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In this note we give a simple, model-independent construction of Chern classes as natural transformations from differential complex K-theory to differential integral cohomology. We verify the expected behavior of these Chern classes with respect to sums and suspension.

1. Statements

Complex K-theory and integral cohomology $H\mathbb{Z}$ are generalized cohomology theories that have unique differential[1] extensions $(\hat{K}, R, I, a, \int)$ and $(\widehat{H\mathbb{Z}}, R, I, a, \int)$ with integration. These extensions are multiplicative in a unique way. We refer to [Bunke and Schick 2010] for a description of the axioms for differential extensions of cohomology theories and a proof of these statements.

The i-th Chern class is a natural transformation of *set-valued* functors

$$c_i : K^0 \to H\mathbb{Z}^{2i}$$

on the category of topological spaces. The product $H\mathbb{Z}^{\mathrm{ev}} := \prod_{i\geq 0} H\mathbb{Z}^{2i}$ is a functor with values in commutative graded rings. We consider the subfunctor

$$H\mathbb{Z}_1^{\mathrm{ev},*} := 1 + \prod_{i\geq 1} H\mathbb{Z}^{2i} \subseteq \prod_{i\geq 0} H\mathbb{Z}^{2i}$$

that takes values in the subgroup of units. The total Chern class

$$c := 1 + c_1 + c_2 + \cdots : K^0 \to H\mathbb{Z}_1^{\mathrm{ev},*}$$

is a natural transformation of *group-valued* functors.

Let $\Omega^*_{\mathrm{cl}}(\dots, K^*) \subseteq \Omega^*(\dots, K^*)$ denote the graded ring valued functors on smooth manifolds of smooth differential forms with coefficients in K^* and its subfunctor of closed forms. We use the powers of the Bott element in K^2 to identify the functors

$$\Omega^0(\dots, K^*) \cong \Omega^{\mathrm{ev}}(\dots) \quad \text{and} \quad \Omega^{-1}(\dots, K^*) \cong \Omega^{\mathrm{odd}}(\dots).$$

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[1] In previous work, we used the term "smooth cohomology" instead of "differential cohomology". We were convinced by D. Freed that the latter is the better name.

We therefore have natural transformations

$$a : \Omega^{\mathrm{odd}} \to \hat{K}^0 \quad \text{and} \quad R : \hat{K}^0 \to \Omega^{\mathrm{ev}}_{\mathrm{cl}},$$

where a only preserves the additive structure, and R is multiplicative.

We consider the symmetric formal power series

$$\widetilde{\mathbf{ch}} := \sum_{i \geq 1} (e^{x_i} - 1) \in \mathbb{Q}[[x_1, x_2, \dots]]$$

in infinitely many variables. We write $\mathbf{ch}_i$ for the homogeneous component of degree i. Then there are polynomials $C_i \in \mathbb{Q}[s_1, s_2, \dots]$ of degree i (where s_i has degree i) such that $C_i(\mathbf{ch}_1, \dots, \mathbf{ch}_i) = \sigma_i$ is the i-th elementary symmetric function in the x_i. The polynomial C_i induces a natural transformation $C_i : \Omega^{\mathrm{ev}} \to \Omega^{2i}$ that maps the even form $\omega = \omega_0 + \omega_2 + \omega_4 + \cdots$, where $\omega_{2k} \in \Omega^{2k}(M)$, to

$$C_i(\omega) := C_i(\omega_2, \dots, \omega_{2i}) \in \Omega^{2i}(M).$$

The following theorem states that the Chern classes have unique lifts to the differential extensions and that these lifts are compatible with the group structures.

Theorem 1.1. (i) *For every $i \geq 1$, there exists a unique natural transformation*

$$\hat{c}_i : \hat{K}^0 \to \widehat{H\mathbb{Z}}^{2i}$$

of set-valued functors on smooth manifolds such that the following diagram commutes:

(1)
$$\begin{tikzcd} \Omega^{\mathrm{ev}} \arrow[r, "C_i"] & \Omega^{2i} \\ \hat{K}^0 \arrow[u, "R"] \arrow[r, "\hat{c}_i"] \arrow[d, "I"] & \widehat{H\mathbb{Z}}^{2i} \arrow[u, "R"] \arrow[d, "I"] \\ K^0 \arrow[r, "c_i"] & H\mathbb{Z}^i \end{tikzcd}$$

