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We prove that a maximal nilpotent subalgebra of a Kac–Moody Lie algebra has an (essentially unique) Euclidean metric whose Laplace operator in the chain complex is scalar on each component of a given degree. Moreover, both the Lie algebra structure and the metric are uniquely determined by this property.

1. Introduction

Let $\mathfrak{g}$ be a real Lie algebra that is either finite-dimensional or has a grading $\mathfrak{g} = \bigoplus_{\boldsymbol{k}\in\mathbb{Z}^n} \mathfrak{g}^{(\boldsymbol{k})}$ such that all the chain spaces

$$C_q^{(\boldsymbol{k})}(\mathfrak{g}) = \bigoplus_{\boldsymbol{k}_1+\cdots+\boldsymbol{k}_q=\boldsymbol{k}} (\mathfrak{g}^{(\boldsymbol{k}_1)} \wedge \cdots \wedge \mathfrak{g}^{(\boldsymbol{k}_q)})$$

are finite-dimensional. (We consider only the case $\mathfrak{g} = \bigoplus_{(k_1,\dots,k_n)\succ(0,\dots,0)} \mathfrak{g}^{(k_1,\dots,k_n)}$ where the notation $(k_1,\dots,k_n) \succ (\ell_1,\dots,\ell_n)$ means that $k_1 \geq \ell_1,\dots,k_n \geq \ell_n$, and $(k_1,\dots,k_n) \neq (\ell_1,\dots,\ell_n)$, and all the spaces $\mathfrak{g}^{(k_1,\dots,k_n)}$ are finite-dimensional.) Suppose that for each value of $\boldsymbol{k}$, some Euclidean structure is fixed for $\mathfrak{g}^{(\boldsymbol{k})}$. Then Euclidean structures arise in all the chain spaces $C_q^{(\boldsymbol{k})}(\mathfrak{g})$, and they give rise to canonical isomorphisms between the chain spaces and the corresponding cochain spaces, $C^q_{(\boldsymbol{k})}(\mathfrak{g}) = (C_q^{(\boldsymbol{k})}(\mathfrak{g}))^*$. Thus, we can regard the boundary and coboundary operators as acting in the same spaces, that is,

$$\partial\colon C_q^{(\boldsymbol{k})}(\mathfrak{g}) \to C_{q-1}^{(\boldsymbol{k})}(\mathfrak{g}) \quad \text{and} \quad \delta\colon C_q^{(\boldsymbol{k})}(\mathfrak{g}) \to C_{q+1}^{(\boldsymbol{k})}(\mathfrak{g}),$$

and to form the *Laplace operators* $\Delta\colon C_q^{(\boldsymbol{k})}(\mathfrak{g}) \to C_q^{(\boldsymbol{k})}(\mathfrak{g})$. Chains (cochains) that are annihilated by Δ are called *harmonic*. The finite-dimensional version of the Hodge–de Rham theory yields the following result.

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Proposition 1. *Every harmonic chain* (*cochain*) *is a cycle* (*cocycle*), *and every homology* (*cohomology*) *class of* $\mathfrak{g}$ *with trivial coefficients is represented by a unique harmonic chain* (*cochain*). *In particular*, *there are canonical isomorphisms*

$$\operatorname{Ker}(\Delta\colon C_q^{(\boldsymbol{k})}(\mathfrak{g}) \to C_q^{(\boldsymbol{k})}(\mathfrak{g})) = H_q^{(\mathbf{k})}(\mathfrak{g}) = H^q_{(\boldsymbol{k})}(\mathfrak{g}).$$

For details, see [Fuchs 1986, Section 1.5.3]

Remark. The results discussed below indicate that not only the kernel but also the whole spectrum of the Laplacian must have significance for the (co)homology. We will return to this matter in subsequent publications.

The first computation of the spectrum of the Laplace operator in the cochain complex of an infinite-dimensional Lie algebra was done by I. M. Gel'fand, B. L. Feĭgin, and D. Fuchs (of this paper) [1978], with a subsequent important correction made by F. V. Weinstein [1993], for a maximal nilpotent subalgebra of the Virasoro algebra; this nilpotent Lie algebra is denoted as $L_1(1)$. It has a basis $\{e_i \mid i > 0\}$ with the commutator operation $[e_i, e_j] = (j-i)e_{i+j}$. We introduce in this algebra a $\mathbb{Z}$-grading and a Euclidean structure, letting $\deg e_i = i$ and $\|e_i\| = 1$. For positive integers $i_1, \dots, i_q$ such that $i_r - i_{r-1} \geq 3$ for $r = 2, \dots, q$, let

$$E(i_1, \dots, i_q) = \sum_{s=1}^{q} \binom{i_s}{3} - \sum_{1 \leq \ell < m \leq q} i_\ell i_m,$$

$$\alpha_r(i_1, \dots, i_q) = \begin{cases} 0 & \text{if } r = 1 \text{ and } i_1 < 3, \\ 1 & \text{if } r = 1 \text{ and } i_1 \geq 3, \\ 0 & \text{if } 1 < r \leq q \text{ and } i_r - i_{r-1} = 3, \\ 1 & \text{if } 1 < r \leq q \text{ and } i_r - i_{r-1} > 3, \end{cases}$$

$$\alpha(i_1, \dots, i_q) = \sum_{r=1}^{q} \alpha_r(i_1, \dots, i_q).$$

It is easy to check that $E(1, 4, 7, \dots, 3q-2) = E(2, 5, 8, \dots, 3q-1) = 0$, and all other values of the function E are positive.

