

For $k\ge 3$, we will show $\psi:\mathbf{B}_{n;k}(\Sigma)\to H$ factors through ψ_Σ, that is, there is an epimorphism $h:H_\Sigma\to H$ such that $h\psi_\Sigma=\psi$. Then H_Σ is isomorphic to H since ψ_Σ also factors through ψ.

Recall the presentation for $\mathbf{B}_{n;k}(\Sigma)$ in Lemma 2.5. The condition (†) implies $\psi(\sigma_i)=q,\ \ \psi(\zeta_j)=t$, and $\psi(\bar\sigma_m)=1$ for all $1\le i\le k-1,\ \ 1\le j\le n$, and $1\le m\le n-1$. Since $k\ge 3$, the relation (CR$_1$) among generators in X_2 is not vacuous and so the relations (CR$_1$) through (CR$_3$) for X_2 and the condition (†) imply

$$[\psi(\mu_r),q]=[\psi(\lambda_r),q]=[\psi(\mu_r),t]=[\psi(\lambda_r),t]=[q,t]=1$$

for all $1\le r\le g$. Also the relation Lemma 2.5(iii) implies

$$[\psi(\bar\mu_r),q]=[\psi(\bar\lambda_r),q]=[\psi(\bar\mu_r),t]=[\psi(\bar\lambda_r),t]=1\quad\text{for all } 1\le r\le g.$$

Thus q and t lie in the center of H. Using this, all other relations in H_Σ can be shown to hold in H. Therefore ψ induces an epimorphism $h:H_\Sigma\to H$. □

Hence H_Σ is the unique quotient group of $\mathbf{B}_{n;k}(\Sigma)$ satisfying (†) for $k\ge 3$. For $k\le 2$, the condition (†) does not uniquely determine a quotient group of $\mathbf{B}_{n;k}(\Sigma)$. To take advantage of representations in analyzing the surface braid group $\mathbf{B}_{0,n}(\Sigma)$, one may prefer a simpler coefficient ring as long as the representation carries enough information. For the classical case, there are also several groups satisfying the condition (∗) if we do not assume they are abelian. For the surface braid groups, we cannot obtain any interesting representation if an abelian coefficient ring is used, as discussed in Section 2. Theorem 4.3(1) says that H_Σ is the simplest quotient group satisfying (†) in the sense that any further quotient of H_Σ violates (†).

We now discuss possible actions of $\mathbf{B}_{0,n}(\Sigma)$ on H_Σ induced from ψ_Σ.

Theorem 4.4. *Let $\psi_\Sigma : \mathbf{B}_{n;k}(\Sigma) \to H_\Sigma$ be the epimorphism defined in Section 3A. Let $\beta \cdot h$ denote any action on $h \in H_\Sigma$ by $\beta \in \mathbf{B}_{0,n}(\Sigma)$ that is induced from ψ_Σ and satisfies the two conditions given above Theorem 4.2. Then*

$$\beta \cdot h = h\chi(\beta)\psi_\Sigma(\beta)$$

for some function $\chi : \mathbf{B}_{0,n}(\Sigma) \to C_{H_\Sigma}(G_\Sigma)$ with the property that

$$(\chi, \psi_\Sigma) : \mathbf{B}_{0,n}(\Sigma) \to C_{H_\Sigma}(G_\Sigma) \rtimes H_\Sigma$$

is a homomorphism, where $C_{H_\Sigma}(G_\Sigma)$ denotes the centralizer of G_Σ in H_Σ.

Proof. By the hypotheses of the action, we have

$$h'(\beta \cdot 1) = \beta \cdot (1h') = (\beta \cdot 1)\beta_\sharp(h') \quad \text{and} \quad \beta_\sharp(h') = \psi_\Sigma(\beta)^{-1}h'\psi_\Sigma(\beta)$$

for all $h' \in G_\Sigma$. By combining these two equations, we have

$$\psi_\Sigma(\beta)^{-1}h'\psi_\Sigma(\beta) = (\beta \cdot 1)^{-1}h'(\beta \cdot 1).$$

and so $(\beta \cdot 1)\psi_\Sigma(\beta)^{-1} \in C_{H_\Sigma}(G_\Sigma)$. Hence $(\beta \cdot 1) = \chi(\beta)\psi_\Sigma(\beta)$ for a function $\chi : \mathbf{B}_{0,n}(\Sigma) \to C_{H_\Sigma}(G_\Sigma)$. Since $\chi(\beta_1\beta_2)\psi_\Sigma(\beta_1\beta_2) = (\beta_1\beta_2) \cdot 1 = \beta_2 \cdot (\beta_1 \cdot 1) = (\chi(\beta_1)\psi_\Sigma(\beta_1))\chi(\beta_2)\psi_\Sigma(\beta_2)$, we have

$$\chi(\beta_1\beta_2) = \chi(\beta_1)\psi_\Sigma(\beta_1)\chi(\beta_2)\psi_\Sigma(\beta_1)^{-1}.$$

This implies that

$$\begin{aligned}(\chi(\beta_1\beta_2), \psi_\Sigma(\beta_1\beta_2)) &= (\chi(\beta_1)\psi_\Sigma(\beta_1)\chi(\beta_2)\psi_\Sigma(\beta_1)^{-1}, \psi_\Sigma(\beta_1\beta_2)) \\ &= (\chi(\beta_1), \psi_\Sigma(\beta_1))(\chi(\beta_2), \psi_\Sigma(\beta_2)).\end{aligned}$$

Therefore $(\chi, \psi_\Sigma) : \mathbf{B}_{0,n}(\Sigma) \to C_{H_\Sigma}(G_\Sigma) \rtimes H_\Sigma$ is a homomorphism. □

The function χ in this theorem behaves like a character of $\mathbf{B}_{0,n}(\Sigma)$. In fact, if $k \geq 2$, it can be shown that $C_{H_\Sigma}(G_\Sigma) = Z(H_\Sigma) = \langle q \rangle \oplus \langle t \rangle$. Hence χ can be any homomorphism from $\mathbf{B}_{0,n}(\Sigma)$ to $Z(H_\Sigma)$. In this case, the representations Ψ_k obtained from ψ are given by $\Psi_k = \chi \otimes \Phi_k$ for some character χ, where Φ_k is the proposed representation in Theorem 3.2.

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