(ii) *The total class $\hat{c} = 1 + \hat{c}_1 + \cdots : \hat{K}^0 \to \widehat{H\mathbb{Z}}^{\mathrm{ev},*}_1$ preserves the group structure.*

Berthomieu [2008] has already constructed lifts of the Chern classes. Our goal is to give a much simpler, model-independent treatment. Further new, but not very deep, points of the theorem above are the assertions about uniqueness and the second statement. Our method of proof is different from Berthomieu's and is in fact a specialization of a general principle used in [Bunke and Schick 2010] and [Bunke 2009a] for the construction of lifts of natural transformations between cohomology functors to their differential refinements.

In the next two paragraphs we connect the differential Chern classes on differential K-theory with previous constructions of differential Chern classes in specific geometric situations.

If $\boldsymbol{V} := (V, h^V, \nabla^V)$ is a hermitian vector bundle with connection over a manifold M, then we have the classes

$$\hat{c}_i^{\mathrm{CS}}(\boldsymbol{V}) \in \widehat{H\mathbb{Z}}^{2i}(M)$$

constructed in [Cheeger and Simons 1985]. In the model of differential K-theory [Bunke and Schick 2009], the geometric bundle is a cycle for a differential K-theory class $[\boldsymbol{V}] \in \hat{K}^0(M)$. We have $\hat{c}_i([\boldsymbol{V}]) = \hat{c}_i^{\mathrm{CS}}(\boldsymbol{V})$.

An even geometric family $\mathscr{E}$ over M (see [Bunke 2009b] for this notion) gives rise to a Bismut superconnection $A(\mathscr{E})$ on an infinite-dimensional Hilbert space bundle $H(\mathscr{E})$ over M. This superconnection

$$A(\mathscr{E}) = D(\mathscr{E}) + \nabla^{H(\mathscr{E})} + \text{higher terms}$$

extends the family of Dirac operators $D(\mathscr{E})$. If the kernel of $D(\mathscr{E})$ is a vector bundle, then it has an induced metric $h^{\ker(D(\mathscr{E}))}$ and connection $\nabla^{\ker(D(\mathscr{E}))}$ obtained from $\nabla^{H(\mathscr{E})}$ by projection. We thus get an induced geometric bundle

$$\mathbf{H}(\mathscr{E}) = (\ker(D(\mathscr{E})), h^{\ker(D(\mathscr{E}))}, \nabla^{\ker(D(\mathscr{E}))})$$

and can define the class $\hat{c}_i^{\mathrm{CS}}(\mathbf{H}(\mathscr{E})) \in \widehat{H\mathbb{Z}}^{2i}(M)$. One goal of [Bunke 2009b], which was not quite achieved there, was to extend this construction to the general case where we do not have a kernel bundle. By assuming that $\mathrm{index}(D(\mathscr{E})) \in K^0(M)$ belongs to the i-th step of the Atiyah–Hirzebruch filtration (that is, that it vanishes after pull-back to any $(i-1)$-dimensional complex), we constructed in that book's 4.1.19 a class $\hat{c}_i(\mathscr{E}) \in \widehat{H\mathbb{Z}}^{2i}(M)$[2] such that $I(\hat{c}_i(\mathscr{E})) = c_i(\mathrm{index}(D(\mathscr{E})))$. On the other hand, the geometric family $\mathscr{E}$ represents a differential K-theory class $[\mathscr{E}, 0] \in \hat{K}^0(M)$ in the model [Bunke and Schick 2009], and we have $I([\mathscr{E}, 0]) = \mathrm{index}(D(\mathscr{E}))$. The class $\hat{c}_i([\mathscr{E}, 0]) \in \widehat{H\mathbb{Z}}^{2i}(M)$ satisfies $I(\hat{c}_i(\mathscr{E})) = c_i(\mathrm{index}(D(\mathscr{E})))$ also and thus gives a second differential refinement of the i-th Chern class of the index of $D(\mathscr{E})$. But in general the class $\hat{c}_i(\mathscr{E})$ differs from $\hat{c}_i([\mathscr{E}, 0])$. This can already be seen on the level of curvatures. Namely, we have

$$R(\hat{c}_i(\mathscr{E})) = R([\mathscr{E}, 0])_{[2i]} \quad \text{and} \quad R(\hat{c}_i([\mathscr{E}, 0])) = C_i(R([\mathscr{E}, 0])),$$

where $\omega_{[2i]}$ denotes the degree-$2i$ component of the form ω. In a sense, this note gives the right answer to the problem considered in [Bunke 2009b].