Theorem 2 [Gel'fand et al. 1978; Weinstein 1993]. *The set of eigenvalues of the Laplace operator* $\Delta\colon C_*(L_1(1)) \to C_*(L_1(1))$ *coincides with the set of numbers* $E(i_1, \dots, i_q)$. *The multiplicity of the eigenvalue* $E(i_1, \dots, i_q)$ *equals* $2^{\alpha(i_1, \dots, i_q)}$. (*Occasional coincidences* $E(i_1, \dots, i_q) = E(i'_1, \dots, i'_{q'})$ *are possible*; *in such cases the multiplicities are added.*)

For a sketch of a proof see [Fuchs 1986, Section 2.3.1(B)].

S. Kumar [1984], using ideas from [Kostant 1963], calculated the spectrum of the Laplacian for a maximal nilpotent subalgebra of the Kac–Moody Lie algebra (see also the related works [Feĭgin 1980] and [Lepowsky 1979]). He noted that actually the Laplace operator is scalar in every homogeneous component with respect to the canonical $\mathbb{Z}^r$-grading (where r is the rank of the algebra), and derived a simple formula relating the eigenvalue to the degree.

Here, we develop a different approach to the calculating the Laplace spectrum. In so doing, we will not only obtain a fairly elementary proof of the Kumar formulas, but in addition prove that the relation between the Laplace eigenvalues and degrees uniquely determines both the Lie algebra structure and the Euclidean structure (which may be regarded as an alternative description of the class of Kac–Moody Lie algebras).

We supply below all the necessary definitions; for the general theory of Kac–Moody Lie algebras, see [Kac 1990].

Let $A = \|a_{ij}\|$ be an $n \times n$ matrix with all the diagonal entries equal to 2 and all nondiagonal entries being nonpositive integers. We assume the matrix A is *symmetrizable*, which means that there exists a diagonal matrix D whose diagonal entries $d_1, \dots, d_n$ are positive integers such that the matrix DA is symmetric. We may also assume the matrix A is *irreducible*, which means that there is no partition of $\{1, \dots, n\}$ into nonempty parts I, J such that $a_{ij} = 0$ for all $i \in I$ and $j \in J$. Let $G = G(A)$ be the (real) Kac–Moody Lie algebra with Cartan matrix A, and let $N = N(A)$ be the corresponding nilpotent Lie algebra. In other words, N has a system of generators $e_1, \dots, e_n$ with the defining set of relations $(\operatorname{ad} e_i)^{-a_{ij}+1} e_j = 0$. The algebra N has a natural $\mathbb{Z}^n$-grading $N = \bigoplus_{(k_1,\dots,k_n)\succ(0,\dots,0)} N^{(k_1,\dots,k_n)}$ where $N^{(k_1,\dots,k_n)}$ consists of linear combinations of commutator monomials of the generators involving precisely k_i letters e_i for $i = 1, \dots, n$. The following statement, proved in Section 2, is our main result; compare with [Kumar 1984, Theorem 2.1].

Theorem 3. *There exist unique Euclidean structures in the spaces $N^{(k_1,\dots,k_n)}$ such that $\|e_i\| = 1$ for $i = 1, \dots, n$ and such that the corresponding Laplace operator $\Delta \colon C_*^{(k_1,\dots,k_n)}(N) \to C_*^{(k_1,\dots,k_n)}(N)$ is the multiplication by*

$$E(k_1, \dots, k_n) = \sum_i d_i k_i - \frac{1}{2} \sum_{i,j} d_i a_{ij} k_i k_j.$$

Moreover, if a $\mathbb{Z}^r$-graded Lie algebra is generated by r generators of degrees

$$(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$$

and the Laplace operator in its cochain complex with respect to some Euclidean structures in the homogeneous components is described by the formulas above, then there exists an isometric isomorphism between this Lie algebra and the nilpotent Kac–Moody Lie algebra N.

Proposition 4. *If $E(k) \neq 0$, then $H_*^{(k)}(N(A)) = 0$.*

This follows from Proposition 1 and Theorem 3.

Proposition 4 is not new: it is essentially contained in [Kac and Kazhdan 1979], which yields a description of a Bernstein–Gel'fand–Gel'fand resolution of the trivial module over a Kac–Moody Lie algebra. This is also a free resolution of the trivial module over $N(A)$.

2. Proof of Theorem 3

2.1. ***The Laplace operator has order 2.*** In the standard calculus, a linear operator $D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a differential operator of order 1 (that is, $D(f) = af' + bf$ where a and b are functions), if the identity

$$D(fg) = D(f)g + D(g)f - D(1)fg$$

holds for any functions f and g. Similarly, an operator of order 2 is characterized by the identity

$$\begin{aligned} &D(fgh) \\ &\quad = D(fg)h + D(fh)g + D(hg)f - D(f)gh - D(g)fh - D(h)fg + D(1)fgh \end{aligned}$$

(and so on, but we do not need operators of orders greater than 2). It is well known that the commutator of operators of order p and q has order $p+q-1$.

In the noncommutative (supercommutative) case of chains/cochains of a Lie algebra (with a Euclidean structure), the notion of a differential order looks slightly different. In particular, the operator $\delta\colon C_*(\mathfrak{g}) \to C_*(\mathfrak{g})$ has order 1, meaning

$$\delta(c_1 \wedge c_2) = \delta(c_1) \wedge c_2 + (-1)^{d_1 d_2} \delta(c_2) \wedge c_1 \quad \text{for } c_i \in C_{d_i}(\mathfrak{g}).$$

However, the operator $\partial\colon C_*(\mathfrak{g}) \to C_*(\mathfrak{g})$ has order 2, which means that

$$\begin{aligned} &\partial(c_1 \wedge c_2 \wedge c_3) \\ &\quad = \partial(c_1 \wedge c_2) \wedge c_3 + (-1)^{d_2 d_3} \partial(c_1 \wedge c_3) \wedge c_2 + (-1)^{d_1(d_2+d_3)} \partial(c_2 \wedge c_3) \wedge c_1 \\ &\qquad - \partial(c_1) \wedge c_2 \wedge c_3 - (-1)^{d_1 d_2} \partial(c_2) \wedge c_1 \wedge c_3 - (-1)^{(d_1+d_2)d_3} \partial(c_3) \wedge c_1 \wedge c_2 \end{aligned}$$

for $c_i \in C_{d_i}(\mathfrak{g})$. Since the Laplace operator is a (super)commutator of ∂ and δ, it also has order 2, and we have the following lemma.

Lemma 5. *The Laplace operator* $\Delta\colon C_*(\mathfrak{g}) \to C_*(\mathfrak{g})$ *has order* 2, *that is*,

$$\begin{aligned} &\Delta(c_1 \wedge c_2 \wedge c_3) \\ &\quad = \Delta(c_1 \wedge c_2) \wedge c_3 + (-1)^{d_2 d_3} \Delta(c_1 \wedge c_3) \wedge c_2 + (-1)^{d_1(d_2+d_3)} \Delta(c_2 \wedge c_3) \wedge c_1 \\ &\qquad - \Delta(c_1) \wedge c_2 \wedge c_3 - (-1)^{d_1 d_2} \Delta(c_2) \wedge c_1 \wedge c_3 - (-1)^{(d_1+d_2)d_3} \Delta(c_3) \wedge c_1 \wedge c_2 \end{aligned}$$

for all $c_i \in C_{d_i}(\mathfrak{g})$.

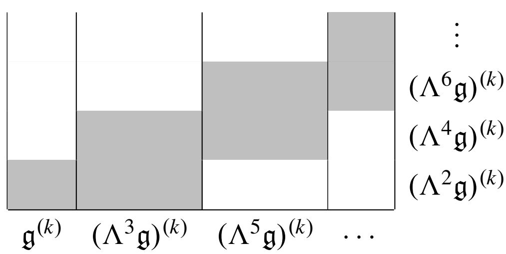
Remark. It is important that Lemma 5 is compatible with Theorem 3 in the sense that if $\mathfrak{g} = N = N(A)$ and $c_i \in C_{d_i}^{(p_i)}(N)$, where $(p_i) = (p_{i1}, \dots, p_{in})$, then every term in the equality of Lemma 5 is $c_1 \wedge c_2 \wedge c_3$ times an approrpiate eigenvalue of Δ, and the equality becomes

$$E(p_1+p_2+p_3) = E(p_1+p_2)+E(p_1+p_3)+E(p_2+p_3)-E(p_1)-E(p_2)-E(p_3),$$

which is true (because E is a polynomial of degree 2).

2.2. ***Construction of a Lie algebra with a given Laplace operator.*** Let A, a_{ij}, D and d_i be the same as in Section 1. We will now construct a graded Lie algebra $\mathfrak{g} = \bigoplus_{(k_1,\dots,k_n)\succ(0,\dots,0)} \mathfrak{g}^{(k_1,\dots,k_n)}$ with Euclidean structures in the summed-over (finite-dimensional) spaces satisfying the conclusion of Theorem 3 (with N replaced by $\mathfrak{g}$). We will see that $\mathfrak{g}$ is unique up to an isometric isomorphism, provided that $\dim \mathfrak{g}^{(1,0,\dots,0)} = \dim \mathfrak{g}^{(0,1,0,\dots,0)} = \cdots = \dim \mathfrak{g}^{(0,\dots,0,1)} = 1$. (Later on, we will see that $\mathfrak{g}$=$N(A)$.)