[2] In [Bunke 2009b] we indexed the Chern classes by their degree, while here we adopt the usual convention.

Finally we discuss odd Chern classes. In topology, the odd Chern classes $c_i^{\mathrm{odd}}: K^{-1} \to H\mathbb{Z}^i$ are related with the even Chern classes by suspension:

$$\begin{tikzcd} \tilde{K}^0(\Sigma M_+) \arrow[r, "c_{(i+1)/2}"] \arrow[d, "\cong"'] & \widetilde{H\mathbb{Z}}^{i+1}(\Sigma M_+) \arrow[d, "\cong"] \\ K^{-1}(M) \arrow[r, "c_i^{\mathrm{odd}}"] & H\mathbb{Z}^i(M). \end{tikzcd}$$

In the smooth context, the suspension isomorphism is replaced by the integration $\int$ along $S^1 \times M \to M$. We have the following odd counterpart of Theorem 1.1.

Theorem 1.2. *For odd $i \in \mathbb{N}$, there are unique natural transformations*

$$\hat{c}_i^{\mathrm{odd}} : \hat{K}^{-1} \to \widehat{H\mathbb{Z}}^i$$

such that

$$\begin{tikzcd} \hat{K}^0(S^1 \times M) \arrow[r, "\hat{c}_{(i+1)/2}"] \arrow[d, "\int"'] & \widehat{H\mathbb{Z}}^{i+1}(S^1 \times M) \arrow[d, "\int"] \\ \hat{K}^{-1}(M) \arrow[r, "\hat{c}_i^{\mathrm{odd}}"] & \widehat{H\mathbb{Z}}^i(M) \end{tikzcd}$$

commutes. The transformation satisfies $I \circ \hat{c}_i^{\mathrm{odd}} = c_i^{\mathrm{odd}} \circ I$.

Let $\pi : W \to B$ be a proper K-oriented map between manifolds. Then we have an Umkehr map $\pi_! : K^*(W) \to K^{*-n}(B)$, where $n = \dim(W) - \dim(B)$. An integral index theorem is an assertion about the Chern classes $c_*(\pi_!(x))$, or $c_*^{\mathrm{odd}}(\pi_!(x))$ for $x \in K^*(W)$, for example, an expression of these classes in terms of the classes $c_*(x)$ or $c_*^{\mathrm{odd}}(x)$, respectively. A prototypical example is given in [Madsen 2009]. The construction of differential lifts of Chern classes makes it possible to ask for geometric refinements of these kinds of results. An example of such a theorem related to the Pfaffian bundle is discussed in [Bunke 2009c].

2. Proofs

Let $\boldsymbol{K}_0 \simeq \mathbb{Z} \times BU$ be a representative of the homotopy type of the classifying space of the functor K^0. By [Bunke and Schick 2010, Proposition 2.1], we may choose a sequence of manifolds $(\mathcal{K}_k)_{k \geq 0}$ together with maps $x_k : \mathcal{K}_k \to \boldsymbol{K}_0$ and $\kappa_k : \mathcal{K}_k \to \mathcal{K}_{k+1}$ such that

(i) $\mathcal{K}_k$ is homotopy equivalent to a k-dimensional CW-complex,

(ii) $\kappa_k : \mathcal{K}_k \to \mathcal{K}_{k+1}$ is an embedding of a closed submanifold,

(iii) $x_k : \mathcal{K}_k \to \boldsymbol{K}_0$ is k-connected, and

(iv) $x_{k+1} \circ \kappa_k = x_k$.