First consider a given graded Lie algebra $\mathfrak{g}$ with Euclidean structures in $\mathfrak{g}^{(k_1,\dots,k_n)}$. Choose an orthonormal basis in each $\mathfrak{g}^{(k)}$, where $(k) = (k_1, \dots, k_n)$; then wedge products of the elements of the bases in $\mathfrak{g}^{(k)}$ form orthonormal bases in the chain spaces $(\Lambda^q\mathfrak{g})^{(k)}$. For a fixed $(k) \succ (0, \dots, 0)$, consider the matrix with rows (columns) labeled by the elements of our orthonormal bases in $(\Lambda^q\mathfrak{g})^{(k)}$ with q even (odd). Let the shaded blocks represent the boundary/coboundary operators in the chain/cochain complexes of $\mathfrak{g}$, and let the unshaded blocks be zero. To illustrate:

$D_{(k)} =$ 

Take two rows or two columns of the matrix $D_{(k)}$ corresponding to basis elements $c \in (\Lambda^q\mathfrak{g})^{(k)}$ for $c' \in (\Lambda^{q'}\mathfrak{g})^{(k)}$ (so q and q' have the same parity) and compute their dot product. If $|q'-q| > 2$, then this dot product is obviously zero. If $|q'-q| = 2$, it is also zero because of the relations $\partial\circ\partial = 0$ and $\delta\circ\delta = 0$. Finally, if $q' = q$, then this dot product is the coefficient at c' in $\Delta(c)$ (and the coefficient at c in $\Delta(c')$).

If the Laplace operator $\Delta : C_*^{(k)} \to C_*^{(k)}$ is multiplication by a positive number λ, then the dot product of every two different rows, as well as of every two different columns, is equal to zero, and the dot-square of every row or column is equal to λ; in other words, the whole matrix $D_{(k)}$ is an orthogonal matrix times $\sqrt{\lambda}$.

We are ready for the construction announced in the beginning of the section. We use the induction with respect to $|k| = k_1 + \cdots + k_n$. We put $\dim \mathfrak{g}^{(0,\dots,0,1,0,\dots,0)} = 1$

and choose (arbitrarily) nonzero vectors $e_1 \in \mathfrak{g}^{(1,0,\dots,0)}, \dots, e_n \in \mathfrak{g}^{(0,\dots,0,1)}$ to have length 1. Take a $(k) = (k_1, \dots, k_n)$ where the k_i are nonnegative integers such that $k_1 + \dots + k_n > 1$. If $E(k) \le 0$, we put $\mathfrak{g}^{(k)} = 0$; let $E(k) > 0$. By the induction hypothesis, the matrix $D_{(k)}$ from above is fully determined, except the bottom left (shaded) block. Away from this block, the dot product of every two distinct rows or columns is zero, and the dot-square of every row or column is equal to $E(k)$. This follows from the identities $\partial\circ\partial = 0$, $\delta\circ\delta = 0$ and also from Lemma 5 and the remark after it, which implies that the Laplace operator $\Delta \colon C_q^{(k)} \to C_q^{(k)}$ with $q \ge 3$ (fully determined) is multiplication by $E(k)$. Thus, the columns of our matrix disjoint from the bottom left box are pairwise orthogonal and have dot-squares $E(k)$. We can construct the missing columns making the whole matrix an orthogonal matrix times $\sqrt{E(k)}$. Since the dot-squares of the rows above the bottom left block are already equal to $E(k)$, the new columns will be confined to this block. Thus, we will have a $\mathfrak{g}^{(k)}$ (with $\dim \mathfrak{g}^{(k)} = \sum_{q \ge 2,\, \text{even}} \dim(\Lambda^q \mathfrak{g})^{(k)} - \sum_{q \ge 3,\, \text{odd}} \dim(\Lambda^q \mathfrak{g})^{(k)}$) with a ready orthonormal basis, and the new box yields a bracket $[\,\cdot\,,\cdot\,] \colon (\Lambda^2 \mathfrak{g})^{(k)} \to \mathfrak{g}^{(k)}$. Moreover, the orthogonality of the columns of the new box to the columns of the box next to the right means precisely that this bracket satisfies the Jacobi identity. (Notice that it could happen that $\sum_{q \ge 2,\, \text{even}} \dim(\Lambda^q \mathfrak{g})^{(k)} = \sum_{q \ge 3,\, \text{odd}} \dim(\Lambda^q \mathfrak{g})^{(k)}$; in this case we do not need any new columns and may simply put $\mathfrak{g}^{(k)} = 0$.)

This completes the construction promised in the beginning of the section; the uniqueness is obvious.

End of the proof. It remains to prove that the Lie algebra $\mathfrak{g}$ of Section 2.2 is $N(A)$. This follows from three remarks.

First, it follows from the construction of Section 2.2 that if $(k_1, \dots, k_n) \succ (0, \dots, 0)$ and $k_1 + \dots + k_n \ge 1$, then the bracket mapping $[\,\cdot\,,\cdot\,] \colon (\Lambda^2 \mathfrak{g})^{(k)} \to \mathfrak{g}^{(k)}$ is onto; hence, $\mathfrak{g}$ (like $N(A)$) is generated by $e_1, \dots, e_n$.

Second, the defining relations $(\operatorname{ad} e_i)^{-a_{ij}+1} e_j = 0$ hold. Indeed, the degree (k) of $(\operatorname{ad} e_i)^{-a_{ij}+1} e_j$ is described by the equalities $k_i = -a_{ij} + 1$, $k_j = 1$ and $k_s = 0$ for $s \ne i, j$. Hence,

$$\begin{aligned} E(k) &= \sum d_i k_i - \tfrac{1}{2} \sum a_{ij} k_i k_j \\ &= d_i(-a_{ij} + 1) + d_j - d_i(-a_{ij} + 1)^2 - d_j - d_i a_{ij}(a_{ij} + 1) \\ &= -d_i a_{ij} + d_i + d_j - d_i a_{ij}^2 + 2 d_i a_{ij} - d_i - d_j + d_i a_{ij}^2 - d_i a_{ij} = 0. \end{aligned}$$

By construction, this means that $\mathfrak{g}^{(k)} = 0$; hence $(\operatorname{ad} e_i)^{-a_{ij}+1} e_j = 0$. Thus, there is a graded epimorphism $N(A) \to \mathfrak{g}$.

Third, it is true that $\dim \mathfrak{g}^{(k)} = \dim N(A)^{(k)}$ for all (k). Indeed, for any (k) with $E(k) \ne 0$, the dimensions $\dim \mathfrak{g}^{(k)}$ are determined inductively from the relation $\sum (-1)^q \dim(\Lambda^q \mathfrak{g})^{(k)} = 0$. A similar relation, $\sum (-1)^q \dim(\Lambda^q N(A))^{(k)} = 0$ (for the same values of (k)), follows from Proposition 4 and the Euler–Poincaré lemma.

In addition to that, $\dim \mathfrak{g}^{(k)} = \dim N(A)^{(k)} = 1$ if $(k) = (0, \dots, 0, 1, 0, \dots, 0)$, and $\mathfrak{g}^{(k)} = N(A)^{(k)} = 0$ if $(k) = (k_1, \dots, k_n) \succ (0)$ with $k_1 + \dots + k_n > 1$, and $E(k) \le 0$. Hence, our epimorphism $N(A) \to \mathfrak{g}$ is actually an isomorphism.

3. Conclusion

3.1. ***Canonical basis in $N(A)$.*** The construction of Section 2.2 shows that a maximal nilpotent subalgebra of a Kac–Moody Lie algebra has a canonical Euclidean metric. The metric depends on the choice of generators of length 1, but the commutator relations do not depend on anything. For example, a maximal nilpotent subalgebra of the rank 2 exceptional Lie algebra G_2 has dimension 6. The Cartan matrix is $A = \left[\begin{smallmatrix} 2 & -1 \\ -3 & 2 \end{smallmatrix}\right]$. There is a basis $\{e_{0,1}, e_{1,0}, e_{1,1}, e_{1,2}, e_{1,3}, e_{2,3}\}$ in $N(A)$ with $\deg e_{i,j} = (i, j)$ and the commutation relations

$$[e_{0,1}, e_{1,0}] = \sqrt{3} e_{1,1}, \quad [e_{0,1}, e_{1,1}] = 2e_{1,2}, \qquad [e_{0,1}, e_{1,2}] = \sqrt{3} e_{1,3},$$
$$[e_{1,0}, e_{1,3}] = \sqrt{3} e_{2,3}, \quad [e_{1,1}, e_{1,2}] = \sqrt{3} e_{2,3}.$$

If we regard this basis as orthonormal, then the Laplace operator in $C_*^{(p,q)}$ is the multiplication by $3p + q - 3p^2 - q^2 + 3pq$.

A more interesting example is provided by the twisted affine Kac–Moody Lie algebra $A_2^{(2)}$ with Cartan matrix $A = \left[\begin{smallmatrix} 2 & -1 \\ -4 & 2 \end{smallmatrix}\right]$. This Lie algebra (after factoring over the one-dimensional center) is embedded into the current Lie algebra $\mathfrak{sl}(3) \otimes \mathbb{R}[t, t^{-1}]$. It is well known that it has a basis e_i such that $[e_i, e_j] = \alpha_{ij} e_{i+j}$, where the numbers α_{ij} depend only on i, j mod 8 (see [Kac 1990, Exercise 8.16]).