Let $u \in K^0(\boldsymbol{K}_0)$ be the universal class represented by the identity map $\boldsymbol{K}_0 \to \boldsymbol{K}_0$. By [Bunke and Schick 2010, Proposition 2.6] we can further choose a sequence $\hat{u}_k \in \hat{K}^0(\mathcal{K}_k)$ such that $I(\hat{u}_k) = x_k^* u$ and $\kappa_k^* \hat{u}_{k+1} = \hat{u}_k$ for all $k \geq 0$. Then by [ibid., Lemma 3.8] and $2j-1 < k$, we have $H^{2j-1}(\mathcal{K}_k, \mathbb{R}) = 0$. We consider the canonical natural transformation $\iota_{\mathbb{R}} : H\mathbb{Z}^* \to H\mathbb{R}^*$ and the de Rham map $\mathrm{Rham} : \Omega^*_{\mathrm{cl}} \to H\mathbb{R}^*$. Since the latter is multiplicative, we have

$$\iota_{\mathbb{R}}(c_i(I(\hat{u}_k))) = C_i(\mathbf{ch}(I(\hat{u}_k))) = C_i(\mathrm{Rham}(R(\hat{u}_k))) = \mathrm{Rham}(C_i(R(\hat{u}_k))).$$

If we choose $k \geq 2i$, then the diagram

$$\begin{array}{ccc} \widehat{H\mathbb{Z}}^{2i}(\mathcal{K}_k) & \xrightarrow{I} & H\mathbb{Z}^{2i}(\mathcal{K}_k) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \iota_{\mathbb{R}}} \\ \Omega^{2i}_{\mathrm{cl}}(\mathcal{K}_k) & \xrightarrow{\mathrm{Rham}} & H\mathbb{R}^{2i}(\mathcal{K}_k) \end{array}$$

is cartesian. Hence for $k \geq 2i$, there exists a unique class $\hat{z}_{i,k} \in \widehat{H\mathbb{Z}}^{2i}(\mathcal{K}_k)$ such that

$$I(\hat{z}_{i,k}) = c_i(I(\hat{u}_k)) \quad \text{and} \quad R(\hat{z}_i) = C_i(R(\hat{u}_k)).$$

Also, we have $\kappa_k^* \hat{z}_{i,k+1} = \hat{z}_{i,k}$. For $k < 2i$, we define $z_{i,k} := (\kappa_k^* \circ \cdots \circ \kappa_{2i-1}^*) z_{i,2i}$.

We now define the natural transformation $\hat{c}_i$. We start with the observation that if $\hat{c}_i$ exists, then it satisfies $\hat{c}_i(\hat{u}_k) = \hat{z}_{i,k}$.

Let $\hat{w} \in \hat{K}^0(M)$. By [ibid., Proposition 2.6] we have $K^0(M) \cong \mathrm{colim}_k[M, \mathcal{K}_k]$, and the underlying class $I(\hat{w}) \in K^0(M)$ can be written as $I(\hat{w}) = f^* x_k^* u$ for some k and $f : M \to \mathcal{K}_k$. We choose a form $\rho \in \Omega^{\mathrm{odd}}(M)$ such that $\hat{w} = f^* \hat{u}_k + a(\rho)$.

We consider a form $\tilde{\rho} \in \Omega^{\mathrm{odd}}([0,1] \times M)$ that restricts to ρ on $\{1\} \times M$ and to 0 on $\{0\} \times M$. We get a class $\tilde{\hat{w}} = \mathrm{pr}_M^* \hat{w} + a(\tilde{\rho}) \in \hat{K}^0([0,1] \times M)$. Note that

$$\tilde{\hat{w}}|_{\{0\} \times M} = f^* \hat{u}_k \quad \text{and} \quad \tilde{\hat{w}}|_{\{1\} \times M} = \hat{w}.$$

If $\hat{c}_i$ exists, then by naturality and the homotopy formula [ibid., (1)], we have

$$\hat{c}_i(\tilde{\hat{w}}|_{\{0\} \times M}) = f^* \hat{z}_{i,k}, \quad \hat{c}_i(\tilde{\hat{w}}|_{\{1\} \times M}) - \hat{c}_i(\tilde{\hat{w}}|_{\{0\} \times M}) = a\Big(\int_{[0,1] \times M/M} R(\hat{c}_i(\tilde{\hat{w}}))\Big).$$

Furthermore, by the commutativity of the upper square in (1), we must require

$$R(\hat{c}_i(\tilde{\hat{w}})) = C_i(R(\tilde{\hat{w}})).$$

Therefore we are forced to define

$$\hat{c}_i(\hat{w}) := f^* \hat{z}_{i,k} + a\Big(\int_{[0,1] \times M/M} C_i(R(\tilde{\hat{w}}))\Big). \tag{2}$$

We see that if $\hat{c}_i$ exists, it is automatically unique.