The basis given in [Kac 1990] is not precisely our canonical basis; to get ours, we need to modify it by some coefficients:

$$e_{8s} = \sqrt{2} \begin{bmatrix} t^{2s} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -t^{2s} \end{bmatrix}, \qquad e_{8s+1} = 2 \begin{bmatrix} 0 & t^{2s} & 0 \\ 0 & 0 & t^{2s} \\ 0 & 0 & 0 \end{bmatrix},$$
$$e_{8s+2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^{2s+1} & 0 & 0 \end{bmatrix}, \qquad e_{8s+3} = \begin{bmatrix} 0 & 0 & 0 \\ t^{2s+1} & 0 & 0 \\ 0 & -t^{2s+1} & 0 \end{bmatrix},$$
$$e_{8s+4} = \sqrt{\frac{2}{3}} \begin{bmatrix} t^{2s+1} & 0 & 0 \\ 0 & -2t^{2s+1} & 0 \\ 0 & 0 & t^{2s+1} \end{bmatrix}, \quad e_{8s+5} = 2 \begin{bmatrix} 0 & t^{2s+1} & 0 \\ 0 & 0 & -t^{2s+1} \\ 0 & 0 & 0 \end{bmatrix},$$
$$e_{8s+6} = 4 \begin{bmatrix} 0 & 0 & t^{2s+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_{8s+7} = \begin{bmatrix} 0 & 0 & 0 \\ t^{2s+2} & 0 & 0 \\ 0 & t^{2s+2} & 0 \end{bmatrix}.$$

The commutation relations of this basis are given by the formula $[e_{8s+i}, e_{8s'+j}] = \alpha_{ij} e_{8(s+s')+(i+j)}$ for $1 \le i \le 8$ and $1 \le j \le 8$, with the 8×8 matrix $\|\alpha_{ij}\|$ being

$$\begin{bmatrix} 0 & 2 & \sqrt{6} & -\sqrt{6} & -2 & 0 & \sqrt{2} & -\sqrt{2} \\ -2 & 0 & 0 & 0 & 2 & -\sqrt{8} & 0 & \sqrt{8} \\ -\sqrt{6} & 0 & 0 & \sqrt{6} & -\sqrt{2} & 2 & -2 & \sqrt{2} \\ \sqrt{6} & 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 2 & -2 & \sqrt{2} & -\sqrt{6} & 0 & 0 & \sqrt{6} & -\sqrt{2} \\ 0 & \sqrt{8} & -2 & 0 & 0 & 0 & 2 & -\sqrt{8} \\ -\sqrt{2} & 0 & 2 & \sqrt{6} & -\sqrt{6} & -2 & 0 & \sqrt{2} \\ \sqrt{2} & -\sqrt{8} & -\sqrt{2} & 0 & \sqrt{2} & \sqrt{8} & -\sqrt{2} & 0 \end{bmatrix}.$$

The natural grading of the Lie algebra $A_2^{(2)}$ is given by the rule that if $-1 \le s \le 6$, then

$$\deg e_{8n+s} = \begin{cases} (4n+s, 2n) & \text{if } s \le 1, \\ (4n+s-2, 2n+1) & \text{if } s > 1. \end{cases}$$

The Laplace operator $\Delta\colon C_*^{(p,q)} \to C_*^{(p,q)}$ with respect to the metric determined by the basis $\{e_i, i > 0\}$ is the multiplication by $4p + q - 4p^2 - q^2 + 4pq$.

3.2. ***Some remarks on the multiplicative structure in $H^*(N(A))$.*** It follows from our results (and actually can be proved directly) that there is a basis in $H^*(N(A))$ represented by uniquely chosen monomial cochains (that is, products of elements of the basis in $C^1(N(A)) = N(A)^*$ dual to our canonical basis). This gives rise to a description of the multiplication in $H^*(N(A))$, which, however, is not very explicit. Let us begin with a couple of simple remarks.

First, it follows from the description above that the multiplication in $H^*(N(A))$ is "square-free": the square of any cohomology class is zero.

Second, every monomial cochain representing a nonzero element of $H^*(N(A))$ should contain at least one factor from $C^1_{(0,\dots,0,1,0,\dots,0)}(N(A))$; this implies that the cohomological length of $H^*(N(A))$ does not exceed the rank of $G(A)$, that is, the size of A.

Third, in the finite-dimensional case, the multiplication in $H^*(N(A))$ satisfies the Poincaré duality: If a nonzero element $\alpha \in H^q(N(A))$ is represented by a monomial cochain $c_{i_1} \dots c_{i_q}$, then the complimentary monomial $c_{j_1} \dots c_{j_r}$, where $q+r = d = \dim N(A)$, also represents a nonzero cohomology class $\beta \in H^r(N(A))$, and $\alpha\beta$ is a nonzero element in $H^d(N(A)) \cong \mathbb{R}$. It follows from the preceding remark that in the finite-dimensional case of rank 2 there are no other nonzero products. (It seems likely that in the infinite-dimensional case of rank 2, the multiplication in $H^*(N(A))$ is trivial.)