Lemma 2.1. *The definition of $\hat{c}_i(\hat{\omega})$ by* (2) *is independent of the choices of $\tilde{\rho}$, ρ and $f : M \to \mathcal{K}_k$.*

Proof. Let us start with a second choice $\tilde{\rho}'$ and write $\tilde{\hat{\omega}}' := \mathrm{pr}_M^* \hat{\omega} + a(\tilde{\rho}')$. Then we can connect $\tilde{\rho}$ with $\tilde{\rho}'$ by a family of such forms, for example, the linear path. This path can be considered as a form $\bar{\rho}$ on $[0, 1] \times [0, 1] \times M$. By construction $\bar{\rho}|_{[0,1]\times\{j\}\times M}$ is constant and has no component in the direction of the first variable for $j = 0, 1$. This implies that

$$R(\tilde{\hat{\omega}}')|_{[0,1]\times\{j\}\times M} = 0. \tag{3}$$

We set $\bar{\hat{\omega}} := \mathrm{pr}_M^* \hat{\omega} + a(\bar{\rho}) \in \hat{K}^0([0, 1] \times [0, 1] \times M)$. By Stokes' theorem we have

$$d \int_{[0,1]\times[0,1]\times M/M} C_i(R((\bar{\hat{\omega}}))) = \int_{[0,1]\times M/M} C_i(R(\tilde{\hat{\omega}}')) - \int_{[0,1]\times M/M} C_i(R(\tilde{\hat{\omega}}))$$

(these are the contributions of the faces $\{j\} \times [0, 1] \times M$) since the integral over the other two faces $[0, 1] \times \{j\} \times M$ vanishes by (3). Since a annihilates exact forms, this implies that

$$a\Big(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{\omega}}))\Big) = a\Big(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{\omega}}'))\Big).$$

Assume now that we have chosen a different ρ'. Then $a(\rho' - \rho) = 0$ so that by the exactness axiom [Bunke and Schick 2010, (2)] there exists a class $\hat{\upsilon} \in \hat{K}^1(M)$ with $R(\hat{\upsilon}) = \rho' - \rho$. Let $\hat{e} \in \hat{K}^1(S^1)$ be a lift of the generator of $K^1(S^1) \cong \mathbb{Z}$ with $R(\hat{e}) = dt$. We consider the form $\tilde{\sigma} \in \Omega^{\mathrm{odd}}([0, 1] \times M)$ with no dt component given by

$$\tilde{\sigma}|_{\{t\}\times M} := \int_{[0,t]\times M/M} R(\hat{e} \times \hat{\upsilon}),$$

where we identify $S^1 \cong \mathbb{R}/\mathbb{Z}$ and view the interval $[0, t]$ as a subset of S^1. Then

$$\tilde{\sigma}|_{\{0\}\times M} = 0, \quad \tilde{\sigma}|_{\{1\}\times M} = \rho' - \rho, \quad d\tilde{\sigma} = dt \wedge \mathrm{pr}_M^* R(\hat{\upsilon}) = R(\hat{e} \times \hat{\upsilon}).$$

We now consider

$$\tilde{\hat{\upsilon}} := \mathrm{pr}_M^* \hat{\omega} + \mathrm{pr}_M^* a(\rho) + a(\tilde{\sigma}) \in \hat{K}^0([0, 1] \times M)$$

and calculate modulo the image of d

$$\begin{aligned}\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{\upsilon}})) &\equiv \int_{S^1\times M/M} C_i(R(\mathrm{pr}_M^*(\hat{\omega})) + \mathrm{pr}_M^* d\rho + R(\hat{e} \times \hat{\upsilon}))\\ &\equiv \int_{S^1\times M/M} C_i(R(\mathrm{pr}_M^*(\hat{\omega})) + R(\hat{e} \times \hat{\upsilon}))\\ &\equiv \int_{S^1\times M/M} C_i(R(\mathrm{pr}_M^*(\hat{\omega}) + \hat{e} \times \hat{\upsilon})).\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Rham}\Big(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{v}}))\Big) &= \mathrm{Rham}\Big(\int_{S^1\times M/M} C_i(R(\mathrm{pr}_M^*(\hat{w})+\hat{e}\times\hat{v}))\Big)\\ &= \int_{S^1\times M/M} \mathrm{Rham}(C_i(R(\mathrm{pr}_M^*(\hat{w})+\hat{e}\times\hat{v})))\\ &= \int_{S^1\times M/M} \imath_{\mathbb{R}}(c_i(I(\mathrm{pr}_M^*(\hat{w})+\hat{e}\times\hat{v}))).\end{aligned}$$

In other words, $\mathrm{Rham}(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{v}})))$ is an integral class, and this implies

$$a\Big(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{v}}))\Big) = 0$$

by [Bunke and Schick 2010, (2)].