Now, let us consider some examples. Let $N(A) = \mathfrak{n}(n)$ be the Lie algebra of (strictly) upper triangular $n \times n$ matrices, associated to the Cartan matrix

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

For this Lie algebra, $\dim \mathfrak{n}(n) = \frac{1}{2}n(n-1)$ and $\dim H^*(\mathfrak{n}(n)) = n!$. The basis in $H^*(\mathfrak{n}(n+1))$ is parametrized by the integral points of the ellipsoid $x_1^2 + \cdots + x_n^2 = x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_1 + \cdots + x_n$, or, still better, by the elements of the Weyl group S_{n+1} whose action on the ellipsoid above is generated by the reflections $s_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, -x_i + x_{i-1} + x_{i+1} + 1, x_{i+1}, \dots, x_n)$ for $i = 1, \dots, n$ (in this formula, x_0 and x_{n+1} are taken to be zero). If $(p_1, \dots, p_n) = \sigma(0, \dots, 0)$ for $\sigma \in S_{n+1}$, then the corresponding cohomology class γ_σ is in $H^\ell(\mathfrak{n}(n+1))$, where ℓ is the length of σ. In this case, $(p_1, \dots, p_n)$ has a unique presentation as the sum $\sum_{s=1}^q \{i_s, j_s\}$ of different points of the form

$$\{i, j\} = (0, \dots, 0, \underset{(i)}{1}, \dots, 1, \underset{(j)}{0}, \dots, 0) \quad \text{for } 1 \le i < j \le n+1$$

and the class γ_σ is represented by the monomial cochain $c_{i_1,j_1} \cdots c_{i_q,j_q}$, where $c_{i,j}$ takes the value 1 on the one-entry matrix $E_{i,j}$ and takes the value 0 on all other such matrices. Moreover, if the presentations $\sigma(0, \dots, 0) = \sum\{i_s, j_s\}$, $\sigma'(0, \dots, 0) = \sum\{i'_t, j'_t\}$ are disjoint and $\sum\{i_s, j_s\} + \sum\{i'_t, j'_t\} = \tau(0, \dots, 0)$, then $\gamma_\sigma\gamma_{\sigma'} = \gamma_\tau$; in all other cases, $\gamma_\sigma\gamma_{\sigma'} = 0$.

For example, there are 6 permutations in S_3: $\sigma_1 = (1, 2, 3)$, $\sigma_2 = (2, 1, 3)$, $\sigma_3 = (1, 3, 2)$, $\sigma_4 = (2, 3, 1)$, $\sigma_5 = (3, 1, 2)$ and $\sigma_6 = (3, 2, 1)$. Accordingly, there are 6 integral points on the ellipse $x^2 + y^2 - x - y - xy = 0$, given by

$$\begin{aligned} &\sigma_1(0,0) = (0,0), \quad &&\sigma_4(0,0) = (1,2) = (0,1) + (1,1), \\ &\sigma_2(0,0) = (1,0), \quad &&\sigma_5(0,0) = (2,1) = (1,0) + (1,1), \\ &\sigma_3(0,0) = (0,1), \quad &&\sigma_6(0,0) = (2,2) = (1,0) + (0,1) + (1,1), \end{aligned}$$

the cohomology of $\mathfrak{n}(3)$ is spanned by

$$\begin{aligned} \gamma_{\sigma_1} = 1 \in H^0(\mathfrak{n}(3)), &\qquad \gamma_{\sigma_2}, \gamma_{\sigma_3} \in H^1(\mathfrak{n}(3)), \\ \gamma_{\sigma_6} \in H^3(\mathfrak{n}(3)), &\qquad \gamma_{\sigma_4}, \gamma_{\sigma_5} \in H^2(\mathfrak{n}(3)), \end{aligned}$$

$\gamma_{\sigma_2}\gamma_{\sigma_4} = -\gamma_{\sigma_3}\gamma_{\sigma_5} = \gamma_{\sigma_6}$, and all other products of cohomology classes of positive dimensions are zero. Similarly for $\mathfrak{n}(4)$ (we write $\sigma_{(ijkl)}$ for the permutation (i, j, k, l)):