If $\tilde{\rho}$ was the path connecting ρ with 0, then we construct the path $\tilde{\rho}'$ from ρ' to 0 by concatenating $\tilde{\rho}$ with $\tilde{\sigma}$ (we may change $\tilde{\rho}$ to ensure a smooth concatenation). Then we get $\tilde{\hat{w}}' := \mathrm{pr}_M^*\,\hat{w} + a(\tilde{\rho}') \in \hat{K}^0([0,1]\times M)$ and

$$\begin{aligned}a\Big(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{w}}'))\Big) &= a\Big(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{w}}))\Big) + a\Big(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{v}}))\Big)\\ &= a\Big(\int_{[0,1]\times M/M} C_i(R(\tilde{\hat{w}}))\Big).\end{aligned}$$

Thus our construction of c_i is independent of the choice of ρ.

Finally we verify that $\hat{c}_i(\hat{w})$ is independent of the choice of $f : M \to \mathcal{K}_k$. If we replace k by $k+1$ and f by $\kappa_k \circ f$, then we obviously get the same result. For two choices $f : M \to \mathcal{K}_k$ and $f' : M \to \mathcal{K}_{k'}$, there exists $k'' \geq \max\{k, k'\}$ such that $\kappa_k^{k''} \circ f$ and $\kappa_{k'}^{k''} \circ f'$ are homotopic. Here $\kappa_i^j : \mathcal{K}_i \to \mathcal{K}_j$ denotes the composition $\kappa_i^j := \kappa_{j-1} \circ \cdots \circ \kappa_i$ for $j > i$. Therefore it remains to show that a choice $f' : M \to \mathcal{K}_k$ homotopic to $f : M \to \mathcal{K}_k$ gives the same result for $\hat{c}_i(\hat{w})$. Let $H : [0,1]\times M \to \mathcal{K}_k$ be a homotopy from f to f'. Then we use H in the construction of $\hat{c}_i(\mathrm{pr}_M^*\,\hat{w}) \in \widehat{H\mathbb{Z}}^{2i}([0,1]\times M)$. If we let $\hat{c}_i'(\hat{w})$ denote the result of the construction based on the choice of f' we have by the homotopy formula

$$\hat{c}_i'(\hat{w}) - \hat{c}_i'(\hat{w}) = a\Big(\int R(\hat{c}_i(\mathrm{pr}_M^*\,\hat{w}))\Big) = a\Big(\int \mathrm{pr}_M^*\,C_i(\hat{w})\Big) = 0. \qquad \square$$

Lemma 2.2. *The construction of $\hat{c}_i$ defines a natural transformation $\hat{c}_i : \hat{K} \to \widehat{H\mathbb{Z}}^{2i}$ of set-valued functors on smooth manifolds.*

Proof. Let $g : N \to M$ be a smooth map between manifolds. Let $\hat{w} \in \hat{K}^0(M)$ and assume that we have constructed $\hat{c}_i(\hat{w})$ using the choices of $f : M \to \mathcal{K}_k$, $\rho \in \Omega^{\mathrm{odd}}(M)$ and $\tilde{\rho} \in \Omega^{\mathrm{odd}}([0,1]\times M)$. Then we construct $\hat{c}_i(g^*\hat{w})$ using the choices $f\circ g : N\to \mathcal{K}_k,\ \ g^*\rho\in\Omega^{\mathrm{odd}}(N)$ and $(\mathrm{id}\times g)^*\tilde\rho\in\Omega^{\mathrm{odd}}([0,1]\times N)$. With these choices we get $(\mathrm{id}\times g)^*\tilde{\hat\omega}=\widetilde{g^*\hat\omega}\in\hat K^0([0,1]\times N)$ and

$$\begin{aligned}g^*\hat c_i(\hat\omega)&=g^*f^*\hat z_{i,k}+g^*a\Big(\int_{[0,1]\times M/M}C_i(R(\tilde{\hat\omega}))\Big)\\&=(f\circ g)^*\hat z_{i,k}+a\Big(\int_{[0,1]\times M/M}C_i(R((\mathrm{id}\times g)^*\tilde{\hat\omega}))\Big)\\&=(f\circ g)^*\hat z_{i,k}+a\Big(\int_{[0,1]\times M/M}C_i(R(\widetilde{g^*\hat\omega}))\Big)=\hat c_i(g^*\hat\omega).\end{aligned}$$

This finishes the proof of Theorem 1.1(i).

To show the part (ii), we consider the natural transformation

$$\hat B:\hat K^0\times\hat K^0\to\widehat{H\mathbb{Z}}^{\mathrm{ev}}$$

given by

$$\hat B(\hat\omega,\hat\upsilon):=\hat c(\hat\omega)\cup\hat c(\hat\upsilon)-\hat c(\hat\omega+\hat\upsilon)\in\widehat{H\mathbb{Z}}^{\mathrm{ev}}(M)\quad\text{for }\hat\omega,\hat\upsilon\in\hat K^0(M).$$

If we apply I, we get

$$\begin{aligned}I(\hat B(\hat\omega,\hat\upsilon))&=I(\hat c(\hat\omega)\cup\hat c(\hat\upsilon))-I(\hat c(\hat\omega+\hat\upsilon))\\&=I(\hat c(\hat\omega))\cup I(\hat c(\hat\upsilon))-I(\hat c(\hat\omega+\hat\upsilon))\\&=c(I(\hat\omega))\cup c(I(\hat\upsilon))-c(I(\hat\omega)+I(\hat\upsilon))=0.\end{aligned}$$

Let $C=1+C_1+C_2+\cdots\in\mathbb{Q}[[s_0,s_1,\dots]]$. Then we have the identity

$$C(s_0+s_0',s_1+s_1',\dots)=C(s_0,s_1,\dots)C(s_0',s_1',\dots).$$

Indeed, if $\widetilde{\mathbf{ch}}=\sum_{i\geq1}(e^{x_i}-1)$, then $C(\mathbf{ch}_1,\dots)=\prod_{i\geq1}(1+x_i)$. If we introduce another set of variables x_i' and set $\widetilde{\mathbf{ch}}'=\sum_{i\geq1}(e^{x_i'}-1)$, then

$$\begin{aligned}C(\mathbf{ch}_1+\mathbf{ch}_1',\mathbf{ch}_2+\mathbf{ch}_2',\dots)&=\prod_{i\geq1}(1+x_i)(1+x_i')\\&=C(\mathbf{ch}_1,\mathbf{ch}_2,\dots)C(\mathbf{ch}_1',\mathbf{ch}_2',\dots).\end{aligned}$$

We now calculate

$$\begin{aligned}R(\hat B(\hat\omega,\hat\upsilon))&=R(\hat c(\hat\omega)\cup\hat c(\hat\upsilon))-R(\hat c(\hat\omega+\hat\upsilon))\\&=R(\hat c(\hat\omega))\cup R(\hat c(\hat\upsilon))-R(\hat c(\hat\omega+\hat\upsilon))\\&=C(R(\hat\omega))\wedge C(R(\hat\upsilon))-C(R(\hat\omega)+R(\hat\upsilon))=0.\end{aligned}$$

Thus $\hat B$ factorizes over the subfunctor $H\mathbb{R}^{\mathrm{odd}}/H\mathbb{Z}^{\mathrm{odd}}\subset H\mathbb{R}/\mathbb{Z}^{\mathrm{odd}}\subset\widehat{H\mathbb{Z}}^{\mathrm{ev}}$, where the inclusion is induced by a. Let $\rho\in\Omega^{\mathrm{odd}}(M)$ and $\tilde\rho:=t\,\mathrm{pr}_M^*\,\rho\in\Omega^{\mathrm{odd}}([0,1]\times M)$.