$$\sigma_{(1234)}(0,0,0) = (0,0,0), \quad \sigma_{(2134)}(0,0,0) = (1,0,0), \quad \sigma_{(1324)}(0,0,0) = (0,1,0),$$
$$\sigma_{(1243)}(0,0,0) = (0,0,1), \qquad \sigma_{(3214)}(0,0,0) = (1,0,0)+(0,1,0)+(1,1,0),$$
$$\sigma_{(2314)}(0,0,0) = (0,1,0)+(1,1,0), \quad \sigma_{(2341)}(0,0,0) = (0,0,1)+(0,1,1)+(1,1,1),$$
$$\sigma_{(3124)}(0,0,0) = (1,0,0)+(1,1,0), \quad \sigma_{(3142)}(0,0,0) = (1,0,0)+(0,0,1)+(1,1,1),$$
$$\sigma_{(2143)}(0,0,0) = (1,0,0)+(0,0,1), \quad \sigma_{(2413)}(0,0,0) = (0,1,0)+(1,1,0)+(0,1,1),$$
$$\sigma_{(1342)}(0,0,0) = (0,0,1)+(0,1,1), \quad \sigma_{(4123)}(0,0,0) = (1,0,0)+(1,1,0)+(1,1,1),$$
$$\sigma_{(1423)}(0,0,0) = (0,1,0)+(0,1,1), \quad \sigma_{(1432)}(0,0,0) = (0,1,0)+(0,0,1)+(0,1,1),$$
$$\sigma_{(3241)}(0,0,0) = (1,0,0)+(0,0,1)+(0,1,1)+(1,1,1),$$
$$\sigma_{(2431)}(0,0,0) = (0,1,0)+(0,0,1)+(0,1,1)+(1,1,1),$$
$$\sigma_{(3412)}(0,0,0) = (0,1,0)+(1,1,0)+(0,1,1)+(1,1,1),$$
$$\sigma_{(4213)}(0,0,0) = (1,0,0)+(0,1,0)+(1,1,0)+(1,1,1),$$
$$\sigma_{(4132)}(0,0,0) = (1,0,0)+(0,0,1)+(1,1,0)+(1,1,1),$$
$$\sigma_{(3421)}(0,0,0) = (0,1,0)+(0,0,1)+(1,1,0)+(0,1,1)+(1,1,1),$$
$$\sigma_{(4231)}(0,0,0) = (1,0,0)+(0,0,1)+(1,1,0)+(0,1,1)+(1,1,1),$$
$$\sigma_{(4312)}(0,0,0) = (1,0,0)+(0,1,0)+(1,1,0)+(0,1,1)+(1,1,1),$$
$$\sigma_{(4321)}(0,0,0) = (1,0,0)+(0,1,0)+(0,0,1)+(1,1,0)+(0,1,1)+(1,1,1).$$

The cohomology classes of the corresponding monomial cochains form a basis in the cohomology:

$$\gamma_{(1234)} = 1 \in H^0(\mathfrak{n}(4)),$$
$$\gamma_{(2134)}, \gamma_{(1324)}, \gamma_{(1243)} \in H^1(\mathfrak{n}(4)),$$
$$\gamma_{(2314)}, \gamma_{(3124)}, \gamma_{(2143)}, \gamma_{(1342)}, \gamma_{(1423)} \in H^2(\mathfrak{n}(4)),$$
$$\gamma_{(3214)}, \gamma_{(2341)}, \gamma_{(3142)}, \gamma_{(2413)}, \gamma_{(4123)}, \gamma_{(1432)} \in H^3(\mathfrak{n}(4)),$$
$$\gamma_{(3241)}, \gamma_{(2413)}, \gamma_{(3412)}, \gamma_{(4213)}, \gamma_{(4132)} \in H^4(\mathfrak{n}(4)),$$
$$\gamma_{(3421)}, \gamma_{(4231)}, \gamma_{(4312)} \in H^5(\mathfrak{n}(4)),$$
$$\gamma_{(4321)} \in H^6(\mathfrak{n}(4)).$$

The multiplication is described by the following relations:

$$\gamma_{(2134)}\gamma_{(1243)} = \gamma_{(2143)};$$
$$\gamma_{(2134)}\gamma_{(2314)} = -\gamma_{(1324)}\gamma_{(3124)} = \gamma_{(3214)},$$

$$\gamma_{(1324)}\gamma_{(1342)} = -\gamma_{(1243)}\gamma_{(1423)} = \gamma_{(1432)};$$
$$\gamma_{(2134)}\gamma_{(2341)} = \gamma_{(3241)},$$
$$\gamma_{(1324)}\gamma_{(2341)} = \gamma_{(2431)},$$
$$\gamma_{(1324)}\gamma_{(4123)} = -\gamma_{(4213)},$$
$$\gamma_{(1243)}\gamma_{(4123)} = -\gamma_{(4132)};$$
$$\gamma_{(1243)}\gamma_{(3412)} = \gamma_{(2314)}\gamma_{(2341)} = -\gamma_{(3421)},$$
$$-\gamma_{(3124)}\gamma_{(2341)} = \gamma_{(1342)}\gamma_{(4123)} = \gamma_{(4231)},$$
$$\gamma_{(2134)}\gamma_{(3412)} = \gamma_{(1423)}\gamma_{(4123)} = \gamma_{(4312)};$$
$$\begin{aligned} -\gamma_{(2134)}\gamma_{(1243)}\gamma_{(3412)} &= -\gamma_{(2134)}\gamma_{(2314)}\gamma_{(2341)} = \gamma_{(1324)}\gamma_{(3124)}\gamma_{(2341)} \\ &= \gamma_{(1324)}\gamma_{(1342)}\gamma_{(4123)} = -\gamma_{(1243)}\gamma_{(1423)}\gamma_{(4123)} \\ &= \gamma_{(3142)}\gamma_{(2413)} = \gamma_{(4321)}. \end{aligned}$$

Although the procedure always determines the multiplication in $H^*(N(A))$, it does not give a satisfactory explicit description even of the ring $H^*(\mathfrak{n}(n))$, for reasons unclear to us.

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