Then we have

$$\hat{B}(\hat{w}+a(\rho),\hat{v})-\hat{B}(\hat{w},\hat{v})=\hat{B}(\mathrm{pr}_M^*\,\hat{w}+a(\tilde{\rho}),\hat{v})|_{\{1\}\times M}-\hat{B}(\mathrm{pr}_M^*\,\hat{w}+a(\tilde{\rho}),\hat{v})|_{\{0\}\times M}.$$

Because $\hat{B}$ takes values in the homotopy invariant subfunctor $H\mathbb{R}^{\mathrm{odd}}/H\mathbb{Z}^{\mathrm{odd}}$, we conclude that $\hat{B}(\hat{w}+a(\rho),\hat{v})=\hat{B}(\hat{w},\hat{v})$. Similarly, we see that $\hat{B}(\hat{w},\hat{v}+a(\rho))=\hat{B}(\hat{w},\hat{v})$. Hence $\hat{B}$ has a factorization over a natural transformation

$$K^0\times K^0\to H\mathbb{R}^{\mathrm{odd}}/H\mathbb{Z}^{\mathrm{odd}}\subset H\mathbb{R}/\mathbb{Z}^{\mathrm{odd}}.$$

Such a natural transformation between homotopy invariant functors on manifolds must be represented by a map of classifying spaces

$$\boldsymbol{K}_0\times\boldsymbol{K}_0\to K(\mathbb{R}/\mathbb{Z},\mathrm{odd}),$$

where $K(\mathbb{R}/\mathbb{Z},\mathrm{odd}):=\bigvee_{i\geq 0}K(\mathbb{R}/\mathbb{Z},2i+1)$ is a wedge of Eilenberg–Mac Lane spaces, that is, by a class in $B\in H^{\mathrm{odd}}(\boldsymbol{K}_0\times\boldsymbol{K}_0;\mathbb{R}/\mathbb{Z})$. Since $\boldsymbol{K}_0$ and therefore $\boldsymbol{K}_0\times\boldsymbol{K}_0$ are even spaces, we know that $H_{\mathrm{odd}}(\boldsymbol{K}_0\times\boldsymbol{K}_0;\mathbb{Z})=0$. Then we have $H^{\mathrm{odd}}(\boldsymbol{K}_0\times\boldsymbol{K}_0;\mathbb{R}/\mathbb{Z})\cong\mathrm{Hom}(H_{\mathrm{odd}}(\boldsymbol{K}_0\times\boldsymbol{K}_0;\mathbb{Z}),\mathbb{R}/\mathbb{Z})=0$ by the universal coefficient formula. We see that $B=0$ and therefore $\hat{B}=0$. This finishes the proof of Theorem 1.1(ii). □

Proof of Theorem 1.2. We let $\hat{e}\in K^1(S^1)$ be, as above, the unique element with $R(\hat{e})=dt$, with $I(\hat{e})=e\in K^1(S^1)$ the canonical generator, and with $\hat{e}|_*=0$ for a basepoint $*\in S^1$. Then we define for odd $i\in\mathbb{N}$ and $\hat{x}\in\hat{K}^{-1}(M)$

$$\hat{c}_i^{\mathrm{odd}}(\hat{x}):=\int\hat{c}_{(i+1)/2}(\hat{e}\times\hat{x}).$$

Note that

$$I\Big(\int\hat{c}_{(i+1)/2}(\hat{e}\times\hat{x})\Big)=\int c_{(i+1)/2}(e\times I(\hat{x})).$$

We have a natural inclusion $\widetilde{H\mathbb{Z}}^*(\Sigma M_+)\subset H\mathbb{Z}^*(S^1\times M)$ since the subspace of classes whose restriction to $\{*\}\times M$ vanishes. Since $e|_*=0$, we see that $e\times I(\hat{x})$ belongs to this subspace. The restriction of $\int$ to this subspace coincides with the suspension isomorphism $\widetilde{H\mathbb{Z}}^{*+1}(\Sigma M_+)\xrightarrow{\sim}H\mathbb{Z}^*(M)$, we have $\int(e\times x)=x$ with inverse $x\mapsto e\times x$. Therefore

$$\int c_{(i+1)/2}(e\times I(\hat{x}))=c_i^{\mathrm{odd}}(I(\hat{x})).$$

In this way we get a natural transformation that has the required property.

Since $\int:\hat{K}^0(S^1\times M)\to\hat{K}^{-1}(M)$ is surjective it is clear that $\hat{c}_i^{\mathrm{odd}}$ is unique. □

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